POWERUNIT

DSP

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Lecture 1

→ Digital signal Processing

* signal: is any physical quantity, that varies with time space or any other independent variable(s).

* Any signal can be represented mathematically by a function of one or more independent variable.

$$f(t)$$

function (physical valuse) ← f ; t → independent variable

* The value of the signal at a specific point is called the amplitude while the variation is called a waveform.

Classifications Of Signals

- Signal values → ① real signal $x(t)$: x is real
  ② Complex signal $x(t) + j\,y(t)$ — real part: $x(t)$, imaginary part: $y(t)$

- Cerntainity

① Deterministic $f(t) = \underbrace{A}\cos(\underbrace{\omega}t)$ → known constant

② Random $x(t)$

random process

$$X \sim N(a_x, \sigma_x^2)$$

- Dimensionlity

1-D

2-D

3-D

- Continuity of independent variable(s)

① continuos (continous time signal) C.T

② Discrete (Discrete time signal) D.T

- Continuity in amplitude

① continous amplitude

② Discrete amplitude

* Anglog signals are continous is in time and amplitude (infinite number of points and infinite possible values).

$f(t) = A\cos(\omega t)$ $\quad 0 \leq t \leq 1$

$-A \leq f(t) \leq A$ → dependening on t and w

* Such signals cannot be represented and processed inside digital computers with limited storge and processing capabilities.

→ to deal with them we use

Discrete time signals

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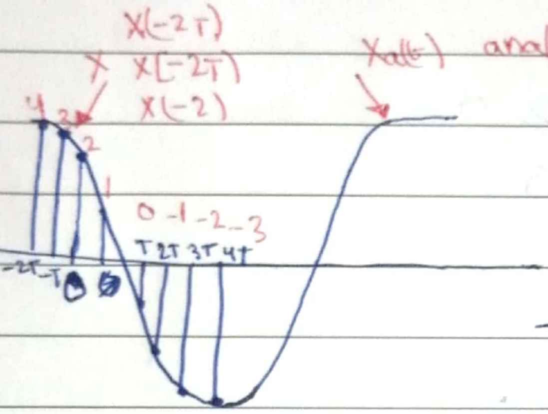
* Discrete Time Signals (DT signals)

- Sources of DT signals are :

① Sampling of continous signals

② Inherent D.T signals in nature

(Taking the reading of tranceedureres and Sensors every some specific period of time).



$$x_a(t)\Big|_{t=nT} = x_a(nT) = x[n]$$ D.S (Sequence)

- where T is the sampling period and $F_r = \frac{1}{T}$ is the sampling frequency in Hz or samples/second

$x(-1)$ $x(-0)$

$x[n] = \{\cdots 4, 3, 2, 0, -1, -2, -3, \cdots\}$ ← ① The sequence Representation of DT signal

$n=-2$ $n=-1$ $n=0$ $n=1$ $n=2$

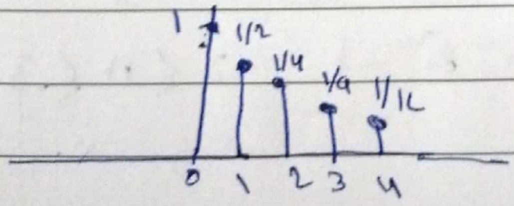
$x[10] \rightarrow x[10T]$ → sampling period

example

$T = 0.1$ second

$x[10T] = x[10 \times 0.1] = x[1]$

② Pictorial Represenation of D.T signal



N O T E B O O K

③ Tabular representation of D.T signal

n	...	-2	-1	0	1	2	3	4	5
X[n]		0	0	0	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$

④ functional representation of DT signal

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

✗ The Length of DT Signals

① Finite ~~tenght~~ length sequences.

a finite length sequence is defind for a finite period of time

$$N_1 \leq n \leq N_2$$

where $N_1 > -\infty$ and $N_2 < \infty$

- The length or duration of the sequence :-

$$N = N_2 - N_1 + 1$$

- A finite length sequence can be considered infinite by padding zeros from both sides.
- A length N sequence is often refered to as N point sequence.

For example if $x[n] = n^2 - 1 \quad -2 \leq n \leq 3$

X[n] is of length 6.

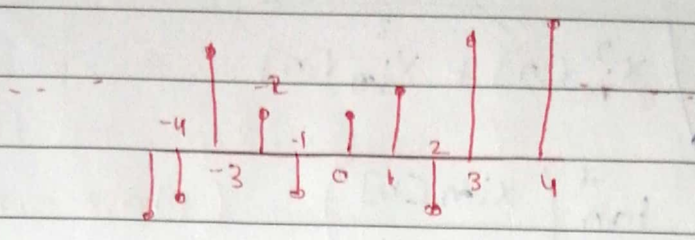
$$x[n] = \{3, 0, \underset{\uparrow}{-1}, 0, 3, 8\}.$$

X[0]

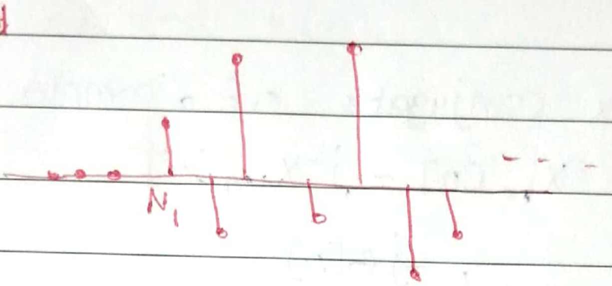
$N = 3 - (-2) + 1 = 6$ samples

② Infinite Length sequence

a. Tow-sided sequence → defined for all $\underline{\underline{n}}$.

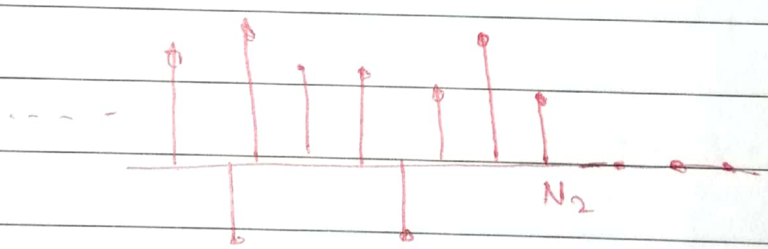


b. Right-Sided sequence → defined for $n \geq N_1$
~~and~~



If $N_1 \geq 0$ → causal sequence

c- Left-Sided sequence → the sequence is defined for $n \leq N_2$



if $N_2 < 0$ → Anti-causal or non-causal

* Complex sequences

① Rectangular for

$$x[n] = x_{re}[n] + jx_{im}[n]$$

$j = \sqrt{-1}$

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② Paler form

$$x[n] = |x[n]| e^{j\theta[n]}$$

where

$$|x[n]| = \sqrt{x_{re}^2[n] + x_{im}^2[n]}$$

$$\theta[n] = \tan^{-1}\left(\frac{x_{im}[n]}{x_{re}[n]}\right)$$ (phase angle)

* The complex conjugate of a complex sequence

$$x^*[n] = x_{re}[n] - j\, x_{im}[n]$$

$$= |x[n]| e^{-j\theta[n]}$$

Elementary Operations on Sequence

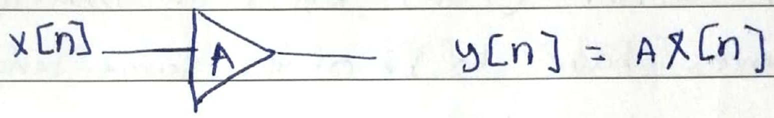
① Scaling → Multiplication by constant

$y[n] = A\ x[n]$

If $|A| > 1$ → Amplification

$|A| < 1$ → Attenuation

- The minus sign (−) means 180° phase shift between $y[n]$ and $x[n]$



Ex

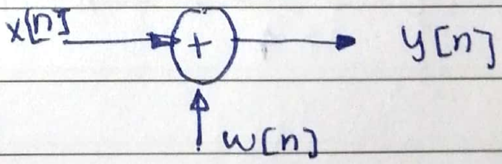
$x[n] = \{6, 5, 3, 0\}$
$n=0 \uparrow$ $w[n] = 0.5\ x[n]$ find $w[n]$

Sol:

$x[n] = \{3, 2.5, 1.5, 0\}$

② Addition: The sequences have to be in the same length, if not, pad with zeros

$y[n] = x[n] + w[n]$



Ex:

$x[n] = \{1, -2, 4\}$
$\uparrow$ (under −2)

$w[n] = \{6, 2, 7, 5, 4\}$
$\uparrow$ (under 6)

find $y[n] = x[n] + w[n]$

Sol: $x[n] = \{1, \underset{\uparrow}{-2}, 4\}$

$w[n] = \{\underset{\uparrow}{6}, 2, 7, 5, 4\}$

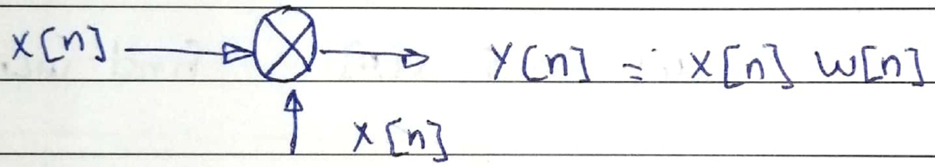
$x[n] = 1 \quad \underset{\uparrow}{-2} \quad 4 \quad 0 \quad 0 \quad 0$

$w[n] = 0 \quad \underset{\uparrow}{6} \quad 2 \quad 7 \quad 5 \quad 4$

$y[n] = \{1, \underset{\uparrow}{4}, 6, 7, 5, 4\}$

③ Multiplication (Point by point multiplication)

- Sequences have to be of the same length if not, ~~add~~ pad with zeros



Ex: $x[n] = \{1, \underset{\uparrow}{-2}, 4\}$ $w[n] = \{\underset{\uparrow}{6}, 2, 7, 5, 4\}$

find $y[n] = x[n]\, w[n]$

$x[n] = 1 \quad \underset{\uparrow}{-2} \quad 4 \quad 0 \quad 0 \quad 0$

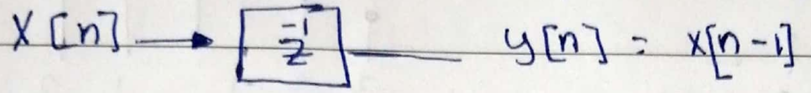
$w[n] = 0 \quad \underset{\uparrow}{6} \quad 2 \quad 7 \quad 5 \quad 4$

$y[n] = \{0, \underset{\uparrow}{-12}, 8, 0, 0, 0\}$

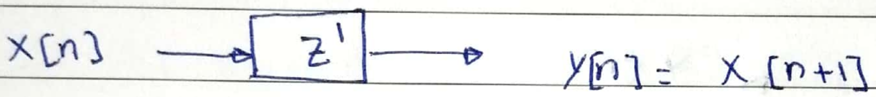
④ Time-Shift

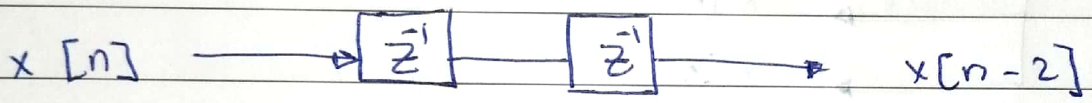
$y[n] = x[n-N_o]$

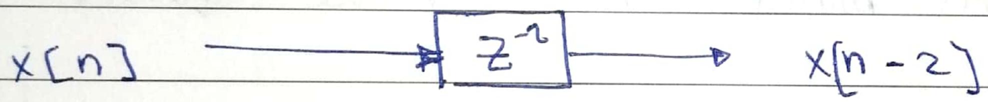
- If $N_o > 0 \rightarrow$ Sequence is shifted to the right by N_o samples (Delay).



- If $N_o < 0 \rightarrow$ Sequence is shifted to the left by N_o sample (time advance)







(both are the same)

Ex: $x[n] = \{1, \underset{\uparrow}{-2}, 4\}$ find $x[n-3]$

Sol:

$y[n] = x[n-3]$

$y[0] = x[-3] = 0$

$y[1] = x[-2] = 0$

$y[2] = x[-1] = 1$

$y[3] = x[3-3] = x[0] = -2$

$y[4] = x[4-3] = x[1] = 4$

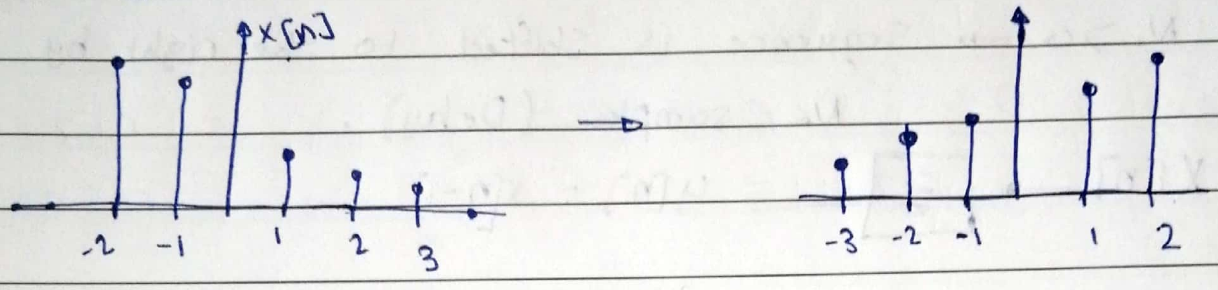
$y[5] = x[5-3] = x[2] = 0$

⋮

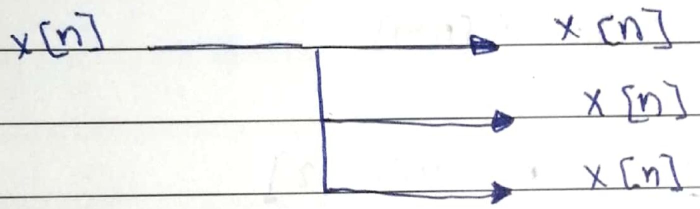
$y[n] = \{\underset{\uparrow}{0}, 0, 1, 2, 4\}$

⑤ Time Reversal

$y[n] = x[-n]$



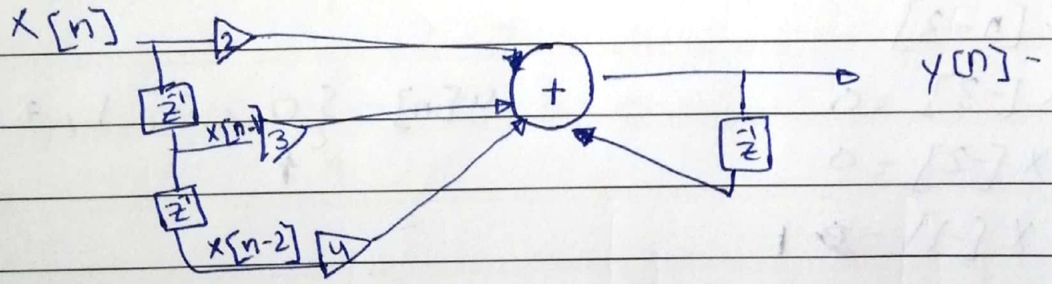
⑥ Branching (pick off point)



- It provids multiple Copies of the sequence

→ Multiple of elementary operations

Ex $y[n] = 2x[n] + 3x[n-1] + 4x[n-2] + y[n-1]$ Difference equation



— Classifications OF Sequences.

** Based on symmetry

- Sequence $x[n]$ is called a conjugate symmetric sequence if $x[n] = x^*[-n]$
- if the sequence is real → $x[n] = x[-n]$ (even sequence)
- A sequence $x[n]$ is called a conjugate-antisymmitric sequence if $x[n] = -x^*[-n]$
- If the sequence is real → $x[n] = - x[-n]$ (odd sequence
- In general any complex sequence can be expressed as a sum of it's conjugate symmitric part and it's conjugate antisymmitric part

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where

$$x_{cs}[n] = \frac{1}{2}(x[n] + x^*[-n])$$

and

$$x_{ca} = \frac{1}{2}(x[n] - x^*[-n])$$

example:

$$x[n] = \{0, 1+j4, -2+j3, \underset{\uparrow}{4-j2}, 5-j6, -j2, 3\}$$

- Determin $x_{cs}[n]$ and $x_{ca}[n]$

- Solution :

$x^*[n] = \{0, 1-j4, -2-j3, \underset{\uparrow n=0}{4+j2}, -5+j6, j2, 3\}$

-3 -2 n = -1 n=1 n = 2 n = 3

$x^*[-n] = \{3, j2, -5+j6, 4+j2, -2-j3, 1-j4, 0\}$

$x_{cs}[n] = \{1.5, 0.5+j3, -3.5+j4.5, \underset{\uparrow}{4}, -3.5-j4.5, 5-j3, 1.5\}$

$x_{as}[n] = \{-1.5, 0.5+j, 1.5-j1.5, \underset{\uparrow}{-j2}, -1.5-j1.5, -0.5+j, 1.5\}$

- For real sequence $x[n]$:-

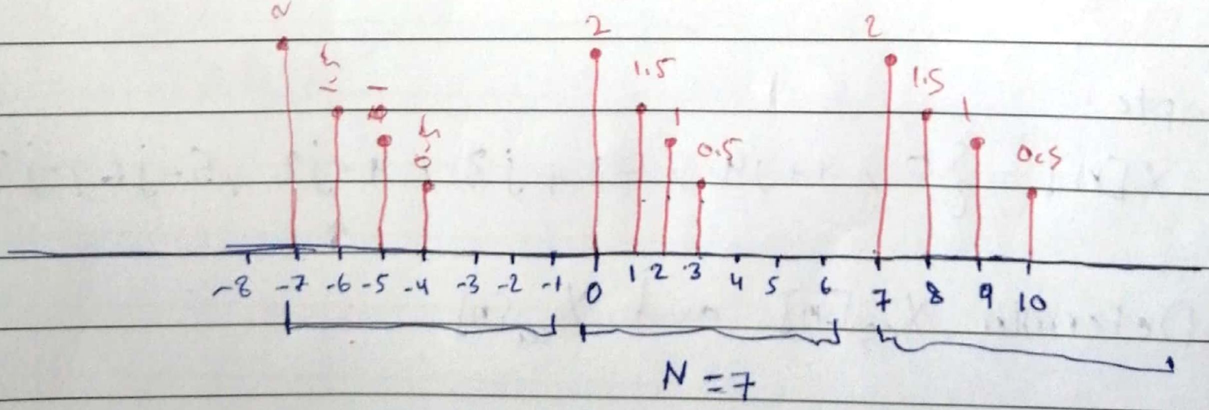
$$x[n] = x_{ev}[n] + x_{od}[n]$$

where $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$

$$x_{od}[n] = \frac{1}{2}(x[n] - x[-n])$$

** Based on periodicity:

- Periodic and Aperiodic sequence
- A sequence $\tilde{x}[n]$ satisfing $\tilde{x}[n] = \tilde{x}[n+KN]$ for all n is called periodic sequence with period N where N is a positive integer and K is any integer



- The fundemental period N_f of a periodic signal is the smallest value of n that satifies the periodicity equation
- The sum of ~~tow of~~ tow or more periodic sequences is also periodic

$\tilde{x}_a[n]$ with period N_a

$\tilde{x}_b[n]$ with Period N_b

$\tilde{y}[n] = \tilde{x}_a[n] + x_b[n]$ then $\tilde{y}[n]$ is periodic with period $LCM(N_a, N_b)$

↳ least common multiple

- $LCM(N_a, N_b) = \dfrac{N_a N_b}{GCD(N_a, N_b)}$

GCD: The greatest common divisor

- likewise, ~~the periodic~~ product of tow or more periodic sequences is again periodic with same formula used for the summation
- example

$N_a = 3$ $N_b = 4$ $N_c = 6$

$\tilde{y}[n] = \tilde{X}_a[n] + \tilde{X}_b[n] + \tilde{X}_c[n]$

$N_y = LCM(3,4,6) = 12$, 24 is $\underline{\underline{\text{not}}}$ N_y

** Energy and power sequences

- The total energy of a sequences x[n] is defiend by

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} | x[n] |^2$$

- A finite sequence with finite sample values has finite energy.
- An infite sequence with finite sample values may or may not have finite enegy

- examples

$x[n] = \{1, 3, \underset{\uparrow}{6}, -2\}$ find E_x

• Sol

$$E_x = \sum_{\infty}^{\infty} | x[n] |^2 = 1+9+36+4 = 50$$

- example

$x[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$ find $\mathcal{E}_x$

• Sol

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$ (has finite amount of energy)

- example $x[n] = \begin{cases} \frac{1}{\sqrt{n}} & n \geq \phi \\ 0 & n \leq 0 \end{cases}$ find $\mathcal{E}_x$

• Sol

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$$ dose not converge, has infite energy

- The ~~average~~ average power of sequence

$$P_x = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$$

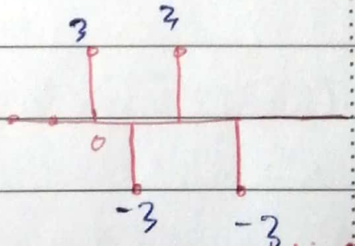
- for ~~a~~ periodic sequences

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

~~example~~

- An infinite energy sequence with finite average power is called power sequence
- A finite energy sequence with zero average power is called energy sequence
- If a sequence has infinite power and ~~a~~ infinite average power ~~Et~~ then it's neither energy nor power sequence.
- example

consider $x[n] = \begin{cases} 3(-1)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$



find P_x ?

Sol

$$P_x = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=-K}^{+K} |x[n]|^2$$

$$= \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=0}^{K} |(3(-1)^n)|^2 = \lim_{K \to \infty} \frac{1}{2K+1} \underbrace{\sum_{n=0}^{K}}\, 9$$

→ adding 9's K+1 times

$$= \lim_{K \to \infty} \frac{9(K+1)}{2K+1} = \frac{9}{2} = 4.5$$ power sequence

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** Other types of classifications

1) A sequence is bound if

$$|x[n]| \leq \beta_x < \infty$$

where β_x is finite positive number

2) Sequence is absolutely summble if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad (\text{finite}$$

3) Sequence is sequare summble if

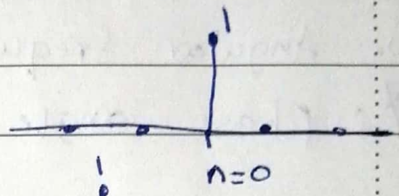
$$\sum_{n=\infty}^{\infty} |x[n]|^2 < \infty \quad (\text{ finite })$$

Typical Sequences And Sequence Represenation

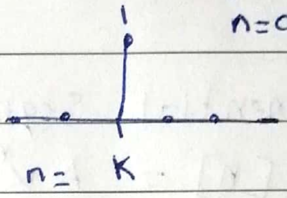
* Some Basic Sequences:

1. Unit sample sequence (discrete-time implulse) or simply unit impulse sequence

- $\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$

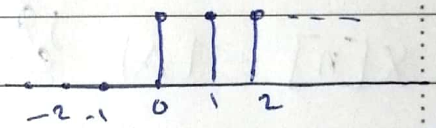


- $\delta[n-k] = \begin{cases} 1 & n=k \\ 0 & n\neq k \end{cases}$

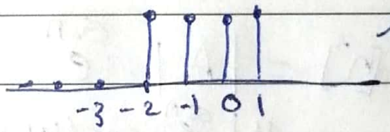


2. Unit step sequence

- $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n<0 \end{cases}$



- $u[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n<k \end{cases}$



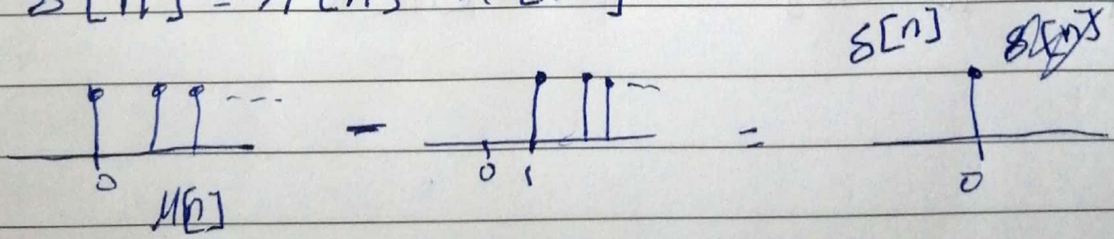
$u[n+2]$

$$u[n] = \sum_{m=0}^{\infty} \delta[n-m] = \sum_{k=-\infty}^{n} \delta[k]$$

$$u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$$

$$u[3] = \sum_{k=-\infty}^{3} \delta[k] = \cancel{\delta}[-1] + \delta[0] + \overset{0}{\cancel{\delta}[1]} + \overset{0}{\cancel{\delta}[2]} + \overset{0}{\cancel{\delta}[3]}$$
$$= 1$$

$$\delta[n] = u[n] - u[n-1]$$



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3. Sinusoidal and Exponential Sequences

- Sinusoidal sequences

$$x[n] = A\cos(\omega_0 n + \phi) \qquad -\infty < n < \infty$$

where A, ω_0 and ϕ are real numbers

A: The amplitude

ω_0: Angular frequency (rad/samples)

ϕ: phase angle

- Exponential sequence

$$x[n] = A\alpha^n \qquad -\infty < n < \infty$$

$$\alpha = e^{\sigma_0 + j\omega_0} \qquad A = |A| e^{j\phi}$$

$$x[n] = |A| e^{j\phi} e^{(\sigma_0 + j\omega_0) n}$$

$$x[n] = |A| e^{\sigma_0 n} e^{j(\omega_0 + \phi)}$$

$$= \underbrace{|A| e^{\sigma_0 n} \cos(\omega_0 n + \phi)}_{x_{re}[n]} + j\,\underbrace{|A| e^{\sigma_0 n} \sin(\omega_0 n + \phi)}_{x_{im}[n]}$$

- Condition for periodicity of sinusoidal sequences

$x_1[n] = A\cos(\omega_0 n + \phi)$

check for periodicity? Is it periodic?

$x[n]_1 = x_1[n+N] \qquad \text{for} -\infty < n < \infty$

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let $x_2[n] = x_1[n+N]$

$x_2[n] = A\cos(\omega_0[n+N]+\phi)$

$x_2[n] = A\cos((\omega_0 n+\phi)+\omega_0 N)$

$\doteq A\left[\cos(\omega_0 n+\phi)\underbrace{\cos(\omega_0 N)}_{1} - \sin(\omega_0 n+\phi)\underbrace{\sin(\omega_0 N)}_{0}\right]$

$\cos(\omega_0 N) = 1$ and $\sin(\omega_0 N) = 0$

$\omega_0 N = 2\pi r$

integer multiples of 2π

$\frac{2\pi}{\omega_0} = \frac{N}{r}$ (rational number)

Example $x_a[n] = \cos(0.45\pi n)$

$x_b[n] = \cos(3.14 n)$

a) $w_a = 0.45\pi$

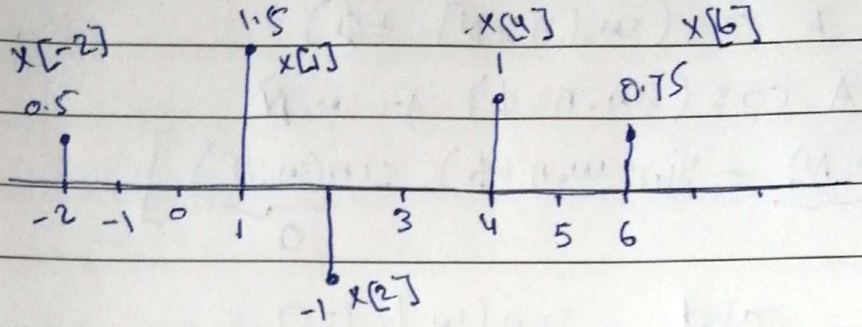
$\frac{2\pi}{0.45\pi} = \frac{200}{45} = \boxed{\frac{40}{9}}$ rational number (periodic)

$N_f = 40$

b) $w_b = 3.14$

$\frac{2\pi}{3.14} = \frac{200\pi}{314} = \frac{100\pi}{157}$ irrational number (Aperiodic)

** Representation of an Arbitrary Sequence



$x[n] = 0.5\,\delta[n+2] + 1.5\,\delta[n-1] + -1\,\delta[n-2] + \delta[n-4] + 0.75\,\delta[n-6]$

$$\boxed{x[n] = \sum_{\substack{\cancel{k=-\infty} \\ k=-\infty}}^{\infty} x[k]\,\delta[n-k]}$$

→ we will use it when we start talking about systems.

The Sampling Theorem :-

let's take a continious time signal $x_a(t) = A\cos(\Omega_0 t + \phi)$

↑ analog

Ω_0 → C.T angular frq. (rad/ sec)

- We would like to sample this signal at tn instants when $t_n = nT = \frac{n}{f_T} = \frac{2\pi n}{\Omega_T}$

T: is the sampling period (sec)
F_T: " " " freq. (Hz)
Ω_T: " " " " (rad/sec)
$\Omega_T = 2\pi f_T$

* $x_a(t)\Big|_{t=nT} = A\cos(\Omega_0 nT + \phi)$

$= A\cos(\Omega_0 Tn + \phi) = A\cos(\omega_0 n + \phi) = x[n]$ D.T signal

$\boxed{\omega_0 = \Omega_0 T}$

ω_0: D.T angular frq. (rad/sample)

Ex: consider Three continious time signals

$g_1(t) = \cos(\underbrace{6\pi}_{\Omega_1 = 2\pi f_1} t)$: $f_1 = 3Hz$

$g_2(t) = \cos(\underbrace{14\pi}_{\Omega_2 = 2\pi f_2} t)$: $f_2 = 7 Hz$

$g_3(t) = \cos(26\pi t)$: $f_3 = 13 Hz$

- The Three signals ~~where~~ were sampled at a sampling rate of 10 Hz, find $g_1[n]$, $g_2[n]$ and $g_3[n]$.

Sol:-

$$T = \frac{1}{F_T} = \frac{1}{10} = 0.1 \text{ sec}$$

→ $g_1[n] = \cos(0.6\pi n)$

→ $g_2[n] = \cos(1.4\pi n) = \cos(2\pi n - 0.6\pi n)$

$\overset{1}{\underbrace{\cos(2\pi n)}}\cos(0.6\pi n) + \overset{0}{\underbrace{\sin(2\cancel{\pi} n)}}\sin(0.6\pi n) = \cos(0.6\pi n) = g_1[n]$

→ $g_3[n] = \cos(2.6\pi n)$

$= \cos(2\pi n + 0.6\pi n)$

$= \overset{1}{\underbrace{\cos(2\cancel{\pi} n)}}\cos(0.6\pi n) - \overset{0}{\sin(2\cancel{\pi} n)}\sin(0.6\pi n)$

$= \cos(0.6\pi n) = g[n]$

* The three sequences are identical,

All cosine waveforms of freq. given by $10k \pm 3$

with K is any non-negative integer gives the same answer as above.

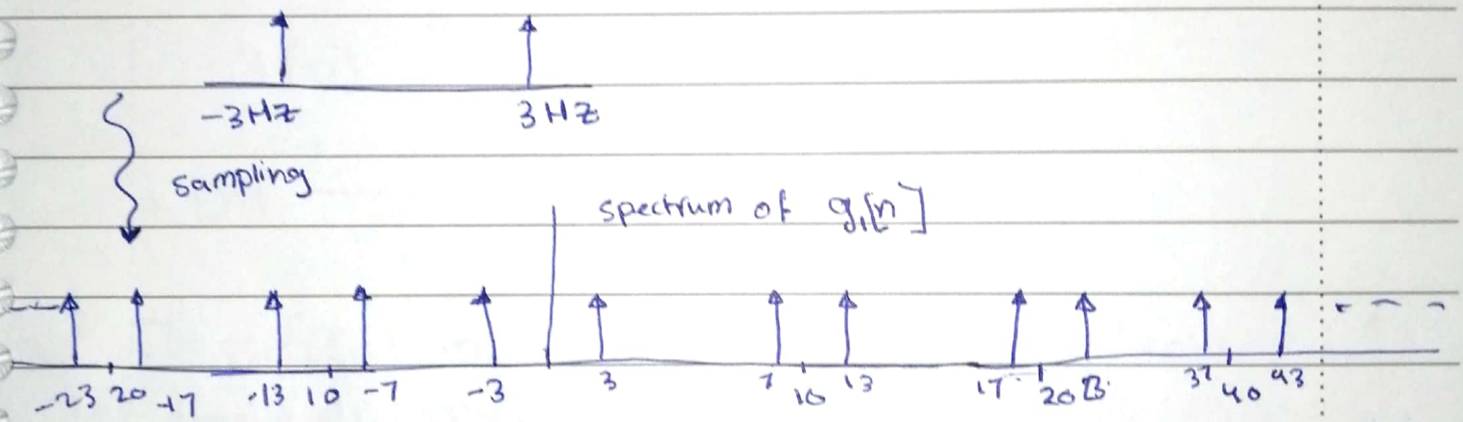
(20 ± 3) (30 ± 3)

Nygreat rate

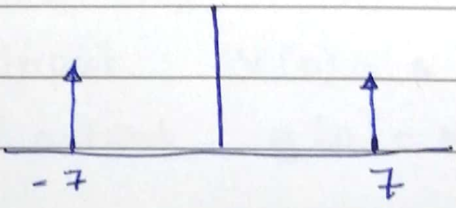
Sampling frq. to be $F_T \geq 2 f_{max}$

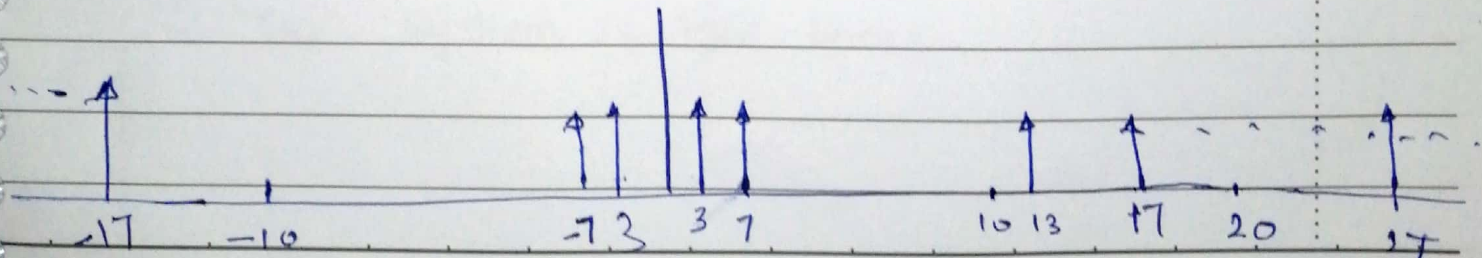
$g_1(t)$

spectrum $g_1(t)$



Spectrum of $g_2(t)$





$g_1[n]$ and $g_2[n]$ identical s-epectrums

- Please do the spectrum of $g_3[n]$ by your
- self and make sure that it's identical to
the spectrum of $g_1[n]$ and $g_2[n]$

freq. of the signal is 7 Hz

$F_T < 2\ f_{max} = 2 \times 7 = 14$

10 Hz $\swarrow$

Discrete Time Systems

The function of a discrete time system is to process agiven input sequence to generate an output sequence.

$$y[n] = T\{x[n]\}$$

where: $T[\,] \rightarrow$ Transformation
$y[n] \rightarrow$ output seq.
$x[n] \rightarrow$ input seq.

$$x[n] \xrightarrow{T} y[n]$$

$$x[n] \longrightarrow \boxed{T[\,]} \rightarrow y[n]$$

Classification Of Discrete Time Systems:

1) linear systems

- if given that $y_1[n]$ and $y_2[n]$ are responses to the input seq. $x_1[n]$ and $x_2[n]$ respectively the for input $x[n] = \alpha x_1[n] + \beta x_2[n]$ then the output $y[n] = \alpha y_1[n] + \beta y_2[n]$

∴ The system is ~~lear~~. linear

Check for linearity

Ex: a) $y[n] = n\, x[n]$

b) $y[n] = x^2[n]$

c) $y[n] = A\,x[n] + B$ where A and B are const.

Solution :

a) $x_1[n] \longrightarrow y_1[n] = n\, x_1[n]$

$x_1[n] \longrightarrow y_2[n] = n\, x_2[n]$

$x[n] = a_1\, x_1[n] + a_2\, x_2[n]$

$y[n] = T[x[n]] = n(a_1 x_1[n] + a_2 x_2[n]$

$\underbrace{y[n] = qn \cdot x_1[n] + a_2 n\, X_2[n]}$ is the same

→ The system is linear

b) $y[n] = x^2[n]$

$x_1[n] \longrightarrow y_1[n] = x_1^2[n]$

$x_2[n] \longrightarrow y_1[n] = x_2^2[n]$

$x[n] = a_1 x_1[n] + a_2 x_2[n] \longrightarrow y[n] = (a_1 x_1[n] + a_2 x[n])^2$

$y[n] = a_1^2 x_1^2[2] + 2a_1 a_2 x_1[n] X_2[n] + a_2^2\, x_2^2[n]$

$\neq a_1^2\, x_1[n] + a_2\, x_2[n]$

non-linear System.

c) $y[n] = A\,x[n] + B$ (Non-linear system)

- please prove that?

2) Time-invariant (shift-Invariant) System

if ~~we have~~

$$x[n] \xrightarrow{T} y[n]$$

$$x[n-k] \longrightarrow y[n-k]$$

∴ Then you have time-invariant system otherwise you have time-variant system.

How to check for Time-invariance?

1) let $x[n] \xrightarrow{T} y[n]$

2) given $x_1[n] = x[n-k]$

$$x_1[n] \xrightarrow{T} y_1[n]$$

3) shift $y[n]$ in step1 by $k \rightarrow y[n-k]$

if $y_1[n] = y[n-k]$ then the system is time-invariant

Example: a) $y[n] = x[n] - x[n-1)$

b) $y[n] = n\,X[n]$

Solution

a) ~~y~~ $x[n] \longrightarrow y[n] = x[n] - x[n-1]$

$x_1[n]$ ~~→~~ $= x[n-k]$

$y_1[n] = x[n-k] - x[n-1-k]$

$y[n-k] = x[n-k] - x[n-k]$

$y_1[n] = y[n-k]$

Time-invariant system

b) ① $y[n] = n\, x[n]$

② let $x_1[n] = x[n-k]$

$y_1[n] = n\; x[n-k]$

③ $y[n-k] =$ ~~y[n=~~ $(n-k)\; x(n-k)$

$y_1[n] \neq y[n-k]$ Time-variant system.

3) Causal System (Realizable system)

In causal D.T system, the n_oth output sample $y[n_o]$ dependce only on input sample $x[n]$ for $n \leq n_o$

$y[n] = T[\, x[n], x[n-1], x[n-2], \ldots\,]$

Ex:

a) $y[n] = x[n] - x[n-1]$ ⟿ causal

b) $y[n] = x[-n]$ ⟿ non-causal

$y[1] = y[-1]$

$y[-1] = y[1]$

4) Stable Systems

A D.T system is stable if and only if for every bounded input, the output is also bounded

If $|\, x[n]\,| \leq \beta_x$ for all n

$|\, y[n]\,| \leq B_y$ where B_x and B_y are finite the constants

- This type of stability is called Bounded input Bounded output (BIBO) Stability criterian

Ex: a) $y[n] = e^{x[n]}$

b) $y[n] = \sum_{k=0}^{n} x[k]$, accumlator

Solution:

a) $|y[n]| = |e^{x[n]}| \le e^{|x[n]|} \le \underbrace{\left\{ e^{\beta_x} \right\}}_{\beta_y}$

$|y[n]| \le \beta_y$ (BIBO) Stable system

(b) $\left|y[n]\right| = \left|\sum_{k=0}^{n} x[k]\right| \le \sum_{k=0}^{n} \cancel{|x[k]|}\, |x[k]|$

$\le \sum_{k=0}^{n} \beta_x$

$|y[n]| \le \sum_{k=0}^{n} \beta_x$

$|y[n]| \le (n+1)\beta_x \ne \beta_y \ne$ costnant

Astable system (non-stable)

5) Static and Dynamic Systems

Memoryless ↳ system with memory

a) $y[n] = a_1 x[n]$ Static

b) $y[n] = x[n] + x[n-1]$ Dynamic - system

c) $y[n] = x[n+1/2] + x[n+1]$ Dynamic

6) Passive and lossless System

if you apply an enery signal to a system

i.e $\sum_{x} = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$

such that

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

→ Passive system

Ex: $y[n] = \alpha x[n-N]$; N is possitive integer

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |\alpha x[n-N]|^2 = |\alpha|^2 \sum_{n=-\infty}^{\infty} |x[n-N]|^2$$

$|\alpha| \leq$ Passive system

$|\alpha| = 1$ lossless systen

$|\alpha| > 1$ Active systen

Linear Time-Invariant System (LTI)

* Impulse and step responses:

$$x[n] \longrightarrow \boxed{T\{\}} \longrightarrow y[n] = T\{x[n]\}$$

$$x[n] = \delta[n] \longrightarrow h[n] = T[\delta[n]]$$

↳ Impulse response

- The impulse response :- The output (response) of the system when the input is a delta.

$$x[n] = u[n] \longrightarrow \boxed{T\{\}} \longrightarrow S[n] = T[u[n]]$$

step response

- $S[n]$: The step response, it is the response of the system when the input is a step

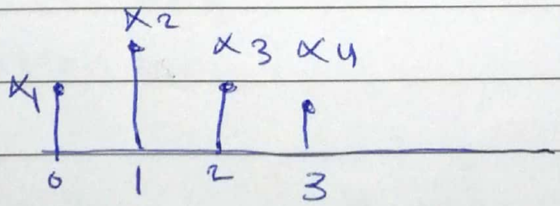
Ex consider

a linear time-invariant D.T system withe an input-output relation

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

find $h[n]$, wich is the impulse response

$$h[n] = \alpha_1 \delta[n] + \alpha_2 \delta[n-1] + \alpha_3 \delta[n-2] + \alpha_4 \delta[n-3]$$



$$h[n] = \{\underset{\uparrow}{\alpha_1}, \alpha_2, \alpha_3, \alpha_4\}$$

FIR system

FIR systems → finite impulse response system

Ex

consider LTI system with an input-output relation:

$y[n] = x[n] + \frac{1}{2} y[n-1]$ find ~~*~~ $h[n]$

given that $y[n] = 0$ for $n < 0$

Solution

$h[n] = \delta[n] + \frac{1}{2} y[n-1]$

$n=0 \rightarrow h[0] = \delta[0] + \frac{1}{2} y[-1] = 1$ $\quad (\frac{1}{2})^0$

$n=1 \rightarrow h[1] = \delta[1] + \frac{1}{2} y[0] = \frac{1}{2}$ $\quad (\frac{1}{2})^1$

$n[2] \rightarrow h[2] = \delta[2] + \frac{1}{2} y[1] = \frac{1}{4}$ $\quad (\frac{1}{2})^2$

$n=3 \rightarrow h[3] = \delta[3] + \frac{1}{2} y[2] = \frac{1}{8}$ $\quad (\frac{1}{2})^3$

$\therefore h[n] = \left(\frac{1}{2}\right)^n , n \geq 0$

$$h[n] = \begin{cases} 0 , & n < 0 \\ (1/2)^n , & n \geq 0 \end{cases}$$

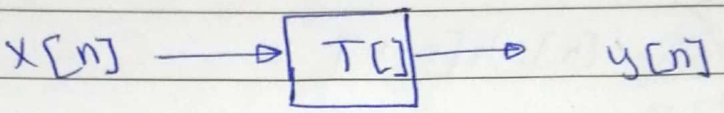
$h[n] = \left(\frac{1}{2}\right)^n u[n]$. IR system

IR system → infinite impulse response system

Time Domain Characterization of LTI Discrete time systems

Any arbitrary D.T signal can be represented as a weigthed sum of shifted deltas:-

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$



$$y[n] = T[x[n]] = T\left[\sum_{K=\infty}^{\infty} x[K]\,\delta[n-K]\right]$$

If the system is linear:-

$$y[n] = \sum_{k=-\infty}^{\infty} x[K]\,T[\delta[n-K]]$$

$$= \sum_{k=-\infty}^{\infty} x[K]\,\delta[(n,K)]$$

$T[\delta[n]] = h[n]$

If the system is also time invariant

$$y[n] = \sum_{k=-\infty}^{\infty} x(K)\,h[n-K] = x[n] \circledast h[n] \qquad T[\delta[n-k]] = h[n-k]$$

$\underbrace{}$ → convolution sum

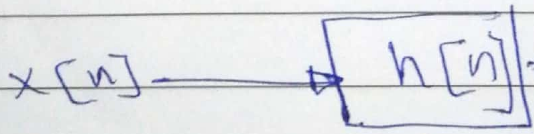
* the same

$$y[n] = \sum_{k=-\infty}^{\infty} h[K]\,x[n-K]$$

commutative operation

$$y[n] = h[n] \circledast x[n] = x[n] \circledast h[n]$$

- Given h[n] ~~and~~ for a system we can find y[n] for any input x[n] by performing the convolution sum



$y[n] = x[n] \circledast h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Ex: $x[n] = \{\underset{\uparrow}{2}, 0, 1, -1, 3\}$ length 5 sequence (N_1)

$h[n] = \{1 \quad 2 \quad 0 \quad -1\}$ length 4 sequence (N_2)

find $y[n] = x[n] \circledast h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[n] = \sum_{k=0}^{4} x[k]h[n-k]$$

$$y[0] = \sum_{k=0}^{4} x[k] \cancel{\Sigma} h[-k] = x[0]h[0] + x[1]\cancel{h[-1]} + x[2]h[2] + h[3]\cancel{x[3]}$$
$$+ x[4]\cancel{h}[-4] = -2 \times 1 = -2$$

$$y[1] = \sum_{k=0}^{4} x[k]\, h[1-k]$$

$$= x[0]h[1] + x[1]h[0] + x[2]\overset{0}{\cancel{h[-1]}} + x[3]\overset{0}{\cancel{h[-2]}} + x[4]\overset{0}{\cancel{h[-3]}}$$
$$= -2 \times 2 + 0 \times 1 = -4$$

$$y[2] = \sum_{k=0}^{4} x[k]\, h[2-k]$$

$$= x[0]\, h[2] + x[1]h[1] + x[2]h[0] + x[3]\cancel{h[-1]} + x[4]\cancel{h[-2]}$$
$$y[2] = (-2)(0) + (0)(2) + (1)(1) = 1$$

⋮

please do the other values

$y[7] = \sum_{k=0}^{4} x[k]\, h[7-k]$ the last value]

$= \cancel{x[0]\, h(7)} + \cancel{x[1]\, h[6]} + \cancel{x[2]\, h[5]} + \cancel{x[3]\, h[4]} + x[4]\, h[3]$

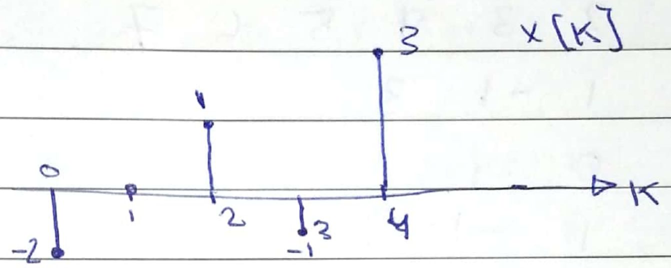
$= 3 \times -1 = -3$

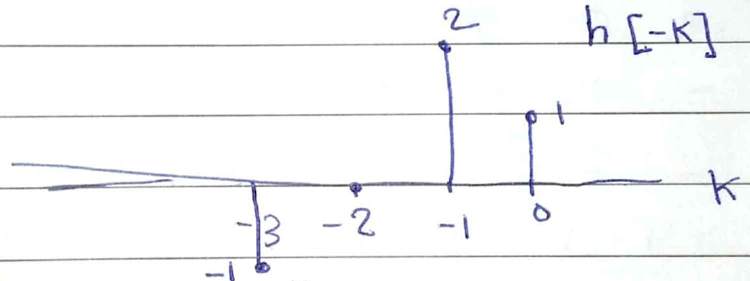
$y[n] = \{\underset{\uparrow}{-2}, \underbrace{-4, 1, 3, 1, 5, 1, -3}\}$

$\uparrow$

$N_1 + N_2 - 1 = M$ y[n] is length M sequence

Graphical Solution:





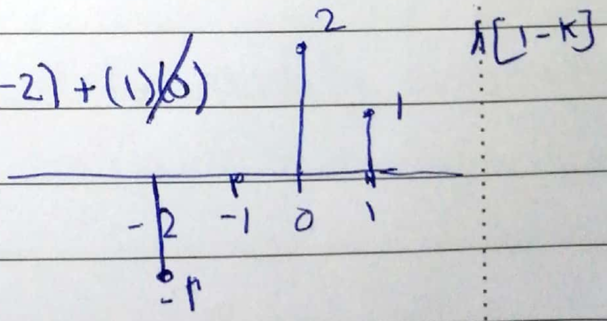
$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$n = 0 \rightarrow y[0] = \cancel{\sum_{k=-\infty}^{\infty} x[0]\, h[n-0]} \sum_{k=-\infty}^{\infty} x[k]\, h[0-k]$

$= -2 \times 1 = -2$

$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[1-k] = (2)(-2) + (1)\cancel{(0)}$

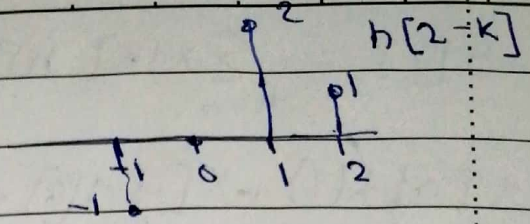
$= (-2)(2) = 4$



Subject Date No.

$y[0] = (1)(1) + (0)(2) + (0)(-2)$

$= 1$



⋮

- As apractice do the rest of the graphical solution

Tabular Method For convolution sum

n	0	1	2	3	4	5	6	7	
x[n]	-2	0	1	-1	3				
h[n]	1	2	0	-1					
	-2	0	1	-1	3				
	-	-4	0	2	-2	6			+
	-	-	0	0	0	0	0		
	-	-	-	2	0	-1	1	-3	

$y[n] = \{ \underset{\uparrow}{-2} \quad -4 \quad 1 \quad 3 \quad 1 \quad 5 \quad 1 \quad -3 \}$

N O T E B O O K

Ex:

$$x[n] = \{-2 \quad 0 \quad \underset{\uparrow}{1} \quad -1 \quad 3\}$$

$$h[n] = \{1, \quad \underset{\uparrow}{2} \quad 0 \quad -1\}$$

$$y[n] = \{-2 \quad -4 \quad 1 \quad \underset{\uparrow}{3}. \quad 1 \quad 5 \quad 1 \quad -3\}$$ (go back to L8)

L9:-

Stability Condintions in Terms of the impulse response

Recall → The BIBO stability criterien.

for every Bounded input $|x[n]| \leq \beta_x < \infty$, for all n

→ the output Bounded $|y[n]| \leq \beta_y < \infty$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]| \; --$$

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]| \cdot \beta_x = \beta_x \underbrace{\sum_{k=-\infty}^{\infty} |h[k]|}_{\beta_y}$$

$$\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$$

↳ finite value

→ The impulse response should be absolutly summable.

BIBO stable system

Causality condition in terms of impulse response

recall

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] \qquad \text{(LTI system)}$$

$$y[n] = \sum_{k=-\infty}^{-1} h[k]\,x[n-k] + \sum_{k=0}^{\infty} h[k]\,x[n-k]$$

$$= \Big[\,\cdots + \cancel{h[-3]\,x[n+3]} + \cancel{h[-2]\,x[n+2]} + \cancel{h[-1]\,x[n+1]}\,\Big] \rightarrow \text{future values of } x[n]$$

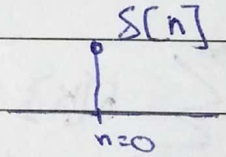
$$+ \Big[h[0]\,x[n] + h[1]\,x[n-1] + h[2]\,x[n-2] + h[3]\,x[n-3]\Big] \rightarrow \text{present and previous values of } x[n]$$

A causal system

$$y[n] = T\big[x[n],\ x[n-1],\ x[n-2]\big]$$

for causality → $h[k] = 0 \quad k < 0$

$$\delta[n] \xrightarrow{T} h[n]$$



- If we apply $x[n] = \delta[n]$ at $n = 0$ for a causal system $y[n]$ should have values for $n = 0$ and above
- No output before you apply an input

The Geometric Series

$$S = \sum_{n=0}^{\infty} \alpha^n$$

compact formula for S

$$S_N = \sum_{n=0}^{N} \alpha^n = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{N-1} + \alpha^N$$

$$S_{N+1} = \sum_{n=0}^{N+1} \alpha^n = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{N-1} + \alpha^N + \alpha^{N+1}$$

$$S_{N+1} = S_N + \alpha^{N+1} = 1 + \alpha\underbrace{[1 + \alpha + \alpha^2 + \cdots + \alpha^N]}_{S_N}$$

$$S_{N+1} = 1 + \alpha S_N = S_N + \alpha^{n+1}$$

$$S_N(1-\alpha) = 1 - \alpha^{N+1} \rightsquigarrow \boxed{S_N = \frac{1-\alpha^{N+1}}{1-\alpha}}$$

As $N \to \infty$

i: S_N converges and has the sum $\frac{1}{1-\alpha}$ if $|\alpha| < 1$

ii: S_N diverges if $|\alpha| \geq 1$

Ex:

consider an LTI D.T system with an impulse response given by $h[n] = \alpha^n u[n]$, $|\alpha| < 1$, Is the system stable? Is the system causal

$$h[n] = \begin{cases} \alpha^n, & n \geq 0 \quad |\alpha| < 1 \\ 0, & n < 1 \end{cases}$$

1) To check for the stability: The impulse response should be absolutely summable

$$S = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} | \alpha^{n} \mu[n] |$$

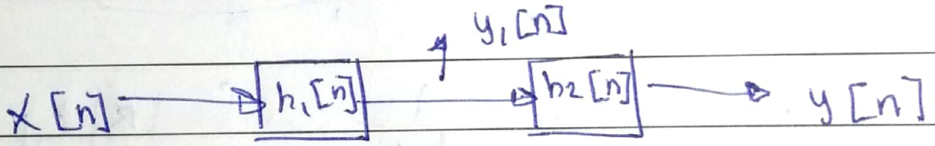
$$= \sum_{n=0}^{\infty} |\alpha^{n}| = \sum_{n=0}^{\infty} |\alpha|^{n}$$

$$S = \frac{1}{1-|\alpha|} \longrightarrow$$ yes, it's stable

2) $h[n] = 0 \quad n < 0 \longrightarrow$ Causal system

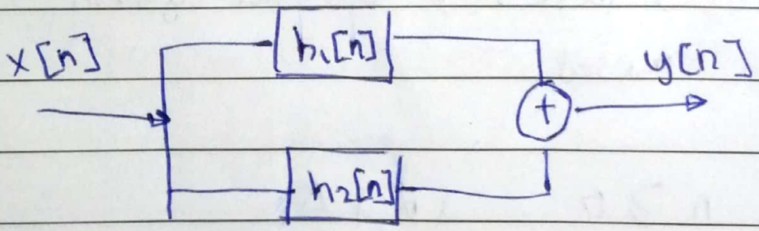
- Cascade and Parallel conections of LTI systems

→ Cascade system.



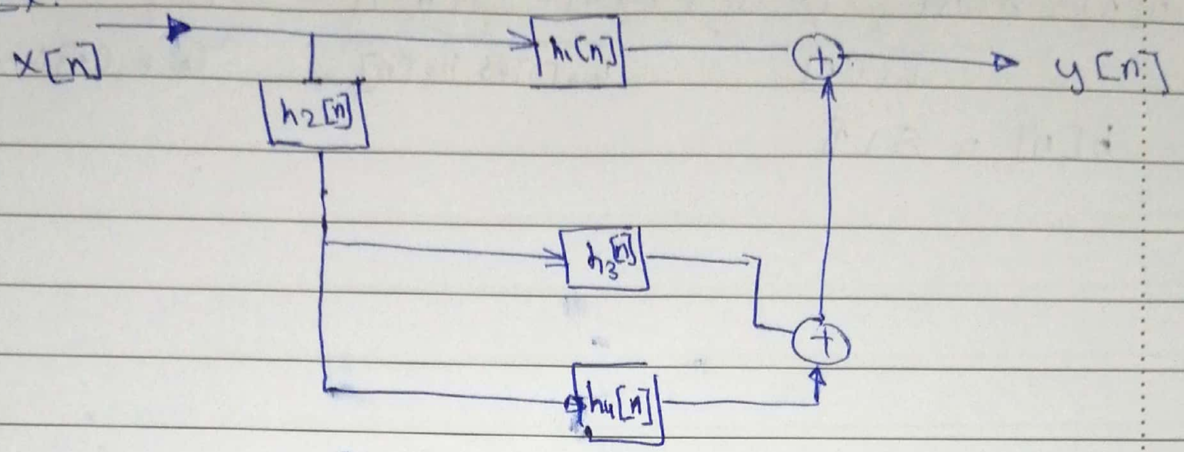
$h[n] = h_1[n] \circledast h_2[n]$ (Prove this result)

→ Parallel conection



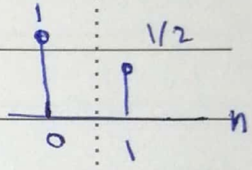
$h[n] = h_1[n] + h_2[n]$ (Prove this result)

Ex:



$$h[n] = h_1[n] + h_2[n] \circledast (h_3[n] + h_4[n])$$
$$= h_2[n] + h_2[n] * h_3[n] + h_2[n] \circledast h_4[n]$$

- Give that $h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$



$$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-11]$$
$$h_3[n] = 2\,\delta[2]$$
$$h_4[n] = -2\left(\frac{1}{2}\right)^n \mu[n]$$

Recall that $x[n] \circledast \delta[n] = x[n]$

and $x[n] \circledast \delta[n-k] = x[n-k]$

$$h_2[n]\, h_3[n] = (\tfrac{1}{2}\delta[n] - \tfrac{1}{4}\delta[n-1] \circledast \delta[n]$$
$$= \delta[n] - \frac{1}{2}\delta[n-1]$$
$$h_2[n]\, h_4[n] = (\tfrac{1}{2}\delta[n] - \tfrac{1}{4}\delta[n-1]) \circledast (-2(\tfrac{1}{2})^n \mu[n])$$
$$= (\tfrac{1}{2})^n \mu[n] + \tfrac{1}{2}(\tfrac{1}{2})^{n-1} \mu[n-1)$$

(common factor $\cancel{\frac{1}{2}}$)

$$= (\tfrac{1}{2})^N (-\mu[n] + \mu[n-1)$$
$$= -(\tfrac{1}{2})^N \delta[n] = -(\tfrac{1}{2})^0 \delta[n] = -\delta[n$$

$$h[n] = \underbrace{\delta[n] + \frac{1}{2}\cancel{\delta[n-1]}}_{h_1[n]} + \underbrace{\cancel{\delta[n]} - \frac{1}{2}\cancel{\delta[n-1]}}_{h_2[n] \circledast h_3[n]} - \underbrace{\cancel{\delta[n]}}_{[h_2[n] \circledast h_3[n]]}$$

$$h[n] = \delta[n]$$