Lecture 1

- SISO Systems : Single input Single output Systems. (ex:

يعتمد على ال propagation path

⇒ Small scale fading, Due to Multipath effect.

Large scale fading

power

$P_A = P$ P_B

Distance

$P_B = \frac{P}{d^{\theta}}$, $\theta > 3$

distance

(amplification or attenuation)

h channel coefficent

Tx - - - - - - - - - - Rx

s transmitted Signal

$y = s.h + w$

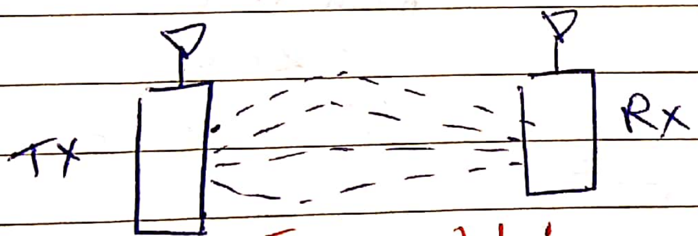
received Signal

(complex)

noise additive gaussian white noise (AGWN)

- h depends on : 1) Small scale fading
  2) Large scale fading / path loss effect.

- Multipath effect:



→ you will receive Multiple replicas of the transmitted Signal.

Time module:

→ $y(t) = \alpha_1 \, s(t - t_1) + \alpha_2 \, s(t - t_2) + \dots$

attenuation

delay

- The key performance metric is SNR

$s.h$ Signal part

w noise part

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→ For now, h is a coefficient and we will treat it as a constant. later on → we will define it as a Random Variable (R.V)

- Noise Characteristics (W)

* $W \sim \underbrace{N}_{\text{normal distribution}} (\underbrace{0}_{\text{mean}}, \underbrace{\sigma^2}_{\text{Variance}})$

* $\underbrace{f_W(X)}_{\text{PdF}} = \dfrac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-X^2/2\sigma^2} \quad , \infty > X > -\infty$

* Noise power $= E[W.W^*]$ or $E[W^2] = \sigma^2 + (E(W))^2$

power is related to the Second moment (expected Value to the X^2)

$\text{Var}(X) = E[X^2] - (E[X])^2$

So ---> $(E[\cancel{X}])^2$

ولانه ال mean = zero

$W \sim N(0, \sigma^2)$

$$\boxed{\text{Noise power} = E[W]^2 = \sigma^2}$$

- Note:-

$E[g(x)] = \int_{-\infty}^{\infty} \underbrace{g(x)}_{\underset{\text{Special case}}{W^2}} \underbrace{f_X(x)}_{\text{pdF}} \, dx$

* Signal power [(S): random variable transmitted]

$$E_s[(S.h)(S.h)^*]$$ وتوقع الصيغ

$$E_s[(S.S^*)(h.h^*)]$$

$$= E_s[|S|^2 . |h|^2]$$ we'll take it out because it's independent.

$$= |h|^2 . \underbrace{E_s[|S|^2]}_{\text{denote it by } P}$$

power transmitted

* Note:
The channel is independent from The signal.

$$\boxed{SNR_{SISO} = \frac{P.|h|^2}{\sigma^2}}$$

→ h is modeled as a complex gaussian R.V.

$h = X + jy$, So $X \sim N(0, \lambda)$
$y \sim N(0, \lambda)$

$|h|^2 = (h.h^*)$

if h is a gaussian R.V, then $|h| \sim$ Rayligh.
positive $(0 \to +\infty)$ R.V

* $|h|^2 \sim$ exponential R.V with mean (λ)

So, SNR is exponential with mean of:

$$\frac{P}{\sigma^2} . \lambda$$

constant ρ

- avg. SNR ($\overline{SNR}$) $= \frac{P \cdot \lambda}{\sigma^2} \triangleq E[SNR]$
- **Outage probability:**

→ $P_{out} = \underbrace{P_r}_{\text{probability:}}\{SNR \leq \underbrace{\gamma_{th}}_{\text{SNR threshold}}\}$

* Outage probability definition: the probability that instantaneous SNR drops below a certain target SNR

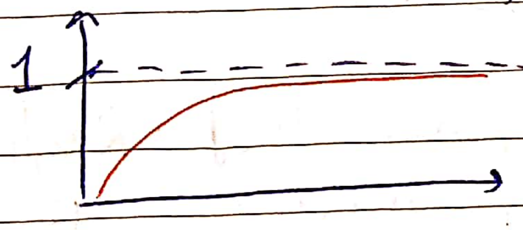
Note CDF: $F_y(x) = P_r\{\overset{R.V}{y} \leq x\}$

PDF: $f_y(x) = \frac{d}{dx}\left(F_y(x)\right)$

$\Rightarrow F_y(x) = \int_{-\infty}^{x} f_y(x)\,dx$

let y be an Exponential R.V

$F_y(x) = \left(1 - e^{-x/\lambda}\right)\cdot u(x)$ → Unit step function



Then the CDF is an increasing monotonic function. with maximum of ONE '1'

- Since SNR is an Exponential R.V, Then

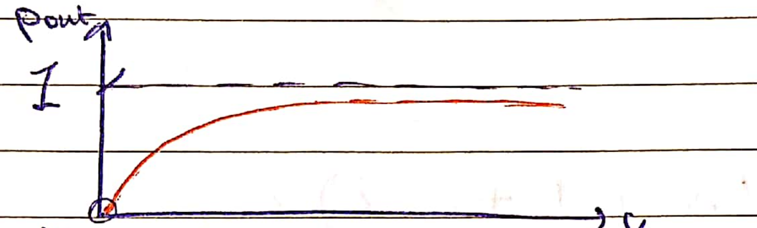
$$P_{out} = P_r\left\{|h|^2 \leq \frac{\gamma_{th}}{\rho}\right\}$$

↝ Recall that $P_r\{|h|^2 \leq x\} = F_{|h|^2}(x)$

So, $$P_{out} = 1 - e^{-\frac{\gamma_{th}}{\rho \cdot \lambda}}$$

$$\boxed{P_{out} = \left(1 - e^{-x/\lambda}\right) \cdot u(x)}$$

plotting P_{out} as a function of γ_{th} :-



@ $\gamma_{th} = \text{Zero}$

⊛ Shannon's Capacity (normalized) bandwidth على

$$C = \log_2(1 + SNR)$$

bit/s/Hz

if $\log \Rightarrow (e/\ln)$, NATs/Second/Hz

⊛ Considering the SISO system:-

$$C = \log_2\left(1 + \frac{P|h|^2}{\sigma^2}\right)$$

↪ ρ

Instantaneous capacity

γ a R.V

we need to average it ⇒ Ergodic capacity $\equiv$ Throughput $(\bar{C})$

$$\boxed{\bar{C} = \underset{|h|^2}{E}\left[\log_2\left(1+\rho|h|^2\right)\right]}$$

$E[|h|^2] = \lambda$ <u>but</u> $E[\log_2(|h|^2)+1] \neq \lambda$

$$\Rightarrow \bar{C} = \int_0^{\infty} \log_2(1+\rho x) \cdot f_{|h|^2}(x) \cdot dx$$

$$= \int_0^{\infty} \log_2(1+\rho x) \cdot \frac{1}{\lambda} e^{-x^2/\lambda} \cdot dx$$

HARD to evaluate

$$\bar{C} \leq \log_2(1+\rho \cdot \lambda)$$

Upper bound

mean of channel variation.

Lecture 2

→ $C = \log_2(1+SNR)$ Shannon's capacity:

* Average throughput Ergodic or average capacity:

Tx ←— h —→ Rx

$$\bar{C} = \underset{|h|^2}{E}\left[\frac{\log_2(1+\rho|h|^2)}{g(|h|^2)}\right]$$

$$= \int_{\cancel{-\infty}}^{\infty} \log_2(1+\rho x) \cdot f_{|h|^2}(x)\,dx$$

Zero

function of :-

$$= \int_0^{\infty} \log_2(1+\rho x) \cdot \frac{1}{\lambda} e^{-x/\lambda} \cdot dx = f(\rho, \lambda)$$

① SNR ② channel variance mean.

Jensen's Inequality:

for a certain function (concave) $g(x)$, the expected value

$$E[g(x)] \leq g(E[x])$$

→ then the expected value now :-

$$\bar{C} = \underset{|h|^2}{E}[\log_2(1+\rho|h|^2)] \leq \log_2(1+\rho E[|h|^2])$$

λ

$\bar{C} \leq \log_2(1+\rho\lambda)$ # Upper bound

Key measures:
1. outage probability:
2. Ergodic capacity:
3. Bit error rate.

Bit Error Rate

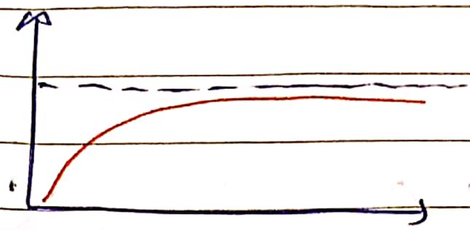
(BER) $\triangleq P_e$ → probability of error.

[1] $P_e = Q(\sqrt{\alpha \cdot SNR})$

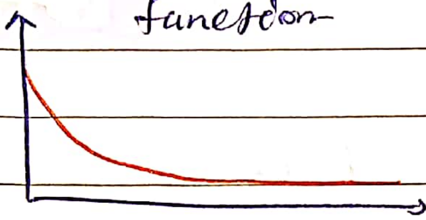
α: depends on the modulation type ($\alpha = 0.5$ or 1) or scheme

[2] $P_e = e^{-\alpha \cdot SNR}$.

Note!



Concave function



Convex function.

Examples on P_e [1]: BPSK ((coherent))

P_e [2]: BFSK ((non-coherent))

⇒ for SISO system:-

Instantaneous Pe:-

[1] $P_e = Q(\sqrt{\alpha \cdot \rho |h|^2})$

[2] $P_e = e^{-\alpha \rho |h|^2}$

⇒ Average BER

Using [1] $\bar{P}_e = E_{|h|^2}[Q(\sqrt{\alpha \rho |h|^2})]$

$$= \int_0^{\infty} Q(\sqrt{\alpha \rho x}) \cdot f_{|h|^2}(x) \cdot dx$$

$$= \int_0^{\infty} Q(\sqrt{\alpha \rho x}) \cdot \frac{1}{\lambda} \cdot e^{-x/\lambda} \cdot dx$$

Also HARD!

Using [2]

$$\bar{P}_e = E_{|h|^2}\left[e^{-\alpha \rho |h|^2}\right] = \int_0^{\infty} e^{-\alpha \rho x} \cdot f_{|h|^2}(x) \cdot dx$$

$$= \int_0^{\infty} e^{-\alpha \rho x} \cdot \frac{1}{\lambda} \cdot e^{-x/\lambda} \cdot dx$$

$$= \int_0^{\infty} \frac{e^{-(\alpha \rho + \frac{1}{\lambda}) x}}{\lambda} \cdot dx$$

$$= \frac{1}{\lambda} \cdot \left. \frac{e^{-(\alpha \cdot \rho + 1/\lambda) x}}{(\alpha \rho + \frac{1}{\lambda})} \right]_0^{\infty}$$

$$= \frac{1}{\lambda} \cdot \frac{1}{\rho \cdot \alpha + \frac{1}{\lambda}} = \boxed{\frac{1}{1 + \rho \alpha \lambda}}$$

Note: moment generating function associated with the random variable X

$$E[e^{tX}] = M_X(t) = MGF(t)$$

$$\left. \frac{\partial M_X(t)}{dt} \right|_{t=0} = \left. E[\underbrace{X \cdot e^{tX}}_{E[X]}] \right|_{t=0}$$

Lecture 3

- MGF of R.v "X"

$\hookrightarrow M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot f(x) \cdot dx$

- If the conditional Bit Error Rate (P_e) is defined as $P_e = e^{-\alpha \rho \cdot |h|^2}$ with mean λ.

- P_e (average) $\Rightarrow \bar{P_e} = E\left[e^{\overbrace{-\alpha\rho}^{t} |h|^2} \right]$

$$= M_{|h|^2}(t) \Big|_{t = -\alpha \cdot \rho} = M_{|h|^2}(-\alpha \cdot \rho)$$

$|h|^2 \rightarrow$ random variable.

- previously, we found $\bar{P_e} = \dfrac{1}{1 + \rho \lambda \cdot \alpha}$ $\alpha \rightarrow$ depends on the modulation scheme.

$\rightarrow$ So, as the mean of the channel power increases, the BER decreases.

- MGF is useful when we have a Q function.

$\Rightarrow$ Question (Exam): if given that $|h|^2$ is a Gama distribution, Find the BER.

+ Gama function will be given.

- $Q(x) \leq e^{-x^2/2}, \quad x > 0.$

if P_e is of the form:

$$P_e = Q\left[\sqrt{\alpha \rho |h|^2}\right] \text{ then}$$

$$\bar{P_e} = E\left[Q\left(\sqrt{\alpha \cdot \rho |h|^2}\right)\right] \rightarrow$$

$\boxed{\text{So}}$ $\bar{P}_e = E[Q(\sqrt{\rho\alpha \cdot |h|^2})]$ can be upper bounded by:

$$\bar{P}_e \leq E\left[e^{\frac{-\alpha\rho|h|^2}{2}}\right] \text{ since}$$

$$Q(x) \leq e^{-x^2/2}, \quad x>0$$

$$Q(\sqrt{x}) \leq e^{-x/2}.$$

- The MGF being evaluated @ $t = \frac{-\alpha\rho}{2}$.

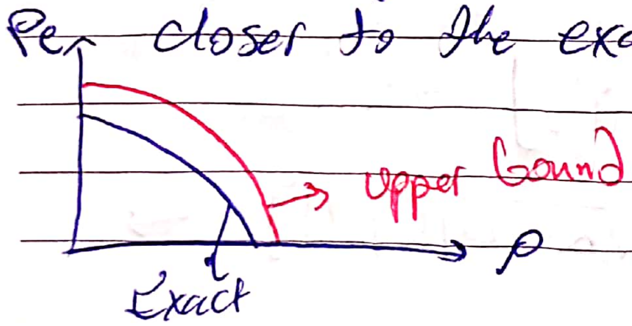
$$\bar{P}_e = E[Q(\sqrt{\alpha\rho|h|^2})] \leq E\left[e^{\frac{-\alpha\rho|h|^2}{2}}\right]$$

$$= M_{|h|^2}\left(\frac{-\alpha\rho}{2}\right)$$

- MGF can be used either the exact Expression when we have conditional exponential type error.
& to find the upper bound when we have a Q type.

- By knowing the BER & its upper bound we can design on the worst case.

- The tighter the upper bound, the better & closer to the exact the design will be.



- Lower bound:

$$E[\log(1+X)] \leq \log(1+E[X])$$

- Gama distribution:

pdf $\leftarrow$ $$f_X(t) = \frac{1}{\lambda} e^{-t/\lambda}$$

$$f_y(t) = \frac{1}{\lambda} e^{-t/\lambda} \cdot t^{\alpha-1}$$ (Generlization)

$$f_y(t)\Big|_{\alpha=1} \text{ then } f_y(t) = f_X(t)$$ if $\alpha = 1$

If α is an Integer The pdf is the sum:

- let $y = \sum_{i=1}^{N} X_i$, $X_i \sim \exp \lambda$

$$f_{X_i}(x) = \frac{1}{\lambda} e^{-x/\lambda} , \quad x \geq 0$$

→ Moment Generating function $M_y(s)$

$$M_y(s) = E[e^{s \cdot y}]$$

$$= E\left[e^{s \cdot \sum_{i=1}^{N} X_i}\right]$$

$$= E[e^{sX_1} \cdot e^{sX_2} \cdot e^{sX_3} \cdot \ldots\ldots e^{sX_N}]$$

$$E[e^{sX_1}] \cdot E[e^{s \cdot X_2}] \cdot \ldots\ldots E[e^{sX_N}]$$

$$(E[e^{sX_1}])^N = \left(\frac{1}{1+\lambda s}\right)^N \#$$

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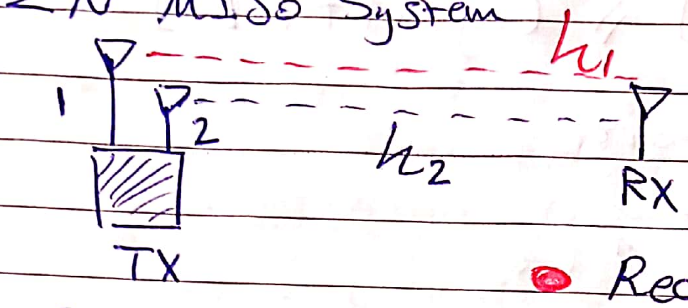
MISO Systems

2N MISO System



2 antennas

- Received Signal

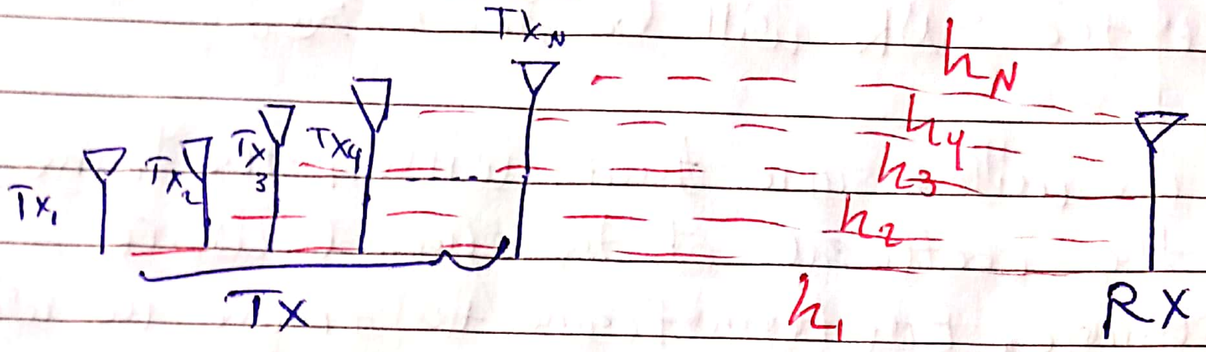
$$y = \underbrace{x_1 h_1 + x_2 h_2}_{\text{Signal Part}} + \underbrace{w}_{\text{noise Power}}$$

- In vector notation:

$$y = \underline{x} \cdot \underline{h}^T + w \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \underline{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

- Transmit beam forming: depends on the transmission scheme. it's the best from performance perspective.
It has the least BER & the lowest outage probability. (optimal Transmit beam forming)
→ receiver has to feedback the channel coefficients to the transmitter.
disadvantage → you need Full channel State information.
more challenging with N-antennas.
(diversity)
Then → The transmitter will select the best channel
* Worst scheme → no feedback.

MISo System



N-antenna

- The received Signal:

$$y = \underbrace{h_1 \cdot X_1 + h_2 \cdot X_2 + h_3 X_3 + \cdots h_N X_N}_{\text{Signal part.}} + \underbrace{w}_{\text{noise part}}$$

$w \sim N(0, \sigma^2)$ ((AGWN))

$$\rightarrow y = \underline{h}^T \underline{X} + w$$

where $\underline{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{bmatrix}$, $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$ transmit Signal Vector.

- CSI: channel State information (feedback) you can test and measure the channel SNR by transmitting a well known signal.
  → measuring channel coefficents.

- The optimal scheme is the transmit beam forming: ((the maximum SNR gain))
  → It requires perfect channel state information.
  $h_1, h_2, \ldots h_N$ (channel gains)
  ↳ channel magnitude & phase.

- design the transmitted signal vector such that the SNR will be maximized.

→ we will assume that within the frame, the coefficents of the channel won't change. But from frame to frame we will repeat the process. block fading channels

- optimal transmit beam former: the signal vector that maximize the SNR at receiver.

$$\underline{\hat{X}} = \frac{1}{\|\underline{h}\|} \cdot \underline{h} \cdot s$$

Some signal / transmit symbol.

norm for normalization

$$\underline{\hat{X}} = \frac{1}{\|\underline{h}\|} \begin{bmatrix} h_1^* \cdot s \\ h_2^* \cdot s \\ \vdots \\ h_N^* \cdot s \end{bmatrix} = \begin{bmatrix} h_1^* \cdot \frac{s}{\|\underline{h}\|} \\ h_2^* \cdot \frac{s}{\|\underline{h}\|} \\ \vdots \\ h_N^* \cdot \frac{s}{\|\underline{h}\|} \end{bmatrix}$$

رح نحطه بالمعادلة

$$y = \underline{h}^T \underline{X} + w \Rightarrow y = \underline{h}^T \cdot \underline{\hat{X}} + w$$

→ Hermitian : Transpose then conjegate :

$$y = \underline{h}^T \cdot \frac{1}{\|\underline{h}\|} \cdot \underline{h}^* \cdot s + w$$

recall : $\|\underline{h}\| = \sqrt{\underline{h}^H \cdot \underline{h}}$

$$\underline{h}^H = (\underline{h}^*)^T = \sqrt{|h_1|^2 + |h_2|^2 + \cdots + |h_N|^2}$$

$= \sqrt{\underline{h}^T \underline{h}^*}$, then we can say that:

$$\|\underline{h}^2\| = \underline{h}^T . \underline{h}^*$$

$$y = \|\underline{h}\| . S + w$$ ((صار, كأنه SISO))

$$SNR = \frac{(\|\underline{h}\|)^2 . P}{\sigma^2}$$

$$SNR_{T.B} = \frac{P}{\sigma^2} . \left(\sum_{i=1}^{N} |h_i|^2 \right)$$ directional transmit beam formings.

T.B: transmit Beam

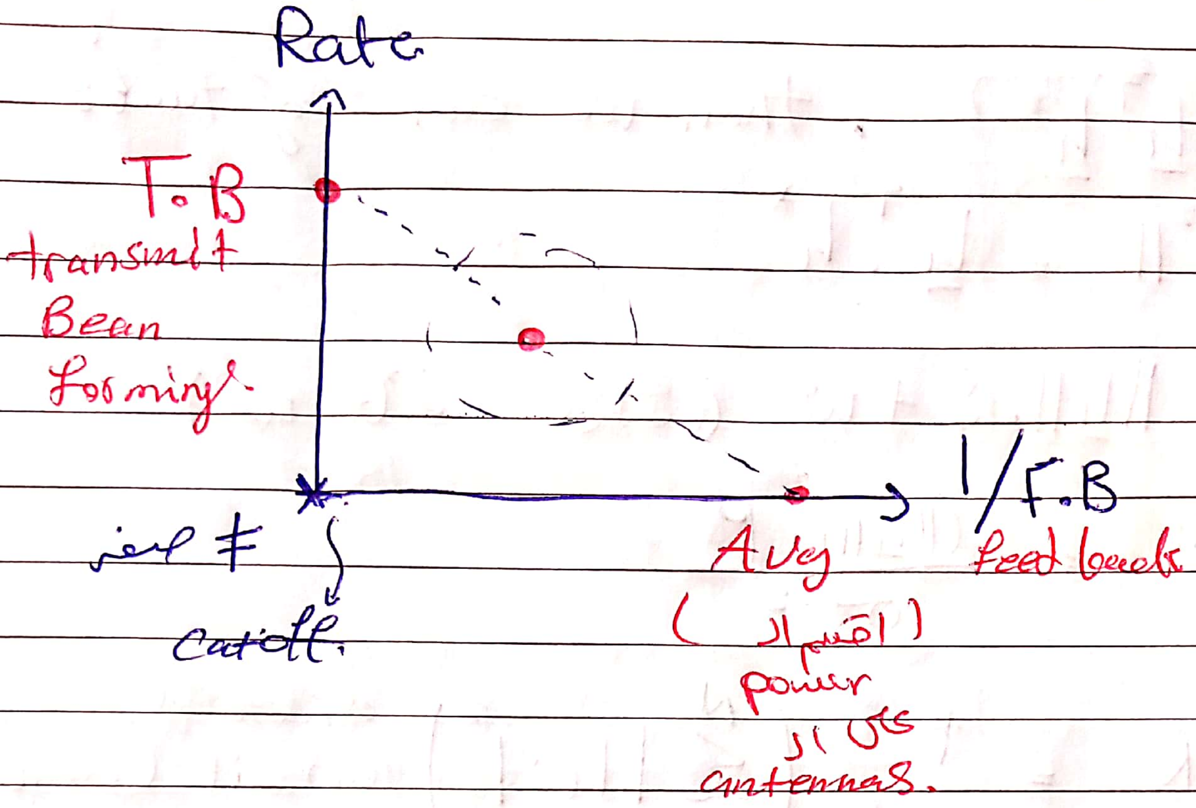
1. The higher N the better SNR. linearly increasing
2. SNR will depend on the sum of channel coefficients.

- The worst case: no information or feedback ((regarding channel coefficients)) ⇝ we will divide the power on the antennas equally.
- If the CSI is available at the transmitter side, let

$$\underline{X} = \begin{bmatrix} \frac{1}{\sqrt{2}} . S \\ \frac{1}{\sqrt{N}} . S \\ \vdots \end{bmatrix}$$

بعد الاشتقاق الي احنا
مش مطالبين فيه ....

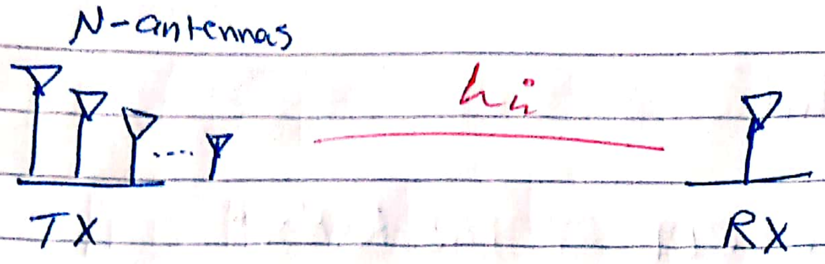
$$\Longrightarrow SNR = \frac{P}{N\sigma^2} \left(\sum_{i=1}^{N} |h_i|^2 \right) \#$$



- We want to find a solution where The rate is close to T.B and complexity is less compaired to T.B.

→ Somewhere in between!

Lecture 5



With transmit beam forming? (perfect CSI)

$$SNR_{optimal} = \frac{P}{\sigma^2} \sum_{i=1}^{N} |h_i|^2$$

optimal transmit beam former

$$\hat{X} = \frac{1}{\|\underline{h}\|} \cdot \underline{h}^* \cdot s$$

If the CSI is not available at the TX, then

$$SNR = \frac{P}{\sigma^2 \cdot N} \cdot \sum_{i=1}^{N} |h_i|^2$$

$$\underline{Y}_{m \times 1} = H_{m \times N} \cdot \underline{X}_{N \times 1} + \underline{W}_{m \times 1}$$

$$Y^T_{1 \times m} \cdot Y_{m \times 1} = (H \cdot \underline{X} + \underline{w})^T \cdot (H \cdot \underline{X} + \underline{w})$$

$$= (\underline{X}^T \cdot H^T + \underline{w})(H \cdot \underline{X} + \underline{w})$$

$$\underbrace{X^T \cdot H^T \cdot H \cdot \underline{X}}_{\text{Symbol correction matrix}} + \underbrace{\cdots}_{\text{noise}}$$

$$X^T \cdot X = \begin{bmatrix} \frac{X_1}{\sqrt{N}} & \circ & \circ & \circ & \circ \\ & & & \circ & \circ \\ & & & & \circ \\ \circ & \circ & & & \\ \circ & \circ & \circ & \circ & \end{bmatrix}$$

→ let $X = \sum_{i=1}^{N} |h_i|^2$

$F_X(t)$: the CDF of the R.V "X"

$|h_i|^2 \sim \exp$ R.V's with mean λ

(independent identically distributed (i.i.d))

→ $$F_X(t) = \frac{\gamma(N, t/\lambda)}{\Gamma(N)} = 1 - \frac{\Gamma(N, t/\lambda)}{\Gamma(N)}$$

$$= 1 - \sum_{k=0}^{N-1} \frac{1}{k!} e^{-x/\lambda} \left(\frac{\gamma}{\lambda}\right)^k$$

→ $\Gamma(\alpha) = \gamma(\alpha, X) + \Gamma(\alpha, X)$

gama function — lower — upper

[1] The outage probability: $P_o = P_r\{SNR < \gamma_{th}\}$

$$= P_r\left\{X < \frac{\gamma_{th}}{\rho}\right\}$$

$$= 1 - \sum_{k=0}^{N-1} \frac{1}{k!} \cdot e^{-\gamma_{th}/\rho\lambda} \cdot \left(\frac{\gamma_{th}}{\rho\lambda}\right)^k$$

to see effects of N-antenna on P_{out}.

[2] Ergodic capacity:

of T.B $\quad \bar{C} = E_X[\log_2(1+\rho X)]$

$$X = \sum_{i=1}^{N} |h_i|^2$$

$$\bar{C} = \int_0^{\infty} \log_2 (1+\rho . t) \, f_X(t) \, dt . dx$$

$$\bar{C} = \int_0^{\infty} \log_2 (1+\rho.t) \, \frac{\partial F_X(t)}{\partial t} . dx$$

this integral is hard to evaluate

Note

$$F_X(t) = 1 - \frac{\Gamma(N, \frac{t}{\lambda})}{\Gamma(N)} = 1 - \sum_{k=0}^{N-1} \left(\frac{t}{\lambda}\right)^k . \frac{1}{k!} e^{-t/\lambda}$$

$$\frac{\partial F_X(t)}{\partial t} = \boxed{f_X(t) = \frac{t^{N-1} . e^{-t/\lambda}}{\lambda^N (N-1)!}} \rightarrow \text{PDF}$$

then $$\bar{C} = \int_0^{\infty} \log_2 (1+\rho t) . \frac{t^{N-1} e^{-t/\lambda}}{\lambda^N (N-1)!} \, dt$$

also hard to be evaluated.

to estimate the value of $\bar{C}$:

$$\bar{C} \leq \log (1+\rho . E[X])$$

$$\hookrightarrow X = \sum_{i=1}^{N} |h_i|^2$$

$$\log_2 (1+\rho \lambda . N)$$

$$E[X] = \sum_{i=1}^{N} E[|h_i|^2]$$

$$= N . X$$

$$\bar{C} = \int_0^\infty \log(1+\rho t)\, f_X(t)\cdot dt$$

$$= \int_0^\infty \log_2(1+\rho t)\, \frac{t^{N-1} e^{t/\lambda}}{\lambda^N (N-1)!}\, dt$$

upper bounded $\leq \log_2\Big(1+\rho\cdot \underbrace{\int_0^\infty \frac{t\cdot t^{N-1} e^{-t/\lambda}}{\lambda^N (N-1)!}\, dt}_{\lambda\cdot N}\Big)$

$\leq \log_2(1+\rho(\lambda\cdot N))$

- the integration is the Expected value:

$$E[X] = E\Big[\sum_{i=1}^{N} |h_i|^2\Big]$$

$$= \sum_{i=1}^{N} E[|h_i|^2] = N\cdot\lambda$$

- $\bar{C} \leq \log_2(1+\rho\lambda\cdot N)$, T.B

If CSI isn't available at the transmitter:

$\bar{C}_{\text{No CSI}} \leq \log_2(1+\rho\lambda)$ it's like SI so

not taking advantage of diversity

3 Bit error rate BER:

$$P_e = Q(\sqrt{\alpha\cdot SNR})$$

↳ conditional: given SNR is fixed.

$$\bar{P}_e = E_{SNR}\big[Q(\sqrt{\alpha\cdot SNR})\big]$$

$$SNR_{T.B} = \rho\cdot X = \rho\cdot \sum_{i=1}^{N} |h_i|^2$$

- $\bar{P}_{e_1} = E_X\left[Q(\sqrt{\alpha_1 \rho X})\right]$, where $X = \sum_{i=1}^{N} |h_i|^2$

$$\bar{P}_{e_1} = \int_0^\infty Q(\sqrt{\alpha \rho t}) \cdot \frac{t^{N-1} e^{-t/\lambda}}{\lambda^N (N-1)!} \, dt$$

no closed form

$\boxed{\text{Note}}$ Recall $Q(x) \leq e^{-x^2/2}$

$$Q(\sqrt{\alpha x}) = e^{-\frac{(\sqrt{\alpha x})^2}{2}} = e^{-\frac{\alpha x}{2}}$$

$$\bar{P}_{e_1} \leq P_{e_2} = M\left(\frac{-\alpha}{2}\right) = E_X\left[e^{-\rho\alpha/2 \cdot X}\right]$$

$$= M_X\left(\frac{-\alpha\rho}{2}\right)$$

P_{e_2}: any exponential type.

- to find $M_X(s)$, $M_X(s) = \int_0^\infty e^{s.t} \cdot f_X(t) \cdot dt$

$$= \int_0^\infty e^{t.s} \cdot \frac{t^{N-1} \cdot e^{-t/\lambda}}{(N-1)! \; \lambda^N} \cdot dt$$

solving this integral is a way ...

(1)

in another way ...

(2)

$$M_X(t) = E\left[e^{s \cdot \sum_{i=1}^{N} |h_i|^2}\right]$$

$$= E\left[e^{s \cdot |h_1|^2}\right] \cdot E\left[e^{s \cdot |h_2|^2}\right] \cdots$$

$$= \left(E\left[e^{s.|h_1|^2} \right] \right)^{2} = \left(\frac{1}{1-\lambda s} \right)^{N}$$

So

$$\leq E_x\left[e^{\frac{-\alpha\rho}{2} x} \right]$$

$$= M_x\left(\frac{-\alpha.\rho}{2} \right) = Pe_2$$

$$= \left(\frac{1}{1+\frac{\lambda\rho}{2}} \right)^{N}$$

if $N=1 \Rightarrow SISO$

$\rho \rightarrow$ SNR

$\lambda \rightarrow$ mean

$$\Downarrow$$

$$\int_0^\infty e^{st} \cdot \frac{1}{\lambda} \cdot e^{-t/\lambda}$$

$$\frac{1}{\lambda} \int_0^\infty e^{(s-1/\lambda).t} \cdot dt$$

$$1/\lambda = \left[\frac{e^{(s-1/\lambda)t}}{(s-1/\lambda)} \right]_0^\infty$$

$$= -1/\lambda \cdot \frac{1}{s-1/\lambda}$$

Lecture 7

- mean $F_X(x)$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, \partial F_X(x).$$

$$X = \exp(1 \times 10^{6})$$

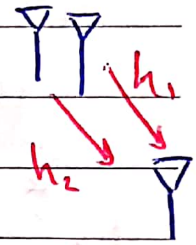
$$g(x) = \log_2(1 + \rho X)$$

mean (g).

- MISO system with T. B

Ex) consider 2×1 MISO system

$$\underline{h} = \begin{bmatrix} 2+j \\ 1+2j \end{bmatrix}$$



① Find the Expression for the received signal @ RX

$$y = \underbrace{[\,2+j \quad 1+2j\,]}_{\underline{h}^T} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \underset{\text{noise}}{w}$$

$$y = \underline{h}^T . X + w$$

② Given that CSI given at Tx, Find the optimal transmit beam vector.

$$\hat{X} = \frac{1}{\|\underline{h}\|} . \underline{h}^* . S$$

$$\hat{X} = \begin{bmatrix} 2-j \\ 1-2j \end{bmatrix} . \frac{S}{\sqrt{10}}$$

$$\frac{1}{\|\underline{h}\|} = \frac{1}{\sqrt{(4+1)+(1+4)}} = \frac{1}{\sqrt{10}}$$

$$\|\underline{h}\| = \sqrt{(\underline{h}^T)^* \cdot \underline{h}}$$

$$= \sqrt{\begin{bmatrix} h_1^* & h_2^* \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}} = \left(|h_1|^2 + |h_2|^2 \right)$$

The received signal provided that we have $\hat{X}$

$$y = \begin{bmatrix} 2+j & 1+2j \end{bmatrix} \begin{bmatrix} \frac{2-j}{\sqrt{10}} \\ \frac{1-2j}{\sqrt{10}} \end{bmatrix} * S + w$$

Rx signal when $\hat{X}$ is used.

$$= \frac{1}{\sqrt{10}} \left(\sqrt{10}\right)^2 \cdot S + w$$

طلعناها برا

هاد الجواب الي بطلع لما اضرب ① ②

$$\boxed{y = \sqrt{10} \cdot S + w}$$

③ Find SNR

$$SNR = \frac{10 \cdot P}{\sigma^2}, \quad \text{if } \frac{P}{\sigma^2} = 1$$

Then The SNR is $\boxed{10}$

(4) The capacity of the system. The bandweidth is normalized.

$$C = \log_2 (1 + 10) = \log_2 (11)$$

Bits per Seconds

(5) Bet error rate for Binary PSK:

$$\underset{BPSK}{BER} = Q(\sqrt{2SNR})$$

$$= Q(\sqrt{20})$$

The upper limit is:

- Shannon's Capacity (it's only valid when X is Gaussian.

entropy: The amount of information you don't have.

→ The measure of ambiguity in the information.

- Capacity reduction → if $X \neq$ gaussian.

→ in optical communication you either transmit or you don't transmit ONOFF Keying.

you restrict X to be positive.

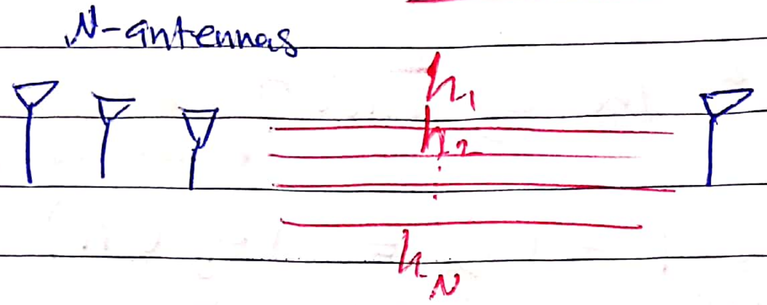
Shannon's Capacity formula will not apply?

Differential entropy?: we study?

$$I(X;y) = h(y) - h(y/x)$$

given

Selection diversity: to select the Best channel.



$$\bar{C} = E_{X^*}\left[\log_2(1+\rho X^2)\right]$$

$$RV's = X_1, X_2, \ldots X_N \quad (iid)$$

$$\text{let } X^* = \text{Max}(X_1, X_2, \ldots X_N)$$

$$F_{X^*}(X) = \Pr\{X^* \leq X\}$$

because transmitting all channel coefficients is (feedback)
Expensive ⇝ we send and feedback the best channel only.

$$= \Pr\{\max(X_1, X_2, \ldots X_N) \leq X\}$$

recall:

$$\Pr\{\max(X_1, X_2) \leq X\}$$

$$= \Pr\{X_1 \leq X, X_2 \leq X\}$$

Independent RV's

$$= \Pr\{X_1 \leq X\} \cdot \Pr\{X_2 \leq X\}$$

$$= F_{X_1}(X) \cdot F_{X_2}(X)$$

So $\longrightarrow$ $Pr\{\max(X_1, X_2, \cdots X_N) \leq X\}$

$$= \prod_{i=1}^{N} F_{X_i}(x) = [F_{X_i}(X)]^N$$

- $\bar{C} = \int_0^\infty \log_2(1+\rho X)\, dF_{X^*}(X)$ $\longrightarrow$ $[F_{X_i}(X)]^N$

$$= \int_0^\infty \log_2(1+\rho \cdot X)\, d[F_{X_i}(X)]^N$$

if $X = 1$
SISO

استخدم
Chain Rule.

Lecture 8

Transmit diversity: MISO system

- Select one antenna to obtain the best SNR.
- The power will be fed to the best channel coefficent & shut down the remaining antennas

$$\rightarrow \text{SNR} = \frac{P}{\sigma^2} \cdot \max_{i=1 \dots N} |h_i|^2 \quad \boxed{\text{i i d}}$$

define $\frac{P}{\sigma^2} = \rho$, $X = \max_{i=1 \dots N} |h_i|^2$

$$F_X(t) = \left[F_{|h_i|^2}(t) \right]^N$$

maximum

from previous lecture

$$F_X(t) = \left(1 - e^{-t/\lambda}\right)^N$$

Exponetial random variable.

$$P_{outage} = Pr\{ \text{SNR} \leq \gamma_{th} \}$$

$$= Pr\{ \rho \cdot X \leq \gamma_{th} \}$$

$$= Pr\{ X \leq \frac{\gamma_{th}}{\rho} \}$$

outage probability $F_X\left(\frac{\gamma_{th}}{\rho}\right) = \left(1 - e^{-\frac{\gamma_{th}}{\rho}}\right)^N$

كل ما زاد عدد ال (N) antennas ال
outage probability will decrease ↓↓

Ergodic capacity

- $\bar{C} = E_X\left[\log_2(1+\rho X)\right], \quad X = \max_{i=1\cdots N} |h_i|^2$

maximum

$$= \int_0^\infty \log_2(1+\rho t) \cdot \frac{d F_X(t)}{dt} \cdot dt$$

CDF

$$F_X(t) = \left(1 - e^{-t/\lambda}\right)^N$$

$$\frac{dF_X(t)}{dt} = \frac{d}{dt}\left[F_{|h_i|^2}(t)\right]^N$$

$$N \cdot \left[F_{|h_i|^2}(t)\right]^{N-1} \cdot f_{|h_i|^2} \rightarrow \text{PDF}$$

$$= N \cdot \left(1 - e^{-t/\lambda}\right)^{N-1} \cdot \frac{1}{\lambda} \cdot e^{-t/\lambda}$$

$$\bar{C} = \int_0^\infty \log_2(1+\rho t) \cdot N\left[\left(1 - e^{-t/\lambda}\right)\right]^{N-1} \cdot \frac{1}{\lambda} e^{-t/\lambda} \cdot dt$$

HARD!

$$\bar{C} \leq \log_2\left(1 + \rho \cdot E[X]\right)$$

$$E[X] = \int_0^\infty t \cdot f_X(t) \cdot dt$$

$$= \int_0^\infty N \cdot \left(1 - e^{-t/\lambda}\right)^{N-1} \cdot \frac{e^{-t/\lambda}}{\lambda} \cdot t \cdot dt$$

⇒ recall

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \cdot b^{n-k}$$

⇐ This Must Equal ONE 1

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

⇓

$$E[X] = \int_0^\infty t \cdot f_X(t) \cdot dt$$

$$= \Psi(N) + E_0$$

E_0 → Euler's constant $= 0.577$

PSI digamma function.

$$\bar{C} \leq \log_2\left(1 + \rho\left(\Psi(N) + E_0\right)\right)$$

upper bound

- **Asymptotic Results**

$\boxed{1}$ $\Psi(N) \sim \log(N)$ (as $N \to \infty$)

$\boxed{2}$ $\bar{C} \sim \log(\log(N))$

in Transmit beam former

$$\bar{C} \sim \log(N)$$

↳ faster than $\log(\log(N))$

and That's the penalty for not having full CSI.

Lecture 9

- Average BER:

$$P_e = E[e^{-\alpha x}]$$

→ for selection diversity:

$$X = \max_{i=1 \dots N} |h_i|^2$$

$$\boxed{f_X(t) = N \cdot (1 - e^{-t/\lambda})^{N-1} \cdot \frac{e^{-t/\lambda}}{\lambda}}$$ Density

$$P_{e_2} = \int_0^\infty e^{-\alpha t} \cdot N \underbrace{(1 - e^{-t/\lambda})^{N-1}}_{\sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \cdot e^{-t/\lambda \cdot k}} \cdot \frac{e^{-t/\lambda}}{\lambda} \cdot dt$$

Exact (monotonic generation function)

$$\rightarrow \sum_{k=0}^{N-1} N \binom{N-1}{k} (-1)^k \cdot \int_0^\infty \frac{e^{-(\alpha + 1/\lambda + k/\lambda) t}}{\lambda} \cdot dt$$

This can be easily evaluated

$$\sum_{k=0}^{N-1} \frac{N}{\lambda} \binom{N-1}{k} (-1)^k \cdot \frac{1}{\frac{\lambda \alpha}{\lambda} + \frac{(k+1)}{\lambda}}$$

$$= \sum_{k=0}^{N-1} N \binom{N-1}{k} (-1)^k \cdot \frac{1}{\lambda \alpha + (k+1)}$$

Scanned with CamScanner

upper bound.

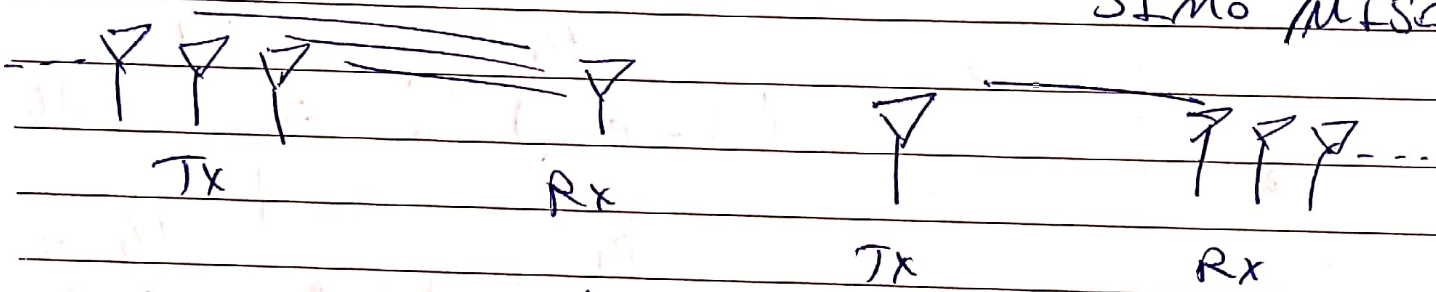
$$P_{e_1} = E_X\left[Q\left(\sqrt{\alpha' \cdot X}\right)\right] \leq E\left[e^{-\frac{\alpha \cdot X}{2}}\right]$$

بدي احط بدل كل α
من ال
old expression
وبحط بداله $\frac{\alpha'}{2}$

$$= P_{e_2} \Big|_{\alpha = \frac{\alpha'}{2}}$$

- you can always select the best Tx antenna

SIMO / MISO

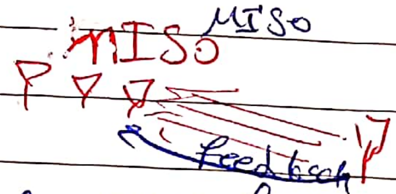


- Same analysis, here we talk about receiving antennas
  ($\bar{C}$, outage probability, ...)

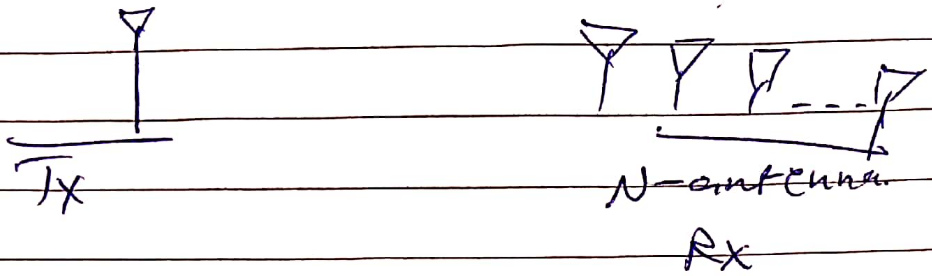
- Advantage:
  Rx : decide which antenna / channel is the best. (no feedback CSI)

* downlink

- Disadvantage on TX, Rx ?
  راح تكون multiple antenna device (without CSI feedback)



- Receive Diversity SIMO



at Rx :

$$i^* = \arg\max_{i=1\cdots M} |h_i|^2$$

$$X = \frac{P}{\sigma^2} \cdot \max_{i=1\cdots M} |h_i|^2$$

X

- Same as before but M instead of N

MIMO

$h_{ij} \rightarrow$ ith Tx antenna to jth

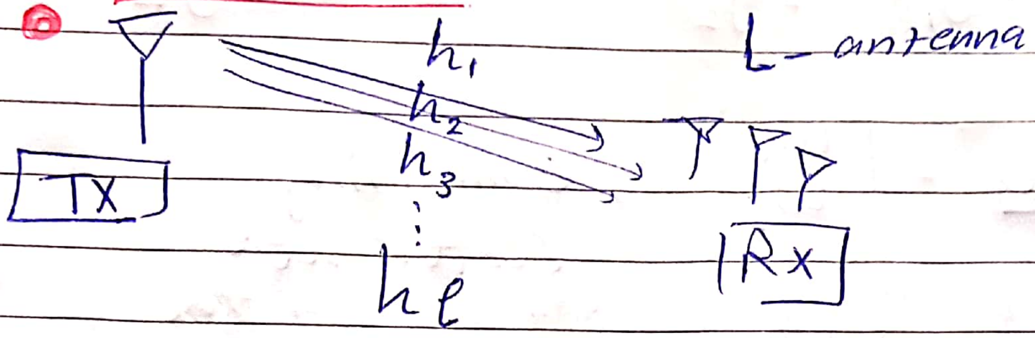
selection مثل SISO

channel matrix [illegible]

- SIMO : no feedback , CSI at Tx is not required.

Maximal Ratio Combining (MRC)

SIMO System



- h_ℓ : the channel coefficient from the transmitter to the ℓ^{th} received antenna.
- S : the transmitted signal @ the ℓ^{th} antenna.
- $\underbrace{y_\ell}_{\text{received}} = S h_\ell + z_\ell$ → noise AGWN $\boxed{N(0, \sigma^2)}$

  → ℓ: takes values from $1 \longrightarrow L$
- In vector form:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_L \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_L \end{bmatrix} . S + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_L \end{bmatrix}$$

Z_ℓ is i.i.d for $\ell = 1 \ldots L$

- Combining the received signals linearly:

$$u = w_1^* y_1 + w_2^* y_2 + \dots + w_L^* y_L$$

Combiner output

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} \text{, Combining coefficients}$$

- In Vector form:

$$u = \underline{w}^H \underline{y}$$

$$u = \underline{w}^H . (\underline{h} . s + \underline{z})$$

channel coefficients.

$$= \underbrace{\underline{w}^H . \underline{h} . s}_{\text{Signal power}} + \underbrace{\underline{w}^H . \underline{z}}_{\text{noise power}}$$

- To quantify the system; define the key performance metrics:

$\boxed{1}$ $SNR = \dfrac{\text{Signal power}}{\text{noise power}}$

→ Signal power: $|\underline{w}^H . \underline{h}|^2 . P$

P = Expected value of the transmitted symbol s.

→ noise power $= E\left[|\underline{w}^H . \underline{z}|^2 \right]$

row vectors

$$= E\left[(\underline{w}^H . \underline{z})^* (\underline{w}^H . \underline{z}) \right]$$

but $(\underline{w}^H \underline{z})^* . (\underline{w}^H . \underline{z}) =$

$$(w_1^* z_1 + w_2^* z_2 + \dots + w_L^* z_L) \cdot (w_1 z_1^* + w_2 z_2^* + \dots + w_L z_L^*)$$

$$= \sum_{i=1}^{L} |w_i|^2 |z_i|^2 + \sum_{i} \sum_{j \atop i \neq j} w_i w_j^* z_i^* z_j \quad \dots (1)$$

- Facts :

* $E\left[|z_i|^2 \right] = \sigma^2$

* $E[z_i z_j] = E[z_i] \cdot E[z_j] = 0$

Independent

They have Zero mean.

Substituting Facts in (1) :

$$\text{noise power} = \sigma^2 \sum_{i=1}^{L} |w_i|^2$$

$$= \sigma^2 (\underline{w}^H . \underline{w}) \quad \#$$

So, $SNR = \dfrac{|\underline{w}^H . \underline{h}|^2 \cdot P}{\underline{w}^H . \underline{w} \, , \sigma^2}$

Choose $\underline{\hat{w}}$ such that $\underline{\hat{w}}$ will maximize The SNR.

((Like transmit beam former $\underline{\hat{X}}$))

- $\hat{w} = \dfrac{h}{\|\underline{h}\|}$

→ The optimal value occures when the combining coefficents are in the same direction of the channel coefficents.

- after plugging the $\underline{\underline{\hat{w}}}$ we get:

$$SNR_{optimal} = \|\underline{h}\|^2 \cdot \frac{P}{\sigma^2}$$ → average SNR

$$\boxed{SNR_{opt} = \left(\sum_{l=1}^{L} |h_l|^2\right) \cdot \frac{P}{\sigma^2}}$$

- The (MRC) gain is equivalent to The (T.B) gain except that we don't need to feedback The channel coefficents and state information.

- We have the same performance analysis for T.B but we'll replace $N \to L$.

There fore, The outage probability & the Ergodic capacity as well as BER for The two types can be computed using T.B analysis

Central Limit theorem (CLT) & Its applications

- Definitions:

A sequance of Random Variables (RVs): X_1, X_2, X_3, ... X; Converges in distribution to a R.v $\boxed{\text{"}X\text{"}}$ if:

$$\lim_{n \to \infty} \underbrace{F_{X_n}(t)}_{} = F_X(t)$$

CDF

for all t at which $F_X(t)$ is continuous.

$$X_n \xrightarrow{d} X$$

Sequance of RV converges to distribution

(Ex) Let $X_1, X_2, \ldots X_n$ be a sequence of R.Vs Such That:

$$F_{X_n}(t) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{nt} & , t > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

$$\lim_{n \to \infty} F_X(t) = \lim_{n \to \infty} \left(1 - \left(1 - \frac{1}{n}\right)^{nt}\right)$$

$$= 1 - \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{nt}$$

$$= 1 - e^{-t}$$

, Exponetial R.V CDF with mean = 1 unity

Central Limit theorem (CLT) & Its applications

Definitions:

A sequance of Random Variables (RVs): $X_1, X_2, X_3, \dots X_n$; Converges in distribution to a R.V $\boxed{\text{"}X\text{"}}$ if:

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$$= 1 - \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{nt}$$

$$= 1 - e^{-t}$$

, Exponetial R.V CDF with mean = 1 unity have the same mean

- let $X_1, X_2, X_3, \ldots, X_N$ be i.i.d sequence R.Vs with mean $E[X_i] = \mu$ and a variance $Var(X_i) = \sigma^2 < \infty$ and define :

must be finite

$$U_N = \frac{\sum_{i=1}^{N} X_i - N.\mu}{\sigma \sqrt{N}}$$

Sequance R.V

→ Then

$$F_{U_N}(t) = \underbrace{\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} . dZ}$$

CDF

↳ $\Phi(t)$: the CDF of standard normal RV $N(0,1)$

- $U_N \xrightarrow{d} N(0,1)$

↳ converges in distribution to normal R.V

- $$\sum_{i=1}^{N} X_i - N.\mu \sim N(\underset{\text{mean}}{0}, \underbrace{\sigma^2 . N}_{\text{variance}})$$

$$\Rightarrow \sum_{i=1}^{N} X_i \sim N(N.\mu, \sigma^2 . N)$$

↳ The sum can be approximated by a normal R.V with $\begin{Bmatrix} \mu' = N.\mu \\ \sigma'^2 = \sigma^2 . N \end{Bmatrix}$

- applications : → Transmit Beam (T.B)
→ Maximal Ratio combining [or] (MRC)

[1] SNR

- $SNR = \rho . \sum_{i=1}^{N} |h_i|^2$, where $\rho = \frac{P}{\sigma^2}$

ρ = average SNR.

$|h_i|^2$ are i.i.d exponetial R.Vs with a mean $E[|h_i|^2] = \Omega$ and a variance

$Var(|h_i|^2) = \Omega^2$

→ according to (CLT):

$$\sum_{i=1}^{N} |h_i|^2 \sim N(\underbrace{N.\Omega}_{\mu'}, \underbrace{\Omega^2 . N}_{\sigma'^2})$$

[2] outage probability

- $P_{out} = Pr\{SNR \leq \gamma_{th}\}$, $SNR = \rho . \sum_{i=1}^{N'} |h_i|^2$

$$= Pr\{\rho . \sum_{i=1}^{N} |h_i|^2 \leq \gamma_{th}\}$$

$$= Pr\{\sum_{i=1}^{N} |h_i|^2 \leq \frac{\gamma_{th}}{\rho}\}$$

PDF of normal distribution R.V

$$\simeq \int_{-\infty}^{\frac{\gamma_{th}}{\rho}} \frac{1}{\sqrt{2\pi \sigma'^2}} . e^{-\frac{(z-\mu')^2}{2\sigma'^2}} . dz$$

$$= \Phi\left(\frac{\frac{\gamma_{th}}{\rho} - \mu'}{\sigma'}\right)$$

- Recall that Gaussian R.V $N(\mu, \sigma^2)$ has a PDF of:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

& a CDF of:

$$\Phi\left(\frac{t-\mu}{\sigma}\right)$$

where $\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}} \cdot dz$

[3] Ergodic Capacity:

- Using Jensen's Inequality, the $\bar{C}$ is upper-bounded by:

$$\bar{C} \leq \log_2\left(1+\rho\, E\left[\sum_{i=1}^{N} |h_i|^2\right]\right)$$

→ & from the CLT:

$$E\left[\sum_{i=1}^{N} |h_i|^2\right] = \mu' = N \cdot \Omega$$

Therefore: $\bar{C} \leq \log_2(1+\rho \cdot \Omega \cdot N)$

$\boxed{4}$ Average Bit Error Rate (BER)

- We have two types P_{e_1} & P_{e_2}:

$\boxed{\text{For } \bar{P}_{e_2}:}$

- $$\bar{P}_{e_2} = E\left[e^{-\alpha . SNR}\right]$$

$$= E\left[e^{-\alpha . \rho . \sum_{i=1}^{N} |h_i|^2}\right], \quad V = \sum_{i=1}^{N} |h_i|^2$$

$$t = -\alpha . \rho$$

$$\simeq E\left[e^{t . V}\right]\Big|_{t=-\alpha\rho}, \quad V \sim N(\mu', \sigma'^2)$$

moment generation function.

$$= M_V(t)\Big|_{t=-\alpha.\rho}$$

But: $$M_V(t) = \exp\left(\mu' t + \sigma'^2 . \frac{t^2}{2}\right)$$

$$\Rightarrow \bar{P}_{e_2} = \exp\left(-N . \Omega . \alpha . \rho + N . \Omega^2 . \frac{\alpha^2 \rho^2}{2}\right)$$

$\boxed{\text{For } \bar{P}_{e_1}:}$

- $$\bar{P}_{e_1} = E\left[Q\sqrt{\alpha . SNR}\right] \leq E\left[e^{-\frac{\alpha}{2} . SNR}\right]$$

$$= E\left[e^{-\frac{\alpha}{2} . \rho . \sum_{i=1}^{N} |h_i|^2}\right]$$

$$= M_N\left(\frac{-\rho\alpha}{2}\right)$$

$$= \exp\left(-N \cdot \Omega \cdot \frac{\rho\alpha}{2} + N \cdot \Omega^2 \cdot \frac{\rho^2\alpha^2}{8}\right)$$

* There will be a project.
* project group must have 4- members.
* MatLab simulation on one of the systems we've studied so far.
* To calculate performance metrics.
* each group must submit members names by the End of this week.

Lecture 12

Extreme Value theory (EVT)

- let $X_1, X_2, \ldots, X_N$ be iid random Variables with common cumulative distribution function (CDF) of $F(x)$. let $X_{(N)} = \max_{i=1,\ldots N} X_i$

→ Then, the limiting CDF of the sequence:

$$\frac{X_{(N)} - a_N}{b_N}$$

normalizing constants.

where:

a_N and $b_N > 0$

→ Must belong to one of the three extreme value distributions $G_i(x)$.

$G_i(x)$:

① Fréchet Distribution.

$$G_1(x) = \begin{cases} 0, & x \leq 0 \\ \exp(-x^{-\alpha}), & x > 0, \alpha > 0 \end{cases}$$

② Weibull Distribution

$$G_2(x) = \begin{cases} \exp[-(-x)^{-\alpha}], & x > 0, \alpha > 0 \\ 1, & x > 0 \end{cases}$$

③ Gumbel Distribution.

$$G_3(x) = \exp(-e^{-x}), \quad -\infty < x < \infty$$

our main focus.

maximal domain of attraction

PDF

- $F(x) \in D(G_3)$ if $F'(x) = f(x) > 0$ and is differentiable for all $x \in (x_1, \xi_1)$ for some x_1, and

$$\lim_{x \to \xi_1} \frac{d}{dx}\left[\frac{1 - F(x)}{f(x)}\right] = 0$$

must be satisfied

→ The normalizing constants a_N & b_N can be obtained as:-

- $a_N = F^{-1}\left(1 - \frac{1}{N}\right)$
- $b_N = F^{-1}\left(1 - \frac{1}{Ne}\right) - F^{-1}\left(1 - \frac{1}{N}\right)$

- Exponential Distribution:

CDF: $F = 1 - \exp\left(-\frac{x}{\text{omega}}\right)$ → Ω: mean of R.V (X)

$$= 1 - e^{-a/\Omega} = 1 - \frac{1}{N \cdot 1}$$

$$a = \ln(N) \cdot \Omega$$

$$a_N = a = \ln(N) \cdot \Omega$$

$$1 - e^{-x/\Omega} = 1 - \frac{1}{N \cdot e}$$

$$x = (\ln(N) + 1)\,\Omega$$

$$X = (\ln(N) + 1) \cdot \Omega$$

$$b_N = b = (\ln(N)+1) \cdot \Omega - \ln(N) \cdot \Omega$$

$$b_N = b = \Omega$$

- Define $X_{(N)} = \max_{i=1 \dots N} |h_i|^2$

where $F_{|h_i|^2}(t) = \left(1 - e^{-t/\Omega}\right) \cdot u(t)$

→ From Extreme Value Theorm:

$$\frac{X_{(N)} - a}{b} \xrightarrow{d} \underbrace{X}_{\text{Gumbel}}$$

converges to the distribution ....

where $\underbrace{F_X(t) = e^{-e^{-t}}}_{\text{CDF of Gumbel}}$, $\infty > t > -\infty$

- $$\lim_{N \to \infty} pr\left\{\frac{X(N) - a}{b} \leq t\right\} = pr\{X \leq t\}$$

$$= e^{-e^{-t}} , \quad \infty > t > -\infty$$

we find the asymptotic CDF of $X(N)$ as follows:

$$pr\{X_{(N)} \leq t\} = pr\left\{b \cdot \frac{(X_{(N)} - a)}{b} + a \leq t\right\}$$

Scanned with CamScanner

approx.

$$\approx pr\{b \cdot X + a \leq t\} = pr\{X \leq \frac{t-a}{b}\}$$

$$= e^{-e^{-\left(\frac{t-a}{b}\right)}} \quad , \infty > t > -\infty$$

- Generalized Gumbel:

$$\rightarrow F_{X(N)}(t) \approx e^{-e^{-\left(\frac{t-a}{b}\right)}} \quad , \infty > t > -\infty$$

as $N \rightarrow \infty$

$\rightarrow a = \sigma \cdot \log(N)$

$\rightarrow b = \sigma$

if a & b = zero we get the original gumbel.

Back to selection diversity: Using the conclusions we obtained

$\boxed{1}$ outage probability

$$P_{out} = pr\{\rho \cdot X_{(N)} \leq \gamma_{th}\}$$

$$= pr\{X_{(N)} \leq \frac{\gamma_{th}}{\rho}\}$$

The maximum

$$= F_{X(N)}\left(\frac{\gamma_{th}}{\rho}\right) \approx e^{-e^{-\left(\frac{\frac{\gamma_{th}}{\rho} - \sigma \cdot \log(N)}{\sigma}\right)}}$$

as $N \rightarrow \infty$

at $t = \frac{\gamma_{th}}{\rho}$

$\boxed{2}$ Ergodic Capacity

- $\bar{C} \leq \log_2 \left(1 + \rho E[X(N)] \right)$

$$E[X(N)] \approx \int_{-\infty}^{\infty} t \cdot \frac{dF_{X(N)}(t)}{dt} \cdot dt$$

$$= \int_{-\infty}^{\infty} t \, \frac{1}{b} \cdot e^{-\left(\frac{t-a}{b}\right)} \cdot e^{-e^{-\left(\frac{t-a}{b}\right)}} \cdot dt$$

$$= a + E_0 \cdot b \quad , E_0 \simeq 0.577215$$

Eular constant

- $\bar{C} \leq \log_2 \left(1 + \rho \left[\Omega \cdot \log(N) + E_0 \Omega \right] \right)$

$\bar{C} \sim O(\log(\log(N)))$ as $N \to \infty$

if $\lim_{x \to a} \frac{g(x)}{f(x)} < \infty$ then $g(x)$ is $O(f(x))$, to calculate the runtime of a function

$\boxed{3}$ Bit Error Rate BER

- $M_{X(N)}(s) = E\left[e^{s \cdot X(N)} \right]$

$$\approx \int_{-\infty}^{\infty} e^{t \cdot s} \cdot \underbrace{\frac{e^{-\left(\frac{t-a}{b}\right)}}{b} \cdot e^{-e^{-\left(\frac{t-a}{b}\right)}}}_{\text{PDF } f_{X(N)}(t)} \cdot dt$$

$$= \underbrace{\Gamma}_{\text{Gamma function}}(1 - b \cdot s) \cdot e^{a \cdot s}$$

- $\bar{P}_{e2} \approx E\left[e^{-\alpha \cdot \rho\, X(N)}\right] = M_{X(N)}(s)\Big|_{s=-\alpha \cdot \rho}$

$$= e^{-\alpha \cdot \rho \Omega \log(N)} \cdot \Gamma(1 + \Omega \cdot \rho \alpha)$$

- $\bar{P}_{e1} \leq E\left[e^{-\frac{\alpha}{2} \cdot \rho \cdot X(N)}\right]$

$$= M_{X(N)}(s)\Big|_{s=-\frac{\alpha \cdot \rho}{2}}$$

Physical Layer Security

- $\text{Alice} \xrightarrow{\text{main channel}} \text{Bob}$

  $\text{Alice} \xrightarrow{\text{Eavesdropping channel}} \text{Eve}$

→ We are transmitting Power $\underline{P}$ at Alice.

→ Eve wants to spy on Bob & Alice.

→ We will study the Strategies that minimize and prevent eavesdropping.

- System components:

  σ_m^2 : noise at the legitimate receiver

  σ_e^2 : noise at the eavesdropping receiver

We want Secure communication, so we define Secrecy capacity.

- Secrecy capacity on AWGN channel (no fading) $h=1$

$$\boxed{C_s = [C_m - C_e]^+}$$

- Cs : the positive difference between the capacity of the main channel and the capacity of the eavesdropping channel.

$$[X]^{+} = \max\{X, 0\}$$

هاد يعني انه اذا القيمة كانت اقل من الصفر ← negative

Cs يرجعلي بدالها صفر.

- $$C_m = \log_2\left(1 + \frac{P}{\sigma_m^2}\right)$$ → SNR main's

main channel capacity

- $$C_e = \log_2\left(1 + \frac{P}{\sigma_e^2}\right)$$ → SNR eve's

eavesdropping channel capacity

⟹ Based on that Cs :

$$C_s = \begin{cases} \log_2\left(1 + \frac{P}{\sigma_m^2}\right) - \log_2\left(1 + \frac{P}{\sigma_e^2}\right), & \sigma_e^2 \geq \sigma_m^2 \\ 0, & \sigma_e^2 < \sigma_m^2 \end{cases}$$

لانه الاصل يطلع سالب (رجعلي الصفر).

- If the main channel is less noisy than the eves dropping, there will be a value.
- If the main channel is noisy & the eves dropping isn't → negative value → return zero.

- Secrecy Capacity under fading effects.

$$\rightarrow C_m = \log_2\left(1 + \underbrace{\frac{P|h_m|^2}{\sigma_m^2}}_{\gamma_m}\right)$$

h_m : channel coefficent from Alice to Bob.

$$\rightarrow C_e = \log_2\left(1 + \underbrace{\frac{P|h_e|^2}{\sigma_e^2}}_{\gamma_e}\right)$$

h_e : channel coefficent from Alice to Eve.

- h_m & h_e are complex gaussian.
$|h_m|$ & $|h_e|$ are Rayleigh
$|h_m|^2$ & $|h_e|^2$ are exponetial.

- $$C_s(\underbrace{\gamma_m, \gamma_e}_{\text{Random Variables}}) = \begin{cases} \log_2(1+\gamma_m) - \log_2(1+\gamma_e), & \gamma_m \geq \gamma_e \\ 0, & \gamma_m < \gamma_e \end{cases}$$

- Two Scenarios → [1] passive Evesdropping
  ↳ [2] Active Evesdropping

[1] Passive Evesdropping

channel state information (CSI) of eve's channel is not known at Alice.
Alice will encode confidintial information at R_s : Secrecy target rate.

(A) If $R_s \leq C_s$:

perfect secrecy can be achieved

(B) If $R_s > C_s$:

Information secrecy theoritic compromized.

- In passive eves dropping, perfect secrecy is not guaranteed since Alice has no Information about Eve's CSI → major deficiency in passive that it can be compromized.

- Two Key performance mitrics
  ↳ [1] Secrecy outage probability.
  [2] Stricly positive secrecy capacity probability.

1 Secrecy outage probability

$$P_{out}(R_s) = Pr\{C_s < R_s\}$$

constant

random variable

$$= 1 - pr\{C_s > R_s\}$$

$$= 1 - pr\left\{\log_2\left[\frac{(1+\gamma_m)}{(1+\gamma_e)}\right] > R_s\right\}$$

$$\log x - \log y = \log\left(\frac{x}{y}\right)$$

$$= 1 - pr\{\gamma_m > 2^{R_s}(1+\gamma_e) - 1\}$$

$$= 1 - \int_0^\infty \left(\int_{2^{R_s}(y+1)-1}^{\infty} f_{\gamma_m}(x) \cdot dx\right) \cdot f_{\gamma_e}(y)\, dy$$

pdf(γ_m) pdf(γ_e)

$$= 1 - \int_0^\infty \left(\cancel{F_{\gamma_m}(\infty)}^{1} - F_{\gamma_m}\left(2^{R_s}(y+1)-1\right)\right] \cdot f_{\gamma_e}(y) \cdot dy$$

$$= 1 - \int_0^\infty f_{\gamma_e}(y) \cdot dy + \int_0^\infty F_{\gamma_m}\left(2^{R_s}(y+1)-1\right) \cdot f_{\gamma_e}(y) \cdot dy$$

$$P_{out}(R_s) = \int_0^\infty \underbrace{F_{\gamma_m}}_{CDF}\left(2^{R_s}(y+1)-1\right) \underbrace{f_{\gamma_e}(y)}_{PdF}\, dy$$

↳ Valid for all communication systems

2 probability of Stricly positive Secrecy capacity (SPSC)

$$\Pr\{C_s > 0\} = 1 - \Pr\{C_s \leq 0\}$$

$$= 1 - P_{out}(0)$$

$$= 1 - \int_0^\infty F_{\gamma_m}\left(2^0(y+1)-1\right) f_{\gamma_e}(y)\, dy$$

$$= \boxed{1 - \int_0^\infty F_{\gamma_m}(y) \cdot f_{\gamma_e}(y) \cdot dy}$$

Exam Question
write the integral represntation
γ_m = MRC ⟶ CDF بتجيب ال
لـ MRC وبتحوله

Ex) let
$$f_{\gamma_m}(x) = \frac{1}{\bar{\gamma}_m} \cdot e^{-\frac{x}{\bar{\gamma}_m}}, \quad x > 0$$

$$f_{\gamma_e}(x) = \frac{1}{\bar{\gamma}_e} e^{-\frac{x}{\bar{\gamma}_e}}, \quad x > 0$$

avg. SNRs
$$\bar{\gamma}_m = \frac{P}{\sigma_m^2} E\left[|h_m|^2\right]$$

$$\bar{\gamma}_e = \frac{P}{\sigma_e^2} E\left[|h_e|^2\right]$$

$$\leadsto P_{out}(R_s) = \int_0^{\infty} \overset{\text{CDF}}{F_{\gamma_m}}\left(2^{R_s}(y+1)-1\right) \overset{\text{Pdf}}{f_{\gamma_e}(y)} \cdot dy$$

$$= \int_0^{\infty} \left[1 - e^{-\left(\frac{2^{R_s}(y+1)-1}{\bar{\gamma}_m}\right)}\right] \cdot \frac{1}{\bar{\gamma}_e} \cdot e^{-y/\bar{\gamma}_e} \cdot dy$$

$$P_{out}(R_s) = 1 - \frac{\bar{\gamma}_m}{\bar{\gamma}_m + 2^{R_s} \cdot \bar{\gamma}_e} \cdot e^{-\left(\frac{2^{R_s}-1}{\bar{\gamma}_m}\right)}$$

لو Random بقى constant

(هيكون لـ Rs = صفر)

$$\Pr\{C_s > 0\} = \frac{\bar{\gamma}_m}{(\bar{\gamma}_m + \bar{\gamma}_e)}$$

SISO channel.

(SPSC)

بس قسمنا البسط والمقام على $\bar{\gamma}_m$

we know that $\Pr\{C_s > 0\} = \dfrac{1}{\dfrac{\bar{\gamma}_e}{\bar{\gamma}_m} + 1}$

also we know that

$\bar{\gamma}_e \propto \dfrac{1}{d^{\alpha}_{A,B}}$ Inversely proportional with distance ((path loss $\leadsto$ power decay with distance))

therefore: ① If $\bar{\gamma}_e \gg \bar{\gamma}_m \Rightarrow$ eve is closer to Alice than Bob.

$\Pr\{C_s > 0\} \approx 0$, $\Pr\{C_s = 0\} \approx 1$

Eve is closer, there is bad Secrecy and Alice should not transmit confidential information.

② If $\bar{\gamma}_m \gg \bar{\gamma}_e$ ⟵ Eve is not closer to Alice compared to Bob ((Alice بعيدة))

$$Pr\{C_s > 0\} \approx 1$$

$$Pr\{C_s = 0\} \approx 0$$

⟶ greater Secrecy is achieved.

* كلما اقترب Alice من Eve زادت

ال eavesdropping ⟹ Secrecy outage probability

ح تزيد.

$$P_{out}(R_s) = 1 - \frac{\bar{\gamma}_m}{\bar{\gamma}_m + 2^{R_s} \cdot \bar{\gamma}_e} \cdot e^{-\left(\frac{2^{R_s}-1}{\bar{\gamma}_m}\right)}$$

$$\boxed{\bar{\gamma}_m \gg \bar{\gamma}_e} \Rightarrow P_{out}(R_s) \approx 1 - e^{-\left(\frac{2^{R_s}-1}{\bar{\gamma}_m}\right)}$$

$$\boxed{\bar{\gamma}_e \gg \bar{\gamma}_m} \Rightarrow P_{out}(R_s) \approx 1$$

ما تقدر تبعث

It's Impossible for Alice to communicate Secure data.

$R_s \longrightarrow \infty \quad , \quad P_{out}(R_s) \longrightarrow 1$

$R_s \longrightarrow 0 \quad , \quad P_{out}(R_s) \longrightarrow \dfrac{\bar{\gamma}_e}{\bar{\gamma}_m + \bar{\gamma}_e}$

↳ بنرجع لل SPSC #

2 Active Eves dropping.

Eve is feeding back everything to Alice.

→ The CSI of Eve's channel is known.

Key performance mitric in this case:

$\boxed{\text{Ergodic Secrecy capacity}}$ Best capacity you can achieve.

$$\bar{C}_s = \int_0^{\infty}\int_0^{\infty} C_s(\gamma_m, \gamma_e)\, f_{\gamma_e}(x_1) \cdot f_{\gamma_m}(x_2) \cdot dx_1\, dx_2$$

$$\boxed{\bar{C}_s = \frac{1}{\ln(2)} \cdot \int_0^{\infty} \frac{F_{\gamma_e}(x)}{1+x} \left(1 - F_{\gamma_m}(x)\right) dx}$$

Exam Question:
integral representation.