LINEAR ALGEBRA

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POWERUNIT

Subject : .... / /

[Inner Product Spaces]

[Definition]

let V be a vector space, let $\langle \ , \ \rangle$ be a map $V \times V \to R$, then V is called an Inner product space under the map $\langle , \rangle$ if the following is satisfied:-

→ $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$

→ $\langle ku, v \rangle = k\langle u, v \rangle$ for all $u, v \in V$

$\langle u, kv \rangle = k\langle u, v \rangle$

→ $\langle u \mp w, v \rangle = \langle u, w \rangle \mp \langle u, v \rangle$

$\langle v, u \mp w \rangle = \langle v, u \rangle \mp \langle v, w \rangle$

→ $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if $v = 0$

Subject :

عشان حتى تفهم

examples: $R^2 = \{(x, y) \quad , x, y \in R\}$

زي ما كنا نثبت ال general vector spaces هون راح يعطينا الصيغة الي

بنجرب فيها mapping وبنعوض بالشروط عليها

لو في السؤال راح تكون صيغة ال mapping الأصلية ونحنا مطلوب منا نثبتها

→ given that $\langle (a_1, a_2), (b_1, b_2) \rangle = a_1 b_1 + a_2 b_2$

the way of mapping ✓

$\langle u, v \rangle = \langle v, u \rangle$

$\langle (a_1, a_2), (b_1, b_2) \rangle \stackrel{?}{=} \langle (b_1, b_2), (a_1, a_2) \rangle$

→ $\langle ku, v \rangle = k \langle u, v \rangle$

so $\langle (ka_1, ka_2), (b_1, b_2) \rangle \stackrel{?}{=} k \langle (a_1, a_2), (b_1, b_2) \rangle$

$ka_1 b_1 + ka_2 b_2 \stackrel{?}{=} k(a_1 b_1 + a_2 b_2)$ → بتوزيع ال k

بطلعوا متساويين ✓ اذا الشرط متحقق ✓

→ $\langle u \mp w, v \rangle = \langle u, v \rangle \mp \langle w, v \rangle$

$\langle (a_1, a_2) \mp (b_1, b_2), (c_1, c_2) \rangle \stackrel{?}{=} \langle (a_1, a_2), (c_1, c_2) \rangle \pm \langle (b_1, b_2), (c_1, c_2) \rangle$

→ $(a_1 \mp b_1) c_1 + (a_2 \mp b_2) c_2 = (a_1 c_1 + a_2 c_2) \pm (b_1 c_1 + b_2 c_2)$

مع التوزيع بنلاقي انه الشرط متحقق ✓ فهو بالدقة بلا تفاوتات

Inner product spaces

examples:-

① let $x = (-1, 2, 3)$, $y = (2, 0, 1) \in \mathbb{R}^2$

find:

① $\langle x, y \rangle \rightarrow$ mapping بنطبق قانون ال

$= -2 + 0 + 3 = 1$

② $\langle x + 2y, z \rangle$, where $z = (0, 1, -2)$

$\langle x, z \rangle + \langle 2y, z \rangle$

$\langle x, z \rangle + 2 \langle y, z \rangle$

$-4 + 2(-2)$

$-4 + -4 = -8$

③ if $\langle x + ky, y \rangle = 4 \rightarrow$ find $\underline{\underline{k}}$

$\langle x, y \rangle + \langle ky, y \rangle$

$\langle x, y \rangle + k \langle y, y \rangle = 4$

$1 + k(5) = 4$

$5k = 3$

$k = \frac{3}{5}$ ✓

▶ Subject :

شو بستفيد من هاي ال map ومن خصائصها؟!

← أولا بقدر أستفيد منها ، مثلاً بقدر أحسب طول ال vector اللي عندي باستخدام فكرة

← بقدر كمان أوجد المسافة بين 2 vectors

خلينا نشوف كيف :-

① Definition:-

let $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, then

norm = the length of vector a = $\|a\| = \sqrt{\langle a, a \rangle}$

ال map بين ال vector ونفسه

② Definition:-

let $u, v \in \mathbb{R}^n$, the distance between u, v

$d(u, v) = \|u - v\| = \sqrt{\langle u-v, u-v \rangle}$

But you should note that :-

If norm of $v = 1$, then v is said to be a unit vector ✓

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example:-

let $u = \left(\frac{1}{\sqrt{2}}, 0, x\right)$, find all values of x such that u is a unit vector.

$\|u\| = 1$

$\sqrt{\langle u, u\rangle} = \sqrt{\frac{1}{2} + 0 + x^2} = 1$ رح أربعهن

$$\frac{1}{2} + x^2 = 1$$

$$x^2 = \frac{1}{2} \longrightarrow x = \sqrt{\frac{1}{2}}$$

Example :-

$u=(3,-1,2)\,,\ v=(1,1,0)\in \mathbb{R}^3$

find $d(u,v)$

$\|u-v\| = \sqrt{\langle u-v, u-v\rangle} \rightarrow u-v=(2,-2,2)$

$= \sqrt{4+4+4} = \sqrt{12}$

$\|u-v\| = \|v-u\|$ وبرهن نكون عارفين انه

$\rightarrow d(v-u) = \|v-u\|$

$v-u = -2, 2, -2 \rightarrow$

$\|v-u\| = \sqrt{4+4+4} = \sqrt{12}$

Scanned with CamScanner

Subject :

example :-

If $u, v \in \mathbb{R}^n$ with $\rightarrow \|u\| = 2$, $\|v\| = 1$, $\langle u, v \rangle = -3$

Find $\|2u - 3v\|$

$= \sqrt{\langle 2u - 3v , 2u - 3v \rangle}$

$\rightarrow \langle 2u - 3v , 2u - 3v \rangle =$

$\langle 2u, 2u - 3v \rangle - \langle 3v, 2u - 3v \rangle =$

$\langle 2u, 2u \rangle - \langle 2u, 3v \rangle - (\langle 3v, 2u \rangle - \langle 3v, 3v \rangle)$

$4\langle u, u \rangle - 6\langle u, v \rangle - (6\langle v, u \rangle - 9\langle v, v \rangle)$

$4 * 4 - 6(-3) - (6(-3) - 9(1))$

$16 + 18 - (-18 - 9)$

$16 + 18 + 18 + 9$

then find the final result and the square root of it

S T A R S N O T E B O

note: If $u \in \mathbb{R}^n$ with $\|u\| \neq 0$, then $\frac{u}{\|u\|}$ is

a unit vector: the norm of $\left(\frac{u}{\|u\|}\right) = \underline{\underline{1}}$

example:

$u = (-1, 2, 3) \in \mathbb{R}^3$ show that $\frac{u}{\|u\|}$ is

a unit vector

$$\|u\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\frac{u}{\sqrt{14}} = \left(\frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

now to get sure:

$$\left\|\frac{u}{\sqrt{14}}\right\| = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = \sqrt{\frac{14}{14}} = 1 \checkmark$$

Definition:-

$u, v \in \mathbb{R}^n$ are called orthogonal vectors

if $\langle u, v \rangle = 0$ $(u \perp v)$

examples:-

$u = (3, x-1, 4)$, $v = (2, 2, -1)$, find x if

$u \perp v$

if $u \perp v$ then $\langle u, v \rangle = 0$

$6 + 2(x-1) - 4 = 0$

$x = 0$

Definition:-

let $u \neq 0 \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$

then, the orthogonal projection of v on u

$= \text{proj}_u v = \dfrac{\langle v, u \rangle}{\langle u, u \rangle} u$ ✓

Theorem:-

$$\langle v - \text{proj}_u v, u \rangle = 0$$

$\therefore$ $v - \text{proj}_u v \perp u$

example:-

let $u = (2, 1, -1)$, $v = (3, 0, 2) \in \mathbb{R}^3$

find ① $\text{proj}_u v = \dfrac{\langle v, u \rangle}{\langle u, u \rangle} u$

$$= \frac{6 + 0 + -2}{4 + 1 + 1} u = \frac{4}{6}(2, 1, -1)$$

② $\text{proj}_v u = \dfrac{\langle u, v \rangle}{\langle v, v \rangle} v =$

$$= \frac{9}{13}(3, 0, 2)$$

Subject :

Definition let w be a subspace of R^n, then

$B = \{y_1, y_2, ---, y_n\}$ is called the orthogonal

basis for w if →

① B is Basis for w

② $\langle y_i, y_j \rangle = 0 \rightarrow y_i \perp y_j$

and if you add ③ $\|y_i\| = 1$ for all $i = 1, 2, 3 ---$
it will become orthonormal basis for w

→ We use G.S.P to find an orthogonal basis

method let $\{x_1, x_2, ---x_m\}$

$y_1 = x_1$

$y_2 = x_2 - \text{proj}_{y_1} x_2$

$y_3 = x_3 - \text{proj}_{y_1} x_3 - \text{proj}_{y_1} x_2$

Subject :

examples:-

let $R = \{(1,0,-1,2),(1,1,0,1),(2,1,1,1)\}$

use G.S.P to find an orthogonal basis:

use the rule, and you'll have these answers:-

$y_1 = x_1 = (1,0,-1,2)$

$y_2 = x_2 - \text{proj}_1 x_2$

$= (\frac{1}{2}, 1, \frac{1}{2}, 0)$

$y_3 = x_3 - \text{proj}_{y_1} x_3 - \text{proj}_{y_1} x_2$

$= (\frac{4}{6}, \frac{-2}{5}, \frac{4}{6}, 0)$

and put them in the space $B = \{y_1, y_2, y_3\}$

note:- $A_{n \times m}$ is called orthogonal matrix

if $AA^T = A^TA = I_n$

احنا بنتذكر من اول شابتر انه الماترکس الي بضربها في A وبتعطيني I

بتكون هي ال A^{-1} ، فما بكون ، بكون $A^{-1} = A^T$ $A \rightarrow$ orthogonal matrix

Subject :

Definition

let $B_1 = \{u_1, u_2, \ldots u_n\}$,

$B_2 = \{v_1, v_2, \ldots v_n\}$

then, the transition matrices from B_1 to B_2

$$P_{B_1}^{B_2} = \left[(u_1)_{B_2} \quad (u_2)_{B_2} \quad \ldots \right]$$

theorem:-

→ $\left(P_{B_1}^{B_2}\right)^{-1} = P_{B_2}^{B_1}$ and $\left(P_{B_2}^{B_1}\right)^{-1} = P_{B_1}^{B_2}$

→ $v \in W$, then $P_{B_1}^{B_2} * (v)_{B_1} = (v)_{B_2}$

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* past papers questions:-

① let $B_1 = \{(1,1), (2,1)\}$, $B_2 = \{(1,3), (2,5)\}$

find ① $P_{B_1}^{B_2}$

implement the rules in the theorem

then $P_{B_1}^{B_2} = \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}$ → استخدم ال linear combinations

② $P_{B_2}^{B_1} = \left(P_{B_1}^{B_2}\right)^{-1} = \frac{1}{1} \begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}$

[Eigenvectors] and [Eigenvalues]

Definition :-

let A be $n \times n$ matrix and let $\lambda \in \mathbb{R}$, then λ is called an Eigenvalues of A if there exists

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \neq 0 \quad \text{such that :-}$$

$$\det(A - \lambda I_n) = 0$$

هاد الحدود بعدين
راح نحكي شو وظيفته ✓

example :-

let $A = \begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix}$, find all eigenvalues of A.

$$\det(A - \lambda I_n) = 0$$

$$\det \begin{bmatrix} 2-\lambda & -1 \\ 2 & 5-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(5-\lambda) + 2 = 0 \rightarrow$$ اعمل معادلة تربيعية وحلها

$$\lambda = 3 \;,\; \lambda = 4$$ ← بطلع معك

properties of eigenvalues

Suppose $Ax = \lambda x$

→ λ^k is an eigenvalue of A^k

→ for any $k \in R$, then $\lambda + k$ is an eigenvalue of $A \mp kI_n$

→ for $k \in R$, $k\lambda$ is an eigenvalue of kA.

example:-

$-2, 1, 3$ are eigenvalues of 3×3 matrix
find the eigenvalues of

① $A - 7I = -9, -6, -4$

② $A^2 + 2I = 6, 3, 1$

③ $(A - I)^2 = 9, 0, 4$

also :- this is the most important
for any $k \in R$, then $\lambda \mp k$ is as mentioned above $A \mp kI$

Subject :

Note that:-

① If A is invertible → all eigenvalues of A are nonzero, and eigenvalues of $A^{-1} = \frac{1}{\lambda \text{ of } A}$

② - summation of all eigenvalues of A = tr(A)
- product of all eigenvalues of A = det(A)

[Reverse Question]

example:-

2, -2 are eigenvalues of $A_{2\times2}$

⇒ 4, 4 are the eigenvalues of A^2

⇒ 4, -4 " " " of 2A → مع اختلاف الترتيب
الي هو اشي عادي ✓

⇒ 1, 5 " " " of A+3I

⇒ 2, 10 " " " of 2(A+3I)

بنحطها بالملاحظة ✓

Subject : .... / /

note that:-

① If A is triangular matrix, then the eigenvalues of A are the entries on the main diagonal

postpapers:-

① $[3, -1, -1, \lambda]$ are eigenvalues of A with $\det(A) = 21$, find $tr(A)$

$\det(A) = 21$

$3 * -1 * -1 * \lambda = 21$

$\lambda = 7$

$tr(A) = 8$

Note that:- the eigenvectors of A relative to λ are the nxn zero vector ~~space~~ x in null $A - \lambda I$, this subspace is called eigenspace of λ

$Eig(\lambda) = null(A - \lambda I)$

[Linear Transformation]

Definition

let V, W be real vector spaces, and let $T: V \to W$ be a map, then T is called a linear transformation

if:

→ $T(u+v) = T(u) + T(v)$ for all $u, v \in V$

→ $T(ku) = kT(u)$ for all $u \in V$, $k \in \mathbb{R}$

لازم هاي الشروط تتحقق عشان اثبت انه transformation

← هاد الشابتر عبارة عن تكرار لشابتر 5 ، بيعطيني طريقة ال ← mapping
اللي هو ببدأ فيها، وأنا بتأكد اذا بتحقق الشروط اولا ، وهاي هي الطريقة اللي
راح اشتغل عليها طول الحل ✓

ولكن كيف بدي احل سؤال هو مش محدد لي فيه صيغة ال mapping ؟