* Probability Introduced Through Set Theory :-

• Set Theory :-

– A Set in a collection of objects called elements.

A Set $A = \{\xi_1, \xi_2 \ldots \xi_n\}$

means that the Set A consist of elements $\xi_1, \xi_2, \ldots, \xi_n$

– A Subset B of a Set A is another set whose elements are also elements of A

– The notation $\xi_i \in A$, means the ξ_i is an element of A.

$\xi_i \notin A$, means that ξ_i is not an element of A.

– The empty or the null set is the set that contains no element. $\emptyset$ or $\{\}$

– The largest set that contain all elements in denoted by S and it called the universal set

– For any universal set with N elements, that are 2^N possible Subset of S.

$S = \{1, 2, 3\}$, $2^3 = 8$ Subsets

$S_1 = \{1\}$ $\quad S_6 = \{2, 3\}$

$S_2 = \{2\}$ $\quad S_7 = \{1, 2, 3\}$

$S_3 = \{3\}$ $\quad S_8 = \emptyset$

$S_4 = \{1, 2\}$

$S_5 = \{1, 3\}$

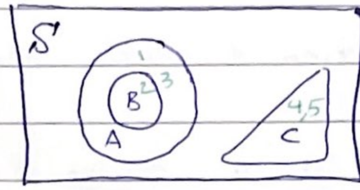
$S = \{H, T\}$

1	2	3	
0	0	0	$\emptyset$
0	0	1	$\{3\}$
0	1	0	$\{2\}$
0	1	1	$\{2, 3\}$
1	0	0	$\{1\}$
1	0	1	
1	1	0	
1	1	1	

* Set operation :

$S = \{1, 2, 3, 4, 5\}$

- Venn Diagram



- Equallity and Differences

- Two sets A & B are equal if all elements in A are presented in B and all elements in B are present in A;

$A \subseteq B$ and $B \subseteq A$, $A = B$

- The difference of Two Sets A and B denoted by A−B (A except B) in the set containing all elements of A that are not present in B.

  - Notice that $A - B \neq B - A$

- Ex : $A = \{0.6 < a \leq 1.6\}$ and $B = \{1.0 \leq b \leq 2.5\}$

Find $C = A - B$ and $D = B - A$

without the inclution

$C = \{0.6 < c < 1.0\}$

$D = \{1.6 < d \leq 2.5$

* Union and Intersection

- Union $C = A \cup B$

all elements of A or B or both

- Intersection $D = A \cap B$

all elements common to both A and B

If $A \cap B = \emptyset$ (disjoint or mutually exclusive)

- $C = A_1 \cup A_2 \cup A_3 \cdots \cup A_N = \bigcup_{n=1}^{N} A_N$

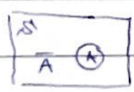
- $A_1 + A_2 + \cdots + A_N \quad \cancel{\prod_{i=1}^{N} A_i} \quad \sum_{i=1}^{N} A_i$

- $A_1 \cdot A_2 \cdot A_3 \cdots A_N \quad \prod_{i=1}^{N} A_i$

* Complement :-

- The complement of A set A, denoted by $\bar{A}$ is the set of all elements in S not in A

$\bar{A} = S - A \qquad \bar{\emptyset} = S \qquad \bar{S} = \emptyset$



$A \cup \bar{A} = S \qquad A \cap \bar{A} = \emptyset$

* ~~Dexxor~~ Morgan's law :

$\overline{A\cup B} = \bar{A}\cap\bar{B}$

$\overline{A\cup B\cup C} = \bar{A}\cap\bar{B}\cap\bar{C}$

$\overline{A\cap B} = \bar{A}\cup\bar{B}$

$\overline{A\cap B\cap C} = \bar{A}\cup\bar{B}\cup\bar{C}$

* Duality Principle :-

- Its states that if in an identity we replace Union by intersection, intersection by union S by $\emptyset$, and $\emptyset$ by S the identity is presented.

Ex:

$A\cup\bar{A} = S$ $\qquad$ $A\cap\bar{A} = \emptyset$

$S = \{H, T\}$

$S = \{f_1, f_2, \ldots, f_6\}$

Event: outcome of an experiment

Sets	Probability
elements	elementary events
A Subset	outcome of an exp.
universal set S	Sample Space S
The empty set	The imposible event

* Definitions of probability :-

1. Axiomatic Definition

- Axiom 1, The probability of an event A, P(A), is a nonnegative number assigned to this event

$P(A) \geq 0$

- Axiom 2, The prob. of the certain event equals 1

$P(S) = 1$

$0 \leq P(A) \leq 1$

- Axiom 3, If the events A & B are mutually exclusive then,

$$P(A \cup B) = P(A) + P(B)$$

if they are not mut. ex
we should subtract
the shared element

ex.

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 3\}$

$B = \{3, 4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5\}$

$A \cap B = \{3\}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{5}{6}$$

- Axiom 3, If the events A & B are mutually exclusive then,

$$P(A \cup B) = P(A) + P(B)$$

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ex.

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 3\}$

$B = \{3, 4, 5\}$

$\neq$

$A \cup B = \{1, 2, 3, 4, 5\}$

$A \cap B = \{3\}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{5}{6}$$

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2. Relative Frequency Defenition

- The probability P(A) of an event A in the limit

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

, where n_A is the number of occurences of A and n is the number of trails

- used in Computer Simulations

3. Classical Defenition

- The probability of an event A is determined a priori without actual experimentation. Its given by

$$P(A) = \frac{N_A}{N}$$

, where N is the number of all possible outcomes and N_A is the outcomes that faverable to the event A

Ex. We roll two dice and we want to find the probability p that the sum of number that show equal 7. P(Sum = 7)

Sol.:

$S = \{2, 3, 4, 5, 6, 7, \dots, 12\}$ → [illegible] النردين   ~~$P(sum = 7) = \frac{1}{11}$~~ X

die 2 \ die 1	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	-	-	-	(2,5)	-
3	(3,1)	(3,2)	(3,3)	(3,4)		
4	-	-	(4,3)	-	-	-
5	-	(5,2)	-	-	-	
6	(6,1)	-	-	-		

$$P(Sum = 7) = \frac{6}{36} = \frac{1}{6} + \frac{1}{6} + \dots + \frac{1}{6} = \frac{1}{6}$$

* Joint and Conditional Probability

- The probability $P(A \cap B)$ is called the joint probability for the two events A and B

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$\rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If A and B are mutually execlusive,

$A \cap B = \emptyset$ and $P(\emptyset) = 0$

$P(A \cup B) = P(A) + P(B)$

* Conditional Probability

- Given some event B with noneZero probability $P(B) > 0$, The conditional probability of an event ~~A&B~~ A, given B

given

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$ (Prob. of A given B)

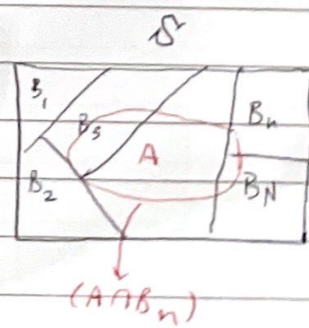
A لما تكون صار B

- If A and B are mutually exclusive $A \cap B = \emptyset$

$$P(A \cap B) = 0 \Rightarrow P(A/B) = 0$$

* Total Probability

- we are given N mutually execlusive events B_n, $n = 1, 2, \dots, N$ whose union is S



$B_m \cap B_n = \emptyset \quad m \neq n = 1, 2, \dots N$

and $\bigcup_{n=1}^{N} B_n = S$

Then

$$P(A) = \sum_{n=1}^{N} P(A/B_n)\,P(B_n)$$, this is known as the total prob. of event A.

• Since $A \cap S = A$:

$$A \cap S = A \cap \left(\bigcup_{n=1}^{N} B_n \right)$$

$$A = \bigcup_{n=1}^{N} (A \cap B_n)$$

$$P(A) = P\left(\bigcup_{n=1}^{N} (A \cap B_n) \right) = \sum_{n=1}^{N} (A \cap B_n)$$

* Bayes Theorem

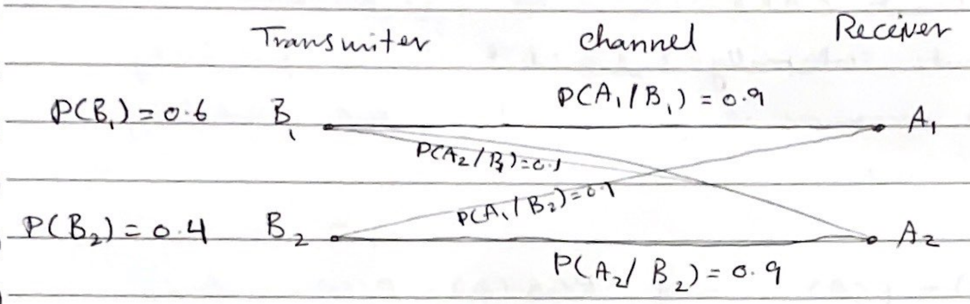
$$P(B_n/A) = \frac{P(B_n \cap A)}{P(A)}$$

$$P(A/B_n) = \frac{P(A \cap B_n)}{P(B_n)}$$

$$P(B_n/A) = \frac{P(A/B_n)\,P(B_n)}{P(A)}$$

$$= \frac{P(A/B_n)\,P(B_n)}{P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \cdots + P(A/B_N)P(B_N}$$

Ex. An elementary binary communication system



- Find $P(A_1)$, $P(A_2)$, $P(B_1/A_1)$, $P(B_2/A_2)$
  $P(B_1/A_2)$ and $P(B_2/A_1)$

$$P(A_1) = P(A_1/B_1)P(B_1) + P(A/B_2)P(B_2)$$
$$= (0.9)(0.6) + (0.1) + (0.4)$$
$$= 0.58$$

$$P(A_2) = P(A_2/B_2)P(B_2)$$
$$= (0.1)(0.6) + (0.9)(0.4) = 0.42$$

$$P(B_1/A_1) = \frac{P(A_1/B_1)P(B_1)}{P(A_1)} = \frac{(0.9)(0.6)}{0.58} = \frac{0.54}{0.58} = 0.931$$

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* Independent Events :-

- Two ~~ind~~ Events (A & B)

• we call events statistically Independent if the probability of ~~simultaneous~~ occurence of one event is not affected by the ~~oth~~ other events.

$$P(A/B) = P(A) \quad \text{and} \quad P(B/A) = P(B)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\boxed{P(A \cap B) = P(A)\,P(B)}$$ check

- Multiple events :-
  - A_1, A_2, and A_3 to be independent events :-

$$P(A_1 \cap A_2) = P(A_1)\,P(A_2)$$
$$P(A_2 \cap A_3) = P(A_2)\,P(A_3)$$
$$P(A_1 \cap A_3) = P(A_1)\,P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$$

• here we have to check if every pair is ind to have this final equation!

- IF N events, $A_1, A_2, \dots, A_N$ are independents then any one of ~~those~~ them is independent of one event formed by unions, intersections and complements of the others.

- for two events that are independent (A_1 and A_2). It results that A_1 is independant of $\bar{A}_2$, $\bar{A}_1$ is independend of A_2 and $\bar{A}_1$ is independet of $\bar{A}_2$

- For three independent events A_1, A_2 and A_3

$$P(A_1 \cap (A_2 \cap A_3)) = P(A_1)\,P(A_2)\,P(A_3)$$
$$= P(A_1)\,P(A_2 \cap A_3)$$

$$P(A_1 \cap (A_2 \cup A_3)) = P(A_1)\,P(A_2 \cup A_3)$$

* Combined Experiments :-

- Combined Sample Space

$S = S_1 \times S_2$ — S_1: Sample space of exp. 1, S_2: sample space of exp 2.

- Ex: If $S_1 = \{H, T\}$

$S_2 = \{1, 2, 3, 4, 5, 6\}$

$$S = S_1 \times S_2$$
$$= \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

$P(H) = \frac{1}{2}$

$P(1) = \frac{2}{12} = \frac{1}{6}$

$P(H,1) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$ ← independent events (joint)

[إذا جا الثاني disjoint ما صير]

* Permutation and Combinations:-

- ordering of r elements taken from n

$$n(n-1)\cdots\underset{}{\overset{(n-(r-1))}{(n-r+1)}} = \frac{n!}{(n-r)!} = P_r^n$$

$r = 1, 2, 3, \ldots, n$

- The order is important!

$\{A, B, C\}$ Two different Combines
$\{B, A, C\}$

no. of choices is $3 \times 2 \times 1$

• لاحظت هون بالترتيب (A,B,C) ~~....~~ مهم كان A الأولى مهمة و B مهمة و C كذلك الأمر
إذا هاي permutation

$$\binom{n}{r} = \frac{P_r^n}{P_r^r} = C_r^n$$

$$= \frac{n!}{(n-r)!\,r!}$$

• أما هون ~~..~~ ما فرقت ABC
عن BAC ... لأن كلهم بنفس
المجموعة إذا هي Combination

- The number $\binom{n}{r}$ is called binomial ~~coe~~ coefficients because ~~$(x+y)^n$~~

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$(x+x)^3 = \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y + \binom{3}{3} x^3 y^0$$

$$\frac{3!}{3!\,6!} = 1 \qquad \frac{3!}{2!\,1!} = 3 \qquad \frac{3!}{1!\,2!} = 3 \qquad \frac{3!}{0!\,3!} = 1$$

$$= y^3 + 3xy^3 + 3x^2y + x^3$$

- $0! = 1$
- $\binom{n}{0} = 1$
- $\binom{n}{n} = 1$
- $\binom{n}{r} = \binom{n}{n-r}$
- $\frac{n!}{(n-r)!\,r!} = \frac{n!}{r!\,(n-r)!}$

* Benoulli Trials :-

- Special type of experiments where :-

1) A and $\bar{A}$ are the only elementary events

$P(A) = P \leftarrow$ proba. ل شرط تحقيق يعني

$P(\bar{A}) = q = (1-P)$

2) The basic experiment is repeated N times

$P(A)$ is observed exactly K times out of N trials

(Bernuli trial)

$$\underbrace{\overset{P(A)\cdot P(A)\cdot P(A)}{A, A, A, \ldots} \overset{P(A)}{A}}_{K \text{ terms}}, \underbrace{\overset{P(\bar{A})\cdot P(\bar{A})}{\bar{A}, \bar{A}}, \ldots, \overset{P(\bar{A})}{\bar{A}}}_{(N-K) \text{ terms}} \quad (\text{one scenario})$$

$P^K (1-P)^{N-K}$ $\binom{N}{K}$

$K = 2, N = 3$

H, H, T
T, H, H
H, T, H

$\binom{3}{2} = \frac{3!}{2!\,3!} = 3 \rightarrow$ 3 sanarios

$$\left[\binom{N}{K} P^K (1-P)^{N-K} \right]$$

if the 2 hits or more must be ~~atleast~~ directly after each other (not Bernolli)

Ex.: A Submarite attempts to sink an aircraft carrier. It will be successful only if two or more torpedos hit the carier. If the Submarine fire three torpedoes and the probability of hit is 0.4 for each torpedo. What is the probability that the carier will be sunk?

H, $\bar{H}$

0 hit $\{\bar{H}, \bar{H}, \bar{H}$

1 hit 3 times $\begin{cases} H, \bar{H}, \bar{H} \\ \bar{H}, H, \bar{H} \\ \bar{H}, \bar{H}, H \end{cases}$

2 hits 3 time $\begin{cases} \bar{H}, H, H \\ H, \bar{H}, H \\ H, H, \bar{H} \end{cases}$

3 hits $\{H, H, H$

Define $A = \{\text{torpedo hits}\}$

$P(A) = 0.4$

$N = 3$

$P(\text{exactly no hit}) = \binom{3}{0}(0.4)^0(0.6)^3$

$\frac{3!}{(3-0)!\,0!} = 1$

$(1)(1)(0.6)^3 = 0.216$

$$P(\text{exactly one hit}) = \binom{3}{1}(0.4)^1(0.6)^2$$

$$= \frac{3!}{(3-1)!\,1!}(0.4)(0.6)^2 = 0.432$$

$$P(\text{exactly 2 hits}) = \binom{3}{2}(0.4)^2(0.6)^1$$

$$= (3)(0.4)^2(0.6) = 0.288$$

$$P(\text{exactly 3 hits}) = \binom{3}{3}(0.4)^3(0.6)^0$$

$$= (1)(0.4)^3 = 0.064$$

$$P\{\text{Carrier will be Sunk}\} = P\{\text{two or more hit}\} = 1 - P\{\text{less than } 2\}$$

if I want the sum for all prob. $= P\{\text{exactly 2 hits}\} + P\{\text{exactly 3 hits}\}$

$$= 0.288 + 0.64 = 0.352$$

$$(p+(1-p))^n = \sum_{r=0}^{n} \binom{n}{r} p^r (1-p)^{n-r}$$

$$= \sum_{r=0}^{3} \binom{3}{r}(0.4)^r(0.6)^{n-r} = \boxed{1}$$

Ex. An airline in a small city has five departs each day. It is known that any given flight has a probability of 0.3 of departing late. For any given day find the probability that:

a) no flight departs late b) all flights depart late

c) three or more depart on time

a) A: departing late $P(A) = 0.3$

$$P(0 \text{ late}) = \binom{5}{0}(0.3)^0(0.7)^5 = 0.16807$$

b)

$$P(\text{all late}) = \binom{5}{5}(0.3)^5(0.7)^0 = 0.00243$$

c) $P\{3 \text{ or more on time}\} = P\{0 \text{ late}\} + P\{1 \text{ late}\} + P\{2 \text{ late}\}$

$$= 0.16807 + \binom{5}{1}(0.3)(0.7)^4 + \binom{5}{2}(0.3)^2(0.7)^3$$

$$= 0.83692$$

• (Other way to the solution)

A = on time $P(A) = 0.7$

$$P(3 \text{ or more on time}) = \binom{5}{3}(0.7)^3(0.3)^2 + \binom{5}{4}(0.7)^4(0.3)^1 + \binom{5}{5}(0.7)^5(0.3)^0$$

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* The Concept of A Random Variable :-

- A random variable is a number $X(S)$ assigned to every outcome S of an experiment.

Ex :

a) In a die experiment, we assign to the six outcomes f_i, $i = 1, 2, \ldots, 6$ the number $X(f_i) = 10i$

$X(f_1) = 10$, $X(f_2) = 20$, $\ldots$, $X(f_6) = 60$

b) Toss two coins

(T,T), (T,H), (H,T), (H,H)

- The Random variable X is in the no. of Heads shown in the experiment

$X(T,T) = 0$ $\quad$ $X(T,H) = 1$

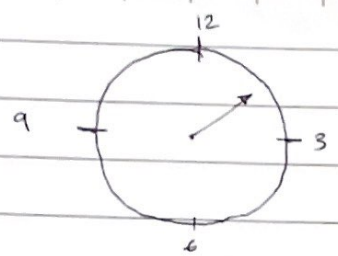
$X(H,T) = 1$ $\quad$ $X(H,H) = 2$

$p(X=0) = \frac{1}{4}$ $\quad$ $p(X=2) = \frac{1}{4}$

$p(X=1) = \frac{2}{4} = \frac{1}{2}$

c) In the experiment of the die, we assign the number 1 to every even outcome and the number zero to every odd outcome, Then

Discrete R.V. $\left\{ \begin{array}{l} X(f_1) = X(f_3) = X(f_5) = 0, \quad X(f_2) = X(f_4) = X(f_6) = 1 \\ p(X=0) = \frac{1}{2} \qquad p(X=1) = \frac{1}{2} \end{array} \right.$



$S = \{0 < S \leq 12\}$

$X(S) = S^2$

$X = \{0 < X \leq 144\}$ Continous R.V

* Conditions for a function to a Random variable :-
  - To be a function not multivalued
  - The Set $\{X \leq x\}$ Shall be an event for every real number x
    R.V ↗ ↖ real number.
  - $P\{X = -\infty\} = 0$ and $P\{X = \infty\} = 0$ not the Same as $P\{X \leq \infty\} = 1$

Ex : Continous curve

- Temperature T is a Countinous R.V, over the range $-60\,F^\circ$ to $+120\,F$

$T = \{-60 \leq t \leq 120\}$ assume that all values are equiprobable

$$P\{0 < t < 50\} = \frac{50 - 0}{120 - (-60)} = \frac{5\cancel{0}}{18\cancel{0}}$$

$$P\{dt\} = \frac{dt}{[120 - (-60)]} = \frac{dt}{180}$$

as $dt \to 0$ $\quad P\{dt\} = 0$
$dt \to 0$