PROBABILITY

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POWERUNIT

Ch0: Mathmatical background

Set theory:

A set is a collection of distinct elements

example :-

$S_1 = \{ 1, 2, 3, 4, 5, 6 \}$

$S_1 = \{$ Hed, Tail $\}$

let $S = \{ x_1, x_2, x_3 - - -, x_n \}$

$x_1 \in S$: x_1 is an element of the set S

$x_n \in S$ x_2 " " " " "

$x_j , j > n$

$x_j \notin S$ x_j is not an element of the set S

- The empty (null) set is the set that contain no elements and is denoted by $\emptyset$

- Universal set (denoted by Ω): it contains all objects that could be conceivably be of interest in a particular context.

Set clasifications:-

① Finite set Ex $A = \{ 1, 2, 3 \}$

$B = \{ 2, 4, 6 \}$

(2) coutebly infinite set $A = \{1, 2, \dots\}$

A set that statisfies certain property P

$A = \{x \mid x \text{ statisfies } P\}$

* the set of even integers

$S = \{K \mid K/2 \text{ is integer}\}$

* the set of all salers between 0 and 1

$S = \{x \mid 1 \geq x \geq 0\}$

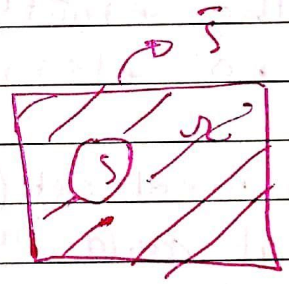
↳ continous , uncountable

→ set operations:

1- The complemet of a set S, with respect to a universal set Ω, is the set

$$\bar{S} = \{x \in \Omega \mid x \notin S\}$$

$$\triangleq \Omega - S$$



$$\bar{\Omega} = \emptyset$$

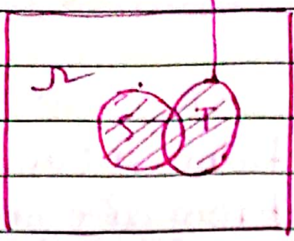
2- union of sets: the union of two sets S and T is the set of all elements that ~~belong~~ belong to S or T (or both) and denoted by $S \cup T$

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$$

$S = \{1, 2\}$ $\quad S \cup T = \{1, 2, 4\}$

$T = \{2, 4\}$

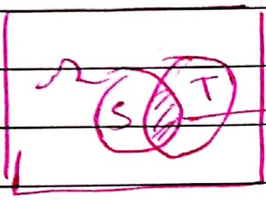
$S \cup T$



3- Intersection of sets

the intersection of two sets S and T ($S \cap T$) is the set of all elements that belongs to both S and T

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$$



$\rightarrow S \cap T$

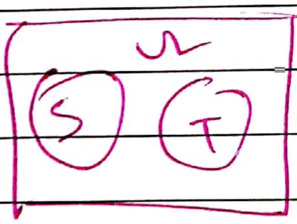
* For infintly many sets $S_1, S_2, \ldots$

$$S_1 \cup S_2 \cup S_3 \cup \cdots = \bigcup_{n=1}^{\infty} S_n = \{x \mid x \in S_n \text{ for some } n\}$$

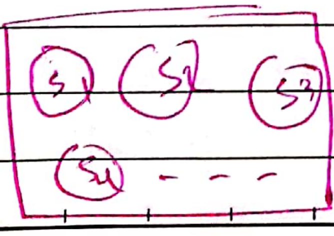
$$S_1 \cap S_2 \cap S_3 \cap \cdots = \bigcap_{n=1}^{\infty} S_n = \{x \mid x \in S_n \text{ for all } n\}$$

4- Tow sets are said to be disjoint if their intersection is empty

S and T are disjoint if $S \cap T = \emptyset$



* several sets are said to be disjoint if no two of them have a common element



$S_1, S_2, \ldots, S_n$ are disjoint if

$$S_i \cap S_j = \emptyset \quad i \neq j$$

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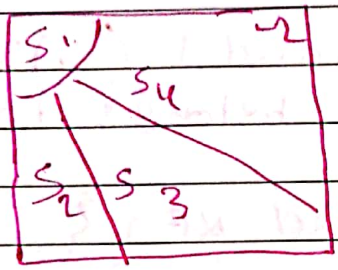
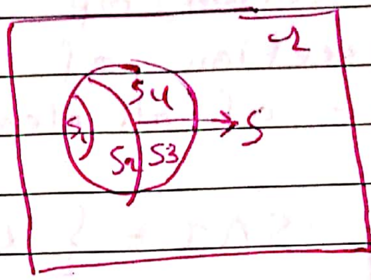
S- Pratitions

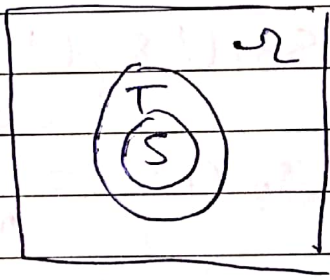
A collection of sets is said to be a partion of set if the sets in the collection are disjoint and their union is S





S is a subset of T if every element of S is also an element of T.

$S \subset T \longleftrightarrow T \supset S$



1- Equality: $S \subset T$ and $T \subset S$, then $S = T$

2- Transitivity: if $S \subset T$ and $T \subset V$, the $S \subset V$

* Algebra of sets:

1- $S \cup T = T \cup S$

2- $S \cap (T \cup A) = (S \cap T) \cup (S \cap A)$

3- $\overline{(\bar{S})} = S$

4- $S \cup \Omega = \Omega$

5- $S \cup (T \cup A) = (S \cup T) \cup A$

6- $S \cup (T \cap A) = (S \cup T) \cap (S \cup A)$

7- $S \cap \bar{S} = \emptyset$

8- $S \cap \Omega = S$

Ex: cosider an experiment involving rolling a 4-faced die

* $\Omega = \{1, 2, 3, 4\} \to$ universal set

* A: is the set of even outcomes

$A = \{2, 4\}$

* B: is the set of all outcomes that are less than 3

$B = \{1, 2\}$

* $A \cap B = \{2\}$

* $A \cup B = \{1, 2, 4\}$

* $A \cap \bar{B} = \{\}$

1- $\bar{B} = \{3, 4\}$

$A \cap \bar{B} = \{4\}$

or we can say (another method)

2- $A \cap \bar{B} = A - B = \{4\}$

* $B \cap \bar{A} = B - A = \{1\}$

→ DE Morgan's law:-

1- $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

2- $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

$\boxed{1-}$ Proof

$x \in \overline{(A \cap B)} \longleftrightarrow x \notin (A \cap B) \longleftrightarrow \left\{ \begin{matrix} x \notin A \\ \text{or} \\ x \notin B \end{matrix} \right\}$

→ x

$\longleftrightarrow \left\{ \begin{matrix} x \in \bar{A} \\ \text{or} \\ x \in \bar{B} \end{matrix} \right\} \longleftrightarrow \cancel{\bar{A} \cup \bar{B}}$ $x \in (\bar{A} \cup \bar{B})$

2 - proof

$x \in \overline{(A \cup B)} \Longleftrightarrow x \notin (A \cup B) \Longleftrightarrow \left\{ \begin{array}{l} x \notin A \\ \text{and} \\ x \notin B \end{array} \right\}$

$\Longleftrightarrow \left\{ \begin{array}{l} x \in \bar{A} \\ \text{and} \\ x \in \bar{B} \end{array} \right\} \Longleftrightarrow x \in (\bar{A} \cap \bar{B})$

Generally

* $\overline{(A_1 \cup A_2 \cup A_3 \cdots)} = \overline{\left(\bigcup_{n=1}^{\infty} A_n \right)} = \bigcap_{n=1}^{\infty} \overline{A_n}$

* $(A_1 \cap A_2 \cap A_3 \cdots) = \overline{\left(\bigcap_{n=1}^{\infty} A_n \right)} = \bigcup_{n=1}^{\infty} \overline{A_n}$

show that

Ex: 1- $\overline{(A \cap (B \cup C))} = (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \bar{C})$

2- $\overline{((A \cap B) \cup (A \cup C))} = \bar{A} \cup (\bar{B} \cap \bar{C})$

Probabistic model (PM)

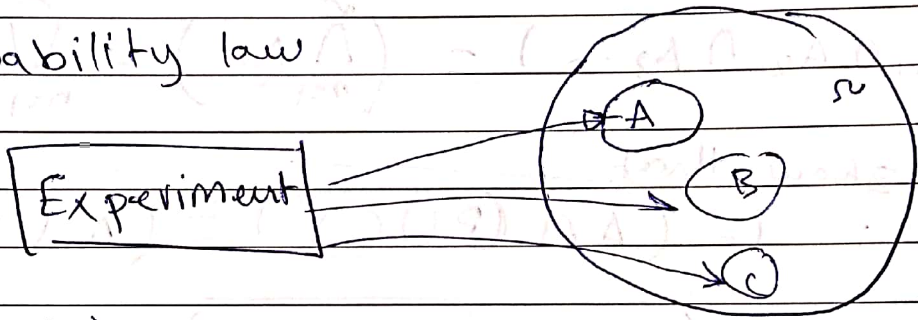
1- Sample space "Ω"
- the set of all possible outcomes of an experiment

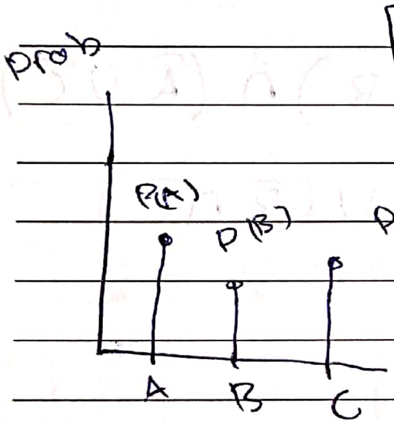
* Tossing a coin twice

1	2
H	H
H	T
T	H
T	T

$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$

2- Probability law

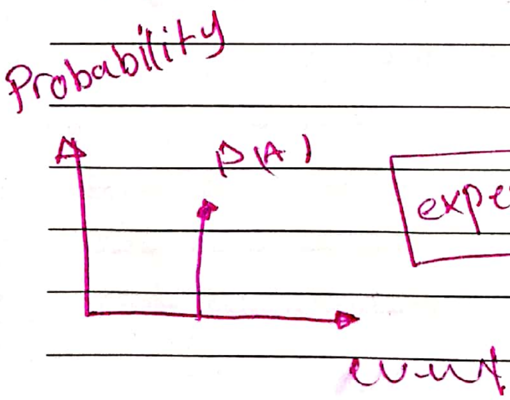


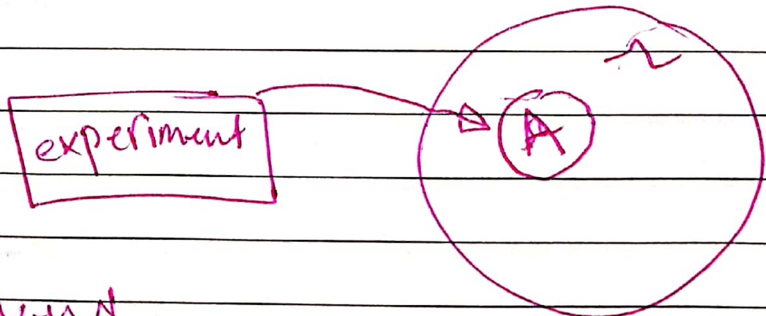


$P(\Omega) = 1$

$1 \geq P(A) \geq 0$

* assigns to a set "A" of possible outcomes, a non negative number "P(A)". P(A) encodes ~~our~~ our beliefes about the liklihood element of "A"





Probability ~~axioms~~ axioms

1- non- negativity

$P(A) \geq 0$

2- normalization

$P(\Omega) = 1$

3- if A and B are disjoint ($A \cap B = \phi$), then

$P(A \cup B) = P(A) + P(B)$

* example: given a 4- sided die

$\Omega = \{1, 2, 3, 4\}$

$P(\Omega) = P(\{1, 2, 3, 4\}) = 1$

$P(\{1\}) = \frac{1}{4}$

$P(\{1, 2\}) = P(\{1\}) + P(\{2\}) = \frac{2}{4} = \frac{1}{2}$

let $A_1, A_2 \cdots A_n$ be disjoint then

$P(A_1 \cup A_2 \cdots A_n) = \sum_{i=1}^{n} P(A_i)$

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Consequances of the Axioms

1- $P(A) \leq 1$ $\qquad \Omega = A \cup \bar{A}$

$P(\Omega) = 1 \longrightarrow P(A \cup \bar{A})$ since they are disjoint

$= P(A) + P(\bar{A}) = 1$

$P(A) = 1 - P(\bar{A}) \rightsquigarrow P(A) \leq 1$ # non-negativity ~~property~~ axioms

2- $P(\emptyset) = 0$

~~$\emptyset = \Omega - \Omega$~~

~~$P(\emptyset) = P(\Omega - \Omega$~~

$\Omega = \Omega \cup \emptyset$

$P(\Omega) = P(\Omega \cup \emptyset)$

$\hookrightarrow$ disjoint

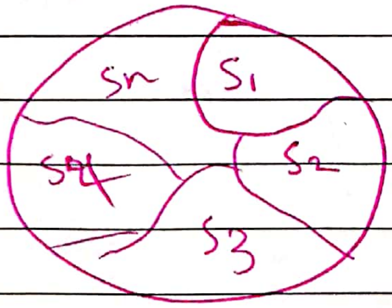
$P(\Omega) = P(A) + P(\emptyset)$

$P(\emptyset) = 0$!!

Discrete probability law

let $S = \{s_1, s_2 --- s_n\}$

$P(\{s_1, s_2, s_3 --- s_n\}) = P(\{s_1\}) + P(\{s_2\}) + ---$
$--- + P(\{s_n\})$

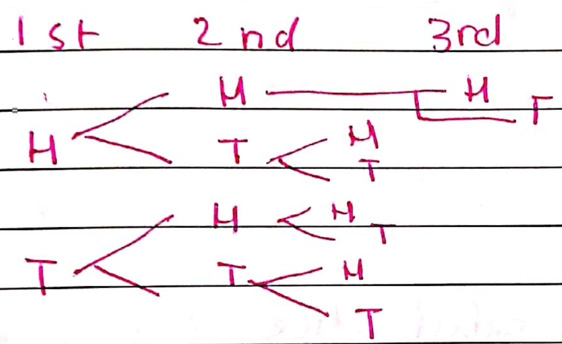


Discrete uniform probability law

- As a special case when $P(s_1) = P(s_1) \dots = P(s_n) = \frac{1}{n}$ then the probability of any event "A" is given by :-

$$P(A) = \frac{\text{\# of elements "A"}}{n}$$

example
Tossing a coin 3 times



$\Omega = \{HHH, HHT, HTH, HHT, THT, TTT, HTT, TTH\}$

let "A" be defined as A = {exactly ~~2~~ 2 Hed ocurr}

$P(A) = P(\{HHT, HTH, THH\})$
$= P(\{HHT\}) + P(\{HTH\}) + P(\{THH\})$

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

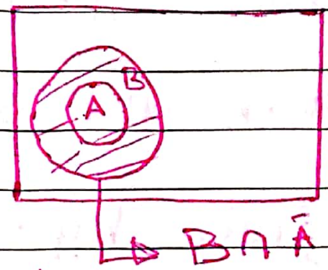
- since all outcomes ocurr equally likely with probability $\frac{1}{8}$

A has 3 elements / Ω has 8 elements

Discrete uniform prob law:-
$P(A) = \frac{3}{8}$

Additional probability laws

1- if $A \subset B$, $P(A) \leq P(B)$



$B = A \cup (B \cap \bar{A})$

$P(B) = P(A \cup (B \cap \bar{A}))$, disjoint

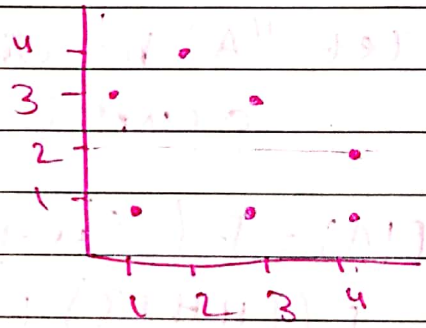
$= P(A) + \underbrace{P(B \cap \bar{A})}_{\geq 0}$

$P(B) \geq P(A)$ #

Ex:

rolling a pair of 4-sided dice

$P(\{\text{sum is even}\}) = \frac{8}{16} = \frac{1}{2}$



$P(\{\text{sum is odd}\}) = 1 - \frac{1}{2} = \frac{1}{2}$

$P(\{1^{st} \text{ roll} = 2^{nd} \text{ roll}\}) = \frac{6}{18}$

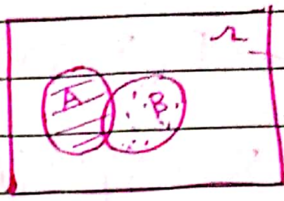
$P(\{\text{at least one roll} = 4\}) = \frac{7}{16}$

→ Additional Probability laws

2- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:

$A \cup B = A \cup (B \cap \bar{A})$



$P(A \cup B) = P(A \cup (B \cap \bar{A}))$

$= P(A) + P(B \cap \bar{A})$ ——— (1)

$P(B) = P((A \cap B) \cup (B \cap \bar{A})) = P(A \cap B) + P(B \cap \bar{A})$ — (2)

from (1) and (2)

$P(A \cup B) - P(A) = P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\boxed{P(A \cup B) \leq P(A) + P(B)}$$

In general

$P(A_1 \cup A_2 \cup A_3 \ldots A_n) \leq \sum_{k=1}^{n} P(A_k)$

$*$ $P(A \cup B \cup C) = P(A) + \underbrace{P(B \cup C)}_{\mathrm{I}} + \underbrace{P((A \cap C) \cup (B \cap A))}_{\mathrm{II}}$

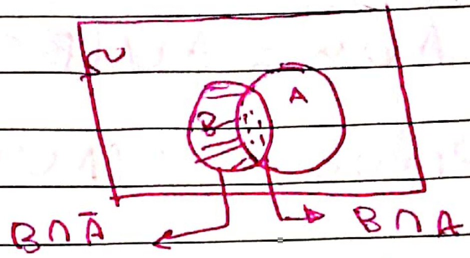
$\mathrm{I} = P(B) + P(C) - P(B \cap C)$

$\mathrm{II} = P(A \cap C) - P(B \cap A) + P(A \cap B \cap C)$

$P(A \cup B \cup C) = P(A) + P(B) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Five Apple

4- $P(B \cap \bar{A})$



$B = (B \cap \bar{A}) \cup (A \cap B)$

$P(B) = P(B \cap \bar{A}) + P(A \cap B)$

$P(B \cap \bar{A}) = P(B) - P(A \cap B)$

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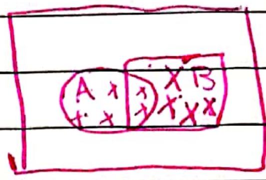
Conditional Probability → the main idea of conditioning

- assuming that all elements from both events are equally likely.



Probability of B

$$P(B) = \frac{6}{9}$$

* What is the probability of "A" given 'B' has occurred?

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{6}$$

* Definition:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \; ; \quad P(B) > 0$$

→ $P(B)$ must be greater than 0 because we can't condition on an empty set.

→ Results from the definition:-

$\boxed{1}$ $P(A/B) \geq 0$

$\boxed{2}$ $P(B/B) = 1$

$\boxed{3}$ $P(\Omega/B) = 1$

$\boxed{4}$ $P(B/\Omega) = P(B)$

$\boxed{5}$ If A_1, A_2 are disjoint events, then:-

$$P((A_1 \cup A_2)/B) = P(A_1/B) + P(A_2/B)$$

proof $\boxed{5}$

- From the defintion: $P((A_1 \cup A_2)/B) = \cancel{P(A_1/B) + P(A_2/B)}$

$$\frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} - \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1/B) + P(A_2/B)$$

// In general.

→ If $A_1, A_2, \ldots, A_n$ are disjoint events then.

$$P(A_1 \cup A_2 \cup \ldots A_n / B) = \sum_{i=1}^{n} P(A_i/B)$$

→ If $A_1, A_2, \ldots A_n$ are not disjoint events then

$$P(A_1 \cup A_2 \cup \ldots A_n / B) \leq \sum_{i=1}^{n} P(A_i/B)$$

example $n=2$ then

$$P((A_1 \cup A_2)/B) \leq P(A_1/B) + P(A_2/B)$$

example:

A conservative design team, call it "c", and an innovative design team call it "n", are asked to seperately design a new product within a month from a past experience we know that:

(a) the probability that team "c" is successful 2/3

(b) " " " " "n" " " 1/2

(c) " " " at least one team is successful equals 3/4

- Assuming that exactly one design is produced, what is the probability that it was done by "n"?

Sol:-

- The sample space for the success/fail for team c, n respectively is $\Omega = \{ss, sf, fs, ff\}$

- The desired probability is

$$P(Fs / \{sF, Fs\}) = \frac{P(Fs \cap \{sF, Fs\})}{P(\{sF, Fs\})} = \frac{P(Fs \cap \{sF, Fs\})}{P(sF) + P(Fs)}$$

- First: Probability that "c" is successful:-

$P(\{ss\}) + P(\{sF\}) = 2/3 \longrightarrow \boxed{1}$

- Second: Probability that N is successful

$P(\{ss\}) + P(\{Fs\}) = 1/2 \longrightarrow \boxed{2}$

- Third: Probability that at least one is successful

$P(\{ss\}) + P(\{sF\}) + P(\{Fs\}) = \frac{3}{4} \longrightarrow \boxed{3}$

- Forth: $P(\Omega) = 1$

$P(\{ss\}) + P(\{sF\}) + P(\{Fs\}) + P(\{FF\}) = 1$

~~Substituting~~

→ substuting $\boxed{3}$ in $\boxed{4}$

$$\frac{3}{4} + P(\{FF\}) = 1 \rightarrow P(\{FF\}) = \frac{1}{4}$$

→ substuting $\boxed{2}$ in $\boxed{3}$

$$P(\{SF\}) + \frac{1}{2} = \frac{3}{4} \longrightarrow P(\{SF\}) = \frac{1}{4}$$

→ substuting $\boxed{1}$ in $\boxed{4}$

$$P(\{FS\}) + \frac{2}{3} + \frac{1}{4} = 1 \longrightarrow P(\{FS\}) = \frac{1}{12}$$

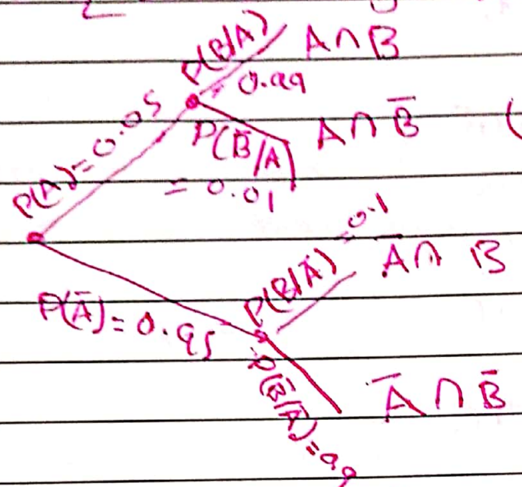
Therefore the desired probability :-

$$\frac{P(\{FS\})}{P(\{SF\}) + P(\{FS\})} = \frac{1/12}{1/4 + 1/12} = \frac{1/12}{4/12} = \frac{1}{4}$$

Radar detection

A = { an Aircraft is flying above }

B = { something has been registered of radar secreen }



$P(A/B) = \dfrac{P(A\cap B)}{P(B)} \longrightarrow P(A\cap B) = P(A/B)\cdot P(B)$
$= P(B/A)\cdot P(A)$

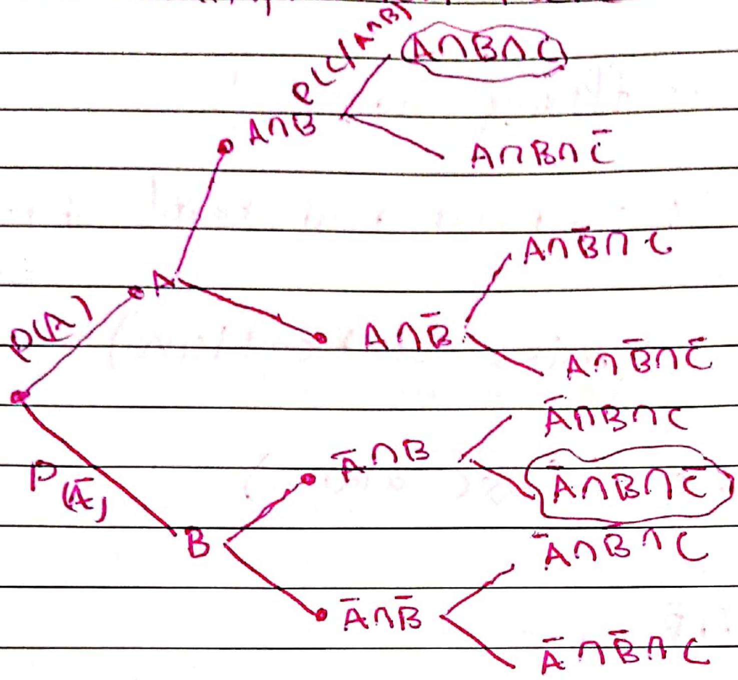
$P(B/A) = \dfrac{P(A\cap B)}{P(A)}$

1- $P(A\cap B) = P(B/A)\cdot P(A) = 0.99 \times 0.05 =$

2- P(missed detection) $= P(A\cap\bar{B}) = P(\bar{B}/A)\cdot P(A)$
$= 0.01 \times 0.05$

3- $P(A/B) = \dfrac{P(A\cap B)}{P(B)}$ (Next lectures we will answer this)

The multiplication rule:-



$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B)$

$P(\bar{A} \cap B \cap \bar{C}) = P(\bar{A}) \cdot P(B/\bar{A}) \cdot P(\bar{C}/\bar{A} \cap B)$

using the defintion

$P(A \cap B \cap \bar{C}) = P(\bar{C}/\bar{A} \cap B) \cdot \underbrace{P(\bar{A} \cap B)}$

$\to P(B/\bar{A}) \cdot P(\bar{A})$

$= P(\bar{C}/\bar{A} \cap B) \cdot P(\bar{B}/A) \cdot P(\bar{A})$

The general form

$$P(A_1 \cap A_2 \cap \dots A_n) = P(A_1) \prod_{i=2}^{n} P(A_i / A_1 \cap A_2 \cap \dots A_{i-1})$$

$n = 3$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2 / A_1) \times P(A_3 / A_1 \cap A_2)$$

example:-

Three cards are drawn from 52-card deck without replacment

Probability that none of these cards is a heart ??

let $A_i = \{$ The ith card is not a heart $\}$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2 / A_1) \cdot P(A_3 / A_1 \cap A_2)$$

$$= \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50}$$

Example:

- Three cards are drwan from 52 card deck without replacement. What is the probability that none of them is a heart.

Solution

let A_i be defined as $A_i = \{$ The i_{th} card is not a heart $\}$

Probability that none of them is not heart is

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2 / A_1) P(A_3 / A_1 \cap A_2)$$

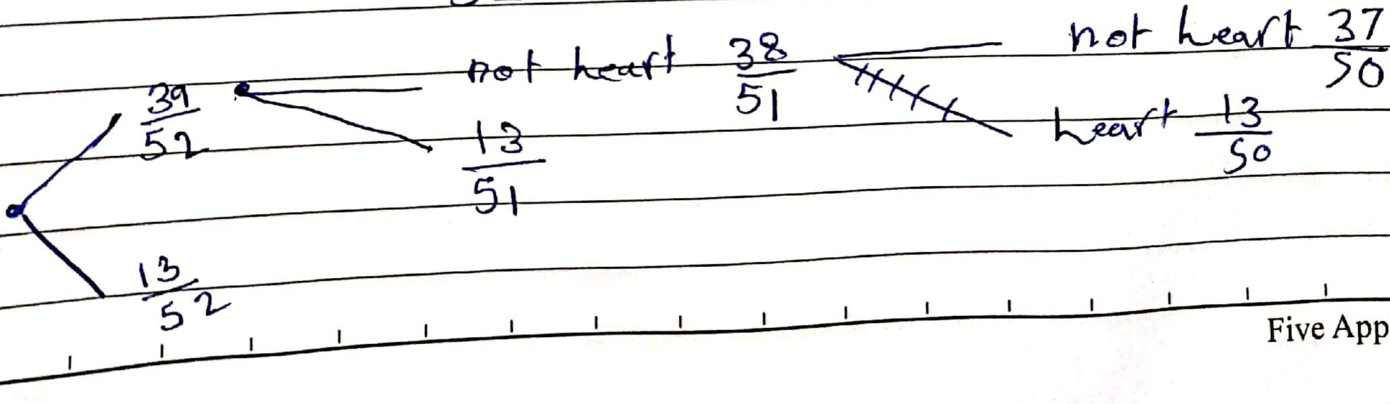
$$= \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50}$$

- What is the probability that the first one is not a heart and the second one is a heart?

$$P(A_1 \cap \bar{A}_2) = P(\bar{A}_2 / A_1) \cdot P(A_1) = \frac{13}{\cancel{3}51} \times \frac{39}{52}$$

- What is the pro. that the first tow cards are not hearts and the third is heart

$$P(A_1 \cap A_2 \cap \bar{A}_3) = \frac{39}{52} \times \frac{38}{51} \times \frac{13}{50}$$ (tree diagram)



Total Probability Theorem :

let $A_1, A_2, \ldots$ Are disjoint events that form a partition of Ω. Then for any event B. We have:

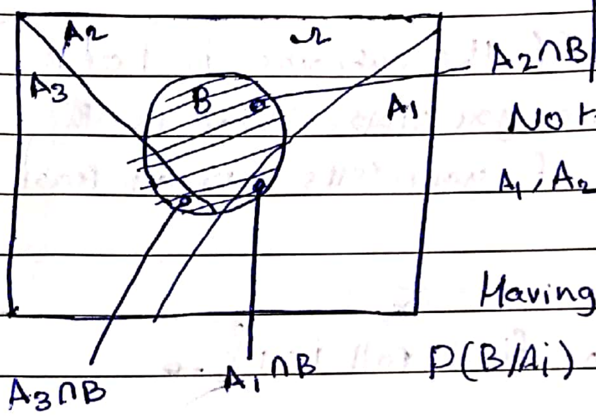
$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) \ldots \rightarrow P(B/A_1).P(A_1) + P(B/A_2).P(A_2)$$

$$+ \rightarrow \sum_{j=1}^{N} P(B/A_i).P(A_i)$$

By using the total probability theorem → we find $P(B)$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

$$= P(B/A_1).P(A_1) + P(B/A_2).P(A_2) + P(B/A_3)P(A_3)$$



Note :-

$A_1, A_2, A_3, \ldots, A_N$ ~~from~~ form a partition of Ω

Having $P(A_i)$ for every i. Having $P(B/A_i)$ for every i → we can find $P(B)$

** From the radar detection example

$$P(B) = P(\{\text{somthing has been registered}\}) = P(B \cap A) + P(B \cap \bar{A})$$

$$= P(B/A).P(A) + P(B/\bar{A})P(\bar{A})$$

$$= 0.99 \times 0.05 + 0.1 \times 0.95 = 0.1445$$

Baye's Rule: $P(A_i/B) = \frac{P(A_i \cap B)}{P(B)}$

$P(A_i \cap B) = P(B/A_i) \cdot P(A_i)$ (Multiplication rule)

$P(B) = \sum_{i=1}^{N} P(B/A_i) \cdot P(A_i)$

by total Probability theorem

→ from the radar detection example

$P(A/B) = \frac{P(B/A) \cdot P(A)}{P(B)} = \frac{0.99 \times 0.05}{0.1445}$

→ Example:

You roll a fair 4-sided die. If the outcome is 1 or 2 then you roll one more. otherwise you stop. What is the probability that the sum total of your rolls is at least 4.

- Solution

Define the event A_i → $A_i = \{$The first roll is $i\}$ →

Note that $P(A_i) = \frac{1}{4}$

Define the event B: The sum total is at least 4

$P(B/A_3) = 0$ - A_3 means the roll is 3

$P(B/A_4) = 1$

$P(B/A_2) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

$P(B/A_3) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(B) = P(B/A_1) \cdot P(A_1) + P(B/A_2) P(A_2) + P(B/A_3) P(A_3) + P(B/A_4) P(A_4)$

→ If the sum is at least 4, What is the Probability that the first roll was 1?

$$P(A_1/B) = \frac{P(B/A_1).P(A_1)}{P(B)} = \frac{\frac{1}{2}\cdot\frac{1}{4}}{\frac{9}{16}} = \frac{2}{9}$$

Independance: Intuitively $P(B/A) = P(B)$ ----→ $\boxed{1}$

↓ the occurrence of event A provides no information about the occurrence of event B.

- From the definition of conditional Probability

$P(B/A) = \frac{P(B\cap A)}{P(A)}$ → $P(B\cap A) = P(A).P(B/A)$

But $P(B/A) = P(B)$

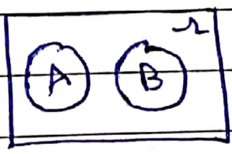
Therefore: $P(B\cap A) = P(A).P(B)$ (Defenition of inependace) ---- $\boxed{2}$

→ From $\boxed{1}$ and $\boxed{2}$ $P(A/B) = P(A)$

→ Tow events are independent if their intersection is equal to their individual Probability.

- Disjoint events cannot be independent. Disjoint events totally dependent.

- When talking about non-zero events



$A\cap B = 0$ $P(A) \times P(B) > 0$

$P(A\cap B) \neq P(A) \times P(B)$

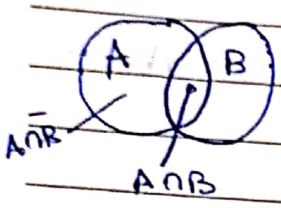
A and B are not independent

- If A and B are independent then :
  (1) $\bar{A}$ and $\bar{B}$ are also indepent
  (2) $\bar{A}$ and B are " "
  (3) A and $\bar{B}$ " " "

• Prove that if A and B are independent, then A and $\bar{B}$ are also independent

we want to show that $P(A\cap\bar{B}) = P(A)\cdot P(\bar{B})$

$$A = (A\cap\bar{B})\cup(B\cap A)$$



$$P(A) = P(A\cap\bar{B}) \cup P(A\cap B)$$
$$P(A) = P(A\cap\bar{B}) + P(A)\cdot P(B)$$
$$P(A)(1-P(B)) = P(A\cap\bar{B})$$
$$P(A\cap\bar{B}) = P(A)\,P(\bar{B})$$

Note : Idependent event cannot be visualised using venn diagram.

Conditional independce

$P(A\cap B/C) = P(A/C)\cdot P(B/C)$ ---- $\boxed{1}$ (conditionallyindependce)

From defention of conditional probability

$$P(A\cap B/C) = \frac{P(B\cap A\cap C)}{P(C)} \quad \text{(Multiplication Rule)}$$

$$P(A\cap B/C) = \frac{\cancel{P(C)}\cdot P(B/C)\cdot P(A/B\cap C)}{\cancel{P(C)}}$$

$P(A\cap B/C) = P(B/C)\cdot P(A/B\cap C)$ ---- $\boxed{2}$

$\boxed{1} = \boxed{2}$

$P(A/C) \cdot P(B/C) = P(B/C) \cdot P(A/B \cap C)$ (assume $P(B/C) > 0$)

$P(A/B \cap C) = P(A/C)$.

The occurrence of B dosen't change are beliefs about $P(A/C)$

- Independence of several events

$A_1, A_2 \dots A_n$ are independence if $P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$ for every subset of $S = \{1, 2 \dots, N\}$

- If $S = \{1, 2\} \rightarrow P(A_1 \cap A_2) = P(A_1) P(A_2)$
- If $S = \{1, 2, 3\} \rightarrow$ To (prove) that all A_1, A_2 and A_3 are independent then:

$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3) \rightarrow [1]$

$P(A_1 \cap A_2) = P(A_1) P(A_2) \rightarrow [2]$

$P(A_1 \cap A_3) = P(A_1) P(A_3) \rightarrow [3]$

$P(A_2 \cap A_3) = P(A_2) P(A_3) \rightarrow [4]$

If all ~~stat~~ true we say that A_1, A_2 and A_3 are pair-wise independent

pair-wise independent dose not imply independence

example 1:- Two successive rolls of a 4-sided die (fair)

let $A_i = \{$ First roll is $i\}$

$B_j = \{$ second roll is $j\}$

1] Are A_i and B_j independent

$P(A_i) = \frac{1}{4}$ $\qquad P(A_i) P(B_j) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$P(B_j) = \frac{1}{4}$ $\qquad P(A_i \cap B_j) = \frac{1}{16}$

Yes, since $P(A_i \cap B_j) = P(A_i) P(B_j)$

$\boxed{2}$ let $A = \{\text{First roll is } 1\}$ $P(A) = \frac{1}{4}$

$B = \{\text{sum of the two rolls is } 5\}$

$B = \{(1,4), (4,1), (2,3), (3,2)\}$

$P(A) * P(B) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$ $P(B) = \frac{4}{16}$

~~P(A) A~~

$P(A \cap B) = 16$

Yes, since $P(A \cap B) = P(A) \cdot P(B)$

$\boxed{3}$ $A = \{\text{The maximum of tow rolls is } 2\}$

$B = \{$ " minimum " " " $\}$

$A = \{(1,2), (2,1), (2,2)\}$ $P(A) = \frac{3}{16}$

$B = \{(2,2), (2,3), (3,2), (4,2), (2,4)\}$ $P(B) = \frac{5}{16}$

$P(A) \times P(B) = \frac{3 \times 5}{16 \times 16} = \frac{15}{256}$

$P(A \cap B) = \frac{1}{16}$

$P(A \cap B) \neq P(B)\, P(A)$ (dependent)

Example 2:- Tow independent fair coin Tosses

$H_1 = \{$ First toss is $H \}$ $H_2 = \{$ second toss is $H\}$

$D = \{$ both tosses have different results $\}$.

- Check if H_1 and H_2 are conditionally Independent given D.
- We need to check if

$P(H_1 \cap H_2 / D) = P(H_1/D)\, P(H_2/D)$

$P(H_1) = \frac{1}{2}$ $P(H_2) = \frac{1}{2}$

$P(H_1 \cap H_2) = \frac{1}{2} \times \frac{1}{2} = P(H_1)\, P(H_2) \rightarrow$ independent

$P(H_1/D) = \frac{1}{2}$

$P(H_2/D) = \frac{1}{2}$

HH	HT
TH	TT

$P(H_1 \cap H_2/D) = 0 \neq P(H_1/D) \cdot P(H_2/D)$

Pair-wise independence does not imply independence

H_1 and H_2 indep.

H_1, D indep. $\rightarrow P(H_1/D) = P(H_1) = \frac{1}{2}$ (from Definition)

H_2, D indep. $\rightarrow P(H_2/D) = P(H_2) = \frac{1}{2}$ (" ")

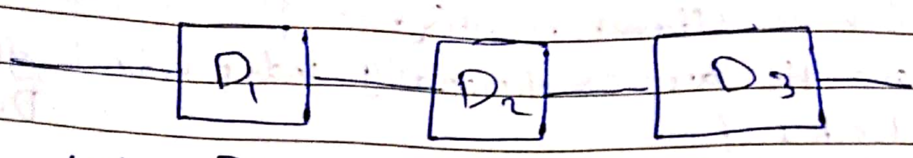
$H_1, H_2, D \rightarrow P(H_1 \cap H_2 \cap D) = 0 \neq P(H_1)\, P(H_2)\, P(D)$

H_1, H_2, D are not indep. $\rightarrow$ but they are pair-wise indep.

Five Apple

Applications of independence

- Reliability



- let P_i denote the probability that D_i is up
- What is the probability that the system is up?

$U_i = D_i$ is up

$P(U_1 \cap U_2 \cap U_3) = P(U_1) \times P(U_2) \times P(U_3)$

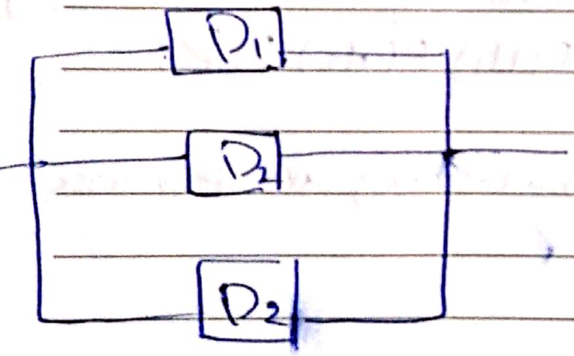
(since devices fail inpendently)

For this system to be up all D_1, D_2, D_3 must be up.

$\rightarrow P_1 \times P_2 \times P_3$

- What is the probability that the system is down

$= 1 - (P_1 \times P_2 \times P_3)$



$P(U_1 \cap U_2 \cap U_3) = P(\bar{U_1} \cap \bar{U_2} \cap \bar{U_3})$

$= 1 - P(\bar{U_1} \cap \bar{U_2} \cap \bar{U_3})$

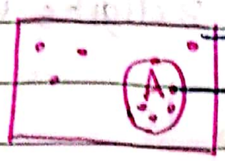
$= 1 - P(\bar{U_1})\ P(\bar{U_2})\ P(\bar{U_3})$

$= 1 - (1 - P(U_1))(1 - P(U_2))(1 - P(U_2))$

$= 1 - P_1 \times P_2 \times P_3$

Probability that the system is up

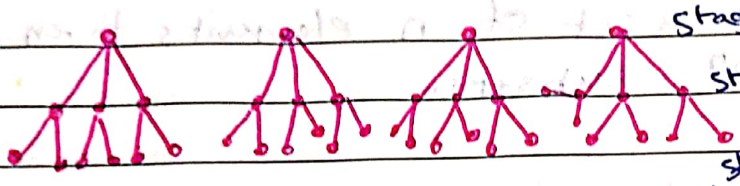
Counting Principle:



→ probability of any element : $\frac{1}{n}$

→ $P(A) = \frac{\text{\# of elements of A}}{\text{\# of elements of } \Omega} = \frac{K}{n}$

- Example : choosing from : 4 shirts, 3 ties, 2 Jackets
- solution : By using tree diagram



stage 1

stage 2

~~stages~~ stage 3

3 stages

First stage : 4 ways (shirt)

second stage :

3 ways (tie)

Third stage :

2 ways (Jacket)

Total number of choices :

$4 * 3 * 2 = 24$

let us define :

n_i : The number of choices at stage i

Total # of choices $= n_1 * n_2 * \ldots n_n$

for n stages

r stage → n is the # of choices at the "i" th stage → total # of choices $= \prod_{i=1}^{r} n_i$

- Example: The # of license plates with two letters followed by 3 digits (repetition allowed)

$\underline{26} \times \underline{26} \times \underline{10} \times \underline{10} \times \underline{10}$

- Same above (but repetition not allowed)

$\underline{26} \times \underline{25} \times \underline{10} \times \underline{9} \times \underline{8}$

Providing that for a set of n elements then we can have 2^n subsets.

- For any element we have
  - elements in the subset
  - element not in the subset

  } 2 choice → Repeated for n stages.

$$\underbrace{2}_{\text{first stage}} \times 2 \times 2 \cdots \times \underbrace{2}_{\text{nth stage}} = 2^n$$

~~example~~

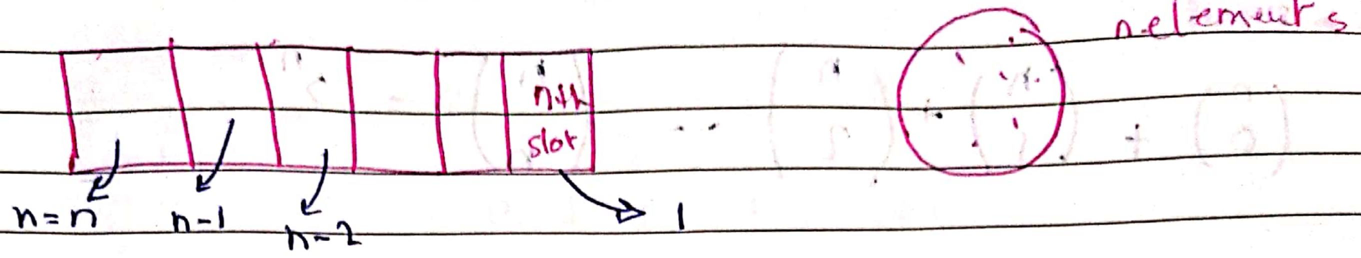
example:

for $S = \{1, 2, 3\} \rightarrow 2^3 = 8$

$\{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{1, 2\},$

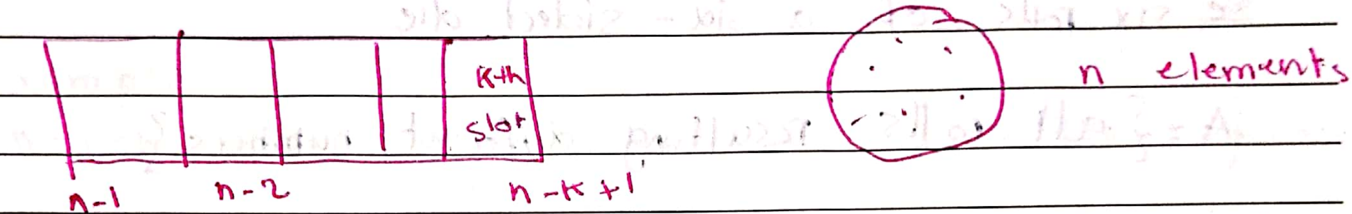
$\{2, 3\}, \{1, 3\}$

* Permutations: the number of ways to order n elements



$\#$ of ways $= n(n-1)(n-2)\ldots\ldots = n!$

K-permutation



$\#$ of ways $= n(n-1)(n-2)\ldots\ldots(n-K+1)$

$$= \frac{n!}{(n-K)!}$$

* Combination:

$\binom{n}{K} = \#$ of K-elements subsets of an n-element set

$$\binom{n}{K} = \frac{n!}{K!(n-K)!}$$

1- $\binom{n}{0} = 1$

2- $\binom{n}{1} = n$

3- $\binom{n}{n-1} = n$

4- $\sum_{K=0}^{n} \binom{n}{K} = 2^n$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

proof

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} \cdots \binom{n}{n} = 2^n$$

Example

~~Se~~ six rolls of a six - sided die

A = { all rolls resulting different numbers }

$$P(A) = \frac{\#\text{ of elements of } A}{\#\text{ of elements of } \Omega} = \frac{6!}{6^6}$$ → # of ways to order 6 numbers

Example

Tossing a coin "n" times. let $P(H) = P$

$P(T) = 1 - P$

$$P(HHTHHT) = P^4 \times (1-P)^2 = P \times P \times (1-P) \times P \times P \times (1-P)$$

$$P(k \text{ heads } (n-k) \text{ Tails}) = p^k (1-P)^{n-k}$$
particular ~~seeq~~ sequnce

$$P(k \text{ heads}) = p^k (1-P)^{n-k} \binom{n}{k}$$

Tossing a coin 3 times

HHH
HHT $\rightarrow p_1 = p\,p(1-p) = p^2(1-p)$
HTH $\rightarrow p_2 = p(1-p)p = p^2(1-p)$
HTT
TTT
THH $\rightarrow p_3 = p.p(1-p) = p^2(1-p)$
THT
THH
TTT

$P(2\text{ heads}) = 3 \times p.p(1-p)$

*Example :-

- n-coin Tosses $\rightarrow P(k$ heads in a particular sequance$)$

$= p^k (1-p)^{n-k}$

$P(k\text{ heads}) = p^k (1-p)^{n-k} \binom{n}{k}$

*Example :

- Given that are 3 heads in 10 tosses. Find the probability that the first tosses are heads. Provided that $P(H) = p$
$P(T) = 1-p$

• Solution:-

Define event $A = \{$ The first two tosses are heads $\}$.
Define event $B = \{$ There are 3 heads in 10 tosses $\}$.
→ we need to find

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad --- \boxed{1}$$

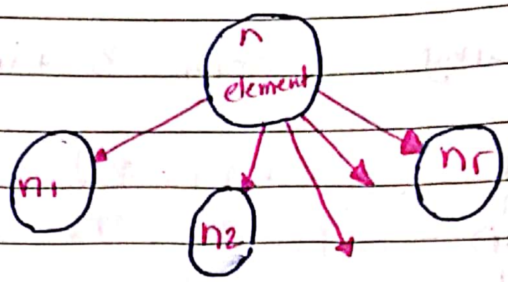
$P(B) = P(\text{Three heads in 10 tosses}) = p^3 (1-p)^7 \binom{10}{3} \quad --- \boxed{2}$

$P(A \cap B) = P(H_1, H_2 \text{ one H in 8 tosses}) = p.p\binom{8}{1}.p(1-p)^7 \quad --- \boxed{3}$

By substituting $\boxed{2}$, $\boxed{3}$ in $\boxed{1}$

$$P(A/B) = \frac{p^3 (1-p)^7 \binom{8}{1}}{p^3 (1-p)^7 \binom{10}{3}} = \frac{8}{\binom{10}{3}}$$

* Partitions :-



$\boxed{1}$ $\binom{n}{n_1}$

$\binom{n-n_1-n_2 \cdots n_{r-1}}{n_r}$ → note that: $n = n_1+n_2+n_3+\cdots$

$$C = \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-n_2 \cdots n_{r-1}}{n_r}$$

$$C = \frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_r!}$$

$\boxed{2}$

$$n! = C \cdot n_1! \cdot n_2! \cdot n_3! \cdots n_r!$$

$$C = \frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_r!}$$

* Example :

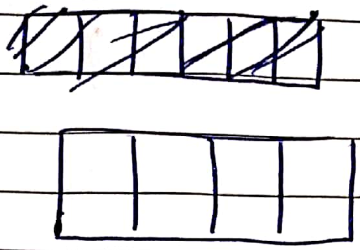
A class contains 12 undergraduate and 4 grad students, we randomly distribute the students on 4 groubs of 4 students. What is the probability that each group has a grad student.

* Solution :-

Define event $A = \{$ each group has a grad student $\}$.

$$P(A) = \frac{\#\text{ of elements of } A}{\#\text{ of elements of } \Omega} = \frac{|A|}{|\Omega|} \quad \boxed{1}$$

$$|\Omega| = \frac{16!}{4!\cdot 4!\cdot 4!\cdot 4!} \quad --- \boxed{2}$$



$\boxed{3}\boxed{3}\boxed{3}\boxed{3}$

Take 4 grad students

$4! \quad --- \boxed{3}$

12 ~~std~~ undergraduate students

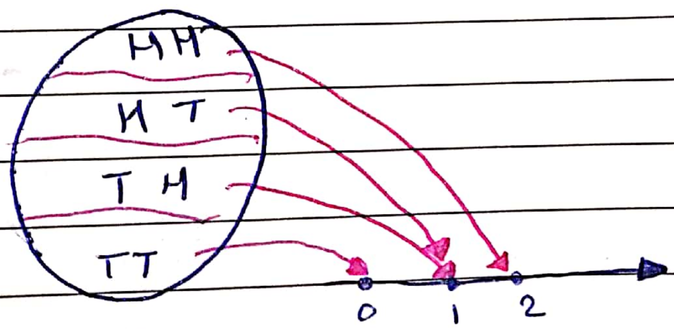
$$\rightarrow \frac{12!}{3!\cdot 3!\cdot 3!} \quad --- \boxed{4}$$

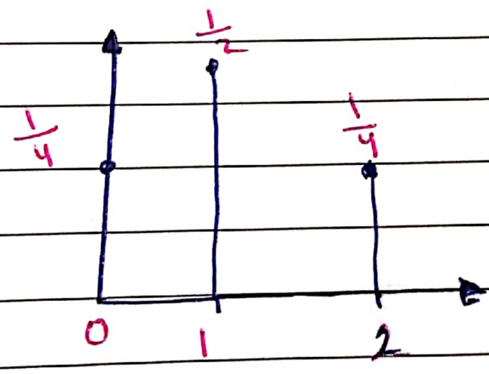
$$* \; |A| = \boxed{3}\cdot\boxed{4} = \frac{4!\cdot 12!}{3!\cdot 3!\cdot 3!} \; \boxed{5} \rightarrow$$ By substituting $\boxed{2}$ and $\boxed{5}$ in $\boxed{1}$

$$* \; P(A) = \frac{\dfrac{4!\cdot 12!}{3!\cdot 3!\cdot 3!}}{\dfrac{16!}{4!\cdot 4!\cdot 4!\cdot 4!}}$$

* Example:-

Tossing a coin twice (fair coin)





# of heads.

→ Random Variable

$X(\Omega)$