Probability

3/3/2021

Mathematical Background

1: **Set**: A set is a collection of distinct objects which are the element of the set.

Ex: $S_1 = \{0, 2, 4, 6\}$ $S_2 = \{H, T\}$

* let $S = \{x_1, x_2, \dots, x_n\}$, $x_1 \in S$: x_1 is an element of the set S, x_1 is in S, x_1 belongs to S.

* Consider x_j, $j > n \rightarrow x_j \notin S$ (x_j is not an element of S)

A] **Empty set (Null Set)**: A set that contains no elements, denoted by "$\emptyset$".

B] **Universal Set**: is a set that contains all objects that could be conceivably of interest in a particular context.

Ex: $S_1 = \{1, 2, 3, 4, 5, 6\}$ "rolling a die"

$S_2 = \{H, T\}$ "Tossing a coin"

Set classification:

1- **Finite Set**: a set that contains a finite number of elements.

ex: $S_1 = \{1, 2, 3\}$ is a Finite set, $S_2 = \{HH, HT, TH, TT\}$ is a Finite set.

2- **Countably infinite Set**: a set that contains infinitely many elements, that can be enumerated in a list.

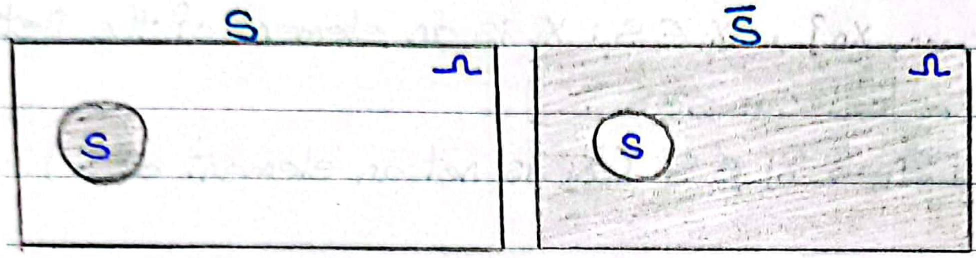
ex: $S_1 = \{0, -2, 2, 4, -4, 6, -6, \dots\}$, $S_2 = \{x \mid x \text{ satisfies some property } P\} \rightarrow S_2 = \{K \mid K/2 \text{ is an integer}\}$

3- **Uncountable Set**: we can't enumerated in a list "لا يمكن إحصاء العدد"

ex: $S_2 = \{x \mid 1 \geq x \geq 0\}$

Set operations :

1- The Complement of a set : The complement of a set "s", with respect to a universal set Ω, is the set $\bar{S} = \{x \in \Omega \mid x \notin S\} \hat{=} \Omega - S$



← Example

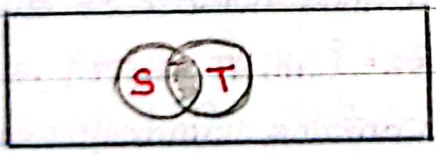
* Note : $\bar{\Omega} = \emptyset$, $\bar{\emptyset} = \Omega$

2- The union of Sets : The union of two sets is a set of all elements that belong to the two sets S or T

$\therefore S \cup T = \{x \mid x \in S \ \underline{\text{or}} \ x \in T\}$

Ex: $S = \{1,2\}$, $T = \{2,4\} \rightarrow S \cup T = \{1,2,4\}$

3- The Intersection of Sets : $S \cap T$ is the intersection of S and T. $\therefore S \cap T = \{x \mid x \in T \ \underline{\text{and}} \ x \in S\}$ is the set of all elements that belong to both S and T.



Ex: $S = \{1,2\}$, $T = \{2,4\}$

$\rightarrow S \cap T = \{2\}$

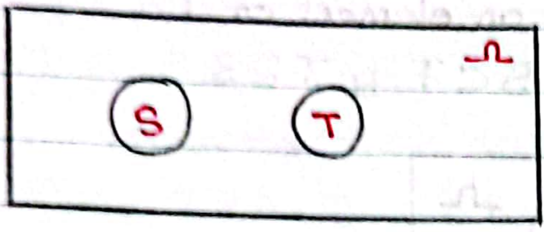
* Note : For infinitely many sets : $S_1, S_2, \dots$,

$S_1 \cup S_2 \cup S_3 \cdots = \bigcup_{n=1}^{\infty} S_n = \{x \mid x \in S_n \text{ for some } n\}$.

* Note : For infinitely many sets : $S_1, S_2, \dots$,

$S_1 \cap S_2 \cap S_3 \cdots = \bigcap_{n=1}^{\infty} S_n = \{x \mid x \in S_n \text{ for all } n\}$.

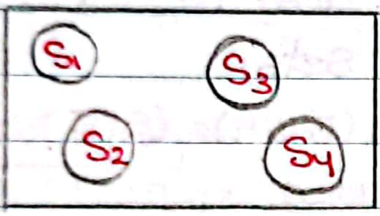
* **Disjoint Sets :** Two sets are said to be disjoint if their intersection is empty. S and T are disjoint if $S \cap T = \emptyset$



S and T are disjoint

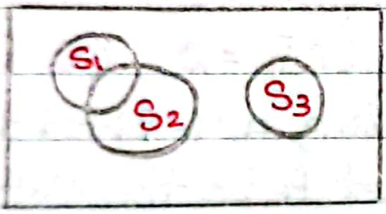
* $S_1, S_2, ---, S_n$ are disjoint if $S_i \cap S_j = \emptyset, i \neq j, i = 1, 2, --, n$
$j = 1, 2, ---, n$

Ex :



S_1, S_2, S_3, S_4 are disjoint

Ex :



$S_1 \cap S_2 \neq \emptyset \longrightarrow$ joint
$S_1 \cap S_3 = \emptyset \longrightarrow$ disjoint
$S_2 \cap S_3 = \emptyset \longrightarrow$ disjoint

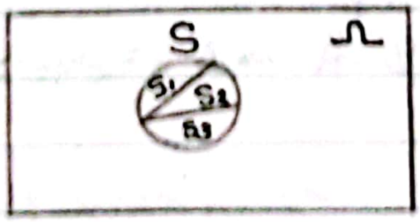
∴ S_1, S_2, S_3 are not disjoint.

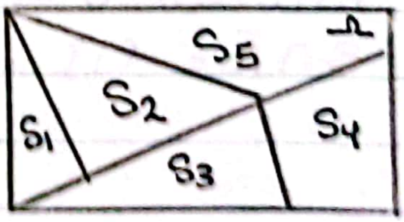
6/3/2021

(Partition) :

Partition : A collection of sets is said to be a partition of sets S if the sets are disjoint and their union is the set "S"

S_1, S_2, S_3 are disjoint ∴ $\bigcup_{i=1} S_i = S$
S_1, S_2, S_3 form a partition of S

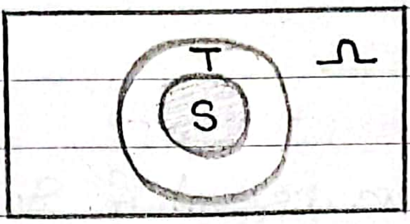




Subsets:

Consider the Sets S and T we say that S is a subset of T, if every element of S is an element of T

$\therefore$ S is subset of T $\triangleq S \subset T \triangleq T \supseteq S$



1] Equality property: if $S \subset T$ and $T \subset S$, then $S = T$

2] Transitivity Property: if $S \subset T$ and $T \subset V$, then $S \subset V$

Algebra of Sets:

① $S \cup T = T \cup S$ ② $S \cap (T \cup A) = (S \cap T) \cup (S \cap A)$

③ $(\bar{S}) = S$ ④ $S \cup \Omega = \Omega$

⑤ $S \cap \Omega = S$ ⑥ $S \cup (T \cup A) = (S \cup T) \cup A = S \cup T \cup A$

⑦ $S \cup (T \cap A) = (S \cup T) \cap (S \cup A)$ ⑧ $S \cap \bar{S} = \emptyset$ ⑨ $\bar{\emptyset} = \Omega$

Ex: Consider an experiment of rolling a 4- Sided die.

① $\Omega = \{1,2,3,4\}$ let A be the Set of all even outcomes and B is the Set of all outcomes less than 3. ② $A = \{2,4\}$, $B = \{1,2\}$

$\boxed{1}$ $A \cap B = \{2\}$ $\boxed{2}$ $A \cup B = \{1,2,4\}$

$\boxed{3}$ $A \cap \bar{B}$ = * $\bar{B} = \Omega - B = \{3,4\} \rightarrow A \cap \bar{B} = \{4\}$

* $\bar{A} = \Omega - A = \{1,3\}$

$\rightarrow A \cap \bar{B} = A - B = \{4\}$

$\boxed{3}$ $B \cap \bar{A} = \{1\} = B - A \rightarrow$ element موجودة في B مش موجودة ب A

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Demorgans Law:

1] $\overline{(A\cap B)} = \bar{A}\cup\bar{B}$

2] $\overline{(A\cup B)} = \bar{A}\cap\bar{B}$

1] Proof : $\overline{(A\cap B)} = \bar{A}\cup\bar{B}$

assume that $x\in\overline{(A\cap B)} \xrightarrow{\text{implies}} x\notin(A\cap B)$

$\xrightarrow{\text{implies}} x\notin A$ or $x\notin B \xrightarrow{\text{implies}} x\in\bar{A}$ or $x\in\bar{B}$

$\xrightarrow{\text{implies}} x\in\bar{A}\cup\bar{B}$ #

* ∴ $\overline{(A_1\cap A_2\cap A_3\cdots)} = \overline{(\bigcap_{i=1}^{\infty} A_i)} = \bigcup_{i=1}^{\infty}\bar{A_i}$

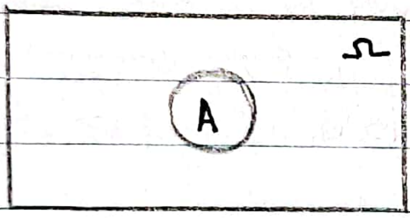
∴ $\overline{(\bigcup_{i=1}^{\infty} A_i)} = \bigcap_{i=1}^{\infty}\bar{A_i}$

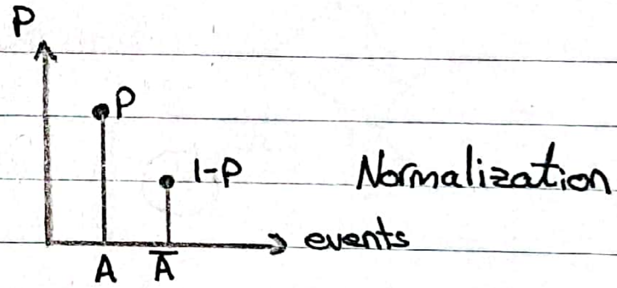
Exercise : show that :

1: $\overline{A\cap(B\cup C)} = (\bar{A}\cup\bar{B})\cap(\bar{A}\cup\bar{C})$

2: $\overline{(A\cap B)\cup(A\cap C)} = \bar{A}\cup(\bar{B}\cap\bar{C})$

* Note :





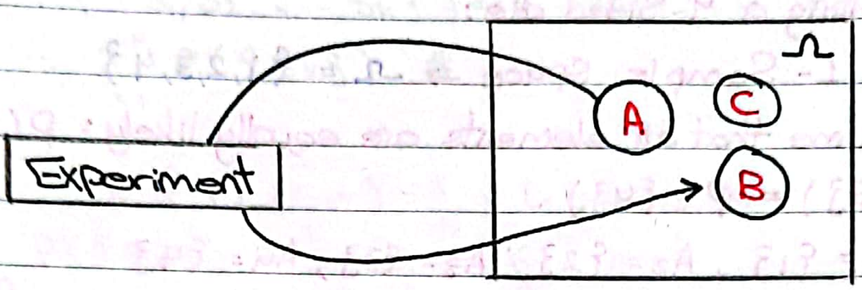
$P(A\cup\bar{A}) = P(\Omega) = 1$

Set و element بين ← belong : Note *

Set و Set بين ← Subset

10 / 3 / 2022

Probabilistic Model (PM) : is a mathematical description of a random experiment



* Steps for PM :

1- Identify the Sample space Ω , Ω is the set of all possible outcomes.

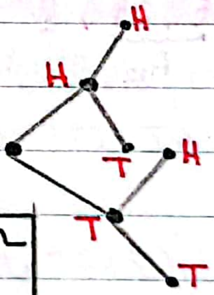
2- The probability law: in which we assign to a set "A", A ⊂ Ω an non negative number "P(A)" that encodes our beliefs about the liklihood of A.

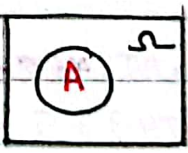
Ex: Tossing a Coin two times.

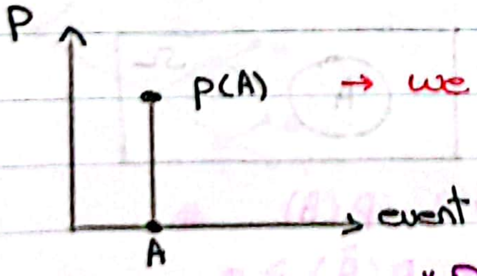
event: A = {one head shows up}

$\Omega = \{HH, HT, TH, TT\}$

$A = \{HT, TH\} \rightarrow A \subset \Omega$







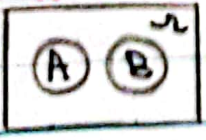
→ we explain how calculate this Later.

* probability Axioms :

1] Non-negativity : $P(A) \geq 0$. ∴ A is event .

2] Normalization : $P(\Omega) = 1$.

3] if A and B are disjoint events, then : $P(A \cup B) = P(A) + P(B)$



* any subset of the Sample space is called an event.

* if $A_1, A_2, \dots A_n$ are disjoint, then $P(A_1 \cup A_2 \cup A_3 \dots A_n)$
$= \sum_{i=1}^{n} P(A_i)$

Ex8 Rolling a 4-Sided die:

1- Sample Space → $\Omega = \{1,2,3,4\}$

we assume that all elements are equally likely: $P(\{1\}) = P(\{2\})$
$= P(\{3\}) = P(\{4\})$.

let $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$, $A_4 = \{4\}$

$P(\Omega) = 1 \longrightarrow \Omega = A_1 \cup A_2 \cup A_3 \cup A_4$, A_1, A_2, A_3, A_4 are disjoint

$\therefore P(\Omega) = 1 = P(A_1) + P(A_2) + P(A_3) + P(A_4)$

$\rightarrow P(A_1) = P(A_2) = P(A_3) = P(A_4) \rightarrow P(\Omega) = P+P+P+P = 1$

$\Rightarrow P = \frac{1}{4} \rightarrow$ probability of A_i

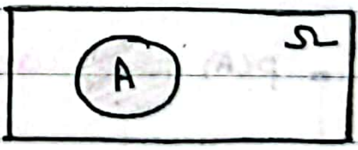
b) $P(\underbrace{\{1,2\}}_{A_{12}})$? $A_{12} = \{1,2\} = \{1\} \cup \{2\} = A_1 \cup A_2$

$\rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2) = \frac{1}{4} + \frac{1}{4}$
$= \frac{2}{4} = \frac{1}{2}$.

* Consequences of Axioms:

1] $P(A) \leq 1$

Proof: $\Omega = A \cup \bar{A}$, A and $\bar{A}$ are disjoint.



$P(\Omega) = 1 = P(A) + P(\bar{A}) \rightarrow P(\bar{A}) = 1 - P(A)$ #

$P(A) = 1 - P(\bar{A}) \rightarrow P(A) \leq 1$ Since $P(\bar{A}) \geq 0$

2] $P(\emptyset) = 0$

$\Omega = \emptyset \cup \Omega$, $\emptyset$ and Ω are disjoint.

$\therefore P(\Omega) = \cancel{1} = P(\emptyset) + \underset{\overset{\downarrow}{1}}{\cancel{P(\Omega)}} \rightarrow P(\emptyset) = 0$

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*Discrete probability law :

* $S = \{s_1, s_2, ... s_k\}$, $S \subset \Omega$, where $\Omega = \{s_1, s_2, --- s_n\}$, $n \geq k$.

$\rightarrow P(S) = P(\{s_1, s_2, --- s_k\}) = P(\{s_1\}) + P(\{s_2\}) + \cdots P(\{s_k\}) = \sum_{i=1}^{k} P(\{s_k\})$ #

* if $P(\{s_i\}) = \frac{1}{N}$, $i = 1, 2, ---, N$ *

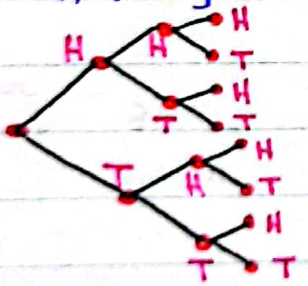
$P(S) = \sum_{i=1}^{k} P(\{s_k\}) = \sum_{i=1}^{k} \frac{1}{N} = \frac{k}{N}$ #

$\therefore P(S) = \frac{\text{\# of elements of } S}{\text{\# of elements of } \Omega}$ $\triangleq$ Discrete uniform prob. law.

↳ this is true as long as all events are equally likely

Ex : Tossing a coin three times, $A = \{$Exactly Two Heads$\}$.
Find $P(A)$? * $P(H) = P(T) = \frac{1}{2}$ → equally likely

1- Sample space $= \Omega =$ {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}



$P(A) = P(\{HHT, HTH, THH\})$

$= P(\{HHT\}) + P(\{HTH\}) + P(\{THH\})$

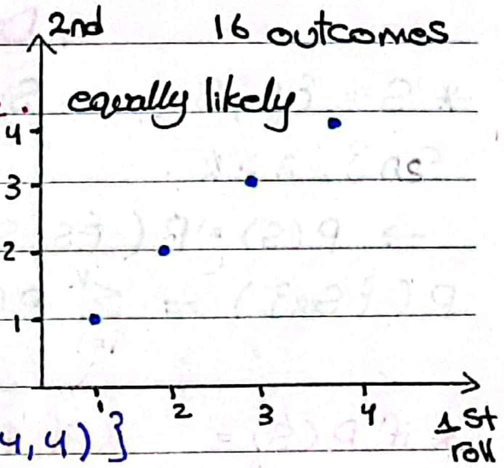
$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$ #

$\therefore P(A) = \frac{\text{\# of } A}{\text{\# of } \Omega} = \frac{3}{8}$ #

This is true because all ↵
events are equally likely

12/3/2022

[redacted]



Ex: Rolling a pair of 4-sided dice.

A- P (sum is even) ?

A- Sample Space:

$\Omega = \{(1,1), (1,2), (1,3), (1,4), (2,1),$
$(2,2), (2,3), (2,4), (3,1), (3,2),$
$(3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

$\therefore P(\text{sum is even}) = \frac{8}{16} = \frac{1}{2}$ → اللي فيهم جمع رقمين
even أو رقمين odd

B- P(Sum is odd) ? $\frac{1}{2}$

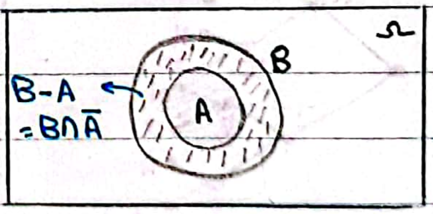
C- P(1st roll = 2nd roll) ? $\frac{4}{16} = \frac{1}{4}$ → diagonal (1,1), (2,2), (3,3), (4,4)

D- P (1st roll > 2nd roll) ? $\frac{6}{16}$ → اللي تحت الـ diagonal

E- P(at least one roll is 4) ? $\frac{7}{16}$

* Additional probability laws *

1] if $A \subset B$, then $P(A) \le P(B)$



∴ proof:

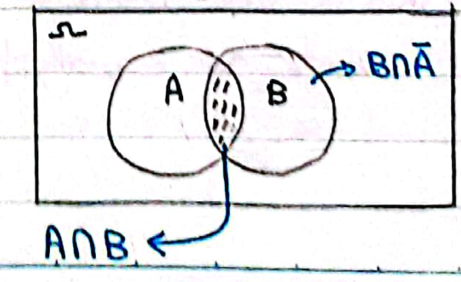
$B = A \cup (\bar{A} \cap B)$

$P(B) = P(A) + \underbrace{P(\bar{A} \cap B)}_{\ge 0} \Rightarrow P(B) \ge P(A)$ #

2] $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

∴ Proof:

↳ الـ General ما يكونوا disjoint.



$B = (A \cap B) \cup (\bar{A} \cap B)$

$P(B) = P(A \cap B) + P(\bar{A} \cap B)$

$\rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$\therefore P(A \cap \bar{B}) = P(A) - P(A \cap B)$

3] $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ * نفس الرسمة اللي ورا مع تغير الأماكن اللي بدنا ياها *

Proof: $A \cup B = A \cup (B \cap \bar{A})$

$P(A \cup B) = P(A) + P(B \cap \bar{A}) = P(A) + P(B) - P(B \cap A)$ *

4] $P(A \cup B) \leq P(A) + P(B)$

↳ المساواة لما يكونوا ال2 sets disjoint

* $P(A \cup B \cup C) = P(A) + P(B \cup C) - \underbrace{P(A \cap (B \cup C))}_{p^*}$

$= P(A) + P(B) + P(C) - P(B \cap C) - p^*$

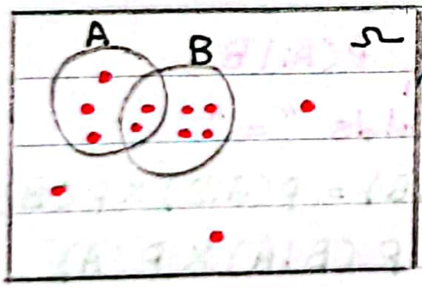
$-p^* = P(\underbrace{(A \cap B)}_{A_1} \cup \underbrace{(A \cap C)}_{A_2}) = -P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \boxed{- P(B \cap C) + P(A \cap B \cap C)}$ → لو ما كانوا disjoint بكون هيك أما لو كانوا disjoint

$\leq P(A) + P(B) + P(C)$. فكل ال intersection بروحوا بضل ← $P(A) + P(B) + P(C)$

السبب ∵ $P(A \cap B \cap C)$ is subset of $P(B \cap C)$ → فناتج طرحهم سالب

"Conditional"



→ $P(\{i\}) = \frac{1}{12}$

* $P(A) = \frac{5}{12}$ * $P(B) = \frac{6}{12}$

* $P(A \cap B) = \frac{2}{12}$

$P(A \text{ given } B) = P(A|B)$ ← * احتمال إنه A يصير إذا B حدث وصار

↪ $= \frac{2}{6} \Rightarrow \frac{P(A \cap B)}{P(B)}$

17/3/2022

Conditional probability

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad P(B) > 0$$

1] $P(A|B) \geq 0$

2] $P(B|B) = \frac{P(B\cap B)}{P(B)} = 1$

3] $P(\Omega|B) = \frac{P(\Omega\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

4] $P(B|\Omega) = \frac{P(B\cap\Omega)}{P(\Omega)} = \frac{P(B)}{1} = P(B)$

5] if A_1 and A_2 are disjoint, then $P(A_1\cup A_2 | B) = P(A_1|B) + P(A_2|B)$

$$\Rightarrow P(A_1\cup A_2|B) = \frac{P((A_1\cup A_2)\cap B)}{P(B)} = \frac{P((A_1\cap B)\cup(A_2\cap B))}{P(B)}$$

$$= \frac{P(A_1\cap B)}{P(B)} + \frac{P(A_2\cap B)}{P(B)} = P(A_1|B) + P(A_2|B)$$

In general: For $A_1, A_2, \ldots A_n$

$$P(A_1\cup A_2\cup \ldots \cup A_n|B) \leq \sum_{i=1}^{n} P(A_i|B)$$

if $A_1, A_2, \ldots A_n$ are disjoint, the Equality holds "="

* Recall that: $P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = P(A|B) \times P(B)$

$P(B|A) = 0.99$ $\qquad\qquad = P(B|A) \times P(A)$

$P(B|\bar{A}) = 0.1$

EX 8 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm with probability 0.10. we assume that an aircraft is present with probability 0.05. what is the probability of no aircraft presence and a false alarm? what is the probability of aircraft presence and no detection?

$\rightarrow P(\bar{A}\cap B)$ $\qquad\qquad \rightarrow P(A\cap\bar{B})$

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Sol. Method 1 : let $A = \{$aircraft is present$\}$, $P(A) = 0.05$, let $\bar{A} = \{$aircraft is not present$\}$, $P(\bar{A}) = 0.95$, let $B = \{$radar generates an alarm$\}$, let $\bar{B} = \{$radar does not generate an alarm$\}$

A- $P(\bar{A} \cap B) \Rightarrow P($no presence, alarm$)$

$= P(B|\bar{A}) \times P(\bar{A})$ → "Multiplication rule"

∴ from question → $P(B|A) = 0.99$

→ $P(B|\bar{A}) = 0.1$

$\Rightarrow P(B|\bar{A}) \times P(\bar{A}) = 0.1 \times 0.95 = 0.095$

B- $P(A \cap \bar{B}) \Rightarrow P($presence, No alarm$)$

$= P(\bar{B}|A) \times P(A) = 0.01 \times 0.05 = 0.0005$

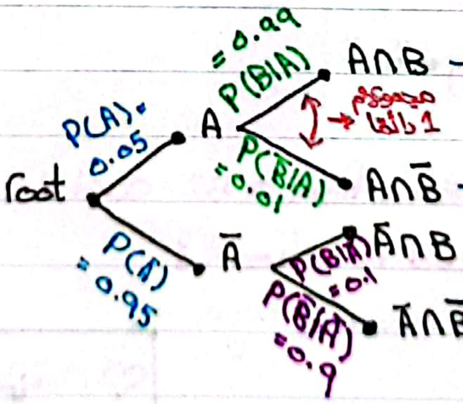
↓

$$* P(A_1 \cup A_2 | B) = P(\underset{\sim}{\overset{B}{A_1}}|B) + P(\underset{\sim}{\overset{\bar{B}}{A_2}}|B) = 1$$

$$\rightarrow P(A_1|A) + P(\bar{A}_1|A) \rightarrow 1 - 0.99$$

$$P(\bar{B}|A) = 1 - 0.99 \hookleftarrow$$

Method 2 :



↱ Prob. of detection

$A \cap B \rightarrow P(A \cap B) = P(A) \times P(B|A) = 0.05 \times 0.99$

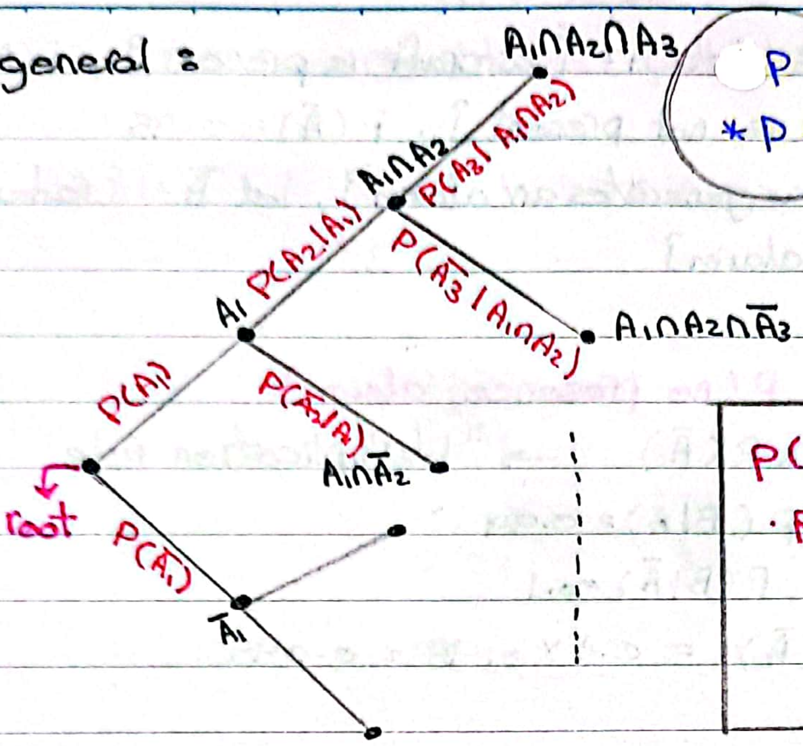
Prob. of Missed detection ↩

$A \cap \bar{B} \rightarrow = 0.05 \times 0.01 = 0.0005$

$\bar{A} \cap B \rightarrow = 0.95 \times 0.1 = 0.095$ → Prob. of false aram

$\bar{A} \cap \bar{B}$ ↳ $= 0.95 \times 0.9$

*In general :



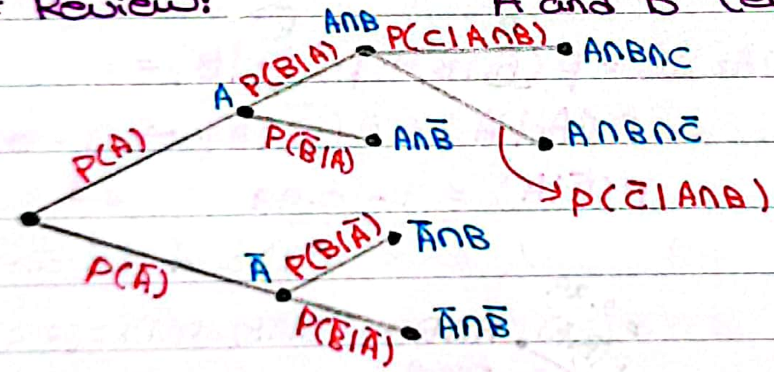
$P(A_1 \cap A_2 \cap A_3) = P(A_1) * P(A_2 | A_1) * P(A_3 | A_1 \cap A_2)$

↓ Multiplication Rule.

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot P(A_4 | A_1 \cap A_2 \cap A_3 \text{ ------ } P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right)$$

20/3/2022

* Review: A and B (events)



$* P(A \cap B \cap \bar{C}) = P(A) * P(B|A) * P(\bar{C} | A \cap B)$

Ex.8 Three cards are drawn from a 52-card deck (without replacement)

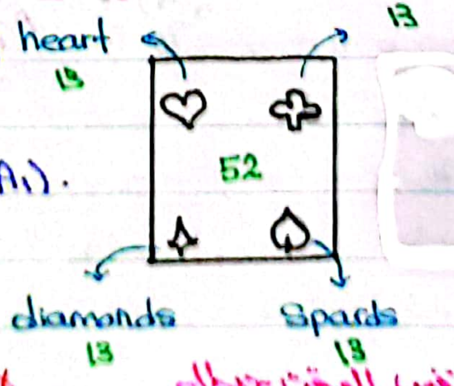
→ P {none of these card is a heart}

*Let A_i = {i^{th} card is not a heart}

, i = 1, 2, 3

→ $P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3 | A_1 \cap A_2)$ → من الـ tree

$= \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50}$



بإني سحبت 2 وما رجعتها [سحبت 1 وما رجعته

* لو سحبت الـ 3 بنفس الوقت حيطلع نفس الناتج *

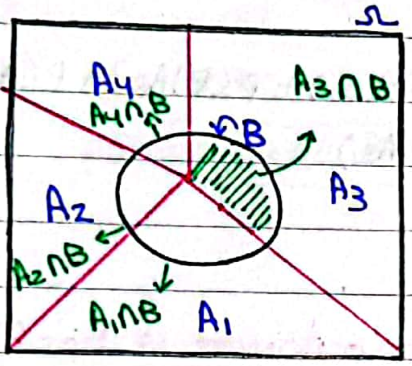
→ P{ 1st + 2nd not a heart and 3rd is }

$$P(A_1 \cap A_2 \cap \bar{A}_3) = \frac{39 * 38 * 13}{52 * 51 * 50}$$

* The Total probability theorem *

Let $A_1, A_2, \dots, A_n$ be disjoint events that form a partition of Ω, and let B be any event, then

$$P(B) = P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + \dots = \sum_{i=1}^{n} P(B|A_i)\,P(A_i)$$

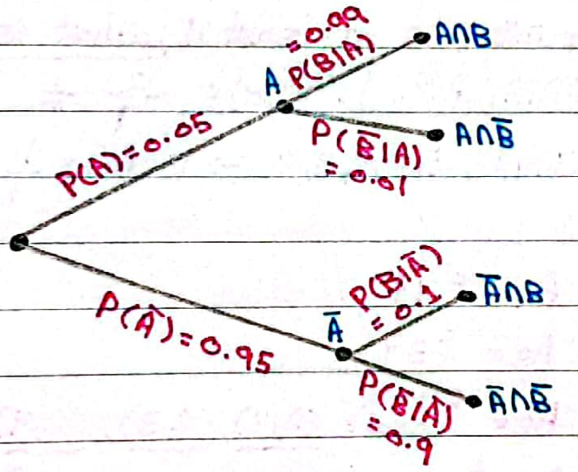


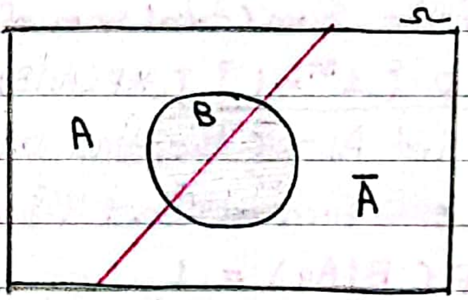
$$* \; B = \bigcup_{i=1}^{n} (B \cap A_i)$$

$$\rightarrow P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i)\,P(A_i).$$

Example : Radar detection : A = {presence}, B = {Alarm}





→ P(B) ??

Sol. $B = (B \cap A) \cup (B \cap \bar{A})$

$$\rightarrow P(B) = P(A) * P(B|A) + P(\bar{A}) * P(B|\bar{A})$$

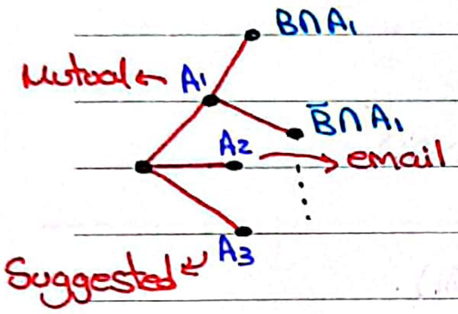
$$= (0.99 * 0.05) + (0.95 * 0.1) = 0.1445$$

→ $P(A|B)$?? ⇒ there is an aircraft given that the alarm was presence.

Sol. $\frac{P(A\cap B)}{P(B)} * \frac{P(A)}{P(A)} = \frac{P(B|A) * P(A)}{P(B)}$ (Baye's rule)

→ $P(A|B) = \frac{P(B|A) * P(A)}{P(B)} = \frac{0.99 * 0.05}{0.1445}$

* Note : "life example : added on facebook"



$B = \{Accepted\}$

→ $P(B) = P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + P(B|A_3) * P(A_3)$

Example: you roll a pair 4- Sided die, If the outcomes is 1 or 2 then you roll once more, otherwise you stop.

→ If the Sum (total sum of your rolls) is at least 4, what is the p {1st = 1} ? $*P(A_1|B) = (P(B|A_1)P(A_1)) / P(B) = \frac{2}{9}$ #

Sol. let $A_i = \{$the first roll is (i)$\}$, $i = 1,2,3,4$ → $P(A_i) = \frac{1}{4}$

→ $B = \{$Sum at least 4$\}$

* $P(B|A_4) = 1$ * $A_4 = \{4\}$
* $P(B|A_3) = 0$ * $A_3 = \{3\}$
* $P(B|A_2) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{3}{4}$ * $A_2 = \{(2,1), \underline{(2,2)}, \underline{(2,3)}, \underline{(2,4)}\}$
* $P(B|A_1) = \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$ * $A_1 = \{(1,1), (1,2), \underline{(1,3)}, \underline{(1,4)}\}$

$*P(B) = \sum_{i=1}^{4} P(B|A_i)P(A_i) = \frac{1}{4}[1+0+\frac{3}{4}+\frac{2}{4}] = \frac{9}{16}$

* Note : الـ disjoint ← بكونوا dependent يعني لو صار event مستحيل يصير الثاني. أمّا الـ union بكونوا Independent لو صار event الثاني ممكن يصير وممكن لا ما بيعتمد عالأول.

24/3/2022

« Independence »:

*if P(B|A) = P(B) then the occurence of "A" provides no information about the occurence of "B"

⟶ B is independent of A.

$$\therefore P(A|B) = \frac{\overset{P(B)}{\cancel{P(B|A)}}P(A)}{\cancel{P(B)}} = P(A).$$

⟶ A is independent of B.

∴ A and B are independent.

*In general: The events A and B are said to be independent if $P(A \cap B) = P(A)P(B)$

proof: $\boxed{P(A|B)} = P(A) \Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$

$$\therefore P(A \cap B) = P(A) * P(B) \quad \#$$

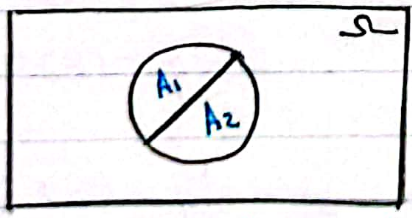
Ex: a fair Coin is tossed two times,

$$P(\underbrace{1^{st}\text{ toss is H}}_{A_1}, \underbrace{2^{nd}\text{ toss is H}}_{A_2}) = P(A_1) \times P(A_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

"tosses are independent"

« Comments for Independence »

1) Disjoint events are not independent - Disjoint events are totally dependent.



$P(A_1 \cap A_2) \neq P(A_1)P(A_2)$

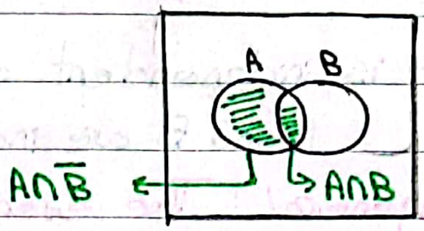
2) independent events can't be visualized using Venn diagram.

3) if A and B are independent, then ① $\bar{A}$ and $\bar{B}$ are independent, ② $\bar{A}$ and B are independent, ③ A and $\bar{B}$ are independent.

Proof (③ للخضراء) : A and B are independent, then $p(A \cap B)$ $= P(A)\,p(B)$

$\rightarrow$ $A = (A \cap B) \cup (A \cap \bar{B})$

disjoint



$p(A) = p(A \cap B) + p(A \cap \bar{B})$ « events are Independent »

$p(A) = p(A) \times p(B) + p(A \cap \bar{B})$, by independence

$p(A \cap \bar{B}) = p(A)(1 - p(B)) = p(A)\,p(\bar{B})$

$\Rightarrow$ A and $\bar{B}$ are independent #

« Conditional independence »

if $p(A \cap B | C) = p(A|C)\,p(B|C)$, then A and B are conditionally independent events.

→ by Multiplication rule

$$\rightarrow p(A \cap B | C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{\cancel{P(C)}\,P(B|C)\,P(A|B \cap C)}{P(\cancel{C})}$$

$$= p(\cancel{B}|C)\,p(A|B \cap C) = p(\cancel{B}|C)\,p(A|C)$$

$$\rightarrow p(A|B \cap C) = p(A|C) \ \#$$

* Independence of several events:

we say that $A_1, A_2, \ldots A_n$ are independent if

$P(\underset{i \in S}{\cap} A_i) = \underset{i \in S}{\prod} P(A_i)$ for every subset of the set "S",

belong ↲ Product ↲

↓

$S = \{1, 2, \ldots n\}$

* if $n = 2$, $S = \{1, 2\}$ $\therefore P(A_1 \cap A_2) = P(A_1) P(A_2)$
* if $n = 3$, (A_1, A_2, A_3), $S = \{1, 2, 3\}$

$$\left.\begin{aligned} \rightarrow P(A_1 \cap A_2) &= P(A_1) P(A_2) \\ P(A_1 \cap A_3) &= P(A_1) P(A_3) \\ P(A_2 \cap A_3) &= P(A_2) P(A_3) \end{aligned}\right\} \text{pairwise independence.}$$

$\therefore P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$

* pairwise independence does not imply independence #

يعني لو تحقق كل ال 3 Pairwise أما اللانهائية ما تحققت فما بكون Independence

Ex: Two successive rolls of a 4-sided die (fair die), let

$A_i = \{1^{st}$ roll is "i"$\}$, $B_j = \{2^{nd}$ roll is "j"$\}$,

A) Are A_i and B_j independent?

Sol. $P(A_i \cap B_j) = P(A_i) P(B_j)$

$\therefore P(A_i) = \frac{1}{4}$, $P(B_j) = \frac{1}{4}$

$\rightarrow P(A_i \cap B_j) = P\{(i,j)\} = \frac{1}{4} * \frac{1}{4} = \frac{1}{16} = P(A_i) P(B_j)$

Yes, they are independent.

B) $A = \{1^{st}$ roll is "1"$\}$, $B = \{$sum of two rolls is 5$\}$

$P(A) = \frac{1}{4}$

* $B = \{\boxed{(1,4)}, (4,1), (3,2), (2,3)\}$

↳ اللي بيعطوا 5 بس.

$P(B) = \frac{4}{16} = \frac{1}{4}$, $P(A \cap B) = \frac{1}{16}$

* $A \cap B = \{(1,4)\}$

↓

$P(A \cap B) = P(A) P(B) \Rightarrow$ A and B are independent

c) A = { max of two rolls is 2 }

B = { min of two rolls is 2 }

A = { (1,2), (2,1), $\underline{\underline{(2,2)}}$ } → $P(A) = \frac{3}{16}$

B = { $\underline{\underline{(2,2)}}$, (2,3), (2,4), (4,2), (3,2) } → $P(B) = \frac{5}{16}$

∴ $P(A \cap B) = \frac{1}{16}$ ∴ $P(A \cap B) \neq P(A)\,P(B)$

$\frac{1}{4} * \frac{1}{4}$ → A and B are not independent

29/3/2022

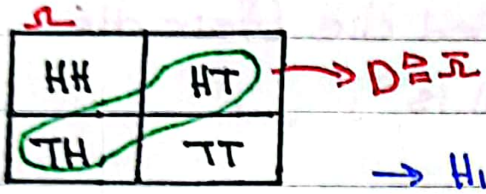
Ex 8 Two independent fair coin tosses

$H_1 = \{1^{st} \text{ toss is } H\}$ $P(H_1) = \frac{1}{2}$

$H_2 = \{2^{nd} \text{ toss is } H\}$ $P(H_2) = \frac{1}{2}$

$D = \{\text{Two Tosses have different outcomes}\}$

Is $P(H_1 \cap H_2 \mid D) = P(H_1 \mid D) P(H_2 \mid D)$?



Sol. $P(H_1 \mid D) = \frac{1}{2}$, $P(H_2 \mid D) = \frac{1}{2}$

$\therefore P(H_1 \cap H_2 \mid D) = 0 \neq P(H_1 \mid D) P(H_2 \mid D)$

→ H_1 and H_2 are not conditionally independent

* Note : $P(H_2 \mid H_1) = \frac{1}{2}$ $\therefore$ H_1 and H_2 are independent

لإنه H_2 مابتأثر ناتجها بـ H_1 .

* H_1 and D are independent (الـ prob مابتغير بوجود أو عدم D)

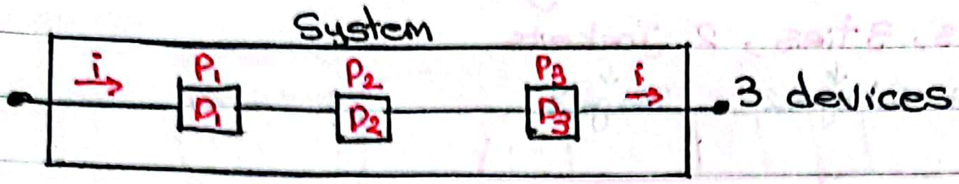
* H_2 and D are independent

→ pairwise independent (H_1, H_2, D)

$\therefore P(H_1 \cap H_2 \cap D) = 0$ → (H_1, H_2, D) are not independent.

$\hookrightarrow \neq P(H_1) P(H_2) P(D)$

* Applications of Independence :



$U_i \triangleq \{D_i \text{ is up}\}, \quad i = 1,2,3$

$P(U_i) = P_i$

$\therefore P\{\text{System is up}\} ? \; P(U_1 \cap U_2 \cap U_3) \rightarrow$ Series لإنهم

لازم كلهم يكونوا "up"

*Devices fail independentally → خراب واحد مادخله بالباقي

$\therefore P(U_1 \cap U_2 \cap U_3) = P(U_1)P(U_2)P(U_3) = P_1P_2P_3$

$\therefore P\{\text{System is down}\} = 1 - P_1P_2P_3$

* Note : 1: up / 0: down : P(down) :

$(1-P_1)P_2P_3$ ← 011

$P_1(1-P_2)P_3$ ← 101

110

100

001

010

000

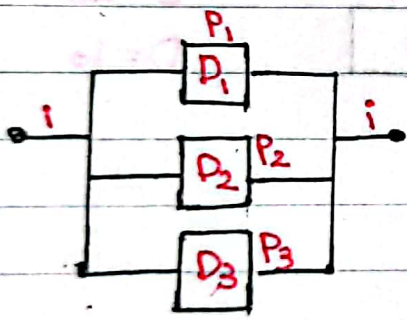
} system down

111 → sys. up

* P(1st D is down | system down)

↓

Exam Question



$U_i \triangleq \{D_i \text{ is up}\}$

$P(U_i) = P_i$

$\therefore P\{\text{system is up}\} = P(U_1 \cup U_2 \cup U_3)$

$= 1 - P(\overline{U_1 \cup U_2 \cup U_3}) = 1 - P(\overline{U_1})P(\overline{U_2})P(\overline{U_3})$ } من القواعد

$\overset{=}{\downarrow}$ $P(\overline{U_1} \cap \overline{U_2} \cap \overline{U_3})$

$= 1 - (1-P_1)(1-P_2)(1-P_3)$

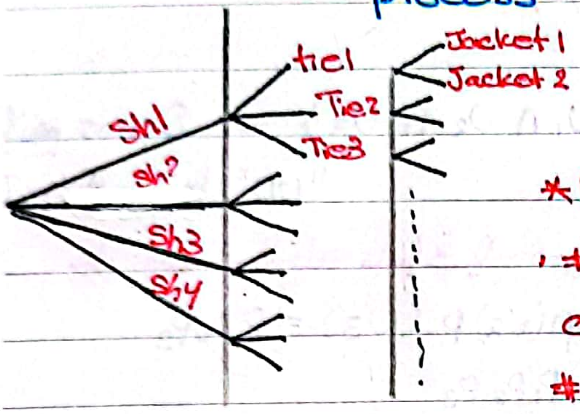
$\therefore P\{\text{System is down}\} = (1-P_1)(1-P_2)(1-P_3)$

3/4/2021

Counting principle

Ex : 4 Shirts, 3 ties, 2 Jackets

$\underbrace{\overset{\text{4 Shirts}}{\downarrow\ \text{Stage 1}}\quad \overset{\text{3 ties}}{\downarrow\ \text{Stage 2}}\quad \overset{\text{2 Jackets}}{\downarrow\ \text{Stage 3}}}_{\text{process}}$



tree diagram

\# of ways = 24 = 4 x 3 x 2

*A process consisting of "r" stages , # of ways the "ith" stage can be completed in n_i

\# ways "ith" which process can be completed $n = \prod_{i=1}^{r} n_i = n_1 n_2 n_3 \cdots n_r$

Ex : Constructing a license plate (2 letters, 3 digits) with repition

$n_1 = 26$	$n_2 = 26$	$n_3 = 10$	$n_4 = 10$	$n_5 = 10$
L	L	D	D	D

L = 26

D = 10

$n = 26 \times 26 \times 10 \times 10 \times 10$

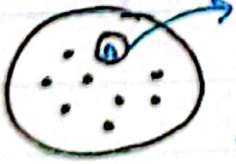
*without repition :

$n = 26 \times 25 \times 10 \times 9 \times 8$

Ex : Total # of subsets that can be made from an n-element set.

*Note : A = {1,2} , Subsets = {1}, {2}, {1,2}, ∅

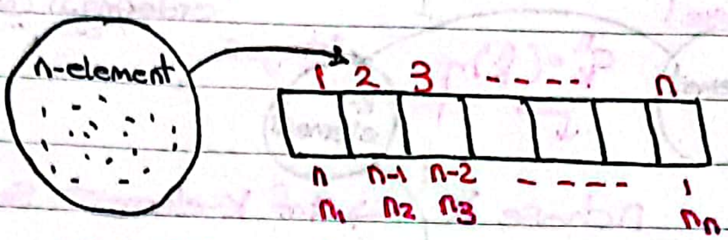
Sol.



for each element → in subset
↳ is not in subset

$\# \text{ of subset} = \underbrace{2 \times 2 \times 2 \times \cdots}_{n \text{ time}} = 2^n$

Permutation :



* Note : $A = \{1,2,3\}$

$P_1(A) = 1,2,3$

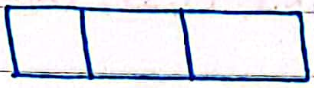
$P_2(A) = 2,3,1$

\# of ways of ordering "n" elements is $= n(n-1)(n-2)$

$= n!$

Ex : $A = \{p, s_1, s_2\}$, $n=3$

Sol.



\# $= n! = 3! = 3 \times 2 \times 1 = 6$

$p \quad s_1 \quad s_2$

$p \quad s_2 \quad s_1$

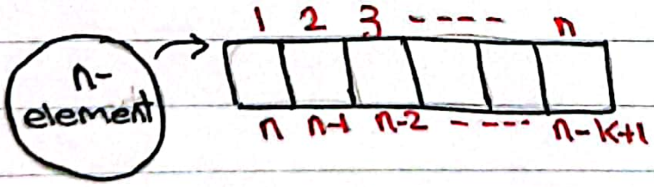
$s_1 \quad p \quad s_2$

$s_2 \quad p \quad s_1$

$s_1 \quad s_2 \quad p$

$s_2 \quad s_1 \quad p$

K- permutation :



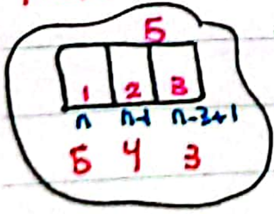
\# of ways $= n(n-1)(n-2)$

--- $(n-k+1) = P_k^n$

↓

$P_k^n = \frac{n!}{(n-k)!}$

"Note"



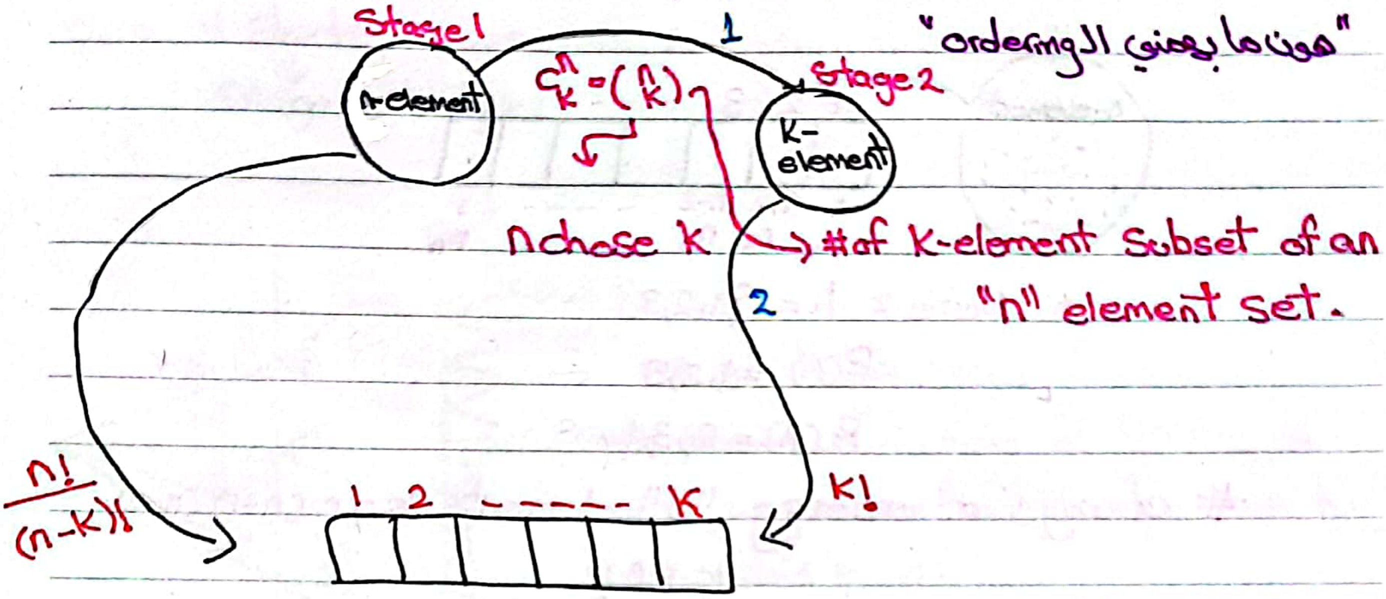
* $n! = n \times (n-1)! = n(n-1)(n-2)!$

→ $n! = \underbrace{n(n-1)(n-2) \ldots\ldots (n-k+1)}_{P_k^n}(n-k)!$

* $P_1^3 = \frac{3!}{2!} = 3$

* Compination :

"هون ما بيعني ال ordering"



n chose K → # of K-element Subset of an "n" element set.

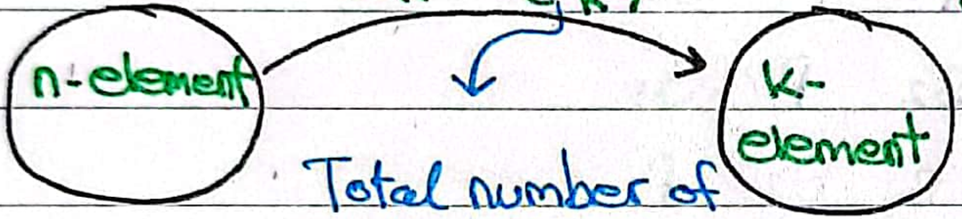
$$\rightarrow \frac{n!}{(n-k)!} = \binom{n}{k}k! \rightarrow \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

* Note: $A = \{1,2,3\} \rightarrow \{1,2\}, \{1,3\}, \{2,3\} \rightarrow \binom{3}{2} = 3$

7/4/ 2022

Combination :

$C_k^n = \binom{n}{k}$ choose subset of n element



Total number of k-element subsets of an "n" element set

Ex: $A = \{x_1, x_2, x_3\}$

Subsets:

$\rightarrow \{x_1\},\{x_2\},\{x_3\}$ $(3) \rightarrow C_1^3$

$\rightarrow \{x_1,x_2\},\{x_1,x_3\},\{x_2,x_3\}$ $(3) \rightarrow C_2^3$

$\rightarrow A, \emptyset$ (2)

① $\binom{n}{0} = \frac{\cancel{n!}}{0!\,\cancel{(n-0)!}} = 1$

② $\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n\cancel{(n-1)!}}{1!\cancel{(n-1)!}} = n$

③ $\binom{n}{n} = \frac{n!}{n!0!} = 1$

④ $\binom{n}{n-1} = \frac{n!}{(n-1)!1!} = n$

⑤ $\sum_{k=0}^{n} \binom{n}{k} = 2^n \rightarrow \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = 2^n$

Ex: a Six Sided die is rolled Six times,

$Pr\{\underbrace{\text{All rolls result in different numbers}}_{A}\}$

Sol. $P(A) = \frac{\text{\# of elements of } A}{\text{\# of elements of } \Omega}$ ∴ $|A|$ = # of elements of A

* $A = \{x_1, x_2, x_3\} \rightarrow |A| = 3$

$= \frac{|A|}{|\Omega|} = \frac{6!}{6^6}$

∴ $|\Omega| = 6\times6\times6\times6\times6\times6 = 6^6$

$|A| = 6! \longrightarrow \{$ (1,2,3,4,5,6)
(6,4,5,3,2,1)
⋮

Ex: a Coin Tossing "n" times, $P(H) = p$ / $P(T) = 1-p$

$P(HH\ TT\ HH\ TT\ TT\ ---T) = ?$ $pp(1-p)(1-p)pp(1-p)---(1-p)$

$= p^4 (1-p)^{n-4}$

general: $P(k \text{ heads}, n-k \text{ tails} \mid \text{for a given sequence}) = p^k (1-p)^{n-k}$

∴ $P(HHH\ TTT \cdots T) = p^3 (1-p)^{n-3}$

∴ $P(HTHHTTT \cdots T) = p^3 (1-p)^{n-3}$

∴ $P(k \text{ Heads}) = \sum_{\text{all sequences}} p^k (1-p)^{n-k} = \underset{\text{Sequences}}{\text{\# of}} \left(p^k (1-p)^{n-k}\right)$

→

*بدي كل ال element اللي فيهم K Heads ما بعطيني ال Sequences ← $\binom{n}{k}$

$\therefore P(K \text{ Heads}) = \binom{n}{k} p^k (1-p)^{n-k}$

Ex : Tossing a coin of "10" times, 1: Pr {There are 3 Heads} ?

* $P(H) = p$ / $P(T) = 1-p$ *

Sol. $\binom{10}{3} p^3 (1-p)^7 \rightarrow 10-3$

2: Pr {first two are Heads, one in 8 Tosses} ?

Sol. $= Pr(H, H, \text{one Head in 8 Tosses})$

$= p * p * \binom{8}{1} p^1 (1-p)^7 \rightarrow 8-1$

Ex : given that there are 3 Heads in 10 Tosses, find Pr {first Two Tosses are Heads}

Sol. let A = {first two tosses are Heads}

B = {3 Heads in 10 Tosses}

$\rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{8\cancel{p^3}\cancel{(1-p)^7}}{\binom{10}{3}\cancel{p^3}\cancel{(1-p)^7}}$

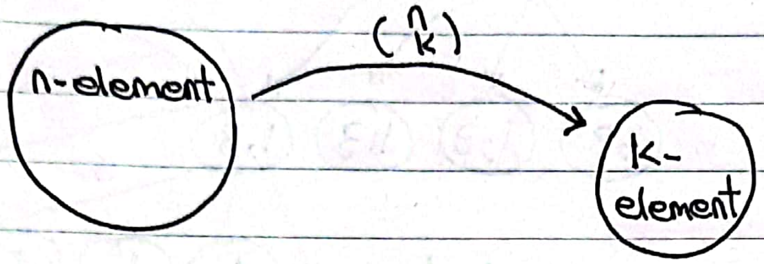
$\therefore P(A \cap B) = P(H, H, \text{one H in 8 Tosses})$

$= p * p * \binom{8}{1} p^1 (1-p)^7$

$= 8p^3(1-p)^7$

Partitions :

*



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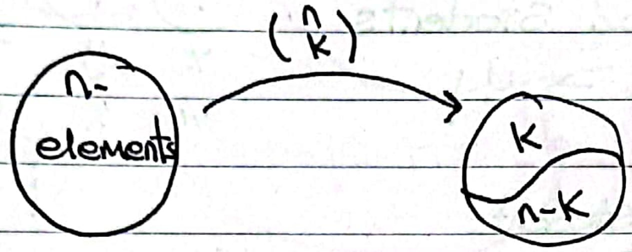

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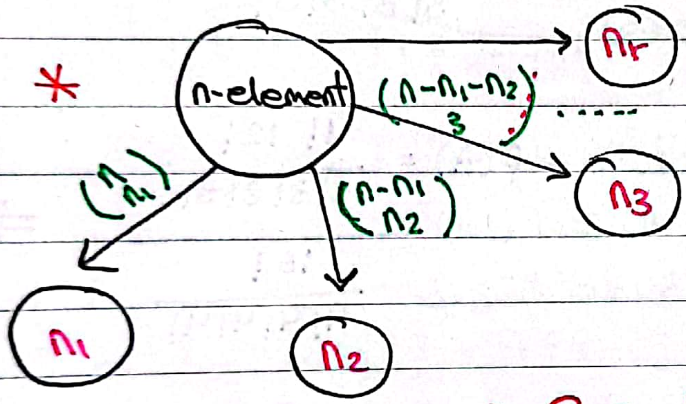


* where $n_1+n_2+\cdots n_r = n$

$C \triangleq$ # of ways $= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3} ----$

$$\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$

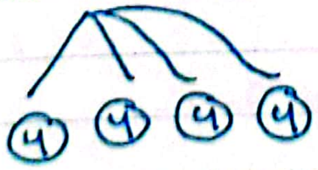
$$\rightarrow C = \frac{n!}{n_1!\,n_2!\,n_3!\,----\,n_r!} \quad \#$$

EX : A class contains 12 ungrad students and 4 grad students. we randomly distribute the students on 4 groups of 4-students, Find the probability that each group has a grad student.

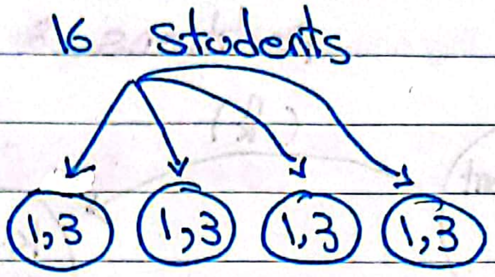
Sol. Define A = { Each group has a grad student }

$$P(A) = \frac{|A|}{|\Omega|}$$

* $|\Omega|$: 16 students $= \frac{16!}{4!4!4!4!} = |\Omega|$



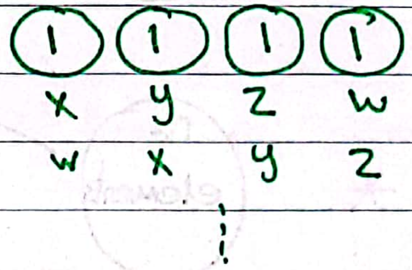
* |A| :



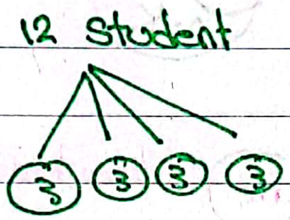
Stage 1 :

4 grad Students

$\rightarrow 4!$



Stage 2 :



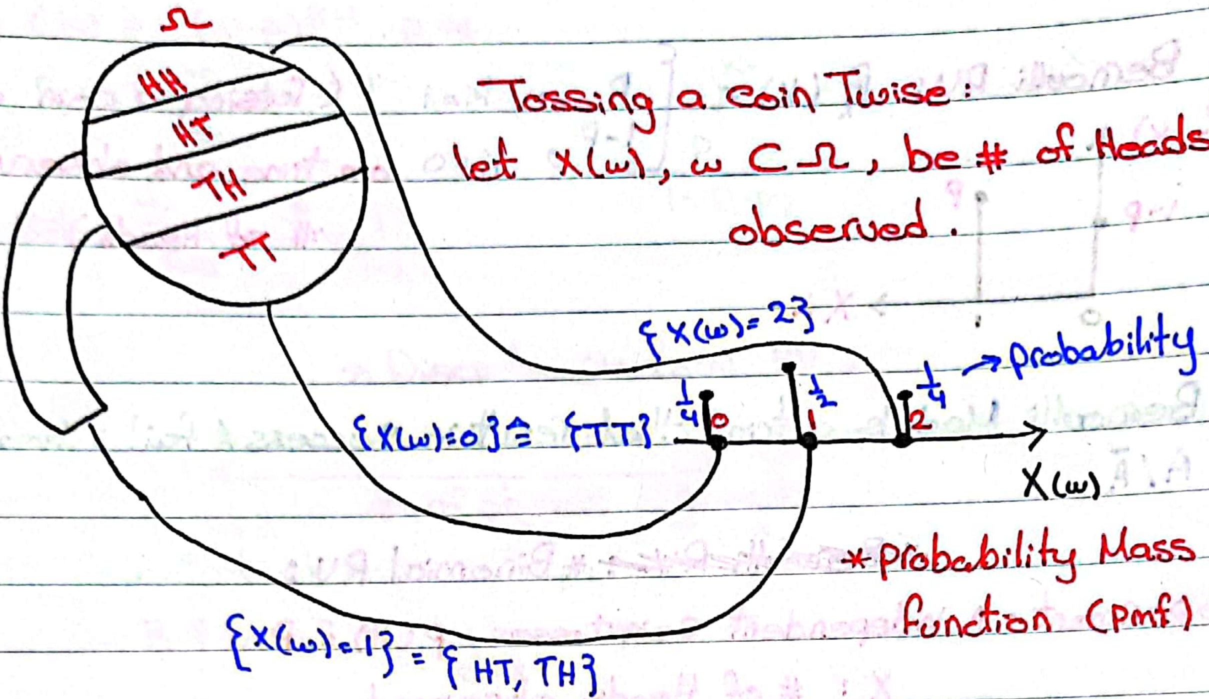
$$= \frac{12!}{3!\,3!\,3!\,3!}$$

$$|A| = 4! * \frac{12!}{3!\,3!\,3!\,3!} \quad \therefore P(A) = \frac{\frac{4!\ 12!}{3!\,3!\,3!\,3!}}{\frac{16!}{4!\,4!\,4!\,4!}} \quad \#$$

12/4/2022

Random Variable (RV):

Tossing a coin Twise:
let $X(\omega)$, $\omega \subset \Omega$, be # of Heads observed.



* Probability Mass function (pmf)

- RV is a real-valued function of the experiment outcome.

  $X(\omega)$, $\omega \subset \Omega$, $X(\omega)$: # of Heads.

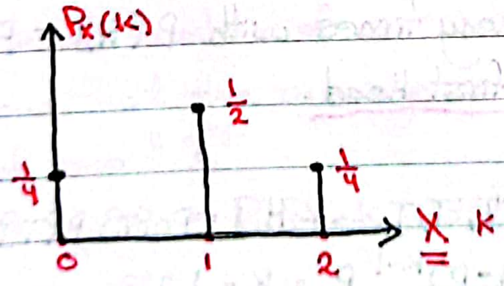
  ↳ مسمّى هكذا للتسهيل

- $Pr\{\omega \subset \Omega, X(\omega)=1\} = Pr\{X=1\} = P_X(1)$

  prob. that the random variable $X=1$ ↲

- $P_X(K) = P_r\{X=K\}$ → "discreet variables"
- $P_X(0) = Pr\{X=0\}$

→ PMF



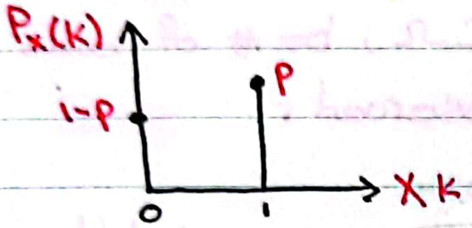
1- $P_X(K) \geq 0$

2- $\sum_{K} P_X(K) = 1$

* دايما الـ discrete random variable بيحتاج pmf .

Common discrete random variable :

- Bernoulli RV: $P_X(K) = \begin{cases} P & , K=1 \\ 1-P & , K=0 \end{cases}$ (Tossing a coin one time and observe the # of Heads)



* Bernoulli Models a trial that result in success / fail , Head / Tail , $A / \bar{A}$

* Binomial RV :

Experiment : n independent coin tosses $P(H) = P$.

X : # of Heads observed.

- $P_X(K) = Pr\{X = K\} = \binom{n}{k} p^k (1-p)^{n-k}$

* if $n=1 \rightarrow$ Bernoulli

14/4/2022

* $\sum_{k=0}^{n} P_X(K) = \sum_{k=0}^{n} \binom{n}{k} \underbrace{p^k}_{a} \underbrace{(1-p)^{n-k}}_{b} = [p+(1-p)]^n = 1^n = 1$

* Note: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

* Geometric RV :

- we Toss a coin infinetally Many times with $P\{H\} = P$.

$\underbrace{X = \text{ \# of Tosses till the first Head.}}_{Rv}$

- $P_X(K) = P_r\{X=K\} = Pr\{T, T, T, T, T, ---- H\} = Pr\{T\} P\{T\} --- P\{H\}$

$= \underbrace{(1-P)(1-P) ----}_{(K-1) \text{ Tails}} P = (1-p)^{K-1} P , K = 1, 2, 3, ---- \infty$

- if $P_X(1) = Pr\{X=1\} = (1-p)^0 . P = P = P\{H\}$

Prob. of gitting H in first Toss

∴ $P_X(2) = P_r\{X=2\} = (1-p).p = p(1-p) = p-p^2$

* $P_X(K) = (1-p)^{K-1}.p$ *

- $\sum_{K=1}^{\infty} P_X(K) = \sum_{K=1}^{\infty}(1-p)^{K-1}p = p\sum_{K=1}^{\infty}(1-p)^{K-1} = p\sum_{K=0}^{\infty}(1-p)^{K}$

$$= \frac{p}{1-(1-p)} = \frac{p}{p} = 1$$

*Note = $\sum_{k=0}^{\infty}(1-x)^k = \frac{1}{1-x}$

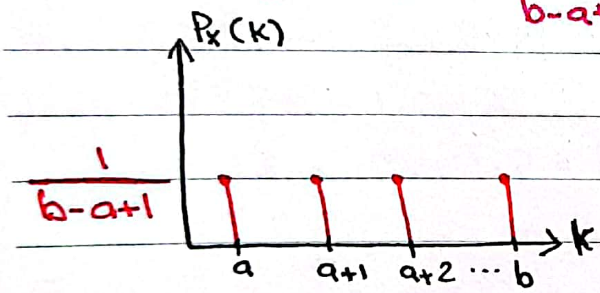
*Discrete Uniform RV :

* let : $\underbrace{a, a+1, a+2, ------, b}_{\# \text{ of objects } = b-a+1}$ (a and b are integers)

X : Selected object (equally likely)

∴ $P_r\{X=a\} = \frac{1}{b-a+1}$ #

∴ $P_r\{X=K\} = \frac{1}{b-a+1}$ #, K= a, a+1, ---- b



* if a=0 , b=n

#of objects = n+1

↳ $P_r\{X=K\} = \frac{1}{n+1}$, K=0,1,...n

*Poisson RV :

X: actual Number of ocurrences during a given time period

- we know that the average # of ocurrences over a given period of time "λ" .

* $P_X(K) = P_r\{X=k\} = e^{-\lambda}\frac{\lambda^k}{k!}$, k=0,1,....∞

* $\sum_{k=0}^{\infty} P_X(K) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}.e^{\lambda} = 1$

*Note : $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

17/4/2022

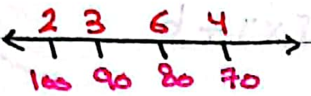
Expected Value of a RV (Mean or Average)

The expected value of R.V X is

$$E[X] \triangleq \underset{\sim}{\mu}_x = \sum_{k} k\,P_x(k)$$

constant not R.V ↵

* Note :



$\therefore \text{Avg} = \frac{2*100+3*90+6*80+4*70}{15}$

$P\{x=100\}$

$$= \boxed{\frac{2}{15}} \times 100 + \boxed{\frac{3}{15}} \times 90 + \frac{6}{15} \times 80 + \frac{4}{15} \times 70$$

$P_k(100).100$ → $P\{x=90\}$

* let $g(x)$ be a function of R.V X, then $E[g(x)] = \sum_{k} g(k)\,\underbrace{P_x(k)}_{\text{pmf of } X}$

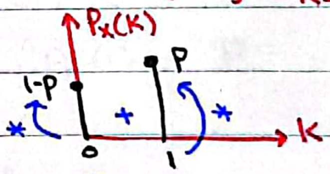
Ex: let $X \sim$ Bernoulli (P), Find $E[X]$ and $E[3X]$

Sol. $E[X] \longrightarrow \sum_{k} k\,P_x(k)$

$= 0 \cdot P_x(0) + 1 \cdot P_x(1)$

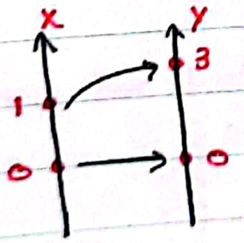
$= 0 \cdot (1-P) + 1 \cdot P = P$

$\therefore P_x(k) = \begin{cases} P, & k=1 \\ 1-P, & k=0 \end{cases}$



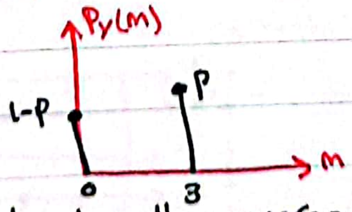
$E[3X] = \sum_{k} 3x\,P_x(k) = 3(0) \cdot P_x(0) + 3(1) \cdot P_x(1) = 3P$

② Find the PMF of Y, $P_Y(m)$



$P_Y(3) = P_r\{Y=3\} = P_r\{X=1\} = P$

$P_Y(0) = P_r\{Y=0\} = P_r\{X=0\} = 1-P$



$E[Y] = 0(1-P) + 3.P = 3P$ #

* you don't need to try this approach unless it (انطلب منك) → Pmf.

Special Cases:

* let $g(X) = aX + b$

Find $E[g(X)] = \sum_{k} (ak+b)\, P_X(k) = \sum_{k} (ak P_X(k) + b\, P_X(k))$

$$= a \sum_{k} k P_X(k) + b \underset{1=}{\boxed{\sum_{k} P_X(k)}} = a E[X] + b$$

$\rightarrow E[aX+b] = a E[X] + b \rightarrow$ Linearity of Expectation

* The n^{th} moment of the R.V X.

$$\underset{g(x)}{} \; E[\underline{X^n}] = \sum_{k} k^n \cdot P_X(k)$$

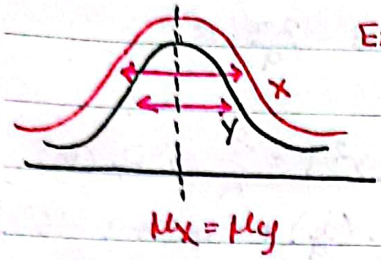
if $n=1 \longrightarrow E[X^n] \triangleq E[X] = \mu_X$ (mean / first moment)

if $n=2 \longrightarrow E[X^n] \triangleq E[X^2] =$ Seconed Moment

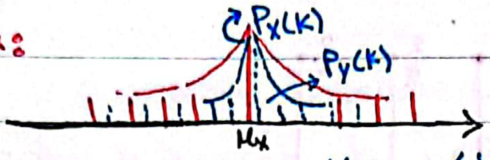
21/4/2021

The varience of R.V :

1: It provides a measure of dispersion of X around its Mean.



Ex:



X متشتتة أكثر
من Y
متباعدة عن ال Mean أكثر من Y.

- let X be a R.V, then $Var(X) = E[(X-\mu_X)^2]$

$$= E[X^2 - 2\mu_X X + \mu_X^2]$$

$$= E[X^2] - E[\underline{\overset{a}{2\mu_X} X}] + E[\overset{\text{Mean}}{\mu_X^2}]$$

$$* E[aX+b] = aE[X]+b$$

$$= E[X^2] - 2\mu_X \cdot \mu_X + \mu_X^2$$

$$\rightarrow Var(X) = E[X^2] - \mu_X^2 \triangleq E[X^2] - (E[X])^2$$

1- $Var(X) \geq 0$ 2- $\sigma_X \triangleq$ Standard deviation, $\sigma_X = \sqrt{Var(X)}$

Ex: let $X \sim$ Berunolli : (p), Find $E[X^n]$, $Var(X)$

Sol. $E[X^n] = \sum_k k^n P_X(k) = \underline{(0)^n P_X(0)} + (1)^n P_X(1) = P$

$\therefore E[X] = P$, $E[X^2] = P$, $E[X^3] = P^0$......

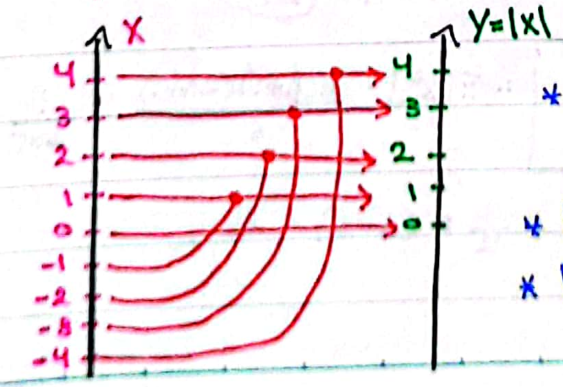
$* Var(X) = E[X^2] - (E[X])^2 = P - P^2 = P(1-P)$ #

Ex: let $Y = |X|$

$$P_X(k) = \begin{cases} \frac{1}{9}, & k \in [-4,4] \\ 0, & \text{other wise (o.w)} \end{cases}$$

Find the pmf of Y ($P_Y(m)$), $E[Y]$ and $Var(Y)$

Sol.



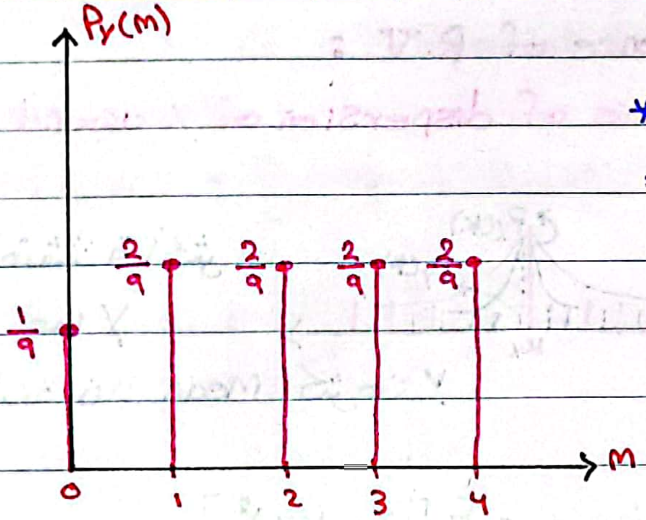
$* P_Y(0) = P_X(0) = \frac{1}{9}$

$* Y=1$ if $X=1$ or $X=-1 \rightarrow P_Y(1) = P_X(1) + P_X(-1)$

$= \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$

$* P_Y(2) = P_X(2) + P_X(-2) = \frac{2}{9}$

$* P_Y(3) = P_Y(4) = \frac{2}{9}$

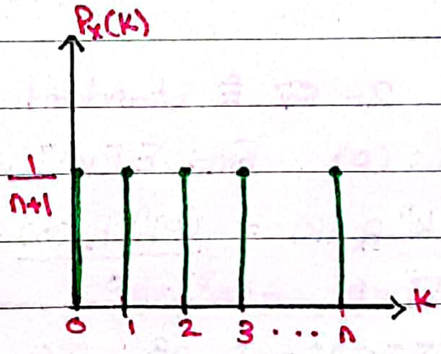


$$* \; E[Y] = \sum_m m P_y(m)$$
$$= 0.(\tfrac{1}{9}) + (1+2+3+4)\tfrac{2}{9}$$
$$= \frac{20}{9} \;\#$$

$$* \; E[Y^2] = \sum_m m^2 P_x(m)$$
$$= 0.(\tfrac{1}{9}) + (1+4+9+16)\tfrac{2}{9}$$
$$= \frac{60}{9}$$

$$* \; Var(Y) = E[Y^2] - (E[Y])^2 = \frac{60}{9} - \left(\frac{20}{9}\right)^2 = \frac{140}{81} \;\#$$

* Mean and Varience of the discrete uniform R.V :



$$* \; P_x(K) = \frac{1}{n+1}$$

$$* \; E[X] = \sum_K K P_x(K)$$
$$= \sum_{k=0}^{n} k . \frac{1}{n+1} = \frac{1}{n+1} \underline{\left(\sum_{k=1}^{n} k\right)}$$

↳ $S = 1+2+3+ \cdots n \Rightarrow S = \frac{n(n+1)}{2}$ from calculus

$$* \; E[x] = \frac{n\cancel{(n+1)}}{2} * \frac{1}{\cancel{n+1}} = \frac{n}{2} \;\#$$

S = ??

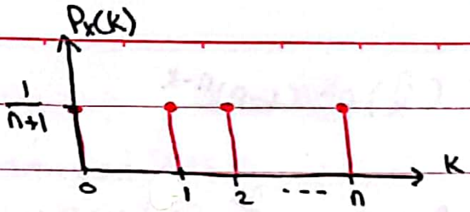
$$* \; E[X^2] = \sum_{k=0}^{n} k^2 . P_x(k) = \sum_{k=0}^{n} k^2 . \left(\frac{1}{n+1}\right) = \frac{1}{n+1} . \boxed{\sum_{k=1}^{n} k^2}$$

$$\# \quad \boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}} \quad \therefore E[X^2] = \frac{n\cancel{(n+1)}(2n+1)}{6} . \frac{1}{\cancel{n+1}} = \frac{n(2n+1)}{6}$$

$$* \; Var(X) = \frac{n(2n+1)}{6} - \left(\frac{n}{2}\right)^2 = \frac{1}{12} n(n+2) \;\#$$

24/4/2022

((Discrete uniform))



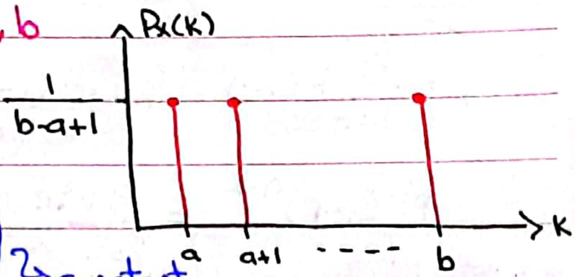
$* \ P_x(K) = \frac{1}{n+1} \Longrightarrow$

$* \ E[x] = \frac{n}{2}$

$* \ Var(x) = \frac{1}{12} n(n+2)$

((general Discrete Uniform))

$$P_x(K) = \begin{cases} \frac{1}{b-a+1} & , K = a, a+1, ---, b \\ 0 & , o.w \end{cases}$$



$* \ E[x] = \sum_{k} K \, P_x(K) = \sum_{k=a}^{b} K \cdot \boxed{\frac{1}{b-a+1}}$ ⤷ constant

$= \frac{1}{b-a+1} \cdot \boxed{\sum_{k=a}^{b} k}$, "a,b are Integers"

↳ $Sum = S = \frac{(\text{first term} + \text{last term})}{2} \cdot \# \text{ of terms}$

$$= (b-a+1) \cdot \frac{(a+b)}{2} \Longrightarrow \boxed{E[x] = \frac{a+b}{2}}$$

$*$ Note: when $k = 0, 1, ---, n \rightarrow P_x(K) = \frac{1}{n+1}$, $Var(x) = \frac{1}{12} n(n+2)$

$*$ for $P_x(K) = \frac{1}{b-a+1}$, $k = a, a+1, ---, b$, $\therefore b-a\cancel{+1} = n\cancel{+1}$

$n = b-a$

$\therefore Var(x) = \frac{1}{12} (b-a)(b-a+2)$

((Binomial RV))

let $X \overset{be}{\sim}$ Binomial (n, p)

$\therefore P_x(K) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, ---, n$

$\therefore E[x] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} k \frac{n!}{k! \, (n-k)!} p^k (1-p)^{n-k}$

$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} p \cdot n \overset{\binom{n-1}{k-1}}{\boxed{\frac{(n-1)!}{(k-1)!(n-k)!}}} \cdot p^{k-1} \cdot (1-p)^{n-k}$

$= n \cdot p \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} \cdot (1-p)^{n-k} = n \cdot p \underbrace{\sum_{r=0}^{n-1} \binom{n-1}{r} p^r (1-p)^{(n-1)-r}}_{=1} = n \cdot p$

$* \ r = k-1$
if $k = 1 \rightarrow r = 0$, if $k = n \rightarrow r = n-1$

CamScanner

$* \; E[x^2] = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$ $\qquad \because k^2 = k^2 + k - k$

$= \sum_{k=0}^{n} (k^2 - k + k) P_x(k) = \sum_{k=0}^{n} (k^2-k) P_x(k) + \boxed{\sum_{k=0}^{n} k P_x(k)} = E[x] = n.p$

$= \underbrace{\sum_{k=0}^{n} (k^2-k) P_x(k)} + n.p = \boxed{n^2p^2 - np^2 + n.p}$

$\hookrightarrow = \sum_{k=0}^{n} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \frac{\cancel{k(k-1)}\, n!}{(n-k)!\, \underbrace{\cancel{k(k-1)}(k-2)!}_{k!}} \cdot p^k (1-p)^{n-k}$

$\binom{n-2}{k-2} \leftarrow$

$= \sum_{k=2}^{n} p^2 . n(n-1) \boxed{\frac{(n-2)!}{(n-k)!\,(k-2)!}} p^{k-2} (1-p)^{n-k}$

↲ لإنه 1 و 0 بيعطوني ناتج = 0

$= n(n-1)p^2 \underbrace{\sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} (1-p)^{n-k}}_{=1} = n(n-1)p^2 = n^2p^2 - np^2$

$* \; Var(x) = E[x^2] - \mu_x^2 = \cancel{n^2p^2} - np^2 + np - \cancel{(np)^2} = \underbrace{np}_{\text{get a head}} \underbrace{(1-p)}_{\text{get a tail}}$ #

8/5/2022

$*$ let x be a R.V, and let $g(x) = ax+b$, then $Var(g(x)) = Var(ax+b)$

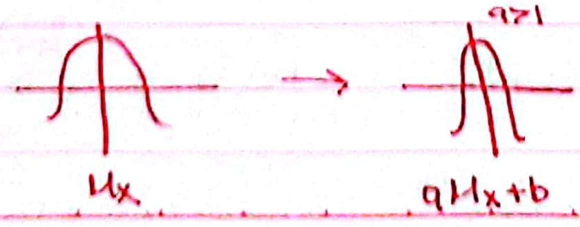
$= \underbrace{E[g^2(x)]}_{I} - \underbrace{(E[g(x)])^2}_{II}$

$\therefore II \rightarrow \therefore (E[ax+b])^2 = (aE[x]+b)^2 = a^2(E[x])^2 + 2abE[x] + b^2$

$\therefore I \rightarrow \therefore E[(ax+b)^2] = E[a^2x^2 + 2abx + b^2] = a^2E[x^2] + 2abE[x] + b^2$

$\therefore Var(ax+b) = I - II = a^2[\underbrace{E[x^2] - (E[x])^2}_{Var(x)}] = a^2 Var(x)$

* ال b ما بتأثر لإنها Shifting، بس ال a بتأثر لإنها Scaling.



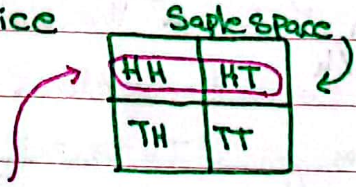
* لو ال Variance كان بساوي صفر فإنه هو مش Random Variable.

∴ Ex: $n=10, P=\frac{1}{3} \rightarrow Var(7X+7)$? سؤال امتحان.

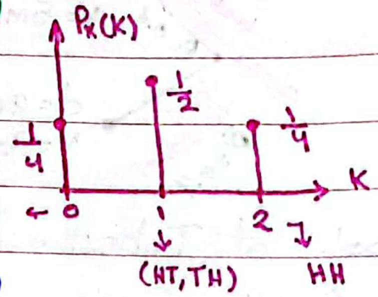
((Conditional PMF))

∴ Tossing a fair coin twice

Saple space



, $X \triangleq$ # of Heads.



A = { 1'st toss is a head }

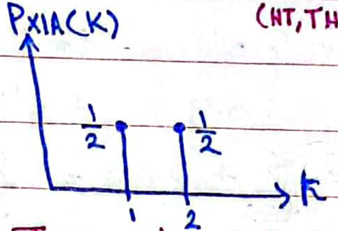
$P_{X|A}(K) = P_r\{X = K|A\}$

→ $P_{X|A}(0) = P_r\{X=0|A\} = 0$

→ $P_{X|A}(1) = P_r\{X=1|A\} = \frac{1}{2}$

→ $P_{X|A}(2) = P_r\{X=2|A\} = \frac{1}{2}$

⟹



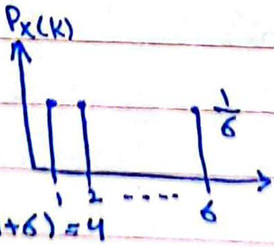
* conditional pmf : $P_{X|A}(K) = \frac{P(X=K \cap A)}{P(A)}$ → The conditional Pmf as a R.V "X" conditional on a particular event "A"

PMF	Conditional PMF
$P_X(K) = P_r\{X=K\}$	$P_{X|A}(K) = P_r\{X=K|A\}$
$\sum_x P_X(K) = 1$	$\sum_x P_{X|A}(K) = 1$
$E[X] = \sum_K K P_X(K)$	$E[X|A] = \sum_K K.P_{X|A}(K)$ → conditional Mean
$E[g(x)] = \sum_K g(K) P_X(K)$	$E[g(x)|A] = \sum_K g(K) P_{X|A}(K)$
$Var(x) = E[X^2] - (E[X])^2$	$Var(X|A) = E[X^2|A] - (E[X|A])^2$

Ex: let x be the roll of a fair Six-Sided die, A = {the roll is even}, ↳ {2,4,6}

1] $E[X|A]$

2] $Var(X|A)$



Sol. find $P_{X|K}(K)$:

⇒ conditional PMF:

$P_{X|A}(K) = P_r\{X=K|A\}$

$= \begin{cases} \frac{1}{3}, & K = 2,4,6 \\ 0, & o.w \end{cases}$

$\frac{P(X=K \cap \text{roll is even})}{P(\text{roll is even})} = \frac{\frac{1}{6}}{\frac{3}{6}}$

2,4,6 ↰

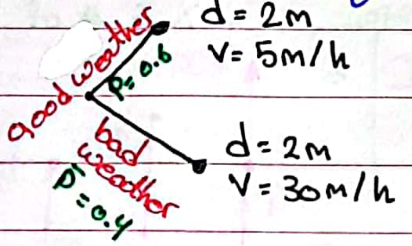
① $\sum_K K P_{X|A}(K) = \frac{1}{3}(2+4+6) = 4$

16 ↰

② $\underline{E[X|A]} - (E[X|A])^2 = \frac{1}{3}(4+16+36) - 16 = \frac{8}{3}$

↳ $\sum_K K^2 P_{X|A}(K) = \frac{1}{3}(2^2+4^2+6^2)$

* Example Defor Conditional PMF :



$\rightarrow T = t_1 = \frac{2}{5}$ h

∴ The mean (average) time to get to class: ?

$\rightarrow T = t_2 = \frac{2}{30} = \frac{1}{15}$ h

Sol. let T be the time to class (T is R.V)

$$\therefore P_T(t) = \begin{cases} 0.6, & t = 2/5 \text{ h} \\ 0.4, & t = 1/15 \text{ h} \end{cases}$$

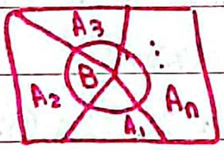
$$\therefore E[T] = \sum_{t \in S} t\,P_T(t) = \frac{2}{5}(0.6) + \frac{1}{15}(0.4) = \frac{4}{15} \text{ h} = 16 \text{ minutes.}$$

$\therefore S = \{\frac{2}{5}, \frac{1}{15}\}$

10/5/2022

(Total Expectation Theorm)

*Note:



$$P(B) = P(A_1 \cap B) + P(B \cap A_2) + \cdots P(B \cap A_n)$$
$$= \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B | A_i)\,P(A_i)$$

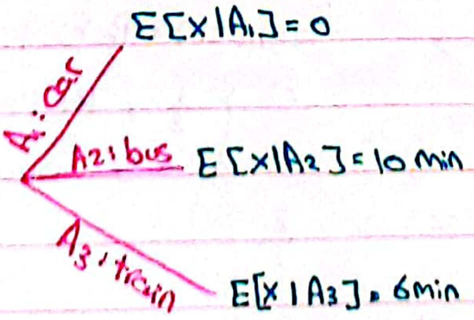
* let X be a R.V, define $B = \{X = K\}$, then $Pr(\{X = K\}) = \sum_{i=1}^{n} Pr\{X = K | A_i\}\,P(A_i) \Rightarrow P_X(K) = \sum_{i=1}^{n} P_{X|A_i}(K)\,P(A_i)$

$$\Rightarrow E[X] = \sum_{K} K\,P_X(K) = \sum_{K}\sum_{i} K\,P_{X|A_i}(K)\,P(A_i) = \sum_{i=1}^{n} \underbrace{\left(\sum_{K} K\,P_{X|A_i}(K)\right)}_{E[X|A_i]} P(A_i)$$

$$\rightarrow E[X] = \sum_{i=1}^{n} E[X | A_i]\,P(A_i)$$

Ex:

X: time delay



$P(A_1) = 0.1$

$P(A_2) = 0.3$

$P(A_3) = 0.6$

$$E[X] = \sum_{i=1}^{n} E[X|A_i]\,P(A_i)$$

الـ Senario ده كان مشروح من قبل.

Mean and Variance of a geometric (P):

* $P_X(K) = Pr\{X=k\} = (1-p)^{k-1} . p \ , \ K=1,2,3,\dots,\infty$

$\therefore X = \#$ of tosses till the first H.

$\rightarrow P(H_1)=P \quad \rightarrow P(T_1H_2) = (1-P).P \quad \rightarrow P(T_1T_1H_3) = (1-P)(1-P)P$

$\therefore E[X]$ and $Var(X)$

* if the first toss is tail, we start from begining :

$X>1 \leftarrow \boxed{X=1}$ T | $(X-1)$: # of tosses remaining till the first H. → يعتبر أيضاً geometric R.V

$\longleftrightarrow$ X

$\therefore P_{\underline{X-1}|\underline{X>1}}(k) = (1-P)^{k-1} p \ , \ k=1,2,----,\infty$ "memorylessness Property"

↳ remaining tosses ↳ First toss is T

→ In general → "$P_{X-n|X>n}$"

* remaining tosses لوحدها geometric ولكن مشروط إن الـ first toss يكون tail

* let X be geometric (P), then $E[X] = \sum_k k P_X(k)$

$\therefore X = (X-1)+1 \rightarrow E[X] = \underbrace{E[X-1]}+1 \quad ---①$

$E[X-1] = \underbrace{E[\cancel{X-1}|X=1]}_{A_1}\overset{0}{} Pr\{X=1\} + \underbrace{E[X-1|X>1]}_{A_2 \,\to\, E[X]} \overbrace{Pr\{X>1\}}^{(1-P)}$ (total theorem)

$= 0 + E[X](1-P) = (1-P)E[X] \quad --②$

$\therefore$ Sub ② in ① $\rightarrow E[X] = 1 + (1-P)E[X] \rightarrow E[X] = \frac{1}{P}$

والـ Sum of probability يكون المفروض $\frac{1}{p}$

* $Var(X) = E[X^2] - (E[X])^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \left(\frac{1-p}{p^2}\right)$ #

↳ * $E[X^2] = E[(1+(X-1))^2] = E[1+2(X-1)+(X-1)^2] =$

$1 + 2E[X-1] + E[(X-1)^2] = 1+2(1-p)\frac{1}{p} + E[X^2].(1-p)$

↳ $E[(X-1)^2] = E[\overset{0}{\cancel{(X-1)^2}}|X=1] + E[(X-1)^2|X>1] Pr\{X>1\}$

$= E[X^2].(1-p)$

→ * $E[X^2] = \frac{2}{p^2} - \frac{1}{p}$

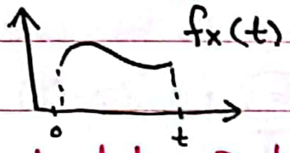
((continuous R.Vs))

Ex: let X be the height of students in a class, $X \in [160 - 190]$ cm,

$P(X = 170.1) = 0 \rightarrow$ $\frac{1}{\infty}$ = infinite many values لإنه بيني

لذلك الحل ناخذ Ranges عشان يطلع معنا ناتج مثل $P(160<X<165)$:

∴Note: $P(160 < X < 190) = 1$



∴ A continuous R.V X can be described by probability density function (PDF) "$f_X(t)$", $P(a \le X \le b) = \int_a^b f_X(t)\,dt$

مع مساواة أو بدون :

$\therefore P(a \le X \le b) = \sum_{k=a}^{b} P_X(k)$ → بالdescrete

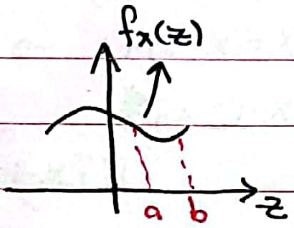
$\therefore P(X \le b) = \int_{-\infty}^{b} f_X(t)\,dt$ / $P(X \ge a) = \int_{a}^{\infty} f_X(t)\,dt = 1 - P(X \le b)$

15/5/2022

Let X be a cont. RV

1) Probability Density function (PDF)

PDF: $f_X(Z)$



a- $f_X(Z) \ge 0$ b- $\int_{-\infty}^{\infty} f_X(z)\,dz = 1$ → $Pr(\infty > X > -\infty)$

c- $\int_a^b f_X(z)\,dz = Pr(b \ge X > a)$

d- Pr(X≤[illegible]) $= \int_{-\infty}^{\alpha} f_X(z)\,dz$ e- $Pr(X > \alpha) = \int_{\alpha}^{\infty} f_X(z)\,dz$

$\downarrow$

$P(X > \alpha) = 1 - P(X \le \alpha)$

$= 1 - \int_{-\infty}^{\alpha} f_X(z)\,dz$

2) a: $E[X] = \int_{-\infty}^{\infty} z\, f_X(z)\,dz$ → بدل الـ Σ في الdiscrete هون نستخدم التكامل

b: $E[g(X)] = \int_{-\infty}^{\infty} g(z)\, f_X(z)\,dz$

c: $Var(X) = E[(X - \mu_X)^2] = E[X^2] - (E[X])^2$

$= \int_{-\infty}^{\infty} z^2 f_X(z)\,dz - \left(\int_{-\infty}^{\infty} z f_X(z)\,dz\right)^2$

3) let $g(x) = ax+b$

a) → $E[ax+b] = a E[x] + b$

b) → $Var(ax+b) = a^2 \, Var(x)$

((Cummolative Distribution function (CDF)

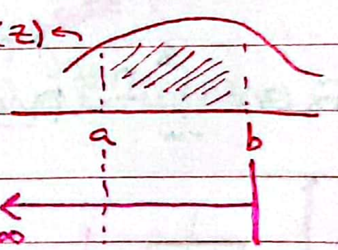
$$CDF: F_x(\gamma) \triangleq Pr(X \leq \gamma) = \int_{-\infty}^{\gamma} f_x(z) dz$$

↲ جطا

$$F_x(\gamma) \triangleq \int_{-\infty}^{\gamma} f_x(z) dz$$

$$\frac{dF_x(\gamma)}{d\gamma} = f_x(\gamma)$$

$$* \frac{d}{dt}\left(\int_{-\infty}^{t} g(z) dz\right) = g(t)$$

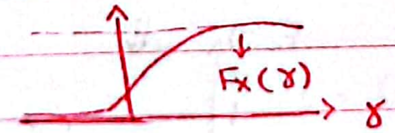


$$\therefore P(b > x > a) = \int_{a}^{b} f_x(z) dz$$

$$= \int_{-\infty}^{b} f_x(z) dz - \int_{-\infty}^{a} f_x(z) dz$$

$$= F_x(b) - F_x(a)$$

$$* F_x(\gamma) = \int_{-\infty}^{\gamma} f_x(z) dz = Pr(X \leq \gamma) \rightarrow$$ a- $F_x(\infty) = 1$ b- $F_x(-\infty) = 0$



← * $F_x(z)$ مستحيل يتناقص

ممكن يثبت بس مايتناقص

(1 اله max الها) monoton non-decreasing

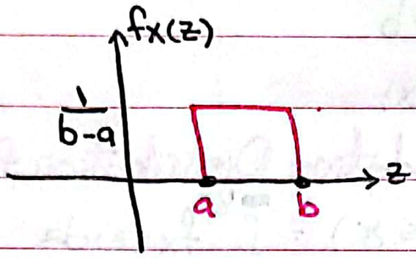
$$* Notes: Pr(X > \gamma) = 1 - Pr(X \leq \gamma) = 1 - F_x(\gamma) \rightarrow CCDF$$

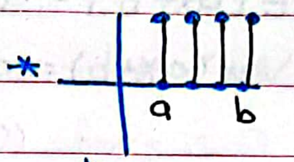
↳ Complementary

17/5/2022

Most common continous R.V. :

1- Uniform RV



* 

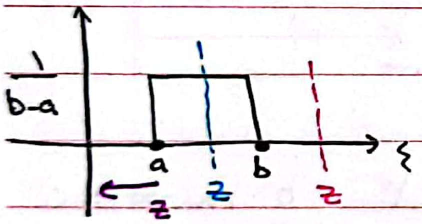
$\sum_{a}^{b} P_X(z) = 1$

$$f_X(z) = \begin{cases} \frac{1}{b-a} & , b > z > a \\ 0 & , O.W \end{cases}$$

CDF :

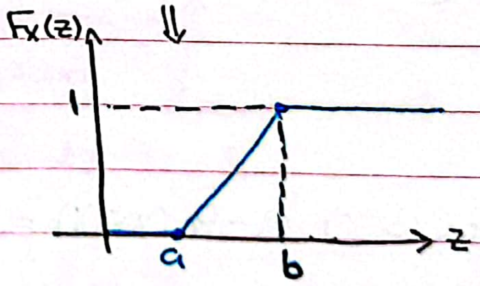
لو بتشرخبط سميها "لا" أو "α" ← (زيتا) zeta

$$F_X(z) = \int_{-\infty}^{z} f_X(\xi)\, d\xi = \begin{cases} 0 & , z < 0 \\ \int_a^z \frac{1}{b-a} & , b > z > a \\ \hookrightarrow = \frac{z-a}{b-a} & \\ 1 & , b < z \end{cases}$$



$\Longrightarrow$

$\Downarrow$



$$* \; E[X] = \int_{-\infty}^{\infty} z f_X(z)\, dz = \int_a^b \frac{z}{b-a} . dz = \frac{1}{b-a} \left(\frac{z^2}{2} \right) \Big|_a^b$$

$$= \frac{1}{2(b-a)} (b^2 - a^2) = \frac{a+b}{2}$$

$$* \; E[x^2] = \int_{-\infty}^{\infty} z^2 f_x(z)\, dz = \int_a^b \frac{z^2}{b-a}\, dz = \frac{z^3}{3(b-a)} \Big|_a^b =$$

$$= \frac{1}{3} \frac{b^3 - a^3}{(b-a)} = \frac{a^2+ab+b^2}{3}$$

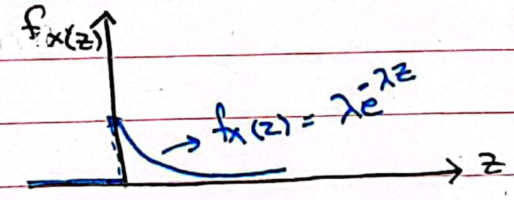
$$\rightarrow Var(x) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12} \quad \#$$

$* \; \overset{P_T}{\bullet} \underset{d}{\rule{3cm}{0.4pt}} \overset{P_r}{\bullet} \quad P_r = P_T \frac{1}{d^2}$ → "الـPower بيقل كل مازادت المسافة"

2- Exponential RV :

rate Parameter

$$f_x(z) = \begin{cases} \lambda e^{-\lambda z} & , z \geq 0 \\ 0 & , o.w \end{cases} = \underline{\lambda} e^{-\lambda z} u(z)$$



$$\therefore F_x(z) = \int_{-\infty}^{z} f_x(\xi)\, d\xi = \int_0^z \lambda e^{-\lambda \xi}\, d\xi = (-1)\, e^{-\lambda \xi} \Big|_0^z = -1\,(e^{-\lambda z} - 1) = 1 - e^{-\lambda z} , z \geq 0$$

$$\rightarrow F_x(z) = (1 - e^{-\lambda z})\, u(z) \quad \#$$

$$* \; F_x(2) = 1 - e^{-2\lambda} = \int_0^2 f_x(\xi)\, d\xi = P(2 > x > 0)$$

$$* \; P(5 > x > 3) = F_x(5) - F_x(3) = (1 - e^{-5\lambda}) - (1 - e^{-3\lambda}) = e^{-3\lambda} - e^{-5\lambda} \quad \#$$

$$* \; E[x] = \int_{-\infty}^{\infty} z f_x(z)\, dz = \int_0^{\infty} z \lambda e^{-\lambda z}\, dz = \lambda \int_0^{\infty} z e^{-\lambda z} dz = \frac{1}{\lambda}$$

$$u \cdot v \Big|_0^{\infty} - \int_0^{\infty} v . du \quad \leftarrow \quad u = z \quad dv = e^{-\lambda z}\, dz$$

$$\frac{du}{dz} = 1 \quad v = \frac{e^{-\lambda z}}{-\lambda}$$

$$= \int_0^{\infty} \frac{e^{-\lambda z}}{\lambda}\, dz = \frac{1}{\lambda^2} \cdot \lambda = \frac{1}{\lambda}$$

* Note : $\frac{d}{d\lambda}\left(\int_0^\infty e^{-\lambda z} . dz = \frac{1}{\lambda}\right) = \lambda \int_0^\infty + z e^{-\lambda z} = +\frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

* $E[X^2] = \int_{-\infty}^{\infty} z^2 f_X(z) = \int_0^\infty (\lambda e^{-\lambda z}) z^2 \cdot dz = \frac{2}{\lambda^2}$

استخدمي نفس ال Note بس تشتقي مرتين

* $Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

* Note : Mean $= \frac{1}{rate}$ → مهمة بتيجي بالامتحان → with rate parameter.
→ with mean parameter.

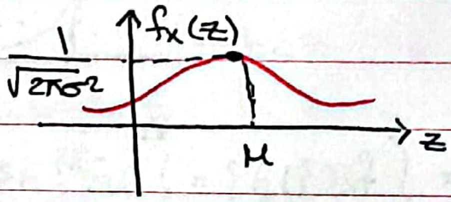
22/5/2022

حفظ

Normal (Gaussian) R.V $X \sim N(\mu, \sigma^2)$ if

pdf

$f_X(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$, $\infty > z > -\infty$



μ: Location Parameter.
σ^2: Scale Parameter.

Symmetric (around μ)
"Table فيها اله"

* Note: $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$

$\Phi(0.1) = \ldots$

$\Phi(0.2) = \ldots$

⋮

normal

∴ CDF of $N(\mu, \sigma^2)$ → Parameters.

$F_X(z) = \int_{-\infty}^{z} f_X(\xi) d\xi$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi \quad , \quad \text{let } y = \frac{\xi-\mu}{\sigma}$$

بالامتحان مطالب تكتبيه لهون.

$$z-\mu \leftarrow \frac{z-\mu}{\sigma}$$

$$= \int_{-\infty}^{\frac{z-\mu}{\sigma}} \frac{1}{\sqrt{2\pi\cancel{\sigma^2}}} e^{-y^2/2} \cancel{\sigma}\, dy = \int_{-\infty}^{\frac{z-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

← بنحسب بالجداول

Special case : if $\mu=0, \sigma^2=1 \rightarrow X \sim N(0,1) \rightarrow$ "Standard Normal"

1- $\int_{-\infty}^{\infty} f_X(z)\, dz = 1$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

زي ما عملناه فوق.

let : $y = \frac{z-\mu}{\sigma}$

Proof :

$$* \text{ let } I = \sqrt{I.I}$$

$$= \sqrt{\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\right)}$$

$$\rightarrow I = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} dx\,dy \left(\frac{1}{2\pi}\right)} = \sqrt{\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{e^{-r^2/2}}{2\pi} r\,dr\,d\theta}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi \cdot \left(e^{-\frac{r^2}{2}} \Big|_{\infty}^{0}\right)} = 1$$

"Polar"

2- $E[X] = \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz = \int_{-\infty}^{\infty} \frac{(\sigma y+\mu)}{\sqrt{2\pi}} e^{-y^2/2} dy$

$y = \frac{z-\mu}{\sigma^2}$

$$= \sigma \boxed{\int_{-\infty}^{\infty} \frac{y\, e^{-y^2/2}}{\sqrt{2\pi}}} dy + \mu \boxed{\int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}} dy = \mu$$

0 = odd funct لأنه تكامل ال 0 =

= 1

$y \rightarrow$ odd $\quad e^{-y^2/2} \rightarrow$ even $\rightarrow$ odd . even = odd additional step ←

3- $Var(X) = \int_{-\infty}^{\infty} (z-\mu)^2 f_x(z)\, dz = \int_{-\infty}^{\infty} \frac{(z-\mu)^2/\sigma^2}{\sqrt{2\pi\sigma^2}/\sigma^2} e^{\frac{-(z-\mu)^2}{2\sigma^2}} dz$

$= \sigma^2 \int_{-\infty}^{\infty} y^2 \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$

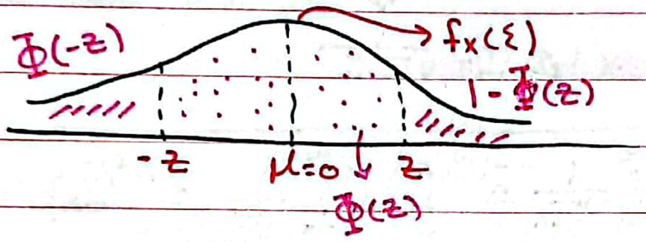
↳ Integration by Parts → $u = y \quad dv = y/e^{-y^2/2}$

$du = 1 \quad v = -e^{-y^2/2}$

⋮

∴ let $X \sim N(0,1)$

$F_X(z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} d\xi$



$\therefore \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} d\xi$

$\therefore \Phi(-z) = 1 - \Phi(z)$

$\therefore \Phi(z) = 1 - \Phi(-z)$

$\therefore \Phi(0) = \frac{1}{2}$

$\therefore \Phi(-1) = 1 - \Phi(1)$

Ex: The annual Snowfall at a Particular area is modeled as $X \sim (\overset{\mu}{60}, \overset{\sigma^2}{400})$, 1] Find the probability that Snowfall is less than 80cm

Sol. Note: $\Pr\{X \le \alpha\} = \Phi\left(\frac{\alpha-\mu}{\sigma}\right) \quad \therefore \Pr\{x \le 6\} = \Phi\frac{(80-60)}{20} = \Phi(1) = 0.8413$

2] Snowfall is at least 90 cm.

$\Pr\{x > 90\} = 1 - \Pr\{x \le 90\} = 1 - \Phi\frac{(90-60)}{20} = 1 - \Phi(\frac{3}{2}) = \Phi(\frac{-3}{2}) = 0.0068$

3] $\Pr\{80 < x \le 90\}$.

$= F_x(90) - F_x(80) = \Phi\frac{(90-60)}{20} - \Phi\frac{(80-60)}{20} = \Phi(\frac{3}{2}) - \Phi(1)$ #

24/5/2022

Conditional PDF / CDF

let X be a continious R.V and let A be an event $f_{X|A}(z)$: The conditional PDF.

$\therefore F_{X|A}(z)$: The conditional CDF.

$f_{X|A}(z) \geq 0$

$$F_{X|A}(z) = \int_{-\infty}^{z} f_{X|A}(\xi)\, d\xi$$

$$\int_{-\infty}^{\infty} f_{X|A}(z) dz = 1$$

$$* \; E[X|A] = \int_{-\infty}^{\infty} z . f_{X|A}(z) dz \quad \text{(conditional mean)}.$$

$$* \; E[g(x)|A] = \int_{-\infty}^{\infty} g(z)\, f_{X|A}(z) dz$$

$\therefore$ Total expectation Theorem:

let $A_1, A_2, \ldots\ldots A_n$ be disjion event (form a partition of Ω)

$$\therefore E[X] = \sum_{i=1}^{n} E[X|A_i] P(A_i)$$

$$\therefore E[X|A_i] = \int_{-\infty}^{\infty} z\, f_{X|A_i}(z) dz$$

$$\therefore E[g(x)] = \sum_{i=1}^{n} E[g(x)|A_i] P(A_i)$$

$$= \sum_{i=1}^{n} \left(\int_{-\infty}^{\infty} g(z) f_{X|A_i}(z) dz \right) P(A_i)$$

Ex: $f_X(z) = \begin{cases} \frac{1}{3} & 1 > z > 0 \\ \frac{2}{3} & 2 > z > 1 \end{cases}$, let $A_1 = \{1 > x > 0\}$, $A_2 = \{2 > x > 1\}$, Find $f_{X|A_1}(z)$:

Sol. $f_{X|A_1}(z) = \frac{d}{dz}\left(F_{X|A}(z) \right) \rightarrow F_{X|A_1}(z) = \frac{P_r\{X \leq z \cap A_1\}}{P(A_1)}$

$\Downarrow$

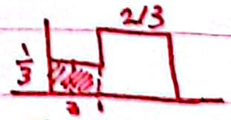
*Note:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Final ans:

$$f_{X|A}(z) = \frac{d}{dz} \frac{F_{X|A_1}(z)}{P(A_1)} = \frac{d}{dz} \frac{F_X(z)}{P(A_1)} = \frac{f_X(z)}{P(A_1)} = \begin{cases} 1, & 1 > z > 0 \\ 0, & o.w \end{cases}$$

$$= \frac{P_r\{X \leq z \cap 1 > X > 0\}}{P(1 > X > 0)}$$



$$= \frac{P_r\{X \leq z\}}{P(1 > x > 0)} = \frac{F_X(z)}{P(1 > x > 0} , 1 > z > 0$$

$$\int_0^1 \frac{1}{3} dz = \frac{1}{3}$$

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$$\therefore f_{X|A_2}(z) = \frac{f_X(z)}{P(A_2)} = \frac{3}{2} f_X(z) = \begin{cases} \frac{3}{2} * \frac{2}{3} = 1 , & 2 > z > 1 \\ 0 , & o.w \end{cases}$$

الحل الـ general ومعروف هيك بنحل
انه يكون كامل = 1

Transformation of R.Vs (Derivation)

let X be a Continous RV, with PDF and CDF $f_X(z)$, $F_X(z)$, respectively.

let $Y = g(X)$: $g(X)$ is a function of X, we are interested in $f_Y(y)$ and $F_Y(y)$

* $E_0 \underset{d}{\text{———}} E_1 \rightarrow E_1 = E_0 \cdot \frac{1}{d^2}$ → Random Variable
ممكن يجي بالامتحان

* دايما بنبلش من الـ Probability من الـ CDF.

obtaining $F_Y(y)$ =

$$F_Y(y) \triangleq Pr\{Y \le y\} , Y = g(X) \triangleq P_r\{g(X) \le y\}$$

$$\triangleq Pr\{X \le g^{-1}(y)\} = F_X(g^{-1}(y))$$

Ex1 : let X be $U \sim [0,1]$ → uniform من 0 لـ 1.

Consider $Y = 3X - 2$, Find $F_Y(y)$: → بنحلها بعين

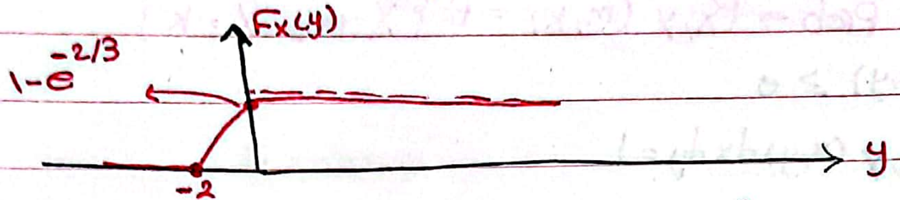
$$F_X(z) = \begin{cases} 0 , & z < 0 \\ z , & 1 > z > 0 \\ 1 , & z \ge 1 \end{cases}$$

Ex1 : let X be exp(1), consider $Y = 3X-2$, Find $F_Y(y)$

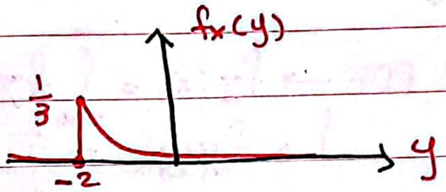
, $F_X(z) = (1 - e^{-z})\,u(z)$.

$F_Y(y) = \Pr\{Y \le y\} = \Pr\{3X-2 \le y\} = \Pr\{X \le \frac{y+2}{3}\} = F_X\left(\frac{y+2}{3}\right)$

$\rightarrow F_Y(y) = \left(1 - e^{-\left(\frac{y+2}{3}\right)}\right) u\left(\frac{y+2}{3}\right)$



$f_Y(y) = \frac{1}{3} e^{-\left(\frac{y+2}{3}\right)}$, $y \ge -2$



EX: X is a cont. RV, $F_X(z)$ is given, Find $f_X(y)$ $F_X(y)$, $Y = ax+b$

Sol. $F_X(y) \triangleq \Pr\{Y \le y\} = \Pr\{ax+b \le y\} = \Pr\{x \le \frac{y-b}{a}\} = F_X\left(\frac{y-b}{a}\right)$

$\rightarrow f_Y(y) = \frac{d}{dy}\left[F_X\left(\frac{y-b}{a}\right)\right] = f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a}$ $(a>0)$

Ex : let x be uniform on $[0,1]$ and let $Y = \sqrt{X}$. we note that for every $y \in [0,1]$ we have: $F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = y^2$

/ $f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{d(y^2)}{dy} = 2y$, $0 \le y \le 1$

أمثلة (نهاية دفتر كوش)

29/5/2022

Joint Distribution

let X and Y be jointly continious R.Vs, then the joint Probability Density function is given by $f_{X,Y}(\underbrace{x,y}_{\hookrightarrow \text{ x,y small}})$

* joint PDF * ↵

* Note: joint Prob → $P_{X,Y}(m,k) = P_r\{X=m, Y=k\}$

1- $f_{X,Y}(x,y) \geq 0$

2- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

* Marginal PDF → $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

↳ $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

* joint CDF *

$$F_{X,Y}(x,y) = P_r\{X \leq x, Y \leq y\} = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(t,s)\,dt\,ds$$

* Marginal CDF → $F_X(x) = \int_{-\infty}^{x} \overbrace{\boxed{\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,dt}}^{\to f_X(s)} ds$

→ $F_Y(y) = \int_{-\infty}^{y} \overbrace{\boxed{\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,ds}}^{\to f_Y(t)} dt$

Notes: 1- $F_{X,Y}(-\infty,\infty) = 0$ 2- $F_{X,Y}(-\infty,y) = 0$

3- $F_{X,Y}(x,-\infty) = 0$ 4- $F_{X,Y}(\infty,\infty) = 1$

5- $F_X(x) = F_{X,Y}(x,\infty)$ → if I have CDF for X, Y

6- $F_Y(y) = F_{X,Y}(\infty,y)$ → if I have CDF for X, Y

7- $P_r\{x_2 > X > x_1, y_2 > Y > y_1\} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dx\,dy$

كشوف كيف وصلنا للعوض

$$= \boxed{F_{X,Y}(x_1,y_1) + F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)}$$

$\int_{-\infty}^{x_2}\int_{y_1}^{y_2} f_{x,y} \ldots$

↳ $\left[\int_{-\infty}^{y_2} - \int_{-\infty}^{y_1}\right]$ ... وهكذا الباقي

2/6/2022

Ex: if X and Y are jointly Rvs, then PDF:

$f_{X,Y}(x,y)$ (joint PDF)

1- $f_{X,Y}(x,y) \geq 0$

2- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

- marginal PDF of X and Y

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

- Joint CDF of X and Y

$$F_{X,Y}(x,y) = P_r\{-\infty < X \leq x \;,\; -\infty < Y \leq y\}$$
$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(t,s)\,ds\,dt$$

* $P_r\{x_2 > X > x_1 \;,\; y_2 \geq Y \geq y_1\} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dx\,dy$

$= F_{X,Y}(x_1,y_1) + F_{X,Y}(x_2,y_2)$

$- F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)$

* $P_r\{x_2 \geq X \geq x_1\} = P_r\{x_2 \geq X \geq x_1 \;,\; \infty > Y > -\infty\}$

$= F_{X,Y}\overset{0}{\cancel{(x_1}}, -\infty) + F_{X,Y}(x_2, \infty) \rightarrow F_X(x_2)$

$- F_{X,Y}(x_1, \infty) - F_{X,Y}\underset{0}{\cancel{(x_2}}, -\infty)$

$F_X(x_1)$

1- $F_{X,Y}(-\infty, \infty) = 0$

2- $F_{X,Y}(x, \infty) = F_X(x)$

3- $F_{X,Y}(\infty, y) = F_Y(y)$

4- $F_{X,Y}(\infty,\infty) = 1$

5- $F_{X,Y}(-\infty, y) = 0$

6- $F_{X,Y}(x, -\infty) = 0$

7- $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$

اشتقاق

8- $P_r\{X > x, Y > y\} = 1 + F_{X,Y}(x,y) - F_X(x) - F_Y(y)$

$\infty > X > x \;,\; \infty > Y > y$

Ex : $f_{X,Y}(x,y) = x e^{-x(y+1)} \underbrace{u(x)}_{x \ge 0} \underbrace{u(y)}_{y \ge 0}$ ∴ $u()$: unit step function

$$- F_{X,Y}(x,y) = \int_{t=0}^{x} \int_{0=s}^{y} f_{X,Y}(t,s)\, ds\, dt$$

$$= \int_{t=0}^{x} \underbrace{\int_{s=0}^{y} t e^{-t(s+1)}\, ds}_{I}\, dt$$

$$I = \int_{s=0}^{y} t e^{-t(s+1)}\, ds = t e^{-t} \int_{s=0}^{y} e^{-ts}\, ds$$

$$= \cancel{t} e^{-t} \left[\frac{e^{-ts}}{\cancel{t}} \right] \Big|_{0}^{y} = - e^{-t} [e^{-ty} - 1]$$

$$= e^{-t} - e^{-t(y+1)} \quad \#$$

$$F_{X,Y}(x,y) = \int_{t=0}^{x} I\, dt = \int_{t=0}^{x} (e^{-t} - e^{-t(y+1)})\, dt = -e^{-t} + \frac{e^{-t(y+1)}}{y+1} \Big|_{t=0}^{x}$$

$$= -e^{-x} + \frac{e^{-x(y+1)}}{y+1} - (-1 + \frac{1}{y+1}) \Rightarrow F_{X,Y}(x,y) = \frac{e^{-x(y+1)} - 1}{y+1} + 1 - e^{-x}, \quad \begin{matrix} x \ge 0 \\ y \ge 0 \end{matrix}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{0}^{\infty} x e^{-x(y+1)}\, dy = x e^{-x} \int_{0}^{\infty} e^{-xy}\, dy =$$

$$\cancel{x} e^{-x} \frac{e^{-xy}}{\cancel{x}} \Big|_{y=\infty}^{0} = e^{-x}(1-0) = e^{-x}$$

∴ $f_X(x) = e^{-x} u(x)$

or

$$F_{X,Y}(x,y) = \frac{e^{-x(y+1)} - 1}{y+1} + 1 - e^{-x}, \quad \begin{matrix} x \ge 0 \\ y \ge 0 \end{matrix}$$

* $F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = 1 - e^{-x}$, $x \ge 0$

$f_X(x) = e^{x}$, $x \ge 0$

* $F_Y(y) = \lim_{x \to \infty} \frac{\overset{0}{\boxed{e^{-x(y+1)}}} - 1}{y+1} + 1 - \overset{0}{\boxed{e^{-x}}} = 1 - \frac{1}{y+1} = \frac{y}{y+1}$, $y \ge 0$

* $f_Y(y) = \frac{d}{dy}\left[1 - \frac{1}{y+1}\right] = 0 + \frac{1}{(y+1)^2}, \ y > 0$

$$= \frac{1}{(y+1)^2} u(y) \ \#$$

or

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \ \#$$

Expectation:

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

if $g(x,y) = aX + by + c$

$$E[g(x,y)] = E[aX+by+c] = aE[X] + bE[Y] + C.$$

7/6/2022

R.V Transformation.

$F_X(z)$ is given , $Y = g(X)$

$F_Y(y) = Pr\{Y \le y\} = Pr\{g(x) \le y\} \triangleq Pr\{X \le h(y)\},$

$h(y) = g^{-1}(y) = F_X(h(y))$

$\rightarrow$ Let $Y = aX + b$

$F_Y(y) = Pr\{Y \le y\} = Pr\{aX + b \le y\} = Pr\{aX \le y - b\} = \begin{cases} Pr\{X \le \frac{y-b}{a}\} = F_X(\frac{y-b}{a}), a>0 \\ Pr\{X > \frac{y-b}{a}\} = 1 - F_X(\frac{y-b}{a}), \ a<0 \end{cases}$ $\leftarrow$

$$f_Y(y) = \begin{cases} \frac{d}{dy} F_X\left(\frac{y-b}{a}\right), \ a > 0 \\ \frac{d}{dy}\left(1 - F_X\left(\frac{y-b}{a}\right)\right), \ a < 0 \end{cases}$$

$$= \begin{cases} f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}, \ a > 0 \\ + f_X\left(\frac{y-b}{a}\right) \cdot \frac{-1}{a}, \ a<0 \end{cases} = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$$

Ex : Let $X \sim \exp(3)$, $F_X(z) = 1 - e^{-3z}$, $z \geq 0$

Find $F_Y(y)$, $Y = 2X+1$

$F_Y(y) = F_X\left(\frac{y-1}{2}\right) = 1 - e^{\frac{-3(y-1)}{2}}$, $y \geq 1$

$f_X(z) = 3e^{-3z}$, $z > 0$

$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|} = f_X\left(\frac{y-1}{2}\right) \cdot \frac{1}{2}$

$= 3e^{-3\left(\frac{y-1}{2}\right)} \cdot \frac{1}{2}$, $y \geq 1$

Expectation :

$g(X,Y)$ is a function of the RVs X and Y

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

$*$ if $g(X,Y) = aX + bY + C$

$E[g(X,Y)] = a E[X] + b E[Y] + C$

Conditioning one RV on another

Let X and Y be jointly distributed RVs with joint PDF $f_{X,Y}(x,y)$, then

$$f_{X|\underset{\downarrow\, =y}{Y}}(x|y) = \frac{f_{X,Y}(x,y) \rightarrow \text{joint}}{f_Y(y)} , \quad \begin{cases} f_{X|Y} > 0 \\ \int_{-\infty}^{\infty} f_{X|Y}(x|y)\, dx = 1 \end{cases}$$

Ex : $f_{X,Y}(x,y) = xe^{-x(y+1)} u(x)\, u(y) \rightarrow f_X(x) = \int_{-\infty}^{\infty} x e^{-x(y+1)} dy$

$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)} \cancel{u(x)}\, u(y)}{e^{-x} \cancel{u(x)}}$

$= xe^{-xy} u(y)$

$\cancel{x}e^{-x} \int_0^{\infty} \underset{\underset{\frac{1}{\cancel{x}}}{\downarrow}}{e^{-xy}}\, dy$

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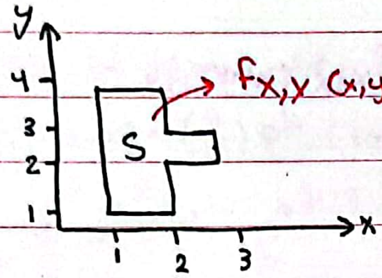
$*f_{y|x=3}(y|3) = 3e^{-3y}u(y)$ $\quad *f_{y|x=0.1}(y|0.1) = 0.1e^{-0.1y}u(y)$

* بيعطيك joint PDF ويطلبك ال CDF, transformation (أسئلة فاينال)

فاينال ↑

Ex:



$\rightarrow f_{x,y}(x,y)$ X and y are jointly uniform.

$$1- f_{x,y}(x,y) = \begin{cases} \frac{1}{\text{Area of S}}, & (x,y) \in S \\ 0, & o.w \end{cases}$$

$$= \begin{cases} \frac{1}{4}, & (x,y) \in S \\ 0, & o.w \end{cases} \leftarrow$$

$$2- f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy = \begin{cases} \int_1^4 \frac{1}{4}dy = \frac{3}{4}, & 2>x>1 \\ \int_2^3 \frac{1}{4}dy = \frac{1}{4}, & 3>x>2 \end{cases}$$

$$* E[X] = \int_{-\infty}^{\infty} x f_x(x)\,dx = \int_1^2 x\cdot\frac{3}{4}dx + \int_2^3 x\cdot\frac{1}{4}dx$$

$$3- f_y(y) = \begin{cases} \int_1^2 \frac{1}{4}dx = \frac{1}{4}, & 2>y>1 \\ \int_1^3 \frac{1}{4}dx = \frac{2}{4}, & 3>y>2 \\ \int_1^2 \frac{1}{4}dx = \frac{1}{4}, & 4>y>3 \end{cases}$$

$$* E[Y] = \int_1^2 y\cdot\frac{1}{4}dy + \int_2^3 y\cdot\frac{2}{4}dy + \int_3^4 y\cdot\frac{1}{4}dy$$

$$4- f_{x|y}(x|y=1.1) = \left.\frac{f_{x,y}(x,y)}{f_y(y)}\right|_{y=1.1} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1, \quad 2>x>1$$

$$5- f_{x|y}(x|2.5) = \left.\frac{f_{x,y}(x,y)}{f_y(y)}\right|_{y=2.5} = \frac{\frac{1}{4}}{\frac{2}{4}}, \quad 3>x>1$$

Conditional Expectation

1- $E[X|y] = E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y=y}(x|y)dx$

2- $E[g(x)|y] = E[g(x)|Y=y] = \int_{-\infty} g(x) f_{X|Y}(x|y)dx$

3- $E[X] = \int_{-\infty}^{\infty} \underbrace{E[X|Y=y]}_{\downarrow} f_X(y)dy$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx\, f_Y(y)dy = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} \underbrace{f_{X|Y}(x|y) f_Y(y)}_{\overset{\downarrow}{f_{X,Y}(x|y)}} dy dx = E[X]$$

Independence

Two continuous RVs are said to be independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \underline{or}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = f_X(x) \quad \underline{\underline{or}}$$

$$F_{X,Y}(x,y) = \Pr\{\underbrace{X \le x}_{A}\overset{}{\underset{\overset{\downarrow}{\cap}}{,}}\underbrace{Y \le y}_{B}\} = \underbrace{\Pr\{X \le x\}}_{F_X(x)} . \underbrace{\Pr\{Y \le y\}}_{F_Y(y)}$$

Consequences of Independent

1- $E[XY] = E[X]E[Y] \rightarrow$ شرط يكونوا Independent فقط

$$E[X.Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_X(x)\, y f_Y(y)\, dxdy$$

$$= \left(\int_{\infty}^{\infty} x f_X(x)dx\right)\left(\int_{-\infty}^{\infty} y f_Y(y)dy\right) = E[X]E[Y].$$

2- $E[g(x)h(y)] = E[g(x)] . E[h(y)]$

لازم يحكيلك, انهم Independent عشان تطبق هالقوانين

3 - $Var(X+Y) = Var(X) + Var(Y)$

$$Var(X_1+X_2+X_3+ \cdots X_n) = \sum_{i=1}^{n} Var(X_i)$$

If $x_1, x_2, \ldots x_n$ are independent $\rightarrow f_{x_1,x_2,x_3,\ldots x_n}(x_1,x_2,x_3,\ldots x_n) = \prod_{i=1}^{n} f_{x_i}(x_i)$

Ex: Var of binomial (n,k).

$* \; P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $P(H) = P$

$X = Y_1 + Y_2 + \cdots Y_n$ (each Y_i: H/T)

$\downarrow$ Bernoulli (P)

$\hookrightarrow$ Binomial $= \sum$ independent Bernoulli (P)

$$Var(X) = \sum_{i=1}^{n} \underbrace{Var(Y_i)}_{\hookrightarrow p.(1-p)} = n.(1-p).p \;\#$$

9/6/2022

Moment generating function (MGF)

The MGF associated with the RV X is given by:

$$M_X(s) = \begin{cases} \int_{-\infty}^{\infty} e^{sz} \underbrace{f_X(z)}_{Pdf} dz, & X \text{ is cont} \\ \sum_{k} e^{sk} \underbrace{P_X(k)}_{Pmf}, & X \text{ is discrete} \end{cases}$$

$\therefore M_X(s) = E[\underbrace{e^{sX}}_{g(x)}]$

Ex: $P_X(k) = \begin{cases} 1/2, & k=2 \\ 1/6, & k=3 \\ 1/3, & k=5 \end{cases}$, Find $M_X(s)$?

Sol. discrete $\rightarrow M_X(s) = \sum_k e^{sk} P_X(k) = e^{2s} P_X(2) + e^{3s} P_X(3) + e^{5s} P_X(5)$

$$= \frac{1}{2} e^{2s} + \frac{1}{6} e^{3s} + \frac{1}{3} e^{5s} \;\#$$

Ex: $X \sim \text{Poisson}(\lambda) \rightarrow P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, \dots \infty$

$$\therefore M_X(s) = \sum_{k=0}^{\infty} e^{sk} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{(\overbrace{e^s . \lambda}^{a})^k}{k!} = e^{-\lambda} . (\overbrace{\sum_{k=0}^{\infty} \frac{a^k}{k!}}^{e^a})$$

$$= e^{-\lambda} e^{e^s . \lambda} = e^{-\lambda + e^s \lambda} = e^{\lambda(e^s - 1)}$$

$$\therefore M_X(S) = e^{\lambda(e^s-1)} \quad \#$$

$$\frac{d}{ds}[M_X(s)]\Big|_{s=0} = \int_{-\infty}^{\infty} z \overset{1}{\cancel{e^{sz}}} f_X(z) dz = E[X]$$

*** Properties of MGF:**

$* \ M_X(s) = \int_{-\infty}^{\infty} e^{sz} f_X(z) \, dz$ (cont): ↑ Proof

① $M_X(0) = \int_{-\infty}^{\infty} f_X(z) dz = 1$ ② $\frac{d}{ds} M_X(s)\Big|_{s=0} = E[X]$

$$\frac{d^n}{ds^n}[M_X(s)]\Big|_{s=0} = E[X^n]$$ ↵ (nth moment)

③ if $Y = aX + b$, $M_Y(s) = e^{sb} M_X(as)$

Proof: $M_Y(s) = E[e^{sY}] = E[e^{s(aX+b)}] = E[e^{\overbrace{sa}^{\omega}X}] e^{sb}$

$= M_X(\omega) e^{sb} = M_X(sa) e^{sb}$

④ if X and Y are independent RVs. then . $M_{X+Y}(s) = M_X(s) M_Y(s)$

Proof: let $Z = X + Y \rightarrow M_Z(s) = E[e^{sZ}] = E[e^{s(X+Y)}]$

$= E[\underbrace{e^{sX}}_{g(x)} \cdot \underbrace{e^{sY}}_{h(y)}] = E[e^{sX}] E[e^{sY}] = M_X(s) M_Y(s)$ #

Ex: Let X and Y be independent Poisson RVs, with means λ and μ, respectively. Define $Z = X + Y$

Sol. $M_Z(s) = E[e^{sZ}] = E[e^{sX} . e^{sY}] = E[e^{sX}] . E[e^{sY}]$

$= M_X(s) M_Y(s) = (e^{\lambda(e^s-1)}) . (e^{\mu(e^s-1)})$

$= e^{\lambda(e^s-1) + \mu(e^s-1)} = e^{\underbrace{(\lambda+\mu)}_{\omega}(e^s-1)}$

$\rightarrow Z \sim$ Piosson $(\omega = \lambda + \mu)$ #