PRPBABILITY

KHALED ALNASER

POWERUNIT

If I have k experiments where exp.1 has n_1 possible outcomes

exp.2 , has n_2 possible outcomes

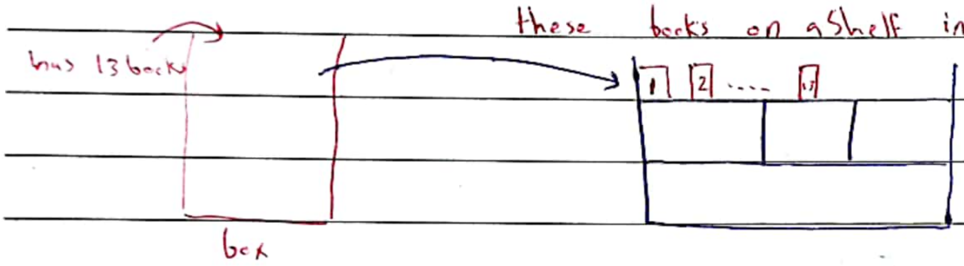
⋮

exp.k has n_k possible outcomes

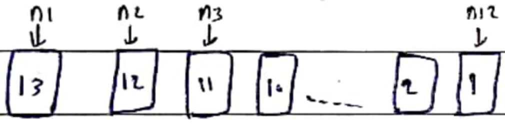
Then the total number of combined outcomes of exp.1 → exp.2 → ... → exp.k is given by: $\boxed{n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k}$

example :

Suppose that Ahmad has 13 books in a box. Further, Ahmad sets out of order these books on a shelf in his home office



Scenario 1: In how many ways can Ahmad order these books?

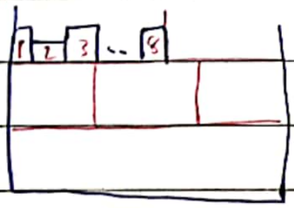


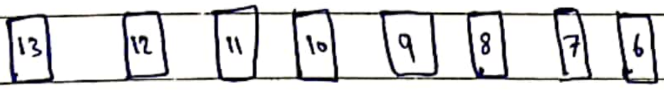
$= (13)(12)(11)(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)$

$= 13!$

$= 6227020800$

Scenario 2: In how many possible ways can Ahmad now arrange 8 books out of his total of 13 books?

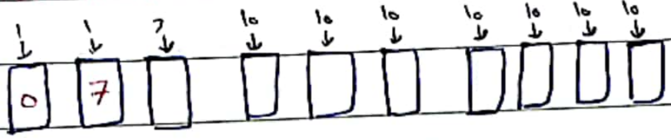




Total # of Possible arragments of 8 books } $= (13)(12)(11)(10)(9)(8)(7)(6)$

$= 51891840$ possible ways

Example: Khaled wants to dial his buddy's phone numbers.



- In how many possible way can Khaled dial a cell phone number in Jordan?

⟹ Total # of possible phone numbers $= (1)(1)(3)(10)(10)(10)(10)(10)(10)(10)$

$= 3 * 10^7$

Scenario: distinct → متميز

$E_1 E_2 D$ لوما عنا ارقام

$E_2 E_1 D$ لكل E بطلع

$D E_1 E_2$ EED

$D E_2 E_1$ DEE $= \frac{3!}{2!1!} = \frac{(3)\cancel{(2)}\cancel{(1)}}{\cancel{(2)}\cancel{(1)}} = \boxed{3}$

$E_1 D E_2$ EDE ↙

$E_2 D E_1$ الحالة هاي ↑

In general, if I have n objects such that n_1 are of type 1

n_2 are of type 2

⋮

n_k are of type k

Then, the total numbers of possible arrangements of these n objects is given by

$$\boxed{\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}} \qquad n_1+n_2+n_3+\cdots+n_k = \boxed{n}$$

Consider:

CONSTITUTION

- In how many ways can one arrange these alphabets?

$$= \frac{12!}{1!\,2!\,2!\,1!\,1!\,3!\,2!\,1!\,1!}$$ ← هيك بنلغي تأثير التكرار

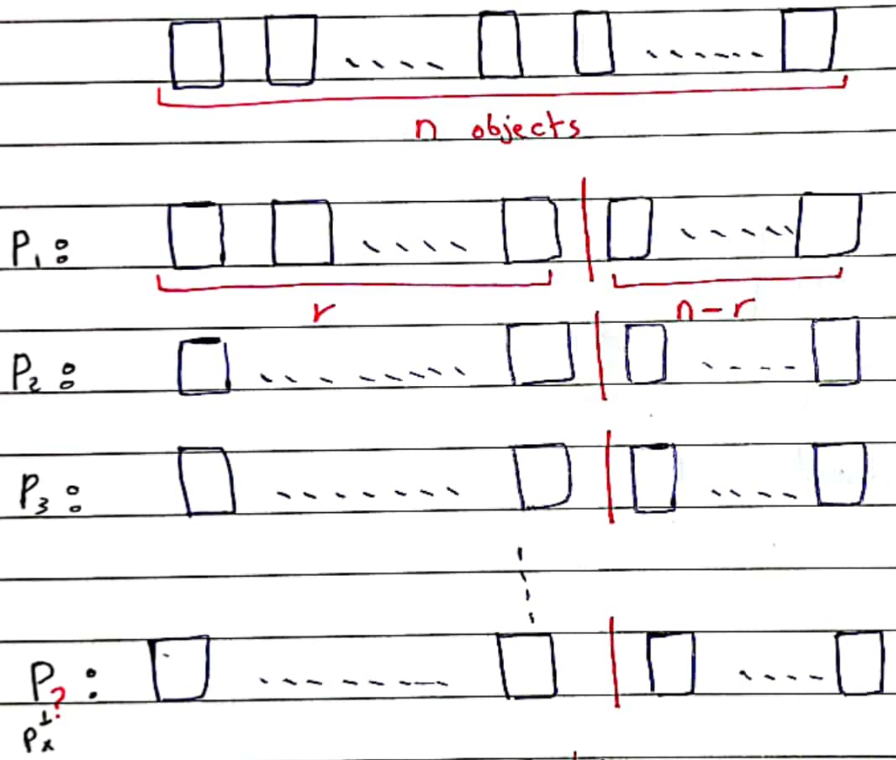
هيك كأنها صبت وحدة من كل حرف

13/10

Scenario 8

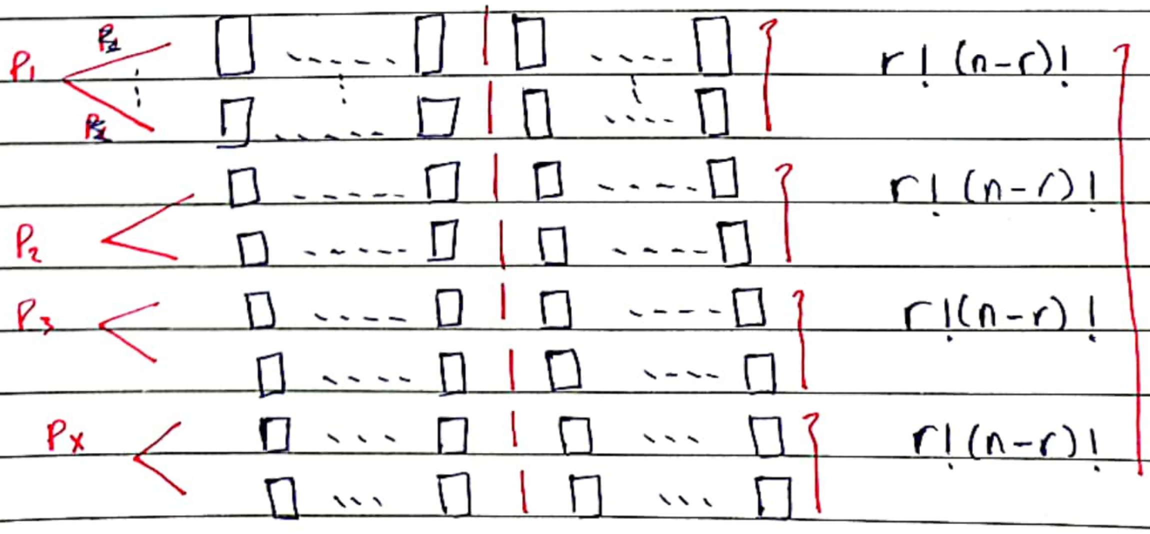
Suppose that I have n distinct object & that I want to split them across two cells:

- One cell containing r objects
- Another cell " n-r "



* In how many ways can I create such a partition?

* هون عني بال Partition انو بدنا نحافظ على نفس العدد بكل Cell وبكل Partition جديد بقعد ابدل مع طالب من ال Cell التانية مع الحفاظ على نفس العدد في ال 2cells .



$r!\,(n-r)!$

$r!\,(n-r)!$

$r!(n-r)!$

$r!\,(n-r)!$

$X \cdot r!(n-r)!$

$$\Rightarrow X \cdot r!(n-r)! \overset{\text{Must}}{=} n! \longrightarrow$$

* زي كأنه كل الطلاب عندي بقاعة وحدة وقاعد برتبهم فيها.

$$\Rightarrow \boxed{X = \frac{n!}{r!(n-r)!}} \longleftarrow \binom{n}{r}$$

In general, If I have n distinct object that I set out to partition across K cells in which

Cell #1 contains n_1 objects

Cell #2 contains n_2 objects

$\vdots$

Cell # K contains n_K objects

Then, the total number of such partitions is given by:

$$\boxed{X = \frac{n!}{n_1!\, n_2! \cdots\cdots n_K!}}$$

$$\downarrow$$

$$\binom{n}{n_1, n_2, \ldots, n_K}$$

16/10

An ordinary deck of playing cards is made up of 52 cards categorized as follows:

◇	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡	A	2	3	4	5	6	7	8	9	10	J	Q	K
♣	A	2	3	4	5	6	7	8	9	10	J	Q	K
♠	A	2	3	4	5	6	7	8	9	10	J	Q	K

- In a poker hand, 5 cards are drawn at random.
- A group of draws yields what we refer to as a Straight "Flush"

أي واحد بطلع من هدول اسمه Flush

All of the Same suit

A	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8
5	6	7	8	9
6	7	8	9	10
7	8	9	10	J
8	9	10	J	Q
9	10	J	Q	K
10	J	Q	K	A

- In how many possible ways can I draw a Straight "Flush"?

→ Total no. of possible draws for a flush $\Big\} = 10 \cdot 4 = 40$

- In how many possible ways can I draw any 5 cards in a Poker hand?

$$\Rightarrow \left.\begin{array}{c}\text{Total \# of}\\ \text{Possible 5-cards}\\ \text{hands}\end{array}\right\} = \frac{52!}{47!\,5!} = \binom{52}{5} = \binom{52}{47}$$

- In how many possible ways can I draw a "Straight"?

* نفس ال Straight flush بس مش من نفس ال Suit

$$\begin{array}{ccccc}\boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6}\\ 4 & 4 & 4 & 4 & 4\end{array} \longrightarrow \left.\begin{array}{c}\text{\# of Such}\\ \text{Possible draws}\end{array}\right\} \begin{array}{l}= (4)(4)(4)(4)\\ = 4^5\end{array}$$

$$\hookrightarrow \left.\begin{array}{c}\text{Total \# of Possible}\\ \text{draws for a straight}\end{array}\right\} = 10 * 4^5$$

- In how many possible ways can I draw a "Full House"?

$$\overbrace{\boxed{Q}\ \boxed{Q}}\ \underbrace{\boxed{3}\ \boxed{3}\ \boxed{3}}$$

$$\Rightarrow 13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{3}$$

"Royal Straight flush"?

④ ← 10 J Q K A

$\boxed{18/10}$

- In how many possible ways can I draw "5 cards hands" all From the Same suit.

$$\left.\begin{array}{c}\text{Total \# of}\\ \text{5-card hands all}\\ \text{From the Same Suit}\end{array}\right\} \begin{array}{l}= 4 \cdot \binom{13}{5} = 5148\\ = 4 \cdot \frac{13!}{5!\,8!}\end{array}$$

Consider throwing 2 die & recording the outcomes.

$$\underset{6}{\boxed{}}\quad\underset{6}{\boxed{}} \longrightarrow \left.\begin{array}{l}\text{Total \#}\\ \text{of Possible}\\ \text{outcomes}\end{array}\right\} = 6\cdot 6 = \boxed{36}$$

2 3 4 5 6 7

(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

(2,1), (2,2), (2,3), (2,4), (2,5), (2,6) 8

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6) 9

(4,1), (4,2), (4,3), (4,4), (4,5), (4,6) 10

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6) 11

(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) 12

* متى طالع شي Even ؟

$$\underset{\text{odd}}{\boxed{3}}\ \underset{\text{odd}}{\boxed{3}} + \underset{\text{even}}{\boxed{3}}\ \underset{\text{even}}{\boxed{3}} = 9+9 = 18$$

$$F_E \sim \frac{\# E}{\# S}$$

100 People

	#Heads	#Tails
1)	45	55
2)	48	52
3)	53	47
4)	40	60
5)	58	42
	⋮	⋮

لو جمعنا هدول وقسمتهم على عددهم

بطلع قريب النص (0.5)

Statistics

$f_E \sim \frac{\# E}{\# S}$

Probability

$P(E) \triangleq \lim_{n \to \infty} f_{E_n}$

$$\underset{\substack{\uparrow \\ \text{Probability of } E}}{P(E)} = \frac{\#\text{ of elements of } E}{\#\text{ of elements in the experement}}$$

Outcome : Any one of possible result of an experiment.

Event : Any collection/Combination of any number of outcomes for agiven experimet.

Sample Space : The set of all possible outcomes for an experiment.

Pokerhand :

$S = \{ \text{all possible 5-card hands} \}$

Find the probability of a "Full House". $E =$ "Full House"

$\# E = 13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{3}$

$\# S = \binom{52}{5}$

$$\longrightarrow P(E) = \frac{\# E}{\# S} = \frac{13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{3}}{\binom{52}{5}}$$

$$= \frac{6}{4165}$$

$\boxed{20/10}$ Axioms of Probability:

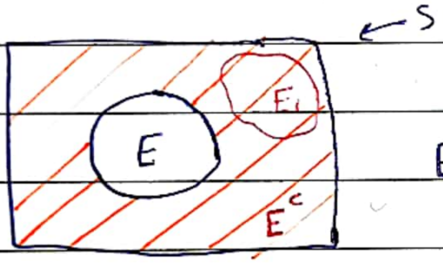
① $$\left.\begin{array}{l} E = \{e_1, e_2, \dots, e_r\} \\ S = \{e_1, e_2, \dots, e_n\} \end{array}\right\} r < n$$

$$P(E) \triangleq \frac{\# E}{\# S}$$

$$\left.\begin{array}{l} - P(E) \geqslant 0 \\ - P(E) \leqslant 1 \end{array}\right\} \longrightarrow 0 \leqslant P(E) \leqslant 1$$

② $$P(S) = \frac{\# S}{\# S} = 1$$

③



$$E^c = \bar{E}$$

$\cup$ - OR

$\cap$ - AND

$$E \cup E^c = S$$

$$P(E \cup E^c) = P(S) = 1$$

$$P(E) + P(E^c) = 1$$

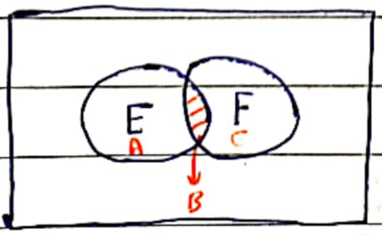
$$\Longrightarrow \boxed{P(E^c) = 1 - P(E)}$$

④ $$P(\emptyset) = \frac{\# \Phi}{\# S} = \Phi \leftarrow \text{zero}$$

$$\underset{\text{Phi}\nearrow}{\Phi} = \{ \quad \}$$

$$- \Phi = S^c \Longrightarrow P(\Phi) = P(S^c) = 1 - P(S) = 1 - 1 = \Phi \leftarrow \text{zero}$$

⑤



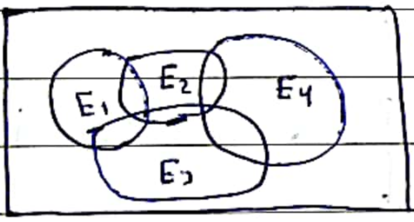
$$P(E \cup F) = P(A) + P(B) + P(C)$$

$$P(E \cup F) = \underbrace{\{P(A) + P(B)\}}_{P(E)} + \underbrace{\{P(C) + P(B)\}}_{P(F)} - P(B).$$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

⑥ $P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(\bigcup_{i=1}^{n} E_i)$

$$= \sum_{i=1}^{n} P(E_i) - \sum_{i<j\cancel{<k}}^{n} P(E_i \cap E_j)\cancel{\cap E_k)}$$



$$+ \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \sum_{i<j<k<l} (E_i \cap E_j \cap E_k \cap E_l)$$

عملنا $i<j<k$ عشان التكرار

$E_1 \cap E_2 \cap E_3$

$E_2 \cap E_1 \cap E_3$

$$+ \cdots\cdots (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n)$$

Principle of inlusion - exclusion

Scenario:

$E = \{e_1, e_2, \ldots, e_n\}$

How many Possible subsets can I get out of set E?

① no. of subsets containing 0 elements : $\binom{n}{0}$
② " " " " 1 elements : $\binom{n}{1}$
③ " " " " 2 elements : $\binom{n}{2}$
⋮
ⓝ " " " " n elements : $\binom{n}{n}$

Total no. of Possible Subsets of E $= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

$$(a+b)^n = \overbrace{(a+b)(a+b)(a+b)\cdots(a+b)}^{n \text{ times}}$$

$$2^n = (1+1)^n = (a+b)^n = \binom{n}{0} a^0 b^n + \binom{n}{1} a b^{n-1} + \cdots + \binom{n}{n} a^n b^0$$

$$\boxed{a = 1 = b} \hookrightarrow \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}$$

23/10

Statistical Independence :

We say that E & F are Statistically independent if the outcome of one does not influence the other & vice versa.

Mathematically ;

E & F are said to be statistically indeendent iff

$$P(E \cap F) = P(E)P(F)$$

Mutually Exclusive events :

We say that events E & F are mutually exclusive if the outcome of one event would preclude the other from happening.

Mathematically Speaking ,

E & F are mutually exclusive iff

$$P(E \cap F) = 0$$

Scenario :

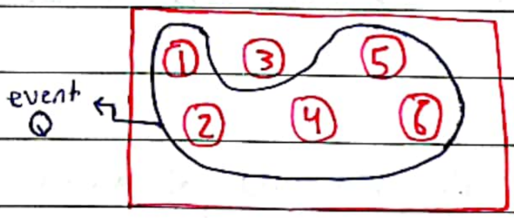
Throw a die & record the outcome

$$S = \{1,2,3,4,5,6\}$$

- I further give the fact that a 3 didn't show up.

* What is the probability that I got an odd outcome.

$S' = \{1,2,3,5,6\}$. $E' = \{1,5\}$



$$S' = \underset{S}{\{1,2,3,4,5,6\}} \cap \underset{Q}{\{1,2,4,5,6\}} = \{1,2,4,5,6\}$$

E = "odd numbers" = {1,3,5}

$Q = \{1,2,4,5,6\}$

$$E' = \underset{E}{\{1,3,5\}} \cap \underset{Q}{\{1,2,4,5,6\}} = \{1,5\}$$

$$P(E') = \frac{\# E'}{\# S'} = \frac{2}{5} = 0.4$$

$$P(E') = \frac{\# E\cap Q}{\# S\cap Q}$$

$$= \frac{\# E\cap Q}{\# Q}$$

$$= \frac{[\# E\cap Q / \# S]}{\# Q \;\; / \# S}$$

$$= \frac{P(E\cap Q)}{P(Q)}$$

E' = It is event E given that Q has happened

$= E|Q$

$$\Longrightarrow \boxed{P(E|Q) = \frac{P(Q\cap E)}{P(Q)}}$$

This is the conditional probability of E given Q

Recall,

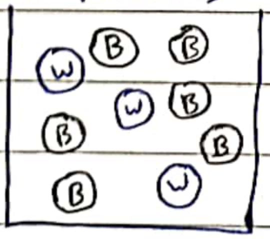
E & F indep $\Longleftrightarrow$ $P(E\cap F) = P(E)P(F)$

$$\begin{tikzcd} \left\{\frac{P(E\cap F)}{P(F)}\right\} = P(E) \arrow[d, bend right] \\ \boxed{P(E|F) = P(E)} \end{tikzcd}$$

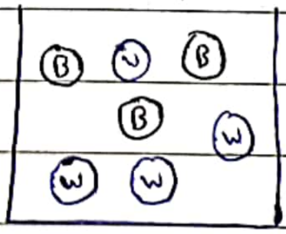
يعني المعلومة الي اجت على F ما اثرت على E بالمطلق.

Scenario:

- What is the probability of a white ball?



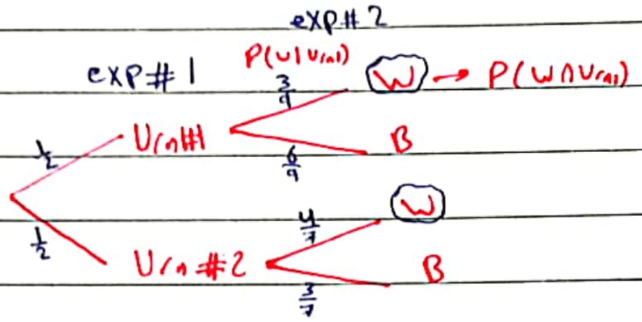
Urn #1



Urn #2

Tree measure :

Tool that is used whenever the outcome of one experiment depends on outcomes of previous experiments.



$\boxed{P(W) = ??}$

$\boxed{25/10}$ - Draw a ball at random.

* What is the probability of a white ball?

they are mutually exclusive

$P(W|Urn_i) = \frac{P(W | Urn\#i)}{P(Urn\#i)}$

$$P(W) = P\{(W \cap Urn_1) \cup (W \cap Urn_2)\}$$
$$= P(W \cap Urn_1) + P(W \cap Urn_2)$$
$$= P(W | Urn_1) P(Urn\#1) + P(W | Urn_2) P(Urn_2)$$
$$= \frac{3}{9} * \frac{1}{2} + \frac{4}{7} * \frac{1}{2}$$
$$= \frac{1}{2}\left[\frac{3}{9} + \frac{4}{7}\right] = \frac{1}{2}\left[\frac{21+36}{63}\right] = \frac{1}{2}\left[\frac{57}{63}\right] = 0.45$$

Scenario: 2 machines at a Casino; A & B

- Pick a machine @ random.
- Play & you won.

تربحك

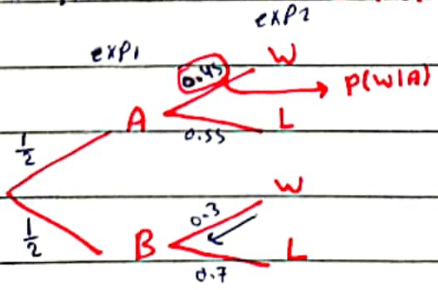
$P(A \text{ pays off}) = 0.45 = P(W | A)$

$P(B \text{ pays off}) = 0.3 = P(W | B)$

* Find the probability that Ibrahim picked B?

P(Ibrahim has picked A) = ? = P(picked A | won)

$\boxed{A}$ $\boxed{B}$



$$P(B|Won) = P(\text{Picked } B \mid \text{Won})$$

$$= \frac{P(\text{Picked } B \cap \text{Won})}{P(W)}$$

$$= \frac{P(\text{Won} \cap \text{picked } B)}{P(W)}$$

$$= \frac{P(W \mid \text{picked } B) \cdot P(B)}{P(W)}$$

$$= \frac{(0.3)(0.5)}{(\frac{1}{2})(0.45) + \frac{1}{2}(0.3)} = \frac{0.3}{0.3+0.45} = \boxed{0.4}$$

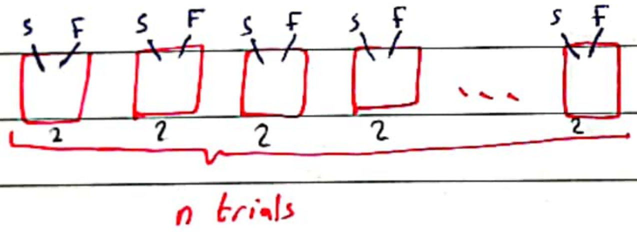
$\boxed{27/10}$

Conduct an experiment over & over, say n times.

At each ~~test~~ trial, I get either a success, S or a failure, F.

Given that $P(S) = p$, $P(F) = q$.

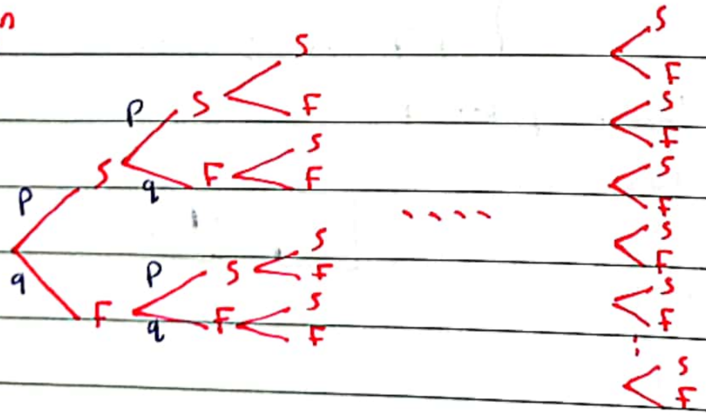
- Find the probability that there will be exactly x successes.



$$\Longrightarrow \#\text{ of Possible outcomes} = \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n$$

$$S = \{\underbrace{(S,S,S,\ldots,S,S)}, \underbrace{(S,S,S,\ldots,S,F)}, \underbrace{(S,S,\ldots,S,F,F)}, \ldots, \underbrace{(F,F,F,\ldots,F)}\}$$

$$\# S = 2^n$$



* لما نعمل tree بيكونوا ال events بيعتمدوا على بعض اما هون ما بيعتمدوا على بعض (Independent).

E & F independent $\Longleftrightarrow P(E \cap F) = P(E)P(F)$

فال tree هون فقط للتوضيح.

$= \binom{n}{x} p^x q^{n-x}$

A vehicle assembly line sets out cars with a batch size of 500, Suppose the probability of any one car coming out defective is $\boxed{0.0025}$. $\rightarrow P(F)$

- what is the probability that there will be exactly 5 cars defective.

$P(\text{exatly 5 defetive cars}) = \binom{500}{5}(0.0025)^5(0.9975)^{495}$

$= (0.9975)^{495}(0.0025)^5 \dfrac{500!}{5!\,495!} = \dfrac{(500)(499)(498)(497)(496)\cancel{(495)}}{5!\,\cancel{495!}}(0.0025)^5(0.9975)^{495}$

$= 7.2201 * 10^{-3}$

(يعني 5 او اكثر)

- What is the probability that there will be 5 defective cars?

$P(\text{5 defective cars}) = P((\text{exactly 5 defetive}) \cup (\text{exactly 6 defective}) \cup \dots (\text{exactly 500 defective}))$

$= P(\text{exactly } 5) + P(\text{exactly } 6\ d) + \dots + P(\text{exactly } 500\ d))$

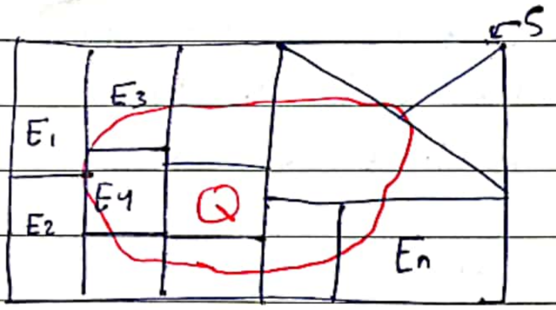
$= \sum_{i=5}^{500} \binom{500}{i}(0.0025)^i(0.9975)^{500-i}$

$= 1 - \sum_{i=0}^{4} \binom{500}{i}(0.0025)^i(0.9975)^{500-i}$

$P(\text{5 cars defective}) = \cancel{1-}\left[\left(\sum_{i=5}^{500}\binom{500}{i}(0.0025)^i(0.9975)^{500-i}\right)\right]^C$

$\boxed{30/10}$

Scenario :



Let S be equal to $S = \bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup \dots \cup E_n$

$\ni E_i \cap E_j = \emptyset \ \forall \ i \neq j$

Furthermore, let Q be some other event defined on the Sample Space

moreover, assume that $P(E_k)$ & $P(\underline{Q|E_k})$ are all given under the Scenario

<u>Task</u>: I want to remeasure relative to Q, need to find $P(E_j|Q) \ \forall j$

$$P(Q) = P(Q \cap S) = P\left(Q \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = P(Q \cap (E_1 \cup E_2 \cup \dots \cup E_n))$$

$$= P((Q \cap E_1) \cup (Q \cap E_2) \cup \dots \cup (Q \cap E_n))$$

Note that because $(Q \cap E_i) \cap (Q \cap E_j) = Q \cap (\underbrace{E_i \cap E_j}_{\phi})$

$$= \phi$$

$$\longrightarrow P(Q) = P(Q \cap E_1) + P(Q \cap E_2) + \dots + P(Q \cap E_n)$$

<u>But</u> $P(Q \cap E_i) = P(Q|E_i)P(E_i) \ \forall i$

$$\longrightarrow P(Q) = P(Q|E_1)\,P(E_1) + \dots + P(Q|E_n)P(E_n)$$

$$\boxed{P(Q) = \sum_{i=1}^{n} P(Q|E_i)\ P(E_i)}$$

Law of total Probability

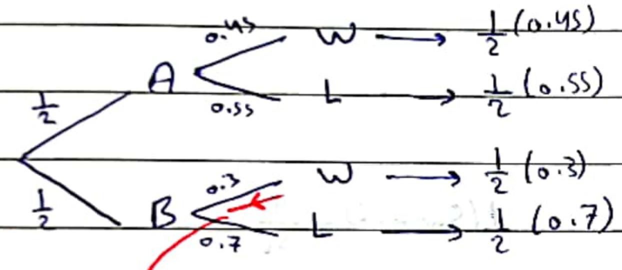
I have all $P(Q|E_j)\ \forall j$ and Commensurate with the task on hand, I want to be able to compute $P(E_L|Q)$ for any value of L.

$$P(E_L|Q) = \frac{P(E_L \cap Q)}{P(Q)} = \frac{P(Q \cap E_L)}{P(Q)}$$

$$= \frac{P(Q|E_L)\,P(E_L)}{P(Q)}$$

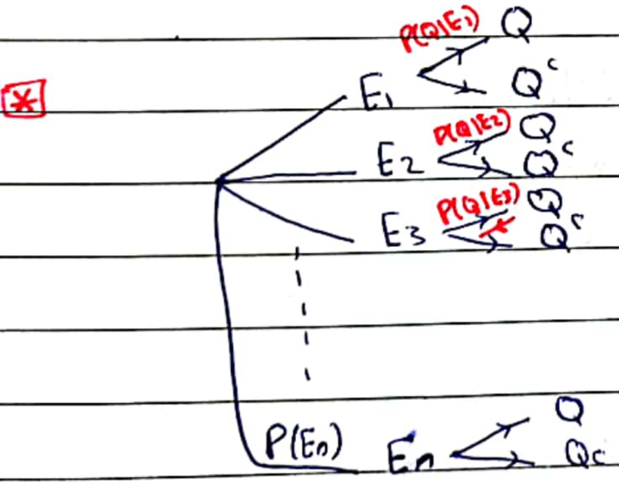
This is Known as Bayes Formula →

$$\boxed{P(E_L|Q) = \frac{P(Q|E_L)\,P(E_L)}{\sum_{i=1}^{n} P(Q|E_i)\,P(E_i)}}$$ For any L



$$\frac{1}{2}\left[0.45 + 0.55 + 0.3 + 0.7\right] = \boxed{1}$$

$P(B|W)$



$P(E_k|Q)$ For any k

$$P(Q) = P(E_1 \cap Q) + P(E_2 \cap Q) + \cdots + P(E_n \cap Q)$$

$$P(Q) = P(Q|E_1)P(E_1) + P(Q|E_2)P(E_2) + \cdots + P(Q|E_n)P(E_n)$$

$$\boxed{P(Q) = \sum_{i=1}^{n} P(Q|E_i)\,P(E_i)}$$

* اذا كانت ال tree معقدة وكبيرة استخدم Bayes Formula احسن.

$\boxed{1/11}$ Introduction to Random Variables:

هعمل مش Random:

$y = x^2 - 1$

$F(x) = 2\cos(3x) + 5$

Random Variable:

- Values occur randomly
- Its is a funtion
  - Has a domain
  - Has arange

$$\begin{array}{l}
\quad\ \ 2 \qquad\ \ 3 \qquad\ \ 4 \qquad\ \ 5 \qquad\ \ 6 \qquad\ \ 7 \\
S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\
\quad (2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\ \ 8 \\
\quad (3,1),(3,2),(3,3),(3,4),(3,6),(3,6)\ \ 9 \\
\quad (4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\ \ 10 \\
\quad (5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\ \ 11 \\
\quad (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\ \ 12
\end{array}$$

$(Sum=2) = \frac{1}{36}$

$(Sum=3) = \frac{2}{36}$

$(Sum=4) = \frac{3}{36}$

$(Sum=5) = \frac{4}{36}$

$(Sum=6) = \frac{5}{36}$

$(Sum=7) = \frac{6}{36}$

$(Sum=8) = \frac{5}{36}$

$(Sum=9) = \frac{4}{36}$

$(Sum=10) = \frac{3}{36}$

$(Sum=11) = \frac{2}{36}$

$P(Sum=12) = \frac{1}{36}$

$$\begin{aligned}
P(\text{Sum is between } 4\&6) &= P\{(Sum=4) \cup (Sum=5) \cup (Sum=6)\} \\
&= P(Sum=4) + P(Sum=5) + P(Sum=6) \\
&= \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{12}{36} = \frac{1}{3}
\end{aligned}$$

$X = \text{"The Sum"} = 4 \leftarrow (2,2)$

$\qquad\qquad\qquad\ \ = 6 \leftarrow (4,2)$

$D(X) \subseteq S$

$R(X) = \{2,3,4,\dots,12\}$

Probability Measure Space:

$(S, P(E), \Sigma)$

$\uparrow$ Collection of events that are measurable

X is said to be a random variable if it is a real-valued measurable function defined on S.

X is measurable iff

$\{\underbrace{s \in S \mid X(s)}\in \Sigma\}$

Scenario:

25	53	61	81
35	53	64	87
40	53	67	90
40	55	68	
40	57	69	
43	57	70	
49	59	70	
50	59	70	
50	59	75	
	59	80	

$\boxed{Avg = 59}$

$\{25, 35, 40, 43, 49, 50, 53, 55,$
$57, 59, 61, 64, 67, 68, 69, 70,$
$75, 80, 81, 87, 90\}$

$avg' = 60.857 \approx 61$ ← هون ال avg تغير لانه بدون تكرار

$\frac{1}{32} * \{1, 1, 3, 1, 1, 2, 3, 1, 2, 4, 1, 1,$ ← عدد تكرار كل واحد
$1, 1, 1, 3, 1, 1, 1, 1, 1\}$

$\Sigma = 1$

Weighting function → $P_X(\cdot)$

Each random variable X, has associated with it a function that fullfils the following

(1) $0 \leq P_X(\cdot) \leq 1$

(2) $\sum_{i=a}^{b} P_X(i) = P(a \leq X \leq b)$

(3) $\sum_{i \in R(X)} P_X(i) = 1$

$\boxed{3/11}$

For any r.v.s:

$$\sum_{i \in R(X)} P_{\mathbb{X}}(i) = 1$$

Conduct an experiment over & over, say n times; At each trial, I either can get a success, $P(S) = p$ or a failure, $P(F) = q$.

* What is the probability of exactly x successes?

X_1 = "Number of successes in n trials"

X_2 = "Number of failures in n trials"

$$S = \{(S,S,\dots,S,S),(S,S,\dots,S,F),(S,S,\dots,S,F,F),\dots,(F,F,\dots,F)\}$$

$$\# S = 2^n$$

[illegible] Success. كم

$$R(x) = \{0,1,2,\dots,n\}$$

$$\boxed{q = 1-p}$$

$$P(\mathbb{X} = x) = \binom{n}{x} p^x q^{n-x} = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P_{\mathbb{X}}(i) = P(X=i) = \binom{n}{i} p^i q^{n-i} \; ; \quad i = 0,1,2,3,\dots,n$$

* Let's investigate the validity of $P_{\mathbb{X}}(i)$ to qualify to be a pmf.

$$\sum_{i \in R(X)} P_{\mathbb{X}}(i) = \sum_{i \in R(X)} P(\mathbb{X} = i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2} + \dots + \binom{n}{n} p^n (1-p)^0 =$$

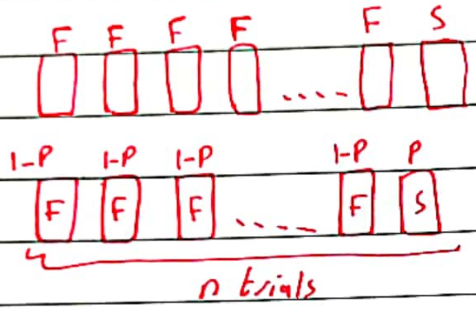
$$= (p + (1-p))^n = 1^n = \boxed{1}$$

Conduct an experiment until I get a Success. At each trial, I either get a Success. At ~~each~~ each trial, I either get a Success S, $P(S)=P$; or a Failure F; $P(F)=1-P$.

X = "Number of trials until Success"

$S=\{S, FS, FFS, FFFS, \ldots\}$

$R(X)=\{1,2,3,4,\ldots\}$



* كل تجربة عن التانية Independent

$P(X=n)=\,?\,=P(n\text{-}1 \text{ failures in preceding } n\text{-}1 \text{ trials}) \cap (\text{Success on last trial})$

$=P(n\text{-}1 \text{ Failures in Preceding } n\text{-}1 \text{ trials}) \cdot P(S)$

$P_X(n)=P(X=n)=(1-P)^{n-1} \cdot P \quad , n=1,2,3,\ldots$

I need to investigate what value I would end up at when taking the Summation:

$$\sum_{n\in R(X)} P_X(n)=\sum_{n=1}^{\infty}(1-P)^{n-1} \cdot P, \quad P=?? = P\sum_{n=1}^{\infty}(1-P)^{n-1} \text{ —— } \circledast$$

let $\boxed{r \triangleq 1-P}$ let $\boxed{m=n-1}$

From $\circledast$

$$P\sum_{n=1}^{\infty}(1-P)^{n-1} = P\sum_{n=1}^{\infty} r^{n-1} = P\sum_{m=0}^{\infty} r^{m}$$

$$= P[1+r+r^2+\ldots]$$

assuming $|r|<1$

$$= P\frac{1}{1-r} = \frac{P}{[\cancel{1}-(\cancel{1}-P)]} = \frac{P}{P} = 1$$

$X \sim$ geometric r.v.

$\boxed{6/11}$

Geometric R.V.

X = "No. of trials Until Success"

$S = \{S, FS, FFS, \ldots\}$

$R(X) = \{1, 2, 3, \ldots\}$

↳ لازم على الأقل محاولة

$$P_X(n) = P(X = n) = (1-P)^{n-1} \cdot P$$

$$\sum_{n \in R(X)} P_X(n) = 1$$

Binomial R.V.

X = "No. of Successes in n trials"

S = { of all Possible n-tuples }

$= \{(\_, \_, \_, \ldots, \_)\}$; $\# S = 2^n$

$R(X) = \{0, 1, 2, \ldots, n\}$

$$P_X(i) = P(X = i) = \binom{n}{i} p^i (1-P)^{n-i}$$

X = "Sum" (Random Variables in grade exp.)

X = "the grade"

Building Blocks:

1- Bernoulli R.V. → One experiment

Success ↙ ↘ Failure

X = "The Outcome"

$$S = \{S, F\} \longrightarrow X = \begin{cases} 1, & \text{if Success} \\ 0, & \text{if Failure} \end{cases}$$

$$\sum_{i \in R(X)} P_X(i) = P + (1-P) = 1$$

2- Geometric R.V.

Scenario :

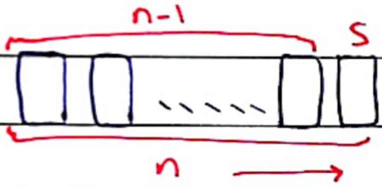
I Conduct an experiment over & over until I get an r^{th} Success

- At each trial, I either get a Success, $P(S) = P$, or a Failure, $P(F) = 1-P$

X = "# of trials Until r^{th} Success"

$S = \{ \underbrace{(S,S,S,\dots,S)}_{r}, \underbrace{(S,S,\dots,F,S)}_{r+1}, \dots \}$

$R(X) = \{ r, r+1, r+2, \dots \}$



بدي n لحد ما اجيب
R^{th} Success

$P_X(n) = P(X=n)$

$= P(\text{I got } r-1 \text{ Success in } 1^{st} (n-1) \text{ trials} \cap \text{Success on last trial})$

$= P(r-1 \text{ Success in } 1^{st}\ n-1 \text{ trials}) \cdot \underset{\nwarrow p}{P(S)}$

$= \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} \cdot p$

$P_X(n) = \binom{n-1}{r-1} p^{r} (1-p)^{n-r}$

To check on validety of R $P_X(n)$

$\sum_{n \in R(X)} P_X(n) = \sum_{n=r}^{\infty} \binom{n-1}{r-1} p^{r} (1-p)^{n-r}$ ***

$$\lim_{\substack{n\to\infty \\ p\to 0 \\ np\to \text{Const.}}} \binom{n}{i} p^i (1-p)^{n-i}$$

$\boxed{8/11}$

$$= \lim_{\substack{n\to\infty \\ p\to 0 \\ np\to\lambda}} \left[\frac{n!}{i!(n-i)!}\right] p^i (1-p)^{n-i}$$

$$\rightarrow \frac{n(n-1)(n-2)\ldots(n-(i-1))\cancel{[(n-i)]}}{i!\cancel{(n-i)!}}$$

$$= \lim_{\substack{n\to\infty \\ p\to 0 \\ np\to\lambda}} \left\{\frac{n(n-1)(n-2)\ldots(n-i+1)}{i!}\right\} \left[\frac{\lambda}{n}\right]^i \frac{(1-\frac{\lambda}{n})^n}{(1-p)^i}$$

$$= \lim_{n\to\infty} \left\{\frac{n^i + \alpha_1 n^{i-1} + \alpha_2 n^{i-2} + \ldots + \alpha_i}{n^i}\right\} \lambda^i \left(1-\frac{\lambda}{n}\right)^n$$

$$= \lim_{n\to\infty} \left\{1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \ldots + \frac{\alpha_i}{n^i}\right\} \frac{\lambda^i}{i!} \left(1-\frac{\lambda}{n}\right)^n$$

$$= \{1\} \frac{\lambda^i}{i!} e^{-\lambda}$$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = ? = e^{-\lambda}$$

$$= \frac{\lambda^i e^{-\lambda}}{i!} ; \quad \lambda > 0, \; i = 0,1,2,\ldots$$

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$$

let's investigate the nature of $\frac{\lambda^i}{i!} e^{-\lambda}$!!!!

(1) $\frac{\lambda^i}{i!} e^{-\lambda} \longrightarrow \geq 0$

(2) $\dfrac{\left[\frac{\cancel{\lambda^{i+1}}}{(i+1)!} \cancel{e^{-\lambda}}\right]}{\left[\frac{\cancel{\lambda^i}}{i!} \cancel{e^{-\lambda}}\right]} = \frac{\lambda}{i+1} \leq 1$

$$0 \leq \frac{\lambda^i}{i!} e^{-\lambda} \leq 1$$

ratio → test

مشان اعرف هل

هو Conv.

ولا لا.

Ratio test

(3) $\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$??

$$e^x = \sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$= e^{-\lambda} e^{\lambda} = \boxed{1}$$

$\therefore \frac{\lambda^i}{i!} e^{-\lambda}$ readily qualifies to be deemed a probability mass function!!!

Poisson

X is said to be a Poisson r.v. with parameter λ

if the probability mass function for which is

given by :

$$\frac{\lambda}{i!} e^{-\lambda} \; ; \quad \lambda > 0 \; ; \quad i = 1, 2, 3, \ldots$$

ما بقدر اعطيه وصف $X =$ " "

هوي هالمثال السيارات الاحتمال

$$\longrightarrow \sum_{i=20}^{500} \binom{500}{i} (0.0025)^i (0.9975)^{500-i}$$

هيك بدل ال Binomial استخدمت

Poisson لانو صارت عندي الحسابات

اسهل و الناتج رح يكون قريب كتير.

$\lambda = np = (500)(0.0025) = 1.25$

$$\approx \sum_{i=20}^{500} \frac{(1.25)^i}{i!} e^{-1.25}$$

$\boxed{10/11}$ Poisson Random Variable :

Ammar has a store that sells grocery. Customers enter Ammar's store at an average rate of 50 customers per week ($\lambda = 50 \frac{\text{customer}}{\text{week}}$)

- Suppose that the probability that Ammar makes any given sale is 0.4 ($P = 0.4$)

Question: What is the probability that Ammar makes no sales in any given month?

X = "# of Sales" ; N = "# of Customers entering Ammar's Store"

$$P(X=0) = P[(X=0 \cap \text{no customers}) \cup (X=0 \cap 1 \text{ customer}) \cup \dots]$$

$$= P(X=0 \cap N=0) + P(X=0 \cap N=1) + P(X=0 \cap N=2) + \dots$$

$$= \sum_{i=0}^{\infty} \underbrace{P(X=0 \cap N=i)}_{P(X=0|N=i)\,P(N=i)}$$

$$= \sum_{i=0}^{\infty} P(X=0|N=i)\,P(N=i)$$

* عدد ال Customers الي دخلوا
المحل Poisson

$$= \sum_{i=0}^{\infty} (1-p)^i \frac{\lambda_1^i}{i!} e^{-\lambda_1}$$

$$\lambda_1 = 50 \frac{\text{Cust}}{\cancel{\text{week}}} \left(4 \frac{\cancel{\text{weeks}}}{\text{month}}\right)$$

$$= 200 \frac{\text{Cust}}{\text{Month}}$$

$$= \sum_{i=0}^{\infty} (1-p)^i \frac{(200)^i}{i!} e^{-200}$$

$$= \sum_{i=0}^{\infty} \frac{[(1-p)(200)]^i}{i!} e^{-200}$$

$$= e^{-200} \sum_{i=0}^{\infty} \frac{[(1-p)(200)]^i}{i!}$$

$$= e^{-200} e^{[(1-p)(200)]} = \cancel{e^{-200}}\,\cancel{e^{200}}\,e^{-200p}$$

$$P(X=0) = e^{-200p} = e^{-200(0.4)} = 1.8 * 10^{-35}$$

$\boxed{13/11}$ Throwing 2 dice & Summing the outcome.

$R(X) = \{2, 3, 4, \dots, 12\}$

X = "the Sum"

$P(X=2) = \frac{1}{36}$ $P(X=9) = \frac{4}{36}$

$P(X=3) = \frac{2}{36}$ $P(X=10) = \frac{3}{36}$

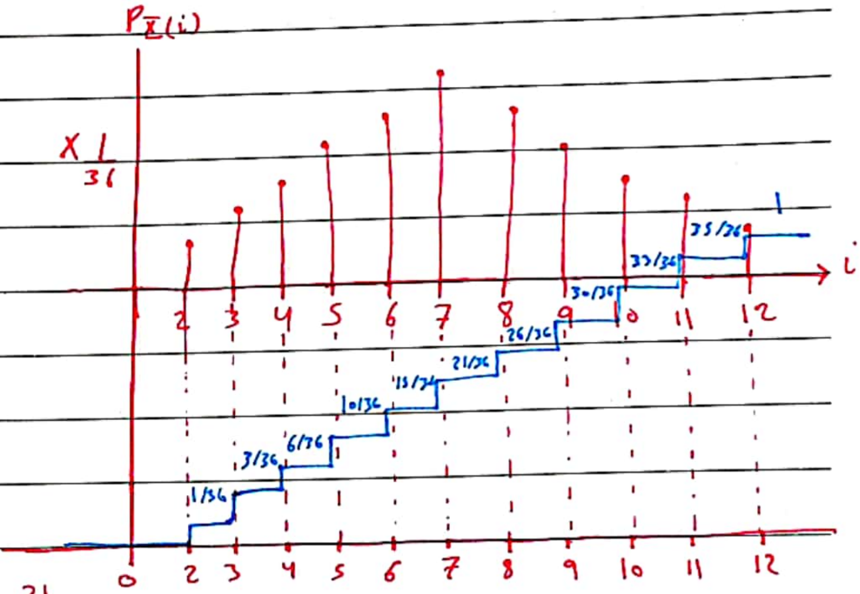
$P(X=4) = \frac{3}{36}$ $P(X=11) = \frac{2}{36}$

$P(X=5) = \frac{4}{36}$ $P(X=12) = \frac{1}{36}$

$P(X=6) = \frac{5}{36}$

$P(X=7) = \frac{6}{36}$

$P(X=8) = \frac{5}{36}$



$P(X<2) = 0$

$P(X \leq 2) = \frac{1}{36}$

$P(X<3) = \frac{1}{36}$

$P(X \leq 3) = \frac{3}{36}$

$P(X<4) = \frac{3}{36}$

$P(X \leq 4) = \frac{6}{36}$

$P(X<5) = \frac{6}{36}$

$P(X \leq 5) = \frac{10}{36}$

$P(X<6) = \frac{10}{36}$

$P(X \leq 6) = \frac{15}{36}$

$P(X<7) = \frac{15}{36}$

$P(X \leq 7) = \frac{21}{36}$

$P(X<8) = \frac{21}{36}$

$P(X \leq 8) = \frac{26}{36}$

$P(X<9) = \frac{26}{36}$

$P(X \leq 9) = \frac{30}{36}$

$P(X<10) = \frac{30}{36}$

$P(X \leq 10) = \frac{33}{36}$

$P(X<11) = \frac{33}{36}$

$P(X \leq 11) = \frac{35}{36}$

$P(X<12) = \frac{35}{36}$

$P(X \leq 12) = \frac{36}{36} = 1$

$F_X(i) \triangleq P(X \leq i)$
↑

Cummulative Distribution Function (CDF)

Each random variable has associated with it a function that accumulates probability values while fulfilling the following properties:

(1) $\lim_{b \to -\infty} F_{\underline{X}}(b) = 0$

(2) $P(a \leqslant X \leqslant b) = F_{\underline{X}}(b) - F_{\underline{X}}(a)$

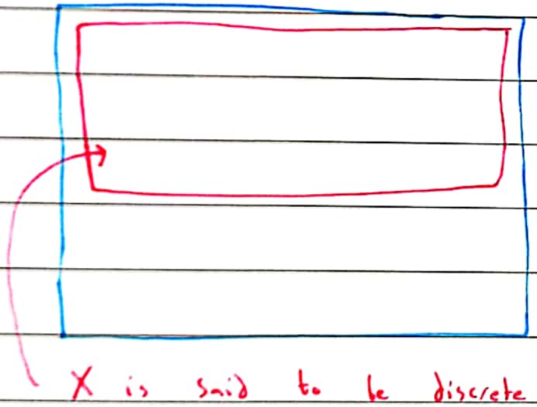
(3) $\lim_{a \to \infty} F_{\underline{X}}(a) = 1$

(4) $F_{\underline{X}}(i) \triangleq P(\underline{X} \leqslant i) = \sum_{i=-\infty}^{i} P_{\underline{X}}(i)$

This Function is called the Cummulative Distribution Function (CDF) for X.

$P(a < X \leqslant b) = F_{\underline{X}}(b) - F_{\underline{X}}(a+1)$

X is said to be a r.v if it is a real-valued measurable for defined ~~es~~ on S.



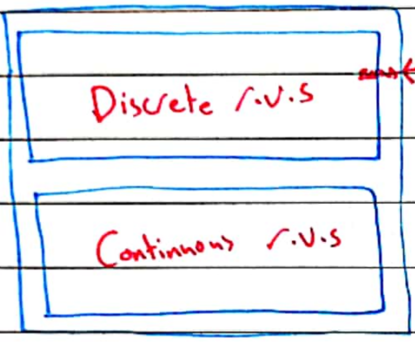
X is said to be discrete r.v. if it is a real-valued function defined on S with a range, R(X), that is countable $\longrightarrow$ 1-1 Correspondence set 0,1,2,3,....

15/11

X is said to be a continuous r.v. if the range, R(X), for which constitutes a ~~Continuous~~. Continuum.

$\hookrightarrow [a,b), (-\infty,\infty), (c,d], \cdots,$

$(-\infty,\infty) \to \mathbb{R}$



X is said to be a r.v. if it is a real-valued measurable for defined on S.

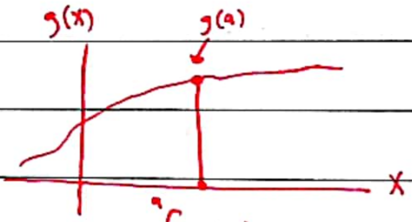
Furthermore, any continuous r.v. possesses a weighting function, $f_X(x)$, the satisfies the following properties:

(1) $f_X(x) \geq 0 \quad \forall x$ (Positive semi-definite). (Non-negative definite)

(2) $P(X \leq x) = \int_{-\infty}^{x} f_X(x)\, dx = F_X(x)$.

(3) $P(a \leq X < b) = \int_{a}^{b} f_X(x)\, dx$

(4) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$



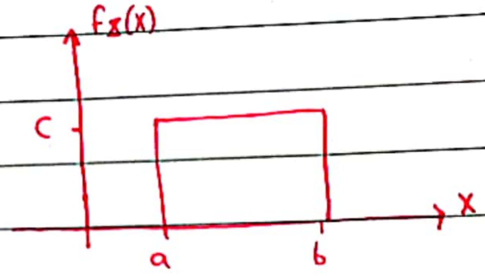
$\int_{a}^{a} g(x)\, dx = 0$

$P(X=c) = \int_{c}^{c} f_X(x)\, dx = 0$

Such a weighting function shall hence for be ~~refer~~ referred to a the probability density function (pdf)

(الاحتمالية متساوية) *equiprobable situation

(1) X is said to be a uniform r.v. if the pdf for which looks like



C-constant

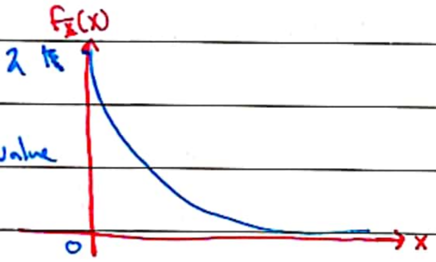
$X \sim$ uniform over (a,b)

C has a particular value !!

To find C, we do normalization :

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \longrightarrow \int_a^b c\,dx = cx\Big|_a^b = c[b-a] \overset{\substack{\text{must}\\ \downarrow}}{=} 1$$

(2) X is said to an exponential r.v. if the pdf for which it looks like



k - has a value

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{elsewhere} \end{cases}$$

To find the value for k, we would need to normalize:

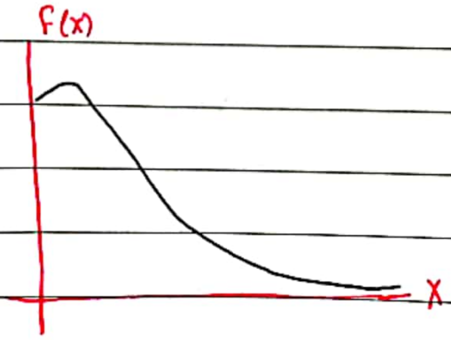
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} k e^{-\lambda x}\,dx = -\frac{k}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = -\frac{k}{\lambda}[0-1]$$

$$= \frac{k}{\lambda} \overset{\text{must}}{=} 1$$

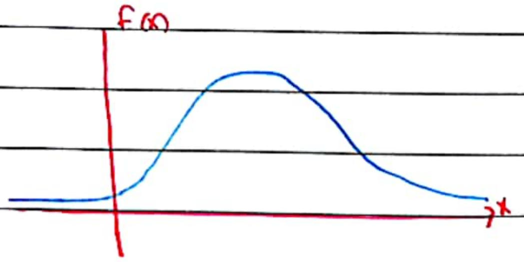
$$\Rightarrow \frac{k}{\lambda} = 1$$

$$\boxed{k = \lambda}$$

- Rayleigh r.v.

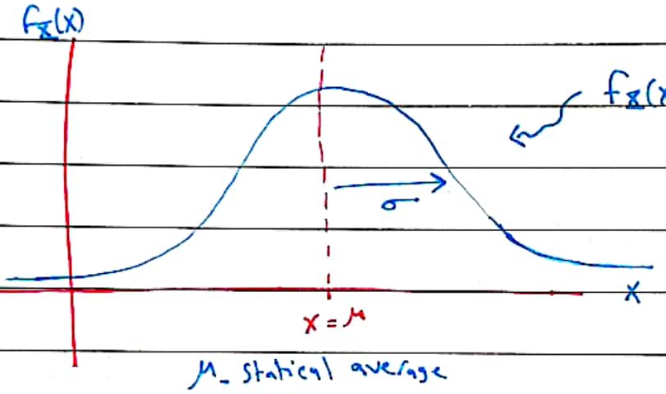


- lognormal
- Normal r.v.



17/11

We say that X is a normal r.v. if the pdf for which is characterized by the following outlook:



$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

تبعثر النقاط

μ - Statical average
- Mean

σ - Standard deviation
- It provides a measure of the degree of Scattering of values of X about the mean value μ.

الها علاقة بالقدرة على اصابة الهدف

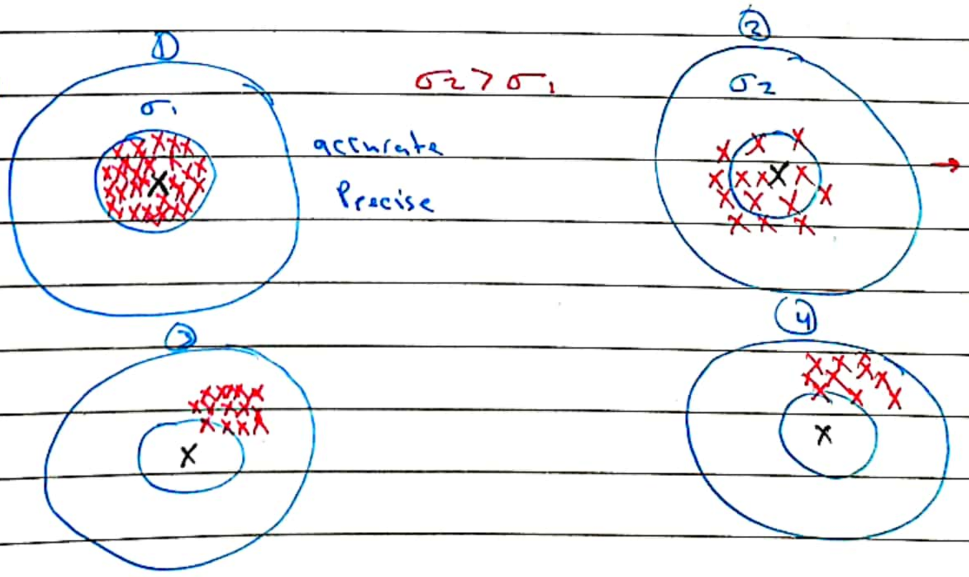
Accurate

Precise → دقة الرميات كلما تجي بمنطقة متقاربة

افضل وحدة بعديها

② بعدين ③ بعدين

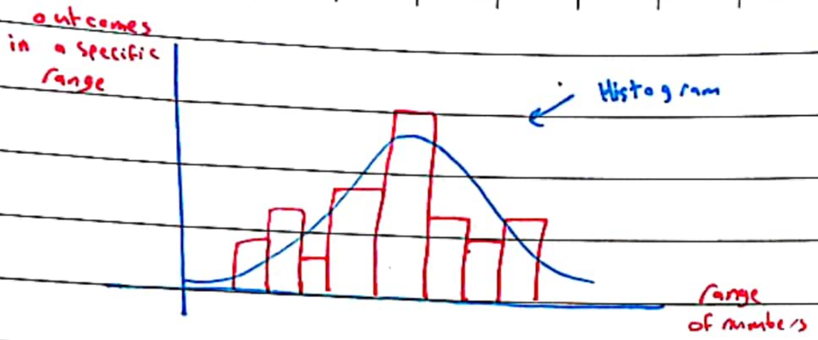
④

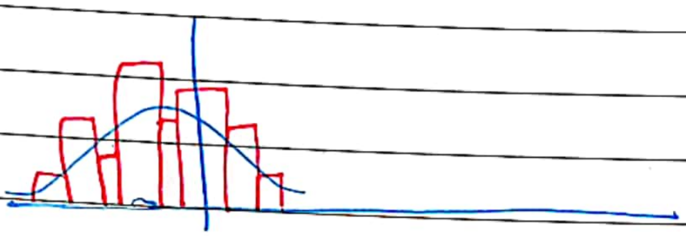


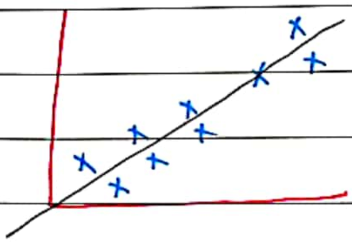
في accuracy
بس مانها Precise

σ لـ ② اكبر من ①

accurate ✓
Precise ✗







* نعمل fit
ال Curve

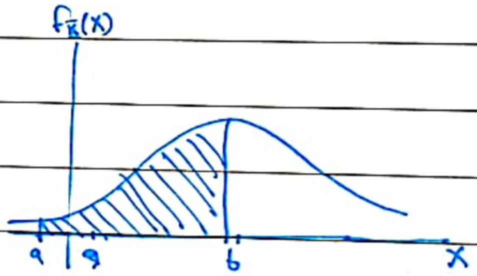
$X \sim N(\mu, \sigma^2)$

$\uparrow$
tilda

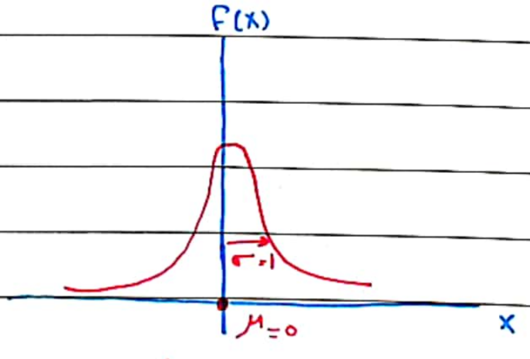
σ^2 — Variance

$X \sim$ normal with parameters μ, σ^2

$$P(a \le X \le b) = \int_a^b \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

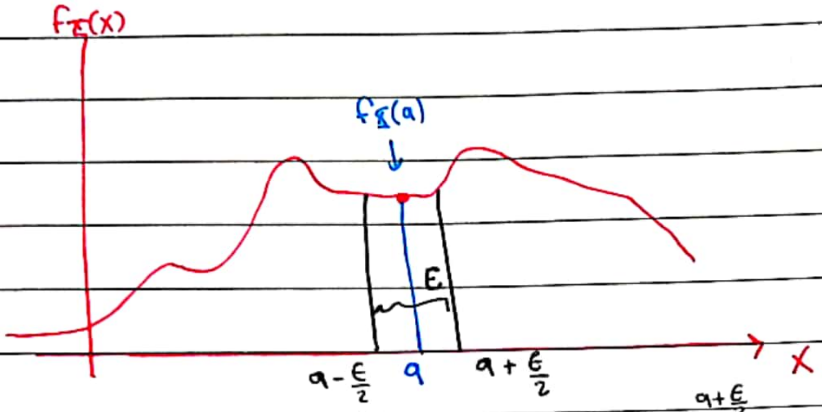


x	f(x)
0 →	
0.5 →	
1 →	
1.5 →	
2 →	
2. →	
⋮	



$\uparrow$
Standard normal

In Math	In Engineering
Normal	Gaussian
Standard Normal	Normal

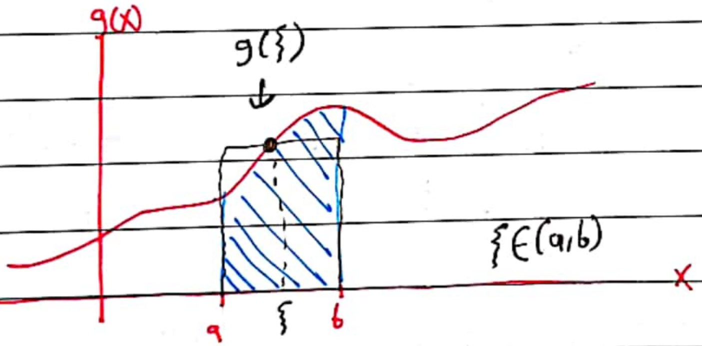


$$P(X=a) = \int_a^b f_X(x)\,dx = 0 = \lim_{\epsilon \to 0} \int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} f_X(x)\,dx = \lim_{\epsilon\to 0} f(\xi)[\epsilon]$$

$$= f(a)\epsilon$$

لما ال ϵ ← 0 بنبطل نميز ال a عن ξ

* Mean value theorem for integrals:



$\xi \in (a,b)$

$$\int_a^b g(x)\,dx = g(\xi)[b-a]$$

→ In this context $P(a)$ is a measure of how close X is to belong near a.
$P(X=a)$

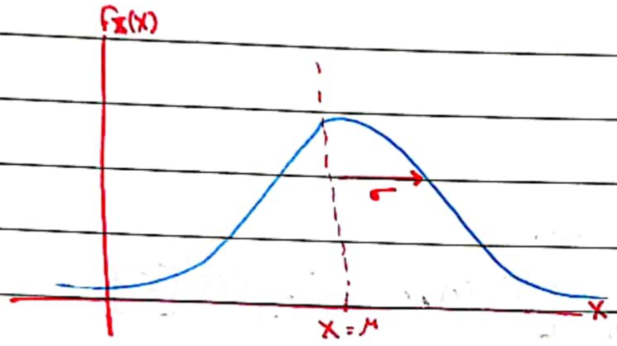
$\boxed{20/11}$

X is said to be a normal r.v. if $f_{\underline{X}}(x)$ is given by

$$f_{\underline{X}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

μ - Statistical average (or mean)

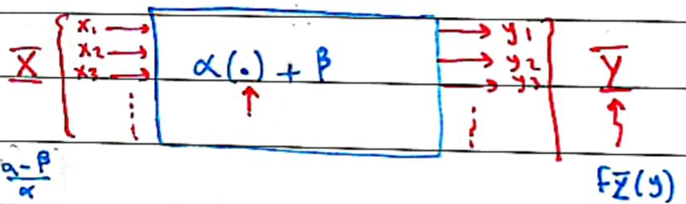
σ - Standard deviation



Let $X \sim N(\mu, \sigma^2)$ & let $Y = \alpha X + \beta$

$f_{\overline{Y}}(y)$

$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$



~~$F_{\underline{Y}}(y)$~~

$$F_{\overline{Y}}(a) = P(\overline{Y} \leqslant a) = \int_{-\infty}^{a} f_{\underline{Y}}(y)\,dy$$

$$= P(\alpha X + \beta \leqslant a) = P\left(X \leqslant \frac{a-\beta}{\alpha}\right) = \int_{-\infty}^{\frac{a-\beta}{\alpha}} f_{\underline{X}}(x)\,dx = F_{\underline{X}}\left(\frac{a-\beta}{\alpha}\right)$$

$$\Longrightarrow F_{\overline{Y}}(a) = F_{\underline{X}}\left(\frac{a-\beta}{\alpha}\right)$$

$$F_Y(a) = \int_{-\infty}^{\frac{a-\beta}{\alpha}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$\frac{\partial F_{\underline{Y}}(a)}{\partial a}$

$f_{\underline{Y}}(a)$

Change of variables

$$\begin{cases} V = \alpha x + \beta = \cancel{\alpha}\left(\frac{a-\cancel{\beta}}{\cancel{\alpha}}\right) + \cancel{\beta} = a \\ x = \frac{V-\beta}{\alpha} \\ dx = \frac{1}{\alpha}\,dV \end{cases}$$

$$F_{\underline{Y}}(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\,\sigma\alpha} e^{-\frac{1}{2}\left(\frac{\frac{V-\beta}{\alpha} - \mu}{\sigma}\right)^2} dV$$

$$F_Y(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\,(\alpha\sigma)} e^{-\frac{1}{2}\left[\frac{v-(\alpha\mu+\beta)}{\alpha\sigma}\right]^2} \partial v$$

$\downarrow$

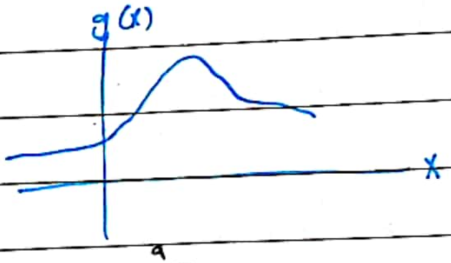
$$\frac{\partial F_Y(a)}{\partial a} = f_Y(a) = \frac{\partial}{\partial a}\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\,(\alpha\sigma)} e^{-\frac{1}{2}\left[\frac{v-(\alpha\mu+\beta)}{\alpha\sigma}\right]^2} \partial v$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,(\alpha\sigma)} e^{-\frac{1}{2}\left[\frac{y-(\alpha\mu+\beta)}{\alpha\sigma}\right]^2}$$

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \xrightarrow{\boxed{Y=\alpha X+\beta}} \frac{1}{\sqrt{2\pi}\,\boxed{\alpha\sigma}} e^{-\frac{1}{2}\left(\frac{y-(\alpha\mu+\beta)}{\boxed{\alpha\sigma}}\right)^2}$$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$N(\mu, \sigma^2)$ $\qquad\qquad\qquad$ $N((\alpha\mu+\beta), (\alpha\sigma)^2)$



$$y(x) = \int_{-\infty}^{a} g(x)\,dx$$

$\downarrow$

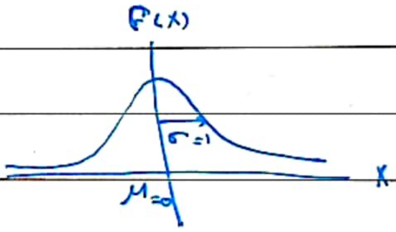
$$\frac{\partial y(x)}{\partial a} = g(a)$$

Corollary:

If I let $\alpha = \frac{1}{\sigma}$ & $\beta = -\frac{\mu}{\sigma}$

① $\mu_y = \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} = 0$

② $\alpha\sigma = \frac{1}{\sigma}\sigma = 1$



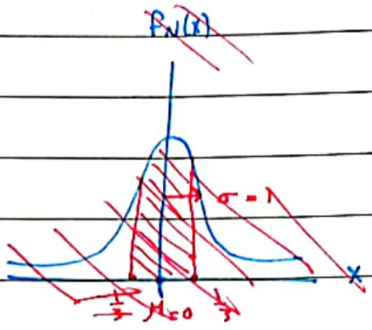
22/11

$$\begin{cases} \mu_y = \alpha \mu_x + \beta \\ \sigma_y = \alpha \sigma_x \end{cases}$$

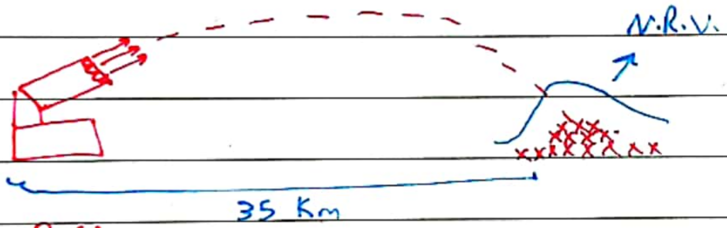
If I let $\alpha = \frac{1}{\sigma_x}$, $\beta = -\frac{\mu_x}{\sigma_x}$

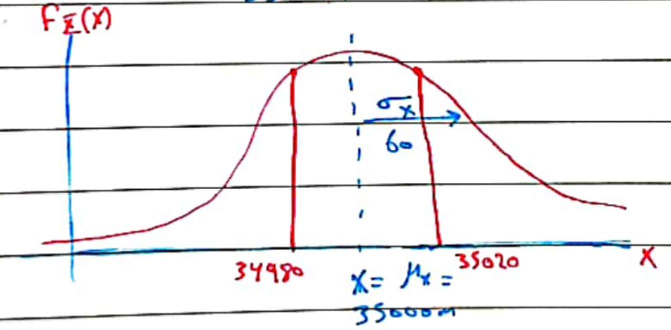
$$\Rightarrow \mu_y = \frac{1}{\sigma_x} \mu_x - \frac{\mu_x}{\sigma_x} = 0$$

$$\sigma_y = \frac{1}{\sigma_x} \cdot \sigma_x = 1$$



A rocket launcher fires projectiles at a target that is 35 Km's away. Suppose that the shells fall within 60 meters from target point. Find the probability that the shells can ~~by~~ be located 20 meters from the target point.





$$P(34980 \leqslant \overline{X} \leqslant 35020) = \int_{34980}^{35020} \frac{1}{\sqrt{2\pi}\, 60} e^{-\frac{1}{2}\left(\frac{X-35000}{60}\right)^{②}} dx$$

② → ما بقدر اكامل سهل بسبب التربيع

لو حكينا عندي 1000 قذيفة بطلع 259 قذيفة

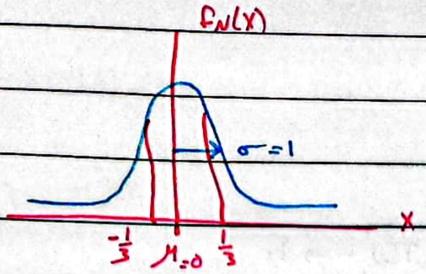
$$P(34980 \leqslant \overline{X} \leqslant 35020) =$$

طرحت μ_x وقسمت على σ_x

$$= P\left(\frac{-20}{60} \leqslant \left[\frac{X-35000}{60}\right] \leqslant \frac{20}{60}\right)$$

من الجدول

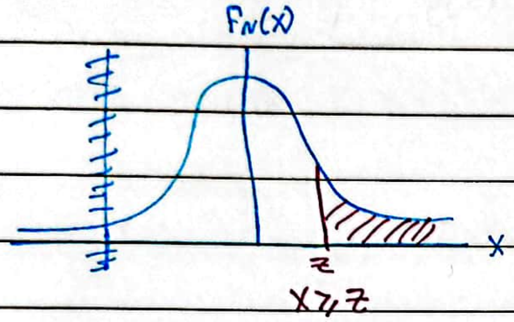
$$= P\left(\underset{-0.33}{-\frac{1}{3}} \leqslant X \leqslant \underset{0.33}{\frac{1}{3}}\right) = 1 - 2 \cdot F_N(0.33) \cong 1 - 2(0.37070) = 0.2586$$



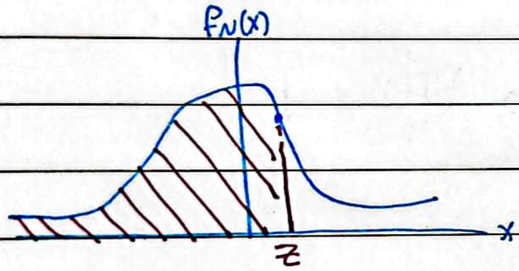
$$\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{\sqrt{2\pi(1)}} \cdot e^{-\frac{1}{2}\boxed{x^2}} \, dx$$

عالجداول:

⊛



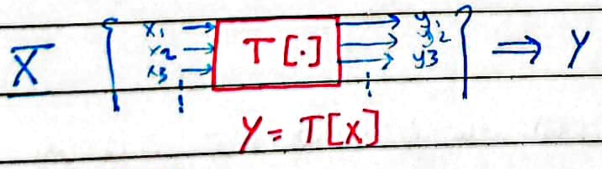
$$P(X \geq Z) = \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$$



$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$$

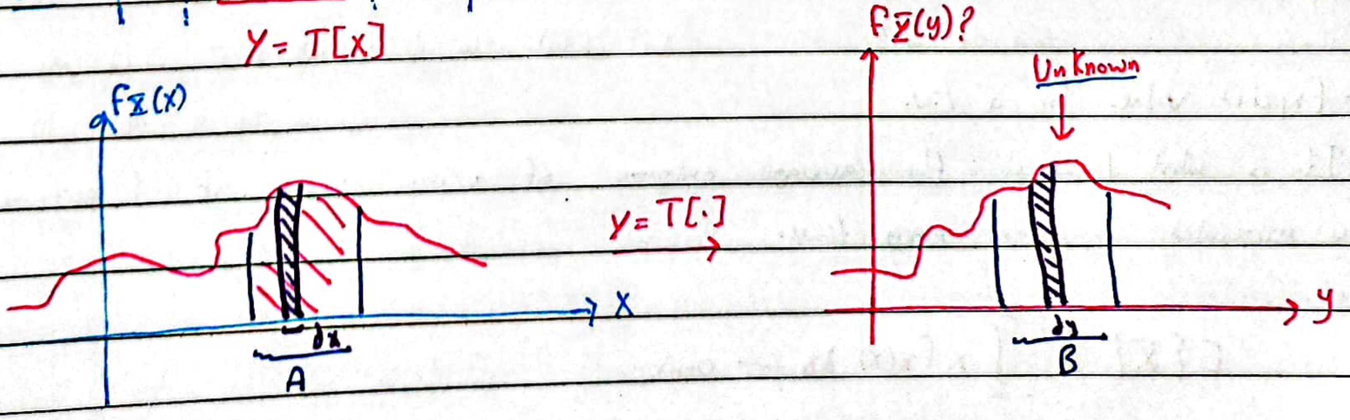
$\boxed{24/11}$ Expectation of a r.v:

- Transformation of random variables.



Assum that $T^{-1}[\cdot]$ exists $\overset{\text{such that}}{\exists}$ $X = T^{-1}[Y]$

ما بقدر اعمل تكامل لانو ال Function مش معروفة



$$P(X \in A) = \int_A f_X(x)\,\partial x$$

$$P(X \in A) \underset{\text{must}}{=} P(Y \in B)$$

$$\partial x \longrightarrow \boxed{T[\cdot]} \longrightarrow \partial y$$

الوحيد الي مش معروف

$$\boxed{f(x)\,\partial x = \overset{\uparrow}{f_Y(y)}\,\partial y} \longrightarrow f_Y(y) = f_X(x)\frac{\partial x}{\partial y}$$

Transf هاد قانون ال ←
$$\boxed{f_Y(y) = f_X(x = T^{-1}(y))\left|\frac{\partial T^{-1}(y)}{\partial y}\right|}$$

Example :

let $X \sim N(\mu, \sigma^2) \longrightarrow \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$X \longrightarrow \boxed{T[\cdot]} \longrightarrow Y$$

$$T[x] = Y = \underline{\alpha X + \beta} \longrightarrow T^{-1}[Y] = X = \frac{y-\beta}{\alpha} \longrightarrow \frac{\partial T^{-1}(y)}{\partial y} = \frac{1}{\alpha}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{\frac{y-\beta}{\alpha}-\mu}{\sigma}\right]^2} \cdot \left|\frac{1}{\alpha}\right|$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma\alpha} e^{-\frac{1}{2}\left[\frac{y-(\alpha\mu+\beta)}{\alpha\sigma}\right]^2}$$

$$X \sim N(\alpha\mu + \beta, (\alpha\sigma)^2)$$

Expectation of a random Variable :

يعني قديش متوقع ال Outcome بالمتوسط من خلال اني اعمل
هاي التجربة اكثر من مرة.

- Expected Value for a r.v.

It is what I expect the average outcome of a r.v. to be if I performed the associated experiment many times.

Def.

$$E\{X\} \triangleq \int_{-\infty}^{\infty} x\, f_X(x)\,\partial x \text{ — cont.}$$

$$E[X] \triangleq \sum_{i \in R(x)} i \cdot P_X(i) \text{ — discrete}$$

$\int_{-\infty}^{\infty} x \, g(x) \, \partial x$ $\qquad\qquad$ $\int_{-\infty}^{\infty} t \cdot g(t) \, \partial t$

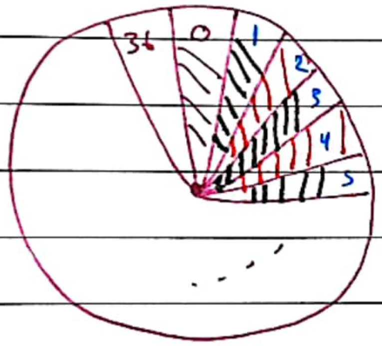
$E[X]$ — 1st moment

— Statistical average

— Mean

$$\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \partial x$$

Analyze Roulette :



Game 1 : you bid a 1$ on a number

~ If your number comes you receive 38 $

~ If complement of your number you lose

- What is the expected net amount that you win ?

X = "Net amount you win" ; $R(X) = \{\overset{\text{اذا خسرت}}{-1}, \overset{\text{اذا فزت}}{35}\}$; $P_{\bar{X}}(i) = \{\frac{36}{37}, \frac{1}{37}\}$

$$E[\bar{X}] = \sum_{i \in R(X)} i \, P_{\bar{X}}(i) = (-1)\left(\frac{36}{37}\right) + (35)\left(\frac{1}{37}\right)$$

$$= -\frac{1}{37}$$

$$= -0.027027 \ \$ = -2.7 ¢ \rightarrow \text{cent}$$

← اني بخسر

Here, I expect to lose 2.7 ¢ per play if I played many times

⇒ play 1000 times

⇓

(1000) (2.7 ¢) = 27 $ خسارة اذا لعب 1000 مرة يتوقع يخسر 27$

او اذا لعب نفس اللعبة 1000 شخص يتوقع مجموعهم يربح الكازينو 27$

$\boxed{27/11}$ Scenario of game 2:

~ you bit a 1$ on a color.

- If your Color comes, you receive 2$.
- If Complement of your color, you to lose.
- If green, you spin again <u>until</u> your color.
  هون معناها بهدف
  - If your color comes, you get your 1$ back
  - If complement, you lose.

باللعبة هاي ممنوع اراهن عالأخضر يا (اسود او احمر)

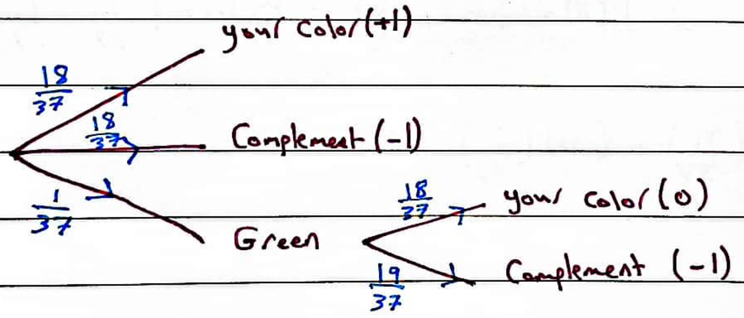
* What is the expeted net amount that you win?

X = "Net amount that you win"

$$E[X] = \sum_{i \in R(X)} i \, P_X(i)$$

$$R(X) = \{-1, 0, +1\}$$

$$P_X(i) = \{ \quad , \quad , \quad \}$$



$$\cancel{E[X] = (-1)\left[\left(\frac{18}{37}\right)+\left(\frac{1}{37}\right)\right] + (1)\left(\frac{18}{37}\right) + (0)}$$

$$\left(\frac{19}{37}\right)$$

$$E[X] = (-1)\left[\left(\frac{18}{37}\right)\overset{\uparrow}{+}\left(\frac{1}{37}\right)\right] + (0)\left[\left(\frac{1}{37}\right)\left(\frac{18}{37}\right)\right] + (1)\left(\frac{18}{37}\right)$$

$$= (-1)\left[\frac{(18)(37)+(19)}{(37)^2}\right] + \frac{18}{37} = -1.388 ¢$$

$$= -0.0138787 \$$$

↲

I expect to lose 1.388 cents per play if I played many times. ⟹ play 1000 times ⟹ you would expect to lose 13.88 $

Let $X_b \sim$ binomial with parameters (n,p)

Find $E[X_b]$

X_b = "# of successes in n trials" $\quad R(X) = \{0,1,2,\dots,n\}$

$$E[X_b] \doteq \sum_{i \in R(x)} i\, P_X(i) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i \left(\frac{n!}{i!(n-1)!}\right) p^i (1-p)^{n-i}$$

عشان [illegible] الصفر بيكون تماما صفر ←

$$= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \underbrace{\frac{(n-1)!}{(i-1)!(n-i)!}\, p^{i-1} (1-p)^{n-i}}_{\binom{n-1}{i-1} p^{i-1}(1-p)^{n-i}}$$

Include a change of variables $\longrightarrow$ $k = i-1 \longrightarrow i = k+1$

$$\Rightarrow np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$= np \left[\underbrace{\binom{n-1}{0} p^0 (1-p)^{n-1} + \binom{n-1}{1} p^1 (1-p)^{n-2} + \dots + \binom{n-1}{n-1} p^{n-1} (1-p)^0}_{(p+(1-p))^{n-1} = 1^{n-1} = 1} \right]$$

$$\boxed{E[X_b] = np}$$

$n = 60$, $p = \frac{1}{2}$

$$E[X] = np = (60)\left(\frac{1}{2}\right) = \boxed{30}$$

Let X be a geometric r.v. with $p(S) = p$

Find $E[X_g]$

X_g = "# of trials until a Success"

$E[X_g]$ – Expected number of trials until a Success.

$$E[X_g] = \sum_{n \in R} n \, P_{\underline{X}}(n) = \sum_{n=1}^{\infty} n \, P(X_g = n)$$

مارح تفرق عندي ← $$= \sum_{n=0}^{\infty} n \cdot (1-P)^{n-1} \cdot P \quad , \quad \text{let } \boxed{q = 1-P}$$

$$= P \sum_{n=0}^{\infty} n \, q^{n-1}$$

$$= P \sum_{n=0}^{\infty} \frac{\partial}{\partial q} \{ q^n \}$$

$$= P \cdot \frac{\partial}{\partial q} \left\{ \sum_{n=0}^{\infty} q^n \right\}$$

$$= P \cdot \frac{\partial}{\partial q} \left\{ \frac{1}{1-q} \right\} \qquad |q| < 1$$

$$= P \left[\frac{0 - (-1)(1)}{[1-q]^2} \right]$$

$$\Rightarrow \quad E[X_g] = P \left[\frac{1}{[1-(1-P)]^2} \right]$$

$$\boxed{E[X_g] = \frac{P}{P^2} = \boxed{\frac{1}{P}}}$$

$\boxed{29/11}$ يمثل متوسط حصول الevent.

let X be a Poisson r.v. with parameter X.

Find $E[X_p]$

$X_p =$ " "

$R(X_p) = \{0,1,2,\ldots\}$

$P_X(i) = \frac{\lambda^i}{i!} e^{-\lambda}$, $\underset{\text{Integer}}{i} \geq 0$, $\lambda > 0$

$$E[X_p] = \sum_{i \in R(X_p)} i \, P_X(i) = \sum_{i \in R(X_p)} i \cdot P(X_p = i)$$

$$= \sum_{i=0}^{\infty} i \cdot \frac{\lambda^i}{i!} e^{-\lambda} = \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)} e^{-\lambda}$$

$$= e^{-\lambda} \cdot \lambda \sum_{i=1} \frac{\lambda^{i-1}}{(i-1)!}$$

$\boxed{\text{Let } k = i-1}$

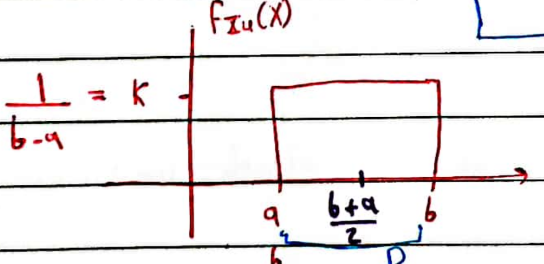
$$= e^{-\lambda} \cdot \lambda \overbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}^{e^\lambda} = e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$$

$$\Longrightarrow \boxed{E[X_p] = \lambda}$$

Let X_u be uniform over (a,b)

Find $E\{X_u\}$ (↑ contin.)

$\boxed{K = \frac{1}{b-a}}$



$$E\{X_u\} = \int_a^b \frac{1}{b-a} \cdot x \, dx = \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b$$

$\text{half}\{D\} = \frac{b-a}{2}$

① $a + \frac{b-a}{2} = \frac{2a+b-a}{2}$
$= \frac{a+b}{2}$

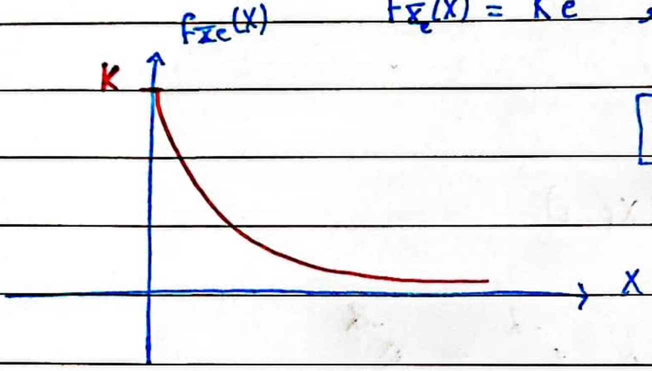
② $b - \frac{b-a}{2} = \frac{2b-b+a}{2} =$
$= \frac{a+b}{2}$

$$= \frac{1}{(b-a)^2}\left[b^2 - a^2\right] = \frac{1}{2\cancel{(b-a)}}(b+a)\cancel{(b-a)} = \frac{b+a}{2}$$

$$\Longrightarrow \boxed{E\{X_u\} = \frac{b+a}{2}}$$

Let X_e be exponentially distributed with parameter $\overset{\text{rate}}{\overset{\uparrow}{\lambda}}$.

$$f_{X_e}(X) = k e^{-\lambda x}, \quad \boxed{x \geq 0}$$



$$\boxed{k = \lambda}$$

$$E\{X_e\} = \int_{-\infty}^{\infty} \lambda x e^{-\lambda x} = \lambda \int_{0}^{\infty} x e^{-\lambda x} dx \longrightarrow \text{By Parts}$$

$$\boxed{E\{X_e\} = \frac{1}{\lambda}}$$

⊛ ① Integration by Parts

② Might need to exploit La'Hopitals rule.

$\boxed{1/12}$ Recall that for two event E & F

بالكتب مرات

$P_r(E,F)$

$\uparrow$ and

$\cap$

$\nearrow P(EF)$

$$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{P(E,F)}{P(F)} = \frac{P(E.F)}{P(F)}$$

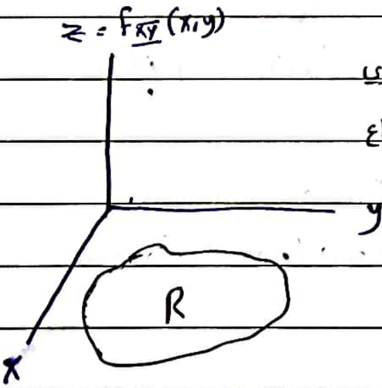
X and Y are said to be jointly distributed random variable if there exists a function i.e $F_{XY}(X,y)$, such that

$$\int_{-\infty}^{\infty}\int \underbrace{F_{XY}(X,y)}\, \partial x\, \partial y = 1$$

Furthere for some region R in $\mathbb{R}^2$:

$$\iint_R F_{XY}(X,y)\, \partial x\, \partial y = P(X, Y \in R)$$

This function $F_{XY}(X,y)$ as such is called the joint probability density function for $\underline{X}$ & $\underline{Y}$



* اكثر من متغير واحد بتغيروا مع بعض
وسلوكهم هنا مرتبط ببعض زي مثلا الضغط الجوي
ودرجة الحرارة والرطوبة بينهم علاقة زي مثلا ارتفاع
درجة الحرارة يؤدي الى ارتفاع الرطوبة بس ممكن
ترتفع الرطوبة مع انخفاض درجة الحرارة

Define :

Conditional density function

$$F_{X|Y}(X|y) \triangleq \frac{F_{XY}(x,y)}{F_Y(y)} \Bigg]\ X \,\&\, Y \text{ are continuous r.v.'s}$$

Like wise :

$$f_{\underline{Y}|\underline{X}} = \frac{f_{\underline{X}\underline{Y}}(x,y)}{P_{\underline{X}}(x)} \Bigg\} \quad X \,\&\, Y \text{ are continuous r.v.'s}$$

Similarly for $\underline{X}$ & $\underline{Y}$ discrete :

$$P_{\underline{X}|\underline{Y}}(i|j) = \frac{P_{\underline{X}\underline{Y}}(i,j)}{P_{\underline{Y}}(j)} \quad \& \quad P_{\underline{Y}|\underline{X}}(j|i) = \frac{P_{\underline{X}\underline{Y}}(i,j)}{P_{\underline{X}}(i)}$$

Let $\underline{X}$ & $\underline{Y}$ be jointly continuous r.v.'s & I seek E to find :

ثبتنا ال Y

a given value for y

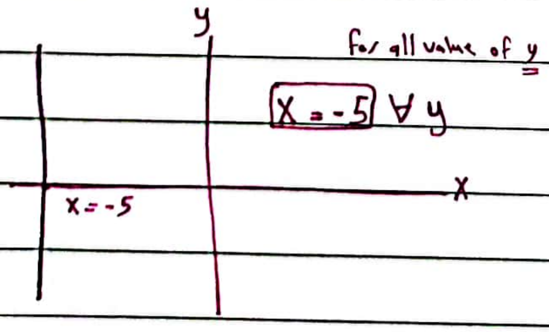
$$\underbrace{E\{\underline{X}|\underline{Y}=y\}}_{} = \int_{-\infty}^{\infty} x \, \underbrace{f_{\underline{X}|\underline{Y}}(x|y)}_{} \, dx$$

هو الX بحلي ال expectation

*بدي استخرج سلوك كل واحد لحاله

$$\begin{tikzcd}
 & P_{\underline{X}\underline{Y}}(i,j) \arrow[dl] \arrow[dr] & & & f_{\underline{X}\underline{Y}}(x,y) \arrow[dl] \arrow[dr] & \\
P_{\underline{X}}(i) & & P_{\underline{Y}}(j) & f_{\underline{X}}(x) & & f_{\underline{Y}}(y)
\end{tikzcd}$$

These are called
the marginal density/mass functions



for all value of y

$\boxed{x=-5}\ \forall y$

Suppose that X & Y are jointly discrete r.v.s behaving according to Some prescribed joint mass function $P_{XY}(i,j)$

$\rightarrow P_{XY}(i,j) = P(X=i, Y=j)$

Suppose that all you have get is $P_{XY}(i,j)$ & I seek to compute $P_X(a)$
$\downarrow$
$P(X=a)$

$$P(X=a) = P(X=a, \forall y) = P(X=a, -\infty < Y < \infty)$$

$\uparrow \qquad \downarrow$

$$P_X(a) = \sum_{y \in R(Y)} P_{XY}(a,y) \longrightarrow$$ ثبتنا ال a ←

$$P_Y(b) = \sum_{X \in R(} P_{XY}(X,b) \longrightarrow$$ هون خلينا ال X متغير

Marginal mass funtions

let X & Y be jointly Continuous according to some f $f_{XY}(x,y)$ then

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$\int_{-\infty}^{\infty} g(x,y)\,dy = f(x)$$

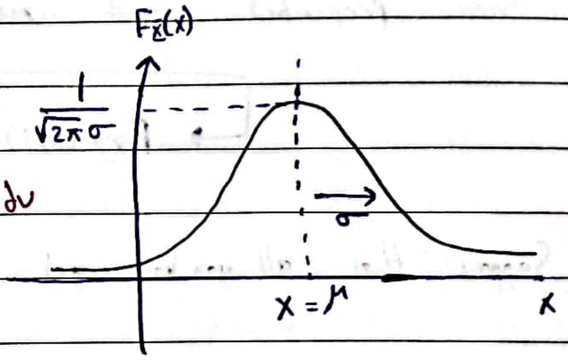
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

Marginal density function

$\boxed{4/12}$ X_N is normal with parameters μ, σ^2

$X_N \sim N(\mu, \sigma^2)$

Task: Find $E\{X_N\}$



$$E\{X_N\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \rightarrow \int e^{v} dv$$

الصورة الناتجة هيك بدنا

If I let $v = -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$

$$dv = -1\left(\frac{x-\mu}{\sigma}\right)\cdot\frac{1}{\sigma}dx = -\frac{(x-\mu)}{\sigma^2}dx$$

بسبب ال term الثاني الي ضفناه

$$= \frac{-\sigma^2}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} \underbrace{-\frac{(x-\mu+\cancel{\mu})}{\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} \mu\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= -\frac{1}{\sqrt{2\pi}\,\cancel{\sigma}}\sigma^{\cancel{2}}\int_{-\infty}^{\infty} e^{v} dv + \frac{1*\mu}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{-\sigma}{\sqrt{2\pi}} e^{v}\Big|_{x=-\infty}^{\infty} + \mu\,\boxed{\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx} = 1$$

$$= \frac{-\sigma}{\sqrt{2\pi}}\cancel{[0-0]} + \mu.1$$

$$\boxed{E\{X_N\} = \mu}$$

Let $f_{XY}(x,y) = \begin{cases} \frac{1}{2} y e^{-xy}, & 0 < x < \infty \;;\; 0 < y < 2 \\ 0, & \text{elsewhere} \end{cases}$

Find $E\{e^{X/2} \mid Y=1\}$

$$= \int_{0}^{\infty} e^{x/2} f_{X|Y}(x|y=1)\, dx$$

$$f_{X|Y}(x|y=1) = \frac{f_{XY}(x,1)}{f_Y(1)} = \frac{\cancel{\tfrac{1}{2}}e^{-x}}{\cancel{1/2}} = e^{-x} \;;\; 0 < x < \infty$$

$$f_Y(y=1) = \int_{0}^{\infty} \frac{1}{2}e^{-x}dx = -\frac{1}{2}e^{-x}\Big|_{0}^{\infty} = -\frac{1}{2}[0-1] = \frac{1}{2}$$

$$E[e^{X/2}|y=1] = \int_0^\infty e^{-x} e^{x/2} \, dx$$

$$= \int_0^\infty e^{-x/2} \, dx = -2 e^{-x/2} \Big|_0^\infty = -2[0-1] = \boxed{2}$$

* ما الو علاقة بالمثال الي قبل

$$E\{e^{X/2}\} = \int_{-\infty}^{\infty} \underset{\uparrow}{e^{x/2}} \, f_{g(x)}(g(x)) \, dg(x)$$

The law of the Unconscious Statistician:

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx$$

$$E\{e^{X/2}\} = \int_{-\infty}^{\infty} e^{x/2} \, f_X(x) \, dx$$

$\boxed{6/12}$ Linearity of expectation operator :

The expected value operator is a linear operator:

$$E\{\alpha X+\beta Y\} \overset{??}{=} \alpha E\{X\} + \beta E\{Y\}$$

In general any function $g(.)$ is called a linear function iff

$$g(\alpha X+\beta y) = \alpha g(X) + \beta g(y)$$

Let X & Y be jointly distributed random variables according to some joint pdf, $f_{XY}(x,y)$.

$$\to E\{\alpha X+\beta Y\} = \iint_{-\infty}^{\infty} (\alpha X \overset{\downarrow}{+} \beta Y)\, f_{XY}(x,y)\, dx\, dy$$

$$= \iint_{-\infty}^{\infty} \alpha x\, f_{XY}(x,y)\, dx\, dy + \iint_{-\infty}^{\infty} \beta y\, f_{XY}\, dx\, dy$$

$$\cancel{= \alpha \iint}$$

$$= \alpha \int_{x=-\infty}^{\infty} x \underbrace{\int_{y=-\infty}^{\infty} f_{XY}(x,y)\, dy}_{f_X(x)}\, dx + \beta \int_{y=-\infty}^{\infty} y \underbrace{\int_{x=-\infty}^{\infty} f_{XY}(x,y)\, dx}_{f_Y(y)}\, dy$$

$$= \alpha \int_{x=-\infty}^{\infty} x\, f_X(x)\, dx + \beta \int_{y=-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \alpha E\{X\} + \beta E\{Y\} \quad \#$$

* Finding Expectation by conditioning :

$$E[X] = \sum_{j \in R(Y)} E[X|Y=j]\, P_Y(j)$$

$$\sum_{j \in R(Y)} E[X|Y=j] \,.\, P_Y(j) = \sum_{j \in R(Y)} \sum_{i \in R(X)} i\, P_{X|Y}(i|j) \,.\, P_Y(j)$$

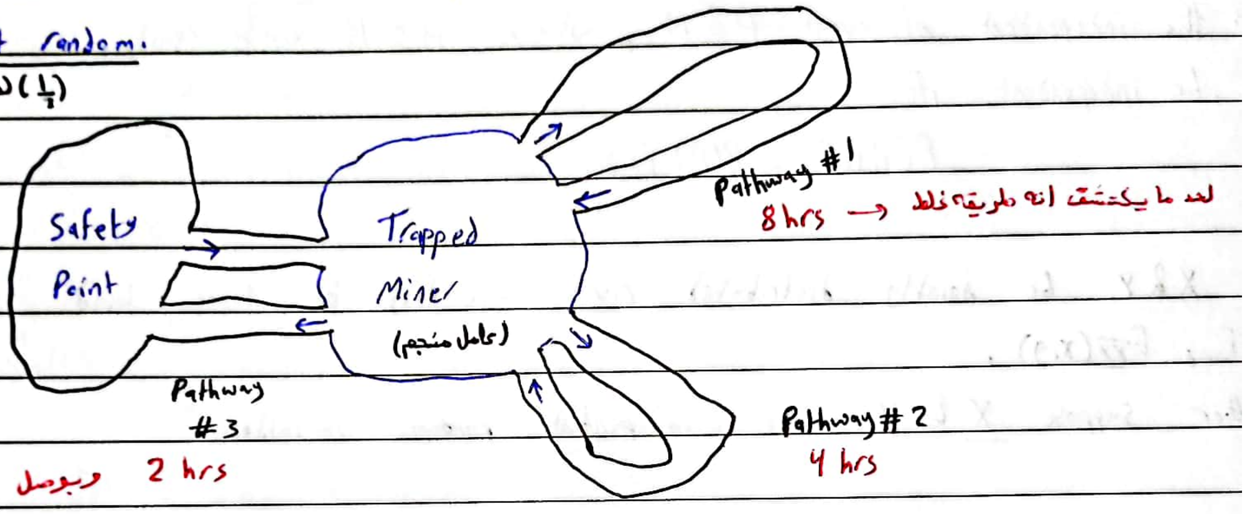
$$= \sum_{j \in R(Y)} \sum_{i \in R(X)} i \cdot \frac{P_{XY}(i,j)}{\cancel{P_Y(j)}}\, \cancel{P_Y(j)}$$

$$= \sum_{i\in R(X)} i \underbrace{\sum_{j\in R(Y)} P_{XY}(i,j)}_{P_X(i)}$$

$$= \sum_{i\in R(X)} i \cdot P_X(i) \triangleq E[X] \quad \#$$

Example : Assume that the miner picks any pathway at random.

($\frac{1}{3}$) لكل واحد



Random Variable X usually has to do with time.

Random Variable Y has to do with Selection (choice)

-What is the expected time that would lead the miner to a Safe exit?

X = "time consumed by miner to reach destination"

Y = "# of particular Pathway"

$$E[X] = \underline{E[X|Y=1]} \cdot P_Y(1) + \underline{E[X|Y=2]}\, P_Y(2) + \underline{E[X|Y=3]}\, P_Y(3)$$

$$E[X] = [8 + E[X]] \cdot \frac{1}{3} + [4 + E[X]] \cdot \frac{1}{3} + [2] \cdot \frac{1}{3}$$

$$E[X]\left[\overbrace{1 - \frac{1}{3} - \frac{1}{3}}^{\frac{1}{3}}\right] = \frac{8}{3} + \frac{4}{3} + \frac{2}{3}$$

$$\Longrightarrow \boxed{E[X] = 14 \text{ hrs}}$$

$\boxed{8/12}$

X & Y are said to be independent r.v.'s iff

$$f_{\underline{XY}}(x,y) = f_{X}(x)\, f_{Y}(y)$$

This is commensurate with our earlier leverage (استندنا) on the independent of event A & B, where A & B were said to be independent iff

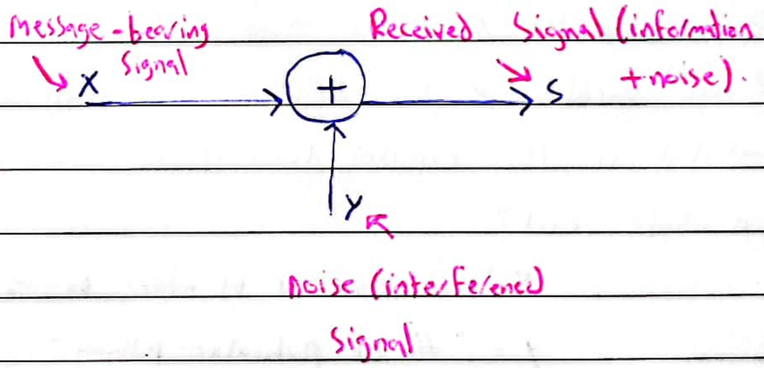
$$P(A \cap B) = P(A)\, P(B)$$

Let X & Y be jointly distributed r.v.'s according to some joint PdF, $f_{\underline{XY}}(x,y)$.

Further suppose X & Y are independent random variables.

Let's consider having a need to add X and Y to form a new r.v., S.

Task: Find the pdf for S, S=X+Y



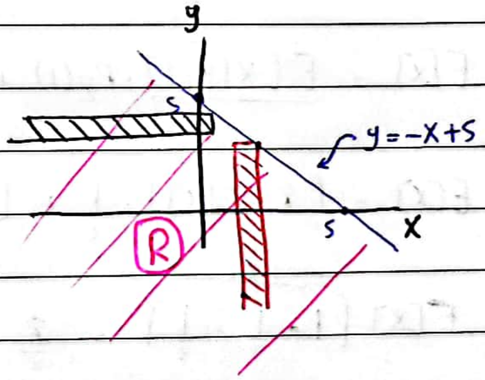
اسم ال r.v

$f_{S}(s)$

القيم الي ممكن اعوضنا متغيرة

$$P(\underbrace{X+Y}_{S} \leq s) = P(S \leq s) = F_S(s)$$

$$P(\underbrace{Y \leq -X+s}_{R}) = F_{Y}(s-x)$$



$$\underset{F_S(s)}{\overset{\|}{P(S\leq s)}} = P(Y \leq -X+s) = \iint_R f_{\underline{XY}}(x,y)\, dx\, dy$$

$$= \iint_{\{R|y\leq -x+s\}} f_{X}(x)\, f_{Y}(y)\, dx\, dy$$

$$F_S(s) = \int_{y=-\infty}^{\infty} f_Y(y) \underbrace{\int_{x=-\infty}^{s-y} f_X(x)\, \partial x}\, \partial y$$

$$F_S(s) = \int_{y=-\infty}^{\infty} f_Y(y)\, F_X(s-y)\, \partial y$$

$$\frac{\partial F_s(s)}{\partial s}$$

$$f_s(s) = \frac{\partial}{\partial s} \left\{ \int_{y=-\infty}^{\infty} f_Y(y)\, F_X(s-y)\, \partial y \right\}$$

$$= \int_{y=-\infty}^{\infty} f_Y(y) \underbrace{\frac{\partial}{\partial s}\{ F_X(s-y) \}}_{f_X(s-y)} \partial y$$

$$\rightarrow f_s(s) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(s-y)\, \partial y$$

$$f_s(s) = f_Y(y) * f_X(x)$$

Also,

$$f_s(s) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(s-x)\, dx$$

Let $X \sim$ Poisson with parameter λ_1

$Y \sim$ Poisson with parameter λ_2

X & Y are added to form a sum, $S = X+Y$

Find $P_S(S)$

$$P_X(i) = \frac{\lambda_1^i}{i!} e^{-\lambda} \qquad P_Y(j) = \frac{\lambda_2^j}{j!} e^{-\lambda_2}$$

$$P_S(S) = \sum_{\text{every } i} P_X(i)\, P_Y(S-i)$$

$$= \sum_{i=0}^{S} \frac{\lambda_1^i}{i!} e^{-\lambda} \cdot \frac{\lambda_2}{(S-i)!} e^{-\lambda_2}$$

$R(X) = \{0, 1, 2, \dots\}$

$R(Y) = \{0, 1, 2, \dots\}$

$$e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{S} \left(\frac{\lambda_1}{\lambda_2}\right)^i \frac{\lambda_2^S}{i!(S-i)!}$$

بدنا نكمل من هون

$\boxed{11/12}$ $\begin{cases} X \sim \text{Poisson with parameter } \lambda_1, \\ Y \sim \text{Poisson with parameter } \lambda_2. \end{cases}$

$S = X + Y$

$P_S(k) = ??$

$$\Rightarrow P_S(k) = P_X(k) * P_Y(k)$$

$$= \sum_{i=0}^{k} \frac{\lambda_1^i}{i!} e^{-\lambda_1} \cdot \frac{\lambda_2^{k-i}}{(k-i)!} e^{-\lambda_2}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!\,\cancel{k!}} \sum k! \frac{\lambda_1^i}{i!} \frac{\lambda_2^{k-i}}{(k-i)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \lambda_1^i \lambda_2^{k-i}$$

$$= e^{-(\lambda_1+\lambda_2)} \frac{\boxed{\sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i}} \longrightarrow (\lambda_1+\lambda_2)^k}{k!}$$

$$P_S(k) = e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^k}{k!} \longrightarrow S \text{ is a Poisson r.v. with } \boxed{\lambda_S = \lambda_1+\lambda_2}$$

⊛ X & Y are both Poisson with Parameters λ_1 & λ_2, respectively

Let $S = X + Y$

Find $P_{X|S}(i \mid s = n)$ لازم نحلها انا

Let X be binomial with parameters (n,p)

$*$Find $E[X_b]$

X = "# of Successes in n trials"

Let $X_i = \begin{cases} 1, & \text{if Success} \\ 0, & \text{otherwise} \end{cases}$

$i = 1,2,3,\ldots,n$

$X_b = X_1 + X_2 + X_3 + \ldots + X_n$ → number of Successes

$$E[X_b] = E[X_1 + X_2 + X_3 + \ldots + X_n]$$
$$= E[X_1] + E[X_2] + E[X_3] + \ldots + E[X_n]$$

But, let's consider the j-th r.v.

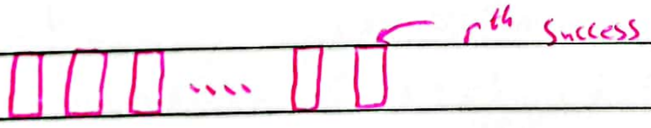
$$E[X_j] = 1 \cdot P(S) + 0 \cdot 0(1-P) = P \ \forall j$$

$$= \underbrace{P + P + \ldots + P}_{n \text{ times}}$$

$$= \boxed{np}$$

Conduct an experiment in however many trials as needed to get an r^{th} success.

Suppose that $P(S) = P$



X = "# of trials until r^{th} Success"

$$R(X) = \{r, r+1, r+2, \ldots\}$$

$$P_X(n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

$$E[X] = \sum_{n=r}^{\infty} n \cdot \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

Let Y_i = "# of trials until Success" ; $i = 1, 2, 3, 4, \ldots, r$

$$X = Y_1 + Y_2 + Y_3 + \cdots + Y_r$$

$$E[X] = E[Y_1 + Y_2 + Y_3 + \cdots + Y_r]$$

$$= E[Y_1] + E[Y_2] + \cdots + E[Y_r]$$

$$E[Y_1] = E[Y_2] = E[Y_3] = \cdots = E[Y_r]$$

$$E[Y_k] = \frac{1}{p} \quad \forall k$$

$$= \underbrace{\frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p}}_{r \text{ times}} = \boxed{\frac{r}{p}}$$

13/12 Moments of random variables :

The n^{th} moment of a random variable is defined as :

$$E\{\bar{X}^n\} = \int_{-\infty}^{\infty} x^n f_{\bar{X}}(x)\, dx$$

$n=1$:

$$E\{\bar{X}^1\} = \int_{-\infty}^{\infty} x^1 f_{\bar{X}}(x)\, dx = \int_{-\infty}^{\infty} x f_{\bar{X}}(x)\, dx \triangleq \mu_x$$

$n=2$:

$$E\{\bar{X}^2\} = \int_{-\infty}^{\infty} x^2 f_{\bar{X}}(x)\, dx \longleftarrow \text{Mean Square value}$$

$$\sqrt{E\{\bar{X}^2\}} = X_{rms}$$

* Cummulant ⟶ $E\{\bar{X}^n\}$ 4th بطلع

Central Moments :

- Moments about the mean :

n-th - Central moment

$$E\{(X-\mu_x)^n\} = \int_{-\infty}^{\infty} (x-\mu_x)^n f_{\bar{X}}(x)\, dx$$

$n=2$

$$E\{(X-\mu_x)^2\} = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_{\bar{X}}(x)\, dx = Var(X) = \sigma_{\bar{X}}^2$$

Thm :

$$E\{(X-\mu_x)^2\} = E\{X^2\} - \mu_x^2$$

$\downarrow$ invoke linearity

$$E\{x^2 - 2\mu_x X + \mu_x^2\} \overset{\downarrow}{=} E\{X^2\} - 2\mu_x \underset{\downarrow}{E\{X\}} + E\{\mu_x^2\}$$

$$= E\{X^2\} - 2\mu_x \mu_x + \mu_x^2$$

$$\boxed{Var(X) = E\{X^2\} - \mu_x^2}$$

$$Var(cX) = ? \longrightarrow Var(Y)$$

Let $Y = cX \longrightarrow \mu_Y = E\{Y\} = E\{cX\} = cE\{X\} = c\mu_X$

$$Var(Y) = E\{(Y-\mu_2)^2\} = E\{Y^2 - 2\mu_Y Y + \mu_Y^2\}$$

$$= E\{Y^2\} - 2\mu_Y E\{Y\} + \mu_Y^2$$

$$= E\{Y^2\} - \mu_Y^2$$

$$= E\{c^2X^2\} - c^2\mu_X^2 = c^2\{E\{X^2\} - \mu_X^2\}$$

$$\boxed{Var(cX) = c^2 Var(X)}$$

Let X & Y be independent random variables with a joint PDF $f_{XY}(x,y)$.

$$\hookrightarrow \quad Var(X+Y) = E\{(X+Y)^2\} - E^2\{X+Y\}$$

$$= E\{X^2 + 2XY + Y^2\} - [E[X]+E[Y]]^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - [E^2[X] + 2E[X]E[Y] + E^2[Y]]$$

$$= \{E[X^2] - \mu_X^2\} + \{E[Y^2] - \mu_Y^2\} + \underbrace{2\{E[XY] - E[X]E[Y]}_{0}$$

$$E[XY] = \iint_{-\infty}^{\infty} xy\, \underset{f_X(x)\quad f_Y(y)}{f_{XY}(x,y)}\, dx\, dy$$

$$= \iint_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$E[XY] = E[X] \cdot E[Y]$$

$$\Longrightarrow Var(X+Y) = Var(X) + Var(Y)$$

$$* \sum_{n=1}^{\infty} n \binom{n-1}{r-1} p^{r} (1-p)^{n-r} = \frac{r}{p}$$

$\boxed{15/12}$

Moment Generating Functions ; mgf :

Def :

$$\Phi_{\underline{X}}(t) \stackrel{\circ}{=} E\{e^{tx}\} = \int_{-\infty}^{\infty} e^{tx} f_{\underline{X}}(x)\, \partial x$$

Moment generating function for r.v. X.

$$\Phi'_{\underline{X}}(t) = \frac{\partial \Phi_{\underline{X}}(t)}{\partial t} = \frac{\partial}{\partial t} E\{e^{tx}\} = \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f_{\underline{X}}(x)\, \partial x = \int_{-\infty}^{\infty} x e^{tx} f_{\underline{X}}(x)\, \partial x$$

$$\Phi''_{\underline{X}}(t) = \frac{\partial^2 \Phi_{\underline{X}}(t)}{\partial t^2} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} x e^{tx} f_{\underline{X}}(x)\, \partial x = \int_{-\infty}^{\infty} x^2 e^{tx} f_{\underline{X}}(x)\, \partial x$$

$$\Phi'''_{\underline{X}}(t) = \frac{\partial^3 \Phi_{\underline{X}}(t)}{\partial t^3} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} x^2 e^{tx} f_{\underline{X}}(x)\, \partial x = \int_{-\infty}^{\infty} x^3 e^{tx} f_{\underline{X}}(x)\, \partial x$$

$$\vdots$$

$$\Phi^{(n)}_{\underline{X}}(t) = \int_{-\infty}^{\infty} x^n e^{tx} f_{\underline{X}}(x)\, \partial x$$

$$\Phi'_{\underline{X}}(t)\Big|_{t=0} = \int_{-\infty}^{\infty} x\, f_{\underline{X}}(x)\, \partial x = E[X] \qquad \Phi''_{\underline{X}}(t)\Big|_{t=0} = E\{X^2\}$$

$$\Phi'''_{\underline{X}}(t)\Big|_{t=(0)} = \int_{-\infty}^{\infty} x^3 f_{\underline{X}}(x)\, \partial x = E\{x^3\} \qquad \Phi^{(n)}_{\underline{X}}(t)\Big|_{t=0} = \int_{-\infty}^{\infty} x^n f_{\underline{X}}(x)\, \partial x = E\{x^n\}$$

$$Var(X) = E\{\underline{X}^2\} - E^2\{\underline{X}\}$$
$$= \Phi''_{\underline{X}}(t=0) - [\Phi'_{\underline{X}}(t=0)]^2$$

X_e is exponential with parameter λ.

$$f_{\underline{X}}(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Find the mgf for X_e.

$$\Phi_{\underline{X}_e}(t) = E\{e^{tX_e}\} = \int_{-\infty}^{\infty} e^{tx} \cdot \lambda e^{-\lambda x} \, dx = \int_{0}^{\infty} \lambda e^{x(t-\lambda)} \, dx$$

$$= \frac{\lambda}{t-\lambda} e^{x(t-\lambda)} \Big|_{x=0}^{\infty}$$

↳ luptale rule سؤ. (الو).

Let X be a binomial r.v. with parameters (n, p) Cont. دايما ال Φ بيكون

Find $\Phi_{\underline{X}_b}(t)$.

X_b = "# of Successes in n trials"
$R(X_b) = \{0, 1, 2, 3, \cdots, n\}$

$$\Phi_{\underline{X}_b}(t) = E[e^{tX_b}] = \sum_{i=0}^{n} e^{ti} P_{\underline{X}_b}(i) = \sum_{i=0}^{n} e^{ti} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} \binom{n}{i} (pe^t)^i (1-p)^{n-i}$$

$$\Phi_{\underline{X}_b}(t) = [pe^t + (1-p)]^n$$

Find $E[X_b]$ & $Var(X_b)$

$$E[X_b] = \Phi'_{\underline{X}_b}(t)\Big|_{t=0} = \frac{\partial}{\partial t}[pe^t + (1-p)]^n = n[p\overset{1}{e^{\cancel{t}}} + (1-p)]^{n-1} \cdot p\overset{1}{e^{\cancel{t}}}$$

$$= n(1)^{n-1} \cdot p = np$$

Remember
$$\Phi'_{\underline{X}}(t) = n[pe^t + (1-p)]^{n-1} \cdot pe^t$$

اشتقاق →

$$E[\underline{X}_b^2] = \Phi''_{\underline{X}_b}(t=0) = n(n-1)\underbrace{[p\overset{1}{e^{\cancel{t}}} + (1-p)]}_{1}^{n-2} \cdot p\overset{1}{e^{\cancel{t}}} \cdot p\overset{1}{e^{\cancel{t}}} + p\overset{1}{e^{\cancel{t}}}\{n\underbrace{[p\overset{1}{e^{\cancel{t}}} + (1-p)]}_{1}^{n-1}\}$$

$$= n(n-1) \cdot p^2 + pn$$

$$E[X_6^2] = np\{(n-1)p+1\}$$

$$Var(X_6) = \Phi''_{X_6}(0) - [\Phi'_{X_6}(0)]^2$$

$$= np[(n-1)p+1] - n^2p^2$$

$$= np\{(\cancel{n}-1)\cancel{p} + 1 - n\cancel{p}\}$$

$$Var(X_6) = np[1-p]$$

Let X & Y be jointly indep. random variables with a joint pdf. $f_{XY}(x,y)$

find $\Phi_{X+Y}(t)$ $S = X+Y$

↓

$$= E\{e^{t(X+Y)}\} = E\{e^{tX} e^{tY}\} = E\{e^{tX}\} . E\{e^{tY}\} = \Phi_X(t)\, \Phi_Y(t)$$

سؤال امتحان + مرات بعطيني ال moment لل random.V بس
ستخدم ال table بعرف هاد اقرب لايش من ال r.v وبكمل حل

$\boxed{18/12}$ Note that there exists a one-to-one correspondence between a pdf/pmf and the corresponding mgf.

* كل ال Pdf/Pmf ال moment واحد
لو عندي Pdf بقدر ارجع لل moment والعكس

For X & Y <u>independent</u> random variables

$$\phi_{X+Y}(t) = \phi_X(t)\,\phi_Y(t) \longleftrightarrow f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

$$f_{x+y}(s) = f_X(s) * f_Y(s)$$

Let X_p be a Poisson r.v. with parameter λ.

Find $\phi_{X_p}(t)$

$$R(X_p) = \{0,1,2,\dots\}$$

$$\phi_{X_p}(t) = E[e^{tX_p}] = \sum_{i\in R(X_p)} e^{ti} P_X(i) = \sum_{i=0}^{\infty} e^{ti} \frac{\lambda^i e^{-\lambda}}{i!}$$

$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{e^{ti}\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{(e^t\lambda)^i}{i!}}$$

$$\phi_{X_p}(t) = e^{-\lambda} e^{\lambda e^t} = e^{\lambda[e^t-1]}$$

* مطلوب مني اعلل لأكثر من نوع $\boxed{r.v}$

① find mean

② Find mean^2

③ Find Var

X & Y indep. r.v.'s

$$\Longrightarrow Var(X+Y) = Var(X) + Var(Y)$$

هاد رح يساوي zero

$$E[XY] = E[X]E[Y]$$

$\uparrow$ X, Y independent

Nonetheless, in general

$$Var(X+Y) = Var(X) + Var(Y) + 2\{E[XY] - E[X]E[Y]\}$$

سؤال امتحان مرات بعطيني ال moment لل random.v بس
ستخدم ال table عالماتا بعرف هاد اقرب لايش من ال r.v وبكمل حل

$\boxed{18/12}$ Note that there exists a one-to-one correspondence between a pdf/pmf and the corresponding mgf.

* كل ال Pdf/Pmf الو moment واحد
لو عندي Pdf بقدر ارجع لل moment والعكس

For X & Y independent random variables

$$\phi_{X+Y}(t) = \phi_X(t)\,\phi_Y(t) \longleftrightarrow f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

$$f_{X+Y}(s) = f_X(s) * f_Y(s)$$

Let X_P be a Poisson r.v. with parameter λ.
Find $\phi_{X_P}(t)$

$$R(X_P) = \{0,1,2,\dots\}$$

$$\phi_{X_P}(t) = E[e^{tX_P}] = \sum_{i \in R(X_P)} e^{ti} P_X(i) = \sum_{i=0}^{\infty} e^{ti} \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{e^{ti} \lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{(e^t \lambda)^i}{i!}}_{u}$$

$$\phi_{X_P}(t) = e^{-\lambda} e^{\lambda e^t} = e^{\lambda[e^t - 1]}$$

* مطلوب مني اعمل لأكثر من نوع ⓡ.ⓥ

① * find mean
② Find mean^2
③ Find Var

X & Y indep. r.v.'s

$$\Longrightarrow Var(X+Y) = Var(X) + Var(Y)$$

هاد رح يساوي Zero

$$\uparrow$$

$$E[XY] = E[X]E[Y]$$

$$\uparrow \quad X, Y \text{ independent}$$

Nonetheless, in general

$$Var(X+Y) = Var(X) + Var(Y) + 2\{\overbrace{E[XY] - E[X]\,E[Y]}\}$$

$\{E\{XY\} - E\{X\}E\{Y\}\} \triangleq \underset{\uparrow}{Cov}(X,Y) \longrightarrow$ It measures the degree of Statistical dependence between X & Y

Covariance of X & Y

عمل قياس لدرجة ال dependence بين X و Y ممكن - او + لو كان

- يعني واحد يتحرك عكس التاني .

$Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$

X, Y independent $\underset{\xleftarrow{??}}{\Longrightarrow}$ $Cov(X,Y) = 0$

Let X be a r.v with a range that has events of equiprobable opportunities

لهم عرفت انه $\frac{1}{3}$ من هاي

$R(X) = \{-1, 0, +1\}$

$P_X(i) = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$

Let $Y = X^2$

$R(Y) = \{0, +1\}$ الامكانين

$P_Y(j) = \{\frac{1}{3}, \frac{2}{3}\}$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$= E[X^3] - E[X]E[X]$

$E[X^3] = (-1)^3 \cdot \frac{1}{3} + (0)^3 \cdot \frac{1}{3} + (1)^3 \cdot \frac{1}{3} =$ Zero

$E[X] = (-1) \cdot \frac{1}{3} + (0) \cdot \frac{1}{3} + (1) \cdot \frac{1}{3} =$ Zero

$Cov(X,Y) = 0 - 0 =$ Zero

* $P_{XY}(i,j) \overset{??}{=} P_X(i) \cdot P_Y(j) \quad \forall\, i,j$

$P_{XY}(-1,0)$, $P_{XY}(0,0)$, $P_{XY}(1,0)$

$P_{XY}(-1,1)$, $P_{XY}(0,1)$, $P_{XY}(1,1)$

Check For :

بتقطعوا بال $\boxed{-1}$

$$P_{\underline{XY}}(-1,1) = P(X=-1 \cap Y=1) = \overset{\uparrow}{P(X=-1)} = \frac{1}{3}$$

$$P_{\underline{X}}(-1) = \frac{1}{3} = P(X=-1) \quad \& \quad P_{\underline{Y}}(1) = P(Y=1) = \frac{2}{3}$$

$$\underset{\downarrow}{P_{\underline{XY}}(-1,1)} \overset{??}{=} P_{\underline{X}}(-1) \cdot P_{\underline{Y}}(1)$$

$$\frac{1}{3} \overset{??}{\cancel{=}} \frac{1}{3} \cdot \frac{2}{3} \Longrightarrow X,Y \text{ are not indepedent}$$

لو طلعها بتساوا بدي احسب ال Independence لل 6 نقاط

$$X,Y \text{ indep.} \Longrightarrow Cov(X,Y) = 0$$

$$\cancel{\Longleftarrow} \quad Cov(X,Y) = 0$$

20/12

Let X be binomial with parameters (n,P)

Y " " " " (m,P)

X and Y are added to produce S = X+Y

Find $P_S(K) \longrightarrow P_S(K) = P_X(K) * P_Y(K)$

* هون ما كان في داعي يحكيلنا انو Independent والـ Poission والـ geometric كمان

Recall that for X & Y independent that

$$\Phi_{X+Y}(t) = \Phi_S(t) = \Phi_X(t)\,\Phi_Y(t)$$

$\Phi_X(t) = [Pe^t + (1-P)]^n$

$$= [\underset{\uparrow}{P}e^t + (1-\underset{\uparrow}{P})]^n\,[\underset{\uparrow}{P}e^t + (1-\underset{\uparrow}{P})]^m$$

الناتج بيعطي binomial

$$\Phi_S(t) = [Pe^t + (1-P)]^{n+m} \longrightarrow$$ this is binomial with Parameters (n+m,P)

Let X be $N(\mu_x, \sigma_x^2)$

Y be $N(\mu_y, \sigma_y^2)$

X & Y are added to produce $S = X+Y$

* هون لازم يحكيلك انه Independent

Find $f_S(S)$ X & Y are indep. r.v.'s

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$f_X(x) = \Phi_X(t) = e^{\mu_x t + \frac{\sigma_x^2 t^2}{2}}$$

$$f_Y(y) = \Phi_Y(t) = e^{\mu_y t + \frac{\sigma_y^2 t^2}{2}}$$

$$\Phi_{X+Y}(t) = \Phi_X(t)\,\Phi_Y(t) = \left\{e^{\mu_x t + \frac{\sigma_x^2 t^2}{2}}\right\}\left\{e^{\mu_y t + \frac{\sigma_y^2 t^2}{2}}\right\}$$

$$= \left\{e^{(\mu_x+\mu_y)t + \frac{(\sigma_x^2+\sigma_y^2)t^2}{2}}\right. *$$

$\Rightarrow$ One can readily conclude that

$$S - N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

$$\Rightarrow f_S(S) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x^2+\sigma_y^2}} e^{-\frac{1}{2}\left(\frac{S-(\mu_x+\mu_y)}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)^2}$$

Let X_b be binomial with parameter (n, p).

Find $Var(X_b)$

① $Var(X_b) = \sum_{i=0}^{n} (i - np)^2 \binom{n}{i} p^i (1-p)^{n-i}$ ← طريقة ①

② $Var(X_b) = \Phi''_{X_b}(0) - [\Phi'_{X_b}(0)]^2 = np(1-p)$ ← طريقة ②

③ Let $X_j = \begin{cases} 1, & \text{if Success} \\ 0, & \text{if failure} \end{cases}$; $j = 1, 2, \ldots, n$ ← طريقة ③

$P(S) = p$

# of Successes in n trials $= X_1 + X_2 + X_3 + \cdots + X_n$

$X_b = X_1 + X_2 + \cdots + X_n$

~~$Var(X_b) = Var(X_1) + Var(X_2) + \cdots + Var(X_n)$~~

$Var(X_b) = Var(X_1 + X_2 + \cdots + X_n)$ ← Indep هدول

$= Var(X_1) + Var(X_2) + \cdots + Var(X_n)$ فبقدر افكهم

$Var(X_k) = (1-p)^2 \cdot p + (0-p)^2 (1-p)$ → Mean Value ل ~~bernoli~~ bernouli

$= (1-p)[(1-p) \cdot p + p^2]$

$= (1-p) \cdot p[(1-p) + p] = (1-p) \cdot p \quad \forall k$

$Var(Y) = E\{(Y - \mu_Y)^2\}$

$\therefore = \underline{p(1-p) + p(1-p) + \cdots + p(1-p)}$

$= np(1-p)$

③ $Var(X_b) = \sum_{k=1}^{n} Var(X_k)$

Statistical Inequalities:

- Classical Inequalities:
1) Markov inequality
2) Chebychev inequality

- other Inequalities:
1) Law of Large numbers
2) Central limit theorem

* نحن لا نلجأ لل Inequality لما يكون ال general Statistics (Mass, density) مش معروفين عندي.

+ ~~Markov~~

1) Markov inequality:

$$P(X \geq a) \leq \frac{E\{X\}}{a} \quad ; \quad a > 0, \quad X \in \mathbb{R}^+$$

Inequalities are used whenever the general Statistics of the underlying process are unknown

Nonetheless, $E[X]$ & $Var(X)$ are provided

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① $P(X \geq a) \leq \frac{E\{X\}}{a}$; $X > 0$, $a > 0$, $R(X) \subseteq \mathbb{R}^+$

$E[X]$, $Var(X)$

Proof:

$$E[x] = \int_{-\infty}^{\infty} x\, f_X(x)\, \partial x = \int_{0}^{\infty} x\, f_X(x)\, \partial x = \int_{0}^{a} \underbrace{x f_X(x)\, \partial x}_{\geq 0} + \int_{a}^{\infty} x\, f_X(x)\, \partial x$$

$$= \int_{0}^{a} x\, f_X(x)\, \partial x + \int_{a}^{\infty} x\, f_X(x)\, \partial x \geq \int_{a}^{\infty} x\, f_X(x)\, \partial x \geq \int_{a}^{\infty} a\, f_X(x)\, \partial x = a \int_{a}^{\infty} f_X(x)\, \partial x = \boxed{a\, P(X \geq a)}$$

لاني اكبر لانه تكامل ال x اكبر من تكامل a

$$\rightarrow E[x] \geq a\, P(x \geq a) \longrightarrow \boxed{P(x \geq a) \leq \frac{E[x]}{a}}$$

Example: Suppose a car factory produces 500 cars an average per week.

$$E[X] = 500$$

X = "# of vehicles produced by the factory each week"

$P(X \geq 1000) = ?$

According to Markov inequality:

$$P(X \geq 1000) \leq \frac{E[X]}{1000} = \frac{500}{1000} = \frac{1}{2}$$

$$\Rightarrow P(X \geq 1000) \leq \frac{1}{2}$$

② Chebychev inequality:

X is produced by means of a random experiment.

$E[X] = \mu$; $Var(X) = \sigma^2$

$$P(|X - \mu| \geq K) \leq \frac{Var(X)}{K^2}$$

Let $Z = (X-\mu)^2$; $Z \geq 0$

$$E[Z] = E[(X-\mu)^2] = Var(X)$$

According to Markov inequality:

$$P(Z \geq K^2) \leq \frac{E[Z]}{K^2}$$

$$P((X-\mu)^2 \geq K^2) \leq \frac{E[(X-\mu)^2]}{K^2}$$

$$P((X-\mu)^2 \geq K^2) \leq \frac{Var(X)}{K^2}$$

$$(X-\mu)^2 \geq K^2 \Longleftrightarrow \underline{-K \geq (X-\mu) \geq K}$$

$$|X-\mu| \geq K$$

$$P(|X-\mu| \geq k) \leq \frac{\sigma_x^2}{k^2}$$

Example: let $E[X] = 500$ as before   نفس المثال السابق

$Var(X) = 100$

Find a bounding probability for $P(400 < X < 600)$

$$P(400 - 500 < X - 500 < 600 - 500)$$

$$\Downarrow$$

$$P(-100 < X - 500 < 100) = ??$$

$$P(|X - 500| < 100) = ??$$ لأني باخذ Complem

$$= 1 - P(|X - 500| \geq 100) > 1 - \frac{\sigma_x^2}{10000}$$

$$= P(|X - 500| < 100) > 1 - \frac{1\cancel{00}}{100\cancel{00}} = \frac{99}{100}$$

③ Law of large numbers:

$\{X_1, X_2, \dots, X_n\}$ ① independent. ② identically distributed.] i.i.d

Such that $E[X_i] = \mu$, $Var(X_i) = \sigma^2$ $\Big\}$ $\forall i$

$\Rightarrow$ For all ~~some~~ $\epsilon > 0$:

$$\lim_{n \to \infty} P\left(\left|\frac{X_1 + X_2 + \dots + X_n}{n} - \mu\right| > \epsilon\right) = 0$$

Proof:

$$\tilde{X} = \frac{X_1 + X_2 + \dots + X_n}{n} \longrightarrow \overset{\overset{\tilde{\mu}}{\Uparrow}}{E[\tilde{X}]}$$

$$= E\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right]$$

$\boxed{27/12}$ $$= \frac{1}{n}\left\{\overset{\mu}{\overset{\downarrow}{E[X_1]}} + \overset{\mu}{\overset{\downarrow}{E[X_2]}} + \dots + \overset{\mu}{\overset{\downarrow}{E[X_n]}}\right\} = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{\cancel{n}}\cdot\cancel{n}\cdot\mu = \mu$$

$$\boxed{\tilde{\mu} = \mu}$$

$$Var(\tilde{X}) = Var\left(\frac{X_1+X_2+\dots+X_n}{n}\right) = \frac{1}{n^2}\left\{Var(X_1+X_2+\dots+X_n\right\} \searrow$$

وزعتهم لانهم independent

$$= \frac{1}{n^2}\left\{\underset{\sigma^2}{\underset{\uparrow}{Var(X_1)}} + \underset{\sigma^2}{\underset{\uparrow}{Var(X_2)}} + \dots + \underset{\sigma^2}{\underset{\uparrow}{Var(X_n)}}\right\}$$

$$\tilde{\sigma}^2 = Var(\tilde{X}) = \frac{\sigma^2}{n}$$

$$\boxed{\tilde{\sigma}^2 = \frac{1}{n^2}\cdot n\cdot\sigma^2 = \frac{\sigma^2}{n}}$$

Employ Chebychev's inquality, we get:

$$P(|\tilde{X}-\tilde{\mu}| \geq \epsilon) \leq \frac{\tilde{\sigma}^2}{\epsilon^2} \Longrightarrow P(|\tilde{X}-\mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

Now, involving the limit $(n \to \infty)$

$$\lim_{n\to\infty} P(|\tilde{X}-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\circled{n}\,\epsilon^2}$$

$n \to \infty$

$$\rightarrow \lim_{n\to\infty} P(|\tilde{X}-\mu| \geq \epsilon) \leq \phi \text{ !!!} \rightarrow$$ لانه ال Prob سالبة

$\Downarrow$

This leads to Contradiction ← متناقض

$\Longrightarrow$ It must be the case that:

$$\lim_{n\to\infty} P(|\tilde{X}-\mu| \geq \epsilon) \overset{must}{\overset{\downarrow}{=}} 0$$

Towards the end:

$$\frac{X_1+X_2+\dots+X_n}{n} \xrightarrow{n\to\infty} \mu \text{ with probability equal to } 1$$

④ Central limit theorem:

For a set of Random variables $\{X_1, X_2, \dots, X_n\}$ ① independent
② identically distributed

with $E[X_i] = \mu \ \forall i$, & $Var(X_i) = \sigma^2 \ \forall i$

then $$\lim_{n\to\infty} P\left(\frac{(X_1+X_2+\dots+X_n) - n\mu}{\sigma\sqrt{n}} \leq a\right) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

Let $X_i = \left\{ \begin{array}{ll} 1, & \text{if Success} \\ 0, & \text{otherwise} \end{array} \right\} \forall i$; $P(S) = P$

$\{X_1, X_2, \dots, X_n\}$ ① Independent
② identically distributed

$$X_b = X_1 + X_2 + \dots + X_n$$

$\mu_{x_i} = E\{X_i\} = P$; $\forall i$

$\sigma^2_{x_i} = Var(X_i) = P(1-P)$

$$\lim_{n\to\infty} P\left(\frac{X_b - n \cdot P}{\sqrt{P(1-P)}\sqrt{n}} \leq a\right) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$\lim_{n\to\infty} P\left(\frac{X_b - E[X_b]}{\sqrt{Var(X_b)}} \leq a\right) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

Now, lets use this to approximate $P(X_b = 20)$ with $n = 40$ & $P = \frac{1}{2}$.

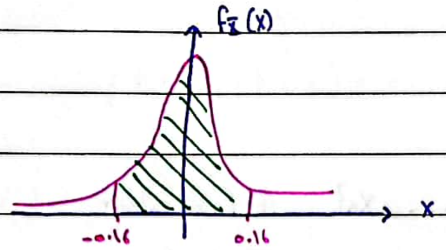
$$P(X_b = 20) = \binom{40}{20}\left(\frac{1}{2}\right)^{20}\left(\frac{1}{2}\right)^{20} = \binom{40}{20}\left(\frac{1}{2}\right)^{40} = 0.125376 \longrightarrow \text{exact.}$$

Contin. [illegible]

$$P(X=20) \simeq P(19.5 \leq X \leq 20.5)$$

$$= P\left(\frac{-0.5}{\sqrt{10}} \leq \frac{X-20}{\sqrt{10}} \leq \frac{0.5}{\sqrt{10}}\right) \longrightarrow 40 \times \frac{1}{2} \times \frac{1}{2} = \boxed{10}$$

$$= P(\overset{-0.16}{-0.158114} \leq X_N \leq \overset{0.16}{0.158114})$$



$\boxed{\text{عالجدول}}$ $$\int_{0.16}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx = 0.4364$$

$= 1 - 2(0.4364)$

$= 0.1272 \rightarrow$ app/ox. *كل ما تزيد ال n ← الرقم هاد سيقترب للرقم الحقيقي

$\boxed{29/12}$ Linear interpolation:

2.78	0.00272
2.79	0.00264

$\boxed{2.7879}$ بدي

Approach 1: ① $\frac{2.79 - 2.7879}{2.79 - 2.78} = 0.21$ ↓

② $0.00272 - 0.00264\,(0.21) = 0.0000168$

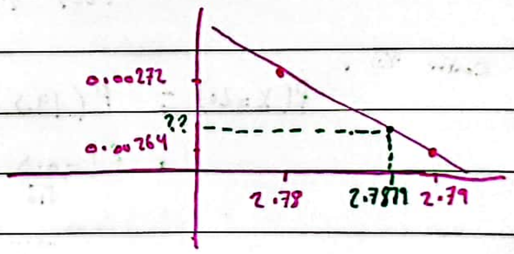
③ $0.00264 + 0.0000168 = \boxed{0.0026568}$

Alternate approach:

① $\frac{2.7878 - 2.78}{2.79 - 2.78} = 0.79$ ↓

② $0.00272 - 0.00264\,(0.79) = 0.0000632$

③ $0.00272 - 0.0000632 = \boxed{0.0026568}$



Confidence interval :

At a cafe, it was notice that the fillings of individual bottles exhibit descripancies. Whereas, the nominal filling level should stand at 300 ml various individual bottles revealed the following filling levels :

300.21	297.67	306.71	298.65
302.00	307.00	300.99	304.15
299.00	300.50	296.65	312.70
298.50	295.00	294.80	293.30
303.15	301.89	300.00	291.99

$$\frac{1}{20}\sum_{i=1}^{20} x_i = \frac{6004.86}{20} = 300.243$$

$$\frac{1}{20}\sum_{i=1}^{20} (x_i - 300.243)^2 =$$

Estimators for statistical entities :

(أدق)

Statistical avg $\longrightarrow$ ① $\mu_x = E[X] = \sum_{i \in R(X)} i \, P_X(i)$

تقدير لل Mean $\hat{\mu}_x = \frac{1}{N}\sum_{i=1}^{N} x_i$

arithmetic avg ↑

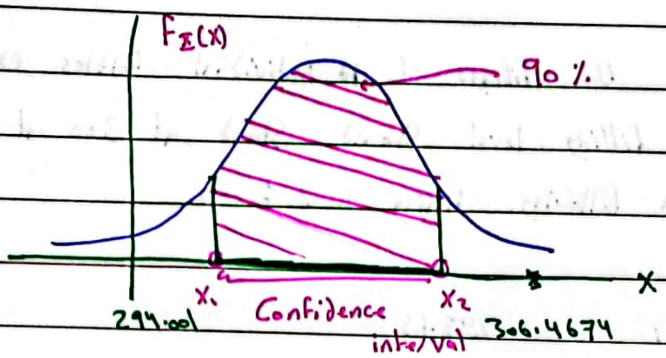
لانه لو عدناها 2, بيطلع ارقام مختلفة estimator for μ_x

② $Var(X) = E[(X-\mu_x)^2] = \sum_{i \in R(X)} (x_i - \mu_x)^2 \, P_X(x_i)$

↑ Statistical Var

$$\hat{\sigma}_x^2 = \frac{1}{N}\sum_{i=1}^{N} (x_i - \hat{\mu}_x)^2$$

Confidence interval :



$$\frac{1}{20}\sum_{i=1}^{20}(x_i - 300.243)^2 = 23.64694$$

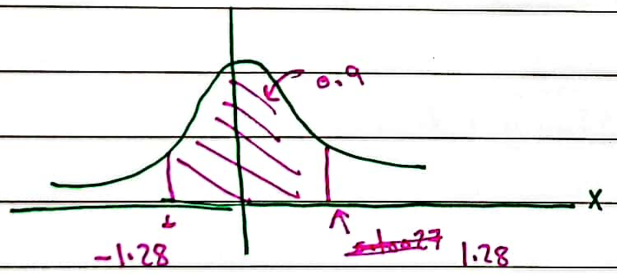
$$\sqrt{\hat{\sigma}_x^2} = 4.86281$$

Objective :

$$P(\overset{??}{x_1} \leq X \leq \overset{??}{x_2}) = 0.9$$

$$P\left(\frac{x_1 - 300.243}{4.86281} \leq \frac{X - 300.243}{4.86281} \leq \frac{x_2 - 300.243}{4.86281}\right)$$

$$P\left(\frac{x_1 - 300.243}{4.86281} \leq X_N \leq \frac{x_2 - 300.243}{4.86281}\right) = 0.9$$



$$1 - 0.9 = 0.1$$

$$\rightarrow 1.28 = \frac{x_2 - 300.243}{4.86281}$$

$$x_2 = (4.86281)(1.28) + 300.243 = 306.4674$$

$$-1.28 = \frac{x_1 - 300.243}{4.86281}$$

$$x_1 = -(1.28)(4.86281) + 300.243 = 294.001$$

Markov chains.

$\boxed{5/1}$ Stochastic Processes:

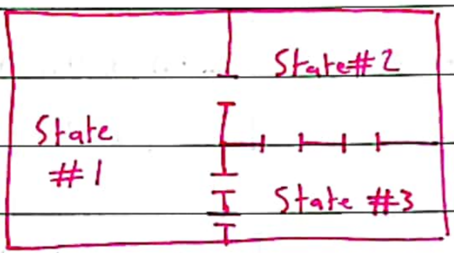
Random process الها علاقة بالزمن يعني

System with a number of states ; $\{X_t \mid X_1, X_2, \dots, X_m\}$

m-number of States

$t = 1, 2, 3, \dots$

*



Assume that at each time cycle the mouse can stay at his current position 40 % of the time. When he moves, he moves into the immediately neighboring available state.

We further assume that the mouse possesses no memory. → *

$$P_t(i \rightarrow j) = P_{ij}(t)$$

We further assume that $P_{ij}(t) = P_{ij}$ (Homogeneous operation) هون ما بعتمد عالزمن لانه ما بتغير مع مرور الوقت

$$P_{ij}(t) = P(X_{t+1} = j \mid X_t = i) \text{ ——— } ①$$

$$* \; P(X_{t+1} = j \mid X_m = i \cup X_m = i-1 \cup X_m = i-2 \cup \dots \cup X_m = 1)$$
$$= P(X_{t+1} = j \mid X_m = i)$$

$P_{11} = 0.4$, $P_{22} = 0.4$, $P_{33} = 0.4$

$$P_{12} = (0.6)\left(\frac{1}{3}\right) = 0.2$$

$$P_{13} = (0.6)\left(\frac{2}{3}\right) = 0.4$$

$P_{21} = (0.6)\left(\frac{1}{3}\right) = 0.2$

* هون لانو الوقت ما بتغير بتضل ال P ثابتين ما بتغيروا.

$P_{23} = (0.6)\left(\frac{2}{3}\right) = 0.4$

$\left(\frac{2}{4}\right) \rightarrow$

$P_{31} = (0.6)\left(\frac{1}{2}\right) = 0.3$

$P_{32} = (0.6)\left(\frac{1}{2}\right) = 0.3$

* حجم المصفوفة بعتمد ع ال States.

$$P = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \begin{matrix} \rightarrow 0.4 + 0.2 + 0.4 = 1 \\ \rightarrow 0.2 + 0.4 + 0.4 = 1 \\ \rightarrow 0.3 + 0.3 + 0.4 = 1 \end{matrix}$$

P ↓ This is called a 1-step transition probability matrix

← This is called a Markov matrix.

ضربنا بحالا

$$P^2 = P.P = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.32 & 0.28 & 0.4 \\ 0.28 & 0.32 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \begin{matrix} \rightarrow 0.32+0.28+0.4=1 \\ \rightarrow 0.28+0.32+0.4=1 \\ \rightarrow 0.3+0.3+0.4=1 \end{matrix}$$

2-step transition Probability matrix

↑ Again, this is a Markov matrix.

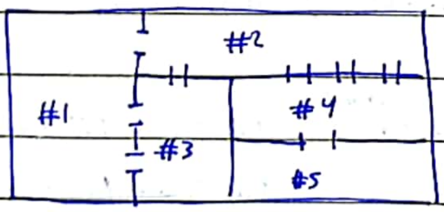
$$P^2.P^2 = P^4 = \begin{bmatrix} 0.32 & 0.28 & 0.4 \\ 0.28 & 0.32 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.32 & 0.28 & 0.4 \\ 0.28 & 0.32 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.3008 & 0.2992 & \overset{0.4}{\cancel{0.3008}} \\ 0.2992 & 0.3008 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \begin{matrix} 0.3008+0.2992+0.4=1 \\ 0.2992+0.3008+0.4= \\ 0.3+0.3+0.4=1 \end{matrix}$$

4-step transition Probability matrix

$P.P^3 \rightarrow$ 3 step ..

~~TB~~ This is, once more a markov matrix.

⊛



→ matrix → 5x5 → بنعبي ب اصفار

لو معتناه بكرراه انه ما يظل بمكانه بتصير من 0.4 ← 0

$\boxed{8/1}$ $\longrightarrow$ $P^4 = P^2 P^2 = \begin{bmatrix} 0.3008 & 0.2992 & 0.4 \\ 0.2992 & 0.3008 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$

$$P^8 = [P^4]^2 = \begin{bmatrix} 0.3 & 0.2999 & 0.4 \\ 0.2999 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

$$P^{16} = \begin{bmatrix} 0.3 & 0.2999 & 0.4 \\ 0.999 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}^2 = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

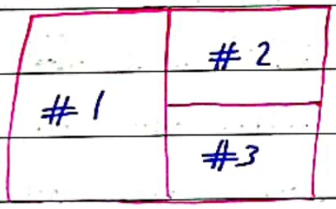
* ال rows صاروا يساووا بعض لما ارفع ال Matrix لرقم كبير.

Initial Distribution Vector:

شو احتمالية الفار يوقع بكل وحدة منهم (الحالة الاولية)
مجموع عناصره = 1

$$\underline{P}^{(0)} = [P_1^{(0)} \; P_2^{(0)} \; P_3^{(0)}]$$

$$P_i^{(0)} = P(X_0 = i), \quad \text{For } i = 1, 2, 3$$



$$\underline{P}^{(0)} = [\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}]$$

$$P_j^{(n)} = P(X_n = j) = ?$$

شو احتمالية اني الاقي ال System بال State=j ← بعد Step (n).

$$= \sum_{i=1}^{m} P(X_n = j \mid X_{n-1} = i)\, P(X_{n-1} = i) \quad ; \quad j = 1, 2, \dots, m$$

$$[P_1^{(n)}, P_2^{(n)}, P_3^{(n)}, \dots, P_m^{(n)}] = [\underline{P_1^{(n-1)}, P_2^{(n-1)}, \dots, P_m^{(n-1)}}] \begin{bmatrix} P_{11} & P_{12} \dots & \boxed{P_{1j}} & \dots & P_{1m} \\ P_{21} & P_{22} \dots & P_{2j} & \dots & P_{2m} \\ \vdots & & \vdots & & \\ P_{m1} & P_{m2} \dots & P_{mj} & \dots & P_{mm} \end{bmatrix}$$

$$\underline{p}^{(n)} = \underline{p}^{(n-1)} P$$

I want to iterate through this eqn.:

$n' = 1$:

$$\underline{p}^{(1)} = \underline{p}^{(0)} P$$

$n' = 2$:

$$\underline{p}^{(2)} = \underline{p}^{(1)} P = \underline{p}^{(0)} PP = \underline{p}^{(0)} P^2$$

$n' = 3$:

$$\underline{p}^{(3)} = \underline{p}^{(2)} P = p^{(0)} P^2 \cdot P = p^{(0)} P^3$$

⋮

هاي بعد n state.

هاي قوة

$n' = n$:

$$\boxed{\underline{p}^{(n)} = p^{(0)} P^n}$$

10/1

$$\underline{p}^{(n)} = \underline{p}^{(n-1)} P \longrightarrow \underline{p}^{(n)} = \underline{p}^{(0)} P^n$$

$$P = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \longrightarrow P^{16} = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

$$\pi = [\pi_1 \quad \pi_2 \quad \pi_3] = [0.3 \quad 0.3 \quad 0.4]$$

[illegible]

$$\underline{p}^{(0)} = \left[\tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{4}\right]$$

$$\lim_{n \to \infty} \underline{p}^{(n)} = \underline{p}^{(n-1)} P = \underline{p}^{(0)} P^n$$

$$= \underline{P}^{(0)} \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

$$= \underline{P}^{(0)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [\pi_1 \; \pi_2 \; \pi_3] = \overbrace{\boxed{[P_1^{(0)} \; P_2^{(0)} \; P_3^{(0)}] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}}^{P_1^{(0)} + P_2^{(0)} + P_3^{(0)} = 1} [\pi_1 \; \pi_2 \; \pi_3] = [\pi_1 \; \pi_2 \; \pi_3] = \underline{\pi}$$

$$\lim_{n \to \infty} \underline{P}^{(n)} = \underline{\pi} \leftarrow \boxed{\text{limiting distribution vector}}$$

* انه لما اضل اضرب المصفوفة بنفسها عدد من المرات بالنهاية رح يثبت .

* مشان اعرفها بدون ما اضربها عدد كبير ← من المرات .

$$\underline{P}^{(n)} = \underline{P}^{(n-1)} P$$

$$\Big\downarrow n \to \infty$$

$$\underline{\pi} = \pi P$$

$$\underline{\pi}[I - P] = [0 \; 0 \; 0]$$

$$[I - P]^T \pi^T = 0 \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.2 & -0.4 \\ -0.2 & 0.6 & -0.4 \\ -0.3 & -0.3 & 0.6 \end{bmatrix}$$

$$[I - P]^T = \begin{bmatrix} 0.6 & -0.2 & -0.3 \\ -0.2 & 0.6 & -0.3 \\ -0.4 & -0.4 & 0.6 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 0.6 & -0.2 & -0.3 \\ -0.2 & 0.6 & -0.3 \\ -0.4 & -0.4 & 0.6 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{0.6} & \cancel{-0.2} & \cancel{-0.3} \\ \cancel{-0.2} & \cancel{0.6} & \cancel{-0.3} \\ \cancel{-0.4} & \cancel{-0.4} & \cancel{0.6} \end{bmatrix}$$

$$\begin{array}{l} \\ R_2 \leftarrow R_2 + \frac{0.2}{0.6}R_1 \leftarrow \\ R_3 \leftarrow R_3 + \frac{0.4}{0.6}R_1 \leftarrow \end{array} \left[\begin{array}{ccc:c} \textcircled{0.6} & -0.2 & -0.3 & 0 \\ -0.2 & 0.6 & -0.3 & 0 \\ -0.4 & -0.4 & 0.6 & 0 \end{array}\right] \leftarrow \text{augmented matrix}$$

$$0 + \left(\frac{0.2}{0.6}\right)(0) = 0$$

① $-0.2 + \left(\frac{0.2}{0.6}\right)(0.6) = 0$

② $0.6 + \left(\frac{0.2}{0.6}\right)(-0.2) = \frac{8}{15}$

③ $-0.3 + \frac{0.2}{0.6}(-0.3) = -\frac{2}{5}$

① $-0.4 + \frac{0.4}{0.6}(0.6) = 0$

② $-0.4 + \frac{0.4}{0.6}(-0.2) = -\frac{8}{15}$

③ $0.6 + \frac{0.4}{0.6}(-0.3) = \frac{2}{5}$

$$\left[\begin{array}{ccc:c} 0.6 & -0.2 & -0.3 & 0 \\ 0 & \frac{8}{15} & -\frac{2}{5} & 0 \\ 0 & -\frac{8}{15} & \frac{2}{5} & 0 \end{array}\right]$$

$$\frac{8}{15}\pi_2 = \frac{2}{5}\pi_3 \longrightarrow \pi_3 = \left(\frac{5}{\cancel{2}_{1}}\right)\left(\frac{\overset{4}{\cancel{8}}}{\cancel{15}_{3}}\right)\pi_2 = \frac{4}{3}\pi_2$$

$$0.6\pi_1 - 0.2\pi_2 - 0.3\pi_3 = 0 \longrightarrow \cancel{0.6}\pi_1 = 0.2\pi_2 + (\overset{0.1}{\cancel{0.3}})\left(\frac{4}{\cancel{3}}\right) = 0.2\pi_2 + 0.4\pi_2 = \cancel{0.6}\pi_2$$

$$\boxed{\begin{array}{l} \pi_3 = \frac{4}{3}\pi_2 \\ \pi_1 = \pi_2 \end{array}} \longrightarrow \text{No Solution exists} \quad \{\pi_2 \rightarrow \text{free parameter}\}$$

$\pi_1 + \pi_2 + \pi_3 = 1$ "لانه كل اشي بدلالة π_2"

$\pi_2 + \pi_2 + \cancel{\frac{4}{3}}\frac{4}{3}\pi_2 = 1$

$2\pi_2 + \frac{4}{3}\pi_2 = 1$

$\frac{6+4}{3}\pi_2 = 1$

$\Rightarrow 10\pi_2 = 3$

$\Rightarrow \boxed{\pi_2 = \frac{3}{10}}$

$$\boxed{\underline{\pi} = [0.3 \quad 0.3 \quad 0.4]}$$

هيك بنقدر نعرف انه بعد مرور وقت
طويل الفار رح يكون غالبًا بـ State $\underline{\underline{3}}$

$\pi_3 = \frac{4}{3}\pi_2 = \frac{4}{3}\left(\frac{3}{10}\right) = 0.4$

$\pi_1 = \pi_2 = \frac{3}{10} = 0.3$

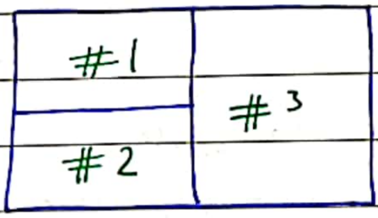
12/1 **Problem:**

expect to (mouse)

Out of n steps, what proportion of them would I ↑ find the object of relevance is state j | that he started out in state i ?

كم نسبة المرات الي لقينا فيها الماوس

با n step في state j



mouse

$N_{ij}^{(n)}$ = "# of times the process entered into state j after n time steps".

mouse

let X_k = "the position of the object of relevance in state j after K time steps".

$$= \begin{cases} 1, & \text{if mouse is in state } j \\ 0, & \text{if not} \end{cases}$$

$$N_{ij}^{(n)} = X_0 + X_1 + X_2 + \dots + X_n$$

$$E[N_{ij}^{(n)}] = E[X_0 + X_1 + X_2 + \dots + X_n]$$
$$= E[X_0] + E[X_1] + E[X_2] + \dots + E[X_n]$$

$$E[X_L] = 1 \cdot P(X_L = 1) + 0 \cdot P(X_L = 0)$$

What is $P(X_L = 1)$ all about ?   اني ابدا ال mouse بعد L من ال State في j ←

$$P(X_L = 1) = P_{ij}^{(L)}$$

$$E[N_{ij}^{(n)}] = P(X_0 = 1) + P(X_1 = 1) + \dots P(X_n = 1)$$
$$= [P_{ij}^{(0)}] + P_{ij}^{(1)} + P_{ij}^{(2)} + \dots + P_{ij}^{(n)}$$

← هاي ح تعطيني ال identity matrix

$$P_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$= [I + P^{(1)} + P^{(2)} + \dots + P^{(n)}]_{ij}$$

← انه ما تحرك من مكانه

$$E[N_{ij}^{(n)}] = [I + P^1 + P^2 + \dots + P^n]_{ij}$$

Problem: Out of n time steps what is the expected proportion of time to find ~~a~~ object in state j | if started out in state i?

answer: $\dfrac{E[N_{ij}^{(n)}]}{n+1}$

كأنه بياخذ ال aver. time

ولأنه $\underline{P}$ عددهم n وضيفنا ال I ف $\boxed{n+1}$

$\boxed{15/1}$

The proportion of time that a move exists in state j given that he started in state i after n time steps was found to be:

$$\frac{E[N_{ij}^{(n)}]}{n+1} = \frac{[I + P^1 + P^2 + \dots + P^n]}{n+1}$$

$$P^0 = I \quad : \quad P_{ij}^{(0)} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases}$$

$$N_{ij}^{(n)} = X_1 + X_2 + \dots + X_k$$

X_K = "Position of Subject of relevance in state j after K time steps".

$$= \begin{cases} 1 , & \text{if present in state } j \\ 0 , & \text{if not} \end{cases}$$

Problem: what proportion of all time can I expect the mouse to be in state j given that he started in i ?

$$\text{all time [illegible]} \leftarrow \lim_{n \to \infty} \frac{E[N_{ij}^{(n)}]}{n+1}$$

In general for a transition matrix with a limiting distribution vector Π, there exists a K Such that:

$$P^k = P^{k+1} = P^{k+2} = \dots = P^n$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

بتضل ال matrix تنعكس .

Not all transition chains have limiting probabilities.

$$\lim_{n\to\infty} \frac{E[N_{ij}^{(n)}]}{n+1} = \lim_{n\to\infty} \frac{[I + P + P^2 + \dots + P^n]_{ij}}{n+1}$$

~~$= \lim_{n\to\infty} [I + P + P^2 + \dots + P^{k-1}]_{ij}$~~

$$= \lim_{n\to\infty} \left\{ \frac{[I + P + P^2 + \dots + P^{k-1}]_{ij}}{\cancel{n+1}} + \left[\frac{P^k + P^{k+1} + \dots + P^n}{n+1}\right]_{ij} \right\}$$

$0 \rightarrow$ رقم ضخم

المصفوفة إلي كل ال rows بيطلعوا زي بعض .

$$= \left\{ 0 + \left[\frac{P^\infty + P^\infty + P^\infty + \dots + P^\infty}{n+1}\right]_{ij} \right\}$$

$$= \frac{n-k}{n+1} P^\infty = \left[\frac{\cancel{n}}{\cancel{n+1}} - \frac{k}{n+1}\right] P_{ij}^\infty$$

1 ← رقم ضخم على رقم ضخم .

$$= \left[1 - \underset{0}{\cancel{\frac{k}{n+1}}}\right] P_{ij}^\infty = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}_{ij} = \pi_j$$

$$\pi = [\pi_1 \; \pi_2 \; \dots \overset{\pi_j}{\vdots} \dots \; \pi_m]$$