LINEAR ALGEBRA

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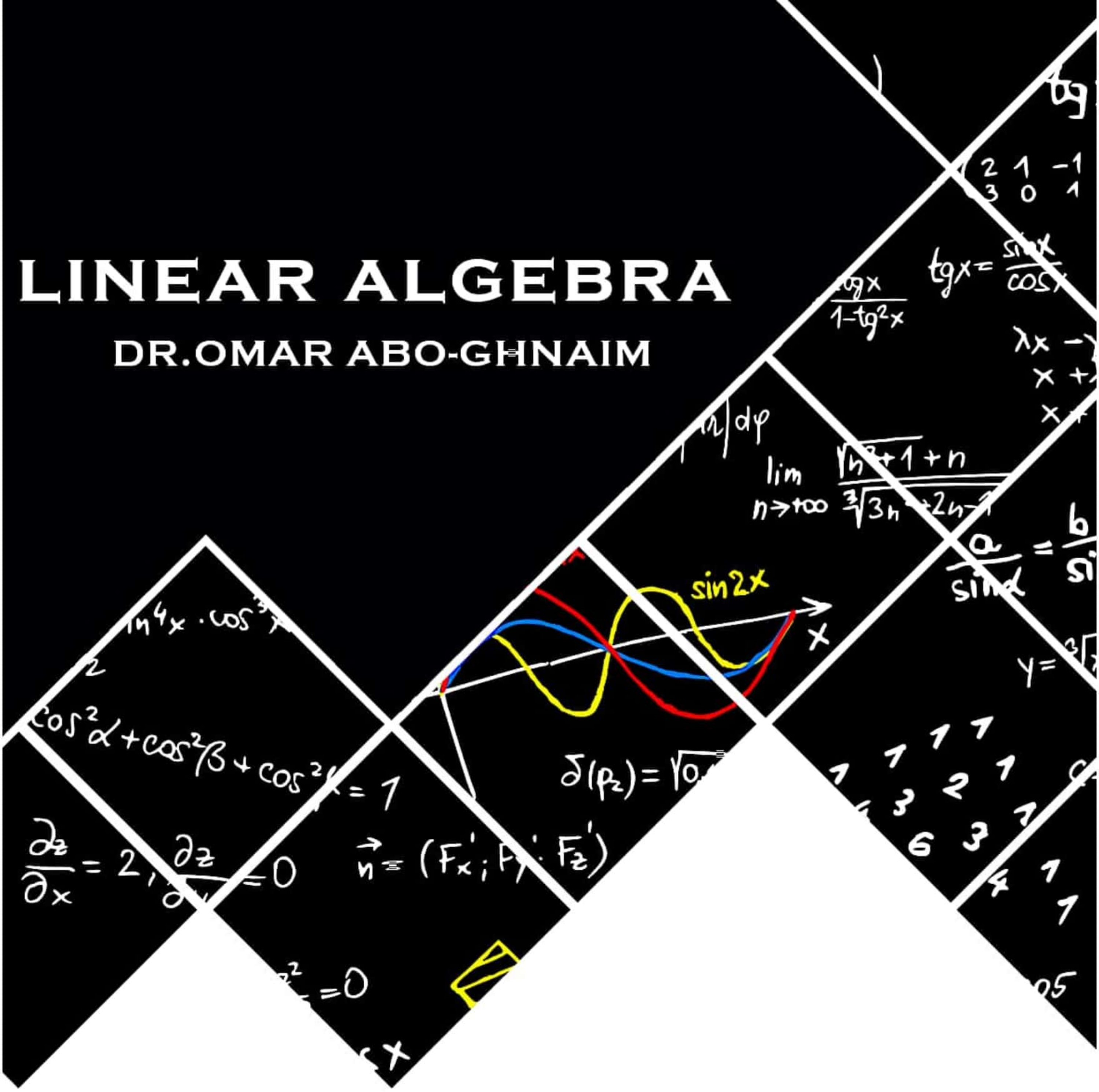

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POWERUNIT

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Ch1 System of Linear Equations and Matrices

1.1

* Def: Linear Equation in variables $x_1, x_2 \cdots x_3$ is an equation that can be expressed in the form $[a_1x_1 + a_2x_2 + \cdots + a_mx_m = b]$ ($x_1, x_2 \cdots$ variables)

* Def: A System of m Linear equations in n unknowns ($x_1, x_2 \cdots x_3$) can be written

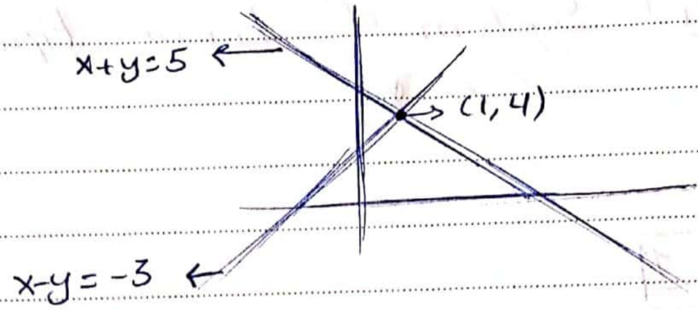
$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

* Def: A solution of the $\boxed{\text{system}}$ is a sequence of n numbers ($s_1, s_2, \cdots, s_n$) such that all equations are satisfied when we substitute ($x_1 = s_1, x_2 = s_2 \cdots x_m = s_m$)

Ex: $\left.\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 + 2x_2 &= 8 \end{aligned}\right]$ $x_1 = 2$ $x_2 = 2$ is a solution
also $x_1 = 5$, $x_2 = -1$ is asolution

Ex: $x+y=5$

$x-y=-3$

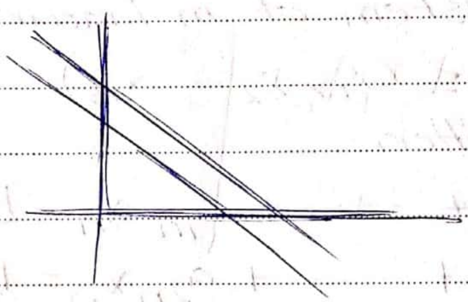


لوبدي اجمع المعادلتين
عشان احكم الحل بطلع

$x=1 \rightarrow$ نقطة التقاطع

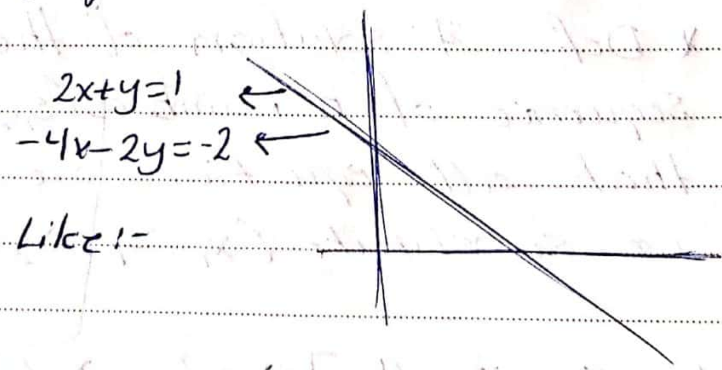
Ex: $x+y=5$ (parallel Lines have no solution)

$x+y=7$



Ex: $2x+y=1$

$-4x-2y=-2$ → they are the same Line on the graph.



they can be solved Like:-

$y=1-2x$

$x=t \rightarrow t \in \mathbb{R}$ (as t is a parameter)

$y=1-2t$

* So the System has infinitly many solutions $\begin{bmatrix} x=0 \\ y=1 \end{bmatrix}$, $\begin{matrix} x=1 \\ y=-1 \end{matrix}$ ----

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$\boxed{\text{Theory}}$: Every system of Linear equations has no solution, exactly one solution, or infinitly many solutions. In case there is a solution, we say the system is consistent. Otherwize, we say the system is inconsistent.

* How to solve the system of Linear equations ⇓

$$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 - x_2 &= 5 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 1 & 4 \\ 2 & -1 & 5 \end{bmatrix}$$

Elimination	Augmented Matrix
	* Elementary row operations
① Multiplying an equation by a nonzero number	* Multiplying a row by a nonzero number
② Interchange two equations	* Interchange two rows
③ Add a multiple of one equation to another equation	* Add a multiple of one row to another row

* We call the three steps done on the augmented matrix (elementary row operations)

Ex:

$x + y + 2z = 9$
$2x + 4y - 3z = 1$
$3x + 6y - 5z = 0$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix} \begin{matrix} -2R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 6 & -22 & -54 \end{bmatrix} \rightarrow 2R_3$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1 & -3 \end{bmatrix} \rightarrow -3R_2 + R_3 \rightarrow R_3$$

$-z = -3 \rightarrow \boxed{z = 3} \longrightarrow$

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$2y - 7z = -17 \xrightarrow{z=3} \boxed{y=2}$

$x + y + 2z = 9 \xrightarrow[y=2]{z=3} \boxed{x=1}$

→ Backward Substitution

* you can make the first nonzero number in the row (one)

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

* We say that this matrix is in echelon form

Leading ones

* This procedure of solution is called Gaussian Elimination Method.

$\boxed{OR}$ leading ones بنصفر الارقام إلي فوق ال وبطلع القيم مباشرةً → This method is called Jordan - Gauss Elimination method.

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow -R_3$$

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$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{array}{l} 7R_3 + R_2 \to R_2 \\ -2R_3 + R_1 \to R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \frac{R_2}{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad -R_2 + R_1 \to R_1$$

So: $z=3$, $y=2$, $x=1$

* This Matrix is in reduced row echelon form

A matrix in reduced echelon form (we get the matrix by solving using Gauss-Jordan Elimination) if all the following are satisfied: $\longrightarrow$

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① If a row does not consist entirely of zeros, then the first nonzero number is 1 (leading ones)

② If there are any rows that consists entirely of zeros, then they are grouped together at the bottom of the matrix ✓

③ In any two successive rows that do not consist entirely zeros, the leading one in the lower row occurs further to the right than the leading ones in the higher row ✓

④ Each column that contains a leading one has zeros every where else in that column.

* A matrix that has the first three properties is said to be in row echelon form.

Examples for
⇒ Matrices in reduced echelon form.

① $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 7 \end{bmatrix}$ ② $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ →

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(3) $\begin{bmatrix} 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(5) $\begin{bmatrix} 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (* = non zero)

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Ex :- Matrices in row echelon form but not in reduced echelon form :-

① $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

② $\begin{bmatrix} 0 & 1 & 0 & 0 & 9 & 6 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

③ $\begin{bmatrix} 1 & * & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ($* \rightarrow$ nonzero)

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Ex: Solve.

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2$$

$$x_1 + x_2 + x_3 - x_4 + x_5 = 0$$

$$4x_4 = 4$$

$$x_3 + x_5 = 3$$

Sol:-

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{bmatrix} \rightarrow \text{Augmented Matrix}$$

أهم شي اتخلص
من إلي تحت
الـ leading ones

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{bmatrix} \rightarrow -R_1 + R_2 \rightarrow R_2$$

ليش ما اتخلصنا منه

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad R_4 \rightarrow R_2, \quad \frac{R_2}{-2}, \quad \frac{R_3}{4}$$

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$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad -R_3 + R_4 \to R_4$$

الصف الخامس ما بيعطي اشي وتجيبه هيك ← $(0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0)$

Backward substitution

→ $x_4 = 1$ l.v

→ $x_3 + x_5 = 3$

$x_3 = 3 - x_5 \quad (x_5 = \text{free variable})$

$x_5 = t \quad , \quad t \in \mathbb{R}$

so → $x_3 = 3 - t$ leading variable

→ $x_1 + x_2 + x_3 + x_4 + x_5 = 2$

$x_1 = 2 - x_2 - x_3 - x_4 - x_5$

$= 2 - x_2 - (3 - t) - 1 - t$

l.v

$x_1 = -2 - x_2 \quad (x_2 = \text{free variable})$

$x_2 = s \quad , \quad s \in \mathbb{R}$

so → $x_1 = -2 - s$

* This system has infinitly many solutions

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* In the last example (x_1, x_3, x_4) were the (Leading variables)

Remark :-

① x_1, x_3, x_4 are called Leading variables [x_2 and x_5 were the free variables]

② We call this solution the general solution.

③ positions of leading ones in row echelon form are called $\boxed{\text{pivot positions}}$

* A column that contains a pivot position is called a pivot column

* In the last example
* pivot columns are column 1, Column 3, column 4

* pivot position for the first leading one is first row, first column

* pivot position for the second leading one is second row, third column

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Ex: find the solution of:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2$$

$$x_1 + x_2 + x_3 - x_4 + x_5 = 0$$

$$4x_4 = 4$$

$$x_3 + x_5 = 3$$

Solu:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right]$$

* Gaussian Elimination

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

→ Row echelon form

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$x_1, x_3, x_4 \to$ leading variables

x_2 and x_5 free variables

$\longrightarrow$ Continuing solution with [Gauss - Jordan Elimination]

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \text{by } 2 \quad -R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to -R_3 + R_1 \to R_1$$

↳ Reduced row echelon form.

$\to x_4 = 1$

$x_3 + x_5 = 3 \to x_5 = t, \ t \in \mathbb{R}$

$x_3 = 3 - t$

$x_1 + x_2 = -2 \to x_2 = s, \ s \in \mathbb{R}$

$x_1 = -2 - s$

Ex: Solve

$$x_1 + x_2 + x_3 = 3$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 + 4x_2 + 6x_3 = 0$$

Solu

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 2 & 4 & 6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & -3 \end{bmatrix} \quad \begin{matrix} -R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -9 \end{bmatrix} \quad -2R_2 + R_3 \to R_3$$

$$0x_1 + 0x_2 + 0x_3 = -9$$
$$0 \neq -9$$

So this System is inconsistent.

has no solution

Ex: for which values of "k" the system:

$x+y=4$

$2x+2y=k$

have exactly one solution? no solution? or infinitly many solutions?

Solu:

$$\begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & k \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & k-8 \end{bmatrix} \quad -2R_1+R_2 \to R_2$$

$0x_1+0x_2=k-8$ $\left(\begin{matrix} k-8 \neq 0 \\ k \neq 8 \end{matrix} \right)$

* If $k \neq 8$, then the system has no solution

* If $k=8$, $\longrightarrow x+y=4$ $\begin{matrix} y=t \\ t \in \mathbb{R} \end{matrix}$ So $x=4-t$

So when $k=8$, the system has infinitly many solutions

Suggested prob for $\boxed{1.1}$ $\left[\begin{matrix} 1, 5, 7, 9, 11 \\ 15, 20, 21, 27 \\ \text{TF ex} \end{matrix} \right.$

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Def: A System of Linear is said to be homogenuous if the constant terms are all zero. That is the system of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Remarks:

① Every hom. System of Linear equations is consistent

Since $x_1=0$, $x_2=0$, — $x_n=0$ is a solution, this solution is called the trival solution

② If there are other solutions, they are called nontrival solutions.

③ The hom. System has two ~~possible~~ possibilites:

Ⓐ The System has only the trival solution

Ⓑ The System has infinitly many solutions

Ex: Solve the following hom. system:

$$\begin{aligned}
2x_1 + 2x_2 - x_3 \qquad\qquad + x_5 &= 0\\
-x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0\\
x_1 + x_2 - 2x_3 \qquad\qquad - x_5 &= 0\\
x_3 + x_4 + x_5 &= 0
\end{aligned}$$

Solu:

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0\\ -1 & -1 & 2 & -3 & 1 & 0\\ 1 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0\\ -1 & -1 & 2 & -3 & 1 & 0\\ 2 & 2 & -1 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_1 + R_2 \to R_2\\ -2R_1 + R_3 \to R_3 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 0 & -3 & 0 & 0\\ 0 & 0 & 3 & 0 & 3 & 0\\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \leftrightarrow R_4\\ \dfrac{R_3}{3} \end{array}$$

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$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \begin{matrix} -R_2 + R_3 \to R_3 \\ \\ \frac{R_4}{-3} \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 + R_4 \to R_4 \\ -R_3 \end{matrix}$$

$$\begin{bmatrix} \boxed{1} & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_4 = 0$

$x_3 + x_4 + x_5 = 0 \xrightarrow{x_4 = 0} x_3 + x_5 = 0$

$x_5 = t$

$x_3 = -t \quad , t \in R$

$x_1 + x_2 - 2x_3 - x_5 = 0 \qquad x_2 = S \ , \ S \in \mathbb{R}$

$x_1 = -S + -2t + t$

$x_1 = -t - S$

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Remark: - A hom. system of Linear equations with more unknowns than equations has infinitely many solutions.

Suggested prob. for $\boxed{1.2}$: 1–13 odd, 14
15–25 odd, 26
27–31 odd
35–39 odd 42
TF Q.

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1.3 Matrices and Matrix Operations

Def: A matrix is a rectangular arrays of numbers

Numbers are called: entries of the matrix

Ex: $\begin{bmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 3 \end{bmatrix}_{3\times 2}$ $\begin{bmatrix} 11 & 1 \\ 79 & 13 \end{bmatrix}_{2\times 3}$

* Size of the matrix = # of rows × # of columns

* A Matrix with only one column is called a column matrix or column vector.

* A Matrix with only one row is called a row matrix or row victor.

* We denote matrices by Capital Letters and numbers by small letter.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$A = [a_{ij}]_{2\times 3}.$$

* An $(n \times n)$ matrix is called a square matrix

Ex $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 6 \\ 7 & 5 & 11 \end{bmatrix}_{3\times3}$

* $n \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \rightarrow \text{Diagonal of the matrix}$$

* Two matrices are equal if they have the same size and their corresponding entries are equal.

$(A = B$ if $a_{ij} = b_{ij})$

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Ex : $A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

$C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}$

$A = B$ if $x = 5$

But $B \neq C$

* if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices with the same size and c is a constant, then

$A \mp B = [a_{ij} \mp b_{ij}]$

$cA = [c\, a_{ij}]$

Ex: if $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ 6 & 4 \\ 1 & 5 \end{bmatrix}$

Then $2A - B = \begin{bmatrix} 2 & -2 \\ -4 & -2 \\ 5 & 5 \end{bmatrix}$

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* Multiplying matrices :-

$$\begin{bmatrix} 2 & -1 & 3 \\ \boxed{1 \quad 4 \quad 2} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 5 \\ 7 \end{matrix}} & \begin{matrix} 4 \\ 6 \\ 4 \end{matrix} \end{bmatrix} = \begin{bmatrix} 20 & 14 \\ 36 & 36 \end{bmatrix}$$

(2×3) (3×2)

* Rules:-

If A is an $(n \times r)$ matrix and B is an $(r \times m)$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}\,b_{2j} + \cdots + a_{ir}\,b_{rj}$$

$$= \sum_{t=1}^{r} a_{it}\, b_{tj}$$

* ith row of the matrix $AB = B \times$ [ith row of the Matrix A]

* jth of the matrix $AB = A \times$ [jth column of Matrix B]

Ex :

$$\begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & 1 \\ -4 & -1 & 3 \\ & & 1 \end{bmatrix}$$

(3×2) (2×3)

$\times$ = (3×3)

* $A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \equiv$ Rows of the matrix A

$$AB = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} * B = \begin{bmatrix} r_1 B \\ r_2 B \\ \vdots \\ r_n B \end{bmatrix}$$

* $B = [b_1 \; ---- \; b_m] \equiv$ Columns of matrix B

$$AB = A[b_1 \; ---- \; b_m] = [Ab_1 \; ---- \; Ab_m]$$

* The system

$$a_{11}x_1 + a_{12}x_2 + ---- + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ---- + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ---- + a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$Ax = b$

Solving the system $\longrightarrow$ finding x : $Ax=b$

Ex: $2x_1 + 3x_2 - x_3 = 0$

$x_1 + x_2 - x_3 = 1$

$2x_1 + x_3 = 5$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

$Ax=b$

* Def: If $A=[a_{ij}]$, then the transpose of A, denoted by

$$A^T = [a_{ji}]$$

Ex: $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 7 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 5 & 7 \end{bmatrix}$.

Ex: $A = \begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 7 \\ 9 & 10 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 & 9 \\ 2 & 5 & 10 \\ -1 & 7 & 0 \end{bmatrix}$

* Def: If A is an $(n \times n)$ matrix, the the trace A is donated by $tr(A)$

$$\equiv \sum_{i=1}^{n} a_{ii}$$

Ex: if $A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 4 & 9 & 3 & 7 \\ 1 & 1 & -9 & 5 \\ 2 & 3 & 4 & 2 \end{bmatrix}$, the $tr(A) = 3$

Sec: $\boxed{1.4}$: Inverses, Algebric Properties of Matrices :

* let A, B and C matrices and a, b and c numbers.

① $A+B = B+A$ (Addition is Commutative)

② $(A+B)+C = A+(B+C)$ (Addition is associative)

③ $A(B \mp C) = AB \mp AC$

④ $(B \mp C)A = BA \mp CA$

⑤ $a(A \mp B) = aA \mp aB$

Remark: Matrix Mult is not commutative.

⑥ $(a \mp b)A = aA \mp bA$

⑦ $(ab)C = a(bC)$

⑧ $a(BC) = (aB)C$

⑨ $(AB)C = A(BC)$.

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① AB is defined but BA is undefined.

Ex: $A_{2\times3}$, $B_{3\times4}$ So $AB_{(2\times4)}$

BA undefined. $B_{3\times4}$ $A_{2\times3}$

② AB and BA have different sizes.

Ex $A_{2\times3}$ $B_{3\times2}$

$AB_{2\times2}$
$BA_{3\times3}$ different sizes.

③ $AB \neq BA$

$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} -1 \\ * \end{bmatrix}$

$BA = \begin{bmatrix} 3 \\ * \end{bmatrix}$

$\neq$

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The Matrix $O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$

* $O + A = A$
* $A + (-A) = O$
  $-A + A = O$

* $ab = ac \rightarrow (a \neq 0)$
  $\rightarrow b = c$

* $ab = 0 \rightarrow a = a = 0$ or $b = 0$

* Cancelation law does not hold for matrices

Ex $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$. Then

$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, but $B \neq C$

Also, $AD = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $A \neq O$ and $D \neq O$

Def: The identity matrix is an $n \times n$ matrix, donated by I_n or $I \equiv \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

Remark: $IA = AI = A$

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 4 & 9 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 9 & -3 \end{bmatrix}$

Theorem: If R is reduced row echelon form of an $n \times n$ matrix A, then neither R has a row of zeros or R is the identity matrix begin with leading one.

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Def: If $\underline{\underline{A}}$ is asquare matrix and there exist $\underline{\underline{B}}$, if the same size with $AB = BA = I$, then we say $\underline{\underline{A}}$ is invertible and $\underline{\underline{B}}$ is called the inverse of $\underline{\underline{A}}$

If no such $\underline{\underline{B}}$ exist, then we say $\underline{\underline{A}}$ is singular

Ex: $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is the inverse of $\underline{\underline{A}}$

So $\rightarrow$ $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex: The matrix $\begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ is singular

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$$\Rightarrow B\begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{bmatrix} \neq I$$

Remark:- ① Matrices with zero column are singular

② Matrices with zero row are singular

Theorem:- If "B" and "A" are both inverses of "A", then $B = C$

Proof: We have $AB = BA = I$

$$AC = CA = I$$

Consider $BAC = IC = C$

$$BAC = BI = B$$

Thus $B = C$

* We denote the inverse of A by A^{-1}

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Theorem: The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if $(ad - bc \neq 0)$

In this case $\rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ex: If $A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$ then find A^{-1}

Solu: $A^{-1} = \frac{1}{-1} \begin{bmatrix} 4 & -9 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$

* If "A" and "B" are invertible, the "AB" is invertible with $(AB)^{-1} = B^{-1}A^{-1}$

Proof: $AB(A^{-1}B^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

* If $A_1, A_2, \text{——}, A_n$ are invertible, then $(A_1 A_2 \text{——} A_n)$ is invertible with
$(A_1 A_2 \text{——} A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \text{——} A_2^{-1} A_1^{-1}$

* If "A" is a square matrix with $AB=I$, then $BA=I$

Remark: ① $A^n = \underbrace{AA---A}_{n \text{ times}}$

② If A is invertible, then

$$A^{-n} = \underbrace{A^{-1}A^{-1}---A^{-1}}_{n \text{ times}}$$

③ $A^0 = I$ → I in Matrices acts like (1) in numerical operations.

④ $(A^n)^m = A^{nm}$

⑤ $(A^n)(A^m) = A^{n+m}$

⑥ If "A" is invertible, then "A^{-1}" is invertible with $(A^{-1})^{-1} = A$

* When Point 6. is correct, then "A^n" is invertible with $(A^n)^{-1} = (A^{-1})^n$
then → $(A^n(A^{-1})^n = (AA^{-1})^n = I^n = I)$

* When Point 6. is correct and $k \neq 0$, then kA is invertible with $(kA)^{-1} = \frac{1}{k} A^{-1}$

* If $p(x) = a_0 + a_1 x + a_2 x^2 + \text{——} + a_n x^n$,
then $P(A) = a_0 I + a_1 A + a_2 A^2 + \text{——} + a_n A^n$

Ex: If $p(x) = 2x^2 - 3x + 4$ and $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$

then find $P(A)$

Solu: $P(A) = 2A^2 - 3A + 4I$

$$= 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix}$$

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Ex: If $P(x) = 2x^2 - 3x + 4$ and $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$

then find $P(A)$.

Solu: $P(A) = 2A^2 - 3A + 4I$

$$= 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix}$$

* If "A" and "B" are invertible, then it is not ~~m~~ necessarly that $A \mp B$ invertible

$$* \ (A+B)^{-1} \neq A^{-1} \mp B^{-1}$$

$$\left(A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \right)$$

* $(AB)^T = B^T A^T$

* $(A \mp B)^T = A^T \mp B^T$

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* $(KA)^T = KA^T$

* If A is invertible, then A^T is invertible with $(A^T)^{-1} = (A^{-1})^T$

Def: A matrix is symmetric if $A = A^T$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 7 & 9 \end{bmatrix}$$ is symmetric

$$A^T = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 7 & 9 \end{bmatrix}$$

Ex: If $(3A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ then

find $A \rightarrow$

Solu: $\frac{1}{3}(A^T)^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

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Ex: If A satisfies $A^2-4A+6I=0$, then show that A is invertible and find A^{-1}

Solu: $A^2-4A+6I=0$

$A^2-4A=-6I$

$-\frac{1}{6}A^2+\frac{4}{6}A=I$

$A\left(-\frac{1}{6}A+\frac{4}{6}I\right)=I$

So A is invertible if

$A^{-1}=\frac{-1}{6}A+\frac{4}{6}I$

Suggested prob for

1.4: 15-23 odd.

31, 36, 39, 43

TF

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Sec: 1.5 : Elementry Matrices and a method for finding A^{-1}

Def: A $n \times n$ matrix is called elementry matrix if it can be obtained from I_n by performing a single elementry row operation.

Ex:

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are elementry matrices

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Theorem: If the elementry matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix results when this same row operation is performed on A.

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Sec 1.6: More on Linear Systems and invertible matrices

<u>Theorem:</u> If "A" is invertible $n \times n$ matrix then for any matrix b ($n \times 1$) matrix, the system $Ax = b$ has exactly one solution $\rightarrow$ ($x = A^{-1}b$)

<u>Proof:</u> ① first we show $A^{-1}b$ is a solution

x نعوض بال $\rightarrow$ $A(A^{-1}b) = (AA^{-1})b = Ib = b$

$\therefore$ So, $A^{-1}b$ is a solution.

* proof that it's the only solution.

Suppose x_0 is solution of $Ax = b$,

then $Ax_0 = b$ $\rightarrow$ $x_0 \rightarrow$ should be equal to $(A^{-1}b)$

$\rightarrow$ Multiply by A^{-1} $\rightarrow$ $A^{-1}Ax_0 = A^{-1}b$

$\rightarrow Ix_0 = A^{-1}b \rightarrow x_0 = A^{-1}b$

<u>Ex:</u> Solve

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 5x_2 + 2x_3 = 3$$
$$x_1 + 8x_3 = 17$$

$\rightarrow$ $Ax = b$ where

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ and A is invertible with $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

Solu: according to the theorem.

$$x = A^{-1}b = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

The solution
$x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

Fundamental theorem: A is $n \times n$ matrix
the following are equivalant

1. A is invertible.
2. $Ax = 0$, has only the trivial solution

⋮

5. $Ax = b$ is consistant for every b.
6. $Ax = b$ has exactly one solution for any b

* prove that $1 \rightarrow 6 \rightarrow 5$

the last theorem

$6 \rightarrow 5$ [if a system has only one solution it should be consistant]

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Proof → $5 \to 1$

① If the system $Ax=b$ is consistant for every b, then in particular

the system $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, ---, $Ax = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

nxn matrix ←

Let $x_1, x_2, \dots x_n$ be solutions of these systems and.

$C = [x_1, x_2 \text{ ------ } x_n]$

Now, $AC = A[x_1, x_2 \text{ ------ } x_n]$

$= [Ax_1 \text{ ------ } Ax_n]$

$= \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} = I$

Thus $A^{-1} = C$ and A is invertible

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Theorem: let A and B be square matrices.
If AB is invertible, then A is invertible, then both A and B are invertible.

Proof: to prove the theorem →
We show B is invertible by showing that the system [$Bx=0$ has only the trivial Solution]

Multiply by A to get

$ABx=0$ → since AB is ~~~~ invertible then the hom. system ~~~~ $ABx=0$ has only $x=0$ solution (trivial solution)

Thus B is invertible

* since B^{-1} is invertible and AB is invertible, then $ABB^{-1}=A$ is invertible

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Ex: What conditions must b_1, b_2, b_3 satisfy in order for the system

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= b_1 \\
x_1 \quad\;\; + x_3 &= b_2 \\
2x_1 + x_2 + 3x_3 &= b_3
\end{aligned}
$$

to be consistant?

Solu:

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \quad \begin{aligned} -R_1 + R_2 &\to R_2 \\ -2R_1 + R_3 &\to R_3 \end{aligned}$$

gives

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \quad -R_2 + R_3 \to R_3$$

gives

$$\begin{bmatrix} 1 & 1 & 2 & (b_1) \\ 0 & -1 & -1 & (b_2 - b_1) \\ 0 & 0 & 0 & (b_3 - 2b_1 - b_2 + b_1) \end{bmatrix}$$

$\rightarrow$ $b_3 - b_2 - b_1 = 0$ $\rightarrow$ so that the system is consistant.

$\boxed{b_3 = b_2 + b_1}$ $\rightarrow$ this is the condition.

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Ex: Consider the system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and }$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

What are the conditions on b_1, b_2, b_3 such that $Ax = b$ is consitant?

Solu: when A is invertible, this system is consistant for any b_1, b_2, b_3.

* Try to solve the example without taking A as an invertible matrix as a known fact

⋮

use the way you used in the last example, so you get

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & (b_1) \\ 0 & 1 & -3 & (b_2 - 2b_1) \\ 0 & 0 & -1 & (b_3 - b_1 + 2b_2 - 4b_1) \end{array}\right]$$

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Thus A is invertible and system $Ax=b$ is consistant for any b_1, b_2, b_3

1.6 suggested problems: 5, 7, 9, 10, 13, 15, 17
18, 19, 21, 22 → TF

Sec 1.7: Diagonal, Triangular, and Symmetric matrices.

Def: A square matrix D in which all entries off the main diagonal are zeros is called a diagonal matrix.

$$\equiv D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ \vdots\ 0 & 0 & 0 & d_n \end{bmatrix}$$

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ are diagonal matrices.

Ex: $\begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}$ is not diagonal

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Remark: D is diagonal matrix then D is invertible if $d_i \neq 0$, $1 \leq i \leq n$

In this case $D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix}$

Also, $D^k = \begin{bmatrix} d_1^k & 0 & 0 \\ 0 & d_2^k & 0 \\ 0 & 0 & d_3^k \end{bmatrix}$

Ex If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then

Find $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$.

and find $A^{-4} = \begin{bmatrix} \frac{1}{16} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{81} \end{bmatrix}$

$\swarrow$

$(A^{-1})^4$

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Ex:

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} =$$

$$\begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} & d_1 a_{14} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} & d_2 a_{24} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} & d_3 a_{34} \end{bmatrix}$$

Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} =$$

$$\begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \\ d_1 a_{41} & d_2 a_{42} & d_3 a_{43} \end{bmatrix}$$

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Def: Upper triangular matrices are of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

$= [a_{ij}]$

$A = [a_{ij}]$ is upper triangular if $a_{ij} = 0$ when $i > j$

② lower triangular matrices are of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix}$$

$A = [a_{ij}]$ is lower triangular if $a_{ij} = 0$ when $i < j$

Ex: $\begin{bmatrix} 4 & 9 & 5 \\ 0 & 3 & 7 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 & 0 & 9 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

are upper triangular matrices

Ex $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 9 & 3 & 0 & 0 & 0 \\ 5 & 7 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$ is a lower traingular matrix

Ex $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ upper and lower triangular

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Remarks:-

① Diagonal matrices are both upper and lower triangular matrices.

② The transpose of an upper triangular matrix is a lower triangular matrix.

③ The transpose of an lower triangular matrix is an upper traiingular matrix.

Ex

$$A = \begin{bmatrix} 2 & 5 & 7 \\ 0 & 9 & 2 \\ 0 & 0 & 1 \end{bmatrix} ; \quad A^T = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 9 & 0 \\ 7 & 2 & 1 \end{bmatrix}$$

④ A Triangular (upper or lower) matrix is invertible if diagonal entries are all nonzero.

⑤ The inverse of an invertible upper triangular matrix is an upper triangular matrix.

⑥ The inverse of ~~a~~ an invertible lower triangular matrix is a lower triangular matrix.

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⑦ Product of upper triangular matrices is an upper triangular.

⑧ Product of lower triangular matrices is a lower triangular.

Ex:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

A is invertible but B is not.

$$AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}$$

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* A is symmetric if $A^T = A$

Ex. $A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix} = A$

So A is symmetric.

* $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i, j.

Theorem: I A and B are symmetric matrices, then:

① A^T is symmetric

② $A \mp B$ is symmetric.

③ cA, cB are symmetric (c is constant)

④ $(AB)^T = B^T A^T = BA$, and thus, the product of symmetric matrices is not necessarly symmetric

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⑤ The product of two symmetric matrices is symmetric if and only if matrices commute.

Ex :
$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{symmetric}} = \underbrace{\begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}}_{\text{not symmetric}}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}}_{\text{symmetric}} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{\text{symmetric}}$$

* Symmetric matrices need not to be invertible

Ex $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ symmetric and not invertible.

* If A is invertible, then A^T is invertible.

$$\left((A^T)^{-1} = (A^{-1})^T \right)$$

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* If A is invertible symmetric, then A^{-1} is symmetric.

$$\left((A^{-1})^T = (A^T)^{-1} = (A)^{-1} = A^{-1} \right)$$

So A^{-1} is symmetric.

* If A is an $m \times n$ matrix, then the matrices AA^T, and A^TA are symmetric.

$\underset{m \times n}{A}\,\underset{n \times m}{A^T}$

Proof $(AA^T) = (A^T)^T A^T = AA^T$
and then AA^T is symmetric.
as same as A^TA (symmetric)

Ex: $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$ then $A^TA =$

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

But $\rightarrow$ $AA^T = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$

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Theorem: If A is invertible matrix, then AA^T and A^TA are also invertible matrices

Proof: If A is invertible, then A^T is invertible.

* The product of two invertible matrices is invertible. Thus AA^T and A^TA are invertible.

Suggested problems: 1-13 odd, 41, 44, 45, TF

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Ch 2. Determinants:-

2.1: Determinants by Cofactor Expansion

* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$

* The number $(ad - bc)$ is called the determinant of A.

denoted by $det(A)$, $|A|$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Ex: $A = \begin{bmatrix} 5 & 2 \\ 3 & 7 \end{bmatrix}$, Then $det(A) = 29$.

$\Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow ad - bc \neq 0$

$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* We want to extend the determinant to square matrices of higher size.

<u>For</u> $A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \\ 4 & 2 & -1 \end{bmatrix} \end{array} = +1 \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} - 3 \begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} -1 & 4 \\ 4 & 2 \end{vmatrix}$

$= 1(-4-4) - 3(1-8) + 2(-2-16)$
$= -23$

* Another way to solve it taking the 3rd row.

$\det(A) = +4 \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + -1 \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix}$

$= \underline{\underline{-23}}$

* " " " " " " the 2nd column

$\det(A) = -3 \begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$

$= \underline{\underline{-23}}$

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* For 4x4 matrices

Ex: $A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & -2 & 1 & 1 \\ 6 & -1 & 7 & 1 \\ 3 & 1 & 2 & -5 \end{bmatrix} \end{array}$

$$\det(A) = (+1)\begin{vmatrix} -2 & 1 & 1 \\ -1 & 7 & 1 \\ 1 & 2 & -5 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 1 & 1 \\ 6 & 7 & 1 \\ 3 & 2 & -5 \end{vmatrix} +$$

$$-(1)\begin{vmatrix} 3 & -2 & 1 \\ 6 & -1 & 1 \\ 3 & 1 & -5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -2 & 1 \\ 6 & -1 & 7 \\ 3 & 1 & 2 \end{vmatrix} =$$

بنكمل الحل لكل واحد كأنه بنشتغل لمصفوفة 3x3 في المثال الأول.

Def: For a matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

The minor matrix if entery a_{ij} is the determinant of the submatrix that remains after deleting ith row and jth column, denoted M_{ij}.

* The number $(-1)^{i+j} M_{ij}$ is denoted C_{ij} and is called the cofactor of the entery a_{ij}.

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Ex: $A = \begin{bmatrix} 1 & 4 & 2 & 5 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ find M_{11} and C_{11}, M_{21}, C_{21}

$$M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} -1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = \underline{\underline{5}}$$

$$C_{11} = \text{cofactor of } a_{11} = (-1)^2 \begin{vmatrix} -1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \underline{\underline{5}}$$

$$M_{21} = \begin{vmatrix} 4 & 2 & 5 \\ 0 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -20$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 4 & 2 & 5 \\ 0 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 20$$

ال M_{ij} و ال C_{ij}
يا يكونوا متساويين
يا مختلفين في الإشارة
بس

Theorem: The determinant of an $n \times n$ matrix can be computed as follows:

for $1 \leq i \leq n$

$$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

for $1 \leq j \leq n$

$$\det(A) = a_{1j} \cdot C_{1j} + a_{2j} \cdot C_{2j} + \cdots + a_{nj} \cdot C_{nj}$$

Ex (Smart choices for rows and columns)

If $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$ then find the $\det(A)$

أكثر عمود في أصفار

$$\det(A) = (+1) \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} + 0 \cdots$$

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Ex: If $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 9 & 2 & 0 \\ -2 & 1 & -2 & -1 \end{bmatrix}$, Then find $\det(A)$

$$\det(A) = 1 \begin{vmatrix} 4 & 0 & 0 \\ 9 & 2 & 0 \\ 1 & -2 & -1 \end{vmatrix} = -8$$

Remark: The determinant of lower triangular or upper triangular matrix equals the product of the entries on the main diagonal as the last example.

*For 3x3 matrices, the determinant for

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ can be solved as follows:

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

J I H B C D

when
$B = a_{11} a_{22} a_{33}$ (multiplying)
$C = a_{12} a_{23} a_{31}$
and so on

$$= -(J + I + H) + (B + C + D)$$

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Ex find det $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \\ 4 & 2 & -1 \end{bmatrix}$

Solu = $\begin{bmatrix} 1 & 3 & 2 & 1 & 3 \\ -1 & 4 & 2 & -1 & 4 \\ 4 & 2 & -1 & 4 & 2 \end{bmatrix}$

32   4   3   -4   24   -4

So $\det = -(32+4+3) + (-4+24+-4) = \underline{\underline{-23}}$

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Sec 2.2: Evaluating determinant by row reduction

* Elementary operations

Remark:

① let A be a square matrix, if A has a row of zeros, or a column of zeros, then $det(A) \equiv 0$

② If A is a square matrix, then $det(A) = det(A^T)$

Theorem: Let A be a $n \times n$ matrix:

① If B is a matrix results when a single row or a single column of A is multiplyed by k, then $det(B) = k\,det(A)$

③ If B is the matrix that results when two rows or two columns of A are interchanged, then $det(B) = -det(A)$

④ If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column then $det(B) = det(A)$

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Ex: ①

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = k \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

②

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+2a_{11} & a_{22}+2a_{12} & a_{23}+2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Ex: If $\begin{bmatrix} a & b & c \\ d & e & f \\ g & e & h \end{bmatrix} = 4$, $\begin{bmatrix} a & b & c \\ 5d & 5e & 5f \\ g+3a & e+3c & h+3c \end{bmatrix}$

then $\begin{bmatrix} g & e & h \\ d & e & f \\ a & b & c \end{bmatrix} = -4$

Remark: Let E be an $n \times n$ matrix $\xrightarrow{}$ elemantry

① If E results from I by multiplying a row by k, then $\det(E) = k$

② " " " " changing two rows, then $\det(E) = -1$

③ " " " " adding a multiple of row to another row, then $\det(E) = 1$

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Ex: $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 5$, $\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$

Remark: If A is a square matrix and have two proportional rows or columns, then $\det A = 0$

Ex: Find $\begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix}$

$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = 0$

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Ex: Find $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = 3 \begin{vmatrix} 0 & 1 & 5 \\ 1 & -2 & 3 \\ 2 & 6 & 1 \end{vmatrix}$

$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$

$-3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$

$= 165$

Ex: Find $\begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{vmatrix}$

not an elemantry operation

-3 Column 1 + Column 4

$= (1)(7)(3)(-26) = ---$

وممكن نضرب الأشياء الي تحت ال (main diag) بس أطول وأصعب.

Ex: find
$$\begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 & 3 \\ \bar{1} & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{bmatrix}$$

$$= (1)\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} = -1\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

$$= -1(-1)(-18) = -18.$$

2.3 : Properties of determinant

Remark: $\det(kA) = k \det(A)$ $\quad (A_{n\times n})$

$$k\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{bmatrix}$$

$$k^3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Ex Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

$\det(A) = 1$, $\det(B) = 8$

$\det(A+B) = 23$

Remark: In general $\det(A+B) \neq \det(A) + \det(B)$

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Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$\det(A) = -2$, $\det(B) = 1$, $\det(C) = -1$

Remark: Let A, B and C be $n \times n$ matrices that differs only in a single row, say r^{th} row and assume that the r^{th} row of C can be obtained by adding r^{th} row of B and r^{th} row of A.

Then $\det(C) = \det(A) + \det(B)$
(The same result holds for columns)

Theorem: If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix then $\det(EB) = \det(E) \times \det(B)$.

Proof: (Easy)

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Remark: $\det(E_1 E_2 \cdots E_n B)$

$$= \det(E_1) \det(E_2) \cdots \det(E_n) \det(B).$$

Theorem: A square matrix A is invertible if and only if $\det(A) \neq 0$

Proof: let (R) be the reduced row echelon form of A

we will first show that $\det A$ and $\det R$ are both zeros or both non zeros.

Let $E_1, E_2, \cdots E_k$ be the elementry matrices such that

$$E_k \cdots E_2 E_1 A = R$$

So, $\det(E_k) \cdots \det(E_1) \det(A) = \det R$ ——— (1)

since the determinant of any elementry matrix is nonzero, then $\det(A)$ and $\det(R)$ are both zeros or both non zeros

If A is invertible, then $R = I$, and $\det R = \det I = 1$ and using (1), we get $\det A \neq 0$

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Now, suppose $\det(A) \neq 0$. Then $\det(R) \neq 0$
Then $\det(R) \neq 0$, So R can't have a row of zeros, thus $R = I$. Then ② becomes

$(E_k \cdots E_2 E_1) A = I$

So A is invertible.

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Ex: Determine whether matrix A is invertible

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix} \end{array}$$

Solu:

$\det(A) = +1(-4) - 2(4) + 3(4) = 0$

then A is not invertible

* $\det(A) = 0$, since ~~[illegible]~~ → 2 * Row1 = Row3

*Theorem: If A and B are square matrices of the same size, then $\det(AB) = \det A \det B$

proof

We have 2 cases, either A is invertible or not.

~~[illegible]~~ If A is not invertible, then AB is not invertible, and $\det(AB) = \det A \det B$

Thus $\det(AB) = 0$

$\det A \det B = 0$, $\det B = 0$

If A is invertible, then A can be written as a product of elementary matrices, we say $A = E_1 E_2 \cdots E_k$

So, $AB = E_1 E_2 \cdots E_k B$

$$\det(AB) = \det(E_1 \cdots E_k B)$$
$$= \det(E_1) \cdots \det(E_k) \det B$$
$$= \det(E_1 \cdots E_k) \det B$$
$$= \det A \det B$$

Ex: Consider $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$

then $AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$

$\det A = 1$, $\det B = -23$

$\det AB = -23$

Theorem: If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

proof: we have $AA^{-1} = I$

then $\det AA^{-1} = \det I = 1$

$\det A \ \det A^{-1} = 1$

$\det A^{-1} = \dfrac{1}{\det A}$.

Ex: If $\det A = 5$, then $\rightarrow \det A^{-1} = \dfrac{1}{5}$

Def: If A is any $n \times m$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & & C_{nn} \end{bmatrix}$$

is called the matrix of cofactors from A

* the transpose of this matrix is called the adjoint of A and denoted by $Adj(A)$

Theorem: If A is invertible, then

$$A^{-1} = \frac{1}{\det A} \times adj(A)$$

Ex: If $A = \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & \end{array} \\ \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix} \end{array}$, then find $Adj(A)$ and A^{-1}.

Solu

$$C_{11} = (-1)^{1+1} \begin{bmatrix} 6 & 3 \\ -4 & 0 \end{bmatrix} = 11$$

$$C_{12} = (-1)^{1+2} \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = 6$$

$C_{13} = -16$ $\quad C_{21} = 4$ $\quad C_{22} = 2$ $\quad C_{23} = 16$

$C_{31} = 12$ $\quad C_{32} = -10$ $\quad C_{33} = 16$

$$\text{Cofactor matrix} = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$\det A = 3(12) + 2(6) + (-1)(-16) = 64$$

$$\text{Thus } A^{-1} = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

* Finding an inverse for large matrices is not reasonable using the cofactor Method, it takes less computations to use last Method. $\rightarrow [A : I] \rightarrow [I, A^{-1}]$

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* Cramer's Rule :

If $AX = b$ is a system if n linear equations in n unknowns such that $\det A \neq 0$, then the system has a unique solution this solution is.

$$x_1 = \frac{\det A_1}{\det A} \qquad x_2 = \frac{\det(A_2)}{\det A} \quad ------ \quad x_n = \frac{\det A_n}{\det A}$$

where A_j is the matrix obtained by replacing the entries in jth column of A by $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Ex: Use cramer's rule to solve.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -2 & -2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$$

$$x_2 + \quad + 2x_3 = 6$$
$$-3x_1 + 4x_2 + 6x_3 = 30$$
$$-x_1 - 2x_2 + 3x_3 = 8$$

$\det A = 1(24) + 2(6+4) = 44$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$$

$\det A_1 = 6(24) + 2(-60-32)$
$= 144 - 184 = -40$

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$x_1 = \frac{\det A_1}{\det A} = \frac{-40}{44}$

$\vdots$

$x_2 = \frac{72}{44}$

$\vdots$

$x_3 = \text{---}$

so $A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}$ $\det A_2 = 72$.

* Gaussian elimination is for more efficient than cramer's law

* Fundamental Theorem: A is $n \times n$ matrix the following
1. A is invertible
2.
3.

$\vdots$

7. $\det(A) \neq 0$

Ex ; If A is a 3×3 matrix with $\det A = 5$, then find

① $\det(2A) = 2^3 \det A = 8(5) = 40$

② $\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{5}$

③ $\det(2A^T) = 2^3 \det A^T = 8 \det A = 40$

④ $\det A^4 = \det(AAAA)$

$= \det(A) \det A \det A \det A$

$= 5^4$

Chapter 4: General vector Spaces

* The set of real numbers → $(\mathbb{R})$
$\equiv \mathbb{R}$ → also is called (Set of scalers)

Def:- (vector space)
let V be an arbitrary nonempty set of objects on which two operations are defined (addition and Scaler multiplication)

* By addition we mean a rule that assigns to each objects $u, v \in V$ a unique object $u+v$ called the sum of u and v

* By scaler multiplication, we mean a rule that assigns to each scaler $k \in \mathbb{R}$ and an object $u \in V$ a unique object ku.

→ We call V a vector space if:- (for any objects $u, v, w \in V$ and any scalers $k, m \in \mathbb{R}$, the following axioms are satisfied:-

① If $u, v \in V$, then $u+v \in V$ (closed under addition)

$\rightarrow$

② For all $u \in V$, for all $k \in \mathbb{R}$, then $ku \in V$
(closed under scaler multiplication)

③ for all $u, v \in V \rightarrow u+v = v+u$.
(addition is commutative)

④ for all $u, v, w \in V$, then $(u+v)+w = u+(v+w)$.
(addition is associative).

⑤ There is an object $0 \in V$, called the zero vector, such that $u+0 = 0+u = u$, for all $u \in V$

⑥ For all $u \in V$, then exist $v \in V$ such that $u+v=0$.
(v is called additive inverse of u)
we denote v by $-u \rightarrow u+(-u)=0$.

⑦ If $u, v \in V$ and $k \in \mathbb{R}$, then $k(u+v) = ku+kv$

⑧ If $u \in V$, $k, m \in \mathbb{R}$, then $(k+m)u = ku+mu$.

(⑦ and ⑧ are called distribution laws).

⑨ for any $k, m \in \mathbb{R}$ and $u \in V$, then $(km)u = k(mu)$

⑩ $1u = u$ for all $u \in V$
$\hookrightarrow \mathbb{R}$

Ex:

$\mathbb{R}^2 = \{(x, y) , x, y \in \mathbb{R}\}$

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$\mathbb{R}^3 = \{(x, y, z) , x, y, z \in \mathbb{R}\}$

المكعب هون بيجي جواها 3 عناصر

$\mathbb{R}^2$ and $\mathbb{R}^3$ are vector spaces with usual addition and usual saler mult.

$\mathbb{R}^2$:

$u = (x_1, y_1)$, $v = (x_2, y_2)$ $\in \mathbb{R}^2 \to (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$ (1)

$k \in \mathbb{R} \to ku = (\underbrace{kx_1}_{\in \mathbb{R}}, \underbrace{ky_1}_{\in \mathbb{R}}) \in \mathbb{R}^2$ (2)

→ * conditions (2) and (4) are satisfied since real numbers with addition are commutitave and associative

(5) * 0 in $\mathbb{R}^2$ is $(0,0) \to u + (0,0) = (x_1, y_1) + (0,0) = (x_1, y_1)$.

(6) If $u = (x_1, y_1) \in \mathbb{R}^2$, then $u + (-x_1, -y_1) = (x_1, y_1) + (-x_1, -y_1) = (0,0) = 0$

Also, $(-x_1, -y_1) + u = 0$

(7) $k((x_1, y_1) + (x_2 + y_2))$

$= k(x_1 + x_2, y_1 + y_2) = (kx_1 + kx_2, ky_1 + ky_2)$
$= (kx_1, ky_1) + (kx_2, ky_2) = k(x_1, y_1) + k(x_2, y_2)$

$= ku + kv.$

* Similarly 8, 9 and 10 are satisfied.

* Elements of the vector space are called vectors.

Ex: $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$ is a vector space

(usual addition and usual scaler multiplication)

$\mathbb{R}^5$ vectors $(x_1, x_2, x_3, x_4, x_5)$
$(1, 4, 2, 0, 5) \in \mathbb{R}^5$

$(1, 4, 2, 9, 3) + (5, 7, 1, -2, 1) = (6, 11, 3, 7, 4)$

$3(1, 2, -1, 4, 7) = (3, 6, -3, 12, 21)$

Ex $V = \{0\}$ with usual addition and " scaler mult. is a vector space.

Ex $V = R^\infty = \{(x_1, x_2, x_3, \dots)\ x_i \in \mathbb{R}\}$

is a vector space.

①

②

⋮

$0 = (0, 0, 0, \dots)$

$-u = (-x_1, -x_2, -x_3, \dots)$

Ex V = Set of all 2X2 matrices $\equiv M_{22}$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in \mathbb{R} \right\}$$

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ M_{22} is a vector space.

$-u = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

Ex M_{mn} is a vector space

Ex $F(-\infty, \infty)$

Ex : $F(-3, 5)$

Ex $F[a, b]$

Ex let $V = \mathbb{R}^2$ with usual addition

If k any real number then

$$k(x, y) = (kx, 0)$$

condition <u>10</u>

(10) $(1u = u)$

$$1(4, 5) = (4, 0) \neq (4, 5)$$

condition 10 is unsatisfied
V with usual addition

and this ~~scale~~ sceler mult is not
a vector space.

Ex $V = \{(x, y, z) \quad x+y+z=0,\ x, y, z \in R\}$

usual addition and usual
scaler mult.

$u = (x_1, y_1, z_1)$ $\qquad$ $x_1+y_1+z=0 \longrightarrow (1)$

$\in V$, the

$v = (x_2, y_2, z_2)$ $\qquad$ $x_2+y_2+z_2=0 \longrightarrow (2)$

$u+v = (x_1+x_2, y_1+y_2, z_1+z_2)$

$= (x_1+y_1+z_1) + (x_2+y_2+z_2)$

$= 0 + 0 \equiv 0$

تذكير

((لقد تم اعطاء تعريف Vector space سابقاً مع بعض الامثلة
وسوف نكمل اليوم هذا الدرس ~~[illegible]~~))

Examples of vector spaces

① $\mathbb{R}^n$ تم الحديث عنه في المحاضرة السابقة
with usual addition & usual scaler multiplication
(One can specify n & get $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, ---$)

② All 2x2 matrices denoted by
M_{22} with usual addition & usual scaler multiplication

$$\equiv \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad a_i \in \mathbb{R} \right\}$$

If $u = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $v = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in M_{22}$, then

$u+v = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$ it is clear that $u+v \in M_{22}$

Also if $k \in \mathbb{R}$, then $ku = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix} \in M_{22}$

The zero element $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{22}$ and

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \ldots$$

; One can check that all other axioms of vector spaces are satisfied.

(3) One can generalize to $M_{mn} \equiv$ set of all $m \times n$ matrices with usual addition and usual scaler multiplication. One can check that M_{mn} is a vector space (Similar to M_{22}).

(4) $V = \{0\}$ (اخذ الصفر لوحده مع الجمع العادي والضرب بالثابت العادي)

$\left(\begin{array}{l} 0+0=0 \\ k \cdot 0 = 0 \end{array} \right)$ It is easy to check that $V = \{0\}$ is a vector space.

(5) The set of all real valued functions defined one $(-\infty, \infty)$ with usual addition & usual scaler multiplication. This set denoted by $F(-\infty, \infty)$

$F(-\infty, \infty)$ is a vector space.

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(5) The set of all real valued functions defined on one $(-\infty,\infty)$ with usual addition & usual scaler multiplication. This set denoted by $F(-\infty,\infty)$

$F(-\infty,\infty)$ is a vector space.

To explain why $F(-\infty,\infty)$ is a vector space. Observe that adding two functions is a function, multiply a function by constant gives a function. Addition of functions is commutative & associative. The zero function $f(x)=0$ plays the zero role. If g is a function, then $g(x)+0=g(x)$. The additive inverse of $f(x)$ is $-f(x)$. Observe that $f(x)+(-f(x))=0$

One can check that all other axioms of vector space are satisfied.

(6) (مشابه لـ (5)) $F(a,b)$ set of all real valued functions defined on (a,b).

$F(a,b)$ is a vector space   يمكن التحقق من ذلك كما في (5)

(7) (مشابه لـ (5)،(6)) $F[a,b]$ set of all real valued functions defined on $[a,b]$

$F[a,b]$ is a vector space   يمكن التحقق من ذلك كما في (5)

---

8) $\mathbb{R}^{\infty}=\{(x_1,x_2,---)\ x_i\in\mathbb{R}\}$ تم الحديث عنه في المحاضرة السابقة
is a vector space.

الآن يتم اعطاء امثلة على مجموعات ليست Vector space

Ex1: (A set that is not a vector space)

Let $V=\mathbb{R}^2$ with usual addition. If $k\in\mathbb{R}$ (scaler), then define $k(u_1,u_2)=(ku_1,0)$

(لاحظ ان الضرب بثابت معرف بطريقة خاصة حيث ان
الاحداثي الاول يضرب بـ k $k(u_1,u_2)=(ku_1,0)$
والاحداثي الثاني يوضع صفر)

V with usual addition & this scaler multiplication is not a vector space

V is not a vector space because $1(u_1,u_2)=(1u_1,0)$
$$=(u_1,0)\neq(u_1,u_2)$$

(لاحظ ان الشرط العاشر غير متحقق $1u=u$
في مثالنا هنا $\leftarrow$ $1u\neq u$)

Ex2 (A set that is not a vector space)

Let $V=\{(x,y):\ x+y=1,\ x,y\in\mathbb{R}\}$

with usual addition and scaler multiplication

(لاحظ ان V في هذا المثال هي عبارة عن مجموعة النقاط من $\mathbb{R}^2$ التي تقع على الخط المستقيم $x+y=1$)

Observe that $v=(1,0) \in V$ (Since $1+0=1$)
$u=(0,1) \in V$ (Since $0+1=1$)

But $u+v=(1,0)+(0,1)=(1,1) \notin V$

(Since $1+1 \neq 1$) (الشرط الاول لم يتحقق)

Thus V is not a vector space

الاسئلة المقترحة 4.1

4-12 + TF Question

4.2 Subspaces

(Vector spaces that Contained in a vector space)

<u>Def:</u> A Subset W of a vector space V is called a Subspace of V if W is ~~a vect~~ itself under the Same operations (same addition and scaler multiplication) is a vector space.

(للتحقق ان W subspace يجب التحقق ان جميع شروط vector space متحققة)

Ex $V = \{(x,0),\ x \in \mathbb{R}\}$ is a Subspace of $\mathbb{R}^2$

للتحقق من ذلك يجب التأكد ان الشروط جميعها تتحقق .

Let $u, v \in V$, then $u = (x_1, 0)$ & $v = (x_2, 0)$ $x_1, x_2 \in \mathbb{R}$

$u + v = (x_1 + x_2, 0) \in V$ $(x_1 + x_2 \in \mathbb{R})$ (الشرط الاول تتحقق)

Let $k \in \mathbb{R}$ then $ku = k(x_1, 0) = (kx_1, 0) \in V$ $(kx_1 \in \mathbb{R})$

(الشرط الثاني يتحقق)

وبالامكان التحقق من ان الشروط العشرة جميعها تتحقق

⋮

(لكن هل هناك طريقة اسهل للتحقق ان مجموعة معينة Subspace . يمكن ذلك بشكل اسرع من خلال النظرية التالية)

Theorem:- Suppose W is a ~~nonempty~~ is a subset of a vector space V. Then W is a subspace of V if the following three Conditions are Satisfied

1. $0 \in W$ (zero vector in W)
2. If $u, v \in W$, then $u + v \in W$ (W is closed under addition)
3. If k is a scaler & $u \in W$, then $ku \in W$. (W is closed under scaler multiplication)

-7-

(من خلال النظرية ... Subspace

(من خلال النظرية السابقة، يمكن ان يتحقق ان مجموعة جزئية Subspace اذا حققت الشروط الثلاثة المذكورة وهذا يجعل عملية التحقق اسهل واسرع)

Ex(1) $W = \{(x,0),\ x \in \mathbb{R}\}$ is a Subspace of $\mathbb{R}^2$

① $(0,0) \in W$ (الشرط الاول متحقق)

② If $u=(x_1,0)$ & $v=(x_2,0) \in W$, then
$u+v=(x_1+x_2,0) \in W$ (الشرط الثاني متحقق)

③ If k is a scaler & $u=(x_1,0) \in W$, then
$ku=(kx_1,0) \in W$ (الشرط الثالث متحقق)

Thus W is a Subspace.

Ex(2) Show that $W=\{(x,y,z) : x+y+z=0\}$ is a Subspace of $\mathbb{R}^3$.

(1) $(0,0,0) \in W$ (Since $0+0+0=0$) (الشرط الاول متحقق)

(2) Suppose $u=(x_1,y_1,z_1)$, $v=(x_2,y_2,z_2) \in W$. Then
$x_1+y_1+z_1=0$ ① and $x_2+y_2+z_2=0$ ②

Now, $u+v=$ ~~…~~ $(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

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Look at $(x_1+x_2)+(y_1+y_2)+(z_1+z_2) \overset{?}{=}$

$$\equiv (x_1+y_1+z_1)+(x_2+y_2+z_2) = 0+0=0$$

from ① from ②

$\rightarrow u+v \in W$

(نريد التحقق هل $u+v \in W$)

(الشرط الثاني تحقق)

Now, $ku = k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1)$

Look at $kx_1+ky_1+kz_1 \overset{?}{=} k(x_1+y_1+z_1) = k\,0$

from ①

$= 0$

(الشرط الثالث تحقق)

Thus W is a subspace.

Ex(3) Is $W=\{(x,y) : x \geq 0, y \geq 0\}$ a subspace of $\mathbb{R}^2$.

Observe that $(1,1) \in W$

But $-1(1,1) = (-1,-1) \notin W$ (الشرط الثالث غير متحقق)

Thus W is not a subspace of $\mathbb{R}^2$.

Linear Algebra 1, Lecture 2

(تكملة من الدرس 4.2
في المحاضرة السابقة:-

4.2 Subspace

تم اعطاء بعض الامثلة التي تكون Subspaces وبعض الامثلة التي ليست Subspaces

نكمل على ذلك

Ex Let $W = \{A \in M_{nn} : A \text{ is Symmetric}\}$ is a Subspace of M_{nn}.

To show that, observe that the zero matrix $O_{n\times n}$ is Symmetric & thus $O_{n\times n} \in W$ (الشرط الاول يتحقق)

$$\left(O_{n\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n} \right)$$

If $A, B \in W$, then $\underbrace{A^T = A \text{ and } B^T = B}$ (لان A و B Symmetric)

Now $(A+B)^T = A^T + B^T = A + B \rightarrow A+B$ Symmetric, $A+B \in W$
(الشرط الثاني يتحقق)

If $k \in \mathbb{R}$ & $A^T = A$, then

$$(kA)^T = kA^T = kA$$

and thus kA is Symmetric i.e $kA \in W$
الشرط الثالث يتحقق

Thus W is a Subspace of M_{nn}.

Ex Let $W = \{A \in M_{nn} : A \text{ is invertible}\}$

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W is not a Subspace, $O_{n\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{n\times n}$ is not invertible

and then $O_{n\times n} \notin W$

(الشرط الاول غير متحقق)

Thus W is not a Subspace of M_{nn}.

Ex Let $V = F(-\infty, \infty)$ (الاقترانات الحقيقية المعرفة على الاعداد) تم الحديث عنه سابقا

1) $W = C(-\infty, \infty)$ = The set of all Continuous functions on الفترة $(-\infty, \infty)$.

W is a Subspace (السبب
0 function is Continuous
Sum of two Cont. is Continuous.
scalar multiplied by Cont. is Cont.)

معلوما = سنتفاوز [illegible]

2) $W = C'(-\infty, \infty)$ = The set of functions with Continuous derivative.

W is a Subspace of V (التأكد مشابه لـ 1)

(كثيرات الحدود)

Ex The Subspace of all polynomials

تذكر Polynomials are functions of the form

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$f(x)$ is a polynomial of degree n.

($f(x) = 2 + x + x^3 + 7x^7$ is a polynomial of degree 7)

The set of all polynomials is denoted by P_∞

(مجموعة جميع كثيرات الحدود يرمز لها بالرمز P_∞)

P_∞ is a subspace of $F(-\infty,\infty)$

توضيح ان P_∞ is a subspace

$f(x)=0$ is the zero polynomial الشرط الاول تحقق

If $p(x)$ & $q(x)$ are polynomials, then their sum is a polynomial الشرط الثاني تحقق

If $k\in\mathbb{R}$ & $p(x)$ is a polynomial, then $k\,p(x)$ is a polynomial الشرط الثالث تحقق

Ex $W=$ The subspace of all polynomials of degree n or less, denoted by P_n

$W \equiv \{ p(x) : p(x)$ poly. of degree n or less $\} \equiv P_n$.

(مثلاً $P_4 \equiv$ كثيرات الحدود التي درجتها اربع واقل (درجتها اربع، ثلاث، اثنان، واحد وصفر))

P_n is a subspace

(الرجاء التحقق ان P_n is a subspace)

Theorem: If $Ax=0$ is a homogenous system of m equations in n unknowns, then the set of solution vectors is a subspace of $\mathbb{R}^n$

المثالان التاليان يوضحان النظرية

Ex $Ax=0 \longleftrightarrow \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longleftrightarrow \begin{matrix} x_1 + 4x_2 = 0 \\ 2x_1 + 9x_2 = 0 \end{matrix}$

Solve $\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 9 & 0 \end{array}\right] \xrightarrow[-2R_1+R_2 \to R_2]{} \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 1 & 0 \end{array}\right] \longrightarrow x_2 = 0 \ \& \ x_1 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus solution set $= \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^2$

(بناءً على النظرية السابقة)

Ex $Ax=0 \longleftrightarrow \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longleftrightarrow \begin{matrix} x_1 - 2x_2 + 3x_3 = 0 \\ 2x_1 - 4x_2 + 6x_3 = 0 \\ 3x_1 - 6x_2 + 9x_3 = 0 \end{matrix}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\longrightarrow x_1 - 2x_2 + x_3 = 0$, $\quad x_3 = t$, $x_2 = s$ $\longrightarrow x_1 = 2x_2 - 3x_3$

$x_1 = 2s - 3t$

$$\longrightarrow \text{Solution set} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

is a subspace of $\mathbb{R}^3$

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Def: A vector w is called a linear Combination of the vectors $v_1, v_2, \ldots, v_r$ if it can be written in the form

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r \quad \text{where } k_1, k_2, \ldots k_r \text{ are scalers}$$

We say w is a linear Combination of $v_1 \ldots v_r$.

Ex $$\underset{w}{(1,1,1)} = \underset{k_1}{2}\underset{v_1}{(1,1,0)} + \underset{k_2}{(-1)}\underset{v_2}{(1,1,-1)}$$

We say $(1,1,1)$ is a linear Combination of ~~[illegible]~~ $(1,1,0)$ & $(1,1,-1)$.

Ex $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

Thus (a,b,c) is a linear Combination of $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$.

Ex Show that $w = (9,2,7)$ is a linear Combination of $u = (1,2,-1)$ and $v = (6,4,2)$ but $(9,2,5)$ is not

Solution:- Is there k_1, k_2 : $k_1(1,2,-1) + k_2(6,4,2) = (9,2,7)$

$$\rightarrow \begin{array}{l} k_1 + 6k_2 = 9 \\ 2k_1 + 4k_2 = 2 \end{array}\Bigg\} \rightarrow 8k_2 = 16 \rightarrow k_2 = 2$$

$$2k_1 + 8 = 2 \rightarrow k_1 = -3$$

$$-k_1 + 2k_2 = 7 \longrightarrow -k_1 + 2k_2 = 7 \xrightarrow[k_2=2]{k_1=-3} -(-3) + 2(2) \overset{?}{=} 7$$

$$7 = 7 \checkmark$$

Thus $k_1 = -3$ & $k_2 = 2$

i.e $w = (9,2,7) = -3u + 2v = -3(1,2,-1) + 2(6,4,2)$

التكملة في الصفحة اللاحقة

But ~~(...)~~, $(9,2,5)$ is not a linear combination of u & v

Is there k_1, k_2 :

$$k_1(1,2,-1) + k_2(6,4,2) = (9,2,5)$$

$$\left.\begin{array}{l} k_1 + 6k_2 = 9 \\ 2k_1 + 4k_2 = 2 \end{array}\right\} \longrightarrow \text{solve to get } k_1 = -3 \text{ \& } k_2 = 2$$

$-k_1 + 2k_2 = 5$ $\qquad -k_1 + 2k_2 = 5 \xrightarrow[k_2=2]{k_1=-3} -(-3)+2(2) \overset{?}{=} 5$

$$7 \overset{?}{=} 5$$

$$7 \neq 5 .$$

Thus $(9,2,5)$ is not a linear Combination

of $u = (1,2,-1)$ & $v = (6,4,2)$.

Lecture 3 Linear I :-

4.2 (We talked last lecture about writing a vector w as a linear combination of vectors $v_1, \dots, v_r$)

Theorem :- If $v_1, v_2, \dots, v_r$ are vectors of a vector space, then

(a) The set W of all linear combination of $v_1, \dots, v_r$ is a subspace of V. Observe that

$$W = \{ c_1 v_1 + c_2 v_2 + \dots + c_n v_n : c_i \in \mathbb{R} \}$$

(b) W is the smallest subspace of V that contains $v_1, v_2, \dots, v_r$.

Ex If $v_1 = (1,1)$, $v_2 = (1,2)$, then

$$W = \{ c_1(1,1) + c_2(1,2) : c_1, c_2 \in \mathbb{R} \}$$

$0(1,1) + 0(1,2) = (0,0) \in W$ $\leftarrow$ مثلاً بأخذ $c_1=0$ & $c_2=0$

$1(1,1) + 1(1,2) = (2,3) \in W$ مثلاً بأخذ $c_1=1$ & $c_2=1$

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Def: If $S=\{v_1, v_2, --- v_r\}$ is a set of vectors in a vector space V, then the subspace W consisting of all linear combination of the vectors of S is called the space spanned by $v_1, ---, v_r$.

We write $W = \text{span}(v_1, v_2, --- v_r)$

Ex $\mathbb{R}^2 = \text{span}\{(1,0),(0,1)\}$

(Any $(a,b) \in \mathbb{R}^2$, then $(a,b) = a(1,0) + b(0,1)$)

Ex $P_n = \text{span}\{1, x, x^2, ---, x^n\}$

Any polynomial $a_0 + a_1 x + \cdots + a_n x^n$

$$= a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n)$$

Ex Determine whether $v_1 = (1,1,2)$, $v_2 = (1,0,1)$ & $v_3 = (2,1,3)$ span $\mathbb{R}^3$

Solution: Let $(b_1, b_2, b_3) \in \mathbb{R}^3$. Does there exist k_1, k_2, k_3 such that $(b_1, b_2, b_3) = k_1 v_1 + k_2 v_2 + k_3 v_3$

$$= k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$$

$\longrightarrow$

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$$k_1 + k_2 + 2k_3 = b_1$$
$$k_1 + 0k_2 + k_3 = b_2$$
$$2k_1 + k_2 + 3k_3 = b_3$$

Is this System Consistent regardless of the values of b_1, b_2 & b_3?

{ ~~What~~

Using results from chapter 1. This system is Consistent iff the Coefficient matrix $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ is invertible

i.e if $\det \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \neq 0$

But $\det \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} = 0$

3/5

Thus v_1, v_2 & v_3 do not span $\mathbb{R}^3$.

<u>Ex</u> Determine whether $v_1 = (1,1,2)$, $v_2 = (1,0,1)$, $v_3 = (0,1,3)$ span $\mathbb{R}^3$

Same work ⋮

$$k_1 + k_2 + 2k_3 = b_1$$
$$k_1 + 0k_2 + k_3 = b_2$$
$$0k_1 + k_2 + 3k_3 = b_3$$

Is this System Consistent regardless of the values of b_1, b_2 & b_3?

The System is Consistent iff the Coefficient matrix $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ is invertible

i.e. $\det \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \neq 0$

But $\det\begin{bmatrix}1&1&2\\1&0&1\\0&1&3\end{bmatrix} = -2$

Thus $v_1=(1,1,2)$, $v_2=(1,0,1)$ & $v_3=(0,1,3)$ span $\mathbb{R}^3$.

Remark: spanning sets are not unique $S_1=\{(1,0),(0,1)\}$ is a spanning set of $\mathbb{R}^2$. Also one can check that $S_2=\{(1,-1),(1,1)\}$ spans $\mathbb{R}^2$.

Suggested Problems for 4.2

1 a, b, c , 2 a-e, 3 a, b, d 4 a, b , 5 a, b

7-10 a, b , 11 a , 12 a, b 13, 18 T.F. Question All except d

10.a) Express $-9-7x-15x^2$ as a linear Combination of

$P_1=2+x+4x^2$, $P_2=1-x+3x^2$, $P_3=3+2x+5x^2$.

Solution: find ~~...~~ k_1, k_2, k_3 such that

$$-9-7x-15x^2=k_1(2+x+4x^2)+k_2(1-x+3x^2)+k_3(3+2x+5x^2)$$

$$-9-7x-15x^2=(2k_1+k_2+3k_3)+(k_1-k_2+2k_3)x+(4k_1+3k_2+5k_3)x^2$$

~~Constant~~ Constant terms are equal $-9=2k_1+k_2+3k_3$

Coefficients of x are equal $-7=k_1-k_2+2k_3$

Coefficients of x^2 are equal $-15=4k_1+3k_2+5k_3$

Find k_1, k_2, k_3 such that

$$\begin{aligned} 2k_1 + k_2 + 3k_3 &= -9 \\ k_1 - k_2 + 2k_3 &= -7 \\ 4k_1 + 3k_2 + 5k_3 &= -15 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$$

Solve this System to get

$k_1 = -2, \quad k_2 = 1, \quad k_3 = -2$

Thus $-9 - 7x - 15x^2 = (-2)p_1 + (1)p_2 + (-2)p_3$

check this out

Q. Section 4.2

2. Use Theorem 4.2.1 to determine which of the following are Subspaces M_{nn}

a) The set of all diagonal matrices $\equiv W$

نذكر بالبداية ان هناك ٣ شروط للتحقق من ان عرض جزئية
من Vectorspace V تكون Subspace W

وهذه الثلاث شروط هي التالي

* W a subset of a vector space V is a Subspace if
1) $0 \in W$
2) If $u, v \in W$, then $u+v \in W$
3) If $k \in \mathbb{R}$ & $u \in W$, then $ku \in W$

الآن نرجع الى a)

1) $0_{M_{n\times n}} \equiv 0\,\text{matrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}$ is a diagonal matrix

& thus $0_{n\times n} \in W$

2) If A & B are diagonal $n\times n$ matrices, then

$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 \\ \vdots & \vdots & & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn} \end{bmatrix}$

يتبع

$$A+B=\begin{bmatrix} a_{11}+b_{11} & & & 0 \\ & a_{22}+b_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn}+b_{nn} \end{bmatrix} \in W$$

الشرط الثاني متحقق

$$kA=\begin{bmatrix} ka_{11} & & & 0 \\ & ka_{22} & & \\ & & \ddots & \\ 0 & & & ka_{nn} \end{bmatrix} \in W$$

الشرط الثالث متحقق

Thus W = set of all diagonal matrices is a subspace of M_{nn}.

b) W = The set of all $n\times n$ matrices A such that $\det(A)=0$.

W is not a subspace the second condition is not satisfied (الشرط الثاني غير متحقق)

A is the diagonal matrix, $A=\begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$ ← (العدد الأول في diagonal صفر وباقي الأعداد 1)

B is the diagonal matrix $B=\begin{bmatrix} 1 & & & & 0 \\ & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$ (جميع الأعداد في diagonal 1 ماعدا العدد الثاني يساوي صفر)

$\det A=0$ & $\det B=0$ ($\det(A)=\det(B)$ = product of elements on the main diagonal)

$$A+B = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 2 & & \\ & & & 2 & \\ 0 & & & & \ddots \\ & & & & & 2 \end{bmatrix}$$

$$\det(A+B) = 1.1.2.2\ldots 2 = 2^{n-2} \neq 0$$

Ths W is not a subspace.

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c) W= The set of all n×n matrices such that $tr(A)=0$

(ملاحظة سهل اثبات ان هذه المجموعة subspace
بالاستعانة ان

① $\begin{cases} tr(A+B) \\ = tr(A)+tr(B) \end{cases}$

② $tr(kA)=k\,tr(A)$ و $\longrightarrow$ هذه الخواص تساعدني اثبات ان W subspace

نترك الحل للطلبة .

d) W= The set of all n×n symmetric matrices

ملاحظة يمكن اثبات ان W subspace
بالاستعانة بالخواص

$(A+B)^T=A^T+B^T$

$(kA)^T=kA^T$

ربما هذا احد الامثلة التي تم حلها في المحاضرات

e) The set of all n×n matrices with $A^T=-A$

(مشابه لـ d)

Lecture 4 Linear I :-

4.3 Linear Independence :-

(ابتداء درس جديد 4.3)

Def : If $S = \{v_1, v_2, \dots v_r\}$ is a set of two vectors or more vectors in a vector space V, then S is said to be linearly independent set if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be linearly dependent.

Ex (مثال توضيحي) $S = \{(1,2,1), (2,1,1), (3,3,2)\}$

Observe that $(3,3,2) = (1)(1,2,1) + (1)(2,1,1)$

Thus $(3,3,2)$ is a linear combination of $(1,2,1)$ & $(2,1,1)$.

So, S is linearly dependent.

Ex $S = \{(1,2), (2,5)\}$ is linearly independent

because $v_1 = (1,2)$ can't be written as a linear combination of $(2,5)$ i.e $(1,2)$ can't be a scaler multiple of $(2,5)$.

* النظرية التالية تعطي طريقة أسهل للتحقق من أن مجموعة متجهات linearly independent.

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Theorem :- A nonempty set $S = \{v_1, v_2, \dots, v_r\}$ in a vector space V is Linearly independent ~~iff~~ if and only if (كان) the only Coefficients satisfying the vector equation

$$k_1 v_1 + k_2 v_2 + \dots + k_r v_r = 0$$

are $k_1 = 0,\ k_2 = 0,\ \dots,\ k_r = 0$.

2/4

Ex Show that $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ are linearly independent.

Solution:- Suppose $k_1 e_1 + k_2 e_2 + k_3 e_3 = 0$

$$k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = 0$$

$$(k_1, k_2, k_3) = \underset{\substack{\downarrow \\ \text{this is the zero vector}}}{0} \equiv (0,0,0)$$

$\longrightarrow k_1 = 0,\ k_2 = 0$ & $k_3 = 0$

Thus e_1, e_2, & e_3 are linearly independent.

Ex: Determine whether the vectors

$$v_1 = (1,-2,3),\quad v_2 = (5,6,-1),\quad v_3 = (3,2,1)$$

are linearly independent in $\mathbb{R}^3$.

Solution: Suppose $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$\rightarrow k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = 0$

$\xrightarrow{\text{تابع}}$

$$k_1 + 5k_2 + 3k_3 = 0$$
$$-2k_1 + 6k_2 + 2k_3 = 0$$
$$3k_1 - k_2 + k_3 = 0$$

(If this system has only the trivial solution ($k_1 = k_2 = k_3 = 0$), then the vectors are linearly independent whereas if the system has non trivial solution, then the vectors are linearly dependent)

So our problem reduces to determining whether this system has only the trivial solution or has a non trivial solution.

One can do that by solving the system (or may be finding the determinant of the coefficient matrix)

If we solve the system ①, the augmented matrix

$$\left[\begin{array}{ccc:c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right]$$ Solve to get $k_1 = -\frac{1}{2}t,\ k_2 = -\frac{1}{2}t,\ k_3 = t$

check this!!

The system has nontrivial solution & thus v_1, v_2 & v_3 are linearly ~~independent~~ dependent.

(✗ Another way to explain linearly dependent:
Since the Coefficient matrix $\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$ is a square matrix, then one can check that the system

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has ~~trivial~~ only the trivial or nontrivial solution by finding the

det. of $\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$, $\det \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix} \equiv 0$

check this!!

Since the det. of the coefficient matrix equal 0, then the homogenous System has a nontrivial solution & thus v_1, v_2 & v_3 are linearly ~~independent~~ dependent

Since in this example the vectors are linearly dependent, then one of them is a linear Combination of the others. From the non-trivial solution w got in (1) we have $k_1 = -\frac{1}{2}t$, $k_2 = -\frac{1}{2}t$, $k_3 = t$, $t \in \mathbb{R}$.

~~If we take t = ... in~~

If we Substitute $k_1 = -\frac{1}{2}t$, $k_2 = -\frac{1}{2}t$, $k_3 = t$ in

$k_1 V_1 + k_2 V_2 + k_3 V_3 = 0$ we get

$(-\frac{1}{2}t) v_1 + (-\frac{1}{2}t) v_2 + t v_3 = 0$ Divide by t

$\rightarrow -\frac{1}{2} v_1 + -\frac{1}{2} V_2 + V_3 = 0$

$\rightarrow V_3 = \frac{1}{2} V_1 + \frac{1}{2} V_2$ We wrote V_3 as a linear Combination of V_1 & V_2.

Lecture 5 Linear I:-
4.3 Linear Independence:-

(4.3 نكمل في الدرس)

Ex Determine whether the vectors

$v_1 = (1, 2, 2, -1)$, $v_2 = (4, 9, 9, -4)$, $v_3 = (5, 8, 9, -5)$

in $\mathbb{R}^4$ are linearly dependent or linearly independent.

Solution:- The linear independence or dependence of these vectors is determined by whether there exist solution

$\exists$ $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$k_1 (1, 2, 2, -1) + k_2 (4, 9, 9, -4) + k_3 (5, 8, 9, -5) = 0$

$$\Longleftrightarrow \begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned} \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \\ -1 & -4 & -5 & 0 \end{array}\right]$$

augmented matrix

Solve this System to get $k_1 = 0, k_2 = 0, k_3 = 0$

(check!)

This System has only the trivial solution & thus v_1, v_2, v_3 are linearly independent.

Ex show that the polynomials

$1, x, x^2, ---, x^n$ form a linearly independent set in P_n.

2/5

Solution:- We should show the only coefficients

satisfying $b_0(1) + b_1(x) + b_2(x^2) + --- + b_n(x^n) \equiv 0$

are $b_0 = 0$ $b_1 = 0$ $---$ $b_n = 0$.

Now, $b_0(1) + b_1(x) + --- + b_n(x^n) = 0$

$\rightarrow$ $b_0 + b_1 x + b_2 x^2 + --- + b_n x^n = 0 \rightarrow$ zero polynomial

coefficient of x $= 0 + 0x + 0x^2 + \cdots + 0x^n$

Constant term

$\rightarrow$ $b_0 = 0$ (Constant terms are equal

$b_1 = 0$ (Coefficients of x are equal)

$\vdots$ (Coefficients of x^n are equal)

$b_n = 0$

Thus ~~1, x, x~~ $1, x, x^2, ---, x^n$ form a set of linearly independent vectors.

Ex show that $e_1 = (1, 0, 0, ---0)$, $e_2 = (0, 1, 0, --0)$ $----$, $e_n = (0, 0, --0, 1)$ form a linearly independent set in $\mathbb{R}^n$

$\xrightarrow{\text{ }}$

Solution:-

Suppose $k_1 e_1 + k_2 e_2 + \cdots + k_n e_n = 0$

$$\longrightarrow (k_1, k_2, \cdots, k_n) = \underbrace{u}_{\text{0 vector}} \equiv (0, 0, \cdots 0)$$

$$\longrightarrow k_1 = 0,\ k_2 = 0, \cdots\ k_n = 0.$$

Thus $e_1, e_2, \cdots, e_n$ form a set of linearly independent vectors in $\mathbb{R}^n$.

Ex Determine whether the polynomials

$$P_1 = 1 - x, \quad P_2 = 5 + 3x - 2x^2, \quad P_3 = 1 + 3x - x^2$$

are linearly independent or linearly dependent in P_2.

Solution: Suppose $k_1 P_1 + k_2 P_2 + k_3 P_3 = 0$

$$\longrightarrow k_1(1-x) + k_2(5+3x-2x^2) + k_3(1+3x-x^2) \equiv 0$$

$$\longrightarrow (k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 \equiv \underbrace{0}_{\text{0 polynomial}}$$

$$\equiv 0 \equiv 0 + 0x + 0x^2$$

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ -1 & 3 & 3 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right]$$

Augmented matrix

تكملة يتبع

This system has a nontrivial solution and thus P_1, P_2, P_3 (check this) (Nontrivial solution $x_1 = 3t$, $x_2 = -t$, $x_3 = 2t$) are linearly ~~independent~~ dependent.

(Another way to determine linearly independent or dep. by looking at the coefficient matrix

$$\begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix}.$$

Since $\det \begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix} = 0$, then the system has nontrivial solution & P_1, P_2, P_3 are linearly dependent)

Remark: (a) A finite set that contains 0 is linearly dependent.

(b) A set with exactly one vector is linearly independent if and only if that vector is not 0.

(c) A set with exactly two vectors is linearly independent if and only if neither vector is a scaler multiple of the other.

Ex The set $\left\{ \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix} \right\}$ is linearly dependent in M_{22} Since it contains the zero vector $\equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Ex The set $S = \{(1,2), (2,4)\}$ is linearly dependent in $\mathbb{R}^2$ since $(2,4)$ is a scaler multiple of $(1,2)$. Indeed $(2,4) = 2(1,2)$ (Using (c) previous remark).

Linear I Lecture 6 :-

4.3 linear independence (نكمل ما تبقى من 4.3)

Theorem :- Let $S = \{v_1, v_2, \ldots, v_r\}$ be a set of vectors in $\mathbb{R}^n$.

If $r > n$, then S is linearly dependent.

(اذا كان عدد المتجهات في $\mathbb{R}^n$ أكبر من البعد n تكون linearly dependent)

Ex Is $S = \{(1,2,-1,5), (2,9,1,7), (3,6,2,1), (0,5,4,3), (6,4,1,0)\}$ linear dependent? Explain.

Solution: The set is linearly dependent, since it has 5 vectors in $\mathbb{R}^4$ $(5 > 4)$.

بالاعتماد على النظرية السابقة.

Suggested Problems for 4.3 are
1, 3, 4, 5, 9, 12, T-F Question a-e.

4.4 Coordinates and basis

Def: A vector space V is said to be finite dimensional if there is a finite set of vectors in V that spans V and is said infinite dimensional if no such set exists.

Ex $\mathbb{R}^2$ is finite dimensional.

$S = \{(1,0), (0,1)\}$ spans $\mathbb{R}^2$

If $(a,b) \in \mathbb{R}^2$ then $(a,b) = a(1,0) + b(0,1)$

and thus ~~S~~ S spans $\mathbb{R}^2$.

Ex Similarly $\mathbb{R}^n$ is finite dimensional, Since

$e_1 = (1,0,0,\dots 0)$, $e_2 = (0,1,0,\dots 0)$, $\dots$, $e_n = (0,0,\dots 0,1)$ span

$\mathbb{R}^n$.

Def: If $S = \{v_1, v_2, \dots, v_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a basis of V if

(a) S spans V

(b) S is linearly independent.

Ex The standard vectors $e_1 = (1,0,\dots 0)$, $e_2 = (0,1,0,\dots 0)$, $\dots$ $e_n = (0,0,\dots 0,1)$ spans $\mathbb{R}^n$ (previous example). Also these vectors are linearly independent (We have seen that in an example in 4.3).

Thus $\{e_1, e_2, \dots, e_n\}$ form a basis of $\mathbb{R}^n$.

This set of vectors ($\{e_1, \dots e_n\}$) is called the standard basis of $\mathbb{R}^n$.

Ex: Let $V = P_n$ & $S = \{1, x, x^2, --- x^n\}$.

S spans P_n if $p(x) = a_0 + a_1 x + -- + a_n x^n \in P_n$, then
$p(x) = a_0(1) + a_1(x) + --- + a_n(x^n)$ and thus S spans P_n.

We have seen (in section 4.3) that S forms a set of linearly independent vectors.

Thus S is a basis of P_n.

This basis $S = \{1, x, x^2, ---, x^n\}$ is called the standard basis of P_n.

3/5

Ex show that the vectors $v_1 = (1,2,1)$, $v_2 = (2,9,0)$ and $v_3 = (3,3,4)$ form a basis for $\mathbb{R}^3$.

Solution: We must show that the vectors are linearly independent and spans $\mathbb{R}^3$

To prove linearly independent, we should show that
$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ has only the trivial solution

i.e
$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 + 0c_2 + 4c_3 &= 0 \end{aligned}$$
We need to show that this system has only the trival solution

To show v_1, v_2, v_3 spans $\mathbb{R}^3$ we must show that any $b = (b_1, b_2, b_3) \in \mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, v_3. i.e $b = (b_1, b_2, b_3) = x_1 v_1 + x_2 v_2 + x_3 v_3$

for some $x_1, x_2, x_3 \in \mathbb{R}$

i.e we have to show the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 9x_2 + 3x_3 &= b_2 \\ x_1 + 0x_2 + 4x_3 &= b_3 \end{aligned}$$ has a solution for any b_1, b_2, b_3.

For $\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 + 0c_2 + 4c_3 &= 0 \end{aligned}$ to have trivial solution & $\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 9x_2 + 3x_3 &= b_2 \\ x_1 + 0x_2 + 4x_3 &= b_3 \end{aligned}$ to be consistent

the coefficient matrix should be invertible.

Coefficient matrix $= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$, But $\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} = -1$ (Check)

Thus v_1, v_2, v_3 are linearly ind. & span $\mathbb{R}^3$, i.e v_1, v_2 & v_3 are basis of $\mathbb{R}^3$.

<u>Ex</u> Show that $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

form a basis of M_{22}. (M_{22} = set of all 2×2 matrices).

<u>Solution:</u> To show linearly independence, consider

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = 0$$

$$\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_4 \end{bmatrix} = \underbrace{0}_{0 \text{ matrix}} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow c_1 = c_2 = c_3 = c_4 = 0$$

Thus M_1, M_2, M_3, M_4 are linearly independent.

To show that $\{M_1, M_2, M_3, M_4\}$ spans M_{22}, Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$.

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= aM_1 + bM_2 + cM_3 + dM_4$.

Thus $\{M_1, M_2, M_3, M_4\}$ spans M_{22}.

Since $\{M_1, M_2, M_3, M_4\}$ is linearly independent & spans M_{22}, then $\{M_1, M_2, M_3, M_4\}$ forms a basis of M_{22}.

<u>Remark</u>: As a generalization of previous example one can take the set of all matrices whose entries are zero except for a single entry of 1. This set form a basis of M_{mn}

Ex For M_{32}, take $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis of M_{32}.

Linear 1 Lecture 7!

4.4 Coordinate and basis

تذكير

(We have seen before that $\mathbb{R}^n$ has the standard basis $e_1, \dots e_n$. Also P_n has the standard basis $1, x, \dots x^n$. Also we have seen that M_{mn} has a standard basis of mn elements)

(A vector space V is finite dimensional if there exists a finite set of vectors than spans V.)

تذكير

Ex $\mathbb{R}^n$, P_n, M_{mn} are finite dimensional

Ex R^∞ & P_∞ are infinite dimensional vector space. (There is no finite set of vectors that spans $\mathbb{R}^\infty$ or P_∞)

النظرية التالية تبين اهمية Basis

Theorem If $S = \{v_1, v_2, \dots v_n\}$ is a basis for a vector space V, then every vector $v \in V$ can be expressed in the form $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ in exactly one way

يتبع →

Ex $S = \{(1,1) \text{ } \& \text{ } (1,2)\}$ is a basis of $\mathbb{R}^2$ & $(0,-1) \in \mathbb{R}^2$ مثال توضيحي

Observe that $(0,-1) = \underset{c_1}{\underline{\underline{(1)}}}(1,1) + \underset{c_2}{\underline{\underline{(-1)}}}(1,2)$

Def:- If $S = \{v_1, v_2, \cdots v_n\}$ is a basis for a vector space V and $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$, then the scalers $c_1, \cdots, c_n$ are called the coordinates of v relative ~~of~~ to the basis S.

The vector $(c_1, c_2, \cdots, c_n)$ in $\mathbb{R}^n$ is called the coordinate vector of v relative to S and denoted by

$$(v)_S = (c_1, c_2, \cdots, c_n)$$

Ex If $V = \mathbb{R}^n$ and S is the standard basis

$$S = \{e_1 = (1,0,0 \cdots, 0),\ e_2 = (0,1,0, \cdots 0),\ \cdots,\ e_n = (0,0, \cdots 0,1)\}$$

Then if $v = (x_1, x_2, \cdots x_n) \in \mathbb{R}^n$, then

$$v = (x_1, x_2, \cdots, x_n) = x_1(1,0, \cdots 0) + x_2(0,1,0, \cdots 0) + \cdots + x_n(0,0, \cdots 0,1)$$

and thus coordinate vector of v relative to S is

$$(v)_S = (x_1, x_2, \cdots, x_n).$$

Ex Let $V = P_n$. The standard basis of P_n is

$S = \{1, x, x^2, ---, x^n\}$. If $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in P_n$, then

$p(x) = a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n)$.

Thus the coordinate of $p(x)$ with respect to S

equals $(p(x))_S = (a_0, a_1, ---, a_n)$.

Ex Let $V = M_{22}$. The standard basis of M_{22} is

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

If $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$, then

$$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus the coordinate of v with respect to S is

$(v)_S = (a, b, c, d)$.

Ex In previous example, we showed that

$S = \{v_1 = (1,2,1),\ v_2 = (2,9,0),\ v_3 = (3,3,4)\}$ form a basis of $\mathbb{R}^3$.

(a) Find the coordinate vector of $v = (5,-1,9)$ relative to S.

Solution: Find c_1, c_2, c_3 such that

$c_1 v_1 + c_2 v_2 + c_3 v_3 = (5,-1,9) = v$

$c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4) = (5,-1,9)$

$\Leftrightarrow$ Find c_1, c_2, c_3 such that

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 + 0c_2 + 4c_3 &= 9 \end{aligned} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{array}\right]$$

Solve this system to get $c_1 = 1$, $c_2 = -1$, $c_3 = 2$.

Thus $(v)_S = (1,-1,2)$. $\left((5,-1,9) = 1(1,2,1) + (-1)(2,9,0) + 2(3,3,4)\right)$

(b) Find the vector v in $\mathbb{R}^3$ whose Coordinate vector relative to S is $(v)_S = (-1,3,2)$.

Solution: $v = (-1)v_1 + (3)v_2 + 2v_3$

$= -1(1,2,1) + (3)(2,9,0) + 2(3,3,4)$

$= (11, 31, 7)$.

Suggested Problems for 4.4 2, 3, 5, 7, 8, 13, 14, 15, 17, 19, TF Question.

4.5 Dimension

ملاحظة: لاحظ ان عدد العناصر في Basis في $\mathbb{R}^n$ يساوي n

في P_n يساوي $n+1$

هل هذا العدد ثابت لا يتغير بتغير Basis

النظرية التالية تبين ذلك

Theorem: All basis for a finite-dimensional vector space have the same number of vectors.

بناءً على ذلك نعطي التعريف التالي.

Def: The dimension of a finite-dimensional vector space V is the number of the elements of a basis of V, denoted by dim V.

Ex $\dim(\mathbb{R}^n) = n$ (standard basis $e_1 \ldots e_n$)

$\dim(P_n) = n+1$ (standard basis $1, x, \ldots x^n$)

$\dim(M_{22}) = 4$ standard basis

$\dim(M_{mn}) = mn$

Linear I Lecture 8:-
4.5 Dimension

نكمل في الدرس 4.5

Theorem Let V be a vector space with $\dim V = n$. Then

(1) If a set in V has more than n elements, then this set is linearly dependent.

(2) If a set in V has less than n elements, then this set does not span V.

Ex The set $S = \{1+x, x+x^2, 1-x+x^2, 1+2x+3x^2\}$ is not linear independent in P_2.

Since ~~dim~~ $\dim(P_2) = 3$ & size of $S = 4$, then according to theorem above S is linearly dependent (or is not linearly independent)

Ex The set $S = \{(1,2,-1,4), (5,1,9,2), (4,3,-1,5)\}$ does not span $\mathbb{R}^4$.

Since $\dim(\mathbb{R}^4) = 4$ & size of $S = 3$, then according to theorem above S does not span $\mathbb{R}^4$.

We have seen before (Section 4.2) the solution set of the homogenous system $Ax = 0$ is a subspace. In the following example we want to find a basis of this subspace.

تذكر

Ex Find a basis and the dimension of the solution space of the homogenous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0\\
5x_3 + 10x_4 \qquad + 15x_6 &= 0\\
2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 2 & 6 & -5 & -2 & 4 & -3 & 0\\ 0 & 0 & 5 & 10 & 0 & 15 & 0\\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & -1 & -2 & 0 & -3 & 0\\ 0 & 0 & 5 & 10 & 0 & 15 & 0\\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{array}\right]$$

Augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & +1 & 2 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{array}\right]$$

$x_6 = 0$, $\quad x_3 + 2x_4 + 3x_6 = 0 \xrightarrow{x_6 = 0} x_3 + 2x_4 = 0$

$x_4 = s \rightarrow x_3 = -2x_4 = -2s$

$x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \xrightarrow[x_2 = r]{x_5 = t} x_1 = -3x_2 + 2x_3 - 2x_5$

$x_5 = t$
$x_2 = r$
$x_3 = 2s$

$\rightarrow x_1 = -3r + 4s - 2t$

$\rightarrow$ Solution $\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6 \end{bmatrix} = \begin{bmatrix} -3r - 4s - 2t\\ r\\ -2s\\ s\\ t\\ 0 \end{bmatrix}$ ·ج

يتبع

$$= \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$r, s, t \in \mathbb{R}$

i.e $(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3,1,0,0,0,0) + s(-4,0,-2,1,0,0)$
$+ t(-2,0,0,0,1,0)$.

Thus the vectors $v_1 = (-3,1,0,0,0,0)$, $v_2 = (-4,0,-2,1,0,0)$ and $v_3 = (-2,0,0,0,1,0)$ span the solution space.

If these vectors are linearly independent, then $S = \{v_1, v_2, v_3\}$ is a basis for the solution space.

* For any homogenous system finding the solution space as we did in this example always produces a linearly independent set of vectors.

Thus $S = \{v_1 = (-3,1,0,0,0,0),\ v_2 = (-4,0,-2,1,0,0),\ v_3 = (-2,0,0,0,1,0)\}$ is a basis of the solution space. Thus the dimension of the solution space equals 3.

يتبع

The solution space ~~is called~~ of the homogenous System $Ax=0$ is called the null space of A.

In last example the nullspace is ~~called~~ span $((-3,1,0,0,0,0), (-4,0,-2,1,0,0), (-2,0,0,0,1,0))$. ~~The dimension~~

<u>Def:</u> The dimension of the nullspace is called the nullity of matrix A.

In previous example the nullspace is span $((-3,1,0,0,0,0), (-4,0,-2,1,0,0), (-2,0,0,0,1,0))$.

Thus the nullity of A equals 3.

Theorem: Let V be a vector space with $\dim V = n$ and let S be a set in V with exactly n vectors. Then
(1) If S is linearly independent, then S spans V.
(2) If S spans V, then S is linearly independent.

So, according to previous Theorem, If $\dim V = n$ & S is a set of n elements, ~~then S is a~~ then to show S is a basis, it is enough to show that S is linearly independent or S ~~is~~ spans V.

Ex: Explain why the vectors $v_1 = (-3,7)$ & $v_2 = (5,5)$ form a basis of $\mathbb{R}^2$.

Solution: Since no vector is a ~~linear com~~ scaler multiple of the other vector, then the two vectors form a linearly independent set in $\mathbb{R}^2$. Since $\dim(\mathbb{R}^2) = 2$, then according to previous Theorem the two vectors form a basis of $\mathbb{R}^2$.

Suggested Problems 3, 5, 8, 9, 10, TF a-e.

[illegible]

Linear 1, lecture 9 4.5 نعطي حل بعض المسائل من الدرس

$\frac{Q_5}{4.5}$ In each part find a basis for the given subspace of $\mathbb{R}^4$ and state its dimension.

a) All vectors of the form $(a, b, c, 0)$

Observe that $(a,b,c,0) = a(1,0,0,0) + b(0,1,0,0) + c(0,0,1,0)$

Thus $\{(a,b,c,0), a, b, c \in \mathbb{R}\} = \text{span}\{(1,0,0,0),$
$(0,1,0,0), (0,0,1,0)\}$

Also observe that $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$ is linearly independent.

Thus $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$ is a basis of all vectors of the form $(a,b,c,0)$ & the dimension of this subspace equals 3.

b) All vectors of the form (a,b,c,d) where $d = a+b$ & $c = a-b$

$= \{(a,b,c,d) : d = a+b, \ c = a-b\}$

$= \{(a, b, a-b, a+b) : a, b \in \mathbb{R}\}$

$= \{ (a,0,a,a) + (0,b,-b,b) \ , \ a,b \in \mathbb{R} \}$

$= \{ a(1,0,1,1) + b(0,1,-1,1) \ ; a,b \in \mathbb{R} \}$

$= \text{Span} \{ \underbrace{(1,0,1,1), (0,1,-1,1)} \}$

These two vectors are linearly independent.

Thus $S = \{ (1,0,1,1), (0,1,-1,1) \}$ is a basis and dim. $= 2$.

$\frac{Q_9}{4.5}$) Find the dimension of each of the following vector spaces.

(a) The vector space of all $n \times n$ diagonal matrices.

$$\equiv \left\{ \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ 0 & & & \ddots \; a_n \end{bmatrix} , \ a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}$$

Observe that $A_1 = \begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & \ddots \; 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & \ddots \; 0 \end{bmatrix}$,

$\ldots \ A_n = \begin{bmatrix} 0 & & 0 \\ & 0 & \\ 0 & & \ddots \; 1 \end{bmatrix}$ form a set of basis of the set of all $n \times n$ diagonal matrices

Thus dimension of this vector space $= n$.

نبدأ بدرس جديد 4.7

4.7 Row space, Column space, and Nullspace

For an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

The vectors (rows of A) $r_1 = [a_{11}\ a_{12}\ \cdots\ a_{1n}]$,

$r_2 = [a_{21}\ a_{22}\ \cdots\ a_{2n}]$, $\ldots$, $r_m = [a_{m1}\ a_{m2}\ \cdots\ a_{mn}]$

are called row vectors of A.

The vectors (column of A) $c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$,

$\ldots$, $c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$ are called the column vectors

of A.

Ex $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$. Then the row vectors of A

are $r_1 = [2\ 1\ 0]$ & $r_2 = [3\ -1\ 4]$.

The column vectors of A are $c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

and $c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

Def: If A is an $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the row space of A, and the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the column space of A.

Ex Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$.

Then $\text{Span}\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}$ is the column space of A. (Note that $\dim(\text{Column space of } A) = 2$)
Subset of $\mathbb{R}^2$

~~And~~ $\text{span}(\{(2,1,0), (3,-1,4)\})$ is the row space of A (Note that dim row space of $A = 2$)

Theorem: A system of linear equations $Ax = b$ is consistent iff b is in the column space of A.

Ex Let $A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}$. Show that $b = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$ is in the column space of A.

تابع $\longrightarrow$

Solution: According to previous Theorem $b = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$ in the Column space of A if the system

$$Ax = b \Leftrightarrow \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$ has

a solution

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right]$$

Augmented matrix

Use Gauss-Elimination to transform the matrix

into $$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 0 & 17 & 51 \end{array}\right]$$

$\longrightarrow x_3 = 3, \; 5x_2 - x_3 = -8 \xrightarrow{x_3 = 3} 5x_2 - 3 = -8$

$\longrightarrow x_2 = -1$

$-x_1 + 3x_2 + 2x_3 = 1 \xrightarrow[x_2 = -1]{x_3 = 3} -x_1 - 3 + 6 = 1$

$-x_1 = -2 \longrightarrow x_1 = 2$.

So $b = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$ in the Column space

and $2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix} = b$.

4.7

Sep

في المحاضرة السابقة تحدثنا عن Column space & Row space

Q) How to find basis for Column & Row space of a matrix

4.7 Row space Column space & Nullspace

Theorem: If a matrix R is in row echelon form, then the row vectors with leading 1's (nonzero row vectors) form a basis for the row space of R, and the Column vectors with leading 1's (leading ones of the row vectors) form a basis for the Column space of R.

Ex Find a basis for ~~the~~ the row and Column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: Since the matrix in row echelon form, then the row vectors with leading ones $r_1 = [1, -2, 5, 0, 3]$, $r_2 = [0, 1, 1, 0, 0]$ & $r_3 = [0, 0, 0, 1, 0]$ form a basis for the row space of R.

Now, the vectors $c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, c_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ & $c_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ form a basis for the column space of R.

(Q What if the matrix not in row echelon form).

Theorem: Elementary row operations don't change the row space of a matrix.

Ex Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Since elementry row operations do not change the row space (Previous Theorem), we change the matrix into row echelon form.

The row echelon form of A is

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the rows with leading ones $r_1 = [1\ -3\ 4\ -2\ 5\ 4]$, $r_2 = [0\ 0\ 1\ 3\ -2\ -6]$ & $r_3 = [0\ 0\ 0\ 0\ 1\ 5]$ form a basis of R and hence form a basis for the matrix A.

For the Column space of a matrix, we have the following Theorem

Theorem:- Suppose ~~If~~ A and B are row equivalent matrices,

A given set of Column vectors of A forms a basis for the column space of A if and only if the Corresponding Column vectors of B form a basis for the Column space of B.

<u>Ex</u> Find a basis for the Column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & -2 & -5 & -4 \end{bmatrix}$$

that Consist Column vectors of A.

<u>Solution</u>: The row echelon form of A is

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Columns that Contain leading ones $c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$c_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, & $c_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ form a basis of the

Column space of R.

Now, according to previous theorem, the Corresponding Columns in A which are $c_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$, $c_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}$ & $c_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$

form a basis for the Column space of A.

(Note that A & R have different Column spaces)

$\underline{Ex}$ The following vectors span a subspace of $\mathbb{R}^4$. Find ~~a basis~~ a subset of these vectors that forms a basis of this subspace.

$v_1 = (1, 2, 2, -1)$ $\quad v_2 = (-3, -6, -6, +3)$

$v_3 = (4, 9, 9, -4)$ $\quad v_4 = (-2, -1, -1, 2)$

$v_5 = (5, 8, 9, -5)$ $\quad v_6 = (4, 2, 7, -4)$

Solution:- If we put these Columns in a matrix $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$, then

$\mathrm{span}(v_1, v_2, v_3, v_4, v_5, v_6)$ = Column space of A.

As in previous example ~~the~~ the basis for the Column space of A are

$\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = v_1$, $\begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} = v_3$ & $\begin{bmatrix} 5 \\ 9 \\ 9 \\ -5 \end{bmatrix} = v_5$

Thus $\{v_1, v_3, v_5\}$ form a basis for $\text{span}(v_1, v_2, v_3, v_4, v_5, v_6)$

Suggested Problems for 4.7, 3, 9, 11, 13, 15

Linear 1 Lecture 11:-

5.1 Eigenvalues and eigenvectors

<u>Def:</u> Suppose A is a square matrix. The values of λ (λ scaler) such that the system $Ax = \lambda x$

$\leftrightarrow Ax - \lambda x = 0 \leftrightarrow (A-\lambda I)x = 0$ or $(\lambda I - A)x = 0$ has a nontrival solution are called the eigenvalues of A.

The eigenvectors of A corresponding to the eigenvalue λ are the <u>nonzero</u> vectors x such that $Ax = \lambda x$.

$\leftrightarrow$ Eigenvectors are the ~~solution~~ nonzero solutions of $(A-\lambda I)x = 0$ or $(\lambda I - A)x = 0$

~~Eigenvectors~~ The nullspace of $A - \lambda I$ (Set of the eigenvectors together with the zero vector) is called the eigenspace correspond to λ.

$\rightarrow$ Now, λ is an eigenvalue if the system $(A-\lambda I)x = 0$ has nontrivial solution $\longleftrightarrow$ $(A-\lambda I)$ is not invertible

$\leftrightarrow$ ~~$A-\lambda I$~~ $\underline{\det(A-\lambda I) = 0}$ or $\det(\lambda I - A) = 0$

eigenvalues ~~eigenvectors~~ هذه تستخدم لإيجاد

نعطي امثلة للتوضيح

eigenvalues & eigenvectors

Ex Find the eigenvalues of $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$.

Solution: According to previous page, eigenvalues ~~the~~ are the values λ such that

$\det(\lambda I - A) = 0$

$\rightarrow \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} \lambda-2 & 2 \\ 3 & \lambda-1 \end{bmatrix} = 0$

$\rightarrow (\lambda-2)(\lambda-1) - 6 = 0$

$\lambda^2 - 3\lambda + 2 - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda-4)(\lambda+1) = 0 \rightarrow \lambda = 4$ or $\lambda = -1$.

* Remark 1) $\det(\lambda I - A) = 0$ is called characteristic equation

In last example $\lambda^2 - 3\lambda - 4 = 0$ is called characteristic equation

2) $\det(\lambda I - A)$ is called characteristic polynomial.

In last example, $\lambda^2 - 3\lambda - 4$ is ~~the~~ a characteristic polynomial

3) Observe that if A is an $n \times n$ matrix, then a characteristic polynomial is of degree n.

-3-

Now, we want to find eigenvectors of previous example.

Ex Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}.$$

Solution: From previous example eigenvalues are
$\lambda = 4$ & $\lambda = -1$.

According to previous page eigenvectors Correspond to $\lambda = 4$ are the non trivial Solution of $(\lambda I - A)x = 0 \overset{\lambda=4}{\Longleftrightarrow} (4I - A)x = 0$

$$4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$$

Solution of $\begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\longrightarrow \begin{array}{l} 2x_1 - 2x_2 = 0 \\ -3x_1 + 3x_2 = 0 \end{array} \Longleftrightarrow \text{Augmented matrix}$$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ -3 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow x_1 - x_2 = 0 \longrightarrow x_1 = x_2$$

take $x_2 = t$ = free, then $x_1 = x_2 = t$

$\rightarrow$ eigenvectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $t \neq 0$

So, Eigenvectors Correspond to $\lambda = 4$ are

$t\begin{bmatrix}1\\1\end{bmatrix}$, $\underline{t \neq 0}$ (We took $t \neq 0$ Since eigenvectors are non zero vectors)

Thus eigenspace Correspond to eigenvalue $\lambda = 4$ is $\text{Span}\{(1,1)\}$. Observe that dim of this eigenspace equals 1.

Now, eigenvectors Correspond to $\lambda = -1$ are the nonzero solutions of $(\lambda I - A)x = 0 \xleftrightarrow{\lambda=-1}$

$((-1)I - A)x = 0$

$$\left(\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} - \begin{bmatrix}2 & 2\\3 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}-3 & -2\\-3 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\rightarrow$ Augmented matrix $\left[\begin{array}{cc|c}-3 & -2 & 0\\-3 & -2 & 0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}-3 & -2 & 0\\0 & 0 & 0\end{array}\right]$

$-3x_1 - 2x_2 = 0$

$\rightarrow -3x_1 = 2x_2$

Take $x_2 = s \rightarrow x_1 = \frac{2s}{-3} = -\frac{2}{3}s$

$\rightarrow$ Eigenvectors are $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}s\\s\end{bmatrix}$, $s \neq 0$

So, eigenvectors Correspond to $\lambda = -1$ are

$s\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$, $s \neq 0$

(One can take $x_2 = 3s$ & thus $-3x_1 = 2(3s)$
$x_1 = -2s$

$\longrightarrow$ eigenvectors $\begin{bmatrix} -2s \\ 3s \end{bmatrix}$, $s \neq 0$

$= s\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $s \neq 0$)

~~Eigenspace This eigenvectors Correspon~~

This eigenspace Correspond to eigenvalue $\lambda = -1$ is $\text{span}\left(\left(-2/3, 1\right)\right)$.

The dim. of this eigenspace equals 1.

5.1 Eigenvalues and eigenvectors

Ex Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Solution: Find λ such that $\det(\lambda I - A) = 0$

$$\to \det \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} = 0$$

$\to \lambda - 2\left(\lambda(\lambda-3)+2\right) = 0$

$\to (\lambda-2)(\lambda^2-3\lambda+2) = 0$

$\to (\lambda-2)(\lambda-1)(\lambda-2) = 0$

$\to \lambda = 2$ repeated & $\lambda = 1$ are the eigenvalues.

($(\lambda-2)(\lambda^2-3\lambda+2)=0$ is called the characteristic equation)

To find the eigenvectors, find nonzero solution of $(\lambda I - A)x = 0$ when $\lambda = 1$

for $\lambda = 1$, $(I-A)x=0$ & $I-A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}$

$$(I-A)x = 0 \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-x_2 + x_3 = 0$

$x_2 = x_3$

يتبع

x_3 = free , $x_3 = t \longrightarrow x_2 = x_3 = t$

~~$x_1 + x_3$~~ First row in previous matrix $x_1 + 2x_3 = 0$

$\longrightarrow x_1 = -2x_3 \xrightarrow{x_3 = t} x_1 = -2t$

$$\longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvectors Correspond to $\lambda = 1$ are $t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $t \neq 0$

The eigenspace Correspond to $\lambda = 1$ is $\text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}$.

The dim. of this eigenspace equals 1.

Now, $\lambda = 2 \rightarrow (\lambda I - A)x = 0 \xrightarrow{\lambda = 2} (2I - A)x = 0$

$$\longrightarrow \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow 2x_1 + 2x_3 = 0 \text{ , } x_2 = \text{free}$$

$x_2 = t$, $x_1 = -x_3 \xrightarrow{x_3 = \text{free}} , x_3 = s \longrightarrow x_1 = -x_3$

$$\xrightarrow{x_3 = s} x_1 = -s \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The eigenvectors Correspond to $\lambda = 2$ are

$s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (not both t,s are zeros)

Thus the eigenspace correspond to $\lambda = 2$ is $\text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$. The dim. of this's eigenspace equals 2.

Ex Find the eigenvalues of the upper triangular matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 0 & 4 \end{bmatrix}$.

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-1 & -2 & -5 \\ 0 & \lambda-3 & -9 \\ 0 & 0 & \lambda-4 \end{bmatrix} \right)$$

$$= (\lambda-1)(\lambda-3)(\lambda-4)$$

$\det(\lambda I - A) = 0 \longrightarrow (\lambda-1)(\lambda-3)(\lambda-4) = 0$

$\longrightarrow \lambda = 1$ or $\lambda = 3$ or $\lambda = 4$.

Theorem If A is an $n \times n$ upper (lower) triangular or diagonal matrix, then the eigenvalues of A are the entries on the main diagonal.

Ex Find the eigenvalues of $A = \begin{bmatrix} 5 & 0 \\ 4 & 9 \end{bmatrix}$

Eigenvalues are 5, 9.

Theorem: A square matrix A is invertible iff $\lambda=0$ is not an eigenvalue of A.

We can add the above Theorem to the Fundamental Theorem.

~~The Fundamental Theorem~~

Theorem (Fundamental Theorem) If A is an $n\times n$ matrix, then the following are equivalent

(1) A is invertible

(2) $Ax=0$ has only the trivial solution.

⋮

(7) $\det(A)\neq 0$

(8) $\lambda=0$ is <u>not</u> an eigenvalue of A.

Suggested Problems 5, 7, 9, 11, 13, 14. TF a-e.

5.2 Diagonalization

نبدأ بدرس جديد

First, want to Compare the matrices A & $P^{-1}AP$ where A & P are $n\times n$ matrices & P is invertible.

Remark: 1) A & $P^{-1}AP$ have the same determinant.

Note that $\det P^{-1}AP = \det P^{-1} \det A \det P$

$= \frac{1}{\det(P)} \det A \, \det(P) = \det A$

So, they have the same determinant

2) A is invertible iff $P^{-1}AP$ is invertible.

<u>Def:</u> If A & B are square matrices, then we say B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$.

<u>Remark:</u> 1) Similar matrices have the same determinant.

2) If B is similar to A, then A is similar to B

$$\left(\begin{aligned} B = P^{-1}AP &\rightarrow \cancel{A = PBP^{-1} = (P^{-1})^{-1}} \\ &\rightarrow PBP^{-1} = PP^{-1}APP^{-1} \\ &\rightarrow PBP^{-1} = IAI \\ &\rightarrow A = PBP^{-1} = (P^{-1})^{-1} B P^{-1} \end{aligned} \right)$$

5.2 Diagonalization

8 تكملة في الدرس

Def: A square matrix A is said to be diagonalizable if it is similar to a diagonal matrix, i.e if there exist invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
In this case we say the matrix P diagonalize A.

(The following Theorem tells us when a matrix is diagonalizable)

Theorem: If A is an $n\times n$ matrix, the following are equivalent.
(a) A is diagonalizable
(b) A has n linearly independent eigenvectors.

~~In this~~ If the linearly independent eigenvectors are $P_1, P_2, \ldots, P_n$, then the matrix $P = [P_1, P_2 \ldots P_n]$, i.e P is the matrix whose Column are the linearly independent eigenvectors.
The matrix P diagonalize A, i.e
$P^{-1}AP = D =$ diagonal matrix

يتبع

The matrix $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_1 & & \\ & & \lambda_1 \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$ where λ_i is the eigenvalue corresponds to the eigenvector P_i.

Ex Find a matrix P that diagonalize

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution:- ~~Ex~~ In previous example we find that the characteristic equation of A is

$$(\lambda-2)^2(\lambda-1) = 0$$

Eigenvalues of A are $\lambda = 2$ & $\lambda = 1$.

In previous example, we found that the eigenvectors correspond to $\lambda = 2$ are $s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

We take $P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Also, we found that the eigenvectors correspond to $\lambda = 1$ are $t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. We take

$P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

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We form the matrix $P = [P_1\ P_2\ P_3] = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

One can compute P^{-1} to get $P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$.

Thus $P^{-1}AP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

<u>Ex</u> In previous example find A^{10}, A^{-8}.

We have $P^{-1}AP = D \rightarrow A = PDP^{-1}$

$$A^{10} = \underbrace{PD\underbrace{P^{-1}P}_{I}D\underbrace{P^{-1}P}_{I}DP^{-1} --- PDP^{-1}}_{10 \text{ times}}$$

$$= P D^{10} P^{-1}$$ (Since D is diagonal, it is easy to compute D^{10})

$$A^{10} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 1^{10} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1028 & 0 & 0 \\ 0 & 1028 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}}_{\text{Multiply to get}}$$

Similarly, one can find A^{-8}.

Theorem: If $\lambda_1, \lambda_2, \ldots \lambda_k$ are distinct eigenvalues of a matrix A, and if $v_1, v_2, \ldots v_k$ are corresponding eigenvectors, then $\{v_1, \ldots, v_k\}$ is a linearly independent set.

Corolary :- An $n\times n$ matrix A with n distinct eigenvalues is diagonalizable.

Ex Is $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 9 & 5 \\ 0 & 0 & 7 \end{bmatrix}$ diagonalizable? Explain.

Solution: ~~From~~ Since A is an upper triangular matrix, then the eigenvalues are the numbers on the main diagonal which are $2, 9$ & 7.
Since A is 3×3 matrix that has 3 distinct eigenvalues, then A is diagonalizable.

Ex: Is $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ diagonalizable? Explain.

Eigenvalues of A are $\lambda = 1$ & $\lambda = 2$

For $\lambda = 2 \rightarrow \lambda I - A = 2I - A = \begin{bmatrix} -1 & -2 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$-x_2 + -3x_3 = 0 \rightarrow x_2 = -3x_3$ & $x_3 = $ free

$x_3 = t \longrightarrow x_2 = -3t$

$-x_1-2x_2-x_3=0 \xrightarrow[x_2=-3t]{x_3=t} -x_1-2(-3t)-(t)=0$

$\rightarrow x_1 = 6t-t=5t$.

$\rightarrow$ eigenvectors correspond to $\lambda=2$

are $t\begin{bmatrix}5\\-3\\1\end{bmatrix}$. We take ~~the~~ eigenvector $\begin{bmatrix}5\\-3\\1\end{bmatrix}$

For $\lambda=1 \rightarrow \lambda I-A = 1I-A = \begin{bmatrix}0&-2&-1\\0&0&-3\\0&0&1\end{bmatrix}$

$x_3=0$ & $-2x_2-x_3=0 \xrightarrow{x_3=0} x_2=0$.

x_1 = free. , say $x_1=t$.

eigenvectors correspond to $\lambda=1$ are $t\begin{bmatrix}1\\0\\0\end{bmatrix}$.

We take eigenvector $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

We have only two ~~eigen~~ linearly independent eigenvectors $\begin{bmatrix}5\\-3\\1\end{bmatrix}$ & $\begin{bmatrix}1\\0\\0\end{bmatrix}$ & thus A is <u>not</u> diagonalizable.

Linear 1 Lecture 14 :-

5.2 Diagonalization:- 5.2 تكملة الدرس

Theorem Suppose λ is an eigenvalue of a matrix A with corresponding eigenvector x. Let k be a positive integer, then λ^k is an eigenvalue of A^k with corresponding eigenvector x.

Ex: Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$. We have seen in previous example that $\lambda=1$ is an eigenvalue of A with corresponding eigenvector $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\lambda=2$ is an eigenvalue of A with corresponding eigenvector $v_2 = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

Find the eigenvalues & eigenvectors of A^5.

Solution: According to previous Theorem,

$\lambda = (1)^5 = 1$ is an eigenvalue of A^5 with corresponding eigenvector $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Also, $\lambda = 2^5 = 32$ is an eigenvalue of A^5 with corresponding eigenvector $v_2 = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

Suggested Problems for S.2 3, 5, 7, 9a+c, 15, 19, 20

TF(a-f).

Chapter 6 : Inner Product Spaces   نبدأ ب Chapter جديد

6.1 Inner Products :-

نذكر في البداية ب Dot Product

Def: If $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2)$ are two vectors, then the dot product of u & v, denoted by $u \cdot v$,

$$\equiv x_1 x_2 + y_1 y_2 + z_1 z_2 .$$

Ex If $u = (1, 2, -1)$, $v = (3, 1, -2)$, then

$$u \cdot v = 1(3) + 2(+1) + (-1)(-2) = 7$$

Observe that dot product takes two vectors to a number.

A generalization of the dot product is what is called the inner product.

Def: An inner product on a real vector space V is a function that associates a real number $\langle u, v \rangle$ with each pair of vectors in V such that the following axioms are satisfied

$$(u, v \longrightarrow \langle u, v \rangle)$$

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For all $u, v, w \in V$ & $k \in \mathbb{R}$

1. $\langle u, v \rangle = \langle v, u \rangle$ (Symmetry axiom)

2. $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ (additive axiom)

3. $\langle ku, v \rangle = k \langle u, v \rangle$ (Homogenity property)

4. $\langle v, v \rangle \geq 0$ & $\langle v, v \rangle = 0$ iff $v = 0$ (positivity property)

~~Hence~~ In this case, we say V is an inner product space.

Ex Euclidean (standard) inner product on $\mathbb{R}^n$.

If $u = (u_1, \ldots, u_n)$ & $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$, then $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$.

To show that this is an inner product, we have to show that the above 4 conditions are satisfied.

1. Observe that $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$
$\langle v, u \rangle = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$
are equal & thus Condition 1 is satisfied

(2) $\langle u+v, w\rangle = (u_1+v_1)w_1 + (u_2+v_2)w_2 + \dots + (u_n+v_n)w_n$

$$= (\underline{u_1w_1} + \underline{\underline{v_1w_1}}) + (\underline{u_2w_2} + \underline{\underline{v_2w_2}}) + \dots + (\underline{u_nw_n} + \underline{\underline{v_nw_n}})$$

$$= \underline{(u_1w_1 + u_2w_2 + \dots + u_nw_n)} + \underline{\underline{(v_1w_1 + v_2w_2 + \dots + v_nw_n)}}$$

$$= \langle u,w\rangle + \langle v,w\rangle$$

Thus Condition 2) is satisfied.

(3) $\langle ku, v\rangle = \langle (ku_1, ku_2, \dots, ku_n), (v_1, v_2, \dots, v_n)\rangle$

$$= ku_1v_1 + ku_2v_2 + \dots + ku_nv_n$$

$$= k(u_1v_1 + u_2v_2 + \dots + u_nv_n)$$

$= k\langle u,v\rangle$ & thus Condition (3) holds

4) $\langle v,v\rangle = v_1v_1 + v_2v_2 + \dots + v_nv_n$

$$= v_1^2 + v_2^2 + \dots + v_n^2 \geq 0$$

If $\langle v,v\rangle = 0$, then $v_1^2 + v_2^2 + \dots + v_n^2 = 0 \longrightarrow v_1 = 0, v_2 = 0 \dots v_n = 0$

Also if $v = 0$, then $\langle v,v\rangle = 0$.

Thus Condition 4) is satisfied

Since the four conditions are satisfied, then V is an inner product space.

4/5

Ex In the Euclidian inner product on $\mathbb{R}^5$

If $u = (1,2,-1,3,4)$ & $v = (2,1,4,2,-1)$,

then $u \cdot v = 1(2) + 2(1) + (-1)(4) + 3(2) + (4)(-1)$

$= 2$

$u \cdot u = (1)^2 + (2)^2 + (-1)^2 + (3)^2 + (4)^2 = 31$

Linear 1, Lecture 15 :-

نكمل من الدرس

6.1 Inner Products

Ex (Example on inner product spaces)

The weighted inner Product on $\mathbb{R}^2$ defined by, Let $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbb{R}^2$,

then $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$.

One can check that the 4 conditions of the inner product are satisfied (Similar to ~~example~~ the Euclidean inner product)

In this example if $u = (2, -1)$ & $v = (3, 4)$,

then $\langle u, v \rangle = 3(2)(3) + 2(-1)(4)$
$= 10$.

Def: If V is an inner product space, then the norm or Length of a vector $v \in V$ is denoted by $\|v\|$ and is defined by

$\|v\| = \sqrt{\langle v, v \rangle}$. Also, we define the distance between two vectors u & v, denoted by

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$$d(u,v) = \|u-v\| = \sqrt{\langle u-v, u-v\rangle}$$

Ex Let $V=\mathbb{R}^5$ with the Euclidean inner product,

~~then if~~ If $u=(u_1, u_2, u_3, u_4, u_5)$, then

$$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u_1^2+u_2^2+u_3^2+u_4^2+u_5^2}.$$

If $u=(u_1,u_2,u_3,u_4,u_5)$ & $v=(v_1,v_2,v_3,v_4,v_5)$, then

$$d(u,v) = \sqrt{\langle u-v, u-v\rangle}$$
$$= \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2+(u_4-v_4)^2+(u_5-v_5)^2}.$$

If $u=(1,-2,3,1,5)$, then

$$\|u\| = \sqrt{1^2+(-2)^2+(3)^2+(1)^2+(5)^2} = \_\_\_\_$$

If $u=(2,1,4,-1,5)$ & $v=(1,-1,3,2,5)$, then

$$d(u,v) = \sqrt{(2-1)^2+(1-(-1))^2+(4-3)^2+((-1)-2)^2+(5-5)^2}$$

Ex Let $V=\mathbb{R}^2$. The inner product is the weighted inner product defined by,

Let $u=(u_1,u_2)$ & $v=(v_1,v_2)$, then

$$\langle u,v\rangle = 2u_1v_1 + 5u_2v_2$$

Then

يتبع

$$\|(3,1)\| = \sqrt{\langle (3,1),(3,1)\rangle} = \sqrt{2(3)(3)+5(1)(1)} = \sqrt{23}.$$

$$d((2,1),(3,5)) = \|(2,1)-(3,5)\| = \|(-1,-4)\|$$
$$= \sqrt{2(-1)^2+5(-4)^2} = \sqrt{82}.$$

Ex An inner product on M_{22}

If $u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ & $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$, then

$$\langle u,v\rangle = \operatorname{trace}(u^T v) = \operatorname{trace}\left(\begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}\right)$$

$$= \operatorname{trace}\left(\begin{bmatrix} u_1v_1+u_3v_3 & u_1v_2+u_3v_4 \\ u_2v_1+u_4v_3 & u_2v_2+u_4v_4 \end{bmatrix}\right) = u_1v_1+u_3v_3+u_2v_2+u_4v_4.$$

i.e $\langle u,v\rangle = u_1v_1+u_2v_2+u_3v_3+u_4v_4$.

One can check that this is an inner product.

Observe that $\|u\|^2 = \langle u,u\rangle = u_1^2+u_2^2+u_3^2+u_4^2$.

If $u = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$ & $v = \begin{bmatrix} 2 & -1 \\ 3 & 7 \end{bmatrix}$, then

$$\langle u,v\rangle = (1)(2)+(2)(-1)+(-1)(3)+(5)(7)$$
$$= 2-2-3+35 = 32$$

$$\|u\| = \sqrt{1^2+(2)^2+(-1)^2+(5)^2} = \sqrt{31}.$$

Ex: An inner product on P_n

If $p(x) = a_0 + a_1x + \cdots + a_n x^n$ & $q(x) = b_0 + b_1 x + \cdots + b_n x^n$, then

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n .$$

This is called the standard inner product.

(One can check that this is an inner product on P_n)

In P_5 if $p(x) = 2 + x + x^3 + 2x^5$

$q(x) = 3 - x + x^2 - x^5$,

then $\langle p, q \rangle = 2(3) + (1)(-1) + 0(1) + (1)(0) + (0)(0) + (2)(-1)$

$= 3$

$\|p\|^2 = \langle p, p \rangle = 2(2) + (1)(1) + (1)(1) + 2(2)$

$= 10$.

$\rightarrow \|p\| = \sqrt{10}$

Ex Another inner product on P_n.

Let p & $q \in P_n$, then

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$$

This is an inner product on P_n

(One can check that it is an inner product)

Ex In P_2 If $p(x) = x^2$, $q(x) = x^2+x$,

then $$\langle p,q\rangle = \int_{-1}^{1} x^2(x^2+x)\,dx$$

$$= \int_{-1}^{1}(x^4+x^3)\,dx$$

$$= \frac{x^5}{5}+\frac{x^4}{4}\Big]_{-1}^{1} = \frac{2}{5}.$$

$$\|p\|^2 = \int_{-1}^{1} x^2x^2dx = \int_{-1}^{1}x^4dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5} \rightarrow \|p\| = \sqrt{\frac{2}{5}}$$

Ex Let $V = C(a,b) \equiv$ Set of all Cont. functions on $[a,b]$.

If $f(x), g(x) \in C(a,b)$, then $\langle f,g\rangle = \int_a^b f(x)\,g(x)$

If $f(x) = e^x$ & $g(x) = x$, then

$$\langle f(x),g\rangle = \int_a^b e^x x\,dx = \int_a^b x e^x\,dx$$

$$\|f\|^2 = \int_a^b e^x e^x dx = \int_a^b e^{2x}dx$$

$$= \frac{1}{2}e^{2x}\Big]_a^b = \frac{1}{2}\left(e^{2b}-e^{2a}\right) \rightarrow \|f\| = \sqrt{\frac{1}{2}(e^{2b}-e^{2a})}.$$

LinearI, Lecture 16

تكملة في الدرس 6.1

6.1 Inner Product

Properties: If $u, v, w \in V$ and $k \in \mathbb{R}$ where V is an inner product space, then

(a) $\langle 0, v \rangle = \langle v, 0 \rangle = 0$

(b) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

(c) $\langle u, kv \rangle = k \langle u, v \rangle$

(d) $\langle u-v, w \rangle = \langle u, w \rangle - \langle v, w \rangle$

(e) $\langle u, v-w \rangle =$ ~~[illegible]~~ $\langle u, v \rangle - \langle u, w \rangle$

Ex If $\|u\| = 2$, $\|v\| = 3$ & $\|u-v\| = 3$, then find $\|2u+3v\|$.

$$\|2u+3v\|^2 = \langle 2u+3v, 2u+3v \rangle$$
$$= \langle 2u+3v, 2u \rangle + \langle 2u+3v, 3v \rangle$$
$$= \langle 2u, 2u \rangle + \langle 3v, 2u \rangle + \langle 2u, 3v \rangle + \langle 3v, 3v \rangle$$
$$= 4 \langle u, u \rangle + 6 \langle v, u \rangle + 6 \langle u, v \rangle + 9 \langle v, v \rangle$$
$$= 4 \|u\|^2 + 12 \langle u, v \rangle + 9 \|v\|^2 \quad \text{——— (1)}$$

We know $\|u\| = 2$, $\|v\| = 3$, we need to find $\langle u, v \rangle$

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But $\|u-v\|=3$

$$\longrightarrow 9 = \|u-v\|^2 = \langle u-v, u-v\rangle$$

$$= \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle$$

$$9 = \|u\|^2 - 2\langle u,v\rangle + \|v\|^2$$

$\|u\|^2 = 4$ & $\|v\|^2 = 9 \longrightarrow 9 = 4 - 2\langle u,v\rangle + 9$

$$\longrightarrow \langle u,v\rangle = 2 .$$

Now back to (1)

$(\|u\|=2)$ $(\langle u,v\rangle = 2)$ $(\|v\| = 3)$

$$\|2u+3v\|^2 = 4\|u\|^2 + 12\langle u,v\rangle + 9\|v\|^2$$

$$= 4(2)^2 + 12(2) + 9(3)^2$$

$$\longrightarrow \|2u+3v\| = \sqrt{4(2)^2 + 12(2) + 9(3)^2} .$$

Suggested Problems for 6.1

1, 9, 11, 17, 19, 21, 27, 28, 37, TF a-f.

نبدأ بدرس جديد

6.2 Angle and orthogonality in Inner Product spaces

(We want to see the relation between $\langle u,v\rangle$, $\|u\|$ & $\|v\|$).

Theorem (Cauchy - Schwarz Inequality) If u and v are vectors in a real inner product space V, then
$|\langle u,v\rangle| \leq \|u\| \, \|v\|$.

Ex In the Euclidian inner product.
If $u=(u_1,u_2,\dots,u_n)$ & $v=(v_1,v_2,\dots,v_n)$, then
$$|\langle u,v\rangle| = |u_1v_1+u_2v_2+\dots+u_nv_n| \leq \sqrt{u_1^2+u_2^2+\dots+u_n^2} \cdot \sqrt{v_1^2+v_2^2+\dots+v_n^2}$$

for instant In $\mathbb{R}^5$

$u=(1,2,-1,3,5)$ & $v=(2,0,1,2,3)$, then

3/4

$$|\langle u,v\rangle| = |1(2)+2(0)+(-1)(1)+3(2)+5(3)|$$
$$= |2-1+6+15| = 22$$

$$\|u\| = \sqrt{(1)^2+(2)^2+(-1)^2+(3)^2+(5)^2} = \sqrt{40} \approx 6.3$$

$$\|v\| = \sqrt{(2)^2+(0)^2+(1)^2+(2)^2+(3)^2} = \sqrt{18} \approx 4.24$$

$$\underset{\downarrow}{22} \leq \underset{\downarrow}{\sqrt{40}}\ \underset{\downarrow}{\sqrt{18}} \approx (6.3)(4.24)$$
$$|\langle u,v\rangle| \leq \|u\| \ \|v\|$$

Remark: Since $|\langle u,v\rangle| \leq \|u\|\,\|v\|$, then

$$-1 \leq \frac{\langle u,v\rangle}{\|u\|\,\|v\|} \leq 1$$

We can define the angle between u & v to be θ such that $\cos(\theta) = \dfrac{\langle u,v\rangle}{\|u\|\,\|v\|}$.

Ex Find the angle between $u=(2,-1,4,5)$ & $v=(2,3,9,7) \in \mathbb{R}^4$ with standard inner product.

Solution: The angle θ between u & v satisfies

$$\cos(\theta) = \frac{|\langle u,v\rangle|}{\|u\|\,\|v\|}$$

$$= \frac{|2(2)+(-1)(3)+4(9)+5(7)|}{\sqrt{(2)^2+(-1)^2+(4)^2+(5)^2}\ \sqrt{(2)^2+(3)^2+(9)^2+(7)^2}}$$

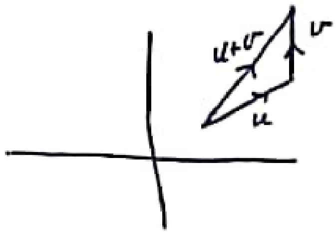
$$= \frac{|4-3+36+35|}{\sqrt{46}\ \sqrt{143}} = \frac{72}{\sqrt{46}\ \sqrt{143}} \approx 0.887$$

($\theta \cong 27.5^\circ$ using Calculator)

Linear 1 Lecture 17

6.2 Angle and Orthogonality in Inner Product Spaces :-

نكمل في الدرس 6.2



Theorem If u, v & w are vectors in a real inner product space V, then

(1) $\|u+v\| \le \|u\| + \|v\|$ (Triangle inequality for vectors)

(2) $d(u,v) \le d(u,w) + d(w,v)$ (Triangle inequality for distance)

Ex: If $u = (1,2,1,3)$ & $v = (2,1,0,4)$ $\in \mathbb{R}^4$ with Euclidian inner product, then

$u+v = (3,3,1,7)$,

$\|u\| = \sqrt{1^2+2^2+1^2+3^2} = \sqrt{15} \approx 3.87$

$\|v\| = \sqrt{2^2+1^2+0^2+4^2} = \sqrt{21} \approx 4.58$

$\|u+v\| = \sqrt{3^2+3^2+1^2+7^2} = \sqrt{68} \approx 8.24$

$\cancel{\|u+v\|}$ $\cancel{8.24 \le (4.58)(3.87)}$

$\cancel{\|u+v\| \le \|u\|\,\|v\|}$

$8.24 \le (3.87)(4.58)$

$\underbrace{8.24}_{\|u+v\|} \le (\underbrace{3.87}_{\|u\|})(\underbrace{4.58}_{\|v\|})$

Def:
Two vectors u & v in an inner Product space V are called orthogonal if $\langle u,v\rangle = 0$.

Ex Let $V = \mathbb{R}^4$ with standard inner product.

$u = (2,-1,2,3)$ & $v = (-3,1,5,-1)$

$\langle u,v\rangle = 2(-3) + (-1)(1) + 2(5) + 3(-1) = 0$

Thus u & v are orthogonal.

Ex: Let $V = M_{22}$ with standard inner product.

$u = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ & $v = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

then $\langle u,v\rangle = (-1)(1) + (0)(2) + (1)(2) + (-1)(1)$
$= 0$

Thus u & v are orthogonal.

Ex Let $V = P_2$ & for $p, q \in P_2$

$\langle p,q\rangle = \int_{-1}^{1} p(x)\, q(x)\, dx.$

يتبع

Let $p(x) = x$ & $q(x) = x^2$. Then

$$\|p\|^2 = \int_{-1}^{1} x \cdot x \, dx = \frac{x^3}{3}\Big|_{-1}^{1} = 2/3 \longrightarrow \|p\| = \sqrt{2/3}$$

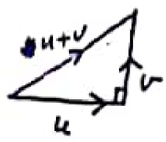
$$\langle p, q \rangle = \int_{-1}^{1} x \cdot x^2 \, dx = \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

Thus $p(x) = x$ & $q(x) = x^2$ are orthogonal.

Theorem (Generalized Pythagoras Theorem) (تعميم لنظرية فيثاغورس)

If u & v are orthogonal vectors in a real inner product space, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$



Ex $u = (1,-1,4)$, $v = (1,5,1)$ are orthogonal in $\mathbb{R}^3$ with Euclidean inner product

$$(\langle u, v \rangle = 1 - 5 + 4 = 0)$$

$u+v = (2,4,5)$, $\|u+v\|^2 = 2^2 + 4^2 + 5^2$
$= 45$

$\|u\|^2 = 18$ ~~Observe~~
$\|v\|^2 = 27$

Observe that $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

Ex In previous example we took $V = P_2$ with

inner $\langle p,q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$.

We have seen, if $p(x) = x$ & $q(x) = x^2$, then

$\langle p, q \rangle = 0$.

$\|p\|^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

$\|q\|^2 = \int_{-1}^{1} x^4 dx = \frac{2}{5}$.

$\|p+q\|^2 = \int_{-1}^{1} (x+x^2)(x+x^2)\, dx$

$= \int_{-1}^{1} (x^2+2x^3+x^4)\, dx = \frac{2}{3} + \frac{2}{5}$.

~~Other~~ So, p & q are orthogonal and

$\|p+q\|^2 = \|p\|^2 + \|q\|^2$.

Suggested Problems 1, 7, 9, 23, 31, 36
6.2

6.3 Gram-Schmidt Process

نبدأ بدرس جديد

Def: A set of vectors in an inner product space is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal.

An orthogonal set in which each vector has norm 1 is called orthonormal.

Ex Let $V = \mathbb{R}^3$ with the Euclidean inner product.

Let $u_1 = (0,1,0)$, $u_2 = (1,0,1)$ and $u_3 = (1,0,-1)$

Then $\langle u_1, u_2 \rangle = 0$, $\langle u_1, u_3 \rangle = 0$ and $\langle u_2, u_3 \rangle = 0$

Thus $S = \{u_1, u_2, u_3\}$ is an orthogonal set

Remark: If v is a nonzero vector, then

$\frac{v}{\|v\|}$ has norm 1.

If a vector u has norm 1, then we say u is a unit vector.

* $u = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit vector since $\|u\| = 1$

يتبع

Ex Let $v = (1,2,9)$, then $\|v\| = \sqrt{1^2+2^2+(9)^2} = \sqrt{86}$

and $\dfrac{v}{\|v\|} = \left(\dfrac{1}{\sqrt{86}}, \dfrac{2}{\sqrt{86}}, \dfrac{9}{\sqrt{86}}\right)$ is a unit vector.

Remark:- If $S = \{u_1, u_2, \dots, u_n\}$ is an orthogonal set,

then $\left\{\dfrac{u_1}{\|u_1\|}, \dfrac{u_2}{\|u_2\|}, \dots, \dfrac{u_n}{\|u_n\|}\right\}$ is an orthonormal

Set.

Ex In previous example we have seen that

$S = \{u_1 = (0,1,0), u_2 = (1,0,1), u_3 = (1,0,-1)\}$ is an

orthogonal set.

Thus $\left\{\dfrac{u_1}{\|u_1\|}, \dfrac{u_2}{\|u_2\|}, \dfrac{u_3}{\|u_3\|}\right\}$

$= \left\{(0,1,0), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\right\}$ is

an orthonormal Set.

Ex: The standard basis $\{e_1 = (1,0,0,\dots 0), e_2 = (0,1,0,\dots 0), \dots$

$e_n = (0,0,\dots 0,1)\}$ is an orthonormal Set in $\mathbb{R}^n$

with the Euclidean inner product.

Theorem 1: Suppose $S = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for an inner product space V. If u is any vector in V, then

$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \cdots + \langle u, v_n \rangle v_n$ and

$(u)_S = (\langle u, v_1 \rangle, \langle u, v_2 \rangle, \ldots, \langle u, v_n \rangle)$

<u>Theorem</u>: If V is an inner product space with $\dim V = n$ & $S = \{u_1, u_2, \ldots, u_n\}$ is an orthonormal set of n vectors, then S is an orthonormal basis of V.

<u>Ex</u> Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $S = \{v_1 = (0,1,0),$ $v_2 = (-\frac{4}{5}, 0, \frac{3}{5})$, $v_3 = (\frac{3}{5}, 0, \frac{4}{5})\}$. It is easy to check that S is an orthnormal set of 3 vectors in $\mathbb{R}^3$. According to previous Theorem S forms a basis of $\mathbb{R}^3$.

Q: Express the vector $u = (1,1,1)$ as a linear combination of the vectors in S and find $(u)_S$.

According to Theorem ① in previous page

$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \langle u, v_3 \rangle v_3$

$\langle u, v_1 \rangle = 1$, $\langle u, v_2 \rangle = -\frac{1}{5}$ & $\langle u, v_3 \rangle = \frac{7}{5}$.

Thus $u = (1) v_1 + (\frac{-1}{5}) v_2 + (\frac{7}{5}) v_3$

$= v_1 + (\frac{-1}{5}) v_2 + (\frac{7}{5}) v_3$.

Thus $(u)_S = (1, -\frac{1}{5}, \frac{7}{5})$

(The importance of applying this theorem is that we do not have to solve system to find $(u)_S$, we can find it using the inner product)

Q: How one can produce an orthogonal or orthnormal basis?

This can be done using what is called the Gram-Schmidt process. We describe it in the following Theorem.

Theorem: 4 (Gram-Schmidt process) Suppose $\{u_1, u_2, \ldots, u_n\}$ is a basis of an inner product space.

Let $v_1 = u_1$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$v_4 = u_4 - \frac{\langle u_4, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_4, v_2 \rangle}{\|v_2\|} v_2 - \frac{\langle u_4, v_3 \rangle}{\|v_3\|} v_3$$

$$\vdots$$

The ~~list of~~ set of vectors $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis of V and the set $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_n}{\|v_n\|} \right\}$ is an orthnormal basis of V.

تكملة الدرس 6.3

Ex Consider $V = \mathbb{R}^3$ with Euclidean inner product.

Let $S = \{(1,1,1), (0,1,1), (0,0,1)\}$ be a basis of $\mathbb{R}^3$. Use the Gram-Schmidt process to transform S to an orthogonal basis. Also, find orthonormal basis.

Solution: $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$.

$$v_1 = u_1 = (1,1,1)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = (0,1,1) - \frac{2}{3}(1,1,1) = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{6/9}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(\frac{-1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= \left(0, \frac{-1}{2}, \frac{1}{2}\right)$$

So, $\{v_1 = (1,1,1),\ v_2 = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right),\ v_3 = \left(0, \frac{-1}{2}, \frac{1}{2}\right)\}$ is an orthogonal basis of $\mathbb{R}^3$.

يتبع

Now, $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$ is an orthonormal basis of $\mathbb{R}^3$

$$\left\{ \frac{(1,1,1)}{\|(1,1,1)\|}, \frac{(-2/3, \frac{1}{3}, \frac{1}{3})}{\|(-2/3, 1/3, 1/3)\|}, \frac{(0, -1/2, 1/2)}{\|(0, -1/2, 1/2)\|} \right\}$$

$$= \left\{ \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(\tfrac{-2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right), \left(0, \tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) \right\}$$

is an orthonormal basis of $\mathbb{R}^3$.

Suggested Problems 1 b,c, 2 b,c, 7, 10, 29, 31

ابداء بدرس جديد

8.1 General linear transformation

<u>Def:</u> Let V and W be vector spaces. If T is a function from V to W ($T: V \longrightarrow W$), then T is called a linear transformation (linear Map) from V to W if the two properties hold for any $u, v \in V$ & $k \in \mathbb{R}$

(a) $T(u+v) = T(u) + T(v)$

(b) $T(ku) = k\,T(u)$.

If $T: \underline{V} \longrightarrow \underline{V}$, then T is called linear operator.

We want to take some examples

Ex $T: V \longrightarrow W$

Defined by $T(v)=0$, the zero transformation.

observe that $T(u+v)=0$, $T(u)=0$, $T(v)=0$ & $T(ku)=0$

~~and~~ Thus $T(u+v)=Tu+Tv$ and $T(ku)=kT(u)$.

So T is a linear transformation

Ex $T: V \longrightarrow V$, defined by $T(v)=v$ for all $v \in V$.

the identity map. Sometimes denoted by I,

$(I: V \longrightarrow V$ with $I(v)=v$ $)$

To show T is linear

$$T(u+v) = u+v = T(u)+Tv$$

$$T(ku) = ku = kT(u)$$

Thus T is linear.

Ex $T: R^2 \longrightarrow R^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(For instance, $T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 19 \\ -3 \end{bmatrix}$)

If $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ & $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, then

$$T(u+v) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right)$$

$$= T(u) + Tv.$$

Now, $T(ku) = T\left(k \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right) = T\left(\begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix} \right) =$

$$= \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix} = \begin{bmatrix} 2kx_1 + 3ky_1 \\ kx_1 - ky_1 \end{bmatrix}$$

$$= k \begin{bmatrix} 2x_1 + 3y_1 \\ x_1 - y_1 \end{bmatrix} = k \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$= k\, T(u).$$

Thus T is linear.

<u>Ex</u> $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ & T defined by

$$T \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

Similar to previous example, we can show that this is a linear transformation.

We can generalize previous examples to

Ex $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ where A is an $m \times n$ matrix

This is a linear transformation & is called ~~the~~ matrix transformation & denoted by

T_A & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ where A is an $m \times n$ matrix

with $T_A(u) = Au$.

For instance If $A = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 5 & 7 \\ 3 & 2 & 9 \end{bmatrix}$,

then $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
with $Tu = Au$

$$\rightarrow T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 5 & 7 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -x_1 + 2x_2 + 4x_3 \\ x_1 + 5x_2 + 7x_3 \\ 3x_1 + 2x_2 + 9x_3 \end{bmatrix}.$$

Remark: These matrix transformations $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T_A(u) = Au$ are linear.

Linear 1 Lecture 20

8.1 Linear transformations:-

نكمل في الدرس
نأخذ بعض الأمثلة

Ex $T: V \longrightarrow V$

defined by $T(v) = cv$ where c is a real number (such as $T v = 4v$)

$T(v+u) = c(v+u) = cv + cu = T(v) + T(u)$

$T(kv) = c(kv) = k(cv) = k\,T(v)$.

Thus T is linear

Ex $T: P_n \longrightarrow P_{n+1}$

$T(p) = x\,p(x)$

$T(p+q) = x(p+q) = xp + xq = T(p) + T(q)$

& $T(kp) = x(kp) = k\,xp = k\,T(p)$.

Thus T is linear.

Ex $T: P_n \longrightarrow P_{n-1}$

$T(p) = p'(x)$

$T(p+q) = (p+q)' = p' + q' = T(p) + T(q)$

$T(kp) = (kp)' = kp' = k\,T(p)$.

Thus T is linear.
T is called the differentiation transformation

Ex Let V be a vector space and $\{v_1, v_2, \ldots v_n\}$ is a basis of V. Then if $u \in V$, we can write

$u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

and $(c_1, c_2, \ldots c_n) = (u)_s$ is called the coordinate of u with respect to S.

We can define the following transformation

$T: V \longrightarrow \mathbb{R}^n$ (V is finit-dim. with $\dim V = n$)

$T(v) = (v)_S$

This is a linear map.

For instant, if $V = P_n$ and $S = \{1, x, \ldots, x^n\}$ the standard basis, then

$T: V \longrightarrow \mathbb{R}^{n+1}$ $\left(\begin{array}{l} a_0 + a_1 x + \cdots + a_n x^n \\ = a_0(1) + a_1(x) + \cdots + a_n(x^n) \end{array} \right)$

$T(a_0 + a_1 x + \cdots + a_n x^n)$

$= (a_0, a_1, \ldots, a_n)$

is a linear transformation.

Ex $T: M_{mn} \longrightarrow M_{nm}$

$T(A) = A^T$

يتبع

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$T(kA) = (kA)^T = k A^T = k\,T(A)$$

Thus T is a linear transformation.

<u>Ex</u> $T: M_{nn} \longrightarrow R$ (<u>Non</u> linear transformation)

$$T(A) = \det(A)$$

$$T(A+B) = \det(A+B) \neq \det(A) + \det(B) = T(A) + T(B)$$

Thus T is not linear.

<u>Ex</u>: $T: R^2 \longrightarrow R^2$ (Non linear transformation)

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{—— } 1$$

But
$$T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \left.\begin{matrix} \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + \\ \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \end{matrix}\right\} \text{—— } 2$$

Observe that $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$ in ①

different than $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$ in ②

i.e. $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \neq T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$T(u+v) \neq T(u) + T(v)$$

Thus T is not linear.

Theorem: (Properties of linear transformations)

Let $T : V \to W$ be a linear transformation. Then

1) $T(0) = 0$

2) $T(-u) = -T(u)$

3) $T(u-v) = T(u) - T(v)$

4) $T(c_1 v_1 + c_2 v_2) = c_1 T v_1 + c_2 T v_2$ (c_1, c_2 Constants)
$v_1, v_2 \in V$.

5) $T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$

$= c_1 T v_1 + c_2 T v_2 + \dots + c_n T v_n$ where $c_1 \dots c_n$ Constants
$v_1, \dots v_n \in V$.