LINEAR ALGEBRA

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POWERUNIT

Chapter 1 :- system of linear equations and matrices.

$\boxed{1.1}$ system of linear equation.

- Definition :- An equation is called linear if the powers of all variables is 1.

ex:- $3x + 2y - z = 1 \longrightarrow$ linear

$x + 2y^2 + 5z = 3 \longrightarrow$ not linear

$\frac{1}{2}x - 5y + \sqrt{2}z = 0 \longrightarrow$ linear

In general :- $a_1x_1 + a_2x_2 ---+ a_nx_n = b \longrightarrow$ liner

$x_1, x_2, ----, x_2$ are called variables or unknowns.

$a_1, a_2 ----, a_n$ are called coefficiants.

Note :- $a_1x_1 + a_2x_2 ---- + a_nx_n = 0$, is called homogeneous linear equation.

Ex) $2x + 3y + z = 2$

$x - 3y + 5z = 1$

System of linear equation

Ex) $5x_1 + 3x_2 + 4x_3 = 0$

$x_1 + 4x_2^2 + 3x_3 = 10$

$x_1 + 4x_2 - x_3 = 4$

is not linear system.

Ex) $5x + 2y = 0$

$x - 3y = 0$

is called homogeneous linear system.

* The solution of the linear system.

Consider the system:-

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \begin{array}{l} 3 \text{ equations} \\ 3 \text{ unknowns.} \end{array}$$

we say $x_1 = s_1$, $x_2 = s_2$, $x_3 = s_3$ is a solution for this system if these values make the three equation true.

and we write (s_1, s_2, s_3) is soultion for the system.

Ex) The pair (3,1) is a solution for the system

$2X - y = 5$

$3X + 2y = 11$

Note :- Every system of liner equations has one solution or infinitely many solutions or no solutions

Ex) find the solution for the system

$3X - y = 3$

$X + y = 5$

answer

$$\begin{aligned} 3X - \cancel{y} &= 3 \\ X + \cancel{y} &= 5 \\ \hline 4X &= 8 \\ X = 2,\ y &= 3 \end{aligned}$$

one solution

Ex)

$$\begin{aligned} -2\cancel{X} + 2y &= 4 \\ 2X + 4y &= 9 \\ \hline 0 &\neq 1 \quad X \end{aligned}$$

No soltion

Ex)

$$\begin{aligned} 3X - 2y &= 9 \\ 6X - 4y &= 18 \\ \hline 0 &= 0 \end{aligned}$$

let $t = y \Rightarrow 3X = 9 + 2t$

$X = 3 + \frac{2}{3}t$

$(3 + \frac{2}{3}t, t)$ where t is any Real number.

infinetly solutions

* Matrix :- is a rectangular array of numbers. The numbers are called the intries in the matrix.

Ex) $A = \begin{bmatrix} 2 & 1 & 5 \\ -3 & 4 & 7 \end{bmatrix} \rightarrow$ row

Column.

الصفوف

$\overset{\uparrow}{2} \times 3$ matrix.

الأعمدة

الصف العمود

$a_{13} = 5$, $a_{23} = 7$, $a_{22} = 4$, $a_{12} = 1$

Q₁:) if $A = [a_{ij}]$, $i = \underbrace{1, 2, 3}_{}$, j , $\underbrace{1, 2}_{}$ 3 2

3×2

when $a_{ij} = 3i - j^2$. find A.

answer:- $A = \begin{array}{c} a_{11} \\ a_{21} \\ \\ \end{array} \begin{bmatrix} 2 & -1 \\ 5 & 2 \\ 8 & 5 \end{bmatrix} \begin{array}{c} a_{12} \\ \\ a_{32} \end{array}$

* How to use matrices to solve linear system :-
(e.r.o)

The following are called elemantry row operations

1) Multiply a raw by a non-zero constant
2) Interchange two rows تبديل صف مكان آخر
3) Add a constant times one row to another.
إضافة مضاعف صف للآخر.

Ex :- use matrices to solve the system.

$$x - 2y + z = -1$$
$$2x + y + 5z = 0$$
$$3x + 2y - z = 9$$

augmented matrix

answer :- leading

$$\begin{array}{cccc} x & y & z & \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 2 & 1 & 5 & 0 \\ 3 & 2 & -1 & 9 \end{array}\right] \end{array}$$

method 3

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 5 & 3 & 2 \\ 0 & 8 & -4 & 12 \end{bmatrix} \begin{array}{l} \\ \rightarrow -2r_1 + r_2 \\ \rightarrow -3r_1 + r_3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & 3/5 & 2/5 \\ 0 & 8 & -4 & 12 \end{bmatrix} \begin{array}{l} \\ \rightarrow \frac{1}{5} r_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 11/5 & -1/5 \\ 0 & 1 & 3/5 & 2/5 \\ 0 & 0 & -44/5 & 44/5 \end{bmatrix} \begin{array}{l} \rightarrow 2r_2 + r_1 \\ \\ \rightarrow -8r_2 + r_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 11/5 & -1/5 \\ 0 & 1 & 3/5 & 3/5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow -\frac{5}{44} r_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} \rightarrow -11 r_3 + r_1 \\ \rightarrow -\frac{3}{5} r_3 + r_2 \\ \end{matrix}$$

Note that the system after elementary row operations:-

$X = 2$, $y = 1$, $z = -1$

$(2, 1, -1)$ is the solution of the system.

1.2 Gaussian Elimination

• we say the matrix is in $\underbrace{\text{reduced row echelon form}}_{\text{شكل صفي مبسط}}^{\text{rref}}$, if the following conditions hold :-

1) if a row doesn't consist entirely of zeroes, then the first non-zero number in the row is $\boxed{1} \longrightarrow$ leading
أول عنصر غير الصفر = 1

2) if there are any rows that consist entirly of zeroes, then they grouped together at the bottom of the matrix.

3) if any two successive rows that do not consist entirely of zeroes, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
إذا كان أول صف بالصفر = 1 بالتالي أول عنصر في الصف الثاني = 1 (عاليمين)

4) each column contains a leading 1 has zeroes everywhere else in that column. كل عناصر العمود = 1 معدا ال leading

Note :- we say the matrix is in row echelon form (ref) if the conditions 1, 2, 3 hold only.

/ /

ex) which of the following matrices are in r.r.e.f

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \times$$

عاليسار
خطأ

$$E = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer: A, B, C, F, H

* we use the procedure called Gauss-Jordan elimination to convert the matrix to r.r.e.f.

1) write in augmented matrix.
2) augmented matrix → r.r.e.f
3) solve the system.

ex. Suppose that the augmented matrix for a linear system in the unknowns x, y, z has been reduced by elementary row operation to:-

$$\begin{bmatrix} {}_{x\leftarrow}1 & 0 & 0 & 5 \\ 0 & 1_{\to y} & 0 & 3 \\ 0 & 0 & \underset{z}{\swarrow}1 & 7 \end{bmatrix} \longrightarrow \text{r.r.e.f}$$

The system has unique solution → (5, 3, 7)

ex. if the augmented matrix for a linear system with unknowns x_1, x_2, x_3, x_4 is:-

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{r.e.f}$$

$x_1 + 2x_2 = 4$

$x_3 + 3x_4 = 5$

$\boxed{0 = 1}$

No solution

ex. if the augmental matrix for a linear system with unknowns x, y, z in r.r.e.f is :-

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

what is the solution of the system?

if # of unknowns > # of equations then the system has ∞ solutions

$x + 3z = 4$
$y + 2z = 5$

let $z = t$ $t \in R$ (كلما اغير t أكون حل جديد.)

$\rightarrow x = 4 - 3t$
$y = 5 - 2t$

$\longrightarrow$ the solution of the system is $(4-3t, 5-2t, t)$ where t is ~~called~~ any real number

- z is called free parameter

# of free parameters = # of unknowns - # of equat. in (r.r.e.f or r.e.f system)

$\frac{Q7}{23}$) Solve the linear system by Gauss-Jordan elimination.

$$3x - y + z + 7w = 13$$
$$-2x + y - z - 3w = -9$$
$$-2x + y - 7w = -8$$

answer:-
$$\begin{bmatrix} 3 & -1 & 1 & 7 & 13 \\ -2 & 1 & -1 & -3 & -9 \\ -2 & 1 & 0 & -7 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/3 & 1/3 & 7/3 & 13/3 \\ -2 & 1 & -1 & -3 & -9 \\ -2 & 1 & 0 & -7 & -8 \end{bmatrix}$$

$$\begin{array}{l} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & 7/3 & 13/3 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 5/3 & -\frac{1}{3} \\ 0 & \frac{1}{3} & 2/3 & -7/3 & 2/3 \end{bmatrix} \begin{array}{l} \\ \rightarrow 2r_1 + r_2 \\ \rightarrow 2r_1 + r_3 \end{array} \end{array}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & 7/3 & 13/3 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & \frac{1}{3} & 2/3 & -\frac{7}{3} & 2/3 \end{bmatrix} \begin{array}{l} \\ \rightarrow 3r_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 4 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 0 & 1 & -4 & 1 \end{bmatrix} \begin{array}{l} \rightarrow \frac{1}{3}r_2 + r_1 \\ \\ \rightarrow -\frac{1}{3}r_2 + r_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & -4 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \end{bmatrix} \begin{array}{l} \\ \rightarrow r_3 + r_2 \\ \\ \end{array}$$

$$x + 4w = 4$$
$$y + w = 0$$
$$z - 4w = 1$$

unknowns > equat.

so ∞ solutions.

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- The system of linear equations is said to be homogenous if the constant terms are all zero.

ex) the system $2x - 3y = 0$
$x + 2y = 0$ is homogenous.

- we say that the linear system is Consistent, if it has at least one solution, and we say it is inconsistent, if it has no solution.

- The homogenous system is Consistent
  we have two cases:
  1) The homogenous system has only the trivial solution
     trivial → كل المتغيرات قيمتها = صفر.
  2) The homogenous system has infinitely many solutions

ex) Solve the system

$x + 2y - z = 0$
$2x - y + 3z = 0$
$3x + y + 2z = 0$

answer:-
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \begin{array}{l} \\ \rightarrow -2r_1 + r_2 \\ \rightarrow -3r_1 + r_2 \end{array} \longrightarrow \left[\begin{array}{cccc} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \rightarrow \frac{-1}{5} r_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x + z = 0$
$y - z = 0$

$z = t$, $t \in R$
$\rightarrow x = -t$, $y = t$
Solution → $(-t, t, t)$ where t is any real number

Note:- A homogenous system of linear equations with more unknowns that equations has infinitely many solutions.

ex) the following system has infinitely many solutions

$$x_1 + 3x_2 - x_3 + x_4 = 0$$
$$2x_1 + x_2 + x_3 + 3x_4 = 0$$
$$x_1 - x_2 + 4x_3 - x_4 = 0$$

why?

The system have 4 unknowns and 3 equations, So by the note the system has infinitely many solutions.

$\frac{Q36}{24}$) solve the following system for x, y, and z

$$\frac{1}{x} + \frac{2}{y} - \frac{4}{z} = 1$$

* Not linear system

$$\frac{2}{x} + \frac{3}{y} + \frac{8}{z} = 0$$

$$-\frac{1}{x} + \frac{9}{y} + \frac{10}{z} = 5$$

answer :- let $\bar{x} = \frac{1}{x}$, $\bar{y} = \frac{1}{y}$, $\bar{z} = \frac{1}{z}$

Linear system
$$\begin{cases} \bar{x} + 2\bar{y} - 4\bar{z} = 1 \\ 2\bar{x} + 3\bar{y} + 8\bar{z} = 0 \\ -\bar{x} + 9\bar{y} + 10\bar{z} = 5 \end{cases}$$

$\frac{Q27}{23}$) Determine the value of a for which the system has No solution, exactly one solution, or infinetly many solutions.

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + (a^2 - 2) = a + 4$$

answer :-

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-2 & a+4 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 2 & -13 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2+10 & a-12 \end{bmatrix} \begin{array}{l} \\ \to -3r_1 + r_2 \\ \to -4r_1 + r_3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-4 & a-2 \end{bmatrix} \begin{array}{l} \\ \\ \to -r_2 + r_3 \end{array}$$

Now, note that the equation 3 became :- $0x + 0y + (a^2-4)z = a-2$

1) if $a=2 \to$ equation 3 became $0=0$ so the system has infinetly many solutions

2) if $a=-2 \to 0=-4 \to$ No solution

3) if $a \neq \mp 2 \to$ in r.r.e.f $\to$ one solution.

Section 1.3 :- Matrices and Matrix operations.

- we Can write the matrix in the form :-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

or as $A = [a_{ij}]_{m \times n}$

- A is called $m \times n$ matrix where $m \times n$ is the size of the matrix.

- a_{ij} is the entry in row i and column j.

ex) if A is 2x3 matrix such that :-

$a_{ij} = 2i - j$ find A,

answer :-

$$A = \begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{0} & \overset{a_{13}}{-1} \\ \underset{a_{21}}{3} & \underset{a_{22}}{2} & \underset{a_{23}}{1} \end{bmatrix}$$

$a_{11} \rightarrow i = 1, \ j = 1$

$a_{21} \rightarrow i = 2, \ j = 1$

* Matrix operations:-

ex) let $A = \begin{bmatrix} 2 & 1 & 5 \\ -1 & 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

find $3A$, $A+B$, $A-2B$

1) $3A = \begin{bmatrix} 6 & 6 & 15 \\ -3 & 12 & 9 \end{bmatrix}$ → ضربنا كل عنصر بثلاثة

2) $\underline{A+B} = \begin{bmatrix} \overset{2+4}{6} & \overset{1-2}{-1} & 6 \\ 0 & 7 & 5 \end{bmatrix}$ +, - → must be the same size.

same size ✓

3) $A - 2B \longrightarrow 2B = \begin{bmatrix} 8 & -4 & 2 \\ 2 & 6 & 4 \end{bmatrix}$

$A - 2B = \begin{bmatrix} -6 & 5 & 3 \\ -3 & -2 & -1 \end{bmatrix}$

- Matrix products → # of of columns in the first matrix = # of rows in the second Matrix.

$$A \times \underline{\underline{B}} = \underline{\underline{B}} \times C$$

ex) if $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}$

find AB and BA <u>if possible</u>

$AB = \underset{2\times\underline{\underline{3}}}{A} \quad B\ \underline{\underline{2}}\times 2$ → not possible

$BA = \underset{2\times 2 /\!/}{B} \quad \underset{2\times 3 /\!/}{A}$ possible

if $C = BA$ then the size of C will be:-

$$\underbrace{2\times 2 \quad 2\times 3} = \boxed{2 \times 3}$$

$$\begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{bmatrix}$$

$C_{11} = [5 \ \ 1] \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 5\times 2 + 1\times -1 = 9$

$C_{12} = 5\times 1 + 1\times 4 = 9$, $C_{13} = 5\times 3 + 1\times 2 = 17$

$C_{21} = 6 - 2 = 4$, $C_{22} = 3 + 8 = 11$

$C_{23} = 9 + 4 = 13$

$$C = \begin{bmatrix} 9 & 9 & 17 \\ 4 & 11 & 13 \end{bmatrix}$$

/ /

ex) if $A = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ 6 & -2 \end{bmatrix}$

find AB and BA if Possible

answer:- $AB = A_{2\times\underline{3}}\ B_{\underline{3}\times 2}$ → Possible

$$AB = \begin{bmatrix} 32 & 18 \\ 23 & 13 \end{bmatrix}$$

$BA = B_{3\times\underline{2}}\ A_{\underline{2}\times 3}$ → Possible

$$BA = \begin{bmatrix} -6 & 28 & 14 \\ -7 & 37 & 14 \\ 10 & 14 & 14 \end{bmatrix}$$

* ضرب صف بعمود

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- Transpose of the matrix (A^T) قلب مصفوفة

ex) let $A = \begin{bmatrix} 3 & 5 & 6 \\ -2 & 4 & 2 \end{bmatrix}$, find A^T

answer :- $A^T = \begin{bmatrix} 3 & -2 \\ 5 & 4 \\ 6 & 2 \end{bmatrix}$

- Note that $A_{2\times3}$ and $A^T_{3\times2}$

ex) if $B = \begin{bmatrix} 3 & 5 & 2 & 4 \\ 1 & 4 & 2 & -3 \\ 6 & 3 & 8 & 9 \end{bmatrix}$, then

$$B^T = \begin{bmatrix} 3 & 1 & 6 \\ 5 & 4 & 3 \\ 2 & 2 & 8 \\ 4 & -3 & 9 \end{bmatrix}$$

- A matrix is called a square matrix if the number of rows equal number of columns.

- if A is a square matrix, then trace of A is the sum of the ~~entries~~ entries on the main diagonal of A.

* Main diagonal in the square matrix = القطر

if $A = [a_{ij}]_{n \times n}$, then $tr(A) = \sum_{i=1}^{n} a_{ij}$

$tr(A) = a_{11} + a_{22} + --- a_{nn}$

ex) if $A = \begin{bmatrix} 6 & 2 & 1 \\ 8 & -3 & 5 \\ 4 & 9 & 5 \end{bmatrix}$ find tr (A)

$tr(A) = 6 - 3 + 5 = 8$

Q) if $A = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 9 \\ 2 & -6 \end{bmatrix}$, find

1) $(A+B)^T \rightarrow$

answer :- $\underbrace{A+B}_{\text{same size}} = \begin{bmatrix} 11 & 14 \\ 3 & -2 \end{bmatrix}$

$(A+B)^T = \begin{bmatrix} 11 & 3 \\ 14 & -2 \end{bmatrix}$

2) $A^T + B^T = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 8 & 2 \\ 9 & -6 \end{bmatrix} = \begin{bmatrix} 11 & 3 \\ 14 & 2 \end{bmatrix}$

3) $tr(AB) \rightarrow$ ~~B~~ $\underset{2\times2}{A}$ $\underset{2\times2}{B} \longrightarrow$ Possible

$$AB = \begin{bmatrix} 34 & 72 \\ 16 & -15 \end{bmatrix}$$

$$tr(AB) = 34 - 15 = 19$$

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section 1.4 :- inverses, Algebraic properties of Matrices (Lecture 1)

* some properties :-

let A, B and C be matrices with sice such that the indicated operations can be performed, then the following hold :-

1) $A+B = B+A$ 2) $A+(B+C) = (A+B)+C$ الترتيب مهم

3) $A(BC) = AB(C)$ 4) $A(B+C) = AB+AC \longrightarrow$

5) $(B+C)A = BA + CA \rightarrow$ الترتيب مهم.

6) $K(A+B) = KA + KB$ where K is Constant.

7) $K(AB) = (KA)B = A(KB)$

8) $A+0 = 0+A = A$

9) $A + (-A) = 0$

10) $KA = 0$ then $K = 0$ or $A = 0$

Q1 :- 1) If A and B are two matrices such that $AB = 0$, then $A = 0$ or $B = 0$

(False)

2) If $A \neq 0$, B and C are matrices such that $AB = AC$, then $B = C$

(False)

- Some properties for transpose :-

1) $(A^T)^T = A$
2) $(A+B)^T = (A)^T + (B)^T$
3) $(kA)^T = kA^T$ where k is constant.
4) $(AB)^T = \underline{\underline{B^T A^T}}$

- let A be a square matrix, we say A is diagonal matrix if all entries Not on the main diagonal are zeroes.

ex) $A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \longrightarrow$ diagonal matrix.

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} \longrightarrow$ diagonal matrix.

- if A is diagonal matrix, we say A is identity matrix if all entries on the main diagonal are 1.

Note :- we use the notation I for identity matrix.

ex) $I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ex) $I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ex) $I = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- we call it identity matrix because it doesn't affect multiplication.

$$A \times I = A$$

- if A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, the A is said to be invertible (or nonsingular) and B is called an inverse of A.

in this case we write $B = A^{-1}$

Note:- if No such matrix can be found, then A is said to be singular.

ex) let $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

Note that:-

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

→ A is invertible and $A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

- Rules for 2X2 matrices :-

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

ليس لها نظاري

1) if $ad - bc = 0$, then A is $\underline{\text{Singular}}$
2) if $ad - bc \neq 0$, then A is invertible

and :- $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

ex) let $C = \begin{bmatrix} 6 & -5 \\ 1 & 2 \end{bmatrix}$, find C^{-1} if exist

answer :- $6 \times 2 - (-5) \times 1 = 12 + 5 = 17 \neq 0$

$$C^{-1} = \frac{1}{17} \begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 2/17 & 5/17 \\ -1/17 & 6/17 \end{bmatrix}$$

Note that $CC^{-1} = C^{-1}C = I$

section 1.4 :- Inverses ; Algebraic properties of Matrices

- linear system in matrices form

ex) write the following system in Matrices form

$$2x - y + 3z = 5$$
$$8x + 2y - z = 1$$

answer:-

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 8 & 2 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

- Note that we can write the system as:-

$$Ax + b$$

$$\begin{bmatrix} 2 & -1 & 3 \\ 8 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Q) if $A^{-1} = \begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $b = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

find the solution for the system $AX = b$

answer:- $AX = b$

$$A^{-1}AX = A^{-1}b$$

$$X = \begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -19 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -19 \end{bmatrix} \longrightarrow x = -1 , y = -19$$

Note :- $A^{-1}A = I$

$IX = X$

- Some Properties of inverse of the matrix:
  if A is invertible an n is nonnegative integer, then,

1) A^{-1} is invertible and $(A^{-1})^{-1} = A$
2) A^{n} is invertible and $(A^{n})^{-1} = (A^{-1})^{n} = A^{-n}$
3) kA is invertible $(k \neq 0)$ and $(kA)^{-1} = \frac{1}{k} A^{-1}$
4) A^{T} is invertible and $(A^{T})^{-1} = (A^{-1})^{T}$
5) if B is invertible matrix, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Q) if $A^{-1} = \begin{bmatrix} 4 & 2 \\ 5 & -1 \end{bmatrix}$, find $(A^{T})^{-1}$

answer:-

$$(A^{T})^{-1} = (A^{-1})^{T} = \begin{bmatrix} 4 & 5 \\ 2 & -1 \end{bmatrix}$$

Q) if $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & -1 & 5 \\ 3 & 4 & 2 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 4 & 1 & 4 \\ 2 & 5 & 7 \\ -1 & 3 & 6 \end{bmatrix}$

and $C = (AB)^{-1}$, find C_{23}

answer $\Rightarrow$ $C = B^{-1}A^{-1} \longrightarrow C_{23} = 2+25+14 = 41$

/ /

$\frac{Q_{53}}{50}$) if A, B and $A+B$ are invertible matrices show that

$$A(A^{-1}+B^{-1})B\,(A+B)^{-1}=I$$

answer:- $A(A^{-1}+B^{-1})B(A+B)^{-1}=(AA^{-1}+AB^{-1})B(A+B)^{-1}$

$$=(I+AB^{-1})B(A+B)^{-1}$$

$$=(B+AB^{-1}B)(A+B)^{-1}$$

$$=(B+AI)(A+B)^{-1}$$

$$=(B+A)(A+B)^{-1}$$

Q_4) let A and B be two matrices of same siee. is the following statement True or false

$$(A+B)^2=(A^2+2AB+B^2)$$

answer:- false

$$(A+B)^2=(A+B)(A+B)=(A^2+\underbrace{AB+BA}_{\neq 2AB}+B^2)$$

Q) if $(A^T+2I)^{-1} = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, find a_{21} and A

answer:

$$(A^T+2I)^{-1} = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$$

$$\rightarrow ((A^T+2I)^{-1})^{-1} = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}^{-1}$$

$$\rightarrow A^T + 2I = \frac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$$

$$\rightarrow A^T = \begin{bmatrix} 2 & -5/2 \\ -1 & 3/2 \end{bmatrix} - 2I$$

$$\rightarrow A^T = \begin{bmatrix} 2 & -5/2 \\ -1 & 3/2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\rightarrow A^T = \begin{bmatrix} 0 & -5/2 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$A = (A^T)^T = \begin{bmatrix} 0 & -1 \\ -5/2 & -\frac{1}{2} \end{bmatrix} \rightarrow a_{21} = \frac{-5}{2}$$

- section 1.5: elementary Matrices and a method for finding A^{-1}

- The two matrices A and B are said to be row equivalent if one can be obtained from the other by elemantry row operations'

ex) let $A = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

A and B are row equivalent, why?

answer: $A = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \rightarrow \frac{1}{3}r_1$

$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow -2r_1 + r_2 = B$

- An nxn matrix E is called elementary matrix if it can be obtained from In by performing a single elementary row operation.

ex) which of the following is elementary matrix.

$$E_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 5 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

تبديل للصف الأول محل الصف الأخير

$$E_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

answer :- E_1, E_3, E_5, E_6

Theorem ① :- if the elementary matrix E results from performing a certain row operation on I_m and if A is $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

ex) $E = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$ عن طريق ضرب I في 5.
لو قمنا بإجراء نفس العملية
على A يكون الناتج EA

$$A = \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 15 & 20 \\ 2 & -1 \end{bmatrix} = 5A$$

* finding the inverse of the matrix :-

To find the inverse for the square matrix $A_{n \times n}$ we use elementary row operations.

$$\begin{bmatrix} A & I_n \end{bmatrix} \longrightarrow \text{matrix with 2 parts}$$

$$\Big\downarrow \text{e.r.o}$$

$$\begin{bmatrix} I_n & A^{-1} \end{bmatrix}$$

* إذا لم نستطع تحويل A إلى I
هذا يعني أن A ليس لها
نظير ضربي.

↓

A is singular.

$\frac{Q_{14}}{59}$) find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

answer:-

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \rightarrow 2r_1 + r_2 \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3/3 & 2/3 & -1/3 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \rightarrow -\frac{1}{3} r_2 \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 4/3 & -1/3 & 2/3 & 0 \\ 0 & 1 & -2/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 7/3 & -4/3 & 2/3 & 1 \end{array}\right] \begin{array}{l} \rightarrow -2r_2 + r_1 \\ \\ \rightarrow -2r_2 + r_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4/3 & -1/3 & 2/3 & 0 \\ 0 & 1 & -2/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -4/7 & 2/7 & 3/7 \end{array}\right] \begin{array}{l} \\ \\ \rightarrow 3/7 r_3 \end{array}$$

$\longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/21 & 6/21 & -12/21 \\ 0 & 1 & 0 & 6/21 & -3/21 & 6/21 \\ 0 & 0 & 1 & -4/7 & 2/7 & 3/7 \end{array}\right] \begin{array}{l} \rightarrow -4/3r_3 + r_2 \\ \rightarrow 2/3 r_3 + r_2 \\ \end{array}$$

$$A^{-1} = \frac{1}{21}\begin{bmatrix} 9 & 6 & 12 \\ 6 & -3 & 6 \\ -12 & 6 & 9 \end{bmatrix}$$

$AA^{-1} = I \rightarrow$ تأكد

* The inverse of matrix as a product of elementary matrices:-

Q9 58) let $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$, write A^{-1} as product of elementary matrices

answer:-

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} = A$$

$$-2r_1 + r_2 \leftarrow \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} = E_1 A$$

$$-2r_2 \leftarrow \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = E_2(E_1 A) \rightarrow$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} E_3(E_2 E_1 A)$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} -2r_1$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} -2r_2$$

$$E_3 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \rightarrow -4r_2 + r_1$$

$\longrightarrow$

Note that :- $E_3 E_2 E_1 A = I$

$$\longrightarrow A^{-1} = E_3 E_2 E_1 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

Note that :-

$$A = (A^{-1})^{-1} = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

Note :- 1) every elementary matrix is invertible.
2) The inverse of elementary matrix is elementary matrix.

* Theorem 2,, :- if A is nxn matrix, then the following are equivalent :-

1) A is invertible matrix
2) The system $AX = 0$ has only trivial solution.
3) The r.r.e.f of A is I_n
4) The matrix A can be written as product of elementary matrices.

Note that:- Assume A is invertible

$\rightarrow A^{-1}$ is exist

$\rightarrow$ if $AX = 0$

then $A^{-1}(AX) = A^{-1}0$

$\rightarrow IX = 0$

$\rightarrow X = 0$

$\rightarrow$ the system has only trivial solution.

Q :- let $A = \begin{bmatrix} 1 & 4 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{bmatrix}$

1) find A^{-1}

$$A^{-1} = [A \vdots I] \xrightarrow{e.r} [I \vdots A^{-1}]$$

$$= \begin{bmatrix} -2 & 1 & 2 \\ 1/2 & -1/2 & 0 \\ 1/4 & 1/4 & -1/2 \end{bmatrix}$$

2) if $AB = \begin{bmatrix} 4 & 3 & 4 & 1 & 2 \\ 1 & 2 & 6 & 3 & 4 \\ 2 & 5 & 1 & -2 & 1 \end{bmatrix}$ find b_{23}

2) $A^{-1}AB = A^{-1}C$

$$B = A^{-1}C$$

$$b_{23} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} = 2-3 = -1$$

3) $AX = b \longrightarrow A^{-1}AX = A^{-1}b$

$$\longrightarrow X = A^{-1}b = \begin{bmatrix} -2 & 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}$$

if $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad b = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}$

$$= \begin{bmatrix} 11 \\ -1 \\ -1 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ -1 \end{bmatrix}$$

$$\longrightarrow x = 11,\ y = -1,\ z = -1$$

Note: $A^0 = I$

$A^{-1} \longrightarrow$ inverse

$A^{-3} \longrightarrow$ اضرب ٣ مرات ثم inverse

او inverse ٣ مرات

Section 1.6) More on linear systems and invertible matrices.

- Theorem 1:- if A is invertible matrix, then for each $n \times 1$ matrix b, the system of equations $AX = b$ has

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Matrices

one solution which is $\bar{X} = A^{-1}b$

size → $n \times 1$

- Theorem 2:- let A and B be square matrices of the same size. if AB is invertible then A and B must be invertible

- Theorem 3:- if A is invertible, then A has unique inverse

Proof:- Assume B and C are two inverses of A, want to show $B = C$

Note that $B = BI = B(AC) = (BA)C = IC = C$

Q, True or false:- or sometimes false

1) let A and B be two matrices, if A is invertible and $AB = 0$, then $B = 0$ True

2) $AB = 0 \rightarrow A^{-1}AB = A^{-1}0$

$\rightarrow IB = 0$

2) let A, B, C be matrices, if A is invertible and $AB = AC$, then $B = C$

True

$AB = AC \longrightarrow A^{-1}AB = A^{-1}AC$ (اضرب بالنظير الضربي)

$\rightarrow IB = IC$

$\rightarrow B = C$

Section 1.7 : Diagonal , Triangular and symmetric matrices

The square matrix is diagonal if all entries not on the main diagonal are zeroes

$A = [a_{ij}]$ is diagonal iff

$a_{ij} = 0$ if $i \neq j$

Note:- if A is diagonal, then A is invertible iff all entries on the main diggonal are not eero.

ex:- $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$ A is invertible

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$A \rightarrow A$ is diagonal but not invertible

- The square matrix A is upper triangular iff all elements bellow the main digonal are zeroes

$A = [a_{ij}]_{n \times n}$ is upper triangular iff $a_{ij} = 0$ for $i > j$

ex) $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is upper triangular

$B = \begin{bmatrix} 3 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 5 & 2 \end{bmatrix}$ ↳ is not upper triangular.

The square matrix A is lower triangular iff all entries above the main diagonal are zeroes

ex) $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 7 & 0 & 4 & 0 \\ 6 & 2 & 1 & 0 \end{bmatrix}$ → lower triangular

if $A = [a_{ij}]$ lower then $a_{ij} = 0$ for $j > i$

* Some Properties:-

1) if A is upper → A^T is lower
2) if A is lower → A^T is upper
3) if $A_{n\times n}$ is upper or lower, then A is invertible iff $a_{ii} \neq 0$ for $i = 1, 2, ----, n$
4) if A is upper and invertible, then A^{-1} is upper
5) if A is lower and invertible then A^{-1} is lower

ex) $A = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 6 & 0 \\ 8 & 9 & 1 \end{bmatrix}$ ~ lower

$A^T = \begin{bmatrix} 3 & 4 & 8 \\ 0 & 6 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ ~ upper

* let A be square matrix, we say A is symmetric if $A = A^T \rightarrow a_{ij} = a_{ji}$

ex) $A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 8 & -4 \\ 3 & -4 & 9 \end{bmatrix}$

$A^T = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 8 & -4 \\ 3 & -4 & 9 \end{bmatrix}$ $A = A^T \rightarrow$ A is ~~symmetric~~
symmetric

* let A be a square matrix, we say A is skew symmetric if $A^T = -A \rightarrow a_{ij} = -a_{ji}$

ex) $A = \begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & 3 \\ -4 & -3 & 0 \end{bmatrix}$, $-A = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & -3 \\ 4 & 3 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & -3 \\ 4 & 3 & 0 \end{bmatrix}$ $A^T = -A$ so it is skew semmetric

___/___/___

* if A and B, are symmetric matrice with same size, then:-

1) A^T is symmetric
2) A+B and A-B are symmetric
3) KA is symmetric

Proof:-

1) since A is symmetric $\rightarrow A = A^T$
but $(A^T)^T = A = A^T \longrightarrow$ ماتغيرت بعد $(A^T)^T$
$\Rightarrow A^T$ symmetric

2) $(A+B)^T = A^T + B^T = A+B$
$\rightarrow A+B$ symmetric ليسل T اي: *
تأثير

$(A-B)^T = A^T - B^T = A-B$
$\rightarrow A-B$ is symmetric

3) $(KA)^T = KA^T = KA$
$\rightarrow KA$ is symmetric

Q) True or false

1) if A is a square matrix, then $A+A^T$ is symmetric

$$(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

True

2) if A is a square matrix, then $A-A^T$ is symmetric

$$(A-A^T)^T = (A^T) - (A^T)^T = A^T - A \neq A - A^T$$

False. $= -(A-A^T)$ skew symmetric

3) if A is a square matrix, then AA^T is symmetric

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

True

Q) if A is invertible symmetric matrix, then A^{-1} is symmetric?

$(A^{-1})^T \ldots$

True.

Chapter 2 : Determinants

section 2.1 :- Determinants by Cofactor expansion.

- If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} is defined to be the determinant of the submatrix, that remain after the ith row and jth column are deleted from A.

- The Cofactor of entry a_{ij} denoted by C_{ij} is defined by: $C_{ij} = (-1)^{i+j} M_{ij}$

ex) let $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 3 & 2 \\ 4 & 6 & 1 \end{bmatrix}$, find M_{21}, M_{22}, M_{32}

answer :- $M_{31} = \begin{bmatrix} 3 & -1 \\ 6 & 1 \end{bmatrix} = 9$

$$M_{22} = \begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} = -2$$

$$M_{32} = \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = 8$$

$C_{21} = (-1)^2 \cdot 9 = -9, \quad C_{22} = (-1)^4 \cdot -2 = -2$

$C_{33} = (-1)^6 \cdot 8 = -8$

* Note :- if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = ad - bc$
$\hookrightarrow |A| = \det(A)$

. if A is nxn Matrix, then

1) $\det(A) = a_{ij} C_{i1} + a_{i2} C_{i2} + \cdots a_{in} C_{in}$

2) $\det(A) = a_{ij} C_{ij} + a_{2j} C_{2j} + \cdots a_{nj} C_{nj}$

ex) let $A = \begin{bmatrix} 2 & 4 & 3 \\ -5 & 6 & 1 \\ 3 & 4 & 2 \end{bmatrix}$, find det(A)

we can use 1 or 2

answer :- By 1), let $i = 1$

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= 2 \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} - 4 \begin{vmatrix} -5 & 1 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} -5 & 6 \\ 3 & 4 \end{vmatrix}$$

$$= 2 \cdot 8 - 4 \cdot -13 + 3 \cdot -38$$

$$= 16 + 52 - 114 = -46$$

* if we choose another "i" $\longrightarrow -46$.

• By ②, let $j=1$

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= 2\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} - (-5)\begin{vmatrix} 4 & 3 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 3 \\ 6 & 1 \end{vmatrix}$$

$$= 2 \cdot 8 + 5 \cdot -4 + 3 \cdot (-14) = \boxed{-46}$$

• By (2), let $j=3$

$$\det(A) = a_{13}C_{13} + a_{23} + a_{33} \cancel{*} C_{33}$$

$$= 3\begin{vmatrix} -5 & 6 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ -5 & 6 \end{vmatrix}$$

$$= 3 \cdot -38 - 1 \cdot 4 + 2 \cdot 32$$

$$= -114 + 4 + 64 = \boxed{-46}$$

*Choose the easier row or Colqumn.

ex) let $A = \begin{bmatrix} 3 & 0 & 7 \\ 5 & 8 & 6 \\ -4 & 0 & 4 \end{bmatrix}$

answer :-

use $\underline{\underline{2}}$, let $j=2$

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= 0 + 8\begin{vmatrix} 3 & 7 \\ -4 & 4 \end{vmatrix} + 0$$

$$= 8 * 40 = 320$$

Q) let $B = \begin{bmatrix} 1 & 2 & 6 & 0 \\ 3 & 2 & 5 & 4 \\ 0 & 6 & 0 & 4 \\ 3 & 0 & 7 & 1 \end{bmatrix}$, find $\det(B)$

answer :-

we will use row 3 $(i=3)$ because it has the largest number of zeroes.

$$\det(B) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$$

$$= 0 - 6\begin{vmatrix} 1 & 6 & 0 \\ 3 & 5 & 4 \\ 3 & 7 & 1 \end{vmatrix} + 0 - 4\begin{vmatrix} 1 & 2 & 6 \\ 3 & 2 & 5 \\ 3 & 0 & 7 \end{vmatrix}$$

$$= -6 * 31 - 4 * -34$$

$$= -186 + 136 = -50$$

/ /

- if A is $n \times n$ diagonal, upper triangular or lower triangular matrix, then

$$\det(A) = \prod_{j=1}^{n} a_{ii} = a_{11} a_{22} a_{33} \cdots a_{nn}$$

$\frac{Q_{30}}{99}$) if $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, find $|A|$

answer:- A is upper triangular

$$|A| = \det(A) = 1 * 2 * 3 * 4 = 24$$

Q) True or false

1) for any square matrix A and scalar k, $\det(kA) = k \det A$

False

2) for any square matrix A and B, $\det(A+B) = \det(A) + \det(B)$

False

section 2.2) evaluating determinants by row reduction

- Properties of det.

1) let A be a square matrix, if A has a row of zeroes or column of zeroes, then det(A) = 0

ex) $A = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 5 & 0 \end{bmatrix}$

det(A) = 0 det(B) = 0

2) let A be a square matrix, then $\det(A) = \det(A^T)$

ex) $A = \begin{bmatrix} 3 & 5 \\ 9 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 3 & 9 \\ 5 & 4 \end{bmatrix}$

$\det(A) = 12 - 45 = -33$ $\det(A^T) = 12 - 45 = -33$

3) if the matrix B results from A by multiply one row, or one column of A by scalar K, then
det(B) = K det(A)

ex) $A = \begin{bmatrix} 2 & 5 \\ 4 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 12 & -9 \end{bmatrix}$

$\det(A) = -6-20 = -26$
$\det(B) = -18-60 = -78 = 3\det(A)$

. if the matrix B results from A by interchange two rows or two Columns, then $\det(B) = -\det(A)$

ex) $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

$\det(A) = 12-10 = 2$, $\det(B) = 10-12 = -2$

. if the matrix B results from A by adding a multiple of one row to another row or adding a multiple of one Column to another, then $\det(A) = \det(B)$

ex) $A = \begin{bmatrix} 6 & 3 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ 14 & 14 \end{bmatrix} \sim -3r_1 + r_2$

$\det(A) = 30+12 = 42$, $\det(B) = 84-42 = 42$

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- if A is a square matrix with two proportional rows or two proportional columns, then det(A) = 0

ex) $A = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \longrightarrow r_2 = \frac{2}{3} r_1$

det(A) = 12 − 12 = 0

Q) if $A = \begin{bmatrix} 3 & 1 & 5 \\ 6 & 4 & -8 \\ 3 & 2 & -4 \end{bmatrix}$, det(A) = 0

- The matrix A is invertible iff $\det(A) \neq 0$

- If A is invertible, then $\det(A)^{-1} = \frac{1}{\det(A)}$

- if A and B are square matrices of same size, then det(AB) = det(A) det(B).

Q) if A and B are 3x3 matrices , $\det(A) = 2$ and $\det(B) = -4$, find:-

1) $\det(AB^{-1}) = \det(A)\det(B^{-1}) = \det(A) = \frac{1}{\det(B)} = \frac{2}{-4} = -\frac{1}{2}$

→ 3x3 matrix

2) $\det(5A) = 5 * 5 * 5 \det(A) = 250$

3) $\det(2A^2B^{-3}) =$

$8 * 4 * \frac{1}{-64} = -\frac{1}{2}$

$\boxed{\hookrightarrow \det(kA) = k^n \det(A)}$ → where n = # of rows

Q) let $A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 3 \\ 2 & 1 & 6 & 1 \\ 2 & 4 & 8 & 11 \end{bmatrix}$ find $\det(A)$

answer:-

$B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 3 \\ 2 & 1 & 6 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim -2r_1 + r_4$

→

$$\det(B) = 1 \cdot \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 2 & 1 & 6 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 2+2 \cdot 4 = 10$$

$\frac{Q28}{105}$) let $A = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

show that $\det(A) = -a_{13} a_{22} a_{31}$

answer :-

$$B = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 13 \end{bmatrix}$$

$\det(B) = a_{31}\, a_{22}\, a_{13}$

$\det(A) = -a_{31}\, a_{22}\, a_{31}$

Section 2.3 :- Cramer's Rule and finding inverse

- let $A = [a_{ij}]_{n \times n}$ be a matrix, we define the matrix
$C = [C_{ij}]_{n \times n}$ where

$C_{ij} = (-1)^{i+j} M_{ij}$

C_{ij} is called the cofactor of the entry a_{ij}
M_{ij} is called the minor of the entry a_{ij}
C is called the Matrix of Cofactor from A.
C^T is called the adjoint of A.

* we write $adj(A)$ to mean the adjoint of A
$adj(A) = C^T$

* if A is invertible matrix, then

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

ex) let $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 5 & 4 \\ 2 & -4 & 1 \end{bmatrix}$, use adjoint method to find A^{-1} (if exist)

answer:- $det(A) = 2 \cdot \begin{vmatrix} 5 & 4 \\ -4 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} -1 & 5 \\ 2 & -4 \end{vmatrix}$

$= \cancel{42}\ 2 \cdot 21 - 1 \cdot -9 + 3 \cdot -6$
$= 42 + 9 - 18 = 33 \qquad det(A) \neq 0 \rightarrow A^{-1}$ exist

Now we need ~~the~~ to find the matrix C

$$C = \begin{bmatrix} 21 & 9 & -6 \\ -13 & -4 & 10 \\ -11 & -11 & 11 \end{bmatrix}$$

$$C_{21} = (-1)^{2+1} M_{21}$$

$$= -1^{2+1} \begin{vmatrix} 1 & 3 \\ -4 & 1 \end{vmatrix} = -1 \times 13 = -13$$

$$\text{adj}(A) = C^T = \begin{bmatrix} 21 & -13 & -11 \\ 9 & -4 & -11 \\ -6 & 10 & 11 \end{bmatrix}$$

$$\rightarrow A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{33} \begin{bmatrix} 21 & -13 & -11 \\ 9 & -4 & -11 \\ -6 & 10 & 11 \end{bmatrix}$$

- Note that :-

$$A \, \text{adj}(A) = \det(A) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det(A) I$$

• Cramer's Rule:-

if the matrix A is inovertible, then the system $AX=b$ has unique solution

ex) Solve the System

$$\begin{aligned} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$$

answer:- $A = \begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}$, $\det(A) = -55$

$\det A \neq 0$ → A is invertible → the system has unique solution → we can use cramer's rule.

$A_x = \begin{bmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{bmatrix}$, $\det(A_x) = +144$

$x = \frac{\det A_x}{\det A} = \frac{-144}{55}$

$A_y = \begin{bmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{bmatrix}$, $\det(A_y) = 61 \rightarrow y = \frac{-61}{55}$

→

$$A_z = \begin{bmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{bmatrix}, \det(A_z) = -230$$

$$z = \frac{230}{55}$$

Q) if $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $\det(A) \neq 0$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} 3b_1 \\ 3b_2 \\ 3b_3 \end{bmatrix}$

find the solution of the system $AX = b$

answer :- by Cramer's Rule

العمود الأول ٣ اضعاف العمود الثاني . ←

$$A_x = \begin{bmatrix} 3b_1 & b_1 & c_1 \\ 3b_2 & b_2 & c_2 \\ 3b_3 & b_3 & c_3 \end{bmatrix}, \det(A_x) = 0$$

$$A_y = \begin{bmatrix} a_1 & 3b_1 & c_1 \\ a_2 & 3b_2 & c_2 \\ a_3 & 3b_3 & c_3 \end{bmatrix}, \det(A_y) = 3\det(A)$$

$$A_z = \begin{bmatrix} a_1 & b_1 & 3b_1 \\ a_2 & b_2 & 3b_2 \\ a_3 & b_3 & 3b_3 \end{bmatrix}, \det(A_z) = 0$$

$$x = \frac{0}{\det(A)} = 0, \quad y = \frac{3\det(A)}{\det A} = 3, \quad z = \frac{0}{\det(A)} = 0$$

Q) let $A = \begin{bmatrix} 2 & 5 & a \\ 1 & 4 & b \\ 0 & 7 & c \end{bmatrix}$, $\det(A) = 8$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

if $x = S_1$, $y = S_2$, $z = S_3$ is the solution of the system $AX = b$, find S_3

answer:- $z = \frac{\det(A_z)}{\det(A)} = S_3$

$$A_z = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 4 & 1 \\ 0 & 7 & 2 \end{bmatrix} \quad \det(A_z) = 2 \cdot \begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 5 & 3 \\ 7 & 2 \end{vmatrix}$$

$$= 2 \cdot 1 - 1 \cdot -11 = 13$$

$$\boxed{S_3 = \frac{13}{8}}$$

Chapter 4 :- General vector spaces

section 4.1 :- Real vector spaces

- The vector space V is a nonempty set with two operations (addition, scalar multiplication). in which the following Conditions hold :-

1) $u+v$ in V whenever u, v in V

2) $u+v = v+u$ for all u, v in V.

3) $u+(v+w) = (u+v)+w$ when u, v, w in V

4) There exists an element 0_V in V such that
$$u+0_V = 0_V+u = u \text{ for all } u \text{ in } V$$

5) for all u in V, there exist $(-u)$ in V such that
$$u+(-u) = (-u)+u = 0_V$$

6) $k(u+v) = ku+kv$, u, v in V , k scalar

~~8~~) ku in V whenever $u \in V$ and $\overset{\text{scalar.}}{k} \in \mathbb{R}$

8) $(K_1+K_2)u = K_1u + K_2u$; $u \in V$, K_1, K_2 scalars

9) $K_1(K_2u) = K_1K_2\,u$

10) $1u = u$

جمع ازواج مرتبة

ex) let $V = R^2$

show that $(R^2, +, \cdot)$ is a vector space

ضرب ب زوج مرتب بثابت

answer:-

1) let $u, V \in R^2 \longrightarrow u = (x_1, y_1)$, $V = (x_2, y_2)$

$u + V = (x_1 + x_2 , y_1 + y_2) \in R^2 = V$

2) $(x_1, y_1) + (x_2, y_2) = (x_2, y_2) + (x_1, y_1)$

3) let $w = (x_3, y_3)$, Note that $(u+V) + w = u + (V + w)$

4) Note that $0_V = (0,0) \in R^2$

$0_V + u = (0,0) + (x_1, y_1) = (x_1, y_1) = u + 0_V$

5) if $u = (x_1, y_1) \longrightarrow -u = (-x_1, -y_1) \in V$

Note that $u + (-u) = (x_1, y_1) + (-x_1, -y_1) =$

$(0,0) = 0_V = (-u) + u$

6) $ku = k(x_1, y_1) = (kx_1, ky_1) \in R^2, k \in R.$

7) it is clear that $k(u+v) = ku + kv$

8) $(k_1 + k_2)u = ((k_1+k_2)x_1, (k_1+k_2)y_1) = k_1u + k_2u$

9) $k_1(k_2u) = k_1(k_2x_1, k_2y_1) = (k_1k_2x_1, k_1k_2y_1) = k_1k_2u$

10) $1u = 1(x_1, y_1) = (x_1, y_1)$

---

ex) let $w = \{(x, 1) : x \in R\}$
show that w is not vector space.

1) let u, v in w
$\rightarrow u = (x_1, 1), v = (x_2, 1)$

Now
$u + v = (x_1, 1) + (x_2, 1) = (x_1 + k_2, 2) \notin w$
w is not vector space

اختل الشرط الاول .

Q) let $V = R^2$

Show that $(R^3, +, \cdot)$ is a vector space.

① هسع الزوجط

Note :- $(R^n, +, \cdot)$ is a vector space.

Q_2) is $w = \{(x, x) : x \in R\}$ a vector space ?

Q_3) is $w = \{(0, y) : y \in R\}$ a vector space?

ex) let $V = M_2(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R \right\}$

$\searrow$ 2×2

show that $(V, +, \cdot)$ is a vector space

matrices جمع

ضرب Matrix
ثابت

answer :- let $A, B \in V \rightarrow A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

$B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

1) $A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \in V$

2) Chapter I

3) Chapter 1

4) $0_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5) if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

6) $kA = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix} \in V$

⋮

etc

Note :- $M_n(R)$ is a vector space for any n.

ex) let $w = \left\{ \begin{bmatrix} a & b \\ 2 & c \end{bmatrix} : a, b, c \in R \right\}$

is w a vector space?

answer:-

No, not a vector space.

$0_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin w$

section 4.2 :- Subspaces

* A ~~subspace~~ subset w of a vector space V is called a ~~sup~~ subspace of V if w is itself a vector space under the addition and scalar multiplication defined on V.

vectorspace ⟵ V

w is vectorspace ⟵ w

inside V

so w is subspace

• Therefore:- ~~supsbac~~ subspace is a vector space inside another vector space.

• if w is a nonempty subet of the vector space V, then W is a subspace of V iff the following hold.

1) if w_1 and w_2 in W, then $w_1 + w_2$ in W

↓ الجمع يكون في w.

2) if k is any scalar and $w_1 \in W$, then $kw_1 \in W$

↓

شروط لإثبات ان w عبارة عن subspace

ex) let $W = \{(X_1, 0) ; X \in R\}$

Show that W is subspace of R^2 ?

answer :-

$R^2 \longrightarrow$ is a vector space

1) let $u = U \in W \rightarrow u = (X_1, 0)$, $U = (X_2, 0)$

$u + U = (X_1, 0) + (X_2, 0) \equiv (X_1 + X_2, 0) \in W$

condition 1 holds.

2) let $w \in W \rightarrow w = (a, 0)$

$Kw = K(a, 0) = (Ka, 0) \in W$

Condition 2 holds

then W is a subspace of R^2.

Q_1 : let $W = \{(X, X) : X \in R\}$

is W subspace of R^2 ?

Q_2 : let $W = \{X, 0, Z) : X, Z \in R\}$, is W subspace of R^3 ?

Q_3 : let $W = \{a, b, C, 1) : a, b, C \in R\}$ is W subspace of R^4 ?

ex) let $M = \{(a, b, c) : c = 2a - b, a, b \in R\}$

is M subspace of R^3 ?

answer:-

$R^3 \longrightarrow$ vector space

$M \rightarrow$ subset of R^3

1) let $u, v \in M \rightarrow u = (a_1, b_1, c_1)$ and $c_1 = 2a_1 - b_1$

$v = (a_2, b_2, c_2)$ and $c_2 = 2a_2 - b_2$

Now:-

$u + v = (a_1, b_1, c_1) + (a_2, b_2, c_2) =$

$(a_1 + a_2, b_1 + b_2, c_1 + c_2)$

$c_1 + c_2 = 2a_1 - b_1 + 2a_2 - b_2 = 2(a_1 + a_2) - (b_1 + b_2)$

$u + v \in M$.

2) let $w \in M \rightarrow w = (x, y, z)$ and $z = 2x - y$

Now:-

$kw = k(x, y, z) = (kx, ky, kz)$

and Note that

$kz = k(2x - y) = 2(kx) - ky$

~~kz~~ $kw \in M$

then, M is subspace.

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/ /

ex) let $w = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b, c \in R \right\}$

show that W is a subspace of $M_{22}(R)$

1) let $A, B \in W \rightarrow A = \begin{bmatrix} a_1, & b_1 \\ c_1 & 0 \end{bmatrix}$

$B = \begin{bmatrix} a_2 & b_2 \\ c_2 & 0 \end{bmatrix}$

$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{bmatrix}$

$A + B \in W$

2) $\overset{\nearrow scalar}{K}A = K \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} = \begin{bmatrix} Ka_1 & Kb_1 \\ Kc_1 & 0 \end{bmatrix} \rightarrow KA \in W$

W is a subspace of $M_{22}(R)$

ex) let $W = \{A \in M_{nn}(R) : A \text{ is upper triangular}\}$

Show that W is a subspace of $M_{nn}(R)$

answer:- 1) let $A, B \in W \rightarrow A$ is upper and B is upper

$\rightarrow A+B \in W$

2) let $A \in W \rightarrow A$ is upper

Note that kA is upper

$\rightarrow kA \in W$

then W is a subspace of $M_{nn}(R)$

Q4) let W be the set of all $n \times n$ matrices A such that $A^T = A$

Show that W is subspace of $M_{nn}(R)$

$A^T = A \rightarrow W = \{A \in M_{nn} : A \text{ is symmetric}\}$

/ /

- By $P_n(\mathbb{R})$ we mean the set of all Polynominals of degree n or less with coefficients from $\mathbb{R}$.
  $\hookrightarrow n \geq 0$

ex) ~~P_3~~ $P_2(R)$: The set of all Poly of degree 2 or less with real coefficients.

Now if $P \in P_2(R) \longrightarrow P = a_0 + a_1 x + a_2 x^2$

- if $P \in P_n(R) = P = a_0 + a_1 x \ldots \; a_n x^n$

- $P_n \equiv P_n(R)$
  $M_{nn} \equiv M_{nn}(R)$

Q) let W be the set of all Polynomial of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where $a_1 = a_2$ show that W is a subspace of P_3.

answer:- since P_3 is a vector space we need only to show that the two conditions

1) let $P, q \in W \longrightarrow P = a_0 + a_1 X + a_2 X^2 + a_3 X^3 \;, a_1 = a_2$
$q = b_0 + b_1 X + b_2 X^2 + b_3 X^3 \;, b_1 = b_2$

$P + q = (a_0 + b_0) + (a_1 + b_1) X + (a_2 + b_2) X^2 + (a_3 + b_3)$

$a_1 + b_1 = a_2 + b_2 \longrightarrow P + q \in W$

$KP = K(a_0 + a_1 X + a_2 X^2 + a_3 X^3) = Ka_0 + (Ka_1)X + (Ka_2)X^2 +$
$(Ka_3)X^3$

Note:- $Ka_1 = Ka_2 \rightarrow KP \in \omega$

then, ω is a subspace.

- let $S = \{U_1, U_2, \dots, U_n\}$ be a subset of the vector space V, let $u \in V$, we say u is linear Combination of S if there exist scalar $K_1, K, \dots K_n$ such that :-

$$u = K_1 U_1 + K_2 U_2 + \dots + K_n U_n.$$

ex) let $S = \{(1,0), (0,1)\}$, let $u = (7,5)$

is u L.C of S?

answer!-

Note that :-

$$(7,5) = 7(1,0) + 5(0,1)$$

$\rightarrow u$ is L.C of S

ex) let $S = \{(1,2), (-1,1)\}$ $u = (3,5)$
is u l.c of s?

answer :-

$(3,5) \overset{?}{=} k_1(1,2) + k_2(-1,1)$

→

$3 = k_1 - k_2$

$5 = 2k_1 + k_2$

$8 = 3k_1 \rightarrow k_1 = \frac{8}{3} \rightarrow k_2 = 5 - 2k_1 = 5 - \frac{16}{3} = \frac{-1}{3}$

$\rightarrow (3,5) = \frac{8}{3}(1,2) - \frac{1}{3}(-1,1)$ → u is L.C of S.

---

ex) let $S = \{(1,1), (2,2)\}$, $u = (4,5)$
is u L.C of s?

answer :-

$u = (4,5) \overset{?}{=} k_1(1,1) + k_2(2,2)$ ...

$\rightarrow 4 = k_1 + 2k_2$

$5 = k_1 + 2k_2$

$-1 = 0$

→ No u is not L.C of s.

Q) let $S = \{(1,1,0), (0,1,1), (1,0,1)\}$

$u = (2,5,3)$ is u L.C of S?

answer:-

$$(2,5,3) \overset{?}{=} k_1(1,1,0) + k_2(0,1,1) + k_3(1,0,1)$$

$$\left.\begin{aligned} 2 &= k_1 + k_3 \cdots ① \\ 5 &= k_1 + k_2 \cdots ② \\ 3 &= k_2 + k_3 \cdots ③ \end{aligned}\right\} \rightarrow$$

إذا استطعنا ايجاد الحل
u L.C of S

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix} -r_1 + r_2$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & 0 \end{bmatrix} -r_1 + r_2 \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$k_3 = 0, \; k_2 = 3, \; k_1 = 2$

$\rightarrow (2,5,3) = 2(1,1,0) + 3(0,1,1) + 0(1,0,1)$

$\rightarrow$ u is L.C of s.

Q) let $S = \{(1,2,-1), (2,3,1), (1,3,-4)\}$,

$u = (5,4,7)$ is u L.C of S?

answer:-

$$(5,4,7) \overset{?}{=} k_1(1,2,-1) + k_2(2,3,1) + k_3(1,3,-4)$$

$$5 = k_1 + 2k_2 + k_3$$
$$4 = 2k_1 + 3k_2 + 3k_3$$
$$7 = -k_1 + k_2 - 4k_3$$

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 3 & 3 & 4 \\ -1 & 1 & -4 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & -1 & 1 & -6 \\ 0 & 3 & -3 & 2 \end{bmatrix} \begin{matrix} \\ \to -2r_1+r_2 \\ r_1+r_2 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 6 \\ 0 & 3 & -3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & -16 \end{bmatrix}$$

$\downarrow$

No solution

u is not L.C of S.

Q) let $S = \{ \underset{P_1}{\underset{\downarrow}{2+X+4X^2}}, \underset{P_2}{\underset{\downarrow}{-1X+3X^2}}, \underset{P_3}{\underset{\downarrow}{3+2X+5X^2}} \}$

let $q = 6+11X+6X^2$

show that q is L.C of S

answer :- $q = k_1 P_1 + k_2 P_2 + k_3 P_3$

$\rightarrow 6+11X+6X^2 = k_1(2+X+4X^2) + k_2(1-X+3X^2)$
$+ k_3(3+2X+5X^2)$

$\rightarrow 6 = 2k_1 + k_2 + 3k_3 \longrightarrow$ الحد الثابت

$11 = k_1 - k_2 + 2k_3 \longrightarrow$ X معامل

$6 = 4k_1 + 3k_2 + 5k_3 \longrightarrow$ X^2 معامل

$$\begin{bmatrix} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 11 \\ 2 & 1 & 3 & 6 \\ 4 & 3 & 5 & 6 \end{bmatrix} \begin{matrix} r_2 \\ r_1 \\ \\ \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 3 & -1 & -16 \\ 0 & 7 & -3 & -38 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 1 & -1/3 & -16/3 \\ 0 & 7 & -3 & -38 \end{bmatrix} \tfrac{1}{3} r_2$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 1 & -1/3 & -16/3 \\ 0 & 0 & -2/3 & -2/3 \end{bmatrix} \rightarrow -7r_2 + r_3 \qquad \rightarrow \begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$k_3 = 1, k_2 = -5, k_1 = 4 \longrightarrow q = 4P_1 - 5P_2 + P_3$

. let V be a vector space, let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of V, we say S spans V if every element in V can be written as L.C of S.

$$V = \text{Span}(S)$$

ex) let $S = \{(1,0), (0,1)\}$

is S spans R^2? ⟶ هل تستطيع كتابة كل عناصر R^2 باستخدام عناصر S على شكل تجميع خطي ؟

answer:-

if $u \in R^2 \rightarrow u = (a,b)$

Now

$$(a,b) \overset{?}{=} k_1(1,0) + k_2(0,1)$$

$$a = k_1$$

$$b = k_2$$

$\rightarrow (a,b) = a(1,0) + b(0,1)$.

$\rightarrow$ S spans R^2

ex) let $S = \{(1,1,1), (3,2,-1), (1,0,-3)\}$

is S spans R^3?

answer:-

if $v \in R^3$, $v = (a,b,c)$

$$(a,b,c) \overset{?}{=} k_1(1,1,1) + k_2(3,2,-1) + k_3(1,0,-3)$$

$$\left.\begin{aligned} a &= k_1 + 3k_2 + k_3 \\ b &= k_1 + 2k_2 \\ c &= k_1 - k_2 - 3k_3 \end{aligned}\right\}$$

إذا كان النظام له حل بدون

اي شروط على a,b,c

↓

S spans R^3

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 1 & 2 & 0 & b \\ 1 & -1 & -3 & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & a \\ 0 & -1 & -1 & b-a \\ 0 & -4 & -4 & c-a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 1 & a-b \\ 0 & -4 & -4 & c-a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & 3a-4b+c \end{bmatrix}$$

if: $3a - 4b + c \neq 0$ → يوجد عناصر لا تحقق المعادلة

The system is consistent

→ S Not spans R^3

or $R^3 \neq \text{Span}(S)$ → يوجد عناصر لا تكتب

باستخدام S

section 4.3: Linear independence

- let $S = \{u_1, v_2, \dots, v_n\}$ is a nonempty set of vectors space V, if the equation $k_1v_1 + k_2v_2 \dots + k_nv_n = 0$

ل الصفر الخايل ل V

has only trivial solution, the S is said to be Linearly independents, otherwise S in linearly dependent.

ex) let $S = \{\overset{\rightarrow v_1}{(1,2)}, \overset{\rightarrow v_2}{(-1,3)}\}$

show that S is L.I set in R^2

answer $k_1(1,2) + k_2(-1,3) = (0,0)$

$$k_1 - k_2 = 0$$

$$2k_1 + 3k_2 = 0$$

$$5k_2 = 0 \longrightarrow k_2 = 0, \ k_1 = 0$$

$\rightarrow$ S is L.I

---

ex) let $S = \{(1,2,1), (-1,1,-3), (1,5,-1)\}$

Determine whether S is L.I or L.d in R^3

answer :-

$$k_1(1,2,1), \ k_2(-1,1,-3), \ k_3(1,5,-1)$$

$$k_1 - k_2 + k_3 = 0$$

$$2k_1 + k_2 + 5k_3 = 0$$

$$k_1 - 3k_2 - k_3 = 0$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 5 & 0 \\ 1 & -3 & -1 & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ The system has Non trivial solution → معادلتين و ٣ متغيرات

→ S is L.d

- Let S be a set with 2 elements on more, we say S is L.d if we can write one element in S as a linear Combination of the others.

Also, we say S is L.I if we cannot write any elements in S as L.C of the other elements in S.

Q1) let $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is S L.I set in M_2?

answer :-

$k_1 + k_3 = 0$

$k_1 + k_2 = 0$

$k_3 = 0$

$k_2 + k_3 = 0$

The system has only Trivial solution

→ S is L.I

$k_3 = 0, \; k_1 = 0, \; k_2 = 0$

Q2) let $S = \{x^2+3,\ 2x^2-x+1,\ x+5\}$

is S L.I set in P_2 ?

answer :-

$$k_1(x^2+3) + k_2(2x^2-x+1) + k_3(x+5) = 0$$

$$k_1 + 2k_2 = 0$$

$$-k_2 + k_3 = 0$$

$$3k_1 + k_2 + 5k_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 3 & 1 & 5 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -5 & 5 & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

The system has infinitely many solution

→ S is L.d

$$v_3 = 2v_1 - v_2$$

Q_3: let $S = \{(-3,0,4), (5,-1,2), (1,1,3)\}$
is S L.I set in R^3?

answer :-

$$k_1(-3,0,4) + k_2(5,-1,2) + k_3(1,1,3) = (0,0,0)$$

$-3k_1 + 5k_2 + k_3 = 0$

$-k_2 + k_3 = 0$

$4k_1 + 2k_2 + 3k_3 = 0$

$$A = \begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix} \longrightarrow AX = 0$$

$$\det(A) = -3 \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} + 4 \begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 15 + 24 = 39$$

$\det(A) \neq 0 \rightarrow$ A is invertible $\rightarrow AX=0$ has trivial solution

$\rightarrow$ S is L.I

$\frac{Q_9}{199}$) For which real values of λ do the following vectors form linearly dependent set in R^3

$u_1 = (\lambda, \frac{-1}{2}, \frac{-1}{2})$, $u_2 = (\frac{-1}{2}, \lambda, -\frac{1}{2})$, $v_3 = (-\frac{1}{2}, \frac{-1}{2}, \lambda)$

→

answer:- $A = \begin{bmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{bmatrix}$

$\det(A) = \lambda^3 - \frac{3}{4}\lambda - \frac{2}{8} = 0$

$8\lambda^3 - 6\lambda - 2 = 0 \rightarrow (\lambda-1)(4\lambda+2)(2\lambda+1) = 0$

$\rightarrow \lambda = 1, -\frac{1}{2}$

if $\lambda = 1$ or $-\frac{1}{2}$ → A is singular → $AX=0$

has Non-trivial solution.

section 4.4 : Coordinates and Basis

- let V be a vector space, let $S = \{v_1, v_2, \dots, v_n\}$ be a subset of V. we say S is basis for V iff :-

1) S is linearly independent
2) S spans V

ex) 1) $S = \{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$

2) $S = \{(1,2),(1,1)\}$ is a basis for $\mathbb{R}^2$

3) $S = \{(1,1),(3,3)\}$ is Not a basis for $\mathbb{R}^2$

why? في عنصر ٣ اضعاف

العنصر التاني فمش linear

ex) Let $S = \{(1,0,0),(0,1,0),(0,0,1)\}$

S is basis for $\mathbb{R}^3$

we call this basis standard basis

Q1: what is the standard basis for $\mathbb{R}^4$?

answer:- $S = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$

Q₂) let $S = \{(1,2,1), (1,0,1), (0,1,1)\}$
Show that s a basis for R^3

ex) let $P_0 = 1$, $P_1 = X$, $P_3 = X^2$
$\rightarrow S = \{P_0, P_1, P_2\} = \{1, X, X^2\}$ is a basis for P_2

↳ standard basis for P_2

answer :- Note that:-

① S spans P_2 ~ why?

if $q \in P_2 \rightarrow q = a_0 + a_1X + a_2X^2$

$\rightarrow q = a_0P_0 + a_1P_1 + a_2P_2$
↓

استطعنا كتابة كل عنصر في P_2 باستخدام عناصر S

so → S spans P_2

② S is L.I ?

$K_0P_0 + K_1P_1 + K_2P_2 = 0$

$\rightarrow K_0 + K_1X + K_2X^2 = 0$

$\rightarrow K_0 = 0, K_1 = 0, K_2 = 0$ → trivial solutio
→ s is L.I

→ from ① and ② s is basis for P_2

/ /

let $S = \{1, x, x^2, x^3\}$

↳ standard basis for $P^{\cancel{3}}_{3}$

ex) let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

S is a basis for M_{22}.

we call this basis standard basis of M_{22}

Q5 :- 207

let $S = \left\{ \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \right\}$ show that S is a basis for M_{22}.

basis but not standard

answer :-

1) S is L.I.

$$k_1 \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + k_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$3k_1 + k_4 = 0$, $6k_1 - k_2 - 8k_3 = 0$, $3k_1 - k_2 - 12k_3 - k_4$

$-6k_1 - 4k_3 + 2k_4 = 0$

⟶

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$$A = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -18 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix}$$

$$\det(A) = 3 \begin{vmatrix} -1 & -8 & 0 \\ -1 & -12 & -1 \\ 0 & -4 & 2 \end{vmatrix} - 1 \begin{vmatrix} 6 & -1 & -8 \\ 3 & -1 & -12 \\ -6 & 0 & -4 \end{vmatrix}$$

$$= 3^{*}12 - 1\cancel{0}12 = 48 \neq 0$$

→ A is invertible → $AX = 0$ has trivial solution

→ S L.I

② S spans M_{22}

$A \in M_{22} \longrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{?}{=} k_1 \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + k_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$$

$$+ k_4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

Complete إذا كان الحل يعتمد على a,b,c,d

spans ← consistent او

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* if $S = \{v_1, v_2, ---, v_n\}$ is a basis for vector space V, and $u \in V$ such that $u = c_1v_1 + c_2v_2 + ---- + c_nv_n$.

Then $c_1, c_2, ---, c_n$ are called the coordinate vector of u relative to s.

and we denote this by:
$(u)_s = (c_1, c_2, ---, c_n)$

ex) let $S = \{(1,0), (0,1)\}$ be a basis for $\mathbb{R}^2$:
find $(u)_s$ where $u = (7,3)$

answer:-
Note that: $(7,3) = 7(1,0) + 3(0,1)$
$\rightarrow u_s = (7,3)$

ex) let $S = \{(1,2), (1,1)\}$ be a basis for $\mathbb{R}^2$
find $(u)_s$ when $u = (2,5)$

answer:- $(2,5) = k_1(1,2) + k_2(1,1)$
$\rightarrow 2 = k_1 + k_2$
$- \underline{5 = 2k_1 + k_2}$
$-3 = -k \rightarrow k_1 = 3 \rightarrow k_2 = -1$
$\rightarrow (2,5) = 3(1,2) - 1(1,1) \longrightarrow (u)_s = (3,-1)$

/ /

section 4.5 Dimension

- let V be a vector space, if V has a basis with n elements, then all bases of V have the same number of element which

x) $S = \{(1,0),(0,1)\}$ basis for $\mathbb{R}^2$

$S = \{(1,2),(1,1)\}$ basis for $\mathbb{R}^2$

↓

عدد عناصر S متساوي

let V be a vector space, the dimension of V is defined to be the number of vectors in a basis of V.
f $\dim V = n$ (finite), we say V is finite dimensional ector space.

:) $\dim(\mathbb{R}^2) = 2$, $\dim(R^3) = 3$, $\dim(R^n) = n$
$\dim(P_n) = n+1$, $\dim(M_{nn}) = n^2$, $\dim(M_{mn}) = mn$

Q: find $\dim(P_2)$, $\dim(P_3)$, $\dim(M_{22})$

answer:-

$\dim(P_2) = 3$, $\dim(P_3) = 4$

$\dim(M_{22}) = 4$

Q: let $S_1 = \{(1,2,1),(1,3,5),(1,1,4),(2,2,7)\}$

$S_2 = \{(1,3,5), (6,2,1)\}$

are S_1, S_2 bases for $\mathbb{R}^3$?

answer:-

- S_1 Not basis $\dim(R^3) = 3$ and S_1 have 4 elements
- S_2 Not basis beacause $\dim(R^3) = 3$ and S_2 has 2 elements.

- let V be finite dimensional vector space and let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V

1) if set S has more than n elements, then S L.d

2) if set S has fewer than n elements, then S doesn't span V.

- let V be n-dimensional vector space; let S be a subset from V with n elements. Then S is a basis for V iff S is L.I or S spans V

/ /

ex) let $S = \{(2,3),(1,5)\}$

is S basis for $\mathbb{R}^2$?

answer:- Note that $\dim(\mathbb{R}^2) = 2$ and S has 2 elements

$\longrightarrow$ we need to check one condition for basis

- S L.I :-

$$c_1(2,3) + c_2(1,5) = (0,0)$$

$\longrightarrow$ $3/2c_1 + c_2 = 0$

$-2/3c_1 + 5c_2 = 0$

$$-7c_2 = 0 \rightarrow c_2 = 0 \rightarrow c_1 = 0$$

trivial solution

$\rightarrow$ S L.I

$\rightarrow$ S basis for $\mathbb{R}^2$

- let V be finite dimensional vector space.

let W be subspace of V, then

1) $\dim(W) \leq \dim(V)$

2) if $\dim(V) = \dim(W) \Rightarrow W = V$

How can we find a basis and dimension for Vector space :-

$\longrightarrow$

ex) find a basis for $\mathbb{R}^3$

answer:-

Note that any element in $\mathbb{R}^3$ has the form of (a,b,c) Now:-

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$\longrightarrow \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$

$\rightarrow \dim \mathbb{R}^3 = 3$

لاحظ ان $\dim(V)$ بيساوي

عدد المتغيرات الحرة (free parameters) في V.

ex) find a basis for M_2

answer:-

$A \in M_2 \longrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\dim M_2 = 4$

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\longrightarrow \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

basis for M_2

ex) let $W = \{(a,b,c,d) : a = 2b, c = 3d\}$

find dim(W) and find a basis for W.

# of free parameters = 2

$\dim(W) = 2$

. Note that if $u \in W$, then $u = (2b, b, 3d, d)$

Now:

$(2b, b, 3d, d) = b(2,1,0,0) + d(0,0,3,1)$

$\rightarrow \{(2,1,0,0), (0,0,3,1)\}$ is a basis for W.

Q) let $M = \{(a,b,c) : a+b+c=0\}$

find dim(M) and find a basis for M.

answer:-

$\dim(M) = 2$ (لأن c بتعتمد على a,b
فهو مش free)

Note that if $v \in M \rightarrow v = (a, b, -a-b)$

Now: $(a, b, -a-b) = a(1,0,-1) + b(0,1,-1)$

$\rightarrow \{(1,0,-1), (0,1,-1)\}$ is a basis for M.

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• Section 4.6 : Change of Basis.

• let V be a vectorspace, let $B = \{v_1, v_2, v - - - , v_n\}$ and $\bar{B} = \{u_1, u_2, - - - u_n\}$ be two basis for V.

we know if $w \in V$, then we can write w as :-

$$w = c_1 v_1 + c_2 v_2 + - - - + c_n v_n$$ (باستخدام ال base الأول)

and we can w as :-

$$w = k_1 u_1 + k_2 u_2 + - - - + k_n u_n$$ (باستخدام ال base الثاني)

Note that :- Coordinate vector of w

$$(w)_B = [c_1, c_2, ..., c_n] \text{ or } (w_B) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$(w)_{\bar{B}} = [k_1, k_2, ..., k_n] \text{ or } (w_B) = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

↓

Coordinate vector

باستخدام ال base الثاني

• is there a relation between $(w)_B$ and $(w)_{\bar{B}}$?

yes, $w_B = P_{\bar{B} \to B} (W)_{\bar{B}}$ (transition matrix from $\bar{B}$ to B)

$w_{\bar{B}} = P_{B \to \bar{B}} (w)_B$ (transition matrix from B to $\bar{B}$)

ex) let $B = \{(1,0) ; (0,1)\}$ and $\bar{B} = \{(1,1), (1,2)\}$ be two bases for R^2, find $P_{\bar{B} \to B}$ and $P_{B \to \bar{B}}$

1) $P_{\bar{B} \to B}$:

$$[B : \bar{B}] \xrightarrow{\text{Gauss.e.}} [I : P_{\bar{B} \to B}]$$

2) $P_{B \to \bar{B}}$

$$[\bar{B} : B] \longrightarrow [I : P_{B \to \bar{B}}]$$

1) $$[B : \bar{B}] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \to P_{\bar{B} \to B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

2) $$[\bar{B} : B] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$P_{B \to \bar{B}} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{\text{Note that: } (P_{B \to \bar{B}})^{-1} = P_{\bar{B} \to B}} \to \text{Rule.}$$

ex) let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\bar{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ be two bases for R^2, let $u = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, find $(u)_B$ and $(u)_{\bar{B}}$ and explain the relation between them.

1) u_B : $\begin{bmatrix} 3 \\ 5 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\rightarrow k_1 = 3, \; k_2 = 5 \rightarrow (u)_B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

2) $u_{\bar{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\left.\begin{aligned} 3 &= c_1 + c_2 \\ 5 &= c_1 + 2c_2 \end{aligned}\right\} \longrightarrow c_1 = 1, \; c_2 = 2, \; (u)_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- if P is the transition matrix from a basis $\bar{B}$ to a basis B for finite dimensional vector space V, then P' is the transition matrix from B to $\bar{B}$.

__/__/__

Q) let $S = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\5\end{bmatrix} \right\}$ and

موجود ≡ $S_2 = \{v_1, v_2, v_3\}$ be a basis for R^3, if

$u = \begin{bmatrix}12\\1\\19\end{bmatrix}$, find $(u)_{S_2}$ given that $P_{S_1 \longrightarrow S_2} = \begin{bmatrix}1&2&1\\-1&1&3\\1&4&2\end{bmatrix}$

answer :-

find $(u)_{S_1}$

$$(u)_{S_2} = P_{S_1 \longrightarrow S_2}(u)_{S_1}$$

to find $(u)_{S_1}$:-

$$\begin{bmatrix}12\\1\\19\end{bmatrix} = k_1\begin{bmatrix}1\\2\\3\end{bmatrix} + k_2\begin{bmatrix}1\\0\\2\end{bmatrix} + k_3\begin{bmatrix}3\\-1\\5\end{bmatrix}$$

$\longrightarrow k_1 = 2,\ k_2 = -1,\ k_3 = 3$

$\longrightarrow (u)_{S_1} = \begin{bmatrix}2\\-1\\3\end{bmatrix}$

$\longrightarrow (u)_{S_2} = \begin{bmatrix}1&2&1\\-1&1&3\\1&4&2\end{bmatrix}\begin{bmatrix}2\\-1\\3\end{bmatrix} = \begin{bmatrix}3\\6\\4\end{bmatrix}$

Q) if $B = \{u_1, u_2\}$ and $\bar{B} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \end{bmatrix} \right\}$ are two bases for R^2, let $w \in \mathbb{R}^2$ such that $(w)_B = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. if $P_{\bar{B} \to B} = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$, find $(w)_{\bar{B}}$ and find w.

the set of all Poly. of degree 1 or less. $\dim(P_1) = 1+1 = 2$

Q 10 / 223) Consider the basis $B = \{P_1, P_2\}$ and $\bar{B} = \{q_1, q_2\}$ for P_1 where $P_1 = 1+2x$, $P_2 = 3-x$, $q_1 = 2-2x$, $q_2 = 4+3x$

a) find transition matrix from $\bar{B}$ to B

b) find transition matrix from B to $\bar{B}$

a) to find $P_{\bar{B} \to B}$ we need to find $(q_1)_B$ and $(q_2)_B$

$(q_1)_B$: $2 - 2x = k_1(1+2x) + k_2(3-x)$

$\to 2 = k_1 + 3k_2$

$3 / -2 = 2k_1 - k_2$

$-4 = 7k_1 \to k_1 = \frac{-4}{7}$, $k_2 = \frac{6}{7} \to (q_1)_B = \begin{bmatrix} -4/7 \\ 6/7 \end{bmatrix}$

- $(q_2)_B$

$4 + 3x = c_1(1+2x) + c_2(3-x)$

$4 = c_1 + 3c_2$

$3/3 = 2c_1 - c_2$

$13 = 7c_1 \to c_1 = 13/7$, $c_2 = 5/7$.

/ /

Section 4.7) Row Space, Column Space, and Null space

, let V be a vector space, $S = \{v_1, v_2, \cdots v_n\}$ is a subset of V, then:

$\text{span}(S) = \{k_1 v_1 + k_2 v_2 + \cdots + k_n v_n : k_1, k_2, \cdots, k_n \in R\}$

Note that span(S) is a subspace of V - why?

let $W = \text{span}(S)$

1) if $w_1, w_2 \in W \rightarrow w_1 = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$

$w_2 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

$\rightarrow w_1 + w_2 = (k_1 + c_1) v_1 + (k_2 + c_2) v_2 + \cdots (k_n + c_n) v_n \in W$

2) $k w_1 = k k_1 v_1 + k k_2 v_2 + \cdots + k k_n v_n \in W$

$\rightarrow$ from ① and ② W is a subspace of V.

$$\longrightarrow \text{let } A= \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{array}\right] \begin{array}{l} \to r_1 \\ \to r_2 \\ \\ \to r_m \end{array} \Bigg\} \to$$ صفوف

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ c_1 & c_2 & c_n \end{array}$$

اعمدة $\longleftarrow$ c_1 c_2 c_n

Then :-

1) row space = Span $\{r_1, r_2, \cdots, r_m\}$

2) Column space = Space $\{c_1, c_2, \cdots, c_n\}$

3) null space = The solution of the system $AX = 0$

ex) let $A = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 2 & -1 & 1 & 3 \\ 3 & 1 & 1 & 6 \end{bmatrix}$, then

1) row space = span $\{(1,2,5,4), (2,-1,1,3), (3,1,1,6)\}$

2) Column Space = span $\{(1,2,3), (2,-1,1), (5,1,1), (4,3,6)\}$

3) null space = the solution of the system

$$x_1 + 2x_2 + 5x_3 + 4x_4 = 0$$

$$2x_1 - x_2 + x_3 + 3x_4 = 0$$

$$3x_1 + x_2 + x_3 + 6x_4 = 0$$

$x_1 =$ ___ , $x_2 =$ ___ , $x_3 =$ ___ , $x_4 =$ ___ "chapter 1"

Note that : row space is a subspace of $\mathbb{R}^4$
Column space is a subspace of $\mathbb{R}^3$

ex) let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

find basis for each of row space, Column space, null space.

answer:-

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) $\{(1,4,5,2),(0,1,1,4/7)$ basis for row space

2) $\{(1,2,-1),(4,1,3)\}$ basis for Column space

لـ الأعمدة التي تحوي leading.

3) to find null space, we solve $AX=0$

$$\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$x_4 = t,\ x_3 = s.\ s,t \in R$

$x_2 + x_3 + 4/7\, x_4 = 0$

$x_1 + 4x_2 + 5x_3 + 2x_4 = 0$

$\rightarrow x_2 = -s - 4/7\, t$

$x_1 = 4(-s - 4/7\, t) - 5s - 2t$

$x_1 = -9s - 30t$

$\rightarrow$ null space $= \left\{\left(-9s - \frac{30}{7}t, -s - 4/7\, t, s, t\right) : t, s \in R\right\}$

basis for null space $= \left\{(-9,-1,1,0), \left(-\frac{30}{7}, -4/7, 0, 1\right)\right\}$

Section 4.8) Rank, Nullity.

- The $\overset{\text{vector space}}{\underline{\text{row space}}}$ and $\overset{\text{vector space}}{\underline{\text{column space}}}$ of the Matrix A have the same dimension

dim(row space) = dim(column space)

- let A be a matrix, then
1) rank(A) = dim(row space)
2) nullity(A) = dim(null space)

- if A is a matrix with n columns, then
rank(A) + nullity A = n

- rank(A) = rank(A^T)

- let A be a matrix, then
  1) rank(A) = # of leading in the matrix result when we write A in r.e.f
  2) nullity (A) = # of the free parameters in the solution of the system AX = 0.

/ /

Q) let A be 5X3 matrix, if rank(A) = 2
Find nullity of A and nullity of A^T

- rank (A) + nullity (A) = 3

$2 + \boxed{} = 3$

nullity (A) = 1

- rank (A^T) + nullity (A^T) = nullity of columns in A^T

→ $2 + \boxed{} = 5$

→ nullity (A^T) = 3

Q) if A is 3X6 matrix, what is the largest value for rank(A) ?

answer:-

$\boxed{3}$ → rank(A) = # of leading and A has at most 3 leading.

Q) if A is 6X3 matrix, what is the largest value for rank(A) ?

answer:- 3 → rank(A) = rank(A^T) and A^T has at most 3 leading.

Chapter 5 section 5.1 Eigenvalues and Eigenvectors

- let A be $n \times n$ matrix, then X is called eigenvector of A if there exist scalar λ such that: $AX = \lambda X$.

In this case λ is called eigenvalue of A and X is called eigenvector corresponding to λ, $X \neq 0$

- If A is an $n \times n$ matrix, then λ is an eigenvalue of A iff:-

$$\det(\lambda I - A) = 0$$

we call this characteristic equation of A.

Ex) let $A = \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix}$, find eigen values of A

- $\lambda I - A = \begin{bmatrix} \lambda-3 & -2 \\ 0 & \lambda-5 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-3)(\lambda-5) = 0$

$\lambda = 3, 5$ are eignen values of A.

Q) let $A = \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$ find the eigenvalues of A

$\lambda I - A = \begin{bmatrix} \lambda-1 & -5 \\ -2 & \lambda+2 \end{bmatrix}$ $\det(\lambda I - A) = ((\lambda-1)(\lambda+2)-10)$

$= \lambda^2 + \lambda - 8 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1+32}}{2}$$

- How we can find the eigenvector of A :-
  - Note that we want to find $X \neq 0$ such that:

identity

$$Ax = \lambda x$$

$$\rightarrow \lambda I X - A X = 0$$

$$(\lambda I - A)\, X = 0$$

So, this mean we need to find the solution of the system $BX = 0$ where $B = \lambda I - A$

- Note Also that the system $BX=0$ has nontrivial solution iff B is singular ($\det B = 0$)

ل حل غير صفري $\det(\lambda I - A) = 0$ ل

ex) find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$$

answer:- $\lambda I - A = \begin{bmatrix} \lambda - 2 & -1 \\ 0 & \lambda - 4 \end{bmatrix}$

$$\det(\lambda I - A) = (\lambda - 2)(\lambda - 4)$$

$\lambda = 2, 4$ an eigenvalues of A.

for $\lambda=2$:-

we need to find the solution of the system
$(\lambda I-A)X=0$ where $\lambda=2$

$$\left[\begin{array}{cc:c} 0 & -1 & 0 \\ 0 & -2 & 0 \end{array}\right]$$

$x_1=t$, $x_2=0$

$$X=\begin{bmatrix} t \\ 0 \end{bmatrix} \longrightarrow X=t\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ is eigenvector of A

for $\lambda=4$

$$\left[\begin{array}{cc:c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$$ تعويض ٤ مكان λ

$2x_1-x_2=0$

$x_1=t \rightarrow x_2=2t$

$$\rightarrow X=\begin{bmatrix} t \\ 2t \end{bmatrix} \longrightarrow X=t\begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ is eigenvector of A

ex) find eigenvalue and eigenvectors of

$$A=\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$\xrightarrow{\text{answer}}$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda-2) \begin{vmatrix} \lambda & 2 \\ -1 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-2)(\lambda^2 - 3\lambda + 2)$$
$$= (\lambda-2)(\lambda-2)(\lambda-1) = 0$$

$\lambda = 2, 2, 1$ are eigenvalues of A.

- To find eigenvectors

for $\lambda = 1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$x_1 + 2x_3 = 0$

$x_3 = t$, $x_2 = t$, $x_1 = -2t$

$$X = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \longrightarrow \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$ eigenvector corresponding to $\lambda = 1$

/ /

Q) let $A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$, find eigenvalues and eigenvectors of A

Q) let $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 4 \end{bmatrix}$, find eigenvalus " " " of A.

. Some notes of characteristic equation

ex) let $P(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$ be the characteristic equation of matrix A.

1) find det(A)

2) Is 0 eigenvalue of A

answer:-

Note that $\det(\lambda I - A) = P(\lambda) = \lambda^3 - 2\lambda^2 + 5$

2) $P(0) = 5 \neq 0 \Rightarrow 0$ is not eigenvalue of A

1) $\det(\lambda I - A) = \lambda^3 - 2\lambda^2 + 5$

$\rightarrow \det(0I - A) = 0 - 2\mathring{0} + 5$

$\rightarrow \det(-A) = 5$

$\rightarrow (-1)^3 \det(A) = 5$

$\det(A) = -5$

- A square matrix A is invertible iff $\lambda=0$ is not eigenvalue of A

- if A is diagonal, upper, lower, then the eigenvalues of A are the entries on the main diagonal

- let $A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 6 & -1 & 2 & 7 \end{bmatrix}$, find eigenvales of A

$\lambda = 1, 3, 5, 7$

invertible or not

- Notes
  1) if λ eigenvalue of A, then λ^k is eigenvalue of A^k $(k>0)$
  2) if λ eigenvalue of A, then $\lambda+c$ is eigenvalue of $A+cI$ → constant
  3) if A is invertible and λ is eigenvalue of A, then $1/\lambda$ is eigenvalue of A^{-1}

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/ /

Section 5.2

- if A and B are square matrices, then we say B is Similar to A, if there is an invertible matrix P such that $B = P^{-1}AP$

- some Properties for similar matrices

  if A and B are similar, then

1) A and B have same determinates.
2) A and B have same trace
3) A and B have same eigenvalues
4) A and B have same rank and nullity
5) A and B have same characteristic equation

Proof:-

1) B similar to A → $B = P^{-1}AP$ for some matrix P

$\to \det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(A)$

2) $B = P^{-1}AP \to tr(B) = tr(P^{-1}AP)$

$= tr(PP^{-1}A)$

$= tr(IA) = tr(A)$

- The square matrix A is said to be diagonaltable if it similar to diagonal matrix

There exist D, P such that :

$D = P^{-1}AP$

where D is diagonal

- How we can know if A is diagonalizable ?

- If A is nxn matrix, then A is diagonalizable iff A has n linearly independent eigenvectors.

ex) let $A = \begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$, is A diagonalizable ? if yes, find D and p such that

$D = P^{-1}AP$

answer:- $\lambda I - A = \begin{bmatrix} \lambda - 3 & -5 \\ 0 & \lambda - 4 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda - 3)(\lambda - 5) = 0 \rightarrow \lambda = 3, \lambda = 5$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$ eigenvector for $\lambda = 3$

$\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ eigenvector for $\lambda = 4$

→

- Note that A is 2X2 and A have 2 eigenvectors

$\longrightarrow$ A is diagonalizable

. To find D, P

$$P=\begin{bmatrix}1&5\\0&1\end{bmatrix},\quad D=\begin{bmatrix}3&0\\0&4\end{bmatrix}$$

$\begin{bmatrix}1\\0\end{bmatrix}$ for $\lambda=3$

$\begin{bmatrix}5\\1\end{bmatrix}$ for $\lambda=4$

ex) let $A=\begin{bmatrix}2&1\\0&2\end{bmatrix}$, is A diagonalizable? if yes, find P and D such $D=P^{-1}AP$

answer:-

$$\lambda I-A=\begin{bmatrix}\lambda-2&-1\\0&\lambda-2\end{bmatrix}$$

$\det(\lambda I-A)=(\lambda-2)(\lambda-2)=0 \rightarrow \lambda=2,2$ eigenvalues

. To find eigenvectors for $\lambda=2$

$$\left[\begin{array}{cc|c}0&-1&0\\0&0&0\end{array}\right]\longrightarrow x_2=0,\ x_1=t\longrightarrow\begin{bmatrix}1\\0\end{bmatrix}\text{ eigenvector}$$

A is not diagonalizable because it is 2X2 ex) let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, is A diagonalizable? if yes, find P and D such that $D = P^{-1}AP$

answer:-

$\lambda = 2, 2, 1$

for $\lambda = 1 \rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is eigenvector

for $\lambda = 2 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ eigenvector

Note that A is 3X3, and A have 3 eigenvectors

→ A is diagonalized

$P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$A^{10} = P D^{10} P^{-1}$

$\searrow$

$D^{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 2^{10} \end{bmatrix}$

Diagonal lé's 8 ↲

مختلف

• if A is $n \times n$ matrix and A have n distinct eigenvalues, then A is diagonalizable → العكس غير صحيح

Q) let $A = \begin{bmatrix} 1 & 3 & 5 & 1 \\ 0 & 2 & 7 & -1 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

is A diagonalizable?

answer:- yes

$\lambda = 1, 2, 8, 7$ are eigenvalues of A

A is 4×4 and have 4 eigenvalues.

• إذا كانوا 1,2,2,7 ← النظرية تفشل.

• let λ_0 be eigenvalue of A, then:-

1) if $V_1, V_2, \dots V_n$ are eigenvectors corresponding to λ_0, then $W_0 = \text{Span}\{V_1, V_2, \dots, N_n\}$ is called eigenspace corresponding to λ_0

$\dim(W_0) = n$ G.m

2) $\dim(W_0)$ is called geometric multiplicity for λ_0

3) The # of times that $(\lambda - \lambda_0)$ appears as a factor ine $\det(\lambda I - A)$ is called Algebric multiplicity for λ_0 A.m

4) if G.m = A.m then A is diagonalizable.

section 6.1 :- Inner products

An inner product on a real vector space V is a function that associates a real number $\langle u,v \rangle$ with each pair of vectors in V in such way the following conditions hold :-

1) $\langle u,v \rangle = \langle v,u \rangle$ for all u,v in V

2) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all u,v,w in V

3) $\langle ku, v \rangle = k \langle u,v \rangle$, $u,v \in V$, k is scalar.

4) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

العنصر المحايد ل V

ex) let $V = R$, u,v in V such that

$f(u,v) \leftarrow$ $u = (u_1, u_2)$, $v = (v_1, v_2)$ and

$\langle u,v \rangle = u_1 v_1 + u_2 v_2 \longrightarrow$ قاعدة الاقتران

Show that $\langle -,- \rangle$ is inner product

التحقق من الشروط:

1) $\langle u,v \rangle = u_1 v_1 + u_2 v_2 = v_1 u_1 + v_2 u_2 = \langle v,u \rangle$

2) let $w \in R^2 \rightarrow w = (w_1, w_2)$

$u + v = (u_1 + v_1, u_2 + v_2)$

$\langle u+v, w \rangle = (u_1+v_1) w_1 + (u_2 + v_2) w_2$

$u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2$

$\langle u, w \rangle + \langle v, w \rangle$

3) $ku = (ku_1, ku_2) \rightarrow \langle ku, v \rangle = k u_1 v_1 + k u_2 v_2 = k(u_1 v_1 + u_2 v_2) =$

$k \langle u, v \rangle$

4) $\langle v,v\rangle = v_1^2 + v_2^2 \geq 0$

if $\langle v,v\rangle = 0 \rightarrow v_1^2 + v_2^2 = 0 \rightarrow v_1 = 0$ and $v_2 = 0$

but $v = \langle v_1, v_2\rangle$

$v = (0,0)$

$\rightarrow \langle -,-\rangle$ is I.P → inner Product

ex) let $V = R^3$, $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$

if $\langle u,v\rangle = u_1v_1 + u_3v_3$

is $\langle -,-\rangle$ I.P ?

answer:-

No

Note that if $u = (0,1,0)$, then $\langle u,u\rangle = 0$

$\rightarrow u \neq 0_V$ but $\langle u,u\rangle = 0$ ← الشرط الرابع لم يتحقق

ex) let $V = P_2$, $p, q \in V$, $p = a_0 + a_1x + a_2x^2$, $q = b_0 + b_1x + b_2x^2$

$\langle p,q\rangle = a_0b_0 + a_1b_1 + a_2b_2$

show that $\langle -,-\rangle$ is I.P

answer:

1) ✓

2) ✓

3) $\langle kp, q\rangle = ka_0b_0 + ka_1b_1 + ka_2b_2 = k(a_0b_0 + a_1b_1 + a_2b_2) =$

$k\langle p,q\rangle$

4) $\langle p,p\rangle = a_0^2 + a_1^2 + a_2^2 \geq 0$

if $\langle p,p\rangle = 0 \rightarrow a_0 = a_1 = a_2 = 0 \rightarrow p = 0$

$\rightarrow \langle -,-\rangle$ is I.P

∘ let V be inner product space, u, v, w are elements in V, k is scalar, then:-

1) $<0, v> = <v, 0> = 0$

↳ عنصر محايد

2) $<u, v+w> = <u, v> + <u, w>$

3) $<u, kv> = k(u, v)$

4) $<u-v, w> = <u, w> - <v, w>$

5) $<u, v-w> = <u, v> - <u, w>$

∘ Let V be inner product space, let V, u in V we define:-

1) $\|u\|^2 = <u, v>$, $\|u\|$ is called the $\underline{\overset{\text{length}}{\text{norm}}}$ of u

2) $d(u, v) = \|u - v\|$

↳ distance between u and v

Note that

1) $\|u\| \geq 0 \quad \forall u \in V$

2) $\|ku\| = |k| \|u\|$, k scalar

↳ قيمة مطلقة.

/ /

Q) Let $\langle u, v \rangle$ be the euclidean inner product on $\mathbb{R}^2$ and let $u=(1,1)$, $v=(3,2)$, $w=(-1,)$, $k=5$

Compute :- $\langle v, w \rangle$, $\|u\|$, $d(u,v)$, $\langle ku, v \rangle$

answer :-

- $\langle v, w \rangle = -3+0=-3$
- $\|u\|^2 = \langle u, u \rangle = 1+1=2$

$\rightarrow \|u\| = \sqrt{2}$

- $d(u,v) = \|u-v\|$

$u-v = (-2,-1)$

$\|u-v\|^2 = \langle u-v, u-v \rangle = -2 \times -2 + -1 \times -1 = 5$

$\rightarrow d(u,v) = \sqrt{5}$

- $\langle ku, v \rangle = k \langle u, v \rangle$

$\langle u, v \rangle = 3+2=5$

$\rightarrow \langle ku, v \rangle = 5 \times 5 = 25$

Q) if $V=\mathbb{R}^2$, $u, v \in \mathbb{R}^2$, $u=(u_1,u_2)$, $v=(v_1,v_2)$ such that $\langle u, v \rangle = 2u_1v_1+3u_2v_2$.

find $\|w\|$ when $w=(4,-2)$

answer :-

$\|w\|^2 = \langle w, w \rangle = 2 \times 4 \times 4 + 3 \times -2 \times -2 = 44$

$\rightarrow \|w\| = \sqrt{44}$

Q) let $V = P_2$, if $\langle -,- \rangle$ is the inner in Page 3 find $\|p\|$, $\langle p,q \rangle$ where

$p = 3 - 2x + 5x^2$, $q = 5 - 2x^2$

answer :-

- $\|p\|^2 = \langle p,p \rangle = 3 \cdot 3 + -2 \cdot -2 + 5 \cdot 5 = 38$

$\longrightarrow \|p\| = \sqrt{38}$

- $\langle p,q \rangle = 3 \cdot 5 + -2 \cdot 0 + 5 \cdot -2 = 5$

Q)) if $\|u\| = 4$, $\|v\| = 3$, $\langle u,v \rangle = 5$

find $\|2u + 5v\|$

answer :-

$\|2u + 5v\|^2 = \langle 2u + 5v, 2u + 5v \rangle$

$= \langle 2u, 2u \rangle + \langle 2u, 5v \rangle + \langle 5v, 2u \rangle + \langle 5v, 5v \rangle$

$= 4 \langle u,u \rangle + \underline{10 \langle u,v \rangle + 10 \langle v,u \rangle} + 25 \langle v,v \rangle$

$= 4 \|u\|^2 + 20 \langle u,v \rangle + 25 \|v^2\|$

$= 4 \cdot 16 + 20 \cdot 5 + 25 \cdot 9 = 239 \rightarrow$

$\longrightarrow \|2u + 5v\| = \sqrt{235}$

$\langle u,v \rangle = \langle v,u \rangle$

Section 6.2 :- Angle and orthogonallity in inner product space:-

∘ (Cauchy-Schwarz inequality)

let V be inner product space, let u,v be vectors in V, then:-

$$|\langle u,v\rangle| \le \|u\| \|v\|$$

Q) Can you find an inner product on vector space V such that $\langle u,V\rangle = 8$, $\|u\| = 2$ and $\|V\| = 3$

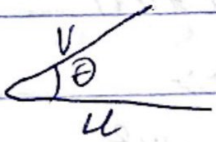
answer:-

No

Note that $8 \not\le 2 * 3$

∘ let V be real inner product space, let u and V be two vectors in V, we define the angle θ between u and V by:-

$$\cos\theta = \frac{\langle u,V\rangle}{\|u\| \|v\|}$$



ex) let $\langle u,V\rangle$ be the euclidean inner product on $\mathbb{R}^3$, if $u=(2,-1,1)$, $V=(3,2,4)$, find θ

answer:-

$$\cos\theta = \frac{\langle u,V\rangle}{\|u\| \|v\|} = \frac{6-2+4}{\sqrt{6}\ \sqrt{29}} = \frac{8}{\sqrt{174}}$$

- if u, v and w are vectors in real inner product space V, then :
  1) $||u+v|| \leq ||u|| + ||v||$
  2) $d(u,v) \leq d(u,w) + d(w,v)$

- let V be real inner product space, let u,v be vectors in V, we say u and v are orthogonal iff $\langle u,v \rangle = 0$ متعامد

ex) let $P = a_0 + a_1X + a_2X^2$, $q = b_0 + b_1X + b_2X^2$ in P_2 if $\langle P, q \rangle = a_0b_0 + a_1b_1 + a_2b_2$ is inner product on P_2. let $P_1 = 2 - 3X + X^2$, $P_2 = 5 + 6X + 8X^2$ are P_1 and P_2 orthogonal?

ex) if $||u|| = 3$, $||v|| = 4$ and u,v orthogonal, find $||u+v||$

answer:-

$$||u+v||^2 = \langle u+v, u+v \rangle = ||u||^2 + 2\langle u,v \rangle + ||v||^2$$
$$= 9 + 0 + 16$$
$$= 25$$

$\rightarrow ||u+v|| = 5.$

/ /

. let $S = \{V_1, V_2, \dots, V_n\}$ be subset of inner product space V, we say S is orthogonal set iff $\langle V_i, V_j \rangle = 0$ for $i \neq j$

ex) let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$
S is an orthogonal set in $\mathbb{R}^3$ ↓

كل عنصر داخله عمودي على جميع العناصر الأخرى داخل S .

. let $S = \{V_1, V_2, \dots V_n\}$ be subset of inner products space V, we say S is an orthogonal set iff the following hold.

1) $\langle V_i, V_j \rangle = 0$ for $i \neq j$, $i, j = 1, 2, \dots, n$

2) $\langle V_i, V_i \rangle = 1$ for $i = 1, 2, \dots, n$

ex) let $S = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$
is S an orthogonal set?

answer:-

$\langle V_1, V_2 \rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} = 0$

$\langle V_1, V_1 \rangle = \frac{1}{2} + \frac{1}{2} = 1$ → مربع الأول + مربع الثاني

$\langle V_2, V_2 \rangle = \frac{1}{2} + \frac{1}{2} = 1$

Yes

o let $\bar{V}$ be inner product space , let $\bar{W}$ be subspace of $\bar{V}$, we define:

$$\bar{W}^{\perp} = \{u \in \bar{V} : <u,w> = 0 \text{ for all } w \in \bar{W}\}$$

↳ orothognal complement of w

⊛ How we can find $w^{\perp}$?

ex) let $\bar{W} = \{(a,b,c,d) : c = 2a+b , d = 4a\}$
find $w^{\perp}$ and find a basis for $w^{\perp}$

answer:-

if $w \in \bar{W} \Rightarrow w = (a,b,2a+b,4a)$

$(a,b,2a+b,4a) = a(1,0,2,4) + b(0,1,10)$

$\rightarrow \{(1,0,2,4),(0,1,1,0)\} \rightarrow$ basis for $\bar{W}$

$$\left[\begin{array}{cccc:c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$x_4 = t$, $x_3 = 5$, $x_2 = -5$, $x_1 = -25-4t$

$\rightarrow$ solution $= \{(-25-4t,-5,5,t) : t,s \in R\}$

Chapter 8) section 8.1 General linear transformations

$\downarrow$

linear map

if $T: V \rightarrow W$ is a function from vector space V to vector space W, then we say T is linear transformation iff the following hold

1) $T(u+v) = T(u) + T(v)$ for all u, v in V
2) $T(ku) = k\,T(u)$

if $V = W$, then T is called linear operator.

ex) let $T: M_n \longrightarrow \mathbb{R}$ such that $T(A) = tr(A)$

show that T is linear transformation.

answer:-

let $A, B \in M_n$

1) $T(A+B) = tr(A+B) = tr(A) + tr(B) = T(A) + T(B)$
2) $T(kA) = tr(kA) = k\,tr(A) = kT(A)$

$\rightarrow$ T is linear transformation.

ex) let V be any vector space, let $T: V \rightarrow V$ such that: $T(u) = u$ for all $u \in V$, then T is linear Transformation, we call T the identity operator on V.

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ex) if $T : V \to W$ such that $T(u) = 0_V$, then T is called zero transformation.

ex) let $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x,y) = (2x+y, y-x)$

show that, T is linear transformation

answer :-

let u, V in $\mathbb{R}^2 \to u = (x_1, y_1)$, $V = (x_2, y_2)$

$u + V = (x_1 + x_2, y_1 + y_2)$

1) $T(u+V) = (2(x_1+x_2) + y_1 + y_2, (y_1 + y_2) - (x_1+x_2))$

$= (2x_1 + y_1 + 2x_2 + y_2, y_1 - x_1 + y_2 - x_2)$

$= T(u) + T(V)$

2) $ku = (kx_1, ky_1)$

$T(ku) = (2kx_1 + ky_1, ky_1 - kx_1) = k(2x_1 + y_1, y_1 - x_1)$

$= kT(u)$

$\to$ T is linear transformation

- let $T : V \to W$ be a linear Transformation then:-

1) Range (T) $= \{Tv : v \in V\}$

2) Kernal (T) $= \{v \in V : Tv = 0_w\}$

ex) let $T: R^2 \longrightarrow R^3$ such that

$T(x,y) = (x+y, x, 2x-y)$ be a linear transformation

is $u = (5,2,1)$, $V = (3,1,4)$ in Range (T)

answer :-

$u = (5,2,1)$:

$x + y = 5$

$x = 2$

$2x - y = 1$

$x = 2$, $y = 3$, Now $2(2) - 3 = 1$

$\rightarrow (5,2,1) \in$ Rang (T)

2) $(x+y, x, 2x-y) = (3,1,4)$

$x+y = 3$, $x = 1$, $2x - y = 4$

$y = 2$, $2(1) - 2 = 0 \neq 4$

$\rightarrow (3,1,4)$ not in range (T)

NoTe : if $T: V \rightarrow W$ is linear transformation, then

1) kar(T) is a subspace of V

2) range (T) is a subspace of W

3) dim(range (T)) = rank (T)

4) dim ($\operatorname{Ker}(T)$) = nullity (T)

5) rank (T) + nullity (T) = dim(V)

Section 8.2: Isomorphism

- if $T: V \rightarrow W$ is a linear transformation then T is said to be one-to-one if T maps distinct vectors in V into distinct vectors in W.

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- if $T: V \rightarrow \underline{\underline{W}}$ is a linear transformation then T is said to be onto if every vector in W is the image of at least one vector in V

- if $T: V \rightarrow W$ is a linear transformation, then T is 1-1 iff $\ker(T) = \{0\} \curvearrowright \{0_V\}$
↓
one-to-one

- Note:- if $T: V \rightarrow W$ is a linear transformation then T is onto iff $\text{range}(T) = W$

Q True or false

if $T: V \rightarrow V$ is 1-1 linear map, then T is onto.

answer:-

yes, $\dim(V) = \dim(\ker T) + \dim(\text{range}\,T)$

· ~~If the~~ if the linear transformation $T: V \rightarrow W$ is both 1-1 and onto, then T is said to be Isomorphism, and the vector spaces V and W are said to be isomorphic.

ex) let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ Such that $T(x,y) = (x-y, 2x+3y)$

Show that T is an isomorphism.

answer:-

we need to show that T is 1-1 and onto, to show that T is 1-1, we find $\ker T$

$T(x,y) = (0,0)$

$\rightarrow (x-y, 2x+3y) = (0,0)$

$\rightarrow x-y = 0$

$2x+3y = 0$

$\rightarrow$ T is 1-1 by theorem 1

but :-

$\dim \mathbb{R}^2 = \dim \ker T + \dim \text{range}\, T$

$2 = 0 + \square$

$\rightarrow \dim \text{range}\, T = \dim \mathbb{R}^2 \rightarrow T$ is onto

$\rightarrow$ T is isomorphism.