Chapter 2

[Determinants]

POWERUNIT

→ Covered Topics :

- Determinants by Cofactor Expansion
- Evaluating Determinants by Row Reduction
- Properties of Determinants; Cramer's Rule

- ✓ Determinants helps to determine whether the matrix is invertible or not.
- ✓ We have much more than one approach to find the determinant, we'll learn each one , and then we'll make a chart to view them all next to each other.

• Finding Determinant by Cofactor Expansion :

To extend the determinant for all sizes use the expansion.

!!ATTENTION : it's often better and easier to you to use this approach (the expansion) to find the determinant for matrices of big sizes especially for those whose sizes are wider than 2×2 or 3×3, it's so easy , and gives the determinant directly.

- ✓ **Pay attention to symbols :**

  Determinant of matrix A = det(A) = |A|

→And when I give you matrix contained in these two orthogonal lines , you should know that I am solving the determinant or telling you to do that , like this one in here :

$\begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix}$ = 14 , and this is the same as det(A) = |A| = 14

For the matrix A = $\begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}$

And now , yeah! Let's go to find the determinant !

❖ The Expansion Approach:

➢ Find the determinant for the following matrix :

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \\ 4 & 2 & -1 \end{bmatrix}$$

$$|A| = +1\begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} -3\begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix} +2\begin{vmatrix} -1 & 4 \\ 4 & 2 \end{vmatrix}$$

كل واحد من هدول الاقواس اللي تحت عبارة عن
إيجاد لل determinant الخاص بكل مصفوفة
صغيرة هون طلعت بعد ال expansion

$$= +1(4\times-1 + -2\times2) -3(-1\times-1 + -4\times2) +2(-1\times4 + -4\times4) = -23$$

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!! But the question is how we computed the determinant of 2×2 matrices ??

ملاحظات :

(1) اصغر حجم ممكن نوصله و احنا بنعمل expansion هو ال 2×2

(2) احنا بنختار صف واحد او عمود واحد بنعمل عليه expansion و احنا بالحل هون اخترنا الصف الأول

(3) ممكن نختار صفوف او أعمدة تسهل علينا الحل اكثر لانها بتحتوي اصفار زي ما راح نشوف بالامثلة قدام

(4) طيب كيف بنوجد ال determinant للمصفوفة اللي حجمها 2×2 ؟

- ذاكرين هاد ؟

Inverse of 2×2 matrix is computed using this :

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ → Then the (ad – bc) is the determinant of the matrix A

!!ATTENTION : we can solve it using the second row (you can take any row or column you want)

→Solving it by the second row :

$$|A| = -3\begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix} +4\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} -2\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$$

→ **What about higher sizes ?!**

➢ Find the determinant of the following matrix :

$$A = \begin{bmatrix} 1 & -2 & -1 & 4 \\ 3 & -2 & 1 & 1 \\ 6 & -1 & 7 & 1 \\ 3 & 1 & 2 & -5 \end{bmatrix}$$

$$|A| = +1\begin{vmatrix} 2 & 1 & 1 \\ -1 & 7 & 1 \\ 1 & 2 & -5 \end{vmatrix} +2\begin{vmatrix} 3 & 1 & 1 \\ 6 & 7 & 1 \\ 3 & 2 & -5 \end{vmatrix} +$$

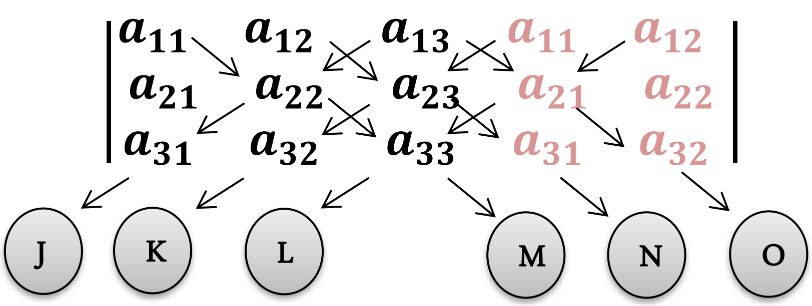
$$-1\begin{vmatrix} 3 & -2 & 1 \\ 6 & -1 & 1 \\ 3 & 1 & -5 \end{vmatrix} -4\begin{vmatrix} 3 & -2 & 1 \\ 6 & -1 & 7 \\ 3 & 1 & 2 \end{vmatrix}$$

.... Then we complete the solution for each side as we did for the 3×3 matrix in the previous example.

❖ Special Way of Finding the Determinant for the 3×3 Matrix :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ can be solved as follows:

→Re-order the matrix as follows:



Such that :

→ $J = a_{13} \times a_{22} \times a_{31}$

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→ $K = a_{11} \times a_{23} \times a_{32}$, and so on .....

✓ Then det(A) = - (J + K + L) + (M + N + O)

○ Example :

(1) Find the determinant of the following matrix : $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \\ 4 & 2 & -1 \end{bmatrix}$

Use the special approach of the 3×3 matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \\ 4 & 2 & -1 \end{bmatrix} \rightarrow \begin{vmatrix} 1 & 3 & 2 & 1 & 3 \\ -1 & 4 & 2 & -1 & 4 \\ -1 & 4 & -1 & -1 & 4 \end{vmatrix} \rightarrow$$

The det of the matrix = - (32 + 4 + 3) + (-4 + 24 + -4) = -23

(2) If A = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 9 & 2 & 0 \\ -2 & 1 & -2 & -1 \end{bmatrix}$, then find det(A)

→ It is bigger than 2×2 and 3×3 , you are now supposed to use the expansion, but you can make a smart choice by choosing the first row for the expansion , because it is almost full of zeros , and this is going to help :

det(A) = 1 $\begin{vmatrix} 4 & 0 & 0 \\ 9 & 2 & 0 \\ 1 & -2 & -1 \end{vmatrix}$ = -8

You can use the special way of finding the (det) for 3×3 matrix in this level ...

(3) If A = $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$, then find det(A)

Use this column for the expansion it is almost full of zeros

det(A) = 1 $\begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ ……… continue as you learned **, this part is the real piece of cake of this material** ":)

❖ Minors and Cofactors :

(1) The Minor or M_{ij} :

The minor matrix of the entry a_{ij} is the determinant of the submatrix that remains after deleting the ith row and jth column.

(2) The Cofactor or C_{ij} :

the cofactor of the entry $a_{ij} = (-1)^{i+j} \times M_{ij}$

- ال minor و ال cofactor لازم يتساووا بالعدد ، و لكن يختلفوا بالإشارة، واحد بطلع سالب و واحد بطلع موجب.

○ **Example:**

A matrix called B = $\begin{bmatrix} 1 & 4 & 2 & 5 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, find M_{11} , C_{11}, M_{21} , C_{21} ....

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هون استخدمنا العمود الأول لل expansion

عشان كله اصفار و راح يسهل علي الحل

M_{11} : $\begin{vmatrix} -1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = 5$

$C_{11} : (-1)^2 M_{11} = 5$

M_{21} : $\begin{vmatrix} 4 & 2 & 5 \\ 0 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4 \begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = -20$

$C_{21} : (-1)^3 M_{21} = 20$

نصيحة أخيرة قبل ما نبلش بالسكشن الجديد!! باغلب الأوقات استخدموا ال expansion عشان تحلوا ال det ما لم يلطب عكس ذلك، لقدام لما نبلش نحل على طرق جديدة.

• Evaluating Determinants by Row Reduction:

✓ **Elementary operations are used here.**

– في هذا السكشن راح نتعلم كيف نحسب ال det عن طريق تحويل الماتركس لِ echelon form ، و هاد الاشي بياخد عدد حسابات اقل من ال expansion ، و طريقة الحل هاي هي خيار ممتاز للمصفوفات ذات الحجم الكبير.

❖ Rule 1:

A- Let A be a square matrix. If A has a row of zeros or a column of zeros, then det(A) = 0.

B- If A is a square matrix and have two proportional rows or columns , then det(A) = 0.

كلمة proportional يعني اذا جمعتهم مع بعض بيعطيني 0

○ **Example:**

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Find the determinant of the next matrix : $\begin{bmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{bmatrix}$

Solution :

Before evaluating the determinant, we notice that if we implement this : ($-2R_1 + R_2 \rightarrow R_2$)

That R_2 is going to be full of zeros in this form : $\begin{bmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{bmatrix}$

– So referring to **rule 1 / A** , we'll find that determinant = 0.

❖ Rule 2:

- Let A be a square matrix. Then $\det(A) = \det(A^T)$./ transpose of A , revise chapter 1 if you don't remember.

❖ Rule 3:

- In this rule I will show you the effect of row operations that are done on some matrix on its determinant.

Rules
If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then det(B) = k det(A)
If B is the matrix that results when two rows or two columns of A are interchanged, then det(B) = – det(A).
If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then det(B) = det(A).

- **Example :**

If $\begin{vmatrix} a & b & c \\ d & e & g \\ h & i & j \end{vmatrix} = 4$, then what is $\begin{vmatrix} h & i & j \\ d & e & g \\ a & b & c \end{vmatrix}$?

Answer is : -4 .

❖ Rule 4: (Dealing with Elementary Matrices)

الفكرة هون ، إنه راح نعتبر هاي القاعدة حالة خاصة من قاعدة رقم 3 ، كيف يعني ؟!

راح نتبع الخطوات التالية :

- راح تكون معطى المصفوفة A على إنها identity matrix ناتج منها عدة elementary matrices
- خلينا نضل متذكرين انه ال elementary matrices ينتجوا من تنفيذ عملية وحدة عال I
- راح يكون عنا مجموعة elementary matrices بدنا نشوف كل وحدة فيهم شو العملية اللي نتجت منها عشان نحدد شو هو ال determinant من خلال تحديد هاي العملية.
- و لنحدد شو هو ال determinant من خلال العملية نتبع جدول القواعد التالي :

↕

Rules
If E results from multiplying a row of In by a nonzero number k, then det(E) = k
If E results from interchanging two rows of In, then det(E) = −1.
If E results from adding a multiple of one row of In to another, then det(E) = 1.

○ **Example:**

1) $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operation:

The second row of I_4 was multiplied by 3.

Then answer is : det(E) = 3

2) $E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Operation:

The first and last rows of I_4 were interchanged.

Then answer is : det(E) = -1

3) $E = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Operation:

7 times the last row of I4 was added to the first row

Then answer is : det(E) = 1

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