Jordan University, Mathematics Department

Linear Algebra, Mid Term Exam III

Answer all Questions. Final answer without supporting work will not receive any credit.

1) (6 points) Solve the following linear system of equations.

$$
\begin{aligned}
x_1 + 2x_2 - x_3 - x_4 + x_5 &= 3 \\
x_1 + x_2 + x_3 + x_4 + 2x_5 &= 0 \\
x_1 - x_2 + 3x_3 + 3x_4 + x_5 &= -4 \\
3x_1 + 2x_2 + 3x_3 + 3x_4 + 4x_5 &= -1
\end{aligned}
$$

2) a) (4 points) If $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 4 & 1 & 2 \end{bmatrix}$, **then find** A^{-1}.

b) (2 points) If A **is a** 5×3 **matrix and** B **is a** 5×6, **then find the size of the matrix** $3A^TB$

3) a) (2 points) For which values of a and b the following system is inconsistent?

$$\begin{aligned} x + 2y + z &= 4 \\ y - z &= 3 \\ (a-3)z &= (b-8) \end{aligned}$$

b) (3 points) Suppose that A is a square matrix with $A^3 + 18A^2 - 6I = 0$. Show that A is invertible and find A^{-1} in terms of A

c) (3 points) Suppose that F and G are two square matrices of the same size. If FG is invertible, then show that G is invertible

Question 1

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & -1 & 3 & 3 & 1 & -4 \\ 3 & 2 & 3 & 3 & 4 & -1 \end{array}\right] \quad \begin{array}{l} -R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ -3R_1 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & -1 & 1 & 3 \\ 0 & -1 & 2 & 2 & 1 & -3 \\ 0 & -3 & 4 & 4 & 0 & -7 \\ 0 & -4 & 6 & 6 & 1 & -10 \end{array}\right] \quad R_2 \div 1$$

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & -1 & 1 & 3 \\ 0 & 1 & 2 & -2 & -1 & 3 \\ 0 & -3 & 4 & 4 & 0 & -7 \\ 0 & -4 & 6 & 6 & 1 & -10 \end{array}\right] \quad \begin{array}{l} 3R_2 + R_3 \to R_3 \\ 4R_2 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -2 & -1 & 3 \\ 0 & 0 & -2 & -2 & -3 & 2 \\ 0 & 0 & -2 & -2 & -3 & 2 \end{array}\right] \quad -R_3 + R_4 \to R_4$$

$$\begin{bmatrix} 1 & 2 & -1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -2 & -1 & 3 \\ 0 & 0 & -2 & -2 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then:

① $-2x_3 - 2x_4 - 3x_5 = 2$

assume $x_4 = t$, $x_5 = s$ for $t, s \in \mathbb{R}$

$-2x_3 - 2t - 3s = 2$

$-2x_3 = 2 + 2t + 3s$

$x_3 = -1 - t - \frac{3}{2}s$

② $x_2 - 2x_3 - 2x_4 - 1x_5 = 3$

$x_2 - 2(-1 - t - \frac{3}{2}s) - 2t - 1s = 3$

$x_2 = 1 - 2s$

③ $x_1 + 2x_2 - 1x_3 - 1x_4 + 1x_5 = 3$

$x_1 = \frac{3}{2}s$

Question 2 (A)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 4 & 1 & 2 \end{bmatrix}$$

Solu:-

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & -3 & 2 & -4 & 0 & 1 \end{array}\right] \quad \begin{array}{l} -R_2 + R_1 \to R_1 \\ 3R_2 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 3 & 1 \end{array}\right] \quad -R_3 \times -1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{array}\right] \quad \begin{array}{l} R_3 + R_2 \to R_2 \\ -R_3 + R_1 \to R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{array}\right] \quad \longrightarrow$$

So the inverse of A is A^{-1} is $\begin{bmatrix} -1 & 2 & 1 \\ 2 & -2 & -1 \\ 1 & -3 & -1 \end{bmatrix}$

Question 2 (B)

A is 5x3 B is 5x6

Find the size of $3A^TB$

① size of $A^T = 3 \times 5$

② size of $A^TB = \underset{3\times5}{A^T} \times \underset{5\times6}{B}$ is → 3×6

then the size of $3A^TB$ is 3×6
(the scalar 3 does not effect the size).

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Question 3 (A)

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & (a-3) & (b-8) \end{array}\right]$$

$(a-3) = (b-8)$

$\boxed{a=3}$

when $(a-3)z = b-8$ turns into $0z = k$

then it is inconsistant

and then when $a=3$

$0 = b-8 \rightarrow b=8$ / but b should not equal

8 to make the system inconsistant and

~~should~~ satisfy my statement above.

then the system has no solution when:

$a=3 \rightarrow B \in \mathbb{R} - \{8\}$

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Question 3 (B)

$A^3 + 18A^2 - 6I = 0$

show that A is invertible and find A^{-1}

solu:

from the equation

$A^3 + 18A^2 - 6I = 0$

$A^3 + 18A^2 = 6I$

$A(A^2 + 18A) = 6I$

$A\left(\frac{A^2}{6} + 3A\right) = \cancel{6}I$

so A is invertible if and only if $A^{-1} = \left(\frac{A^2}{6} + 3A\right)$

for $AA^{-1} = I$

and A^{-1} here should be $\left(\frac{A^2}{6} + 3A\right)$ so when the equation is satisfied A is invertible and the inverse of $A = A^{-1} = \left(\frac{A^2}{6} + 3A\right)$

\ \

~~Ques~~

Question 3 (c)

F and G are two square matrices of the same size when FG ~~is~~ is invertible, then show that G is invertible.

(1) first we show that G is invertible by showing that the system $GX=0$ has only the trivial solution:

multiply by F

$FGx=0$
so when FG is invertible, then the homogenous system $FGx=0$ has the only trivial solution $(x=0)$

thus $\rightarrow$ G is invertible