mathmatical background:-

* Set theory:-

A set: is a collection of objects which are the elements of a set (uniqe objects)

ex: $S_1 = \{1, 2, 3, 4, 5, 6\}$ → rolling a die
ex: $S_2 = \{H, T\}$ → Tossing a coin

* let $S = \{x_1, x_2 \dots x_n\}$

Then $x \in S$ →
- x is in S
- x is an element in S
- x belongs to S

* for any x_j, $j > n$
  Then $x_j \notin S$
  ↳ x_j is not in S

* Definitions:-

1. Empty (null) set ($\varnothing$): is a set that contains no elements → or { }

2. universal set (Ω): is a set that contains all the probabilities of the experament ← omega

ex: Ω for rolling a ~~dice~~ die:
$S_{dice} = \{1, 2, 3, 4, 5, 6\}$

* Classification of sets :-

1. finite set : Number of elements
is finite → الها عدد

2. Countably infinite set :

infinite number of elemets BUT countable
→ can be enumarated in a list

ex: $S_{\text{even ints}} = \{ \dots -2, 0, 2 \dots \}$

3. uncountably infinite set : infinite number
of elements that can not be enumerated
in a list

ex: $S = \{ x \in S , \ 17 x 7 0 \}$

* Set operations :-

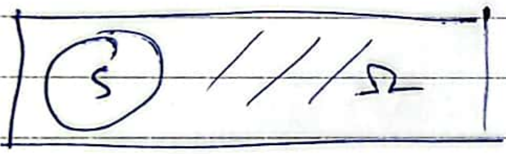
1. The complement of a set :

it is all the elements in the universal set but not in the set S

* universal set must be defiened *

$\bar{S} = \Omega - S$

$\bar{S} = \{ x \in \Omega \text{ and } x \notin S \}$



$S = \{2, 4\}$
$\Omega = \{1, 2, 3, 4\}$
$\bar{S} = \{1, 3\}$

2. union of sets :

↳ The union of two sets S_1, S_2 is the set of all elements that belongs to S_1 or S_2

↳ each element in S_1 or in S_2 belongs to the union set $S_1 \cup S_2$

$S_1 = \{1, 2, 3, 4\}$, $S_2 = \{2, 6, 4, 7\}$

$S_1 \cup S_2 = \{1, 2, 3, 4, 6, 7\} = \{ x \mid x \in S_1 \text{ or } x \in S_2 \}$

→

*Continue → Set operations

3. the $\boxed{\text{intersection}}$ of set S_1, S_2 is the set that contains all the elements that belongs to S_1 and S_2

$S_1 = \{1, 2, 3, 4\}$ $\quad S_2 = \{2, 6, 4, 7\}$

$S_1 \cap S_2 = \{x \mid x \in S_1 \text{ and } x \in S_2\} = \{2, 4\}$

4. Disjoint sets:

Two sets are said To be disjoint if and only if the intersection ~~of~~ is $\emptyset$ → null set

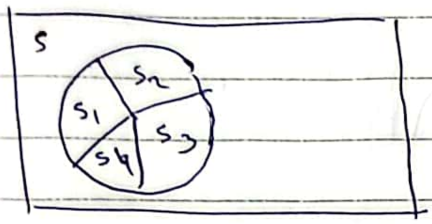
$S_1 \cap S_2 = \emptyset = \{\}$

S and T are disjoint
U and R are not disjoint
$U \cap R \neq \emptyset$

*in general → $S_1, S_2, S_3 \ldots S_n$ are disjoint if

$S_i \cap S_j = \emptyset \quad i \neq j, \; i = 1, \ldots n$
$\qquad\qquad\qquad\qquad j = 1, \ldots n$

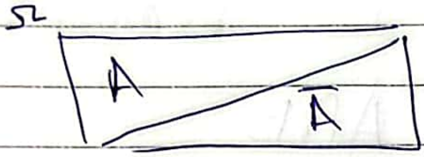
* **Partition:** a Collections of sets are called to be a partition if the sets are disjoint and Thier union is the set S



$S_1 \cap S_2 \cap S_3 \cap S_4 = \emptyset \rightarrow$ disjoint

$S_1 \cup S_2 \cup S_3 \cup S_4 = S$

Then S_1, S_2, S_3, S_4 form a Partition of S



$A \cap \bar{A} \rightarrow \emptyset$

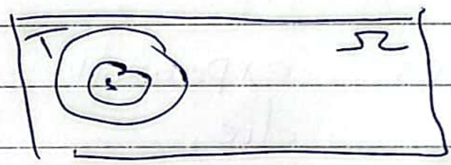
$A \cup \bar{A} \rightarrow \Omega$

$A, \bar{A}$ form ~~are~~ a partition of the universal set Ω

* **Subsets:** A set S is a subset of The set T if every element in S is in T
Then $S \rightarrow$ subset of T

$S \subset T$

subset annotation



properties

1. equality property: $\rightarrow$ $S \subset T$ and $T \subset S$

2. Transitivity property:

if $S \subset T$ and $T \subset V$ then $S \subset V$

* Algebra of sets

1) $S \cup T = T \cup S$

2) $S \cap (T \cup A) = (S \cap T) \cup (S \cap A)$

3) $\bar{\bar{S}} = S \rightarrow$ double complement

4) $S \cap \Omega = S$

5) $S \cap \bar{S} = \emptyset$

6) $S \cup \Omega = \Omega$

AB, $A \times B \rightarrow A \cap B$

7) $(S \cup T) \cup A = S \cup (T \cup A) = S \cup T \cup A$

8) $S \cup (T \cap A) = (S \cup T) \cap (S \cup A)$

* examples: experament of rolling 4 sided die,

$A = \{2, 4\} \leftarrow$ A $\rightarrow$ the set of all even outcomes

$B = \{1, 2\}$ B $\rightarrow$ // // // // outcomes That are less that 3

$A \cap B$ or $A \times B$ or $AB = \{2\}$

$A \cup B = \{1, 2, 4\}$

$A \cap \bar{B} = \{4\} \rightarrow A - B$ → هي نفسها

$B \cap \bar{A} = B - A = \{1\}$

~~B~~ demargans law:

$\boxed{1}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$ → proof:

suppose that $x \in \overline{A \cup B}$

Then $x \notin A \cup B$

↳ $x \notin A$ and $x \in B$

↳ $x \in \bar{A}$ or $x \in \bar{B}$

↳ $x \in \bar{A} \cap \bar{B}$

$\boxed{2}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$

↳ proof: H.w

* bonus in first *
whatis mathmatical induction ~~to~~ difinition

* generalization

⇒ In ~~generat~~ general

$\boxed{1}$ $\overline{(A_1 \cap A_2 \cap A_3 \dots A_n)} = \bar{A}_1 \cup \bar{A}_2 \cup \dots A_n$

$$\overline{\left(\bigcap_{i=1}^{\infty} A_i\right)} = \bigcup_{i=1}^{\infty} \bar{A}_1$$

$\boxed{2}$ $\overline{\left(\bigcup_{i=1}^{\infty} A_i\right)} = \bigcap_{i=1}^{\infty} \bar{A}_i$

prob

النموذج الاحتمالي

* The ~~Probatistic~~ probabalistic model → PM

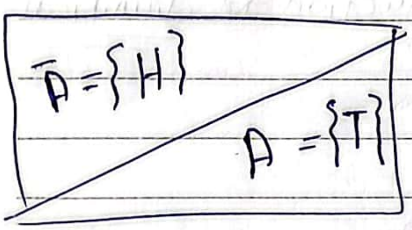
is the mathmatical representation of an experement.

↳ elements of PM:

1) The sample space → is Ω for aspecific experament

2) Probability law → it ~~assistours~~ assigns a nonnegative value (probabilities) To The lirelihood of an event

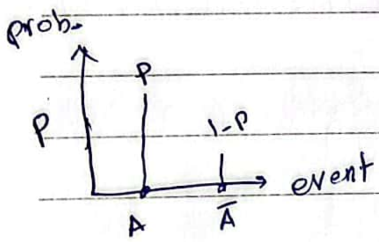
ex: Tossing the coin experament:



1] $\Omega=\{H,T\}$ → sample space

2] we need $P(A)$ and $P(\bar{A})$
$\Rightarrow P(\{H\})$ and $P(\{T\})$

assign an nonnegative value ← suppose that $P(A)=P$ $1 \geq P > 0$
Then $P(\bar{A})=1-P$



prob.

تلاتة ←

Probability Axioms

1] non negativity: probability of an event A $(P(A) \geq 0)$

2] Normalization: $P(\Omega) = 1 \rightarrow 100\%$

3] if A_1 and A_2 are disjoint (mutually exclusive)

then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

$A_1, A_2 \ldots A_n$ are disjoint then

$P(\bigcup_{i=1}^{n}) = \sum_{i=1}^{n} P(A_i)$ → the upper bound for the probability

*examples: rolling a fiar 4-sided die:

$\Omega = \{1,2,3,4\} \Rightarrow$ using Axioms ~~num~~ #2:

let $A_i = \{i\}$, $i = 1,2,3,4$

A_1, A_2, A_3, A_4 are partitions of Ω

then $\Omega = A_1 \cup A_2 \cup A_3 \cup A_4$

$P(\Omega) = P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1$

$P(A_1) + P(A_2) + P(A_3) + P(A_4) = 1$

it is a fair die

then $P + P + P + P = 4P = 1$

$P = \frac{1}{4}$ for each element

$B = \{\text{odd outputs}\}$ of $A = \{1, 3\}$

$P(B) \rightarrow ??$

if $A_1 = \{1, 3\}$ and $A_2 \{2, 4\}$

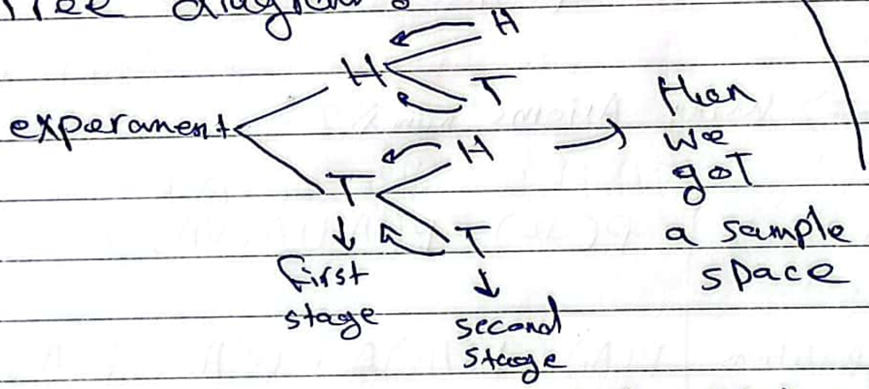
$P(A_1 \cup A_2) = 1 = P(A_1) + P(A_2)$ and it is fair

Then $2P = 1$ $\quad P(B) = \frac{1}{2}$

ex: Tossing a fair ~~coin~~ coin two times:

$\Omega = \{HH, HT, TH, TT\}$

Tree diagram:



if it is fair then $P(H) = P(T) = \frac{1}{2}$

$\Rightarrow P(HH) = P(TH) = P(TT) \rightarrow$ disjoint events $= \frac{1}{4}$

P_B(of getting only one head) $\rightarrow B = \{HT, TH\}$

$= P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

also $P(\{HT, TH\}) + P(\{TT, HH\}) = P(\Omega) = 1$

and it is fair so $2P = 1 \rightarrow P = \frac{1}{2}$

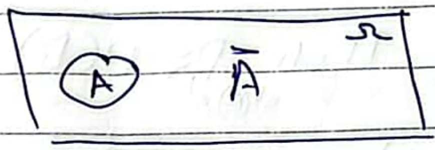
axioms 2 and 3

توابع/تبعات

prob

* Consequenses of Axioms: → probability نبني على ال Axioms

[1] $P(A) \leq 1$



$\Omega = A \cup \bar{A}$

$P(\Omega) = 1$

$P(A \cup \bar{A}) = P(A) + P(\bar{A}) = P(\Omega) = 1$ axiom *3

we have $P(A) = 1 - P(\bar{A})$ → axiom 1 Non negative then the upper bound is 1 to get a non negative value

[2] $P(\emptyset) = 0$

$\Omega = \Omega \cup \emptyset$

$\cancel{P(\Omega)} = \cancel{P(\Omega)} + P(\emptyset)$

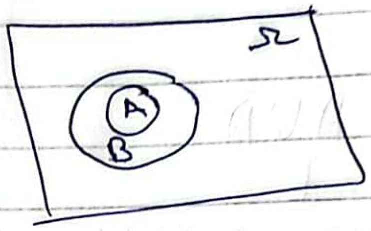
$P(\emptyset) = \text{zero}$

prob

additional probabilities laws:

subset

1. if $A \subset B$, then $P_{(A)} \leq P(B)$



$B = \underbrace{A \cup (B \cap \bar{A})}_{\text{disjoint}}$

$P(B) = \underbrace{P(A)}_{\text{less or equal}} + \underbrace{P(B \cap \bar{A})}_{\geq \text{zero}}$ // axiom #3

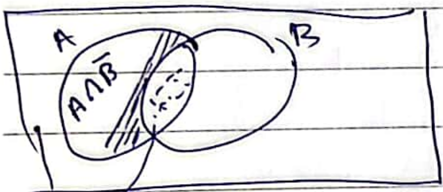
if $P(B \cap \bar{A}) = \text{zero}$

Then $P(B) \geq P(A)$

general formula without conditions

2. * for any 2 events A and B,

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$



* axiom 3

then $A = (A \cap \bar{B}) \dot{\cup} (A \cap B)$

$P(A) = P(A \cap \bar{B}) + P(A \cap B)$

Then $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

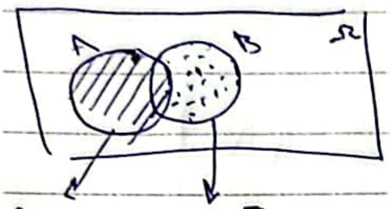
Note

$P\left(\text{any disjoint events}\right) = 0$

* try to apply the special case above on the general case

* $P(A\cap B) \rightarrow$ if A is subset of B then = $P(A)$

$\boxed{3}$ $P(A\cup B) = P(A) + P(B) - P(A\cap B)$



A $\quad$ $B\cap\bar{A}$

هي نفسها $(B-A)$

$\therefore A\cup B = A\cup(B\cap\bar{A})$ $\quad$ النقطة الي قبلها

↳ $P(A\cup B) = P(A) + P(B\cap\bar{A})$

↳ $= P(A) + P(B) - P(A\cap B)$

* $P(A\cup B) \leq P(A) + P(B)$ $\quad$ متى بكونوا متساويات؟

the equality holds if A and B are disjoint

← اي ايفينتس

* for any events A_1, A_2, A_3

↳ $P(A_1\cup A_2\cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1\cap A_2) - P(A_1\cap A_3) - P(A_2\cap A_3) + P(A_1\cap A_2\cap A_3)$

proof: $P(A_1\cup A_2\cup A_3) = P(A_1\cup(A_2\cup A_3))$ $\quad$ نقطة $\boxed{3}$

$P(A_1\cup A_2\cup A_3) \leq \sum_{i=1}^{3} P(A_i) \rightarrow$ can be generalized

↳ $P(A_1\cup A_2\cup A_3) \leq P(A_1) + P(A_2) + P(A_3) + P(A_1\cap A_2\cap A_3)$

لإنه بالقانون فوق هاد قيمة قليلة $\quad$ لا هو موجب

واقل من $P(\cap$ of any 2 events$)$

* Discreat probability laws

Let $\Omega = \{s_1, s_2, \dots s_n\}$

and s is a subset of $\Omega = \{s_1, s_2 \dots s_k\}$ $k \leq n$

$P(S) = P(\{s_1, s_2 \dots, s_k\})$

↳ $P(\{s_1\} \cup \{s_2\} \dots s_k)$

↳ $P(\{s_1\}) + P(\{s_2\}) + \dots P(\{s_k\})$

↳ $\sum_{i=1}^{k} P(\{s_i\})$

$\boxed{\text{this role for non uniform}}$

→ uniform

* if all elements are equally likdy, then

$P(\{s_i\}) = \frac{1}{n}$ $\quad i=1, \quad i = 1, 2, 3 \dots n$

$P(S) = \sum_{i=1}^{k} \frac{1}{n} = \frac{k}{n}$ → probability of s in the sample space Ω

* generalization:-

$$\frac{\text{num of elements of } A}{\text{num of elements of } \Omega} = P(A) = \frac{|A|}{|\Omega|}$$

$\boxed{|A|}$ → this symbol refores to the cardinality of a set (number of elements in the set)

prob

Ex: Tossing a ~~cointt~~ fair coin 3 Times

where $A = \{2 \text{ heads } \xrightarrow{\text{exactly}} \text{ showed up}\}$

$\Omega = \{\underbrace{HHH}_{\substack{\text{3 heads} \\ \times}}, \underbrace{HHT}_{\substack{\checkmark \\ \text{2 heads}}}, \underbrace{HTH}_{\checkmark}, \underbrace{HTT}_{\times}, \underbrace{THH}_{\checkmark}, \underbrace{THT}_{\times}, \underbrace{TTH}_{\times}, \underbrace{TTT}_{\times}\}$

all outcomes are equally likely
→ fair coin

$A\{HHT, HTH, THH\}$

$P(A) = \frac{|A|}{|\Omega|} = \frac{3}{8}$

Ex: rolling a pair of fair 4-sided ~~dice~~ die Twice

$$\Omega = \begin{Bmatrix} (1,1), (1,2), \dots (1,4) \\ (2,1), (2,2), \dots (2,4) \\ \vdots \\ (4,1), (4,2), \dots (4,4) \end{Bmatrix} \to 4^2 \to 16 \text{ elemet}$$

→ using Tree diagram

$|\Omega| = 16 \to P((i,j)) = \frac{1}{16}$ where $i, j = 1, 2, \dots 4$

* for complex Ω then counting roles are much ~~easear~~ ~~easyer~~ better

continue →

Five Apple

Contine for two example

1] $P($ sum of two rolls is eaven $)$

$P(\underbrace{\text{sum of two rolls is eaven}}_{\text{event } A})$

$= \frac{|A|}{|\Omega|} = \frac{8}{16} = \frac{1}{2}$

2] $P(\underbrace{\text{sum of two rolls is odd}}_{\text{event } \bar{A}})$

$= 1 - P(A) = \frac{1}{2}$

3] $P(\underbrace{\text{sum of the rolls if the tow pairs are the same}}_{\text{event } B})$

$\frac{|B|}{|\Omega|} = \frac{4}{16} = \frac{1}{4}$

4] $P(\text{first roll} > \text{second roll}) = \frac{6}{16} = \frac{3}{8}$

5] $P(\underbrace{\text{at least one of the rolls is 4}}_{\text{event } D})$

$= \frac{7}{16}$ → $A_1 \rightarrow$ first is 4
$A_2 \rightarrow$ second is 4

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$
$= \frac{1}{4} + \frac{1}{4} - \frac{1}{16} = \frac{7}{16}$

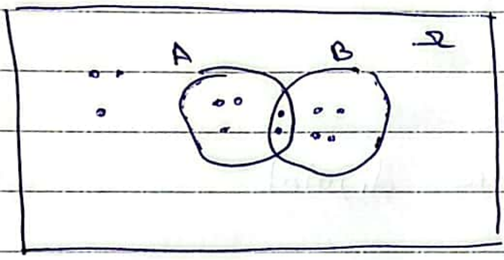
* Conditional probability :

↳ The concept of conditioning

Let A and B be events → $A, B \subset \Omega$, (subset)
then we are interrested in the Probability of A given that B has occured

↳ $P(A \mid B)$ → Conditional probability

الهم علاقة ببعض (نفس الـ Ω)



$|\Omega| = 12$

$P(A) = \frac{5}{12}$

$P(B) = \frac{6}{12}$

B has occured the my new sample space is B then $|\Omega'| = 6$ → $P(A \mid B) = \cancel{P(A \cap B)} = \frac{2}{6} = \frac{1}{3}$

B ↲

* refering To Ω $\quad P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$P(B) \neq$ zero $\quad$ لإنه وقتها يعني $B = \emptyset$
Then the new $\Omega = \emptyset$

prob

$\cancel{X}$

↳ $P(A \cap B) = P(A) \times P(B|A)$

↓ $= P(B) \times P(A|B)$

$\boxed{\text{multiplicatio role}}$
and it is valid
for any 2 events

Axioms: $\boxed{1}$ $P(A|B) \geq 0$

$\boxed{2}$ $P(\Omega|B) = 1$ Because B is new Ω

$\boxed{3}$ $P(B|\Omega) = P(B)$

$\boxed{4}$ if A_1 and A_2 are disjoint

then $P((A_1 \cup A_2)|B) = P(A_1|B) + P(A_2|B)$

← الجايتها من المعادلة فوق

* for any ~~2~~ events $P(\text{their union} | B) \leq \sum_{i=1}^{n} (P(A_i|B))$

The equality holds
if they are
dis joint

* lets go back to the multiplication rule

$P(A_1 \cap A_2 \cap A_3) = \cancel{P(A_1) \times P(A_2 \cap A_3 | A_1)}$

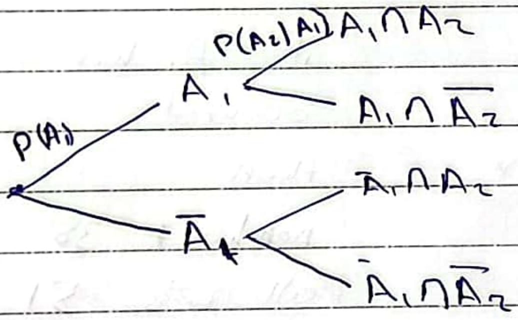
$P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2)$

$= \cancel{P(A_1)} \cdot \frac{\cancel{P(A_2 \cap A_1)}}{\cancel{P(A_1)}} \cdot \frac{P(A_3 \cap A_2 \cap A_1)}{\cancel{P(A_2 \cap A_1)}}$ ✓ proofed

* in general

$$P\left(\bigcap_{i=1}^{N} A_i\right) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2) \dots P\left(A_n \Big| \bigcap_{i=1}^{N-1} A_i\right)$$

* in tree diagram



..... and so on for events A_n

* the answer is to multiply the probabilities *

then $P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|P(A_2|A_1))$

* this can be used without ~~mathematical~~ mathmatics

* $P(A_1)$ can be expressed as conditional

↳ $P(A_1|\Omega)$

prob

Ex: Three cards are drawn from a 52-card deck <u>without replacement</u>

hearts 13	clubs 13
diamonds 13	spades 13

→ well shuffled

1- P(non of these cards is a heart).

let us define $A_i \rightarrow$ card is not a heart

$A_i = A_1 \cap A_2 \cap A_3$

$P(A) = P(A_1) * P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2)$

$= \frac{39}{52} * \frac{38}{51} \times \frac{37}{50}$

if A_1 has occured then non heart 38, all cards 51 and so on....

* لازم نمشي ب Tree لأن كل event بعتمد على الي قبله

2- P(only the third card was a heart)

$B = A_1 \cap A_2 \cap \overline{A_3}$

applying the role above

$= \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{\cancel{37}}{\cancel{50}} \frac{13}{50}$

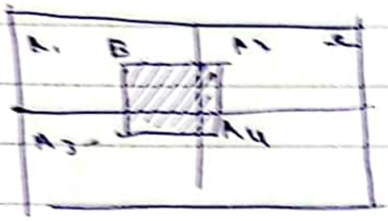
Five Apple

prob.

* Total probability ~~thar~~ theorem ~~theorem~~

let $A_1, A_2 \dots A_n$ be a partition of a sample space Ω and let B be an event, then

$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)$$



$$B = (B \cap A_1) \cup (B \cap A_2) \cup \dots (B \cap A_n)$$

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)$$

↳ $P(A_i|B)$???

↳ $\dfrac{P(A_i \cap B)}{P(B)} = \dfrac{P(B|A_i) \cdot P(A_i)}{P(B)}$

↳ $\dfrac{P(B|A_i) \cdot P(A_i)}{\sum_{i=1}^{n} P(B \cap A_i)} \rightarrow \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)$

important for some applications

* This is called Base Rule *

Prob.

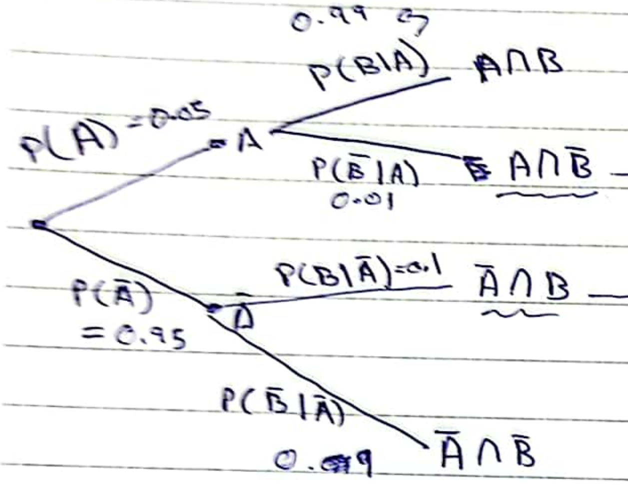
event B

Ex: if an air crafit present, $P(\text{getting a signal}) = 0.99$

if an air craft is not present, $P(\text{false alarm}) = 0.1$

↳ $P(\underbrace{\text{air craft presents}}_{\text{event } A}) = 0.05$

↳ tree diagram



$A \cap \bar{B}$ → missed detection $= P(A) \cdot P(\bar{B}|A)$
$= 0.05 * 0.01 = 0.005$

$\bar{A} \cap B$ → False alarms $= P(\bar{A} \cap B)$
$= P(B|\bar{A}) \cdot P(\bar{A}) = 0.1 \times 0.95 = 0.095$

↳ $P(B) = P(\text{getting alarm signal})$

$$P(\underbrace{(B \cap A) \cup (B \cap \bar{A})}_{\text{disjoint}})$$

$$P(B \cap A) + P(B \cap \bar{A}) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$$

$$= 0.99 \times 0.05 \times 0.1 \times 0.95$$

Note

Find $P(A_2)$ → second card is non heart

$P(A_3)$ → third card is non heart

→

$$\overset{A}{\text{p(air craft presente}}\text{, given that } \overset{\bar{B}}{\text{No alarm}} \text{ has been generated)}$$

$$P(A|\bar{B}) = \frac{P(\bar{B}|A) \cdot P(A)}{P(\bar{B})} = \frac{0.01 * 0.95}{1 - P(B)}$$

لـ الفرع الي قبل

Ex: you roll a fair 4 - sided die

if the out come is 1 or 2
you roll once more (one additional time)
, otherwise you ~~stop~~ stop

1 - p ({sum of your rolls is at least 4})

↳ define $A_i = \{$ the first roll is "i" $\}$, $i = 1, 2, 3, 4$

then $P(A_i) = \frac{1}{4}$

↳ define $B = \{$ the sum is at least 4 $\}$

$$P(B) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

ProB.

Continue---

$P(B|A_4) = 1$

$P(B|A_3) = 0$

$P(B|A_2) = \frac{3}{4}$

$P(B|A_1) = \frac{2}{4}$

$P(B) = \frac{1}{4}\left(1+0+\frac{5}{4}\right) = \frac{9}{16}$

Part B:

$P(\underbrace{1^{st} \text{ roll is } 1}_{A_1} \text{ given that } \underbrace{\text{sum is at least } 4}_{B})$

$= P(A_1|B)$

$= \frac{P(B|A_1) \cdot P(A_1)}{P(B)} = \frac{\frac{2}{\cancel{4}} \cdot \frac{1}{\cancel{4}}}{\frac{9}{\cancel{16}}} = \frac{2}{9}$

* Indepenemnce :

we say that A is independeant (ind.) of B if the occurrence of B does not provide information about the likelihood of A

↳ if A is independeant of B

then:

$$P(A|B) = P(A)$$

ex
* واحد رسب بالزرقا
شو دخلك انت ترسب؟

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}} \rightarrow = P(A)$$

لإنه ما دخلهم
فصارت ضرب الـ intersection

$$\Rightarrow P(A \cap B) = P(B|A) \cdot P(A) \longrightarrow \text{multiplication rule}$$

$$\rightarrow P(B|A) \cdot P(\cancel{A}) = P(A|\cancel{B}) \cdot P(B)$$

$P(A)$ لـ

$$P(B|A) = P(B) \longrightarrow \text{independent}$$

prob.

Ex: A fair coin is tossed twice.

1- $P(\underbrace{1^{st} \text{ toss is head}}_{A}, \underbrace{2^{nd} \text{ toss is head}}_{B})$

comma means independent

$\Omega = \{HH, HT, TH, TT\}$

$P(1st \text{ is head}) = \frac{1}{2}$

$P(2^{nd} \text{ is head}) = \frac{1}{2}$

$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

* interesting remark

Disjoint events are totally dependent, they are not independent

↳ if A and B are ind. then:

$$\left.\begin{array}{l} A \text{ and } \bar{B} \text{ are ind} \\ \bar{A} \text{ and } B \text{ are ind} \\ \bar{A} \text{ and } \bar{B} \text{ are ind} \end{array}\right\}$$

important
سؤال ابشان
علم ~~[illegible]~~
بالإمتحان

Proof

1] if A and B are ind, then A and $\bar{B}$ are ind.

we want to show that $P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$

↳ $A = (A \cap B) \cup (A \cap \bar{B})$ → dep نفترض انهم

من السؤال

$P(A) = P(A \cap B) + P(A \cap \bar{B})$

$P(A) = P(A) \cdot P(B) + P(A \cap \bar{B})$

$P(A \cap \bar{B}) = P(A)(1 - P(B)) = P(A) \cdot P(\bar{B})$

عامل مشترك

↳ $P(A)(1 - P(B)) = P(A \cap \bar{B})$

باقي نفس الاشي ← مهم للامتحان

Conditional independance:

we say that A and B are conditionally independent (given C) if

↳ $P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$ → 1

Proof $P(A \cap B \mid C) = \dfrac{P(A \cap B \cap C)}{P(C)} = \dfrac{\cancel{P(C)} \cdot P(B \mid C) \cdot P(A \mid B \cap C)}{\cancel{P(C)}}$

$= P(B \mid C) \cdot P(A \mid B \cap C)$ → 2

From 1 and 2 $P(A \mid C) = P(A \mid B \cap C)$

↳ لو صارت عالفاضي، مابتأثر

Ind. of several events

We say that $A_1, A_2, A_3 \cdots A_n$ are ind.

if $P(\bigcap_{i \in S} A_i)$ = the product of all $P(A_i)$ for $i \in S$

every subset of S

We care about the subsets that contains more that 2 elements or exactly 2

$S = \{1, 2\}$ A_1 and A_2

$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$

~~$P(A_1 \cap A_2 \cap A_3)$~~

$S = \{1, 2, 3\} \rightarrow A_1, A_2, A_3 \rightarrow$ subsets:

$\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$

$\vdots$

$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$

if all the above are true then A_1 and A_2 and A_3 are perwise ind.

* اذا أيّ وحدة ما تحققت ← mentually ind.

Ex: Two rolls of a fair 4-sided die

let $A_i = \{ 1^{st} \text{ roll is } i \}$

$B_j = \{ 2^{nd} \text{ roll is } j \}$

are A_i and B_j ind.?

$P(A_i) = \frac{1}{4}$

$P(B_j) = \frac{1}{4}$

$P(A_i \cap B_j) = \frac{1}{16} = P(A_i) \cdot P(B_j)$

A_i and B_j are ind.

↳ let $A = \{ 1^{st} \text{ roll is } 1 \}$

$B = \{ \text{sum of two rolls is } 5 \} = \{ (1,4) \ (4,1), (2,3) \ (3,2) \}$

$P(A) = \frac{1}{4}$

$P(B) = \frac{1}{4}$

$P(A \cap B) = \{ (1,4) \} = \frac{1}{16} = P(A) \cdot P(B)$

Continue

3- $A = \{$ the max of the two rolls is 2 $\} = \{(1,2)(2,1)(2,2)\}$

$B = \{$ the minimum of the two rolls is 2 $\} =$
$\{(2,3)(2,4)(3,2)(4,2),(2,2)\}$

$P(A) = \frac{3}{16}$ $\quad P(B) = \frac{5}{16}$

$P(A \cap B) = \frac{1}{16} \longrightarrow$ = intersection [illegible] $\neq P(A) \cdot P(B)$
$\{2,2\}$

↳ then A and B are not ind

Ex: Two ind fair coin tosses

$h_1 = \{$ the first toss is H $\}$

$h_2 = \{$ the second is H $\}$

$D = \{$ Two tosses have different outcome $\}$

are h_1 and h_2 conditionally ind. given d ?

$P(h_1 \cap h_2 \mid D) = P(h_1 \mid D) \cdot P(h_2 \mid D)$???

$\Omega = \{HH, TT, HT, TH\} \xrightarrow{\text{ander conditioning}} \Omega' = \{TH, HT\}$

$P(H_1 \mid D) = \frac{1}{2}$ → reffering to new Ω
$P(H_2 \mid D) = \frac{1}{2}$

$P(H_1 \cap H_2 \mid D) = 0 \neq P(H_1 \mid D) \cdot P(H_2 \mid D)$

then they are not ind. ander conditionning

H_1 and H_2 are ind → [illegible]

~~Continue~~ continue...

$H_1, H_2, D \rightarrow$ are they ind.? condition بدون كلام

the are not ind.

proof.

$H_1 = \{HH, HT\}$

$H_2 = \{HH, TH\}$ $\qquad H_1 \cap H_2 \cap D = \emptyset$

then prob = zero

$D = \{TH, HT\}$ $\qquad$ but $P(H_1) \cdot P(H_2) \cdot P(D) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \neq$ zero

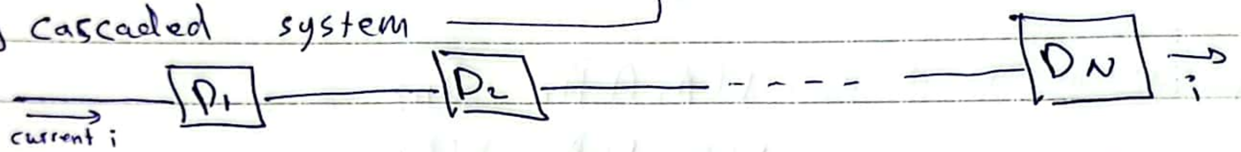
---

سؤال امتحان

* Applications of ind.

يعني مشبوكين serial

I] cascaded system



Let $A_i = \{D_i$ is up$\}$

$P(A_i) = P_i$

↳ Find $P($system is up$)$

↳ means that all devices are up

then: $A_1 \cap A_2 \cap \dots \cap A_n$

↳ each device fails indipindanty of other device

then $A_1, A_2 \dots A_n$ are ind

↳ $P(\text{system is up}) = P(A_1 \cap A_2 \dots A_n) = P(A_1) P(A_2) \dots P(A_n)$

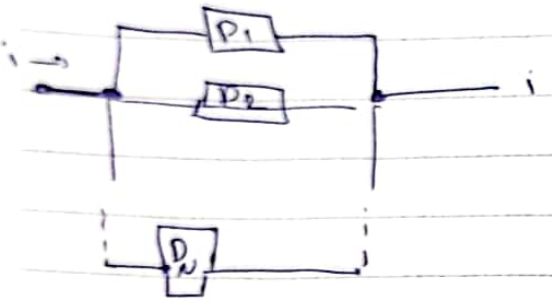
$= P_1 \cdot P_2 \cdot P_3 \dots P_n$

↳ $P(\text{sysem is down}) = 1 - P(\text{sytem is up})$

proof $\rightarrow P(\text{sytem is down}) = P(\bar{A}_1 \cup \bar{A}_2 \dots \bar{A}_n)$ $\qquad$ Apply demorgans low

prob

2] Parallel sytem



$P(A_i) = P(D_i \text{ is up}) = P_i$

↳ sytem up → any device ic up
↳ sytem down → ~~any of~~ all the devices are down

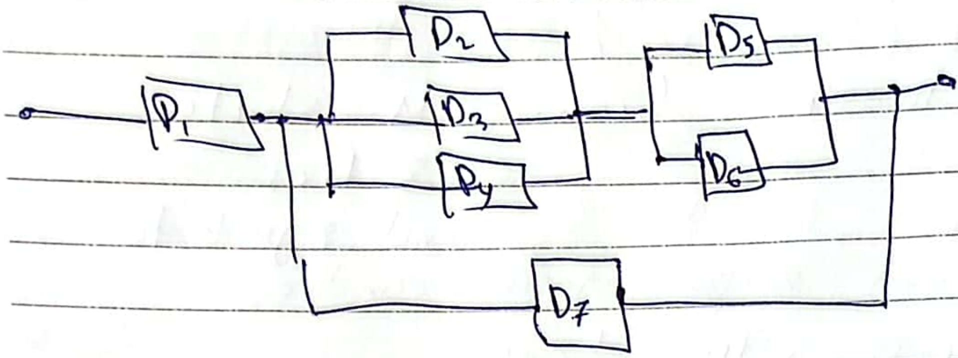
$P(\text{system is down}) = P(\bar{A}_1 \cap \bar{A}_2 \dots \bar{A}_n)$
$= P(\bar{A}) \cdot P(\bar{A}_2) \cdot P(\bar{A}_n)$
$= (1-P_1) \cdot (1-P_2) \dots \cdot (1-P_n)$

P(sytem is ~~down~~ up) = 1 - (system is down)

another aproache

$= P(A_1 \cup A_2 \cup A_2 \dots A_n)$
$= 1 - P(\overline{A_1 \cup A_2 \dots A_n})$
⋮
= 1 - system is down

* suggested example for the first exam :-



$P(D_i \text{ is up}) = P$

Hint : ~~take~~ .
فكك الـ system

Sol :

مش داخل
بال first

Prof.

Counting principle :

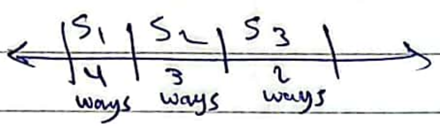
Ex: choosing from : 4 - shirts,
3 ties
and 2 jackets

in how many ways a person can dress

⇒ the process → tree diagram

you have the count all the leafs

the process ~~cou~~ can viewed as stages



↳ then num of ways = $4 \times 3 \times 2$ = 24 way

* multiply the stages *

⇒ Conclusion :

1. divide into stages
2. multiply the stages (ways of each stag)

Prob.

Continue....

* In general:

lets define n_i as the number of Choices in stage i

↳ the total number of Choices in which a process that cantains r stages can be done is given by:

$n = n_1 \cdot n_2 \cdots n_r$

$$\boxed{n = \prod_{i=1}^{r} n_i}$$

Ex: the number of licenses ~~licenses~~ plates with two letters followed by 3 digits:

⇒ Case I: repitition is allowed:

$\underline{\quad} \; \underline{\quad} \; \underline{\quad} \; \underline{\quad} \; \underline{\quad}$
$L_1 \quad L_2 \quad D_1 \quad D_2 \quad D_3$

E letters 26
Num of digits 10

we have 5 stages:

$n = L_1 * L_2 * D_1 * D_2 * D_3$

$= 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$

→

prob.

Continue ....

Case 2: repitition is NOT allowed:

$$\underset{l_1}{\underline{26}} \quad \underset{l_2}{\underline{25}} \quad \underset{d_1}{\underline{10}} \quad \underset{d_2}{\underline{9}} \quad \underset{d_3}{\underline{8}}$$

we still have 5 stages But:

each time we reduce our sample space

then $n = 26 \cdot 25 \cdot 10 \cdot 9 \cdot 8$

Ex: Total num of subsets that can be made out of an N element sets:

Num of stages $\longrightarrow n$

* for each element there are 2 choices
1- inside the subset
2- out sied the subset

then each stage $\rightarrow$ 2 ways

$2 \times 2 \times 2$ ...... n times

$\longrightarrow 2^n$

example: $S = \{1, 2\}$

$\{1\}, \{2\}, \{1, 2\}, \{\} \rightarrow 4$ subsets $\rightarrow 2^2$

Five Apple

* permutations:

→ the total number of ways of ordering n elements

n-elements set

↳ think of it like Bus that contains n seats

N_1	N_2		N_n

* we want to fill the seats

↳ for the first seat we have n ways
↳ // // second seat // // $n-1$ ways
⋮
↳ // // last // // // 1 ways

if we considered each seat as a stage

then → $n \cdot (n-1) \cdot (n-2) \cdots \cdot 1$

$= n!$ ways

Continue

example :

we have 2 students and a prof.

$\Omega = \{ s_1, s_2, p \} \rightarrow$ find num of ways to order them

num of ways $= 3!$

$$\left.\begin{matrix} s_1 & s_2 & p \\ s_1 & p & s_2 \\ s_2 & s_1 & p \\ s_2 & p & s_1 \\ p & s_1 & s_2 \\ p & s_2 & s_1 \end{matrix}\right\} = 6 = 1 \times 2 \times 3$$

* K-permutations :

n-element set $\Longrightarrow$ then we have bus of n seats

n	$n-1$	.....	$n-K+1$

K elements

the num of ways $= n(n-1)(n-2) \cdots (n-K+1)$

* we know that $\underbrace{n(n-1)(n-2) \cdots (n-K+1)}_{\text{Term 1}} \underbrace{(n-K) \cdots}_{\text{Term 2}}$

Term 1 $= n!$

Term 2 $= (n-K)!$

Then num of ways $= \dfrac{n!}{(n-K)!} \rightarrow P^n_K \Longrightarrow$ K permutations

Continue...

Ex: as before

$\Omega = \{S_1, S_2, P\}$

order them on a Bus of 2 seats

1	2

$N = 3 \longrightarrow$ elements
$K = 2 \longrightarrow$ seats

P	S_1
P	S_2
S_1	S_2
S_1	P
S_2	S_1
S_2	P

$$P_K^n = \frac{n!}{(n-K)!} = \frac{3!}{1!} = 6$$

---

* Combitations: $\longrightarrow$ we use the term C_n^K
or $\binom{n}{K} \longrightarrow$ Choose K elemnt from n element

definition:

1- the number of ways of selecting K elements from an n- element set

2- the total number of subsets that Contains K elements that can be made from an n element set

$\longrightarrow$

prob.

Continue on Combinations:

↳ the formula of n choose k C_n^k or $\binom{n}{k}$

$C_n^k \Rightarrow$

$\cancel{P_n^k}$ $\binom{n}{k} \cdot k! \rightarrow \binom{n}{k} = \frac{n!}{(n-k)! * k!}$

$\binom{n}{zero}$ = one empty set then = 1

$\binom{n}{1} = n$

$\binom{n}{n} = 1$ try on the formula above

$\binom{n}{n-1} = n$

* in general $\Rightarrow \binom{n}{r} = \binom{n}{n-r}$ → try to proove it using the formula

↳ $\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} \text{----} + \binom{n}{n} = 2^n$

Because it is the total num of subsets → 2^n

Prob.

Continue....

Ex: fair six-sided die is rolled six times

find:

$P(\underbrace{\text{all rolls result in different outcomes}}_{A})$:-

$|\Omega| = 6^6$

$P(A) = \frac{|A|}{|\Omega|}$

$|A| = P^n_k = P^n_n = \frac{n!}{(n-n)!} = \frac{6!}{1!} = 6!$

then $P(A) = \frac{6!}{6^6}$

proofe $\Rightarrow$ $A = \{1,2,3,4,5,6\} \rightarrow$ and all the orders of this set

then $\Rightarrow$ order $\Rightarrow P^n_n$

* remember: how many ways we can order a set then the solution is using permetation.

prob.

Continues:-

Ex: Tossing ~~a~~ an unfair coin n times, $P(H) = P$
$P(T) = 1 - P$

Find:
P(getting exactly K heads)

we have $|\Omega| = 2^n$

Seq 1: $\underbrace{H\ H\ H \dots}_{K \text{ times}},\ \underbrace{T\ T\ T \dots}_{(n-K) \text{ times}}$

$P(\text{seq } 1) : P(\text{heads}) \cdot P(\text{Tails})$ ⟹ each toss is independant
$= \boxed{P^K \cdot (1-P)^{n-K}}$

this is P for each sequance

$P(K \text{ heads}) = P(\text{seq}1) + P(\text{seq}2) \dots + P(\text{seq}m)$

$= m(P(\text{seq})) = M \cdot P^K \cdot (1-P)^{n-K}$

we need to find m

$m = {}^{n}K$ why ??

we need to find in how many ways we ~~can~~ can choose K slots and put Heads in it

| K | K | n | n |

| n | K | K | n |

⋮

and soon

Continue.... prob.

$P(K \text{ heads}) = m \cdot p^k \cdot (1-p)^{n-k}$

$= {}^{n}_{k} \cdot p^k \cdot (1-p)^{n-k}$

* lets vertify that :

Tossing an unfair coin 3 times

P(getting exactly 2 heads)

$|\Omega| = 2^3 = 8$

HHH	TTT
HHT	TTH
HTH	THT
HTT	THH

all possible ~~out comos~~ outcomes

n of seq = 3

$P(HHT) = P(HTH) = P(THH) = p^2 \cdot (1-p)$

$P(2 \text{ heads}) = {}^{3}_{2} \cdot p^2 \cdot (1-p)$

2 of 3

Prob.

Ex: a coin is tossed 10 times

$P(H) = p \Rightarrow P(T) = 1 - p$

find:

A) P(exactly 3 Hs)

B) P(first 2 tosses are head given that there are exactly 3 Hs)

A) $n = 10$, $k = 3$

P(three heads in 10 tosses)

$= \binom{10}{3} \cdot p^3 \cdot (1-p)^7$

B) A = {first two tosses are heads}

B = {exactly 3 heads} → ~~the~~ ~~previous~~

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{P(H^{1st} \cap H^{2nd} \cap \text{one head of the remaining } 8)}{P(3\,Hs)}$$

$$\frac{\cancel{p} \cdot \cancel{p} \cdot \cancel{p} \cdot \cancel{(1-p)^7} \cdot \binom{8}{1}}{\binom{10}{3} \cdot \cancel{p^3} \cdot \cancel{(1-p)^7}} = \frac{8}{\binom{10}{3}}$$

Prob.

* Partitions OR Partitioning

n-element Set

(n_1) subset 1, (n_2) 2, (n_3) 3, . . . , (n_i) i

$\Rightarrow n_1 + n_2 + n_3 \dots + n_i = n$

num of ways to choose:

$n_1 \Rightarrow \binom{n}{n_1}$

$n_2 \Rightarrow \binom{n-n_1}{n_2}$

$n_3 \Rightarrow \binom{n-n_1-n_2}{n_3}$

$n_r \Rightarrow \binom{n-n_1-n_2 \dots - n_{r-1}}{n_r}$

* the process is to partition the set into subsets

then: the total number of ways the process can be completed:

$$\boxed{= \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \dots = \frac{n!}{n_1!\, n_2! \dots n_r!}}$$

num of ways that we can partition a set

* note * Combinations is a special case of partitions where we have exactly 2 subsets $\Rightarrow \binom{n}{k} \cdot \binom{n-k}{n-k} = \binom{n}{k} \cdot 1 = \binom{n}{k}$

Ex: 16 students are distributed on 8 groups each of which contains 2 students only:

$\boxed{16 \quad \Omega}$ $\rightarrow$ (2) (2) (2) . . . . (2)

subset 1   2   3   8

num of partitions or ways:

$$\frac{16!}{(2!)^8} = \frac{16!}{2^8} = \frac{16!}{256}$$

Ex: class contains 12 under-grad and 4 grad. students, the students are distributed on 4 groups of 4 students

event A

find P(each group has a grad. student)

Ω = {all possible partitions} $= \frac{16!}{4! \cdot 4! \cdot 4! \cdot 4!}$

$|\Omega|$

Continue....

Stage 1 : one grad. student in each group

$\# \text{ of ways} = \frac{4!}{(1!)^4} = 4!$

Stage 2 : 12 under grad. ⇒ 3 in each group

$\# \text{ of ways} = \frac{12!}{(3!)^4}$

$|A| = 4! \times \frac{12!}{(3!)^4}$

$P(A) = \frac{|A|}{|\Omega|}$

Random Variables

Ex: Tossing a coin Twices

$\Omega = \{HH, HT, TH, TT\}$

↳ lets defin a function of the experemental outcome

$X(\omega)$, # of heads observed, ~~ω is a subset~~

$$X(\omega) = \begin{cases} 0\ Hs \rightarrow \{T, T\} \\ 1\ Hs \rightarrow \{HT, TH\} \\ 2\ Hs \rightarrow \{HH\} \end{cases}$$

numerical mapping

Five Apple

→

Prob.

Continue....

~~$P(X(0))$~~

$P(X(\omega)=0) = P(\{TT\}) = \frac{1}{4}$ → if it is a fair coin

$P(X(\omega)=1) = P(\{TH, HT\}) = \frac{1}{2}$

$P(X(\omega)=2) = P(\{HH\}) = \frac{1}{4}$

* if we dropped omega (ω) then X will be a variable → ~~since~~ it is a random variable → because X represents an experament

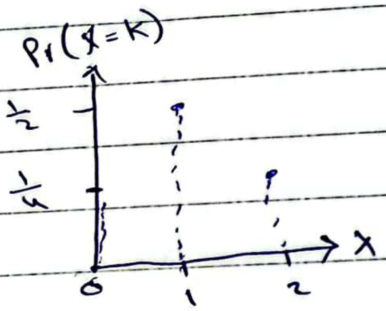
جملة عالزبالة

X is not random and not a variable → But it defines a random experament and it is a function

→ $P(X=0) = \frac{1}{4}$

$P(X=1) = \frac{1}{2}$

$P(X=2) = \frac{1}{4}$



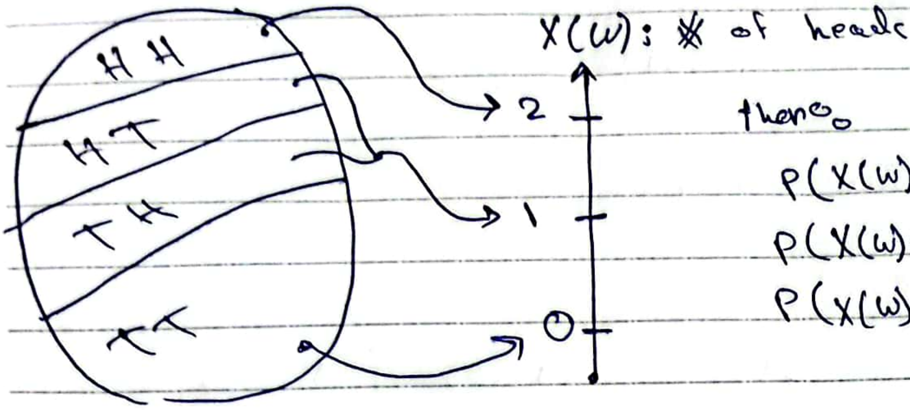
X is discreat random variable ⟹ discreat values

→

Prob.

* Random Variables * $\xRightarrow{\text{shortcut}}$ (RV)

$\Rightarrow$ a random variable is a real valued function of the experimental outcome



then:

$P(X(\omega)=0) = P(\{TT\}) = \frac{1}{4}$

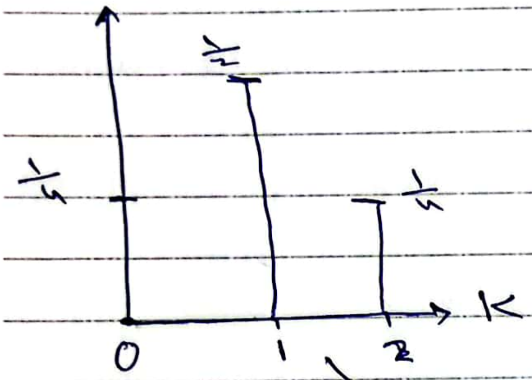
$P(X(\omega)=1) = P(\{HT, TH\}) = \frac{1}{2}$

$P(X(\omega)=2) = P(\{HH\}) = \frac{1}{4}$

$P(X=K) \xRightarrow{\text{هي نفسها}} P_X(K) \rightarrow$ probability mass function (PMF)

$\Downarrow$

discribes an experiment



* Discreat Random variable: is a RV that has ~~PMF~~ PMF $\rightarrow$ X takes discreat values

* $P_X(K)$ satisfies the Axioms:

$\rightarrow P_X(K) \geq 0$

$\rightarrow \sum_K P_X(K) = 1$
$\hookrightarrow$ for all K values

Five Apple

Prob.

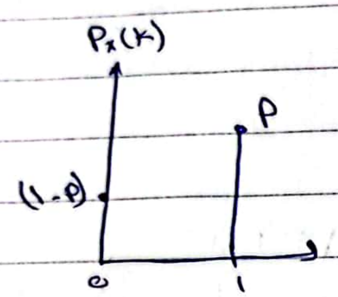
Common discreat RV's

[1] ⇒ * the simplest one → Bernoulli RV:

the experament → tossing a coin one time
$P(H) = P$

X → * of heads observed → possible values for $k \to 0$ or 1

PMF:



$$P_x(K) = \begin{cases} P, & x = 1 \\ 1-P, & x = 0 \end{cases}$$

* if X is Bernoulli distributed with parameter P:

$X \sim$ ~~Ber~~ Bernoulli(P)

$\sim$ ⇒ distributed

P → ~~gett~~ getting a head

* the Bernoulli RV is used to model events that result in head / tail, ~~Sucesses~~ success / fail, $A / \bar{A}$

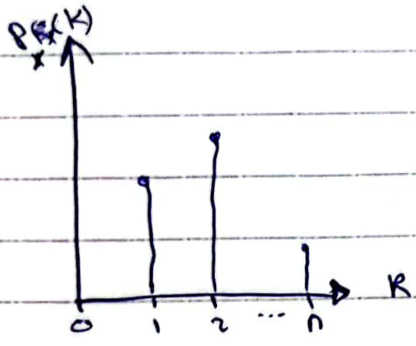
Continue...   Prob.

2] Binomial RV

Experiments: Tossing a coin n times and observing # of H's.

$X \to$ # of heads observed

$$P_X(K) = P\{X=K\} = \binom{n}{K} p^K (1-p)^{n-K}$$



$\to X \sim \text{Binomial}(n, p)$

$\boxed{\text{* always } \sum_{K} P_X(K) = 1 \text{ *}}$

* we are interested in total number of successes

* Bernoulli is a special case of Binomial *

3] * Binomial theorem: * Check *

$$(x+y)^n = \sum_{K=0}^{n} \binom{n}{K} x^K y^{n-K}$$

Prob.

3 Geometric RV $\rightarrow$ One Parameter

$X \sim \text{geometric}(P)$

Experament: Tossing a coin infinitly many Times

X: # Tosses till the first H

OR $\rightarrow$ Doing an experamint ~~out~~ till the first success

* sending a packet 20 times till we get the first ~~first~~ feedback

$$P(X=K) = P(\underbrace{TTT\ldots}_{K-1 \text{ times}}, \underset{K\text{-th}}{H})$$

$$= (1-P)^{K-1} \cdot P \quad , \quad K = 1, 2, 3 \ldots \infty$$

① $\rightarrow P(X=1) = (1-P)^0 \cdot P \rightarrow P$

② $\rightarrow P(X=2) = P(T \cap H) = (1-P) \cdot P$

ind.

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad , 1 > x > 0}$$

proof: $\sum_{K} p(K) = 1 \rightarrow \sum_{K=1}^{\infty} P(X=K)$

$$= \sum_{K=1}^{\infty} P(1-P)^{K-1} = P\left(\sum_{K=1}^{\infty} (1-P)^{K-1}\right)$$

geometric series

$$= P\left[(1-P)^0 \cdot (1-P)^1 \cdot (1-P)^2 \ldots \cdot (1-P)^{\infty}\right] = P\left(\frac{1}{1-(1-P)}\right) = \frac{P}{P} = 1$$

Continue → RV

we assume that all objects are equaly likly

4 Discreat uniform RV

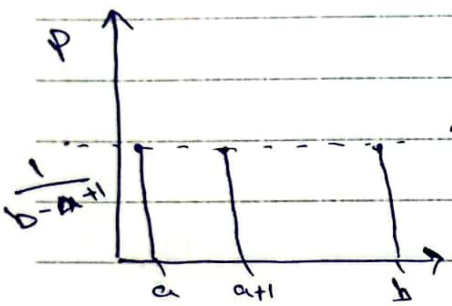
~~Imagine~~ we have (a) (a+1) (a+2) .... (B)

the number of objects between A and B

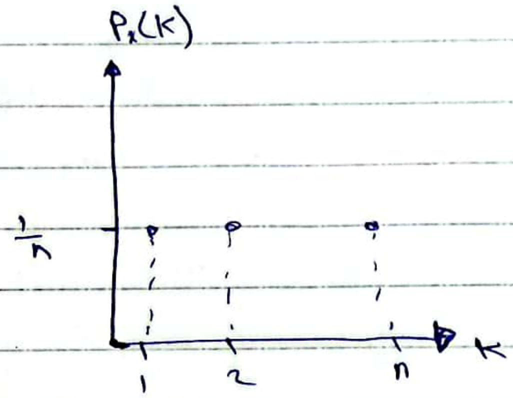
~~inclusive~~ b-a+1 → a and b are included

* a and B are integers *

$$p(X=k) = \frac{1}{\#\text{poss}} = \frac{1}{B-a+1} , \quad k = a, a+1 \dots b$$



if a=1
b=n



5 poisson RV

The experament : the AVg # of occurances over a given time period is λ, we are interested in the actual # of occurences

→ X : actual # of occurences

$$P_x(k) = p(X=k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!} \quad \to \lambda \text{ is the AVg}$$

$$\to P_x(0) = e^{-\lambda}$$

$$\to \cancel{P_x} \; P(X \geq 1) = 1 - P_x(0) = 1 - e^{-\lambda}$$

→

Five Apple

Continue....

Prob.

$$\sum_{K} P_x(K) = 1$$

taylor ~~taylor~~ series

$$\sum_{K=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^K}{K!} = e^{-\lambda} \underbrace{\sum_{K=0}^{\infty} \frac{\lambda^K}{K!}}$$

$$= e^{-\lambda} \cdot e^{\lambda} = e^{0} = 1$$

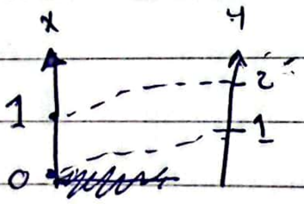
Transformation of discreat RV.

Let X be a discreat RV with pmf $P_x(K)$, and let y be some function $y = f(X)$, we are interested in the pmf of y.

Ex: $X \sim$ Bernoulli (P)

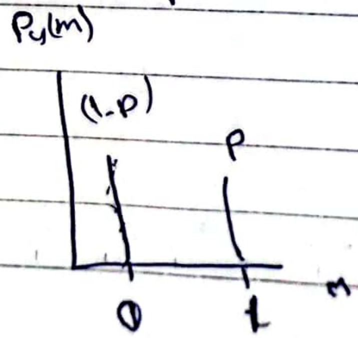
↳ Find pmf of $y = X^2 + 1$

↳ First step:



$P_y(m) = P(y=m)$:

$$= \begin{cases} P_y(1) = P_x(0) = 1-P, & m=0 \\ P_y(2) = P_x(1) = P, & m=1 \end{cases}$$

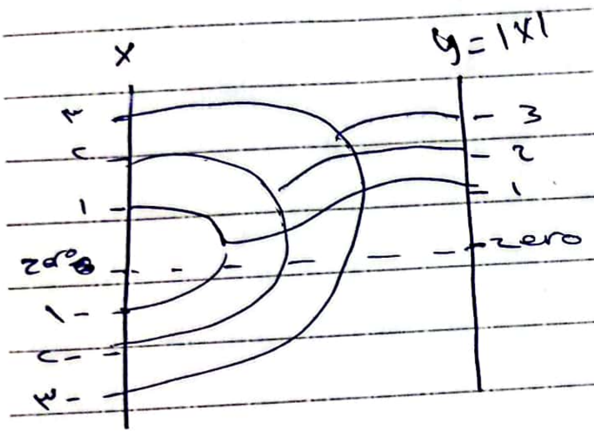


Continue - - -   Prob

Ex $X \sim \text{uniform}[-3, 3]$

$$P_X(K) = \frac{1}{3-(-3)+1} = \frac{1}{7}$$

Find pmf of $y = |x|$



$P(y=0) = P(X=0) = \frac{1}{7}$

$P(y=1) = P(X=1 \text{ or } X=-1) = P(X=1) + P(X=-1) = \frac{2}{7}$

$\therefore P_y(2) = \frac{2}{7}$

$\therefore P_y(3) = \frac{2}{7}$

$$P_y(m) = \begin{cases} \frac{1}{7}, & m=0 \\ \frac{2}{7}, & m=1,2,3 \\ \text{zero}, & \text{not } 1,2,3,0 \end{cases}$$

Prob

* the expected value of a ~~discrete~~ discrete RV

Expected value / mean / Avarage

$\underbrace{\text{Expected value / mean / Avarage}}_{\text{they are all the same.}}$

$E[X]$ $\downarrow$ Expected Value of X

RV

Avg of $E[X] = \sum_{K} K \cdot P_X(K)$

All values that x can take

$= \sum_{K} K \cdot P_r\{X=K\}$

↳ let $g(X)$ → Be a function of ~~a~~ RV X

$E[g(X)] = \sum_{K} g(K) \cdot P_X(K)$

* Valid for any functon of X *

Special case: if $g(X) = aX+b$ a, b are constants

$E[g(X)] = a \cdot E[X] + b$

↳ Linearity of expectation

PROOF →

Prob

* $g(x) = ax + b$

$$E[g(x)] = \sum_{k} g(k) \cdot P_x(k)$$

$$= \sum_{k} (ak+b) \cdot P_x(k)$$

$\sum_{k} P_x(x) = 1$

$$= \sum_{k} ak \cdot P_x(k) + \sum_{k} b \, P_x(k)$$

$$= a \cdot \sum_{k} k \cdot P_x(k) + b \cdot 1$$

$$= a \cdot E[x] + b$$

Ex: let $X \sim$ Bernoulli (p)

find $E[x]$

$$= \sum_{k} k \, P_x(k)$$

$$= 0 \cdot P_x(0) + 1 \cdot P_x(1)$$

zero + p

$= p$

Proof

* another special case:

$g(X) = X^n$, $n >$ zero $\rightarrow$ integer

$E[X^n] \rightarrow n^{th}$ moment of X

if $n=1 \Rightarrow E[X] \rightarrow$ first moment

if $n=2 \Rightarrow E[X^2] \rightarrow$ second moment

* we are interested in the first and the second to find the Variance

$\hookrightarrow E[X^n] = \sum_{k} k^n \cdot P_X(k)$

ex: let $X \sim$ Bernoulli(P), then

$E[X^n] = \sum_{K} K^n \cdot P_X(K) = \cancel{0^n \cdot P_X(0)}^{\,0} + (1)^n P_X(1) = P$ *

* Variance ال و Mean كال Proof سؤال → second exam *

لواحد من ال discrete RV

Proof

* Variance of a discrete R.V

Variance y > Variance x

لإنه الداتا مشتتة أكتر عند y

Variance: is a measure of dispersion of x around its mean (expected value)

$\rightarrow X - \mu_X \longrightarrow \mu_X = E[X]$

then $E[X - \mu_X] = \text{zero} \longrightarrow$ $\boxed{\text{كالعادي}}$

$* \; E[(X-\mu_X)^2] \rightarrow (X-\mu_X)^2 = X^2 - 2 \cdot X \cdot \mu_X + \mu_X^2$

$E[X^2 - 2 \cdot X \cdot \mu_X + \mu_X^2] = E[X^2] - 2 \cdot \mu_X \cdot E[X] + \mu_X^2$

$2 \cdot \mu_X \cdot \mu_X = 2\mu_X^2$

$= E[X^2] - \mu_X^2$

$= E[X^2] - (E[X])^2$

then Variance of x = 2^{nd} moment $-$ mean^2

it is also the 2^{nd} central moment

يؤخذ طرحه منه ال mean

Prob

* Note the n^{th} central moment:

$E[(X-\mu_X)^n] \rightarrow n^{th}$ central moment

* observations *

1] $Var(X) \geq 0 \rightarrow$ لانه تربيع

2] $\sigma_X = \sqrt{Var(X)} \rightarrow$ standard deviation

* to find variance:

1- find mean
2- find second moment
3- // variance

1] $X \sim Bernoulli(P)$:

$E[X] = P$
$E[X^n] = P$

$Var(X) = P - P^2 = P(1-P)$

2] $X \sim \text{Binomial}\ (n, p)$

$$p_X(k) = \binom{n}{k} p^k \cdot (1-p)^{n-k} \quad , \quad k = 0, 1, 2, \dots, n$$

$$E[X] = \sum_k k \cdot p_X(k)$$

$$= \sum_{k=0}^{n} k \binom{n}{k} \cdot p^k (1-p)^{n-k} \longrightarrow \text{Proof}$$

$$\boxed{= n \cdot p}$$

$$\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} n \cdot \binom{n-1}{k-1} \cdot p^k (1-p)^{n-k}$$

$$= n \cdot p \cdot \underbrace{\sum_{k=0}^{n} \binom{n-1}{k-1} \cdot p^{(k-1)} \cdot (1-p)^{n-k}}_{\text{مجموع كل ال probabilities} \ = 1}$$

$$= n \cdot p \cdot 1$$

$$= n \cdot p$$

اثباتها عالصفحة دي

$$\sum_{k=0}^{n} \underbrace{k \binom{n}{k}}_{\text{Term I}} p^k (1-p)^{n-k}$$

Term I: $k \cdot \binom{n}{k} =$

$$= \frac{k \cdot n!}{(n-k)! \cdot k!}$$

$$= \frac{n!}{(n-k)! \cdot (k-1)!}$$

$$= n \times \frac{(n-1)!}{(n-k)! \cdot (k-1)!} \rightarrow \binom{n-1}{k-1}$$

$$= n \cdot \binom{n-1}{k-1}$$

Continue.... proof

$$\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} \cdot (1-p)^{(n-1)-(k-1)}$$

let $k-1=m$

$m=k-1$

for $k=1 \rightarrow m=0$

for $k=n \rightarrow m=n-1$

let $n-1=n'$

$$\underbrace{\sum_{m}^{n'} \binom{n'}{m} p^{m} \cdot (1-p)^{n'-m}}$$

↳ the probability when $X=m$

the sumation is the total summation of the all probabilities $=1$

Continue of the previose example → Binomial

$E[X]=n.p$

$$E[X^2]=\sum_{k=0} k^2 \binom{n}{k} p^k (1-p)^{n-k} \rightarrow$$

اثبتها ←

hint : $k^2=k(k-1)+k$

$Var(X)=n.P(1-P)$

كمل على باقي انواع ال RV's

Continue.... Prob

* $X \sim$ discrete uniform

$$P_x(k) = \begin{cases} \dfrac{1}{b-a+1}, & k = a, a+1 \ldots b \\ 0, & \text{other wise} \end{cases}$$

$$E[X] = \sum_{k=a}^{b} k \cdot \frac{1}{b-a+1}$$

$$= \frac{1}{b-a+1} \cdot \sum_{k=a}^{b} k$$

arithmatic series $\sum_{k=1}^{n} = \dfrac{n \cdot (n+1)}{2}$

$$\boxed{\sum_{k=a}^{b} k = \frac{(a+b)}{2} * (b-a+1)} \longrightarrow \text{general formula}$$

$$= \frac{1}{\cancel{b-a+1}} \cdot \frac{(a+b)}{2} \cdot \cancel{(b-a+1)}$$

$$E[X] = \frac{(a+b)}{2} \longrightarrow \text{the mean for discrete uniform RV}$$

* this formula is true for $K = a, a+1, a+2 \ldots b$ *
incremented by 1
each time

Continue...

$$E[X^2] = \sum_{k=a}^{b} k^2 \cdot P_x(k) = \sum_{k=a}^{b} k^2 \cdot \frac{1}{b-a+1}$$

$$= \frac{1}{b-a+1} \cdot \sum_{k=a}^{b} k^2 \longrightarrow \boxed{\text{search for it}}$$

$$\text{Var}(X) = E[X^2] - \left(\frac{a+b}{2}\right)^2 = \frac{1}{12} \cdot (b-a) \cdot (b-a+2)$$

Continue... Prob

$X \sim \text{Poisson}(\lambda) \rightarrow P_X(K) = \frac{\lambda^K \cdot e^{-\lambda}}{K!}, \quad K = 0, 1, 2 \ldots$

$E[X] = \sum_{K=0}^{\infty} K \cdot \frac{\lambda^K e^{-\lambda}}{K!}$

$= \sum_{K=1}^{\infty} \frac{\lambda^K \cdot e^{-\lambda}}{(K-1)!} = \sum_{K=1}^{\infty} \frac{\lambda^{K-1} \cdot e^{-\lambda} \cdot \lambda}{(K-1)!}$

$= \lambda \cdot e^{-\lambda} \sum_{K=1}^{\infty} \frac{\lambda^{K-1}}{(K-1)!} \rightarrow m = K-1, \quad K=1 \rightarrow m=0$

$= \lambda \cdot e^{-\lambda} \cdot \underbrace{\sum_{m=0}^{\infty} \frac{\lambda^m}{m!}}_{e^{\lambda}}$

$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda}$

$= \lambda$

$E[X^2] = \sum_{K=0}^{\infty} K^2 \cdot \frac{\lambda^K \cdot e^{-\lambda}}{K!}$

$= \sum_{K=0}^{\infty} \frac{\cancel{K(K-1)} \cdot \lambda^K \cdot e^{-\lambda}}{\cancel{K(K-1)} \cdot (K-2)!} + \sum_{K=0}^{\infty} K \cdot \frac{\lambda^K \cdot e^{-\lambda}}{K!}$

$= \lambda$ mean

من الفرع الي قبل

$= \sum_{K=0}^{\infty} \frac{\lambda^K \cdot e^{-\lambda}}{(K-2)!}$

$= \lambda^2 \cdot e^{-\lambda} \cdot \underbrace{\sum_{K=0}^{\infty} \frac{\lambda^{K-2}}{(K-2)!}}_{e^{\lambda}} = \lambda^2 + \lambda$

$\text{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$

Prob.

* Note:

let X be a discrete R.V

$g(X) = aX + b$

$E[aX+b] = aE[X]+b$ → this formula is valid for leniar Rv only

$$\boxed{\text{* } Var(aX+b) = a^2 \cdot Var(X)}$$

Proof: $y = aX+b \rightarrow var(y) = E[y^2] - E[y]^2$

$$= \underbrace{E[(aX+b)^2]}_{\text{Term 1}} - (\underbrace{E[aX+b]}_{\text{Term 2}})^2$$

* term 1: $E[a^2X^2 + 2abX + b^2]$

$= a^2 \cdot E[X^2] + 2abE[X] + b^2$

* Term 2: $= (aE[X]+b)^2$

$= a^2 \cdot E[X]^2 + 2ab \cdot E[X] + b^2$

Term 1 - term 2 = $Var(y) = var(aX+b)$

$$Var(y) = a^2(E[X^2] - E[X]^2) = a^2 \cdot Var(X)$$

Five Apple

Prob

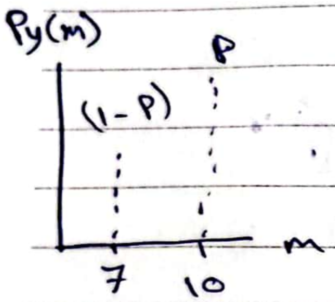
Continue...

Ex: let $X \sim$ Bernoulli

Define $y = g(X) = 3X + 7$

find $E[y]$ and $var(y)$

$$\left.\begin{aligned} * \; E[y] &= 3E[x] + 7 = 3P + 7 \\ * \; Var(y) &= a^2 . var(X) = 9 . P(1-P) \end{aligned}\right\} \rightarrow \text{method } 1$$



method 2

$$E[y] = \sum_m m . P_y(m) = 7 - 7P + 10P = 7 + 3P$$

$$E[y^2] = 7^2(1-p) + 10^2 p \rightarrow 49 - 49.P + 100p$$

$$= 49 + 51.P$$

$$E[y]^2 = 49 + 42P + 9P^2$$

$$E[y^2] - E[y]^2 = 9P - 9P^2 = 9P.(1-P)$$

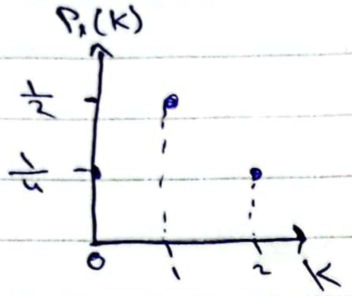
Continue. . . .   Prob

Conditional PMF

experament : Tossing the coin twice

HH	HT
TH	TT

X : # of heads $\rightarrow$ Binomial $(2, \frac{1}{2})$



event $A \rightarrow$ {1st toss is a head}

the conditional PMF given that A has occured is denoted by :

$$P_{X|A}(K)$$

$A = \{HH, HT\}$

$P_{X|A}(1) = Pr\{X=1 \mid A\} = \frac{1}{2}$

$P_{X|A}(2) = Pr\{X=2 \mid A\} = \frac{1}{2}$

$P_{X|A}(K) = 0 \;, K \neq 1$ or 2

$$P_{X|A}(K) = \begin{cases} \frac{1}{2} \;, & K=1 \\ \frac{1}{2} \;, & K=2 \\ 0 \;, & \text{other wise} \end{cases}$$

ProB

Continue...

* properties of conditional PMF

| * conditioning a RV on an event * |

1] $P_{X|A}(K) \geq 0$

2] $\sum_{K} P_{X|A}(K) = 1$

3] conditional mean / avg: $E[X|A] = \sum_{K} K \cdot P_{X|A}(K)$

4] $Var(X|A) = E[X^2|A] - E[X|A]^2$

$\hookrightarrow \sum_{K} K^2 \cdot P_{X|A}(K)$

* in general:

→ $E[g(X)|A] = \sum_{K} g(K)\, P_{X|A}(K)$ → mean

→ $Var(g(X)|A) = E[g^2(X)|A] - E[g(X)|A]^2$

EX: let X be the roll of a fair six-sided die.

Define $A = \{\text{roll is even}\}$

* ⇒ Find $E[X|A]$ $\boxed{1}$, $Var[X|A]$ $\boxed{2}$

* في سؤال proof كال mean وال variance لأنواع ال RV

* اثبت انه ال mean = كذا

Continue...

Prob

$\Rightarrow$ continue on the prev. example

$P_X(K) = Pr\{X=K\} = \frac{1}{6}$, $K = 1,2,3,4,5,6$

$P_{X|A}(K) = Pr\{X=K \mid A\}$ → finding PMF of X|A

$= Pr\{X=K \mid \text{Roll is even}\}$

$$= \begin{cases} \frac{1}{3} , & K = 2,4,6 \\ 0 , & \text{other wise} \end{cases}$$

$E[X|A] = \sum_K K \cdot P_{X|A}(K)$ → finding the mean

$= 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} = \frac{1}{3} \cdot 12 = 4$

$E[X^2|A] = \sum_K K^2 \cdot P_{X|A}(K) = (2^2 + 4^2 + 6^2) \cdot \frac{1}{3} = \frac{56}{3}$ → finding second moment

$Var(X|A) = E[X^2|A] - E[X|A]^2$ → Calculating the variance

$= \frac{56}{3} - 16$

$= \frac{56}{3} - \frac{48}{3} = \frac{8}{3}$

★ اهم اشي تعرف تحدد نوع ال RV

example for the second exam

Tossing a coin 1000 times → X: # num of heads → Binomeal

find the avg number of heads

$P(H) = 0.175$

$E[X] = n.P = 1000 \times 0.175$

$var(X) = n \cdot P \cdot (1-P)$

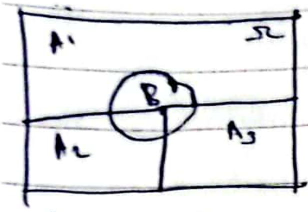
⋮

and so on

Five Apple

Continue . . .   Prob

* Total expectation theorem *



A_1, A_2, A_3 are partitions of Ω so they are disjoint

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B)$$

* general *

$$B = (A_1 \cap B) \cup (A_2 \cap B) \dots \cup (A_n \cap B)$$

n = number of the partition

$$\Rightarrow P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \dots + P(A_n \cap B)$$

$$P(B) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i) \Longrightarrow \text{Define } B = \{X = K\}$$

$$P\{X=K\} = \sum_{i=1}^{n} P_r\{X=K \mid A_i\} \cdot P(A_i)$$

$$\boxed{P_X(K) = \sum_{i=1}^{n} P_{X|A_i}(K) \cdot P(A_i)}$$

$$E[X] = \sum_K K \cdot P_X(K)$$

$$= \sum_K K \sum_{i=1}^{n} P_{X|A_i}(K) \cdot P(A_i)$$

$\rightarrow$ Continue

Continue...

Prob

$$= \sum_{i=1}^{n} P(A_i) \cdot \underbrace{\sum_{K} K \cdot P_{X|A_i}(K)}_{E[X|A_i]}$$

$$\Rightarrow E[X] = \sum_{i=1}^{N} E[X|A_i] \cdot P(A_i)$$

Total Expectation Theorem

X = total ~~#~~ amount of delay to arrive to the school

you can go via three ways → train / Bus / Car

each way have a specific delay and a probability

so the avg amount of delay:

$$E[X] = \sum_{i=1}^{n} E[X|A_i] P(A_i)$$

where A is the event ~~go~~ going by one of the three ways

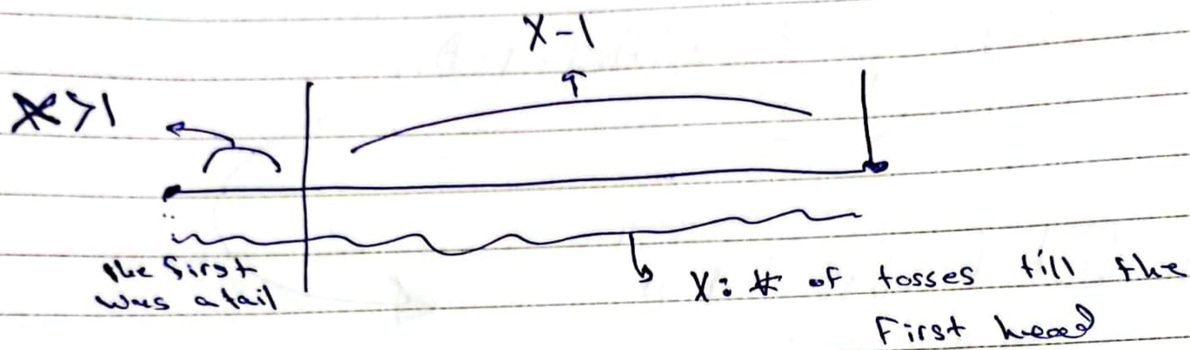
example on Total Expectation Theorem

Continue on RV. : Prob

* Geometric RV (Revisited)

X: # of tosses till the first head

$P_X(K) = (1-p)^{K-1} \cdot p \ , \ K = 1, 2, 3, \dots n$



↳ $P_{X-1 \mid \underbrace{X>1}_{\text{event } A}}(K) = (1-p)^{K-1} \cdot p \ , k = 1, 2, \dots$

↳ in general : $P_{X-m \mid X>m}(K) = (1-p)^{K-1} \cdot p \ , \ k = 1, 2, 3, 4 \dots$

ما بتغير لإنه بعد ال Tail لما في عدد ال Tosses
عشان نلاقي ال First H

واسمها memorylessness property of a geometric RV

↲ لإنه ما بهم الي صار قبل

Prob

Remark on the prev. lecture

let $X \sim$ geometric

$P_X(K) = (1-p)^{K-1} \cdot p \quad , K = 1, 2, 3 \ldots$

event $A = \{X > 1\}$

then: $P_{X-1|A}(K) = (1-p)^{K-1} \cdot p$

memorylessness property

☆ in general:

event $A = \{X > m\}$

then $P_{X-m|A}(K) = (1-p)^{K-1} \cdot p$

$X-m \rightarrow$ # of tosses till the first head

$X > m \rightarrow$ the first m tosses resulted in Tails

Ex: ~~what is the probability that the first head resulted in the third toss given that the first was a tail~~

1) $P_{X-1|X>1}(3) = Pr\{X-1=3 \mid X>1\} = (1-p)^{3-1} \cdot p$

2) $P_{X-20|X>20}(4) = Pr\{X-20=4 \mid X>20\} = (1-p)^{4-1} \cdot p$

~~$E[X]$~~

let $X \sim$ geometric(p)

$$E[X] = \sum_{k=1} k \cdot (1-p)^{k-1} \cdot p = \frac{1}{p} \quad \#$$

Proof

$$X = (X-1)+1$$

event $A_1 = \{X=1\}$

$A_2 = \{X>1\}$

$$E[X] = E[X-1] + 1 \quad \ldots\ (1)$$

$$\to E[X-1] = \underbrace{E[X-1 \mid X=1] \cdot Pr\{X=1\}}_{\text{zero}} + E[X-1 \mid X>1] \cdot p\{X>1\}$$

geometric

$$\to E[X-1] = \text{zero} + E[X] \cdot (1-p) \quad \ldots\ (2)$$

2 in 1 $\quad E[X] = E[X] \cdot (1-p) + 1$

$\vdots$

$$E[X] = \frac{1}{p}$$

Five Apple

Prob

Second moment

* $E[X^2] = E[((X-1)+1)^2]$

$= E[(X-1)^2] + 2E[X-1] + 1$

$= E[(X-1)^2 \mid X=1] \cdot P_r\{X=1\} + \underbrace{E[(X-1)^2 \mid X>1]}_{E[X^2] \to \text{geometric}} \cdot (1-P)$

zero

then $E[X^2] = E[X^2] \cdot (1-P) + 2E[X-1] + 1$

from ~~calculating~~ prev. calculations

$\to E[X^2] = E[X^2] \cdot (1-P) + 2\dfrac{(1-P)}{P} + 1$

$E[X^2] = \dfrac{2(1-P)}{P^2} + 1$

$= \dfrac{2}{P^2} - \dfrac{1}{P}$

$\boxed{\text{then } \quad var(X)} = E[X^2] - E[X]^2 = \dfrac{1}{P^2}(1-P)$ #

for geometric RV

second أول محاضرة في الـ Prob

h.w ~~e~~questions:

7 cards

* $P(3 \text{ cards are aces}) = \dfrac{\binom{4}{3} \cdot \binom{48}{4}}{\binom{52}{7}}$

→ 2 stages → 3 aces and any 4 cards

* $P(2 \text{ Kings}) = \dfrac{\binom{4}{2} \cdot \binom{48}{5}}{\binom{52}{7}}$

* the prev. solutions are the simplest ones *

* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

→ $P(3 \text{ aces and } 2 \text{ Kings})$

$= \dfrac{\binom{4}{3} \cdot \binom{4}{2} \cdot \binom{44}{2}}{\binom{52}{7}}$

* ~~Continue~~ Continueus RV's :

Let x be the tempreture in a given area where the values of X
$X \in [-3, 38]$

$\boxed{X: \text{ is a continueses RV.}}$

possible values for X is ∞ → defined over a range

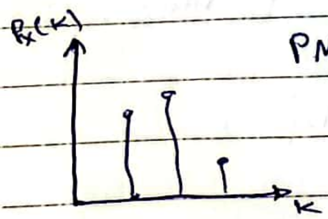
$Pr\{X = 7\}$ assuming that all the elements are equally likely.

$$= \frac{1}{\infty \to |\Omega|} = \text{zero}$$

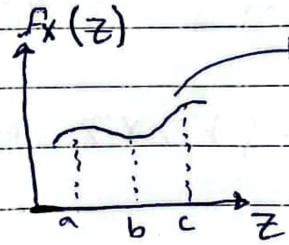
$$Pr\{X = \text{any real number}\} = \frac{1}{\infty} = \text{zero}$$

* $Pr\{b > X > a\}$ → in continuous RV we are interested in non eaqaulity

→ if x is discrete:



→ if x is continueas



→ probability Density Function (PDF)

* $f_X(Z)$ must be ≥ 0
* must be above the X axis

$Pr\{X = a\} = \text{zero}$
$Pr\{X = b\} = \text{zero}$

→ Continue ---

Continue ....

Prob

* Continue on PDF

$$P_r\{b > X > a\} = \int_a^b f_x(z)\,dz$$ → Calculate the area between 2 values

$$P_r\{c > X > b\} = \int_b^c F_x(z)\,dz$$

* A Continuous RV "X" is charecterized by a probability Density function (PDF) → $(f_x(z))$

* $f_x(z)$ is not the probability but its inegration is.

* $f_x(z)$ must satisfies the axioms :

1] $f(z) \geq$ zero

2] $\int_{-\infty}^{\infty} f_x(z)\,dz = 1 = P_r\{\infty > X > -\infty\}$

3] $\int_a^b f_x(z)\,dz = P_r\{b > X > a\}$

4] $\int_a^{\infty} f_x(z)\,dz = P_r\{\infty > X > a\} = P_r\{X > a\}$

5] $\int_{-\infty}^{b} f_x(z)\,dz = P_r\{b > X > -\infty\} = P_r\{X < b\}$

** $f_x(z)$ → if z is a number then the prob. is zero. if z is a range the the prob is the integral between that range.

Continue ...

Prob

* Mean and Variance of Cont. RV

→ the mean (expected value) of the cont. RV X is :

↳ $E[X] = \int_{-\infty}^{\infty} z \, f_X(z) \, dz$

Discrete $= \sum_k k \cdot P_X(k)$

cont $= \int_{-\infty}^{\infty} z \cdot f_X(z) \, dz$

→ the n^{th} momeunt of X is

↳ $E[X^n] = \int_{-\infty}^{\infty} z^n \, f_X(z) \, dz$

→ if $g(X)$ is a function of X, then :

↳ $E[g(X)] = \int_{-\infty}^{\infty} g(z) \cdot f_X(z) \, dz$

→ Variance of (X)

↳ $Var(X) = E[(X - \mu_X)^2] \rightarrow \mu_X = E[X]$

$= \int_{-\infty}^{\infty} z^2 f_X(z) \, dz - \left(\int_{-\infty}^{\infty} z \, f_X(z) \, dz\right)^2$

after simplofication

$Var(X) = E[X^2] - (E[X])^2$

Continue...

Prob

* let $g(x) = ax + b$, then

$E[aX+b] = aE[X] + b$

$Var(aX+b) = a^2 Var(X)$

valid for discrete and cont. RV

* Cumulative Distribution Function (CDF)

→ we use $F_X(z)$ to denote the CDF the RV X

capital F

just a dummy variable

* $F_X(z) = P_r\{X \leq z\} = \int_{-\infty}^{z} f_X(T)\, dt$

CDF

PDF * small f *

→ $F_X(b) = P_r\{X \leq b\}$

then → $P_r\{b > x > a\} = \int_a^b f_X(z)\, dz$

$= \int_{-\infty}^{b} f_X(z)\, dz - \int_{-\infty}^{a} f_X(z)\, dz$

$= F_X(b) - F_X(a)$

* the equality here does not change anything *

Five Apple