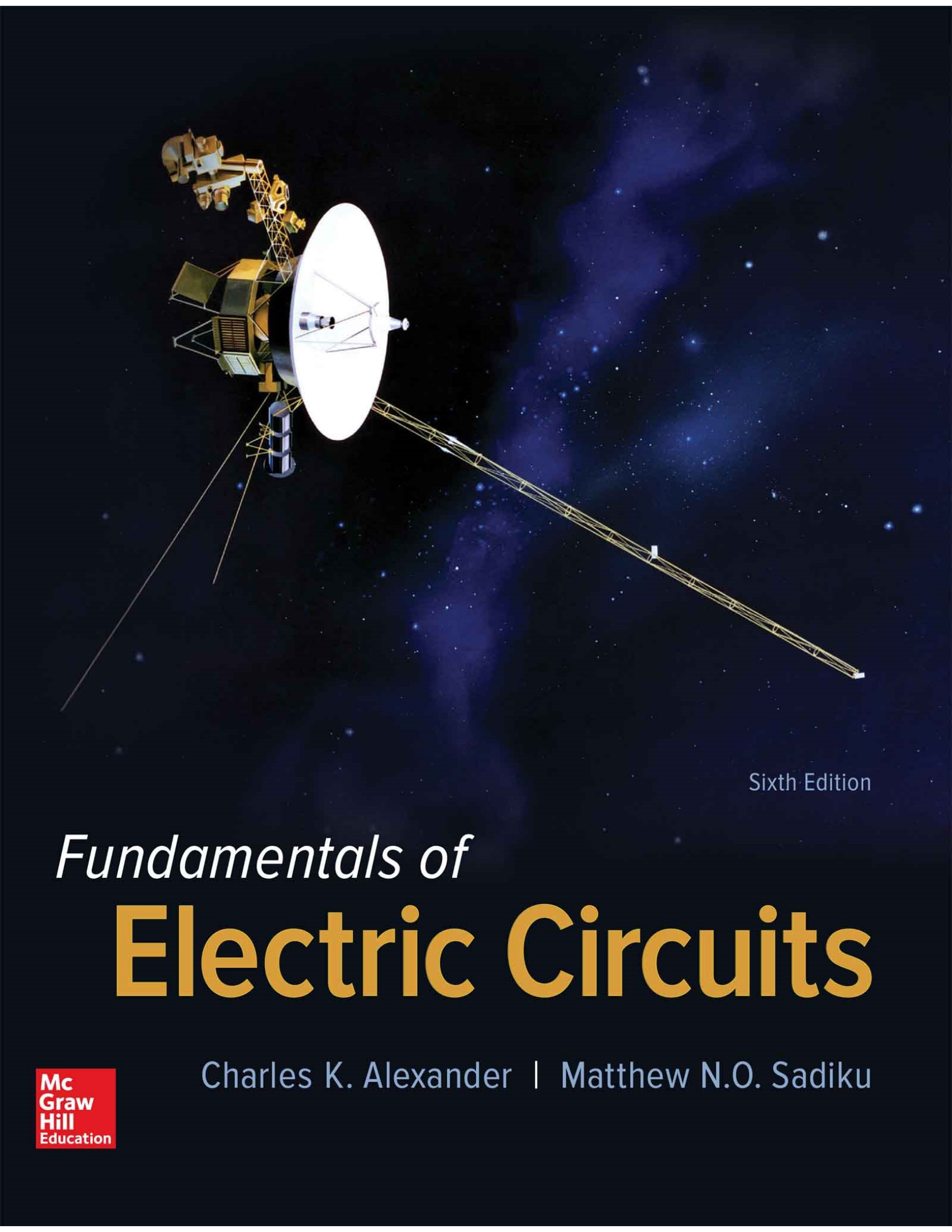

Sixth Edition

# *Fundamentals of*
# Electric Circuits

Charles K. Alexander | Matthew N.O. Sadiku

Mc Graw Hill Education

sixth edition

# Fundamentals of Electric Circuits


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Mc Graw Hill Education

FUNDAMENTALS OF ELECTRIC CIRCUITS, SIXTH EDITION

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This book is printed on acid-free paper.

1 2 3 4 5 6 7 8 9 0 DOW/DOW 1 0 9 8 7 6 5

ISBN 978-0-07-802822-9
MHID 0-07-802822-1

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Design: *Studio Montage, Inc.*
Content Licensing Specialist: *Lorraine Buczek*
Cover Image: *Courtesy NASA/JPL - Caltech*
Compositor: *MPS Limited*
Printer: *R. R. Donnelley*

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**Library of Congress Cataloging-in-Publication Data**

Alexander, Charles K., author.
Fundamentals of electric circuits / Charles K. Alexander, Department of Electrical and Computer Engineering, Cleveland State University, Matthew N. O. Sadiku, Department of Electrical Engineering, Prairie View A&M University. — Sixth edition.
pages cm
Includes index.
ISBN 978-0-07-802822-9 (alk. paper) — ISBN 0-07-802822-1 (alk. paper)
1. Electric circuits. I. Sadiku, Matthew N. O., author. II. Title.

TK454.A452 2017
621.3815—dc23 2015035301

The Internet addresses listed in the text were accurate at the time of publication. The inclusion of a website does not indicate an endorsement by the authors or McGraw-Hill Education, and McGraw-Hill Education does not guarantee the accuracy of the information presented at these sites.

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Dedicated to our wives, Kikelomo and Hannah, whose understanding and support have truly made this book possible.

Matthew
and
Chuck

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# Preface

In keeping with our focus on space for covers for our book, we have chosen the NASA Voyager spacecraft for the sixth edition. The reason for this is that like any spacecraft there are many circuits that play criti - cal roles in their functionality. The beginning of the Voyager 1 and 2 odyssey began on August 20, 1977, for Voyager 2 and on September 5, 1977, for Voyager 1. Both were launched from NASA's Kennedy Space Center in Florida. The Voyager 1 was launched on a faster orbit so it eventually became the first man-made object to leave our solar system. There is some debate over whether it has actually left the solar system, but it certainly will at some point in time. Voyager 2 and two Pioneer spacecraft will also leave the solar system at some point in time.

Voyager 1 is still functioning and sending back data, a truly signifi- cant achievement for NASA engineers. The design processes that make the Voyager operate so reliably are based on the fundamentals discussed in this textbook. Finally, space is vast so that Voyager 1 will fly past other solar systems; the odds of actually coming into contact with some- thing are so remote that it may virtually fly through the universe forever! For more about Voyager 1, go to NASA's website: www.nasa.gov/.

## Features

### New to This Edition

We have added learning objectives to each chapter to reflect what we believe are the most important items to learn from each chapter. These should help you focus more carefully on what you should be learning.

There are more than 580 revised end-of-chapter problems, new end- of-chapter problems, and revised practice problems. We continue to try and make our problems as practical as possible.

In addition, we have improved Connect for this edition by increasing the number of problems available substantially. Now, professors may select from more than a thousand problems as they build thier online homework assignments.

We have also built SmartBook for this edition. With SmartBook, stu - dents get the same text as the print version, along with personalized tips on what to study next, thanks to SmartBook's adaptive technology.

### Retained from Previous Editions

A course in circuit analysis is perhaps the first exposure students have to electrical engineering. This is also a place where we can enhance some of the skills that they will later need as they learn how to design. An important part of this book is our 121 ***design a problem*** problems. These problems were developed to enhance skills that are an impor - tant part of the design process. We know it is not possible to fully de- velop a student's design skills in a fundamental course like circuits. To fully develop design skills a student needs a design experience

normally reserved for their senior year. This does not mean that some of those skills cannot be developed and exercised in a circuits course. The text already included open-ended questions that help students use creativity, which is an important part of learning how to design. We already have some questions that are open-ended but we desired to add much more into our text in this important area and have developed an approach to do just that. When we develop problems for the student to solve our goal is that in solving the problem the student learns more about the theory and the problem solving process. Why not have the students design problems like we do? That is exactly what we do in each chapter. Within the normal problem set, we have a set of problems where we ask the student to design a problem to help other students better understand an important concept. This has two very important results. The first will be a better understanding of the basic theory and the second will be the enhancement of some of the student's basic design skills. We are making effective use of the principle of learning by teaching. Essentially we all learn better when we teach a subject. Designing effective problems is a key part of the teaching process. Students should also be encouraged to develop problems, when appropriate, which have nice numbers and do not necessarily overemphasize complicated mathematical manipulations.

A very important advantage to our textbook, we have a total of 2,481 Examples, Practice Problems, Review Questions, and End-of-Chapter Problems! Answers are provided for all practice problems and the odd numbered end-of-chapter problems.

The main objective of the sixth edition of this book remains the same as the previous editions—to present circuit analysis in a manner that is clearer, more interesting, and easier to understand than other circuit textbooks, and to assist the student in beginning to see the "fun" in engineering. This objective is achieved in the following ways:

- **Chapter Openers and Summaries**
  Each chapter opens with a discussion about how to enhance skills which contribute to successful problem solving as well as successful careers or a career-oriented talk on a subdiscipline of electrical engineering. This is followed by an introduction that links the chapter with the previous chapters and states the chapter objectives. The chapter ends with a summary of key points and formulas.
- **Problem-Solving Methodology**
  Chapter 1 introduces a six-step method for solving circuit problems which is used consistently throughout the book and media supplements to promote best-practice problem-solving procedures.
- **Student-Friendly Writing Style**
  All principles are presented in a lucid, logical, step-by-step manner. As much as possible, we avoid wordiness and giving too much detail that could hide concepts and impede overall understanding of the material.
- **Boxed Formulas and Key Terms**
  Important formulas are boxed as a means of helping students sort out what is essential from what is not. Also, to ensure that students clearly understand the key elements of the subject matter, key terms are defined and highlighted.

- **Margin Notes**
  Marginal notes are used as a pedagogical aid. They serve multiple uses such as hints, cross-references, more exposition, warnings, reminders not to make some particular common mistakes, and problem-solving insights.
- **Worked Examples**
  Thoroughly worked examples are liberally given at the end of every section. The examples are regarded as a part of the text and are clearly explained without asking the reader to fill in missing steps. Thoroughly worked examples give students a good understanding of the solution process and the confidence to solve problems themselves. Some of the problems are solved in two or three different ways to facilitate a substantial comprehension of the subject material as well as a comparison of different approaches.
- **Practice Problems**
  To give students practice opportunity, each illustrative example is immediately followed by a practice problem with the answer. The student can follow the example step-by-step to aid in the solution of the practice problem without flipping pages or looking at the end of the book for answers. The practice problem is also intended to test a student's understanding of the preceding example. It will reinforce their grasp of the material before the student can move on to the next section. Complete solutions to the practice problems are available to students on the website.
- **Application Sections**
  The last section in each chapter is devoted to practical application aspects of the concepts covered in the chapter. The material covered in the chapter is applied to at least one or two practical problems or devices. This helps students see how the concepts are applied to real-life situations.
- **Review Questions**
  Ten review questions in the form of multiple-choice objective items are provided at the end of each chapter with answers. The review questions are intended to cover the little "tricks" that the examples and end-of-chapter problems may not cover. They serve as a self test device and help students determine how well they have mastered the chapter.
- **Computer Tools**
  In recognition of the requirements by ABET® on integrating computer tools, the use of *PSpice, Multisim, MATLAB, KCIDE for Circuits*, and developing design skills are encouraged in a student-friendly manner. *PSpice* is covered early on in the text so that students can become familiar and use it throughout the text. Tutorials on all of these are available onConnect. *MATLAB* is also introduced early in the book.
- **Design a Problem Problems**
  Finally, *design a problem* problems are meant to help the student develop skills that will be needed in the design process.
- **Historical Tidbits**
  Historical sketches throughout the text provide profiles of important pioneers and events relevant to the study of electrical engineering.

- **Early Op Amp Discussion**
  The operational amplifier (op amp) as a basic element is introduced early in the text.
- **Fourier and Laplace Transforms Coverage**
  To ease the transition between the circuit course and signals and systems courses, Fourier and Laplace transforms are covered lucidly and thoroughly. The chapters are developed in a manner that the interested instructor can go from solutions of first-order circuits to Chapter 15. This then allows a very natural progression from Laplace to Fourier to AC.
- **Four-Color Art Program**
  An interior design and four-color art program bring circuit drawings to life and enhance key pedagogical elements throughout the text.
- **Extended Examples**
  Examples worked in detail according to the six-step problem solving method provide a road map for students to solve problems in a consistent fashion. At least one example in each chapter is developed in this manner.
- **EC 2000 Chapter Openers**
  Based on ABET's skill-based CRITERION 3, these chapter openers are devoted to discussions as to how students can acquire the skills that will lead to a significantly enhanced career as an engineer. Because these skills are so very important to the student while still in college as well after graduation, we use the heading, *"Enhancing your Skills and your Career."*
- **Homework Problems**
  There are 580 new or revised end-of-chapter problems and changed practice problems which will provide students with plenty of practice as well as reinforce key concepts.
- **Homework Problem Icons**
  Icons are used to highlight problems that relate to engineering design as well as problems that can be solved using *PSpice, Multisim, KCIDE,* or *MATLAB*.

## Organization

This book was written for a two-semester or three-quarter course in linear circuit analysis. The book may also be used for a one-semester course by a proper selection of chapters and sections by the instructor. It is broadly divided into three parts.

- Part 1, consisting of Chapters 1 to 8, is devoted to dc circuits. It covers the fundamental laws and theorems, circuits techniques, and passive and active elements.
- Part 2, which contains Chapter 9 to 14, deals with ac circuits. It introduces phasors, sinusoidal steady-state analysis, ac power, rms values, three-phase systems, and frequency response.
- Part 3, consisting of Chapters 15 to 19, are devoted to advanced techniques for network analysis. It provides students with a solid introduction to the Laplace transform, Fourier series, Fourier transform, and two-port network analysis.

The material in the three parts is more than sufficient for a two-semester course, so the instructor must select which chapters or sections to cover. Sections marked with the dagger sign (†) may be skipped, explained briefly, or assigned as homework. They can be omitted without loss of continuity. Each chapter has plenty of problems grouped according to the sections of the related material and diverse enough that the instructor can choose some as examples and assign some as homework. As stated earlier, we are using three icons with this edition. We are using to denote problems that either require *PSpice* in the solution process, where the circuit complexity is such that *PSpice* or *Multisim* would make the solution process easier, and where *PSpice* or *Multisim* makes a good check to see if the problem has been solved correctly. We are using to denote problems where *MATLAB* is required in the solution process, where *MATLAB* makes sense because of the problem makeup and its complexity, and where *MATLAB* makes a good check to see if the problem has been solved correctly. Finally, we use to identify problems that help the student develop skills that are needed for engineering design. More difficult problems are marked with an asterisk (*).

Comprehensive problems follow the end-of-chapter problems. They are mostly applications problems that require skills learned from that particular chapter.

## Prerequisites

As with most introductory circuit courses, the main prerequisites, for a course using this textbook, are physics and calculus. Although familiarity with complex numbers is helpful in the later part of the book, it is not required. A very important asset of this text is that ALL the mathematical equations and fundamentals of physics needed by the student, are included in the text.

## Acknowledgments

We would like to express our appreciation for the loving support we have received from our wives (Hannah and Kikelomo), daughters (Christina, Tamara, Jennifer, Motunrayo, Ann, and Joyce), son (Baixi), and our extended family members. We sincerely appreciate the invaluable help given us by Richard Rarick in helping us make the sixth edition a significantly more relevant book. He has checked all the new and revised problems and offered advice on making them more accurate and clear.

At McGraw-Hill, we would like to thank the following editorial and production staff: Raghu Srinivasan, global brand manager; Vincent Bradshaw, product developer; Nick McFadden, marketing manager; and Melissa Leick, content project manager.

The sixth edition has benefited greatly from the many outstanding individuals who have offered suggestions for improvements in both the text as well as the various problems. In particular, we thank Nicholas Reeder, Professor of Electronics Engineering Technology, Sinclair Community College, Dayton, Ohio, and Douglas De Boer, Professor of Engineering, Dordt College, Sioux Center, Iowa, for their detailed and careful corrections and suggestions for clarification which have

contributed to making this a better edition. In addition, the following have made important contributions to this edition (in alphabetical order):

Zekeriya Aliyazicioglu, *California State Polytechnic University—Pomona*
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Peter LoPresti, *University of Tulsa*
Robert Norwood, *John Brown University*
Aaron Ohta, *University of Hawaii—Manoa*
Salomon Oldak, *California State Polytechnic University—Pomona*
Hesham Shaalan, *U.S. Merchant Marine Academy*
Surendra Singh, *University of Tulsa*

Finally, we sincerely appreciate the feedback received from instructors and students who used the previous editions. We want this to continue, so please keep sending us e-mails or direct them to the publisher. We can be reached at c.alexander@ieee.org for Charles Alexander and sadiku@ieee.org for Matthew Sadiku.

C. K. Alexander and M. N. O. Sadiku

## Supplements

### Instructor and Student Resources

Available on Connect are a number of additional instructor and student resources to accompany the text. These include complete solutions for all practice and end-of-chapter problems, solutions in *PSpice* and *Multisim* problems, lecture PowerPoints®, and text image files. In addition, instructors can use COSMOS, a complete online solutions manual organization system to create custom homework, quizzes, and tests using end-of-chapter problems from the text.

### Knowledge Capturing Integrated Design Environment for Circuits (*KCIDE for Circuits*)

This software, developed at Cleveland State University and funded by NASA, is designed to help the student work through a circuits problem in an organized manner using the six-step problem-solving methodology in the text. *KCIDE for Circuits* allows students to work a circuit problem in *PSpice* and *MATLAB*, track the evolution of their solution, and save a record of their process for future reference. In addition, the software automatically generates a Word document and/or a PowerPoint presentation. The software package can be downloaded for free.

It is hoped that the book and supplemental materials supply the instructor with all the pedagogical tools necessary to effectively present the material.

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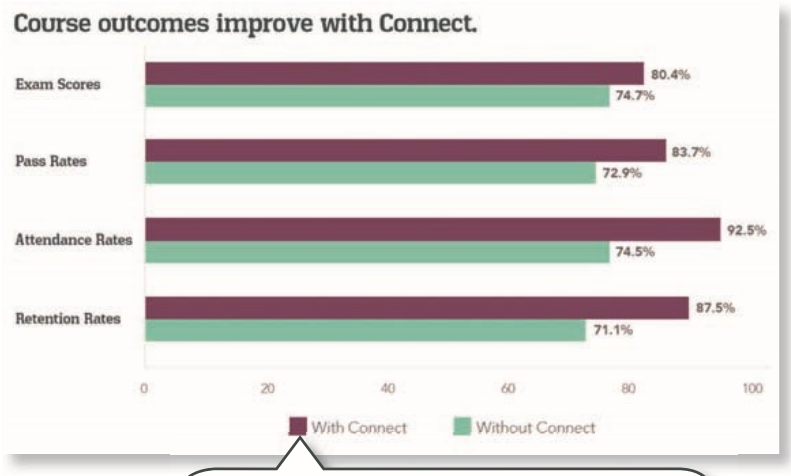


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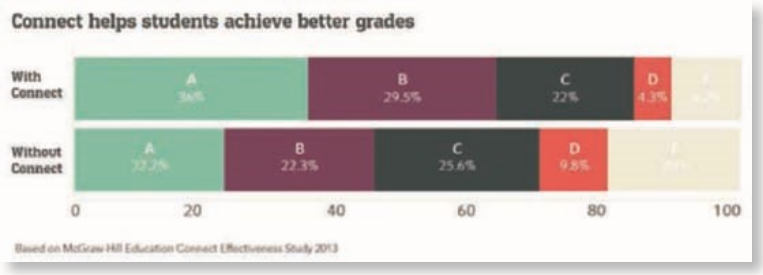


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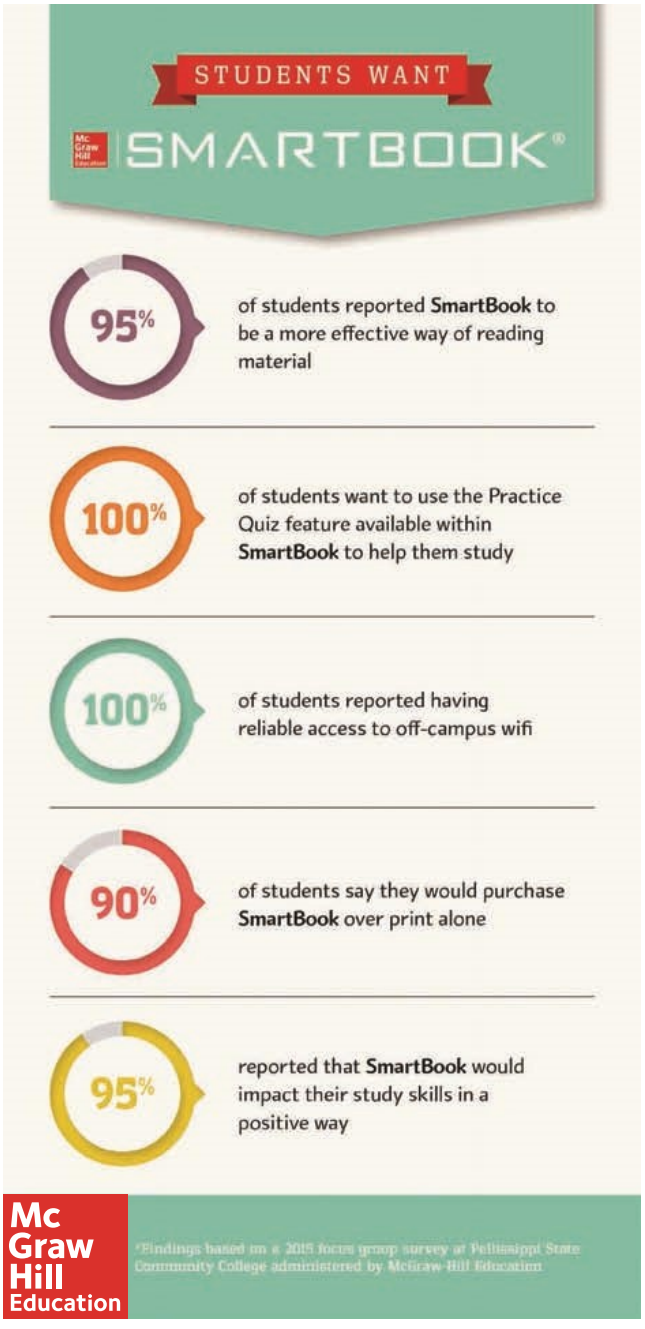

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Dr. Alexander has secured funding for two centers of research at Ohio University and Cleveland State University. He has been the director of three additional research centers at Temple and Tennessee Tech and has obtained research funding of approximately $100 million (in today's dollars). He has served as a consultant to 23 private and governmental organizations including the Air Force and the Navy.

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Dr. Alexander has authored many publications, including a workbook and a videotape lecture series, and is coauthor of *Fundamentals of Electric Circuits*, currently in its fifth edition, *Engineering Skills for Career Success, Problem Solving Made* ALMOST *Easy*, the fifth edition of the *Standard Handbook of Electronic Engineering,* and *Applied Circuit Analysis,* all with McGraw-Hill. He has delivered more than 500 paper, professional, and technical presentations.

Dr. Alexander is a Life Fellow of the IEEE and served as its president and CEO in 1997. In addition he has held several volunteer positions within the IEEE during his more than 45 years of service. This includes serving from 1991 to 1999 on the IEEE board of directors.

He has received several local, regional, national, and international awards for teaching and research, includingan honorary Doctor of Engineering degree, Fellow of the IEEE, the IEEE-USA Jim Watson Student Professional Awareness Achievement Award, the IEEE Undergraduate Teaching Award, the Distinguished Professor Award, the Distinguished Engineering Education Achievement Award, the Distinguished Engineering Education Leadership Award, the IEEE Centennial Medal, and the IEEE/RAB Innovation Award.

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Matthew N. O. Sadiku

**Matthew N. O. Sadiku** received his PhD from Tennessee Technological University, Cookeville. From 1984 to 1988, he was an assistant professor at Florida Atlantic University, where he did graduate work in computer science. From 1988 to 2000, he was at Temple University, Philadelphia, Pennsylvania, where he became a full professor. From 2000 to 2002, he was with Lucent/Avaya, Holmdel, New Jersey, as a system engineer and with Boeing Satellite Systems as a senior scientist. He is currently a professor at Prairie View A&M University.

Dr. Sadiku is the author of more than 240 professional papers and over 60 books, including *Elements of Electromagnetics* (Oxford University Press, 6th ed., 2015), *Numerical Techniques in Electromagnetics with MATLAB* (CRC, 3rd ed., 2009), and *Metropolitan Area Networks* (CRC Press, 1995). Some of his books have been translated into French, Korean, Chinese (and Chinese Long Form in Taiwan), Italian, Portuguese, and Spanish. He was the recipient of the 2000 McGraw-Hill/Jacob Millman Award for outstanding contributions in the field of electrical engineering. He was also the recipient of Regents Professor award for 2012 to 2013 by the Texas A&M University System.

His current research interests are in the areas of numerical modeling of electromagnetic systems and computer communication networks. He is a registered professional engineer and a fellow of the Institute of Electrical and Electronics Engineers (IEEE) "for contributions to computational electromagnetics and engineering education." He was the IEEE Region 2 Student Activities Committee Chairman. He was an associate editor for *IEEE Transactions on Education* and is a member of the Association for Computing Machinery (ACM).

Fundamentals of

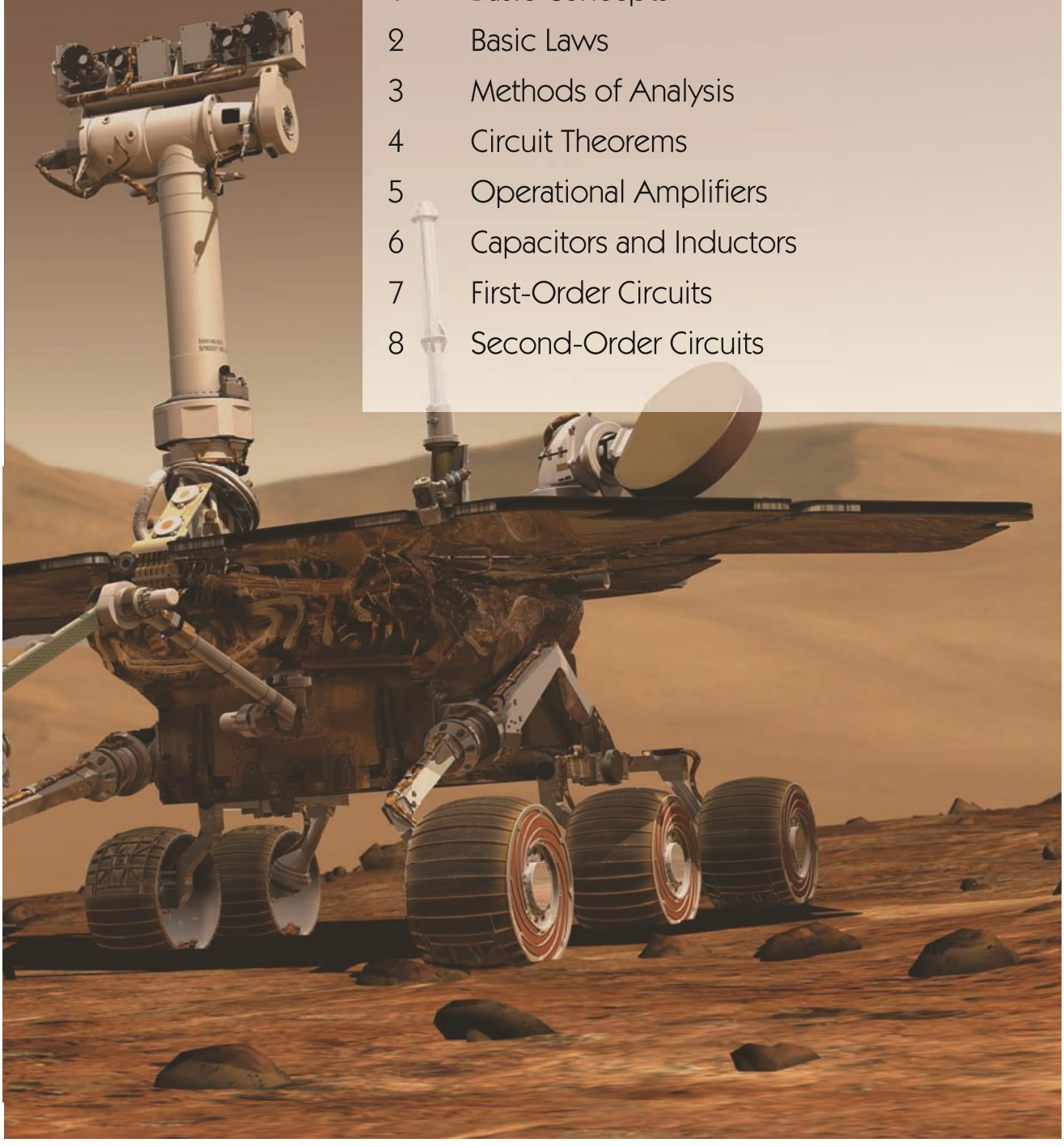
# Electric Circuits

PART ONE

# DCCi rcuits

OUTLINE


1 Basic Concepts
2 Basic Laws
3 Methods of Analysis
4 Circuit Theorems
5 Operational Amplifiers
6 Capacitors and Inductors
7 First-Order Circuits
8 Second-Order Circuits

NASA

chapter

# 1

# Basic Concepts

*Some books are to be tasted, others to be swallowed, and some few to be chewed and digested.*

—Francis Bacon

## Enhancing Your Skills and Your Career

### ABET EC 2000 criteria (3.a), *"an ability to apply knowledge of mathematics, science, and engineering."*

As students, you are required to study mathematics, science, and engineering with the purpose of being able to apply that knowledge to the solution of engineering problems. The skill here is the ability to apply the fundamentals of these areas in the solution of a problem. So how do you develop and enhance this skill?

The best approach is to work as many problems as possible in all of your courses. However, if you are really going to be successful with this, you must spend time analyzing where and when and why you have difficulty in easily arriving at successful solutions. You may be surprised to learn that most of your problem-solving problems are with mathematics rather than your understanding of theory. You may also learn that you start working the problem too soon. Taking time to think about the problem and how you should solve it will always save you time and frustration in the end.

What I have found that works best for me is to apply our six-step problem-solving technique. Then I carefully identify the areas where I have difficulty solving the problem. Many times, my actual deficiencies are in my understanding and ability to use correctly certain mathematical principles. I then return to my fundamental math texts and carefully review the appropriate sections, and in some cases, work some example problems in that text. This brings me to another important thing you should always do: Keep nearby all your basic mathematics, science, and engineering textbooks.

This process of continually looking up material you thought you had acquired in earlier courses may seem very tedious at first; however, as your skills develop and your knowledge increases, this process will become easier and easier. On a personal note, it is this very process that led me from being a much less than average student to someone who could earn a Ph.D. and become a successful researcher.

Photo by Charles Alexander

## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Understand the different units with which engineers work.
2. Understand the relationship between charge and current and how to use both in a variety of applications.
3. Understand voltage and how it can be used in a variety of applications.
4. Develop an understanding of power and energy and their relationship with current and voltage.
5. Begin to understand the volt-amp characteristics of a variety of circuit elements.
6. Begin to understand an organized approach to problem solving and how it can be used to assist in your efforts to solve circuit problems.

## 1.1 Introduction

Electric circuit theory and electromagnetic theory are the two fundamental theories upon which all branches of electrical engineering are built. Many branches of electrical engineering, such as power, electric machines, control, electronics, communications, and instrumentation, are based on electric circuit theory. Therefore, the basic electric circuit theory course is the most important course for an electrical engineering student, and always an excellent starting point for a beginning student in electrical engineering education. Circuit theory is also valuable to students specializing in other branches of the physical sciences because circuits are a good model for the study of energy systems in general, and because of the applied mathematics, physics, and topology involved.

In electrical engineering, we are often interested in communicating or transferring energy from one point to another. To do this requires an interconnection of electrical devices. Such interconnection is referred to as an *electric circuit*, and each component of the circuit is known as an *element.*

An **electric circuit** is an interconnection of electrical elements.

A simple electric circuit is shown in Fig. 1.1. It consists of three basic elements: a battery, a lamp, and connecting wires. Such a simple circuit can exist by itself; it has several applications, such as a flashlight, a search light, and so forth.

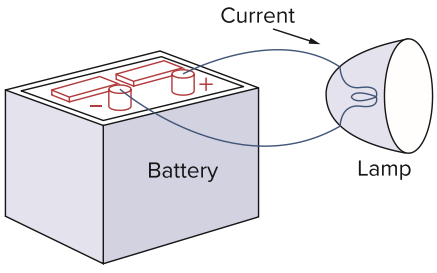



**Figure 1.1**
A simple electric circuit.

A complicated real circuit is displayed in Fig. 1.2, representing the schematic diagram for a radio receiver. Although it seems complicated, this circuit can be analyzed using the techniques we cover in this book. Our goal in this text is to learn various analytical techniques and computer software applications for describing the behavior of a circuit like this.

Electric circuits are used in numerous electrical systems to accomplish different tasks. Our objective in this book is not the study of various uses and applications of circuits. Rather, our major concern is the analysis of the circuits. By the analysis of a circuit, we mean a study of the behavior of the

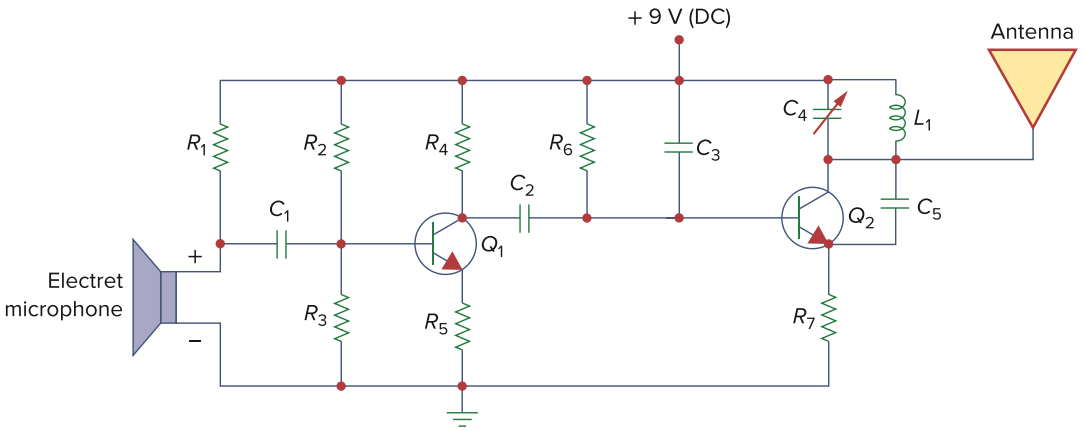



**Figure 1.2**
Electric circuit of a radio transmitter.

circuit: How does it respond to a given input? How do the interconnected elements and devices in the circuit interact?

We commence our study by defining some basic concepts. These concepts include charge, current, voltage, circuit elements, power, and energy. Before defining these concepts, we must first establish a system of units that we will use throughout the text.

## 1.2 Systems of Units

As electrical engineers, we must deal with measurable quantities. Our measurements, however, must be communicated in a standard language that virtually all professionals can understand, irrespective of the country in which the measurement is conducted. Such an international measurement language is the International System of Units (SI), adopted by the General Conference on Weights and Measures in 1960. In this system, there are seven base units from which the units of all other physical quantities can be derived. Table 1.1 shows six base units and one derived unit (the coulomb) that are related to this text. SI units are commonly used in electrical engineering.

One great advantage of the SI unit is that it uses prefixes based on the power of 10 to relate larger and smaller units to the basic unit. Table 1.2 shows the SI prefixes and their symbols. For example, the following are expressions of the same distance in meters (m):

600,000,000 mm  600,000 m  600 km

**TABLE 1.1**

Six basic SI units and one derived unit relevant to this text.

| Quantity | Basic unit | Symbol |
|---|---|---|
| Length | meter | m |
| Mass | kilogram | kg |
| Time | second | s |
| Electric current | ampere | A |
| Thermodynamic temperature | kelvin | K |
| Luminous intensity | candela | cd |
| Charge | coulomb | C |

**TABLE 1.2**

The SI prefixes.

| Multiplier | Prefix | Symbol |
|---|---|---|
| $10^{18}$ | exa | E |
| $10^{15}$ | peta | P |
| $10^{12}$ | tera | T |
| $10^{9}$ | giga | G |
| $10^{6}$ | mega | M |
| $10^{3}$ | kilo | k |
| $10^{2}$ | hecto | h |
| 10 | deka | da |
| $10^{-1}$ | deci | d |
| $10^{-2}$ | centi | c |
| $10^{-3}$ | milli | m |
| $10^{-6}$ | micro | $\mu$ |
| $10^{-9}$ | nano | n |
| $10^{-12}$ | pico | p |
| $10^{-15}$ | femto | f |
| $10^{-18}$ | atto | a |

## 1.3 Charge and Current

The concept of electric charge is the underlying principle for explaining all electrical phenomena. Also, the most basic quantity in an electric circuit is the *electric charge*. We all experience the effect of electric charge when we try to remove our wool sweater and have it stick to our body or walk across a carpet and receive a shock.

**Charge** is an electrical property of the atomic particles of which matter consists, measured in coulombs (C).

We know from elementary physics that all matter is made of fundamental building blocks known as atoms and that each atom consists of electrons, protons, and neutrons. We also know that the charge $e$ on an electron is negative and equal in magnitude to $1.602 \times 10^{-19}$ C, while a proton carries a positive charge of the same magnitude as the electron. The presence of equal numbers of protons and electrons leaves an atom neutrally charged.

The following points should be noted about electric charge:

1. The coulomb is a large unit for charges. In 1 C of charge, there are $1/(1.602 \times 10^{-19}) = 6.24 \times 10^{18}$ electrons. Thus realistic or laboratory values of charges are on the order of pC, nC, or $\mu$C.[1]
2. According to experimental observations, the only charges that occur in nature are integral multiples of the electronic charge $e = -1.602 \times 10^{-19}$ C.
3. The *law of conservation of charge* states that charge can neither be created nor destroyed, only transferred. Thus, the algebraic sum of the electric charges in a system does not change.

We now consider the flow of electric charges. A unique feature of electric charge or electricity is the fact that it is mobile; that is, it can be transferred from one place to another, where it can be converted to another form of energy.

When a conducting wire (consisting of several atoms) is connected to a battery (a source of electromotive force), the charges are compelled to move; positive charges move in one direction while negative charges move in the opposite direction. This motion of charges creates electric current. It is conventional to take the current flow as the movement of positive charges. That is, opposite to the flow of negative charges, as Fig. 1.3 illustrates. This convention was introduced by Benjamin Franklin (1706–1790), the American scientist and inventor. Although we now know that current in metallic conductors is due to negatively charged electrons, we will follow the universally accepted convention that current is the net flow of positive charges. Thus,

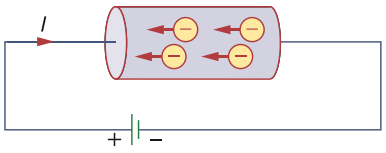



**Figure 1.3**
Electric current due to flow of electronic charge in a conductor.

A convention is a standard way of describing something so that others in the profession can understand what we mean. We will be using IEEE conventions throughout this book.

**Electric current** is the time rate of change of charge, measured in amperes (A).

Mathematically, the relationship between current $i$, charge $q$, and time $t$ is

$$i \triangleq \frac{dq}{dt} \tag{1.1}$$

[1] However, a large power supply capacitor can store up to 0.5 C of charge.

## Historical

**Andre-Marie Ampere** (1775–1836), a French mathematician and physicist, laid the foundation of electrodynamics. He defined the electric current and developed a way to measure it in the 1820s.

Born in Lyons, France, Ampere at age 12 mastered Latin in a few weeks, as he was intensely interested in mathematics and many of the best mathematical works were in Latin. He was a brilliant scientist and a prolific writer. He formulated the laws of electromagnetics. He invented the electromagnet and the ammeter. The unit of electric current, the ampere, was named after him.

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where current is measured in amperes (A), and

$$1 \text{ ampere} = 1 \text{ coulomb/second}$$

The charge transferred between time $t_0$ and $t$ is obtained by inte grating both sides of Eq. (1.1). We obtain

$$Q \triangleq \int_{t_0}^{t} i\, dt \tag{1.2}$$

The way we define current as $i$ in Eq. (1.1) suggests that current need not be a constant-valued function. As many of the examples and problems in this chapter and subsequent chapters suggest, there can be se veral types of current; that is, charge can vary with time in several ways.

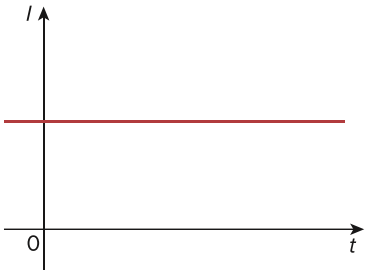
There are different ways of looking at direct current and alternating current. The best definition is that there are two ways that current can flow: It can always flow in the same direction, where it does not re verse direction, in which case we ha ve *direct current* (dc). These currents can be constant or time v arying. If the current flows in both directions, then we have *alternating current* (ac).

A **direct current** (dc) flows only in one direction and can be constant or time varying.

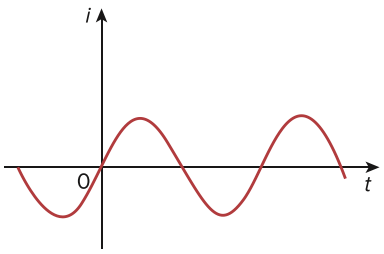
By convention, we will use the symbol $I$ to represent a constant current. If the current v aries with respect to time (either dc or ac) we will use the symbol $i$. A common use of this w ould be the output of a rectifier (dc) such as $i(t) = |5 \sin(377t)|$ amps or a sinusoidal current (ac) such as $i(t) = 160 \sin(377t)$ amps.

An **alternating current** (ac) is a current that changes direction with respect to time.

An e xample of alternating current (ac) is the current you use in your house to run the air conditioner , refrigerator , w ashing machine, and other electric appliances. Figure 1.4 depicts tw o common e xamples of




(a)




(b)

**Figure 1.4**
Two common types of current: (a) direct current (dc), (b) alternating current (ac).

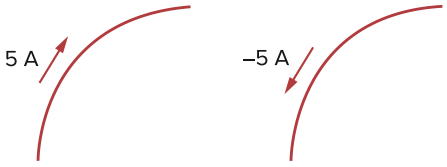



**Figure 1.5**
Conventional current flow: (a) positive current flow, (b) negative current flow.

dc (coming from a battery) and ac (coming from your home outlets). We will consider other types later in the book.

Once we define current as the movement of charge, we expect current to have an associated direction of flow. As mentioned earlier, the direction of current flow is conventionally taken as the direction of positive charge movement. Based on this convention, a current of 5 A may be represented positively or negatively as shown in Fig. 1.5. In other words, a negative current of −5 A flowing in one direction as shown in Fig. 1.5(b) is the same as a current of +5 A flowing in the opposite direction.

## Example 1.1

How much charge is represented by 4,600 electrons?

**Solution:**
Each electron has $-1.602 \times 10^{-19}$ C. Hence 4,600 electrons will have $-1.602 \times 10^{-19}$ C/electron $\times$ 4,600 electrons $= -7.369 \times 10^{-16}$ C

## Practice Problem 1.1

Calculate the amount of charge represented by 6.667 billion protons.

**Answer:** $1.0681 \times 10^{-9}$ C.

## Example 1.2

The total charge entering a terminal is given by $q = 5t \sin 4\pi t$ mC. Calculate the current at $t = 0.5$ s.

**Solution:**

$$i = \frac{dq}{dt} = \frac{d}{dt}(5t \sin 4\pi t) \text{ mC/s} = (5 \sin 4\pi t + 20\pi t \cos 4\pi t) \text{ mA}$$

At $t = 0.5$,

$$i = 5 \sin 2\pi + 10\pi \cos 2\pi = 0 + 10\pi = 31.42 \text{ mA}$$

## Practice Problem 1.2

If in Example 1.2, $q = (10 - 10e^{-2t})$ mC, find the current at $t = 1.0$ s.

**Answer:** 2.707 mA.

## Example 1.3

Determine the total charge entering a terminal between $t = 1$ s and $t = 2$ s if the current passing the terminal is $i = (3t^2 - t)$ A.

**Solution:**

$$Q = \int_{t=1}^{2} i\,dt = \int_{1}^{2} (3t^2 - t)\,dt$$

$$= \left(t^3 - \frac{t^2}{2}\right)\Big|_1^2 = (8 - 2) - \left(1 - \frac{1}{2}\right) = 5.5 \text{ C}$$

**Practice Problem 1.3**

The current flowing through an element is

$$i = \begin{cases} 4\text{ A}, & 0 < t < 1 \\ 4t^2\text{ A}, & t > 1 \end{cases}$$

Calculate the charge entering the element from $t = 0$ to $t = 2$ s.

**Answer:** 13.333 C.

## 1.4 Voltage

As explained briefly in the previous section, to move the electron in a conductor in a particular direction requires some work or energy transfer. This work is performed by an external electromotive force (emf), typically represented by the battery in Fig. 1.3. This emf is also known as *voltage* or *potential difference*. The voltage $v_{ab}$ between two points $a$ and $b$ in an electric circuit is the energy (or work) needed to move a unit charge from $b$ to $a$; mathematically,

$$v_{ab} \triangleq \frac{dw}{dq} \tag{1.3}$$

where $w$ is energy in joules (J) and $q$ is charge in coulombs (C). The voltage $v_{ab}$ or simply $v$ is measured in volts (V), named in honor of the Italian physicist Alessandro Antonio Volta (1745–1827), who invented the first voltaic battery. From Eq. (1.3), it is evident that

1 volt = 1 joule/coulomb = 1 newton-meter/coulomb

Thus,

> **Voltage** (or **potential difference**) is the energy required to move a unit charge from a reference point (−) to another point (+), measured in volts (V).

Figure 1.6 shows the voltage across an element (represented by a rectangular block) connected to points $a$ and $b$. The plus (+) and minus (−) signs are used to define reference direction or voltage polarity. The $v_{ab}$ can be interpreted in two ways: (1) Point $a$ is at a potential of $v_{ab}$ volts higher than point $b$, or (2) the potential at point $a$ with respect to point $b$ is $v_{ab}$. It follows logically that in general

$$v_{ab} = -v_{ba} \tag{1.4}$$

For example, in Fig. 1.7, we have two representations of the same voltage. In Fig. 1.7(a), point $a$ is +9 V above point $b$; in Fig. 1.7(b), point $b$ is −9 V above point $a$. We may say that in Fig. 1.7(a), there is a 9-V *voltage drop* from $a$ to $b$ or equivalently a 9-V *voltage rise* from $b$ to $a$. In other words, a voltage drop from $a$ to $b$ is equivalent to a voltage rise from $b$ to $a$.

Current and voltage are the two basic variables in electric circuits. The common term *signal* is used for an electric quantity such as a current or a voltage (or even electromagnetic wave) when it is used for conveying

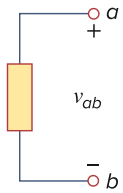



**Figure 1.6**
Polarity of voltage $v_{ab}$.

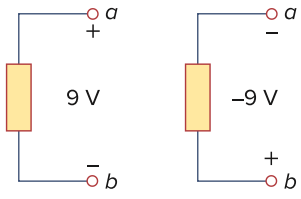



**Figure 1.7**
Two equivalent representations of the same voltage $v_{ab}$: (a) Point $a$ is 9 V above point $b$; (b) point $b$ is −9 V above point $a$.

## Historical

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**Alessandro Antonio Volta** (1745–1827), an Italian physicist, invented the electric battery—which provided the first continuous flow of electricity—and the capacitor.

Born into a noble family in Como, Italy, Volta was performing electrical experiments at age 18. His invention of the battery in 1796 revolutionized the use of electricity. The publication of his work in 1800 marked the beginning of electric circuit theory. Volta received many honors during his lifetime. The unit of voltage or potential difference, the volt, was named in his honor.

information. Engineers prefer to call such variables signals rather than mathematical functions of time because of their importance in communications and other disciplines. Like electric current, a constant voltage is called a *dc voltage* and is represented by $V$, whereas a sinusoidally time-varying voltage is called an *ac voltage* and is represented by $v$. A dc voltage is commonly produced by a battery; ac voltage is produced by an electric generator.

Keep in mind that electric current is always *through* an element and that electric voltage is always *across* the element or between two points.

## 1.5 Power and Energy

Although current and voltage are the two basic variables in an electric circuit, they are not sufficient by themselves. For practical purposes, we need to know how much *power* an electric device can handle. We all know from experience that a 100-watt bulb gives more light than a 60-watt bulb. We also know that when we pay our bills to the electric utility companies, we are paying for the electric *energy* consumed over a certain period of time. Thus, power and energy calculations are important in circuit analysis.

To relate power and energy to voltage and current, we recall from physics that:

**Power** is the time rate of expending or absorbing energy, measured in watts (W).

We write this relationship as

$$p \triangleq \frac{dw}{dt} \tag{1.5}$$

where $p$ is power in watts (W), $w$ is energy in joules (J), and $t$ is time in seconds (s). From Eqs. (1.1), (1.3), and (1.5), it follows that

$$p = \frac{dw}{dt} = \frac{dw}{dq} \cdot \frac{dq}{dt} = vi \tag{1.6}$$

or

$$\boxed{p = vi} \tag{1.7}$$

The power $p$ in Eq. (1.7) is a time-varying quantity and is called the *instantaneous power*. Thus, the power absorbed or supplied by an element is the product of the voltage across the element and the current through it. If the power has a + sign, power is being delivered to or absorbed by the element. If, on the other hand, the power has a − sign, power is being supplied by the element. But how do we know when the power has a negative or a positive sign?

Current direction and voltage polarity play a major role in determining the sign of power. It is therefore important that we pay attention to the relationship between current $i$ and voltage $v$ in Fig. 1.8(a). The voltage polarity and current direction must conform with those shown in Fig. 1.8(a) in order for the power to have a positive sign. This is known as the *passive sign convention*. By the passive sign convention, current enters through the positive polarity of the voltage. In this case, $p = +vi$ or $vi > 0$ implies that the element is absorbing power. However, if $p = -vi$ or $vi < 0$, as in Fig. 1.8(b), the element is releasing or supplying power.

**Passive sign convention** is satisfied when the current enters through the positive terminal of an element and $p = +vi$. If the current enters through the negative terminal, $p = -vi$.

Unless otherwise stated, we will follow the passive sign convention throughout this text. For example, the element in both circuits of Fig. 1.9 has an absorbing power of +12 W because a positive current enters the positive terminal in both cases. In Fig. 1.10, however, the element is supplying power of +12 W because a positive current enters the negative terminal. Of course, an absorbing power of −12 W is equivalent to a supplying power of +12 W. In general,

$$+\text{Power absorbed} = -\text{Power supplied}$$

In fact, the *law of conservation of energy* must be obeyed in any electric circuit. For this reason, the algebraic sum of power in a circuit, at any instant of time, must be zero:

$$\boxed{\sum p = 0} \tag{1.8}$$

This again confirms the fact that the total power supplied to the circuit must balance the total power absorbed.

From Eq. (1.6), the energy absorbed or supplied by an element from time $t_0$ to time $t$ is

$$w = \int_{t_0}^{t} p\,dt = \int_{t_0}^{t} vi\,dt \tag{1.9}$$

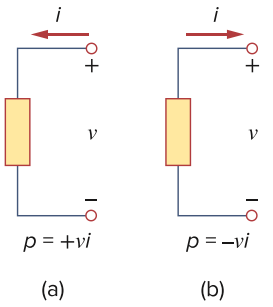



**Figure 1.8**
Reference polarities for power using the passive sign convention: (a) absorbing power, (b) supplying power.

When the voltage and current directions conform to Fig. 1.8(b), we have the *active sign convention* and $p = +vi$.

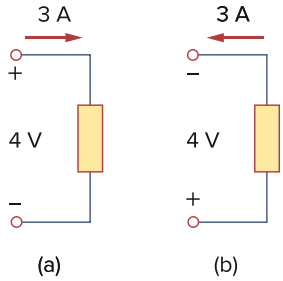



**Figure 1.9**
Two cases of an element with an absorbing power of 12 W: (a) $p = 4 \times 3 = 12$ W, (b) $p = 4 \times 3 = 12$ W.

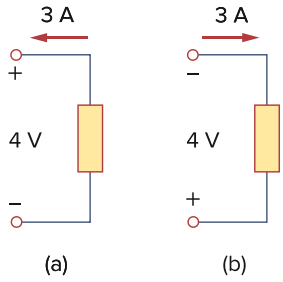



**Figure 1.10**
Two cases of an element with a supplying power of 12 W: (a) $p = -4 \times 3 = -12$ W, (b) $p = -4 \times 3 = -12$ W.

**Energy** is the capacity to do work, measured in joules (J).

The electric power utility companies measure energy in watt-hours (Wh), where

$$1 \text{ Wh} = 3{,}600 \text{ J}$$

## Example 1.4

An energy source forces a constant current of 2A for 10 s to flow through a light bulb. If 2.3 kJ is given off in the form of light and heat energy, calculate the voltage drop across the bulb.

**Solution:**
The total charge is

$$\Delta q = i\,\Delta t = 2 \times 10 = 20 \text{ C}$$

The voltage drop is

$$v = \frac{\Delta w}{\Delta q} = \frac{2.3 \times 10^3}{20} = 115 \text{ V}$$

## Practice Problem 1.4

To move charge $q$ from point $b$ to point $a$ requires 25 J. Find the voltage drop $v_{ab}$ (the voltage at $a$ positive with respect to $b$) if: (a) $q = 5$ C, (b) $q = -10$ C.

**Answer:** (a) 5 V, (b) −2.5 V.

## Example 1.5

Find the power delivered to an element at $t = 3$ ms if the current entering its positive terminal is

$$i = 5\cos 60\pi\, t \text{ A}$$

and the voltage is: (a) $v = 3i$, (b) $v = 3\, di/dt$.

**Solution:**
(a) The voltage is $v = 3i = 15 \cos 60\pi\, t$; hence, the power is

$$p = vi = 75\cos^2 60\pi\, t \text{ W}$$

At $t = 3$ ms,

$$p = 75\cos^2(60\pi \times 3 \times 10^{-3}) = 75\cos^2 0.18\pi = 53.48 \text{ W}$$

(b) We find the voltage and the power as

$$v = 3\frac{di}{dt} = 3(-60\pi)5\sin 60\pi\, t = -900\pi \sin 60\pi\, t \text{ V}$$

$$p = vi = -4500\pi \sin 60\pi\, t \cos 60\pi\, t \text{ W}$$

At $t = 3$ ms,

$$p = -4500\pi \sin 0.18\pi \cos 0.18\pi \text{ W}$$

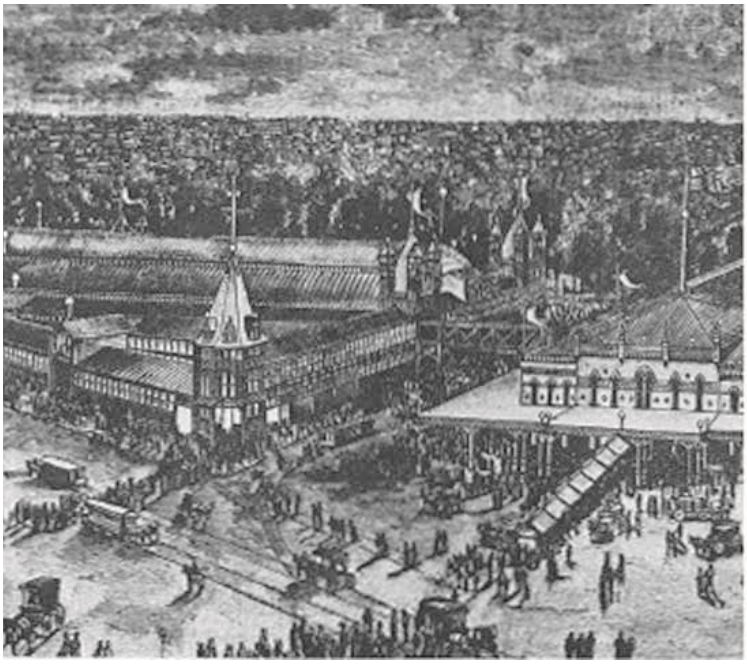
$$= -14137.167 \sin 32.4^\circ \cos 32.4^\circ = -6.396 \text{ kW}$$

## Historical

**1884 Exhibition** In the United States, nothing promoted the future of electricity like the 1884 International Electrical Exhibition. Just imagine a world without electricity, a world illuminated by candles and gaslights, a world where the most common transportation was by walking and riding on horseback or by horse-drawn carriage. Into this world an exhibition was created that highlighted Thomas Edison and reflected his highly developed ability to promote his inventions and products. His exhibit featured spectacular lighting displays powered by an impressive 100-kW "Jumbo" generator.

Edward Weston's dynamos and lamps were featured in the United States Electric Lighting Company's display. Weston's well known collection of scientific instruments was also shown.

Other prominent exhibitors included Frank Sprague, Elihu Thompson, and the Brush Electric Company of Cleveland. The American Institute of Electrical Engineers (AIEE) held its first technical meeting on October 7–8 at the Franklin Institute during the exhibit. AIEE merged with the Institute of Radio Engineers (IRE) in 1964 to form the Institute of Electrical and Electronics Engineers (IEEE).



Source: IEEE History Center

## Practice Problem 1.5

Find the power delivered to the element in Example 1.5 at $t = 5$ ms if the current remains the same but the voltage is: (a) $v = 2i$ V,

(b) $v = \left(10 + 5\int_0^t i\,dt\right)$ V.

**Answer:** (a) 17.27 W, (b) 29.7 W.

## Example 1.6

How much energy does a 100-W electric bulb consume in two hours?

**Solution:**

$$w = pt = 100\ (\text{W}) \times 2\ (\text{h}) \times 60\ (\text{min/h}) \times 60\ (\text{s/min})$$
$$= 720{,}000\ \text{J} = 720\ \text{kJ}$$

This is the same as

$$w = pt = 100\ \text{W} \times 2\ \text{h} = 200\ \text{Wh}$$

## Practice Problem 1.6

A home electric heater draws 10 A when connected to a 115 V outlet. How much energy is consumed by the heater over a period of 6 hours?

**Answer:** 6.9 k watt-hours

## 1.6 Circuit Elements

As we discussed in Section 1.1, an element is the basic building block of a circuit. An electric circuit is simply an interconnection of the elements. Circuit analysis is the process of determining voltages across (or the currents through) the elements of the circuit.

There are two types of elements found in electric circuits: *passive* elements and *active* elements. An active element is capable of generating energy while a passive element is not. Examples of passive elements are resistors, capacitors, and inductors. Typical active elements include generators, batteries, and operational amplifiers. Our aim in this section is to gain familiarity with some important active elements.

The most important active elements are voltage or current sources that generally deliver power to the circuit connected to them. There are two kinds of sources: independent and dependent sources.

An ideal independent source is an active element that provides a specified voltage or current that is completely independent of other circuit elements.

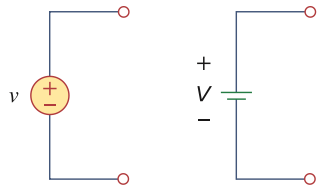



**Figure 1.11**
Symbols for independent voltage sources: (a) used for constant or time-varying voltage, (b) used for constant voltage (dc).

In other words, an ideal independent voltage source delivers to the circuit whatever current is necessary to maintain its terminal voltage. Physical sources such as batteries and generators may be regarded as approximations to ideal voltage sources. Figure 1.11 shows the symbols for independent voltage sources. Notice that both symbols in Fig. 1.11(a) and (b) can be used to represent a dc voltage source, but only the symbol in Fig. 1.11(a) can be used for a time-varying voltage source. Similarly, an ideal independent current source is an active element that provides a specified current completely independent of the voltage across the source. That is, the current source delivers to the circuit whatever

voltage is necessary to maintain the designated current. The symbol for an independent current source is displayed in Fig. 1.12, where the arrow indicates the direction of current $i$.

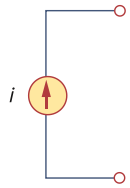



**Figure 1.12**
Symbol for independent current source.

An **ideal dependent** (or **controlled**) **source** is an active element in which the source quantity is controlled by another voltage or current.

Dependent sources are usually designated by diamond-shaped symbols, as shown in Fig. 1.13. Since the control of the dependent source is achieved by a voltage or current of some other element in the circuit, and the source can be voltage or current, it follows that there are four possible types of dependent sources, namely:

1. A voltage-controlled voltage source (VCVS).
2. A current-controlled voltage source (CCVS).
3. A voltage-controlled current source (VCCS).
4. A current-controlled current source (CCCS).

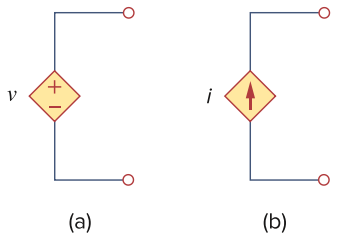



**Figure 1.13**
Symbols for: (a) dependent voltage source, (b) dependent current source.

Dependent sources are useful in modeling elements such as transistors, operational amplifiers, and integrated circuits. An example of a current-controlled voltage source is shown on the right-hand side of Fig. 1.14, where the voltage $10i$ of the voltage source depends on the current $i$ through element $C$. Students might be surprised that the value of the dependent voltage source is $10\,i$ V (and not $10\,i$ A) because it is a voltage source. The key idea to keep in mind is that a voltage source comes with polarities $(+\ -)$ in its symbol, while a current source comes with an arrow, irrespective of what it depends on.

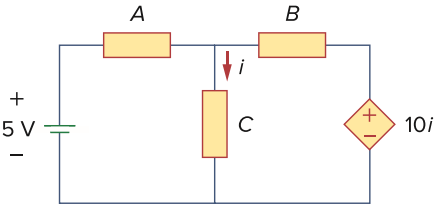



**Figure 1.14**
The source on the right-hand side is a current-controlled voltage source.

It should be noted that an ideal voltage source (dependent or independent) will produce any current required to ensure that the terminal voltage is as stated, whereas an ideal current source will produce the necessary voltage to ensure the stated current flow. Thus, an ideal source could in theory supply an infinite amount of energy. It should also be noted that not only do sources supply power to a circuit, they can absorb power from a circuit too. For a voltage source, we know the voltage but not the current supplied or drawn by it. By the same token, we know the current supplied by a current source but not the voltage across it.

## Example 1.7

Calculate the power supplied or absorbed by each element in Fig. 1.15.

**Solution:**
We apply the sign convention for power shown in Figs. 1.8 and 1.9. For $p_1$, the 5-A current is out of the positive terminal (or into the negative terminal); hence,

$$p_1 = 20(-5) = -100 \text{ W} \qquad \text{Supplied power}$$

For $p_2$ and $p_3$, the current flows into the positive terminal of the element in each case.

$$p_2 = 12(5) = 60 \text{ W} \qquad \text{Absorbed power}$$

$$p_3 = 8(6) = 48 \text{ W} \qquad \text{Absorbed power}$$

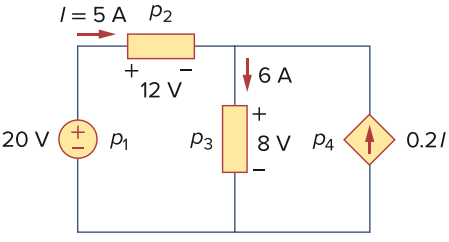



**Figure 1.15**
For Example 1.7.

For $p_4$, we should note that the voltage is 8 V (positive at the top), the same as the voltage for $p_3$ since both the passive element and the dependent source are connected to the same terminals. (Remember that voltage is always measured across an element in a circuit.) Since the current flows out of the positive terminal,

$$p_4 = 8(-0.2I) = 8(-0.2 \times 5) = -8 \text{ W} \qquad \text{Supplied power}$$

We should observe that the 20-V independent voltage source and $0.2I$ dependent current source are supplying power to the rest of the network, while the two passive elements are absorbing power. Also,

$$p_1 + p_2 + p_3 + p_4 = -100 + 60 + 48 - 8 = 0$$

In agreement with Eq. (1.8), the total power supplied equals the total power absorbed.

## Practice Problem 1.7

Compute the power absorbed or supplied by each component of the circuit in Fig. 1.16.

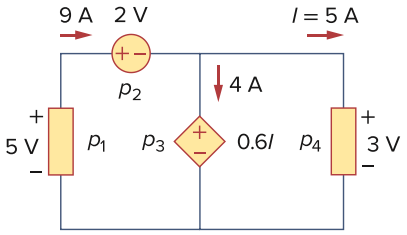



**Figure 1.16**
For Practice Prob. 1.7.

**Answer:** $p_1 = -45$ W, $p_2 = 18$ W, $p_3 = 12$ W, $p_4 = 15$ W.

## 1.7 †Applications[2]

In this section, we will consider two practical applications of the concepts developed in this chapter. The first one deals with the TV picture tube and the other with how electric utilities determine your electric bill.

### 1.7.1 TV Picture Tube

One important application of the motion of electrons is found in both the transmission and reception of TV signals. At the transmission end, a TV camera reduces a scene from an optical image to an electrical signal. Scanning is accomplished with a thin beam of electrons in an iconoscope camera tube.

At the receiving end, the image is reconstructed by using a cathode-ray tube (CRT) located in the TV receiver.[3] The CRT is depicted in Fig. 1.17. Unlike the iconoscope tube, which produces an electron beam of constant intensity, the CRT beam varies in intensity according to the incoming signal. The electron gun, maintained at a high potential, fires the electron beam. The beam passes through two sets of plates for vertical and horizontal deflections so that the spot on the screen where the beam strikes can move right and left and up and down. When the electron beam strikes the fluorescent screen, it gives off light at that spot. Thus, the beam can be made to "paint" a picture on the TV screen.

[2] The dagger sign preceding a section heading indicates the section that may be skipped, explained briefly, or assigned as homework.

[3] Modern TV tubes use a different technology.

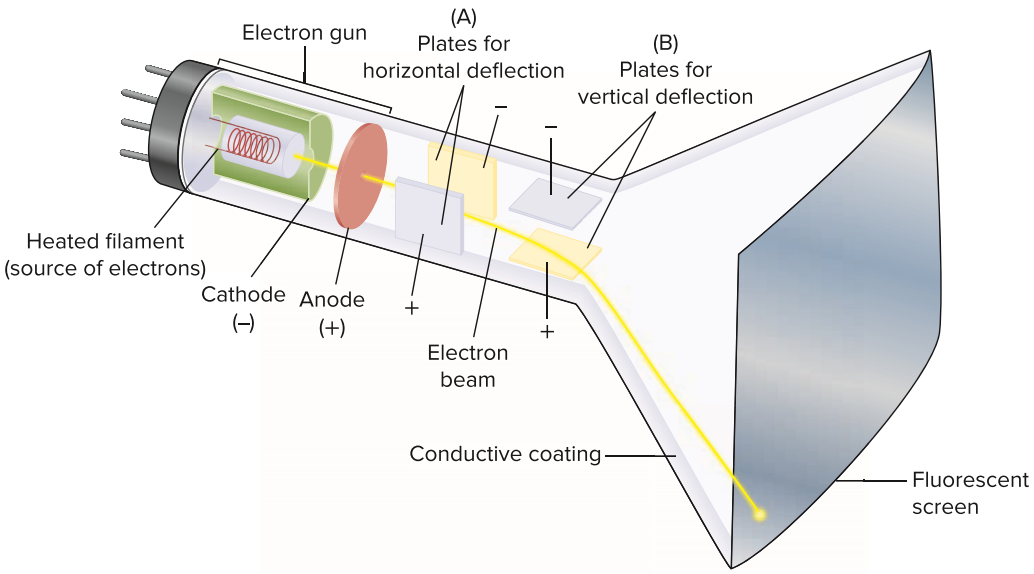



**Figure 1.17**
Cathode-ray tube.

## Historical

### Karl Ferdinand Braun and Vladimir K. Zworykin

**Karl Ferdinand Braun** (1850–1918), of the University of Strasbourg, invented the Braun cathode-ray tube in 1879. This then became the basis for the picture tube used for so many years for televisions. It is still the most economical device today, although the price of flat-screen systems is rapidly becoming competitive. Before the Braun tube could be used in television, it took the inventiveness of **Vladimir K. Zworykin** (1889–1982) to develop the iconoscope so that the modern television would become a reality. The iconoscope developed into the orthicon and the image orthicon, which allowed images to be captured and converted into signals that could be sent to the television receiver. Thus, the television camera was born.

## Example 1.8

The electron beam in aTV picture tube carries $10^{15}$ electrons per second. As a design engineer, determine the voltage $V_o$ needed to accelerate the electron beam to achieve 4 W.

**Solution:**
The charge on an electron is

$$e = -1.6 \times 10^{-19}\ \text{C}$$

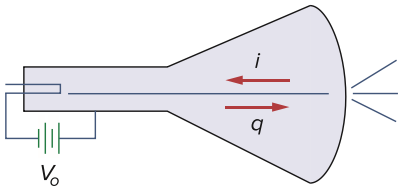



**Figure 1.18**
A simplified diagram of the cathode-ray tube; for Example 1.8.

If the number of electrons is $n$, then $q = ne$ and

$$i = \frac{dq}{dt} = e\frac{dn}{dt} = (-1.6 \times 10^{-19})(10^{15}) = -1.6 \times 10^{-4}\ \text{A}$$

The negative sign indicates that the current flows in a direction opposite to electron flow as shown in Fig. 1.18, which is a simplified diagram of the CRT for the case when the vertical deflection plates carry no charge. The beam power is

$$p = V_o i \qquad \text{or} \qquad V_o = \frac{p}{i} = \frac{4}{1.6 \times 10^{-4}} = 25{,}000\ \text{V}$$

Thus, the required voltage is 25 kV.

## Practice Problem 1.8

If an electron beam in a TV picture tube carries $10^{13}$ electrons/second and is passing through plates maintained at a potential difference of 30 kV, calculate the power in the beam.

**Answer:** 48 mW.

### 1.7.2 Electricity Bills

The second application deals with how an electric utility company charges their customers. The cost of electricity depends upon the amount of energy consumed in kilowatt-hours (kWh). (Other factors that affect the cost include demand and power factors; we will ignore these for now.) However, even if a consumer uses no energy at all, there is a minimum service charge the customer must pay because it costs money to stay connected to the power line. As energy consumption increases, the cost per kWh drops. It is interesting to note the average monthly consumption of household appliances for a family of five, shown in Table 1.3.

**TABLE 1.3**

Typical average monthly consumption of household appliances.

| Appliance | kWh consumed | Appliance | kWh consumed |
|---|---|---|---|
| Water heater | 500 | Washing machine | 120 |
| Freezer | 100 | Stove | 100 |
| Lighting | 100 | Dryer | 80 |
| Dishwasher | 35 | Microwave oven | 25 |
| Electric iron | 15 | Personal computer | 12 |
| TV | 10 | Radio | 8 |
| Toaster | 4 | Clock | 2 |

## Example 1.9

A homeowner consumes 700 kWh in January. Determine the electricity bill for the month using the following residential rate schedule:

Base monthly charge of $12.00.
First 100 kWh per month at 16 cents/kWh.
Next 200 kWh per month at 10 cents/kWh.
Over 300 kWh per month at 6 cents/kWh.

**Solution:**
We calculate the electricity bill as follows.

$$\begin{aligned}
\text{Base monthly charge} &= \$12.00\\
\text{First 100 kWh @ \$0.16/k Wh} &= \$16.00\\
\text{Next 200 kWh @ \$0.10/k Wh} &= \$20.00\\
\text{Remaining 400 kWh @ \$0.06/k Wh} &= \$24.00\\
\text{Total charge} &= \$72.00
\end{aligned}$$

$$\text{Average cost} = \frac{\$72}{100 + 200 + 400} = 10.2 \text{ cents/kWh}$$

## Practice Problem 1.9

Referring to the residential rate schedule in Example 1.9, calculate the average cost per kWh if only 350 kWh are consumed in July when the family is on vacation most of the time.

**Answer:** 14.571 cents/kWh.

## 1.8 †Problem Solving

Although the problems to be solved during one's career will vary in complexity and magnitude, the basic principles to be follo wed remain the same. The process outlined here is the one developed by the authors over many years of problem solving with students, for the solution of engi - neering problems in industry, and for problem solving in research.

We will list the steps simply and then elaborate on them.

1. Carefully **define** the problem.
2. **Present** everything you know about the problem.
3. Establish a set of **alternative** solutions and determine the one that promises the greatest likelihood of success.
4. **Attempt** a problem solution.
5. **Evaluate** the solution and check for accuracy.
6. Has the problem been solved **satisfactorily**? If so, present the solution; if not, then return to step 3 and continue through the process again.

1. *Carefully* ***define*** *the problem* . This may be the most important part of the process, because it becomes the foundation for all the rest of the steps. In general, the presentation of engineering problems is

somewhat incomplete. You must do all you can to make sure you understand the problem as thoroughly as the presenter of the problem understands it. Time spent at this point clearly identifying the problem will save you considerable time and frustration later. As a student, you can clarify a problem statement in a textbook by asking your professor. A problem presented to you in industry may require that you consult several individuals. At this step, it is important to develop questions that need to be addressed before continuing the solution process. If you have such questions, you need to consult with the appropriate individuals or resources to obtain the answers to those questions. With those answers, you can now refine the problem, and use that refinement as the problem statement for the rest of the solution process.

2. ***Present*** *everything you know about the problem.* You are now ready to write down everything you know about the problem and its possible solutions. This important step will save you time and frustration later.

3. *Establish a set of* ***alternative*** *solutions and determine the one that promises the greatest likelihood of success*. Almost every problem will have a number of possible paths that can lead to a solution. It is highly desirable to identify as many of those paths as possible. At this point, you also need to determine what tools are available to you, such as *PSpice* and *MATLAB* and other software packages that can greatly reduce effort and increase accuracy. Again, we want to stress that time spent carefully defining the problem and investigating alternative approaches to its solution will pay big dividends later. Evaluating the alternatives and determining which promises the greatest likelihood of success may be difficult but will be well worth the effort. Document this process well since you will want to come back to it if the first approach does not work.

4. ***Attempt*** *a problem solution.* Now is the time to actually begin solving the problem. The process you follow must be well documented in order to present a detailed solution if successful, and to evaluate the process if you are not successful. This detailed evaluation may lead to corrections that can then lead to a successful solution. It can also lead to new alternatives to try. Many times, it is wise to fully set up a solution before putting numbers into equations. This will help in checking your results.

5. ***Evaluate*** *the solution and check for accuracy.* You now thoroughly evaluate what you have accomplished. Decide if you have an acceptable solution, one that you want to present to your team, boss, or professor.

6. *Has the problem been solved* ***satisfactorily****? If so, present the solution; if not, then return to step 3 and continue through the process again.* Now you need to present your solution or try another alternative. At this point, presenting your solution may bring closure to the process. Often, however, presentation of a solution leads to further refinement of the problem definition, and the process continues. Following this process will eventually lead to a satisfactory conclusion.

Now let us look at this process for a student taking an electrical and computer engineering foundations course. (The basic process also applies to almost every engineering course.) Keep in mind that although the steps have been simplified to apply to academic types of problems, the process as stated always needs to be followed. We consider a simple example.

Solve for the current flowing through the 8-Ω resistor in Fig. 1.19. **Example 1.10**

**Solution:**

1. *Carefully* ***define*** *the problem.* This is only a simple example, but we can already see that we do not know the polarity on the 3-V source. We have the following options. We can ask the professor what the polarity should be. If we cannot ask, then we need to make a decision on what to do next. If we have time to work the problem both ways, we can solve for the current when the 3-V source is plus on top and then plus on the bottom. If we do not have the time to work it both ways, assume a polarity and then carefully document your decision. Let us assume that the professor tells us that the source is plus on the bottom as shown in Fig. 1.20.

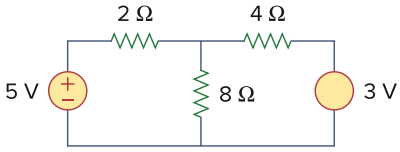



**Figure 1.19**
Illustrative example.

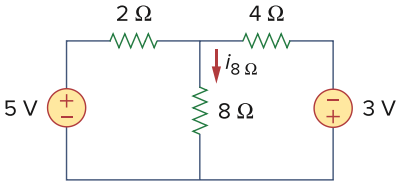



**Figure 1.20**
Problem definition.

2. ***Present*** *everything you know about the problem.* Presenting all that we know about the problem involves labeling the circuit clearly so that we define what we seek.

   Given the circuit shown in Fig. 1.20, solve for $i_{8\Omega}$.

   We now check with the professor, if reasonable, to see if the problem is properly defined.

3. *Establish a set of* ***alternative*** *solutions and determine the one that promises the greatest likelihood of success* . There are essentially three techniques that can be used to solve this problem. Later in the text you will see that you can use circuit analysis (using Kirchhoff's laws and Ohm's law), nodal analysis, and mesh analysis.

   To solve for $i_{8\Omega}$ using circuit analysis will eventually lead to a solution, but it will likely take more work than either nodal or mesh analysis. To solve for $i_{8\Omega}$ using mesh analysis will require writing two simultaneous equations to find the two loop currents indicated in Fig. 1.21. Using nodal analysis requires solving for only one unknown. This is the easiest approach.

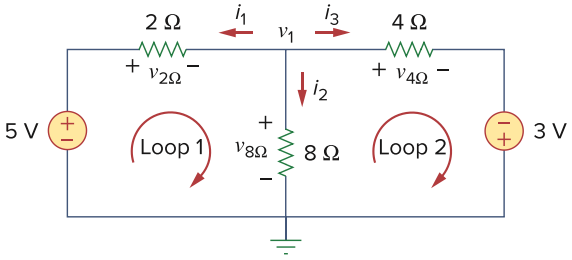



**Figure 1.21**
Using nodal analysis.

Therefore, we will solve for $i_{8\Omega}$ using nodal analysis.

4. ***Attempt*** *a problem solution.* We first write down all of the equations we will need in order to find $i_{8\Omega}$.

$$i_{8\Omega} = i_2, \qquad i_2 = \frac{v_1}{8}, \qquad i_{8\Omega} = \frac{v_1}{8}$$

$$\frac{v_1 - 5}{2} + \frac{v_1 - 0}{8} + \frac{v_1 + 3}{4} = 0$$

Now we can solve for $v_1$.

$$8\left[\frac{v_1 - 5}{2} + \frac{v_1 - 0}{8} + \frac{v_1 + 3}{4}\right] = 0$$

$$\text{leads to } (4v_1 - 20) + (v_1) + (2v_1 + 6) = 0$$

$$7v_1 = +14, \qquad v_1 = +2 \text{ V}, \qquad i_{8\Omega} = \frac{v_1}{8} = \frac{2}{8} = \mathbf{0.25\ A}$$

5. ***Evaluate*** *the solution and check for accuracy*. We can now use Kirchhoff's voltage law (KVL) to check the results.

$$i_1 = \frac{v_1 - 5}{2} = \frac{2 - 5}{2} = -\frac{3}{2} = -1.5 \text{ A}$$

$$i_2 = i_{8\Omega} = 0.25 \text{ A}$$

$$i_3 = \frac{v_1 + 3}{4} = \frac{2 + 3}{4} = \frac{5}{4} = 1.25 \text{ A}$$

$$i_1 + i_2 + i_3 = \mathbf{-1.5 + 0.25 + 1.25 = 0} \qquad \text{(Checks.)}$$

Applying KVL to loop 1,

$$\begin{aligned} -5 + v_{2\Omega} + v_{8\Omega} &= -5 + (-i_1 \times 2) + (i_2 \times 8) \\ &= -5 + [-(-1.5)2] + (0.25 \times 8) \\ &= \mathbf{-5 + 3 + 2 = 0} \qquad \text{(Checks.)} \end{aligned}$$

Applying KVL to loop 2,

$$\begin{aligned} -v_{8\Omega} + v_{4\Omega} - 3 &= -(i_2 \times 8) + (i_3 \times 4) - 3 \\ &= -(0.25 \times 8) + (1.25 \times 4) - 3 \\ &= \mathbf{-2 + 5 - 3} = 0 \qquad \text{(Checks.)} \end{aligned}$$

So we now have a very high degree of confidence in the accuracy of our answer.

6. *Has the problem been solved* ***satisfactorily****? If so, present the solution; if not, then return to step 3 and continue through the process again.* This problem has been solved satisfactorily.

The current through the 8-Ω resistor is 0.25 A flowing down through the 8-Ω resistor.

**Practice Problem 1.10** Try applying this process to some of the more difficult problems at the end of the chapter.

## 1.9 Summary

1. An electric circuit consists of electrical elements connected together.
2. The International System of Units (SI) is the international measurement language, which enables engineers to communicate their results. From the seven principal units, the units of other physical quantities can be derived.

3. Current is the rate of charge flow past a given point in a given direction.

$$i = \frac{dq}{dt}$$

4. Voltage is the energy required to move 1 C of charge from a reference point (−) to another point (+).

$$v_{ab} = \frac{dw}{dq}$$

5. Power is the energy supplied or absorbed per unit time. It is also the product of voltage and current.

$$p = \frac{dw}{dt} = vi$$

6. According to the passive sign convention, power assumes a positive sign when the current enters the positive polarity of the voltage across an element.
7. An ideal voltage source produces a specific potential difference across its terminals regardless of what is connected to it. An ideal current source produces a specific current through its terminals regardless of what is connected to it.
8. Voltage and current sources can be dependent or independent. A dependent source is one whose value depends on some other circuit variable.
9. Two areas of application of the concepts covered in this chapter are the TV picture tube and electricity billing procedure.

## Review Questions

**1.1** One millivolt is one millionth of a volt.

(a) True (b) False

**1.2** The prefix *micro* stands for:

(a) $10^6$ (b) $10^3$ (c) $10^{-3}$ (d) $10^{-6}$

**1.3** The voltage 2,000,000 V can be expressed in powers of 10 as:

(a) 2 mV (b) 2 kV (c) 2 MV (d) 2 GV

**1.4** A charge of 2 C flowing past a given point each second is a current of 2 A.

(a) True (b) False

**1.5** The unit of current is:

(a) coulomb (b) ampere
(c) volt (d) joule

**1.6** Voltage is measured in:

(a) watts (b) amperes
(c) volts (d) joules per second

**1.7** A 4-A current charging a dielectric material will accumulate a charge of 24 C after 6 s.

(a) True (b) False

**1.8** The voltage across a 1.1-kW toaster that produces a current of 10 A is:

(a) 11 kV (b) 1100 V (c) 110 V (d) 11 V

**1.9** Which of these is not an electrical quantity?

(a) charge (b) time (c) voltage
(d) current (e) power

**1.10** The dependent source in Fig. 1.22 is:

(a) voltage-controlled current source
(b) voltage-controlled voltage source
(c) current-controlled voltage source
(d) current-controlled current source

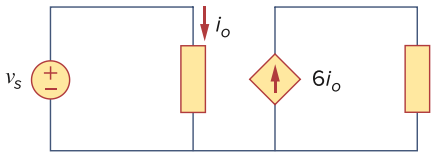



**Figure 1.22**
For Review Question 1.10.

*Answers: 1.1b, 1.2d, 1.3c, 1.4a, 1.5b, 1.6c, 1.7a, 1.8c, 1.9b, 1.10d.*

# Problems

## Section 1.3 Charge and Current

**1.1** How much charge is represented by these number of electrons?

(a) $6.482 \times 10^{17}$

(b) $1.24 \times 10^{18}$

(c) $2.46 \times 10^{19}$

(d) $1.628 \times 10^{20}$

**1.2** Determine the current flowing through an element if the charge flow is given by

(a) $q(t) = (3)$ mC

(b) $q(t) = (4t^2 + 20t - 4)$ C

(c) $q(t) = (15e^{-3t} - 2e^{-18t})$ $n$C

(d) $q(t) = 5t^2(3t^3 + 4)$ pC

(e) $q(t) = 2e^{-3t} \sin(20\pi t)$ $\mu$C

**1.3** Find the charge $q(t)$ flowing through a device if the current is:

(a) $i(t) = 3$ A, $q(0) = 1$ C

(b) $i(t) = (2t + 5)$ mA, $q(0) = 0$

(c) $i(t) = 20\cos(10t + \pi/6)$ $\mu$A, $q(0) = 2$ $\mu$C

(d) $i(t) = 10e^{-30t} \sin 40t$ A, $q(0) = 0$

**1.4** A total charge of 300 C flows past a given cross section of a conductor in 30 seconds. What is the value of the current?

**1.5** Determine the total charge transferred over the time interval of $0 \leq t \leq 10$ s when $i(t) = \frac{1}{2}t$ A.

**1.6** The charge entering a certain element is shown in Fig. 1.23. Find the current at:

(a) $t = 1$ ms (b) $t = 6$ ms (c) $t = 10$ ms

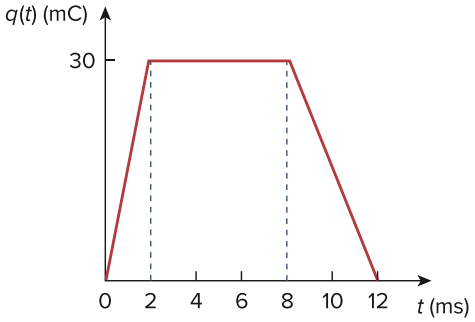



**Figure 1.23**
For Prob. 1.6.

**1.7** The charge flowing in a wire is plotted in Fig. 1.24. Sketch the corresponding current.

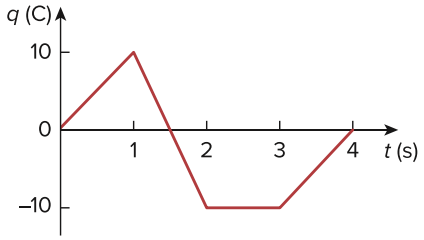



**Figure 1.24**
For Prob. 1.7.

**1.8** The current flowing past a point in a device is shown in Fig. 1.25. Calculate the total charge through the point.

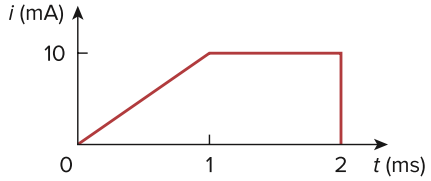



**Figure 1.25**
For Prob. 1.8.

**1.9** The current through an element is shown in Fig. 1.26. Determine the total charge that passed through the element at:

(a) $t = 1$ s (b) $t = 3$ s (c) $t = 5$ s

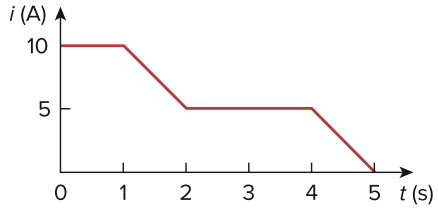



**Figure 1.26**
For Prob. 1.9.

## Sections 1.4 and 1.5 Voltage, Power, and Energy

**1.10** A lightning bolt with 10 kA strikes an object for 15 $\mu$s. How much charge is deposited on the object?

**1.11** A rechargeable flashlight battery is capable of delivering 90 mA for about 12 h. How much charge can it release at that rate? If its terminal voltage is 1.5 V, how much energy can the battery deliver?

**1.12** If the current flowing through an element is given by

$$i(t) = \begin{cases} 3t\text{A}, & 0 \leqq t < 6 \text{ s} \\ 18\text{A}, & 6 \leqq t < 10 \text{ s} \\ -12\text{A}, & 10 \leqq t < 15 \text{ s} \\ 0, & t \geqq 15 \text{ s} \end{cases}$$

Plot the charge stored in the element over $0 < t < 20$ s.

**1.13** The charge entering the positive terminal of an element is

$$q = 5 \sin 4\pi t \text{ mC}$$

while the voltage across the element (plus to minus) is

$$v = 3 \cos 4\pi t \text{ V}$$

(a) Find the power delivered to the element at $t = 0.3$ s.

(b) Calculate the energy delivered to the element between 0 and 0.6 s.

**1.14** The voltage $v(t)$ across a device and the current $i(t)$ through it are

$$v(t) = 20 \sin (4t) \text{ V and } i(t) = 10(1 + e^{-2t}) \text{ mA}$$

Calculate:

(a) the total charge in the device at $t = 1$ *s*, $q(0) = 0$.

(b) the power consumed by the device at $t = 1$ s.

**1.15** The current entering the positive terminal of a device is $i(t) = 6e^{-2t}$ mA and the voltage across the device is $v(t) = 10di/dt$V.

(a) Find the charge delivered to the device between $t = 0$ and $t = 2$ s.

(b) Calculate the power absorbed.

(c) Determine the energy absorbed in 3 s.

## Section 1.6 Circuit Elements

**1.16** Figure 1.27 shows the current through and the voltage across an element.

(a) Sketch the power delivered to the element for $t > 0$.

(b) Fnd the total energy absorbed by the element for the period of $0 < t < 4$s.

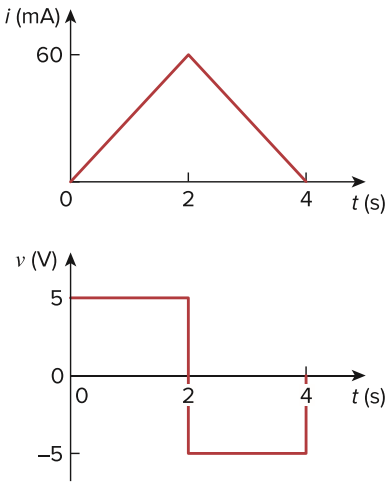



**Figure 1.27**
For Prob. 1.16.

**1.17** Figure 1.28 shows a circuit with four elements, $p_1 = 60$ W absorbed, $p_3 = -145$ W absorbed, and $p_4 = 75$ W absorbed. How many watts does element 2 absorb?

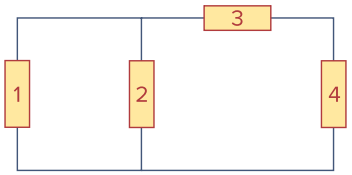



**Figure 1.28**
For Prob. 1.17.

**1.18** Find the power absorbed by each of the elements in Fig. 1.29.

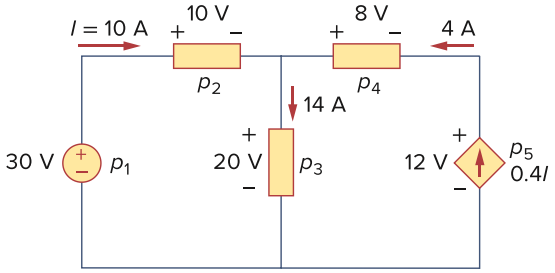



**Figure 1.29**
For Prob. 1.18.

**1.19** Find $I$ and the power absorbed by each element in the network of Fig. 1.30.

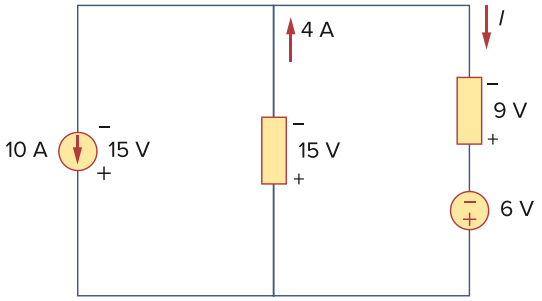



**Figure 1.30**
For Prob. 1.19.

**1.20** Find $V_o$ and the power absorbed by each element in the circuit of Fig. 1.31.

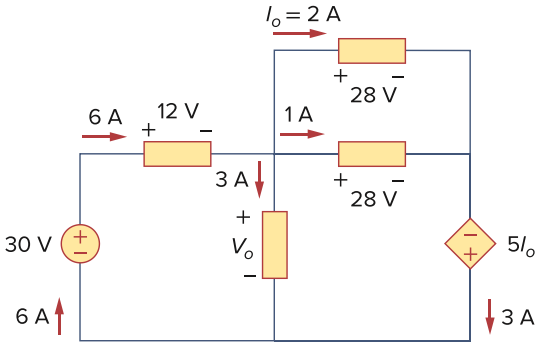



**Figure 1.31**
For Prob. 1.20.

## Section 1.7 Applications

**1.21** A 60-W incandescent bulb operates at 120 V. How many electrons and coulombs flow through the bulb in one day?

**1.22** A lightning bolt strikes an airplane with 40 kA for 1.7 ms. How many coulombs of charge are deposited on the plane?

**1.23** A 1.8-kW electric heater takes 15 min to boil a quantity of water. If this is done once a day and power costs 10 cents/kWh, what is the cost of its operation for 30 days?

**1.24** A utility company charges 8.2 cents/kWh. If a consumer operates a 60-W light bulb continuously for one day, how much is the consumer charged?

**1.25** A 1.2-kW toaster takes roughly 4 minutes to heat four slices of bread. Find the cost of operating the toaster twice per day for 2 weeks (14 days). Assume energy costs 9 cents/kWh.

**1.26** A cell phone battery is rated at 3.85 V and can store 10.78 watt-hours of energy.

(a) How much average current can it deliver over a period of 3 hours if it is fully discharged at the end of that time?

(b) How much average power is delivered in part (a)?

(c) What is the ampere-hour rating of the battery?

**1.27** A constant current of 3 A for 4 hours is required to charge an automotive battery. If the terminal voltage is $10 + t/2$V, where $t$ is in hours,

(a) how much charge is transported as a result of the charging?

(b) how much energy is expended?

(c) how much does the charging cost? Assume electricity costs 9 cents/kWh.

**1.28** A 150-W incandescent outdoor lamp is connected to a 120-V source and is left burning continuously for an average of 12 hours per day. Determine:

(a) the current through the lamp when it is lit.

(b) the cost of operating the light for one non-leap year if electricity costs 9.5 cents per kWh.

**1.29** An electric stove with four burners and an oven is used in preparing a meal as follows.

Burner 1: 20 minutes Burner 2: 40 minutes

Burner 3: 15 minutes Burner 4: 45 minutes

Oven: 30 minutes

If each burner is rated at 1.2 kW and the oven at 1.8 kW, and electricity costs 12 cents per kWh, calculate the cost of electricity used in preparing the meal.

**1.30** Reliant Energy (the electric company in Houston, Texas) charges customers as follows:

Monthly charge $6

First 250 kWh @ $0.02/kWh

All additional kWh @ $0.07/kWh

If a customer uses 2,436 kWh in one month, how much will Reliant Energy charge?

**1.31** In a household, a business is run for an average of 6 h/day. The total power consumed by the computer and its printer is 230 W. In addition, a 75-W light runs during the same 6 h. If their utility charges 11.75 cents per kWh, how much do the owners pay every 30 days?

## Comprehensive Problems

**1.32** A telephone wire has a current of $20\mu$A flowing through it. How long does it take for a charge of 15 C to pass through the wire?

**1.33** A lightning bolt carried a current of 2 kA and lasted for 3 ms. How many coulombs of charge were contained in the lightning bolt?

**1.34** Figure 1.32 shows the power consumption of a certain household in 1 day. Calculate:

(a) the total energy consumed in kWh,

(b) the average power over the total 24 hour period.

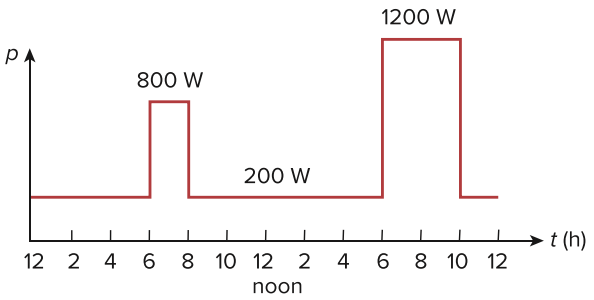



**Figure 1.32**
For Prob. 1.34.

**1.35** The graph in Fig. 1.33 represents the power drawn by an industrial plant between 8:00 and 8:30 a.m. Calculate the total energy in MWh consumed by the plant.

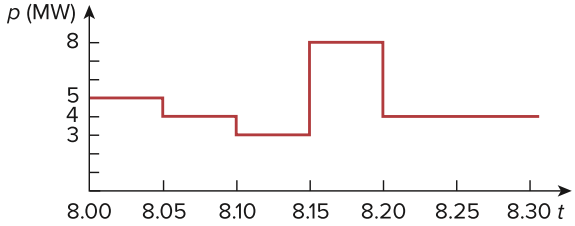



**Figure 1.33**
For Prob. 1.35.

**1.36** A battery can be rated in ampere-hours (Ah) or watt hours (Wh). The ampere hours can be obtained from the watt hours by dividing watt hours by a nominal voltage of 12 V. If an automobile battery is rated at 20 Ah:

(a) What is the maximum current that can be supplied for 15 minutes?

(b) How many days will it last if it is discharged at a rate of 2 mA?

**1.37** A total of 2 MJ are delivered to an automobile battery (assume 12 V) giving it an additional charge. How much is that additional charge? Express your answer in ampere-hours.

**1.38** How much energy does a 10-hp motor deliver in 30 minutes? Assume that 1 horsepower = 746 W.

**1.39** A 600-W TV receiver is turned on for 4 h with nobody watching it. If electricity costs 10 cents/kWh, how much money is wasted?

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chapter

# 2

# Basic Laws

*There are too many people praying for mountains of difficulty to be removed, when what they really need is the courage to climb them!*

—Unknown

## Enhancing Your Skills and Your Career

**ABET EC 2000 criteria (3.b), *"an ability to design and conduct experiments, as well as to analyze and interpret data."***

Engineers must be able to design and conduct experiments, as well as analyze and interpret data. Most students have spent many hours performing experiments in high school and in college. During this time, you have been asked to analyze the data and to interpret the data. Therefore, you should already be skilled in these two activities. My recommendation is that, in the process of performing experiments in the future, you spend more time in analyzing and interpreting the data in the context of the experiment. What does this mean?

If you are looking at a plot of voltage versus resistance or current versus resistance or power versus resistance, what do you actually see? Does the curve make sense? Does it agree with what the theory tells you? Does it differ from expectation, and, if so, why? Clearly, practice with analyzing and interpreting data will enhance this skill.

Since most, if not all, the experiments you are required to do as a student involve little or no practice in designing the experiment, how can you develop and enhance this skill?

Actually, developing this skill under this constraint is not as difficult as it seems. What you need to do is to take the experiment and analyze it. Just break it down into its simplest parts, reconstruct it trying to understand why each element is there, and finally, determine what the author of the experiment is trying to teach you. Even though it may not always seem so, every experiment you do was designed by someone who was sincerely motivated to teach you something.

## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Know and understand the voltage current relationship of resistors (Ohm's law).
2. Understand the basic structure of electrical circuits, essentially nodes, loops, and branches.
3. Understand Kirchhoff's voltage and current laws and their importance in analyzing electrical circuits.
4. Understand series resistances and voltage division, and parallel resistances and current division.
5. Know how to convert delta-connected circuits to wye-connected circuits and how to convert wye-connected circuits to delta-connected circuits.

## 2.1 Introduction

Chapter 1 introduced basic concepts such as current, voltage, and power in an electric circuit. To actually determine the values of these variables in a given circuit requires that we understand some fundamental laws that govern electric circuits. These laws, known as Ohm's law and Kirchhoff's laws, form the foundation upon which electric circuit analysis is built.

In this chapter, in addition to these laws, we shall discuss some techniques commonly applied in circuit design and analysis. These techniques include combining resistors in series or parallel, voltage division, current division, and delta-to-wye and wye-to-delta transformations. The application of these laws and techniques will be restricted to resistive circuits in this chapter. We will finally apply the laws and techniques to real-life problems of electrical lighting and the design of dc meters.

## 2.2 Ohm's Law

Materials in general have a characteristic behavior of resisting the flow of electric charge. This physical property, or ability to resist current, is known as *resistance* and is represented by the symbol $R$. The resistance of any material with a uniform cross-sectional area $A$ depends on $A$ and its length $\ell$, as shown in Fig. 2.1(a). We can represent resistance (as measured in the laboratory), in mathematical form,

$$R = \rho \frac{\ell}{A} \tag{2.1}$$

where $\rho$ is known as the *resistivity* of the material in ohm-meters. Good conductors, such as copper and aluminum, have low resistivities, while insulators, such as mica and paper, have high resistivities. Table 2.1 presents the values of $\rho$ for some common materials and shows which materials are used for conductors, insulators, and semiconductors.

The circuit element used to model the current-resisting behavior of a material is the *resistor*. For the purpose of constructing circuits, resistors are

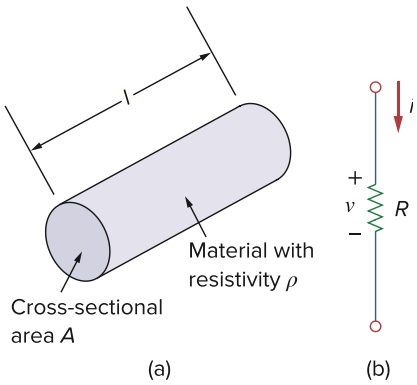


**Figure 2.1**
(a) Resistor, (b) Circuit symbol for resistance.

**TABLE 2.1**

Resistivities of common materials.

| Material | Resistivity (Ω•m) | Usage |
|---|---|---|
| Silver | $1.64 \times 10^{-8}$ | Conductor |
| Copper | $1.72 \times 10^{-8}$ | Conductor |
| Aluminum | $2.8 \times 10^{-8}$ | Conductor |
| Gold | $2.45 \times 10^{-8}$ | Conductor |
| Carbon | $4 \times 10^{-5}$ | Semiconductor |
| Germanium | $47 \times 10^{-2}$ | Semiconductor |
| Silicon | $6.4 \times 10^{2}$ | Semiconductor |
| Paper | $10^{10}$ | Insulator |
| Mica | $5 \times 10^{11}$ | Insulator |
| Glass | $10^{12}$ | Insulator |
| Teflon | $3 \times 10^{12}$ | Insulator |

usually made from metallic alloys and carbon compounds. The circuit symbol for the resistor is shown in Fig. 2.1(b), where $R$ stands for the resistance of the resistor. The resistor is the simplest passive element.

Georg Simon Ohm (1787–1854), a German ph ysicist, is credited with finding the relationship between current and voltage for a resistor. This relationship is known as *Ohm's law*.

Ohm's law states that the voltage $v$ across a resistor is directly proportional to the current $i$ flowing through the resistor.

That is,

$$v \propto i \tag{2.2}$$

Ohm defined the constant of proportionality for a resistor to be the resistance, $R$. (The resistance is a material property which can change if the internal or external conditions of the element are altered, e.g., if there are changes in the temperature.) Thus, Eq. (2.2) becomes

$$\boxed{v = iR} \tag{2.3}$$

## Historical

**Georg Simon Ohm** (1787–1854), a German physicist, in 1826 experimentally determined the most basic law relating voltage and cur - rent for a resistor. Ohm's work was initially denied by critics.

Born of humble beginnings in Erlangen, Bavaria, Ohm threw himself into electrical research. His efforts resulted in his famous law. He was awarded the Copley Medal in 1841 by the Royal Society of London. In 1849, he was given the Professor of Physics chair by the University of Munich. To honor him, the unit of resistance was named the ohm.

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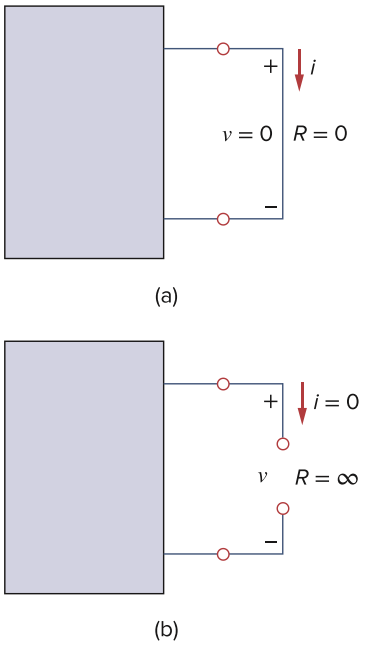


**Figure 2.2**
(a) Short circuit ($R = 0$), (b) Open circuit ($R = \infty$).

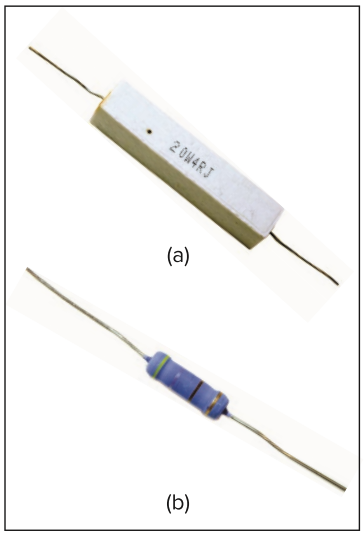


**Figure 2.3**
Fixed resistors: (a) wirewound type, (b) carbon film type.
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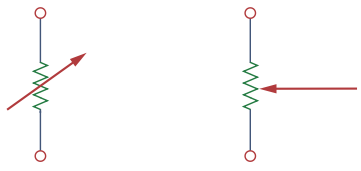


**Figure 2.4**
Circuit symbol for: (a) a variable resistor in general, (b) a potentiometer.

which is the mathematical form of Ohm's law. $R$ in Eq. (2.3) is measured in the unit of ohms, designated Ω. Thus,

> The *resistance R* of an element denotes its ability to resist the flow of electric current; it is measured in ohms (Ω).

We may deduce from Eq. (2.3) that

$$R = \frac{v}{i} \tag{2.4}$$

so that

$$1\ \Omega = 1\ \text{V/A}$$

To apply Ohm's law as stated in Eq. (2.3), we must pay careful attention to the current direction and voltage polarity. The direction of current $i$ and the polarity of voltage $v$ must conform with the passive sign convention, as shown in Fig. 2.1(b). This implies that current flows from a higher potential to a lower potential in order for $v = iR$. If current flows from a lower potential to a higher potential, $v = -iR$.

Since the value of $R$ can range from zero to infinity, it is important that we consider the two extreme possible values of $R$. An element with $R = 0$ is called a *short circuit*, as shown in Fig. 2.2(a). For a short circuit,

$$v = iR = 0 \tag{2.5}$$

showing that the voltage is zero but the current could be anything. In practice, a short circuit is usually a connecting wire assumed to be a perfect conductor. Thus,

> A **short circuit** is a circuit element with resistance approaching zero.

Similarly, an element with $R = \infty$ is known as an *open circuit*, as shown in Fig. 2.2(b). For an open circuit,

$$i = \lim_{R \to \infty} \frac{v}{R} = 0 \tag{2.6}$$

indicating that the current is zero though the voltage could be anything. Thus,

> An **open circuit** is a circuit element with resistance approaching infinity.

A resistor is either fixed or variable. Most resistors are of the fixed type, meaning their resistance remains constant. The two common types of fixed resistors (wirewound and composition) are shown in Fig. 2.3. The composition resistors are used when large resistance is needed. The circuit symbol in Fig. 2.1(b) is for a fixed resistor. Variable resistors have adjustable resistance. The symbol for a variable resistor is shown in Fig. 2.4(a). A common variable resistor is known as a *potentiometer* or *pot* for short, with the symbol shown in Fig. 2.4(b). The pot is a three-terminal element with a sliding contact or wiper. By sliding the wiper, the resistances between the wiper terminal and the fixed terminals vary. Like fixed resistors, variable resistors can be of either wire wound or composition type, as shown in Fig. 2.5. Although resistors like those in Figs. 2.3 and 2.5 are used in circuit designs, today most circuit components including resistors are either surface mounted or integrated, as typically shown in Fig. 2.6.

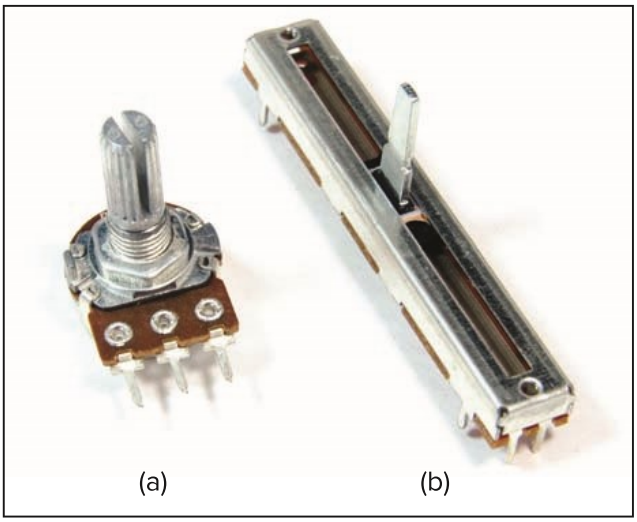


**Figure 2.5**
Variable resistors: (a) composition type, (b) slider pot.
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It should be pointed out that not all resistors obey Ohm's law. A resistor that obeys Ohm's law is known as a *linear* resistor. It has a constant resistance and thus its current-voltage characteristic is as illustrated in Fig. 2.7(a): Its *i*-*v* graph is a straight line passing through the origin. A *nonlinear* resistor does not obey Ohm's law. Its resistance varies with current and its *i*-*v* characteristic is typically shown in Fig. 2.7(b). Examples of devices with nonlinear resistance are the light bulb and the diode. Although all practical resistors may exhibit nonlinear behavior under certain conditions, we will assume in this book that all elements actually designated as resistors are linear.

A useful quantity in circuit analysis is the reciprocal of resistance $R$, known as *conductance* and denoted by $G$:

$$G = \frac{1}{R} = \frac{i}{v} \tag{2.7}$$

The conductance is a measure of how well an element will conduct electric current. The unit of conductance is the *mho* (ohm spelled backward) or reciprocal ohm, with symbol ℧, the inverted omega. Although engineers often use the mho, in this book we prefer to use the siemens (S), the SI unit of conductance:

$$1\ \text{S} = 1\ ℧ = 1\ \text{A/V} \tag{2.8}$$

Thus,

**Conductance** is the ability of an element to conduct electric current; it is measured in mhos (℧) or siemens (S).

The same resistance can be expressed in ohms or siemens. For example, 10 Ω is the same as 0.1 S. From Eq. (2.7), we may write

$$i = Gv \tag{2.9}$$

The power dissipated by a resistor can be expressed in terms of $R$. Using Eqs. (1.7) and (2.3),

$$p = vi = i^2R = \frac{v^2}{R} \tag{2.10}$$

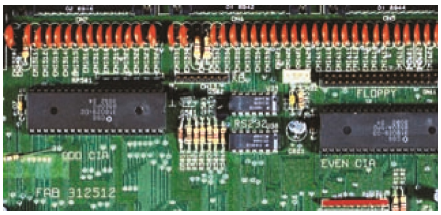


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**Figure 2.6**
Resistors in an integrated circuit board.

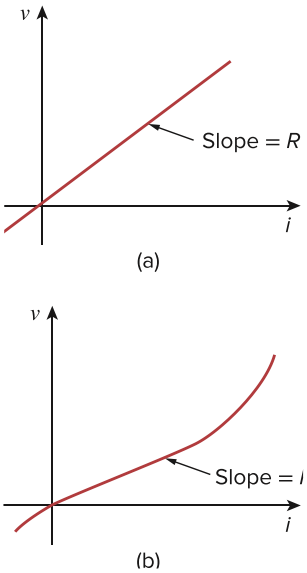


**Figure 2.7**
The *i*-*v* characteristic of: (a) a linear resistor, (b) a nonlinear resistor.

The power dissipated by a resistor may also be expressed in terms of $G$ as

$$p = vi = v^2G = \frac{i^2}{G} \tag{2.11}$$

We should note two things from Eqs. (2.10) and (2.11):

1. The power dissipated in a resistor is a nonlinear function of either current or voltage.
2. Since $R$ and $G$ are positive quantities, the power dissipated in a resistor is always positive. Thus, a resistor always absorbs power from the circuit. This confirms the idea that a resistor is a passive element, incapable of generating energy.

## Example 2.1

An electric iron draws 2 A at 120 V. Find its resistance.

**Solution:**
From Ohm's law,

$$R = \frac{v}{i} = \frac{120}{2} = 60\ \Omega$$

## Practice Problem 2.1

The essential component of a toaster is an electrical element (a resistor) that converts electrical energy to heat energy. How much current is drawn by a toaster with resistance 15 Ω at 110 V?

**Answer:** 7.333 A.

## Example 2.2

In the circuit shown in Fig. 2.8, calculate the current $i$, the conductance $G$, and the power $p$.

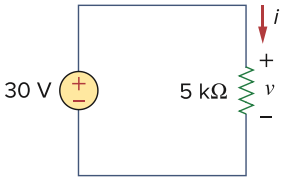


**Figure 2.8**
For Example 2.2.

**Solution:**
The voltage across the resistor is the same as the source voltage (30 V) because the resistor and the voltage source are connected to the same pair of terminals. Hence, the current is

$$i = \frac{v}{R} = \frac{30}{5 \times 10^3} = 6\ \text{mA}$$

The conductance is

$$G = \frac{1}{R} = \frac{1}{5 \times 10^3} = 0.2\ \text{mS}$$

We can calculate the power in various ways using either Eqs. (1.7), (2.10), or (2.11).

$$p = vi = 30(6 \times 10^{-3}) = 180\ \text{mW}$$

or

$$p = i^2R = (6 \times 10^{-3})^2 5 \times 10^3 = 180\ \text{mW}$$

or

$$p = v^2G = (30)^2 0.2 \times 10^{-3} = 180\ \text{mW}$$

### Practice Problem 2.2

For the circuit shown in Fig. 2.9, calculatethe voltage $v$, the conductance $G$, and the power $p$.

**Answer:** 30 V, 100 $\mu$S, 90 mW.

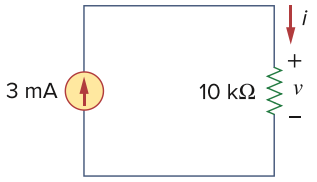


**Figure 2.9**
For Practice Prob. 2.2

### Example 2.3

A voltage source of 20 sin $\pi t$ V is connected across a 5-kΩ resistor. Find the current through the resistor and the power dissipated.

**Solution:**

$$i = \frac{v}{R} = \frac{20 \sin \pi t}{5 \times 10^3} = 4 \sin \pi t \text{ mA}$$

Hence,

$$p = vi = 80 \sin^2 \pi t \text{ mW}$$

### Practice Problem 2.3

A resistor absorbs an instantaneous power of 30 cos $^2$ $t$ mW when connected to a voltage source $v = 15 \cos t$ V. Find $i$ and $R$.

**Answer:** 2 cos $t$ mA, 7.5 kΩ.

## 2.3 †Nodes, Branches, and Loops

Since the elements of an electric circuit can be interconnected in se veral ways, we need to understand some basic concepts of netwrk topology. To differentiate between a circuit and a network, we may regard a network as an interconnection of elements or devices, whereas a circuit is a netw ork providing one or more closed paths. The con vention, when addressing network topology, is to use the w ord netw ork rather than circuit. We do this even though the word network and circuit mean the same thing when used in this context. In network topology, we study the properties relating to the placement of elements in the network and the geometric configuration of the network. Such elements include branches, nodes, and loops.

A **branch** represents a single element such as a voltage source or a resistor.

In other w ords, a branch represents an y two-terminal element. The circuit in Fig. 2.10 has five branches, namely, the 10-V voltage source, the 2-A current source, and the three resistors.

A **node** is the point of connection between two or more branches.

A node is usually indicated by a dot in a circuit. If a short circuit (a connecting wire) connects tw o nodes, the tw o nodes constitute a single node. The circuit in Fig. 2.10 has three nodes $a$, $b$, and $c$. Notice that

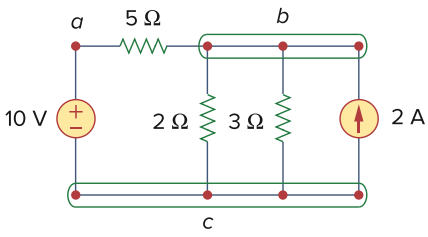


**Figure 2.10**
Nodes, branches, and loops.

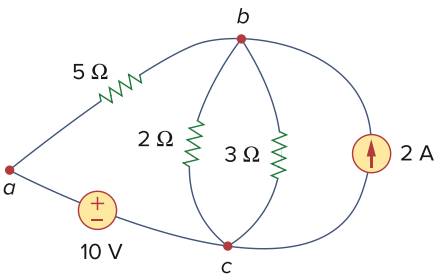


**Figure 2.11**
The three-node circuit of Fig. 2.10 is redrawn.

the three points that form node $b$ are connected by perfectly conducting wires and therefore constitute a single point. The same is true of the four points forming node $c$. We demonstrate that the circuit in Fig. 2.10 has only three nodes by redrawing the circuit in Fig. 2.11. The two circuits in Figs. 2.10 and 2.11 are identical. However, for the sake of clarity, nodes $b$ and $c$ are spread out with perfect conductors as in Fig. 2.10.

A loop is any closed path in a circuit.

A loop is a closed path formed by starting at a node, passing through a set of nodes, and returning to the starting node without passing through any node more than once. A loop is said to be *independent* if it contains at least one branch which is not a part of any other independent loop. Independent loops or paths result in independent sets of equations.

It is possible to form an independent set of loops where one of the loops does not contain such a branch. In Fig. 2.11, $abca$ with the 2 Ω resistor is independent. A second loop with the 3 Ω resistor and the current source is independent. The third loop could be the one with the 2Ω resistor in parallel with the 3Ω resistor. This does form an independent set of loops.

A network with $b$ branches, $n$ nodes, and $l$ independent loops will satisfy the fundamental theorem of network topology:

$$b = l + n - 1 \tag{2.12}$$

As the next two definitions show, circuit topology is of great value to the study of voltages and currents in an electric circuit.

Two or more elements are in **series** if they exclusively share a single node and consequently carry the same current.
Two or more elements are in **parallel** if they are connected to the same two nodes and consequently have the same voltage across them.

Elements are in series when they are chain-connected or connected sequentially, end to end. For example, two elements are in series if they share one common node and no other element is connected to that common node. Elements in parallel are connected to the same pair of terminals. Elements may be connected in a way that they are neither in series nor in parallel. In the circuit shown in Fig. 2.10, the voltage source and the 5-Ω resistor are in series because the same current will flow through them. The 2-Ω resistor, the 3-Ω resistor, and the current source are in parallel because they are connected to the same two nodes $b$ and $c$ and consequently have the same voltage across them. The 5-Ω and 2-Ω resistors are neither in series nor in parallel with each other.

## Example 2.4

Determine the number of branches and nodes in the circuit shown in Fig. 2.12. Identify which elements are in series and which are in parallel.

**Solution:**
Since there are four elements in the circuit, the circuit has four branches: 10 V, 5Ω, 6Ω, and 2 A. The circuit has three nodes as identified in Fig. 2.13. The 5-Ω resistor is in series with the 10-V voltage source because the same current would flow in both. The 6-Ω resistor is in parallel with the 2-A current source because both are connected to the same nodes 2 and 3.

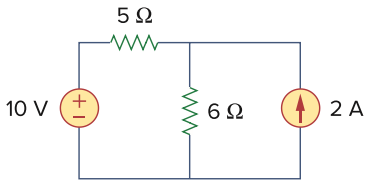


**Figure 2.12**
For Example 2.4.

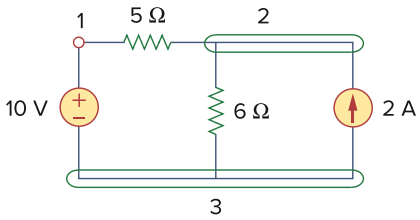


**Figure 2.13**
The three nodes in the circuit of Fig. 2.12.

## Practice Problem 2.4

How many branches and nodes does the circuit in Fig. 2.14 have? Identify the elements that are in series and in parallel.

**Answer:** Five branches and three nodes are identified in Fig. 2.15. The 1-Ω and 2-Ω resistors are in parallel. The 4-Ω resistor and 10-V source are also in parallel.

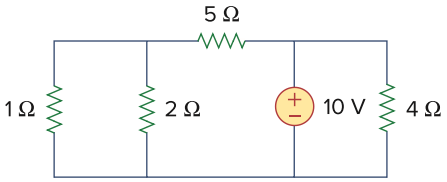


**Figure 2.14**
For Practice Prob. 2.4.

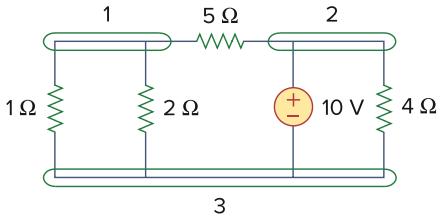


**Figure 2.15**
Answer for Practice Prob. 2.4.

## 2.4 Kirchhoff's Laws

Ohm's law by itself is not sufficient to analyze circuits. However, when it is coupled with Kirchhoff's two laws, we have a sufficient, powerful set of tools for analyzing a large variety of electric circuits. Kirchhoff's laws were first introduced in 1847 by the German physicist Gustav Robert Kirchhoff (1824–1887). These laws are formally known as Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL).

Kirchhoff's first law is based on the law of conservation of charge, which requires that the algebraic sum of charges within a system cannot change.

Kirchhoff's current law (KCL) states that the algebraic sum of currents entering a node (or a closed boundary) is zero.

Mathematically, KCL implies that

$$\sum_{n=1}^{N} i_n = 0 \tag{2.13}$$

where $N$ is the number of branches connected to the node and $i_n$ is the $n$th current entering (or leaving) the node. By this law, currents entering a node may be regarded as positive, while currents leaving the node may be taken as negative or vice versa.

## Historical

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**Gustav Robert Kirchhoff** (1824–1887), a German physicist, stated two basic laws in 1847 concerning the relationship between the currents and voltages in an electrical network. Kirchhoff's laws, along with Ohm's law, form the basis of circuit theory.

Born the son of a lawyer in Konigsberg, East Prussia, Kirchhoff entered the University of Konigsberg at age 18 and later became a lecturer in Berlin. His collaborative work in spectroscopy with German chemist Robert Bunsen led to the discovery of cesium in 1860 and rubidium in 1861. Kirchhoff was also credited with the Kirchhoff law of radiation. Thus, Kirchhoff is famous among engineers, chemists, and physicists.

To prove KCL, assume a set of currents $i_k(t)$, $k = 1, 2, \ldots$, flow into a node. The algebraic sum of currents at the node is

$$i_T(t) = i_1(t) + i_2(t) + i_3(t) + \cdots \tag{2.14}$$

Integrating both sides of Eq. (2.14) gives

$$q_T(t) = q_1(t) + q_2(t) + q_3(t) + \cdots \tag{2.15}$$

where $q_k(t) = \int i_k(t)\,dt$ and $q_T(t) = \int i_T(t)\,dt$. But the law of conservation of electric charge requires that the algebraic sum of electric charges at the node must not change; that is, the node stores no net charge. Thus, $q_T(t) = 0 \rightarrow i_T(t) = 0$, confirming the validity of KCL.

Consider the node in Fig. 2.16. Applying KCL gives

$$i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0 \tag{2.16}$$

since currents $i_1$, $i_3$, and $i_4$ are entering the node, while currents $i_2$ and $i_5$ are leaving it. By rearranging the terms, we get

$$i_1 + i_3 + i_4 = i_2 + i_5 \tag{2.17}$$

Equation (2.17) is an alternative form of KCL:

> The sum of the currents entering a node is equal to the sum of the currents leaving the node.

Note that KCL also applies to a closed boundary. This may be regarded as a generalized case, because a node may be regarded as a closed surface shrunk to a point. In two dimensions, a closed boundary is the same as a closed path. As typically illustrated in the circuit of Fig. 2.17, the total current entering the closed surface is equal to the total current leaving the surface.

A simple application of KCL is combining current sources in parallel. The combined current is the algebraic sum of the current supplied by the individual sources. For example, the current sources shown in

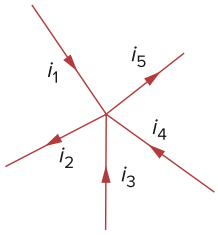



**Figure 2.16**
Currents at a node illustrating KCL.

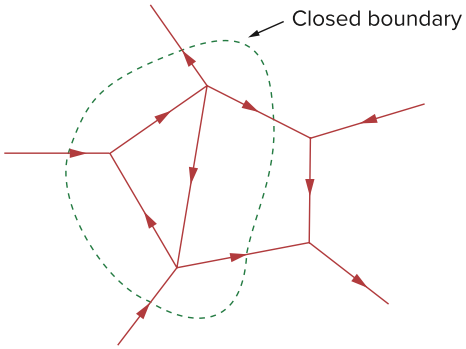



**Figure 2.17**
Applying KCL to a closed boundary.

Two sources (or circuits in general) are said to be equivalent if they have the same *i-v* relationship at a pair of terminals.

Fig. 2.18(a) can be combined as in Fig. 2.18(b).The combined or equivalent current source can be found by applying KCL to node $a$.

$$I_T + I_2 = I_1 + I_3$$

or

$$I_T = I_1 - I_2 + I_3 \tag{2.18}$$

A circuit cannot contain two different currents, $I_1$ and $I_2$, in series, unless $I_1 = I_2$; otherwise KCL will be violated.

Kirchhoff's second law is based on the principle of conservation of energy:

**Kirchhoff's voltage law (KVL)** states that the algebraic sum of all voltages around a closed path (or loop) is zero.

Expressed mathematically, KVL states that

$$\sum_{m=1}^{M} v_m = 0 \tag{2.19}$$

where $M$ is the number of voltages in the loop (or the number of branches in the loop) and $v_m$ is the $m$th voltage.

To illustrate KVL, consider the circuit in Fig. 2.19. The sign on each voltage is the polarity of the terminal encountered first as we travel around the loop. We can start with any branch and go around the loop either clockwise or counterclockwise. Suppose we start with the voltage source and go clockwise around the loop as shown; then voltages would be $-v_1$, $+v_2$, $+v_3$, $+v_4$, and $-v_5$, in that order. For example, as we reach branch 3, the positive terminal is met first; hence, we have $+v_3$. For branch 4, we reach the negative terminal first; hence, $-v_4$. Thus, KVL yields

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0 \tag{2.20}$$

Rearranging terms gives

$$v_2 + v_3 + v_5 = v_1 + v_4 \tag{2.21}$$

which may be interpreted as

$$\text{Sum of voltage drops} = \text{Sum of voltage rises} \tag{2.22}$$

This is an alternative form of KVL. Notice that if we had traveled counterclockwise, the result would have been $+v_1$, $-v_5$, $+v_4$, $-v_3$, and $-v_2$, which is the same as before except that the signs are reversed. Hence, Eqs. (2.20) and (2.21) remain the same.

When voltage sources are connected in series, KVL can be applied to obtain the total voltage. The combined voltage is the algebraic sum of the voltages of the individual sources. For example, for the voltage sources shown in Fig. 2.20(a), the combined or equivalent voltage source in Fig. 2.20(b) is obtained by applying KVL.

$$-V_{ab} + V_1 + V_2 - V_3 = 0$$

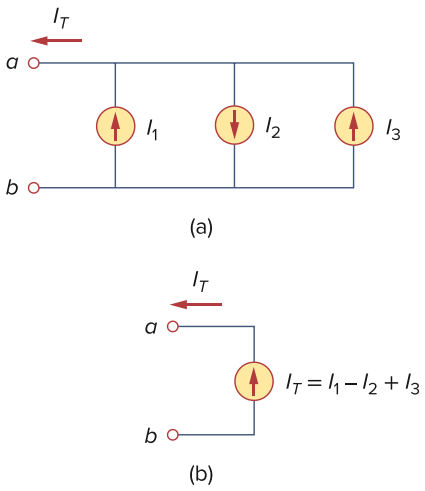



**Figure 2.18**
Current sources in parallel: (a) original circuit, (b) equivalent circuit.

KVL can be applied in two ways: by taking either a clockwise or a counterclockwise trip around the loop. Either way, the algebraic sum of voltages around the loop is zero.

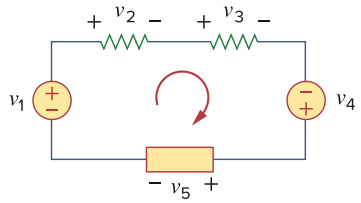



**Figure 2.19**
A single-loop circuit illustrating KVL.

or

$$V_{ab} = V_1 + V_2 - V_3 \tag{2.23}$$

To avoid violating KVL, a circuit cannot contain tw o different voltages $V_1$ and $V_2$ in parallel unless $V_1 = V_2$.

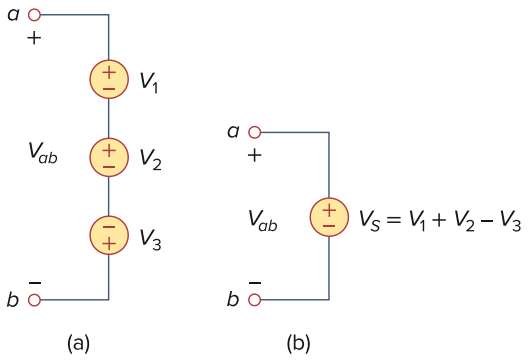



**Figure 2.20**
Voltage sources in series: (a) original circuit, (b) equivalent circuit.

## Example 2.5

For the circuit in Fig. 2.21(a), find voltages $v_1$ and $v_2$.

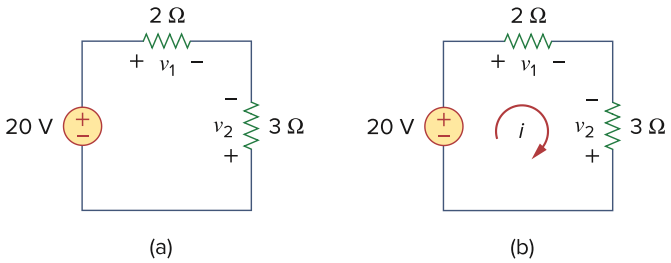



**Figure 2.21**
For Example 2.5.

**Solution:**
To find $v_1$ and $v_2$ we apply Ohm's law and Kirchhoff's voltage law. Assume that current $i$ flows through the loop as shown in Fig. 2.21(b). From Ohm's law,

$$v_1 = 2i, \qquad v_2 = -3i \tag{2.5.1}$$

Applying KVL around the loop gives

$$-20 + v_1 - v_2 = 0 \tag{2.5.2}$$

Substituting Eq. (2.5.1) into Eq. (2.5.2), we obtain

$$-20 + 2i + 3i = 0 \quad \text{or} \quad 5i = 20 \quad \Rightarrow \quad i = 4\ \text{A}$$

Substituting $i$ in Eq. (2.5.1) finally gives

$$v_1 = 8\ \text{V}, \qquad v_2 = -12\ \text{V}$$

## Practice Problem 2.5

Find $v_1$ and $v_2$ in the circuit of Fig. 2.22.

**Answer:** 16 V, −8 V.

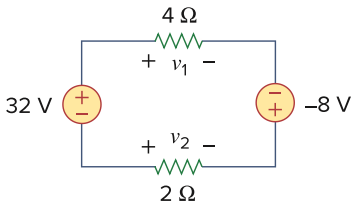



**Figure 2.22**
For Practice Prob. 2.5.

## Example 2.6

Determine $v_o$ and $i$ in the circuit shown in Fig. 2.23(a).

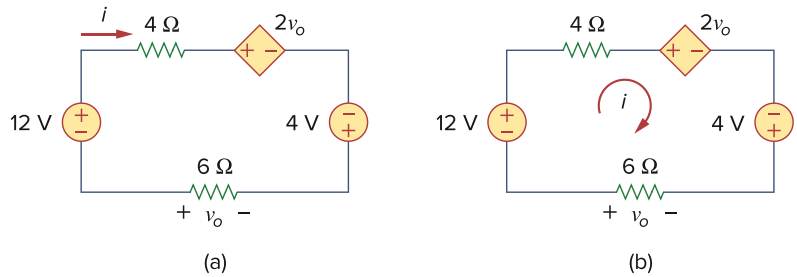



**Figure 2.23**
For Example 2.6.

**Solution:**
We apply KVL around the loop as shown in Fig. 2.23(b). The result is

$$-12 + 4i + 2v_o - 4 + 6i = 0 \qquad \textbf{(2.6.1)}$$

Applying Ohm's law to the 6-Ω resistor gives

$$v_o = -6i \qquad \textbf{(2.6.2)}$$

Substituting Eq. (2.6.2) into Eq. (2.6.1) yields

$$-16 + 10i - 12i = 0 \quad \Rightarrow \quad i = -8 \text{ A}$$

and $v_o = 48$ V.

## Practice Problem 2.6

Find $v_x$ and $v_o$ in the circuit of Fig. 2.24.

**Answer:** 20 V, −10 V.

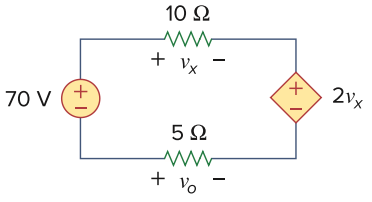



**Figure 2.24**
For Practice Prob. 2.6.

## Example 2.7

Find current $i_o$ and voltage $v_o$ in the circuit shown in Fig. 2.25.

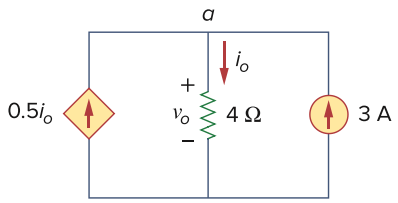



**Figure 2.25**
For Example 2.7.

**Solution:**
Applying KCL to node $a$, we obtain

$$3 + 0.5i_o = i_o \quad \Rightarrow \quad i_o = 6 \text{ A}$$

For the 4-Ω resistor, Ohm's law gives

$$v_o = 4i_o = 24 \text{ V}$$

## Practice Problem 2.7

Find $v_o$ and $i_o$ in the circuit of Fig. 2.26.

**Answer:** 20 V, 10 A.

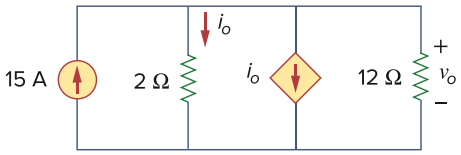



**Figure 2.26**
For Practice Prob. 2.7.

## Example 2.8

Find currents and voltages in the circuit shown in Fig. 2.27(a).

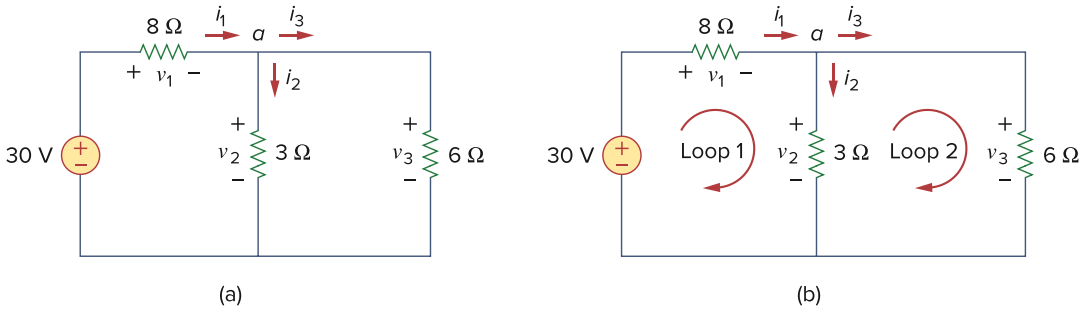



**Figure 2.27**
For Example 2.8.

**Solution:**
We apply Ohm's law and Kirchhoff's laws. By Ohm's law,

$$v_1 = 8i_1, \qquad v_2 = 3i_2, \qquad v_3 = 6i_3 \tag{2.8.1}$$

Since the voltage and current of each resistor are related by Ohm's law as shown, we are really looking for three things: $(v_1, v_2, v_3)$ or $(i_1, i_2, i_3)$. At node $a$, KCL gives

$$i_1 - i_2 - i_3 = 0 \tag{2.8.2}$$

Applying KVL to loop 1 as in Fig. 2.27(b),

$$-30 + v_1 + v_2 = 0$$

We express this in terms of $i_1$ and $i_2$ as in Eq. (2.8.1) to obtain

$$-30 + 8i_1 + 3i_2 = 0$$

or

$$i_1 = \frac{(30 - 3i_2)}{8} \tag{2.8.3}$$

Applying KVL to loop 2,

$$-v_2 + v_3 = 0 \quad \Rightarrow \quad v_3 = v_2 \tag{2.8.4}$$

as expected since the two resistors are in parallel. We express $v_1$ and $v_2$ in terms of $i_1$ and $i_2$ as in Eq. (2.8.1). Equation (2.8.4) becomes

$$6i_3 = 3i_2 \quad \Rightarrow \quad i_3 = \frac{i_2}{2} \tag{2.8.5}$$

Substituting Eqs. (2.8.3) and (2.8.5) into (2.8.2) gives

$$\frac{30 - 3i_2}{8} - i_2 - \frac{i_2}{2} = 0$$

or $i_2 = 2$ A. From the value of $i_2$, we now use Eqs. (2.8.1) to (2.8.5) to obtain

$$i_1 = 3 \text{ A}, \quad i_3 = 1 \text{ A}, \quad v_1 = 24 \text{ V}, \quad v_2 = 6 \text{ V}, \quad v_3 = 6 \text{ V}$$

### Practice Problem 2.8

Find the currents and voltages in the circuit shown in Fig. 2.28.

**Answer:** $v_1 = 6$ V, $v_2 = 4$ V, $v_3 = 10$ V, $i_1 = 3$ A, $i_2 = 500$ mA, $i_3 = 2.5$ A.

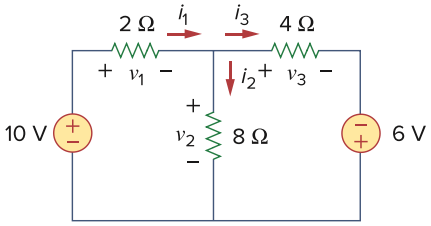



**Figure 2.28**
For Practice Prob. 2.8.

## 2.5 Series Resistors and Voltage Division

The need to combine resistors in series or in parallel occurs so frequently that it warrants special attention. The process of combining the resistors is facilitated by combining two of them at a time. With this in mind, consider the single-loop circuit of Fig. 2.29. The two resistors are in series, since the same current $i$ flows in both of them. Applying Ohm's law to each of the resistors, we obtain

$$v_1 = iR_1, \qquad v_2 = iR_2 \tag{2.24}$$

If we apply KVL to the loop (moving in the clockwise direction), we have

$$-v + v_1 + v_2 = 0 \tag{2.25}$$

Combining Eqs. (2.24) and (2.25), we get

$$v = v_1 + v_2 = i(R_1 + R_2) \tag{2.26}$$

or

$$i = \frac{v}{R_1 + R_2} \tag{2.27}$$

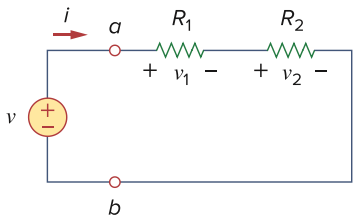



**Figure 2.29**
A single-loop circuit with two resistors in series.

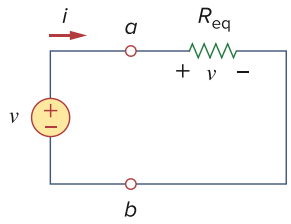



**Figure 2.30**
Equivalent circuit of the Fig. 2.29 circuit.

Notice that Eq. (2.26) can be written as

$$v = iR_{eq} \tag{2.28}$$

implying that the two resistors can be replaced by an equivalent resistor $R_{eq}$; that is,

$$R_{eq} = R_1 + R_2 \tag{2.29}$$

Thus, Fig. 2.29 can be replaced by the equivalent circuit in Fig. 2.30.The two circuits in Figs. 2.29 and 2.30 are equivalent because they exhibit the same voltage-current relationships at the terminals *a-b*. An equivalent circuit such as the one in Fig. 2.30 is useful in simplifying the analysis of a circuit. In general,

The **equivalent resistance** of any number of resistors connected in series is the sum of the individual resistances.

Resistors in series behave as a single resistor whose resistance is equal to the sum of the resistances of the individual resistors.

For $N$ resistors in series then,

$$R_{eq} = R_1 + R_2 + \cdots + R_N = \sum_{n=1}^{N} R_n \tag{2.30}$$

To determine the voltage across each resistor in Fig. 2.29, we substitute Eq. (2.26) into Eq. (2.24) and obtain

$$v_1 = \frac{R_1}{R_1 + R_2} v, \qquad v_2 = \frac{R_2}{R_1 + R_2} v \tag{2.31}$$

Notice that the source voltage $v$ is divided among the resistors in direct proportion to their resistances; the larger the resistance, the larger the voltage drop. This is called the *principle of voltage division*, and the circuit in Fig. 2.29 is called a *voltage divider*. In general, if a voltage divider has $N$ resistors ($R_1, R_2, \ldots, R_N$) in series with the source voltage $v$, the $n$th resistor ($R_n$) will have a voltage drop of

$$v_n = \frac{R_n}{R_1 + R_2 + \cdots + R_N} v \tag{2.32}$$

## 2.6 Parallel Resistors and Current Division

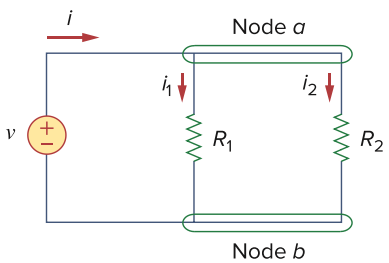



**Figure 2.31**
Two resistors in parallel.

Consider the circuit in Fig. 2.31, where two resistors are connected in parallel and therefore have the same voltage across them. From Ohm's law,

$$v = i_1 R_1 = i_2 R_2$$

or

$$i_1 = \frac{v}{R_1}, \qquad i_2 = \frac{v}{R_2} \tag{2.33}$$

Applying KCL at node $a$ gives the total current $i$ as

$$i = i_1 + i_2 \tag{2.34}$$

Substituting Eq. (2.33) into Eq. (2.34), we get

$$i = \frac{v}{R_1} + \frac{v}{R_2} = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{v}{R_{eq}} \tag{2.35}$$

where $R_{eq}$ is the equivalent resistance of the resistors in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{2.36}$$

or

$$\frac{1}{R_{eq}} = \frac{R_1 + R_2}{R_1 R_2}$$

or

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \tag{2.37}$$

Thus,

The **equivalent resistance** of two parallel resistors is equal to the product of their resistances divided by their sum.

It must be emphasized that this applies only to tw o resistors in parallel. From Eq. (2.37), if $R_1 = R_2$ then $R_{eq} = R_1/R_2$.

We can extend the result in Eq. (2.36) to the general case of a circuit with $N$ resistors in parallel. The equivalent resistance is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N} \tag{2.38}$$

Note that $R_{eq}$ is always smaller than the resistance of the smallest resistor in the parallel combination. If $R_1 = R_2 = \cdots = R_N = R$, then

$$R_{eq} = \frac{R}{N} \tag{2.39}$$

For e xample, if four 100- Ω resistors are connected in parallel, their equivalent resistance is 25 Ω.

It is often more convenient to use conductance rather than resistance when dealing with resistors in parallel. From Eq. (2.38), the equi valent conductance for $N$ resistors in parallel is

$$G_{eq} = G_1 + G_2 + G_3 + \cdots + G_N \tag{2.40}$$

where $G_{eq} = 1/R_{eq}$, $G_1 = 1/R_1$, $G_2 = 1/R_2$, $G_3 = 1/R_3$, … , $G_N = 1/R_N$. Equation (2.40) states:

The **equivalent conductance** of resistors connected in parallel is the sum of their individual conductances.

This means that we may replace the circuit in Fig. 2.31 with that in Fig. 2.32. Notice the similarity between Eqs. (2.30) and (2.40). The equivalent conductance of parallel resistors is obtained the same w ay as the equi valent resistance of series resistors. In the same manner , the equivalent conductance of resistors in series is obtained just the same

Conductances in parallel behave as a single conductance whose value is equal to the sum of the individual conductances.

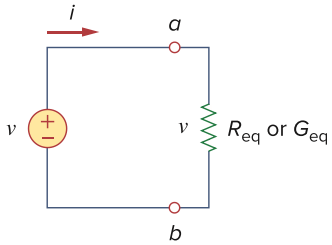



**Figure 2.32**
Equivalent circuit to Fig. 2.31.

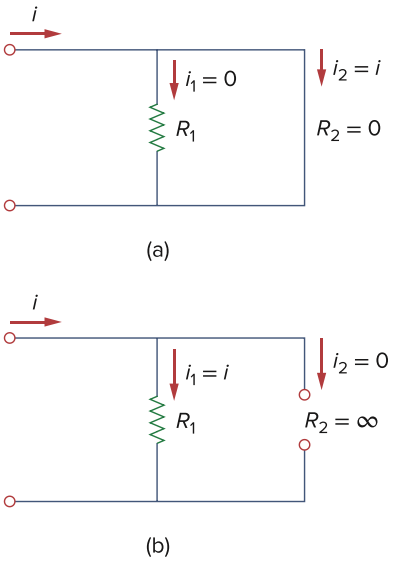



**Figure 2.33**
(a) A shorted circuit, (b) an open circuit.

way as the resistance of resistors in parallel. Thus, the equivalent conductance $G_{eq}$ of $N$ resistors in series (such as shown in Fig. 2.29) is

$$\frac{1}{G_{eq}} = \frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3} + \cdots + \frac{1}{G_N} \tag{2.41}$$

Given the total current $i$ entering node $a$ in Fig. 2.31, how do we obtain current $i_1$ and $i_2$? We know that the equivalent resistor has the same voltage, or

$$v = iR_{eq} = \frac{iR_1R_2}{R_1 + R_2} \tag{2.42}$$

Combining Eqs. (2.33) and (2.42) results in

$$i_1 = \frac{R_2\, i}{R_1 + R_2}, \qquad i_2 = \frac{R_1\, i}{R_1 + R_2} \tag{2.43}$$

which shows that the total current $i$ is shared by the resistors in inverse proportion to their resistances. This is known as the *principle of current division*, and the circuit in Fig. 2.31 is known as a *current divider*. Notice that the larger current flows through the smaller resistance.

As an extreme case, suppose one of the resistors in Fig. 2.31 is zero, say $R_2 = 0$; that is, $R_2$ is a short circuit, as shown in Fig. 2.33(a). From Eq. (2.43), $R_2 = 0$ implies that $i_1 = 0$, $i_2 = i$. This means that the entire current $i$ bypasses $R_1$ and flows through the short circuit $R_2 = 0$, the path of least resistance. Thus when a circuit is short circuited, as shown in Fig. 2.33(a), two things should be kept in mind:

1. The equivalent resistance $R_{eq} = 0$. [See what happens when $R_2 = 0$ in Eq. (2.37).]
2. The entire current flows through the short circuit.

As another extreme case, suppose $R_2 = \infty$, that is, $R_2$ is an open circuit, as shown in Fig. 2.33(b). The current still flows through the path of least resistance, $R_1$. By taking the limit of Eq. (2.37) as $R_2 \to \infty$, we obtain $R_{eq} = R_1$ in this case.

If we divide both the numerator and denominator by $R_1R_2$, Eq. (2.43) becomes

$$i_1 = \frac{G_1}{G_1 + G_2}\, i \tag{2.44a}$$

$$i_2 = \frac{G_2}{G_1 + G_2}\, i \tag{2.44b}$$

Thus, in general, if a current divider has $N$ conductors ($G_1, G_2, \ldots, G_N$) in parallel with the source current $i$, the $n$th conductor ($G_n$) will have current

$$i_n = \frac{G_n}{G_1 + G_2 + \cdots + G_N}\, i \tag{2.45}$$

In general, it is often convenient and possible to combine resistors in series and parallel and reduce a resisti ve network to a single *equivalent resistance* $R_{eq}$. Such an equi valent resistance is the resistance between the designated terminals of the netw ork and must e xhibit the same *i-v* characteristics as the original network at the terminals.

## Example 2.9

Find $R_{eq}$ for the circuit shown in Fig. 2.34.

**Solution:**
To get $R_{eq}$, we combine resistors in series and in parallel. The 6- Ω and 3-Ω resistors are in parallel, so their equivalent resistance is

$$6\ \Omega \parallel 3\Omega = \frac{6 \times 3}{6 + 3} = 2\ \Omega$$

(The symbol ‖ is used to indicate a parallel combination.) Also, the 1-Ω and 5-Ω resistors are in series; hence their equivalent resistance is

$$1\ \Omega + 5\ \Omega = 6\ \Omega$$

Thus the circuit in Fig. 2.34 is reduced to that in Fig. 2.35(a). In Fig. 2.35(a), we notice that the tw o 2-Ω resistors are in series, so the equivalent resistance is

$$2\ \Omega + 2\ \Omega = 4\ \Omega$$

This 4-Ω resistor is now in parallel with the 6-Ω resistor in Fig. 2.35(a); their equivalent resistance is

$$4\ \Omega \parallel 6\ \Omega = \frac{4 \times 6}{4 + 6} = 2.4\ \Omega$$

The circuit in Fig. 2.35(a) is now replaced with that in Fig. 2.35(b). In Fig. 2.35(b), the three resistors are in series. Hence, the equivalent resistance for the circuit is

$$R_{eq} = 4\ \Omega + 2.4\ \Omega + 8\ \Omega = 14.4\ \Omega$$

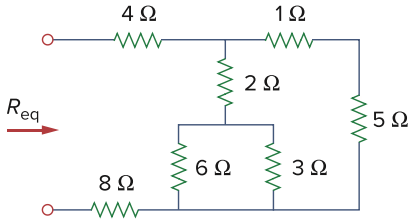



**Figure 2.34**
For Example 2.9.

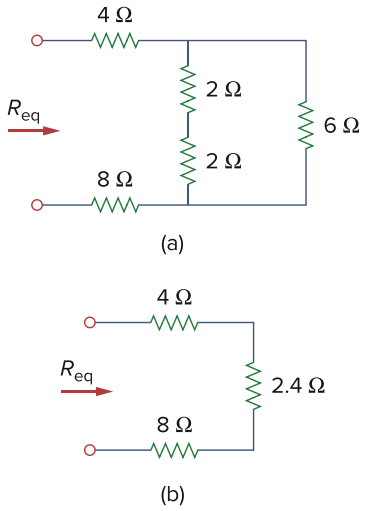



**Figure 2.35**
Equivalent circuits for Example 2.9.

## Practice Problem 2.9

By combining the resistors in Fig. 2.36, find $R_{eq}$.

**Answer:** 11 Ω.

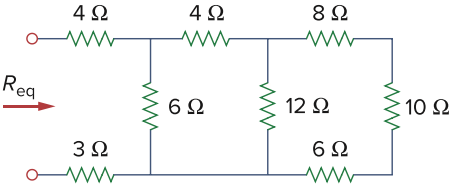



**Figure 2.36**
For Practice Prob. 2.9.

## Example 2.10

Calculate the equivalent resistance $R_{ab}$ in the circuit in Fig. 2.37.

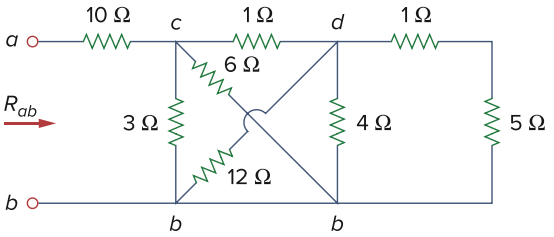



**Figure 2.37**
For Example 2.10.

**Solution:**

The 3-Ω and 6-Ω resistors are in parallel because they are connected to the same two nodes $c$ and $b$. Their combined resistance is

$$3\ \Omega \parallel 6\ \Omega = \frac{3 \times 6}{3 + 6} = 2\ \Omega \qquad \textbf{(2.10.1)}$$

Similarly, the 12-Ω and 4-Ω resistors are in parallel since the y are connected to the same two nodes $d$ and $b$. Hence

$$12\ \Omega \parallel 4\ \Omega = \frac{12 \times 4}{12 + 4} = 3\ \Omega \qquad \textbf{(2.10.2)}$$

Also the 1-Ω and 5-Ω resistors are in series; hence, their equivalent re - sistance is

$$1\ \Omega + 5\ \Omega = 6\ \Omega \qquad \textbf{(2.10.3)}$$

With these three combinations, we can replace the circuit in Fig. 2.37 with that in Fig. 2.38(a). In Fig. 2.38(a), 3-Ω in parallel with 6-Ω gives 2-Ω, as calculated in Eq. (2.10.1). This 2-Ω equivalent resistance is now in series with the 1-Ω resistance to give a combined resistance of $1\Omega + 2\ \Omega = 3\ \Omega$. Thus, we replace the circuit in Fig. 2.38(a) with that in Fig. 2.38(b). In Fig. 2.38(b), we combine the 2-Ω and 3-Ω resistors in parallel to get

$$2\ \Omega \parallel 3\ \Omega = \frac{2 \times 3}{2 + 3} = 1.2\ \Omega$$

This 1.2-Ω resistor is in series with the 10-Ω resistor, so that

$$R_{ab} = 10 + 1.2 = 11.2\ \Omega$$

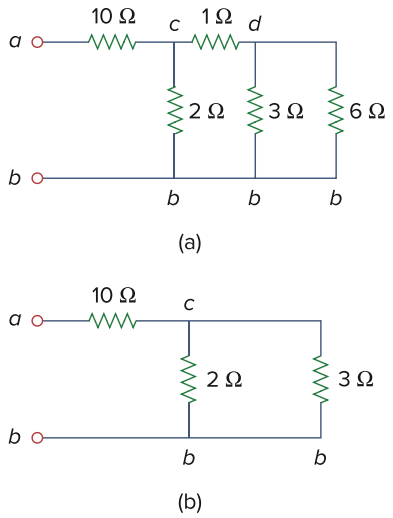



**Figure 2.38**
Equivalent circuits for Example 2.10.

## Practice Problem 2.10

Find $R_{ab}$ for the circuit in Fig. 2.39.

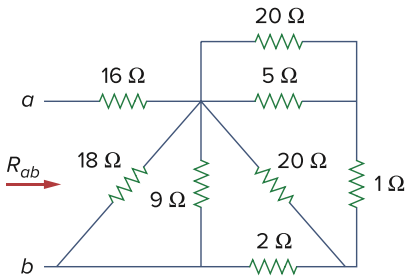



**Figure 2.39**
For Practice Prob. 2.10.

**Answer:** 19 Ω.

## Example 2.11

Find the equivalent conductance $G_{eq}$ for the circuit in Fig. 2.40(a).

**Solution:**
The 8-S and 12-S resistors are in parallel, so their conductance is

$$8\ \text{S} + 12\ \text{S} = 20\ \text{S}$$

This 20-S resistor is now in series with 5 S as shown in Fig. 2.40(b) so that the combined conductance is

$$\frac{20 \times 5}{20 + 5} = 4\ \text{S}$$

This is in parallel with the 6-S resistor. Hence,

$$G_{eq} = 6 + 4 = 10\ \text{S}$$

We should note that the circuit in Fig. 2.40(a) is the same as that in Fig. 2.40(c). While the resistors in Fig. 2.40(a) are expressed in siemens, those in Fig. 2.40(c) are expressed in ohms. To show that the circuits are the same, we find $R_{eq}$ for the circuit in Fig. 2.40(c).

$$R_{eq} = \frac{1}{6} \Big\| \left( \frac{1}{5} + \frac{1}{8} \,\|\, \frac{1}{12} \right) = \frac{1}{6} \,\Big\|\, \left( \frac{1}{5} + \frac{1}{20} \right) = \frac{1}{6} \,\Big\|\, \frac{1}{4}$$

$$= \frac{\frac{1}{6} \times \frac{1}{4}}{\frac{1}{6} + \frac{1}{4}} = \frac{1}{10}\ \Omega$$

$$G_{eq} = \frac{1}{R_{eq}} = 10\ \text{S}$$

This is the same as we obtained previously.

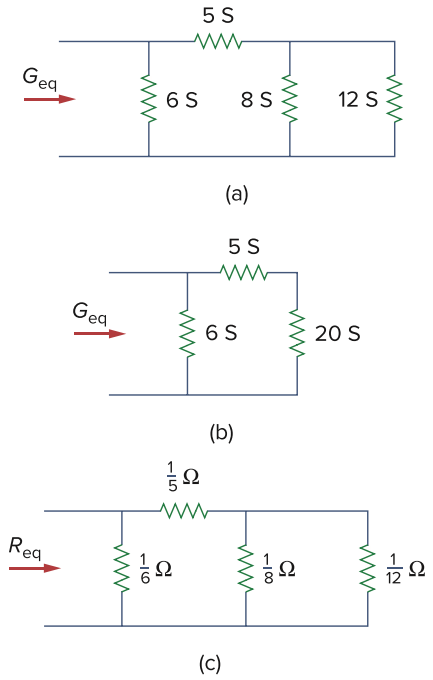



**Figure 2.40**
For Example 2.11: (a) original circuit, (b) its equivalent circuit, (c) same circuit as in (a) but resistors are expressed in ohms.

## Practice Problem 2.11

Calculate $G_{eq}$ in the circuit of Fig. 2.41.

**Answer:** 8 S.

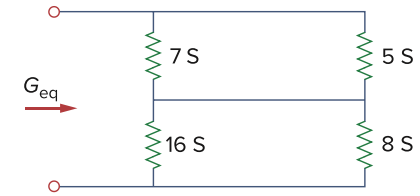



**Figure 2.41**
For Practice Prob. 2.11.

## Example 2.12

Find $i_o$ and $v_o$ in the circuit shown in Fig. 2.42(a). Calculate the power dissipated in the 3-Ω resistor.

**Solution:**
The 6-Ω and 3-Ω resistors are in parallel, so their combined resistance is

$$6\ \Omega \,\|\, 3\ \Omega = \frac{6 \times 3}{6 + 3} = 2\ \Omega$$

Thus, our circuit reduces to that shown in Fig. 2.42(b). Notice that $v_o$ is not affected by the combination of the resistors because the resistors are

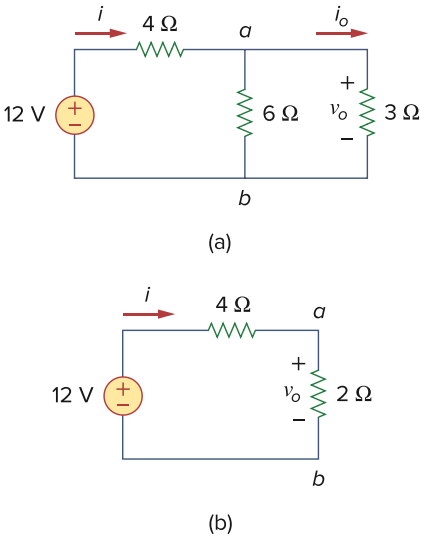



**Figure 2.42**
For Example 2.12: (a) original circuit, (b) its equivalent circuit.

in parallel and therefore have the same voltage $v_o$. From Fig. 2.42(b), we can obtain $v_o$ in two ways. One way is to apply Ohm's law to get

$$i = \frac{12}{4+2} = 2\text{ A}$$

and hence, $v_o = 2i = 2 \times 2 = 4$ V. Another way is to apply voltage division, since the 12 V in Fig. 2.42(b) is divided between the 4-Ω and 2-Ω resistors. Hence,

$$v_o = \frac{2}{2+4}(12\text{ V}) = 4\text{ V}$$

Similarly, $i_o$ can be obtained in two ways. One approach is to apply Ohm's law to the 3-Ω resistor in Fig. 2.42(a) now that we know $v_o$; thus,

$$v_o = 3i_o = 4 \quad \Rightarrow \quad i_o = \frac{4}{3}\text{ A}$$

Another approach is to apply current division to the circuit in Fig. 2.42(a) now that we know $i$, by writing

$$i_o = \frac{6}{6+3}i = \frac{2}{3}(2\text{ A}) = \frac{4}{3}\text{ A}$$

The power dissipated in the 3-Ω resistor is

$$p_o = v_o i_o = 4\left(\frac{4}{3}\right) = 5.333\text{ W}$$

## Practice Problem 2.12

Find $v_1$ and $v_2$ in the circuit shown in Fig. 2.43. Also calculate $i_1$ and $i_2$ and the power dissipated in the 12-Ω and 40-Ω resistors.

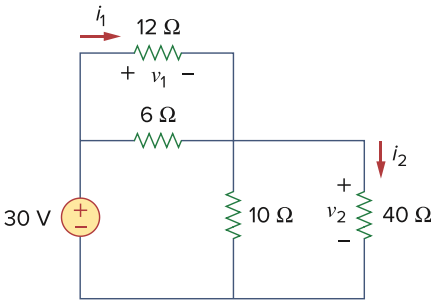



**Figure 2.43**
For Practice Prob. 2.12.

**Answer:** $v_1 = 10$ V, $i_1 = 833.3$ mA, $p_1 = 8.333$ W, $v_2 = 20$ V, $i_2 = 500$ mA, $p_2 = 10$ W.

## Example 2.13

For the circuit shown in Fig. 2.44(a), determine: (a) the voltage $v_o$, (b) the power supplied by the current source, (c) the power absorbed by each resistor.

**Solution:**
(a) The 6-kΩ and 12-kΩ resistors are in series so that their combined value is 6 + 12 = 18 kΩ. Thus the circuit in Fig. 2.44(a) reduces to that shown in Fig. 2.44(b). We now apply the current division technique to find $i_1$ and $i_2$.

$$i_1 = \frac{18{,}000}{9{,}000 + 18{,}000}(30\text{ mA}) = 20\text{ mA}$$

$$i_2 = \frac{9{,}000}{9{,}000 + 18{,}000}(30\text{ mA}) = 10\text{ mA}$$

Notice that the voltage across the 9-kΩ and 18-kΩ resistors is the same, and $v_o = 9{,}000i_1 = 18{,}000i_2 = 180$ V, as expected.
(b) Power supplied by the source is

$$p_o = v_o i_o = 180(30)\ \text{mW} = 5.4\ \text{W}$$

(c) Power absorbed by the 12-kΩ resistor is

$$p = iv = i_2(i_2 R) = i_2^2 R = (10 \times 10^{-3})^2\,(12{,}000) = 1.2\ \text{W}$$

Power absorbed by the 6-kΩ resistor is

$$p = i_2^2 R = (10 \times 10^{-3})^2\,(6{,}000) = 0.6\ \text{W}$$

Power absorbed by the 9-kΩ resistor is

$$p = \frac{v_o^2}{R} = \frac{(180)^2}{9{,}000} = 3.6\ \text{W}$$

or

$$p = v_o i_1 = 180(20)\ \text{mW} = 3.6\ \text{W}$$

Notice that the power supplied (5.4 W) equals the power absorbed (1.2 + 0.6 + 3.6 = 5.4 W). This is one way of checking results.

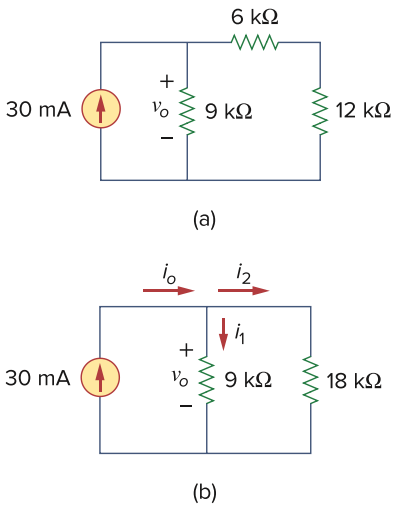



**Figure 2.44**
For Example 2.13: (a) original circuit, (b) its equivalent circuit.

## Practice Problem 2.13

For the circuit shown in Fig. 2.45, find: (a) $v_1$ and $v_2$, (b) the power dissipated in the 3-kΩ and 20-kΩ resistors, and (c) the power supplied by the current source.

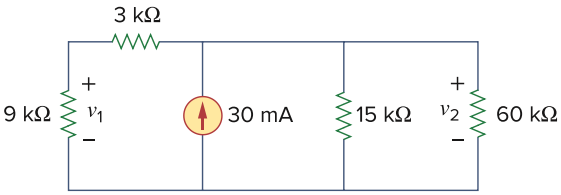



**Figure 2.45**
For Practice Prob. 2.13.

**Answer:** (a) 135 V, 180 V, (b) 2.025 W, 540 mW, (c) 5.4 W.

## 2.7 †Wye-Delta Transformations

Situations often arise in circuit analysis when the resistors are neither in parallel nor in series. For example, consider the bridge circuit in Fig. 2.46. How do we combine resistors $R_1$ through $R_6$ when the resistors are neither in series nor in parallel? Man y circuits of the type sho wn in Fig. 2.46 can be simplified by using three-terminal equivalent networks. These are the wye (Y) or tee (T) netw ork shown in Fig. 2.47 and the delta (Δ) or pi (Π) network shown in Fig. 2.48. These networks occur by themselves or as part of a lar ger network. They are used in three-phase networks, electrical filters, and matching networks. Our main interest

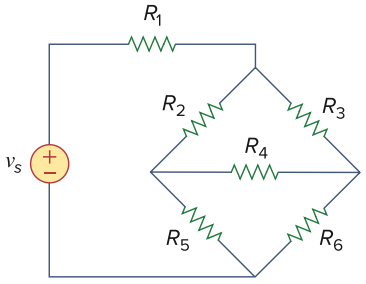



**Figure 2.46**
The bridge network.

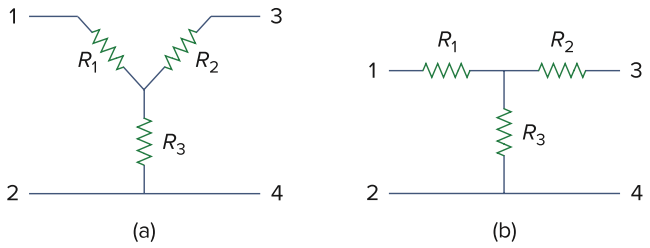



**Figure 2.47**
Two forms of the same network: (a) Y, (b) T.

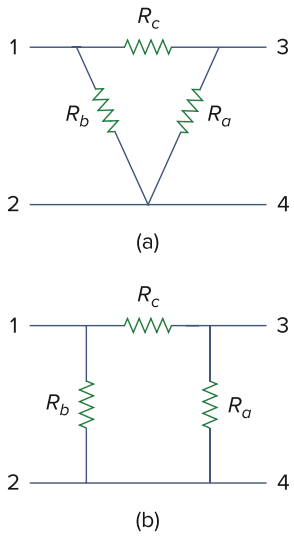



**Figure 2.48**
Two forms of the same network: (a) Δ, (b) Π.

here is in how to identify them when they occur as part of a network and how to apply wye-delta transformation in the analysis of that network.

## Delta to Wye Conversion

Suppose it is more convenient to work with a wye network in a place where the circuit contains a delta configuration. We superimpose a wye network on the existing delta network and find the equivalent resistances in the wye network. To obtain the equivalent resistances in the wye network, we compare the tw o networks and mak e sure that the resistance between each pair of nodes in the Δ (or Π) network is the same as the resistance between the same pair of nodes in the Y (or T) network. For terminals 1 and 2 in Figs. 2.47 and 2.48, for example,

$$R_{12}(\mathrm{Y}) = R_1 + R_3 \tag{2.46}$$

$$R_{12}(\Delta) = R_b \| (R_a + R_c)$$

Setting $R_{12}(\mathrm{Y}) = R_{12}(\Delta)$ gives

$$R_{12} = R_1 + R_3 = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c} \tag{2.47a}$$

Similarly,

$$R_{13} = R_1 + R_2 = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c} \tag{2.47b}$$

$$R_{34} = R_2 + R_3 = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c} \tag{2.47c}$$

Subtracting Eq. (2.47c) from Eq. (2.47a), we get

$$R_1 - R_2 = \frac{R_c(R_b - R_a)}{R_a + R_b + R_c} \tag{2.48}$$

Adding Eqs. (2.47b) and (2.48) gives

$$\boxed{R_1 = \frac{R_b R_c}{R_a + R_b + R_c}} \tag{2.49}$$

and subtracting Eq. (2.48) from Eq. (2.47b) yields

$$\boxed{R_2 = \frac{R_c R_a}{R_a + R_b + R_c}} \tag{2.50}$$

Subtracting Eq. (2.49) from Eq. (2.47a), we obtain

$$\boxed{R_3 = \frac{R_a R_b}{R_a + R_b + R_c}} \tag{2.51}$$

We do not need to memorize Eqs. (2.49) to (2.51). To transform a Δ network to Y, we create an extra node $n$ as shown in Fig. 2.49 and follow this conversion rule:

> Each resistor in the Y network is the product of the resistors in the two adjacent Δ branches, divided by the sum of the three Δ resistors.

One can follow this rule and obtain Eqs. (2.49) to (2.51) from Fig. 2.49.

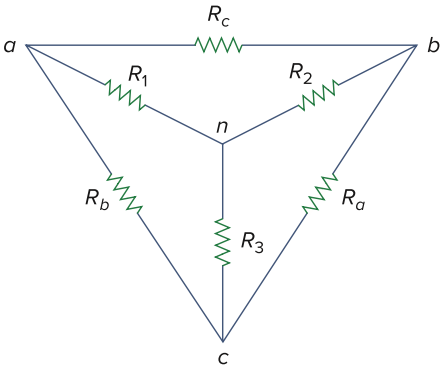



**Figure 2.49**
Superposition of Y and Δ networks as an aid in transforming one to the other.

## Wye to Delta Conversion

To obtain the conversion formulas for transforming a wye network to an equivalent delta network, we note from Eqs. (2.49) to (2.51) that

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{R_aR_bR_c(R_a + R_b + R_c)}{(R_a + R_b + R_c)^2} = \frac{R_aR_bR_c}{R_a + R_b + R_c} \tag{2.52}$$

Dividing Eq. (2.52) by each of Eqs. (2.49) to (2.51) leads to the follo w-ing equations:

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \tag{2.53}$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \tag{2.54}$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} \tag{2.55}$$

From Eqs. (2.53) to (2.55) and Fig. 2.49, the con version rule for Y to Δ is as follows:

> Each resistor in the Δ network is the sum of all possible products of Y resistors taken two at a time, divided by the opposite Y resistor.

The Y and Δ networks are said to be *balanced* when

$$R_1 = R_2 = R_3 = R_Y, \qquad R_a = R_b = R_c = R_\Delta \tag{2.56}$$

Under these conditions, conversion formulas become

$$R_Y = \frac{R_\Delta}{3} \quad \text{or} \quad R_\Delta = 3R_Y \tag{2.57}$$

One may w onder wh y $R_Y$ is less than $R_\Delta$. Well, we notice that the Y-connection is like a "series" connection while the Δ-connection is like a "parallel" connection.

Note that in making the transformation, we do not take anything out of the circuit or put in anything new. We are merely substituting different but mathematically equivalent three-terminal network patterns to create a circuit in which resistors are either in series or in parallel, allo wing us to calculate $R_{eq}$ if necessary.

## Example 2.14

Convert the Δ network in Fig. 2.50(a) to an equivalent Y network.

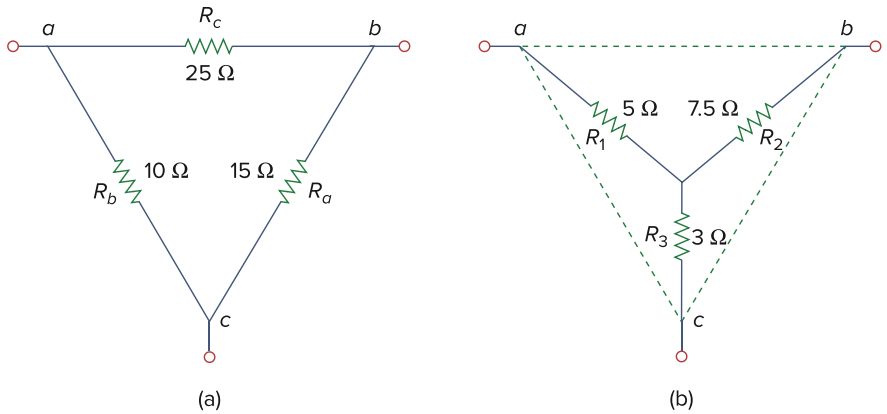



**Figure 2.50**
For Example 2.14: (a) original Δ network, (b) Y equivalent network.

**Solution:**

Using Eqs. (2.49) to (2.51), we obtain

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} = \frac{10 \times 25}{15 + 10 + 25} = \frac{250}{50} = 5\ \Omega$$

$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c} = \frac{25 \times 15}{50} = 7.5\ \Omega$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c} = \frac{15 \times 10}{50} = 3\ \Omega$$

The equivalent Y network is shown in Fig. 2.50(b).

## Practice Problem 2.14

Transform the wye network in Fig. 2.51 to a delta network.

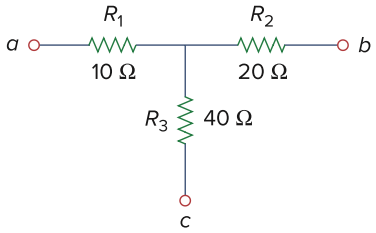



**Figure 2.51**
For Practice Prob. 2.14.

**Answer:** $R_a = 140\ \Omega$, $R_b = 70\ \Omega$, $R_c = 35\ \Omega$.

## Example 2.15

Obtain the equivalent resistance $R_{ab}$ for the circuit in Fig. 2.52 and use it to find current $i$.

**Solution:**

1. **Define.** The problem is clearly defined. Please note, this part normally will deservedly take much more time.
2. **Present.** Clearly, when we remove the voltage source, we end up with a purely resistive circuit. Since it is composed of deltas and wyes, we have a more complex process of combining the elements together .

We can use wye-delta transformations as one approach to find a solution. It is useful to locate the wyes (there are two of them, one at $n$ and the other at $c$) and the deltas (there are three: *can, abn, cnb*).

3. **Alternative.** There are different approaches that can be used to solve this problem. Since the focus of Sec. 2.7 is the wye-delta transformation, this should be the technique to use. Another approach would be to solve for the equivalent resistance by injecting one amp into the circuit and finding the voltage between $a$ and $b$; we will learn about this approach in Chap. 4.

   The approach we can apply here as a check would be to use a wye-delta transformation as the first solution to the problem. Later we can check the solution by starting with a delta-wye transformation.

4. **Attempt.** In this circuit, there are two Y networks and three Δ networks. Transforming just one of these will simplify the circuit. If we convert the Y network comprising the 5-Ω, 10-Ω, and 20-Ω resistors, we may select

$$R_1 = 10\ \Omega, \qquad R_2 = 20\ \Omega, \qquad R_3 = 5\ \Omega$$

Thus from Eqs. (2.53) to (2.55) we have

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} = \frac{10 \times 20 + 20 \times 5 + 5 \times 10}{10}$$

$$= \frac{350}{10} = 35\ \Omega$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} = \frac{350}{20} = 17.5\ \Omega$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} = \frac{350}{5} = 70\ \Omega$$

With the Y converted to Δ, the equivalent circuit (with the voltage source removed for now) is shown in Fig. 2.53(a). Combining the three pairs of resistors in parallel, we obtain

$$70 \| 30 = \frac{70 \times 30}{70 + 30} = 21\ \Omega$$

$$12.5 \| 17.5 = \frac{12.5 \times 17.5}{12.5 + 17.5} = 7.292\ \Omega$$

$$15 \| 35 = \frac{15 \times 35}{15 + 35} = 10.5\ \Omega$$

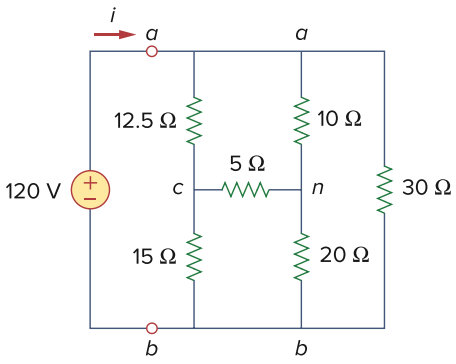



**Figure 2.52**
For Example 2.15.

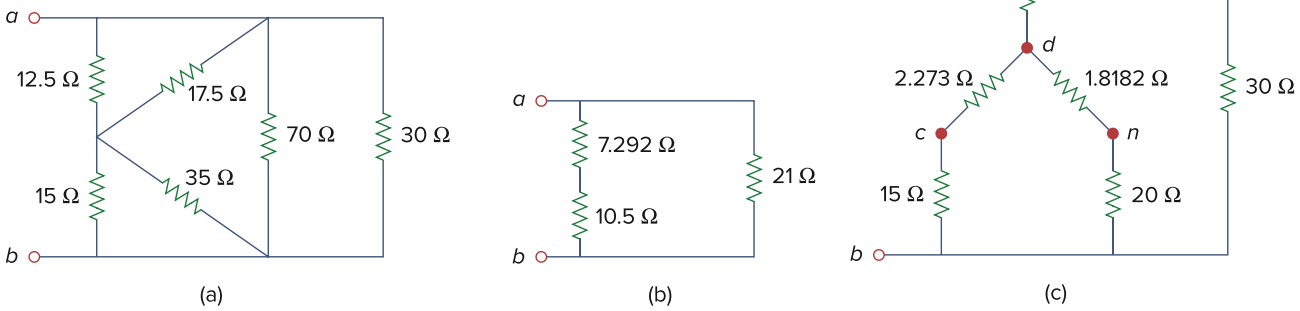



**Figure 2.53**
Equivalent circuits to Fig. 2.52, with the voltage source removed.

so that the equivalent circuit is shown in Fig. 2.53(b). Hence, we find

$$R_{ab} = (7.292 + 10.5) \parallel 21 = \frac{17.792 \times 21}{17.792 + 21} = \mathbf{9.632\ \Omega}$$

Then

$$i = \frac{v_s}{R_{ab}} = \frac{120}{9.632} = \mathbf{12.458\ A}$$

We observe that we have successfully solved the problem. Now we must evaluate the solution.

5. **Evaluate.** Now we must determine if the answer is correct and then evaluate the final solution.

   It is relatively easy to check the answer; we do this by solving the problem starting with a delta-wye transformation. Let us transform the delta, *can*, into a wye.

   Let $R_c = 10\ \Omega$, $R_a = 5\ \Omega$, and $R_n = 12.5\ \Omega$. This will lead to (let $d$ represent the middle of the wye):

$$R_{ad} = \frac{R_c R_n}{R_a + R_c + R_n} = \frac{10 \times 12.5}{5 + 10 + 12.5} = 4.545\ \Omega$$

$$R_{cd} = \frac{R_a R_n}{27.5} = \frac{5 \times 12.5}{27.5} = 2.273\ \Omega$$

$$R_{nd} = \frac{R_a R_c}{27.5} = \frac{5 \times 10}{27.5} = 1.8182\ \Omega$$

   This now leads to the circuit shown in Figure 2.53(c). Looking at the resistance between $d$ and $b$, we have two series combination in parallel, giving us

$$R_{db} = \frac{(2.273 + 15)(1.8182 + 20)}{2.273 + 15 + 1.8182 + 20} = \frac{376.9}{39.09} = 9.642\ \Omega$$

   This is in series with the 4.545-Ω resistor, both of which are in parallel with the 30-Ω resistor. This then gives us the equivalent resistance of the circuit.

$$R_{ab} = \frac{(9.642 + 4.545)30}{9.642 + 4.545 + 30} = \frac{425.6}{44.19} = \mathbf{9.631\ \Omega}$$

   This now leads to

$$i = \frac{v_s}{R_{ab}} = \frac{120}{9.631} = \mathbf{12.46\ A}$$

   We note that using two variations on the wye-delta transformation leads to the same results. This represents a very good check.

6. **Satisfactory?** Since we have found the desired answer by determining the equivalent resistance of the circuit first and the answer checks, then we clearly have a satisfactory solution. This represents what can be presented to the individual assigning the problem.

## Practice Problem 2.15

For the bridge network in Fig. 2.54, find $R_{ab}$ and $i$.

**Answer:** 60 Ω, 4 A.

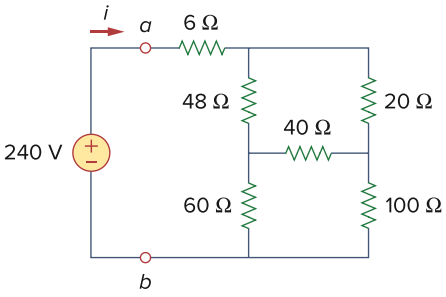



**Figure 2.54**
For Practice Prob. 2.15.

## 2.8 †Applications

Resistors are often used to model de vices that con vert electrical ener - gy into heat or other forms of energy. Such devices include conducting wire, light bulbs, electric heaters, stoves, ovens, and loudspeakers. In this section, we will consider two real-life problems that apply the concepts developed in this chapter: electrical lighting systems and design of dc meters.

### 2.8.1 Lighting Systems

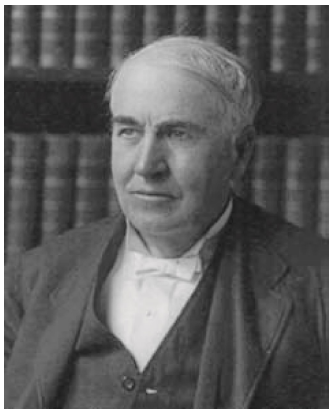
Lighting systems, such as in a house or on a Christmas tree, often con- sist of $N$ lamps connected either in parallel or in series, as sho wn in Fig. 2.55. Each lamp is modeled as a resistor . Assuming that all the lamps are identical and $V_o$ is the power-line voltage, the voltage across each lamp is $V_o$ for the parallel connection and $V_o/N$ for the series con- nection. The series connection is easy to manufa cture b ut is seldom used in practice, for at least two reasons. First, it is less reliable; when a lamp fa ils, all the lamps go out. Second, it is harder to maintain; when a lamp is bad, one must test all the lamps one by one to detect the faulty one.

So far, we have assumed that connect- ing wires are perfect conductors (i.e., conductors of zero resistance). In real physical systems, however, the resistance of the connecting wire may be appreciably large, and the modeling of the system must include that resistance.

## Historical

**Thomas Alva Edison** (1847–1931) was perhaps the greatestAmerican inventor. He patented 1093 inventions, including such history-making inventions as the incandescent electric bulb, the phonograph, and the first commercial motion pictures.

Born in Milan, Ohio, the youngest of seven children, Edison re- ceived only three months of formal education because he hated school. He was home-schooled by his mother and quickly began to read on his own. In 1868, Edison read one of Faraday's books and found his calling. He moved to Menlo Park, New Jersey, in 1876, where he man - aged a well-staffed research laboratory. Most of his inventions came out of this laboratory. His laboratory served as a model for modern re- search organizations. Because of his diverse interests and the over - whelming number of his inventions and patents, Edison began to estab - lish manufacturing companies for making the devices he invented. He designed the first electric power station to supply electric light. Formal electrical engineering education began in the mid-1880s with Edison as a role model and leader.



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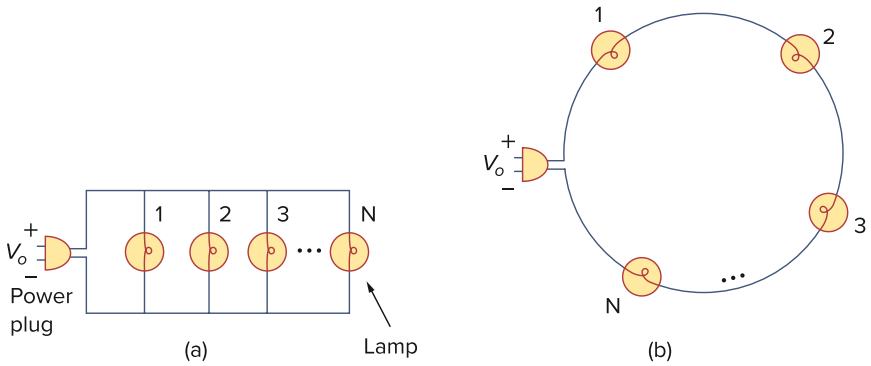



**Figure 2.55**
(a) Parallel connection of light bulbs, (b) series connection of light bulbs.

## Example 2.16

Three light bulbs are connected to a 9-V battery as shown in Fig. 2.56(a). Calculate: (a) the total current supplied by the battery, (b) the current through each bulb, (c) the resistance of each bulb.

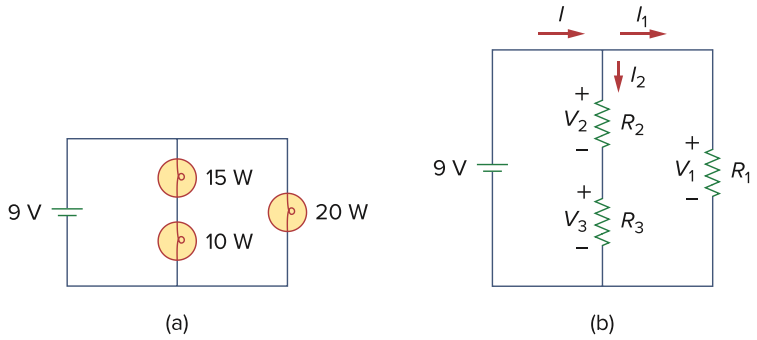



**Figure 2.56**
(a) Lighting system with three bulbs, (b) resistive circuit equivalent model.

**Solution:**

(a) The total power supplied by the battery is equal to the total power absorbed by the bulbs; that is,

$$p = 15 + 10 + 20 = 45 \text{ W}$$

Since $p = VI$, then the total current supplied by the battery is

$$I = \frac{p}{V} = \frac{45}{9} = 5 \text{ A}$$

(b) The bulbs can be modeled as resistors as shown in Fig. 2.56(b). Since $R_1$ (20-W bulb) is in parallel with the battery as well as the series combination of $R_2$ and $R_3$,

$$V_1 = V_2 + V_3 = 9 \text{ V}$$

The current through $R_1$ is

$$I_1 = \frac{p_1}{V_1} = \frac{20}{9} = 2.222 \text{ A}$$

By KCL, the current through the series combination of $R_2$ and $R_3$ is

$$I_2 = I - I_1 = 5 - 2.222 = 2.778 \text{ A}$$

(c) Since $p = I^2R$,

$$R_1 = \frac{p_1}{I_1^2} = \frac{20}{2.222^2} = 4.05\ \Omega$$

$$R_2 = \frac{p_2}{I_2^2} = \frac{15}{2.777^2} = 1.945\ \Omega$$

$$R_3 = \frac{p_3}{I_3^2} = \frac{10}{2.777^2} = 1.297\ \Omega$$

**Practice Problem 2.16**

Refer to Fig. 2.55 and assume there are six light bulbs that can be connected in parallel and six different light bulbs that can be connected in series. In either case, each light bulb is to operate at 40 W. If the voltage at the plug is 115 V for the parallel and series connections, calculate the current through and the voltage across each bulb for both cases.

**Answer:** 115 V and 347.8 mA (parallel), 19.167 V and 2.087 A (series).

### 2.8.2 Design of DC Meters

By their nature, resistors are used to control the flow of current. We take advantage of this property in several applications, such as in a potentiometer (Fig. 2.57). The word *potentiometer*, derived from the words *potential* and *meter*, implies that potential can be metered out. The potentiometer (or pot for short) is a three-terminal device that operates on the principle of voltage division. It is essentially an adjustable voltage divider. As a voltage regulator, it is used as a volume or level control on radios, TVs, and other devices. In Fig. 2.57,

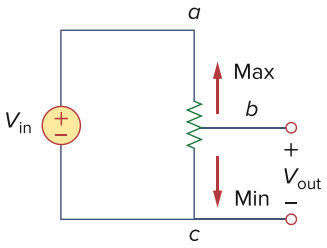



**Figure 2.57**
The potentiometer controlling potential levels.

$$V_{\text{out}} = V_{bc} = \frac{R_{bc}}{R_{ac}} V_{\text{in}} \tag{2.58}$$

where $R_{ac} = R_{ab} + R_{bc}$. Thus, $V_{\text{out}}$ decreases or increases as the sliding contact of the pot moves toward $c$ or $a$, respectively.

Another application where resistors are used to control current flow is in the analog dc meters—the ammeter, voltmeter, and ohmmeter, which measure current, voltage, and resistance, respectively. Each of these meters employs the d'Arsonval meter movement, shown in Fig. 2.58. The movement consists essentially of a movable iron-core coil mounted on a pivot between the poles of a permanent magnet. When current flows through the coil, it creates a torque which causes the pointer to deflect. The amount of current through the coil determines the deflection of the pointer, which is registered on a scale attached to the meter movement. For example, if the meter movement is rated 1 mA, 50 Ω, it would take 1 mA to cause a full-scale deflection of the meter movement. By introducing additional circuitry to the d'Arsonval meter movement, an ammeter, voltmeter, or ohmmeter can be constructed.

An instrument capable of measuring voltage, current, and resistance is called a *multimeter* or a *volt-ohm meter* (VOM).

Consider Fig. 2.59, where an analog voltmeter and ammeter are connected to an element. The voltmeter measures the voltage across a *load* and

A load is a component that is receiving energy (an energy sink), as opposed to a generator supplying energy (an energy source). More about loading will be discussed in Section 4.9.1.

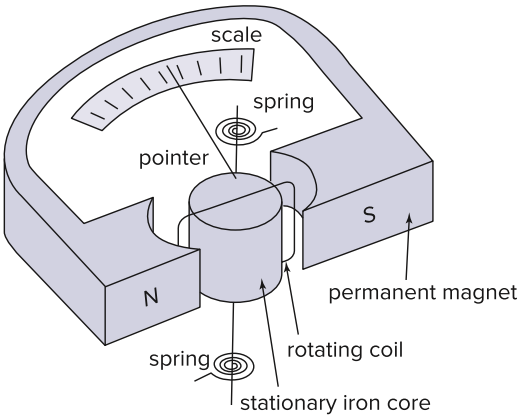



**Figure 2.58**
A d'Arsonval meter movement.

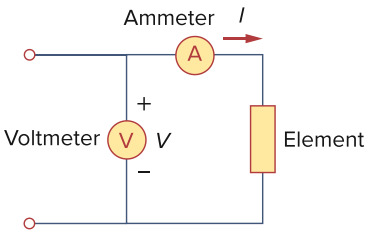
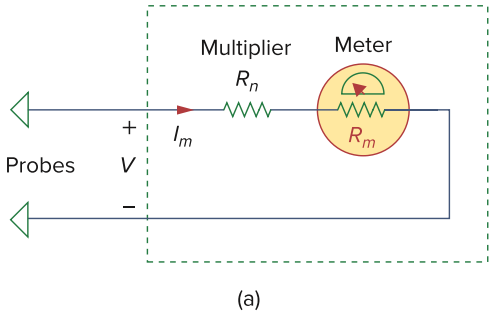


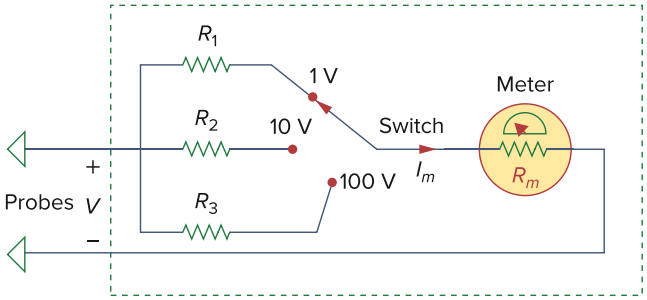

**Figure 2.59**
Connection of a voltmeter and an ammeter to an element.

is therefore connected in parallel with the element. As shown in Fig. 2.60(a), the voltmeter consists of a d'Arsonval movement in series with a resistor whose resistance $R_m$ is deliberately made very large (theoretically, infinite), to minimize the current drawn from the circuit. To extend the range of voltage that the meter can measure, series multiplier resistors are often connected with the voltmeters, as shown in Fig. 2.60(b). The multiple-range voltmeter in Fig. 2.60(b) can measure voltage from 0 to 1 V, 0 to 10 V, or 0 to 100 V, depending on whether the switch is connected to $R_1$, $R_2$, or $R_3$, respectively.

Let us calculate the multiplier resistor $R_n$ for the single-range voltmeter in Fig. 2.60(a), or $R_n = R_1$, $R_2$, or $R_3$ for the multiple-range voltmeter in Fig. 2.60(b). We need to determine the value of $R_n$ to be connected in series with the internal resistance $R_m$ of the voltmeter. In any design, we consider the worst-case condition. In this case, the worst case occurs when the full-scale current $I_{fs} = I_m$ flows through the meter. This should also correspond




(a)




(b)

**Figure 2.60**
Voltmeters: (a) single-range type, (b) multiple-range type.

to the maximum voltage reading or the full-scale voltage $V_{fs}$. Since the multiplier resistance $R_n$ is in series with the internal resistance $R_m$,

$$V_{fs} = I_{fs}(R_n + R_m) \tag{2.59}$$

From this, we obtain

$$R_n = \frac{V_{fs}}{I_{fs}} - R_m \tag{2.60}$$

Similarly, the ammeter measures the current through the load and is connected in series with it. As shown in Fig. 2.61(a), the ammeter consists of a d'Arsonval movement in parallel with a resistor whose resistance $R_m$ is deliberately made very small (theoretically, zero) to minimize the voltage drop across it. To allow multiple ranges, shunt resistors are often connected in parallel with $R_m$ as shown in Fig. 2.61(b). The shunt resistors allow the meter to measure in the range 0–10 mA, 0–100 mA, or 0–1 A, depending on whether the switch is connected to $R_1$, $R_2$, or $R_3$, respectively.

Now our objective is to obtain the multiplier shunt $R_n$ for the single-range ammeter in Fig. 2.61(a), or $R_n = R_1, R_2$, or $R_3$ for the multiple-range ammeter in Fig. 2.61(b). We notice that $R_m$ and $R_n$ are in parallel and that at full-scale reading $I = I_{fs} = I_m + I_n$, where $I_n$ is the current through the shunt resistor $R_n$. Applying the current division principle yields

$$I_m = \frac{R_n}{R_n + R_m} I_{fs}$$

or

$$R_n = \frac{I_m}{I_{fs} - I_m} R_m \tag{2.61}$$

The resistance $R_x$ of a linear resistor can be measured in two ways. An indirect way is to measure the current $I$ that flows through it by connecting an ammeter in series with it and the voltage $V$ across it by connecting a voltmeter in parallel with it, as shown in Fig. 2.62(a). Then

$$R_x = \frac{V}{I} \tag{2.62}$$

The direct method of measuring resistance is to use an ohmmeter. An ohmmeter consists basically of a d'Arsonval movement, a variable resistor or potentiometer, and a battery, as shown in Fig. 2.62(b). Applying KVL to the circuit in Fig. 2.62(b) gives

$$E = (R + R_m + R_x)I_m$$

or

$$R_x = \frac{E}{I_m} - (R + R_m) \tag{2.63}$$

The resistor $R$ is selected such that the meter gives a full-scale deflection; that is, $I_m = I_{fs}$ when $R_x = 0$. This implies that

$$E = (R + R_m)\, I_{fs} \tag{2.64}$$

Substituting Eq. (2.64) into Eq. (2.63) leads to

$$R_x = \left(\frac{I_{fs}}{I_m} - 1\right)(R + R_m) \tag{2.65}$$

As mentioned, the types of meters we have discussed are known as *analog* meters and are based on the d'Arsonval meter movement. Another type of meter, called a *digital meter*, is based on active circuit elements

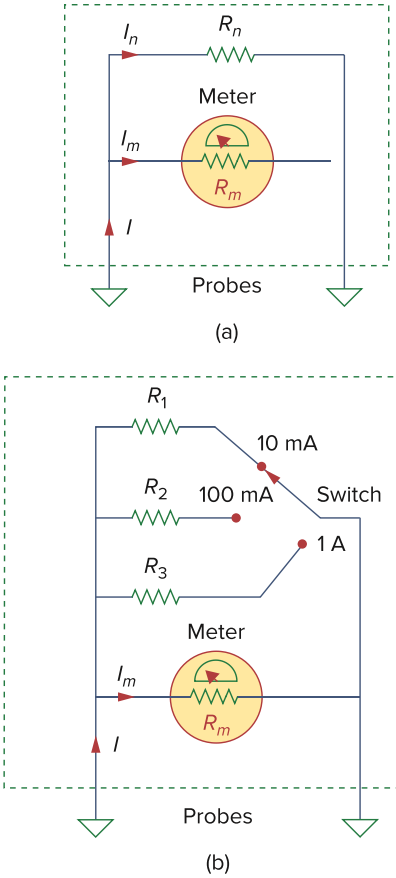



**Figure 2.61**
Ammeters: (a) single-range type, (b) multiple-range type.

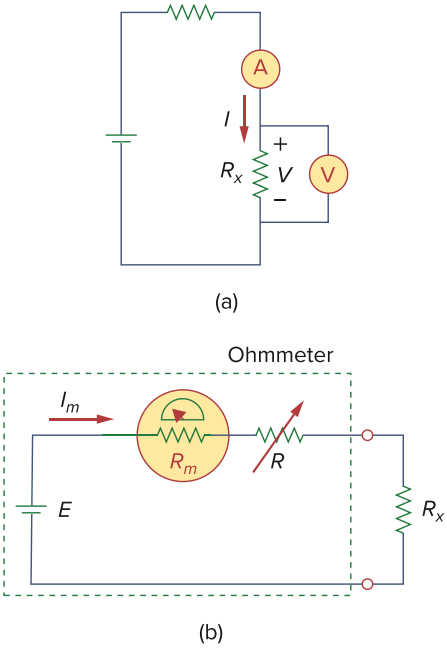
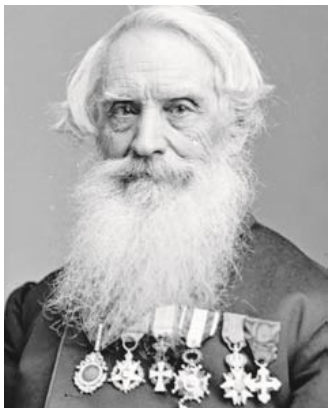



**Figure 2.62**
Two ways of measuring resistance: (a) using an ammeter and a voltmeter, (b) using an ohmmeter.

## Historical



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**Samuel F. B. Morse** (1791–1872), an American painter, invented the telegraph, the first practical, commercialized application of electricity.

Morse was born in Charlestown, Massachusetts, and studied at Yale and the Royal Academy of Arts in London to become an artist. In the 1830s, he became intrigued with developing a telegraph. He had a working model by 1836 and applied for a patent in 1838. The U.S. Senate appropriated funds for Morse to construct a telegraph line between Baltimore and Washington, D.C. On May 24, 1844, he sent the famous first message: "What hath God wrought!" Morse also developed a code of dots and dashes for letters and numbers, for sending messages on the telegraph. The development of the telegraph led to the invention of the telephone.

such as op amps. For example, a digital multimeter displays measurements of dc or ac voltage, current, and resistance as discrete numbers, instead of using a pointer deflection on a continuous scale as in an analog multimeter. Digital meters are what you would most likely use in a modern lab. However, the design of digital meters is beyond the scope of this book.

## Example 2.17

Following the voltmeter setup of Fig. 2.60, design a voltmeter for the following multiple ranges:
(a) 0–1 V (b) 0–5 V (c) 0–50 V (d) 0–100 V
Assume that the internal resistance $R_m = 2\ \text{k}\Omega$ and the full-scale current $I_{\text{fs}} = 100\ \mu\text{A}$.

**Solution:**
We apply Eq. (2.60) and assume that $R_1$, $R_2$, $R_3$, and $R_4$ correspond with ranges 0–1 V, 0–5 V, 0–50 V, and 0–100 V, respectively.
(a) For range 0–1 V,

$$R_1 = \frac{1}{100 \times 10^{-6}} - 2000 = 10{,}000 - 2000 = 8\ \text{k}\Omega$$

(b) For range 0–5 V,

$$R_2 = \frac{5}{100 \times 10^{-6}} - 2000 = 50{,}000 - 2000 = 48\ \text{k}\Omega$$

(c) For range 0–50 V,

$$R_3 = \frac{50}{100 \times 10^{-6}} - 2000 = 500{,}000 - 2000 = 498\ \text{k}\Omega$$

(d) For range 0–100 V,

$$R_4 = \frac{100\ \text{V}}{100 \times 10^{-6}} - 2000 = 1{,}000{,}000 - 2000 = 998\ \text{k}\Omega$$

Note that the ratio of the total resistance $(R_n + R_m)$ to the full-scale voltage $V_{\text{fs}}$ is constant and equal to $1/I_{\text{fs}}$ for the four ranges. This ratio (given in ohms per volt, or $\Omega/\text{V}$) is known as the *sensitivity* of the voltmeter. The larger the sensitivity, the better the voltmeter.

### Practice Problem 2.17

Following the ammeter setup of Fig. 2.61, design an ammeter for the following multiple ranges:
(a) 0–1 A (b) 0–100 mA (c) 0–10 mA
Take the full-scale meter current as $I_m = 1$ mA and the internal resistance of the ammeter as $R_m = 50\ \Omega$.

**Answer:** Shunt resistors: 50 mΩ, 505 mΩ, 5.556 Ω.

## 2.9 Summary

1. A resistor is a passive element in which the voltage $v$ across it is directly proportional to the current $i$ through it. That is, a resistor is a device that obeys Ohm's law,
$$v = iR$$
where $R$ is the resistance of the resistor.
2. A short circuit is a resistor (a perfectly conducting wire) with zero resistance ($R = 0$). An open circuit is a resistor with infinite resistance ($R = \infty$).
3. The conductance $G$ of a resistor is the reciprocal of its resistance:
$$G = \frac{1}{R}$$
4. A branch is a single two-terminal element in an electric circuit. A node is the point of connection between two or more branches. A loop is a closed path in a circuit. The number of branches $b$, the number of nodes $n$, and the number of independent loops $l$ in a network are related as
$$b = l + n - 1$$
5. Kirchhoff's current law (KCL) states that the currents at any node algebraically sum to zero. In other words, the sum of the currents entering a node equals the sum of currents leaving the node.
6. Kirchhoff's voltage law (KVL) states that the voltages around a closed path algebraically sum to zero. In other words, the sum of voltage rises equals the sum of voltage drops.
7. Two elements are in series when they are connected sequentially, end to end. When elements are in series, the same current flows through them ($i_1 = i_2$). They are in parallel if they are connected to the same two nodes. Elements in parallel always have the same voltage across them ($v_1 = v_2$).
8. When two resistors $R_1$ ($= 1/G_1$) and $R_2$ ($= 1/G_2$) are in series, their equivalent resistance $R_{eq}$ and equivalent conductance $G_{eq}$ are
$$R_{eq} = R_1 + R_2, \qquad G_{eq} = \frac{G_1 G_2}{G_1 + G_2}$$
9. When two resistors $R_1$ ($= 1/G_1$) and $R_2$ ($= 1/G_2$) are in parallel, their equivalent resistance $R_{eq}$ and equivalent conductance $G_{eq}$ are
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}, \qquad G_{eq} = G_1 + G_2$$

10. The voltage division principle for two resistors in series is

$$v_1 = \frac{R_1}{R_1 + R_2} v, \qquad v_2 = \frac{R_2}{R_1 + R_2} v$$

11. The current division principle for two resistors in parallel is

$$i_1 = \frac{R_2}{R_1 + R_2} i, \qquad i_2 = \frac{R_1}{R_1 + R_2} i$$

12. The formulas for a delta-to-wye transformation are

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}, \qquad R_2 = \frac{R_c R_a}{R_a + R_b + R_c}$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

13. The formulas for a wye-to-delta transformation are

$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}, \qquad R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}$$

$$R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$$

14. The basic laws covered in this chapter can be applied to the problems of electrical lighting and design of dc meters.

## Review Questions

**2.1** The reciprocal of resistance is:

(a) voltage (b) current
(c) conductance (d) coulombs

**2.2** An electric heater draws 10 A from a 120-V line. The resistance of the heater is:

(a) 1200 Ω (b) 120 Ω
(c) 12 Ω (d) 1.2 Ω

**2.3** The voltage drop across a 1.5-kW toaster that draws 12 A of current is:

(a) 18 kV (b) 125 V
(c) 120 V (d) 10.42 V

**2.4** The maximum current that a 2W, 80 kΩ resistor can safely conduct is:

(a) 160 kA (b) 40 kA
(c) 5 mA (d) 25 $\mu$A

**2.5** A network has 12 branches and 8 independent loops. How many nodes are there in the network?

(a) 19 (b) 17 (c) 5 (d) 4

**2.6** The current $I$ in the circuit of Fig. 2.63 is:

(a) −0.8 A (b) −0.2 A
(c) 0.2 A (d) 0.8 A

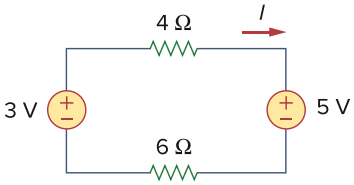



**Figure 2.63**
For Review Question 2.6.

**2.7** The current $I_o$ of Fig. 2.64 is:

(a) −4 A (b) −2 A (c) 4 A (d) 16 A

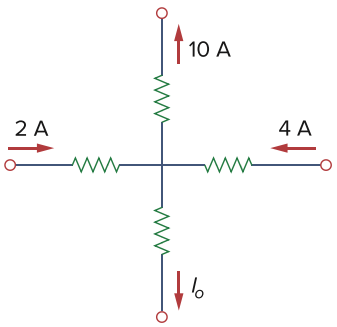



**Figure 2.64**
For Review Question 2.7.

**2.8** In the circuit in Fig. 2.65, $V$ is:

(a) 30 V (b) 14 V (c) 10 V (d) 6 V

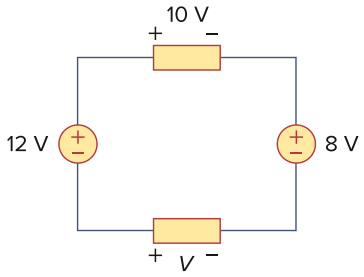



**Figure 2.65**
For Review Question 2.8.

**2.9** Which of the circuits in Fig. 2.66 will give you $V_{ab} = 7$ V?

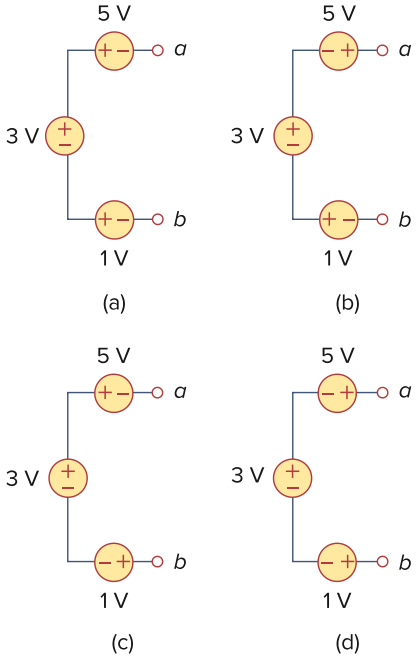



**Figure 2.66**
For Review Question 2.9.

**2.10** In the circuit of Fig. 2.67, a decrease in $R_3$ leads to a decrease of, select all that apply:

(a) current through $R_3$

(b) voltage across $R_3$

(c) voltage across $R_1$

(d) power dissipated in $R_2$

(e) none of the above

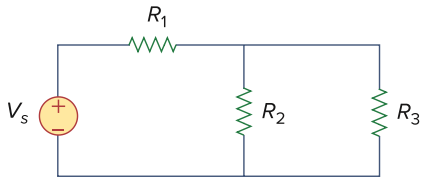



**Figure 2.67**
For Review Question 2.10.

*Answers: 2.1c, 2.2c, 2.3b, 2.4c, 2.5c, 2.6b, 2.7a, 2.8d, 2.9d, 2.10b, d.*

## Problems

### Section 2.2 Ohm's Law

**2.1** Design a problem, complete with a solution, to help students to better understand Ohm's law. Use at least two resistors and one voltage source. Hint, you could use both resistors at once or one at a time, it is up to you. Be creative.

**2.2** Find the hot resistance of a light bulb rated 60 W, 120 V.

**2.3** A bar of silicon is 4 cm long with a circular cross section. If the resistance of the bar is 240 Ω at room temperature, what is the cross-sectional radius of the bar?

**2.4** (a) Calculate current $i$ in Fig. 2.68 when the switch is in position 1.

(b) Find the current when the switch is in position 2.

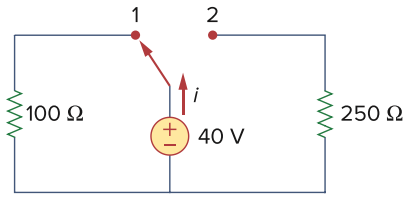



**Figure 2.68**
For Prob. 2.4.

## Section 2.3 Nodes, Branches, and Loops

**2.5** For the network graph in Fig. 2.69, find the number of nodes, branches, and loops.

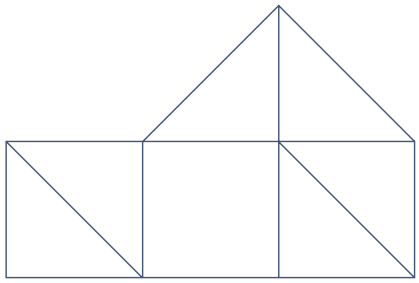


**Figure 2.69**
For Prob. 2.5.

**2.6** In the network graph shown in Fig. 2.70, determine the number of branches and nodes.

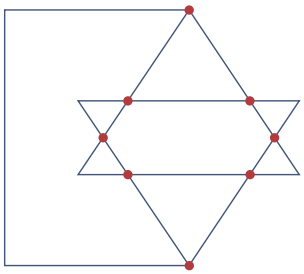


**Figure 2.70**
For Prob. 2.6.

**2.7** Determine the number of branches and nodes in the circuit of Fig. 2.71.

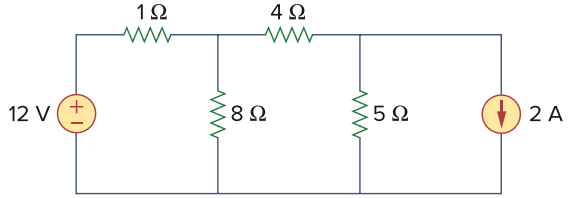



**Figure 2.71**
For Prob. 2.7.

## Section 2.4 Kirchhoff's Laws

**2.8** Design a problem, complete with a solution, to help other students better understand Kirchhoff's Current Law. Design the problem by specifying values of $i_a$, $i_b$, and $i_c$, shown in Fig. 2.72, and asking them to solve for values of $i_1$, $i_2$, and $i_3$. Be careful to specify realistic currents.

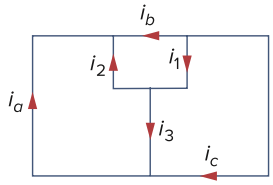



**Figure 2.72**
For Prob. 2.8.

**2.9** Find $i_1$, $i_2$, and $i_3$ in Fig. 2.73.

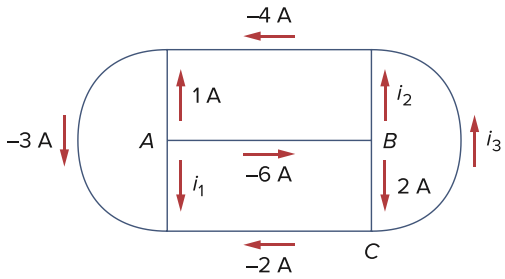



**Figure 2.73**
For Prob. 2.9.

**2.10** Determine $i_1$ and $i_2$ in the circuit of Fig. 2.74.

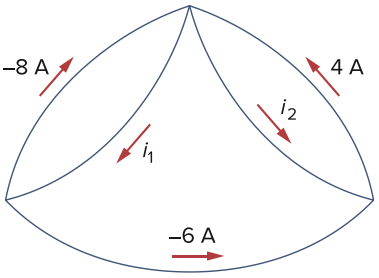



**Figure 2.74**
For Prob. 2.10.

**2.11** In the circuit of Fig. 2.75, calculate $V_1$ and $V_2$.

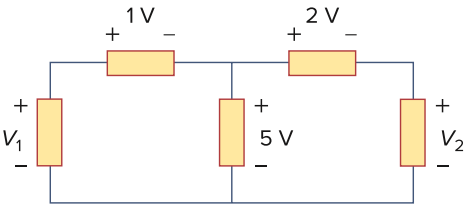



**Figure 2.75**
For Prob. 2.11.

**2.12** In the circuit in Fig. 2.76, obtain $v_1$, $v_2$, and $v_3$.

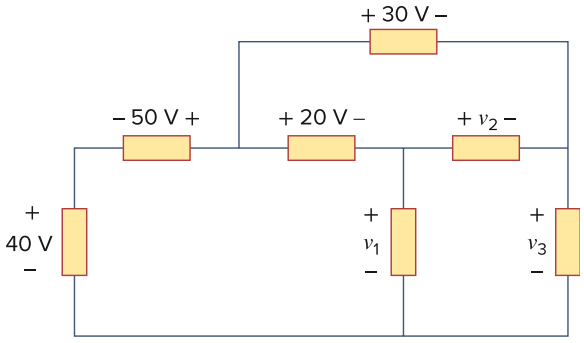



**Figure 2.76**
For Prob. 2.12.

**2.13** For the circuit in Fig. 2.77, use KCL to find the branch currents $I_1$ to $I_4$.

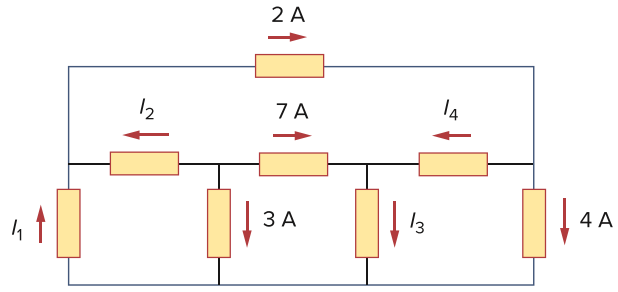



**Figure 2.77**
For Prob. 2.13.

**2.14** Given the circuit in Fig. 2.78, use KVL to find the branch voltages $V_1$ to $V_4$.

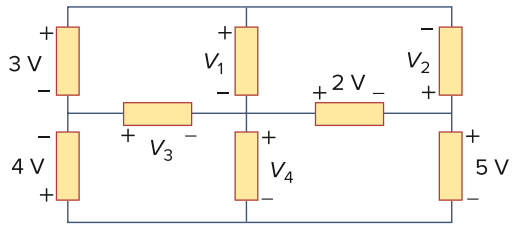



**Figure 2.78**
For Prob. 2.14.

**2.15** Calculate $v$ and $i_x$ in the circuit of Fig. 2.79.

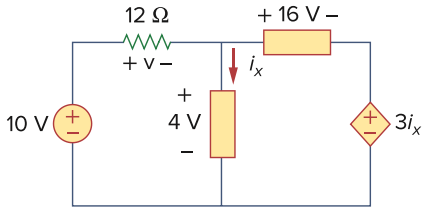



**Figure 2.79**
For Prob. 2.15.

**2.16** Determine $V_o$ in the circuit in Fig. 2.80.

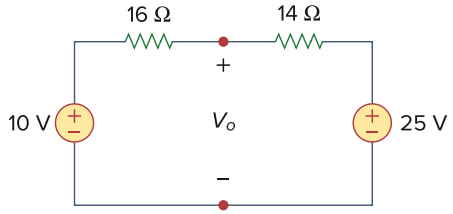



**Figure 2.80**
For Prob. 2.16.

**2.17** Obtain $v_1$ through $v_3$ in the circuit of Fig. 2.81.

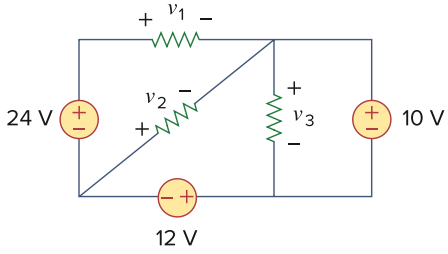



**Figure 2.81**
For Prob. 2.17.

**2.18** Find $I$ and $V$ in the circuit of Fig. 2.82.

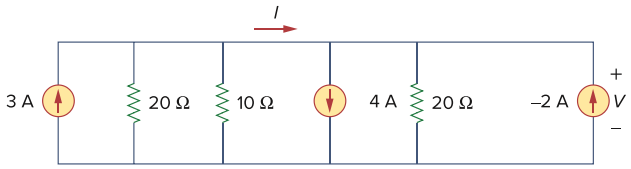



**Figure 2.82**
For Prob. 2.18.

**2.19** From the circuit in Fig. 2.83, find $I$, the power dissipated by the resistor, and the power supplied by each source.

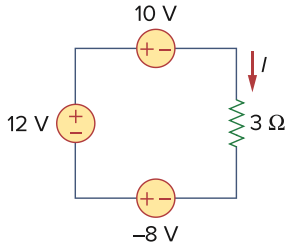



**Figure 2.83**
For Prob. 2.19.

**2.20** Determine $i_o$ in the circuit of Fig. 2.84.

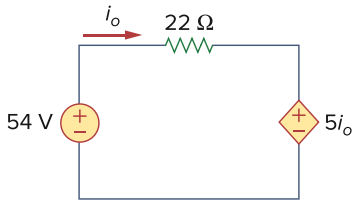



**Figure 2.84**
For Prob. 2.20.

**2.21** Find $V_x$ in the circuit of Fig. 2.85.

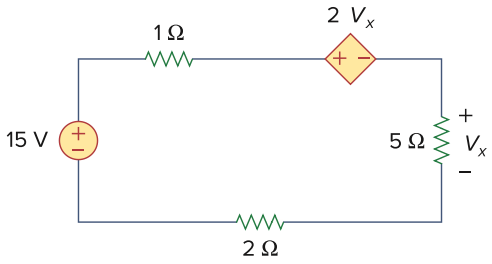



**Figure 2.85**
For Prob. 2.21.

**2.22** Find $V_o$ in the circuit in Fig. 2.86 and the power absorbed by the dependent source.

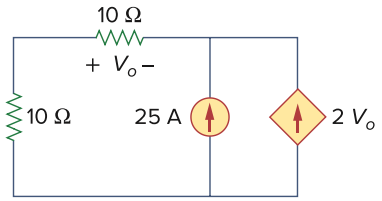



**Figure 2.86**
For Prob. 2.22.

**2.23** In the circuit shown in Fig. 2.87, determine $V_x$ and the power absorbed by the 60-Ω resistor.

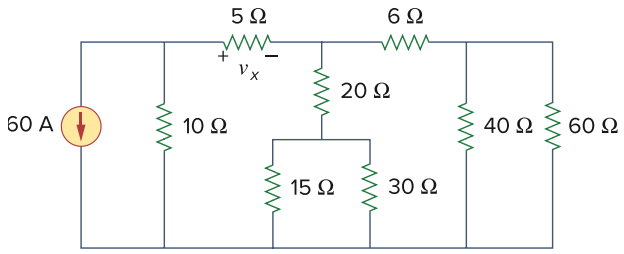



**Figure 2.87**
For Prob. 2.23.

**2.24** For the circuit in Fig. 2.88, find $V_o/V_s$ in terms of $\alpha$, $R_1$, $R_2$, $R_3$, and $R_4$. If $R_1 = R_2 = R_3 = R_4$, what value of $\alpha$ will produce $|V_o/V_s| = 10$?

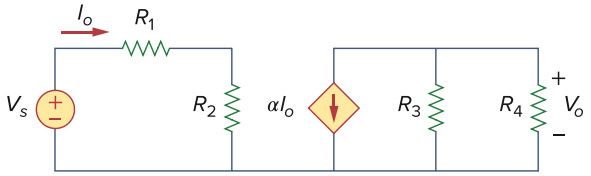



**Figure 2.88**
For Prob. 2.24.

**2.25** For the network in Fig. 2.89, find the current, voltage, and power associated with the 20-kΩ resistor.

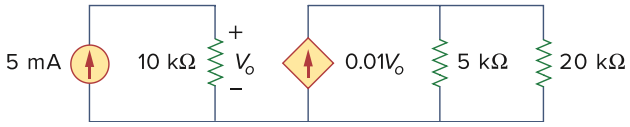



**Figure 2.89**
For Prob. 2.25.

## Sections 2.5 and 2.6 Series and Parallel Resistors

**2.26** For the circuit in Fig. 2.90, $i_o = 3$ A. Calculate $i_x$ and the total power absorbed by the entire circuit.

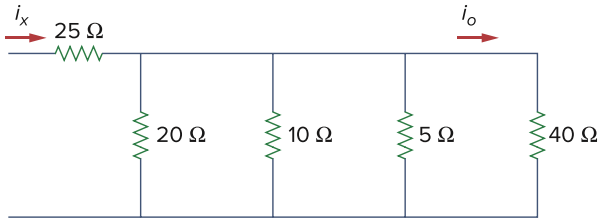



**Figure 2.90**
For Prob. 2.26.

**2.27** Calculate $I_o$ in the circuit of Fig. 2.91.

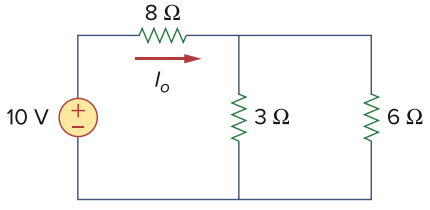



**Figure 2.91**
For Prob. 2.27.

**2.28** Design a problem, using Fig. 2.92, to help other students better understand series and parallel circuits.

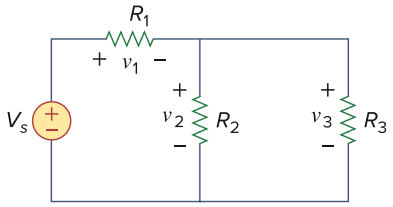



**Figure 2.92**
For Prob. 2.28.

**2.29** All resistors (R) in Fig. 2.93 are 10 Ω each. Find $R_{eq}$.

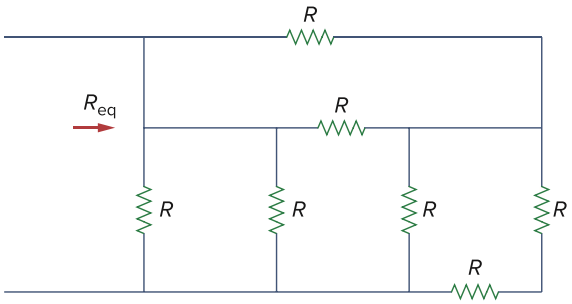



**Figure 2.93**
For Prob. 2.29.

**2.30** Find $R_{eq}$ for the circuit in Fig. 2.94.

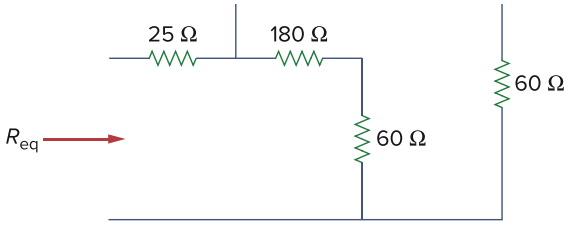



**Figure 2.94**
For Prob. 2.30.

**2.31** For the circuit in Fig. 2.95, determine $i_1$ to $i_5$.

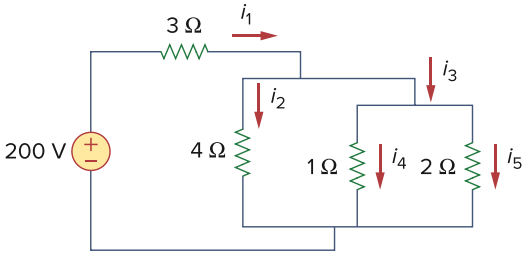



**Figure 2.95**
For Prob. 2.31.

**2.32** Find $i_1$ through $i_4$ in the circuit in Fig. 2.96.

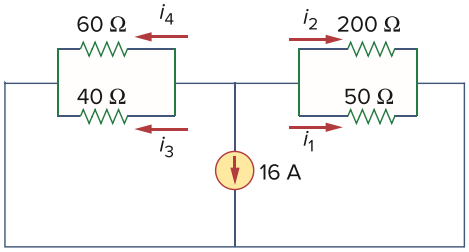



**Figure 2.96**
For Prob. 2.32.

**2.33** Obtain $v$ and $i$ in the circuit of Fig. 2.97.

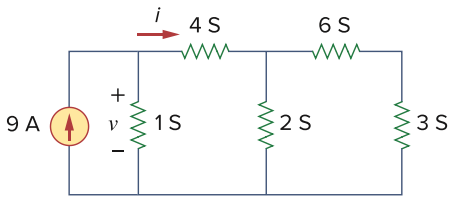



**Figure 2.97**
For Prob. 2.33.

**2.34** Using series/parallel resistance combination, find the equivalent resistance seen by the source in the circuit of Fig. 2.98. Find the overall absorbed power by the resistor network.

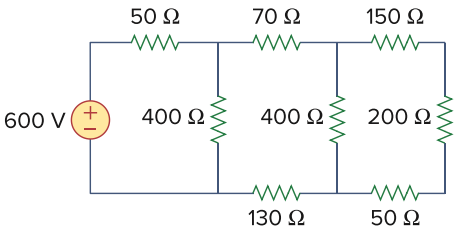



**Figure 2.98**
For Prob. 2.34.

**2.35** Calculate $V_o$ and $I_o$ in the circuit of Fig. 2.99.

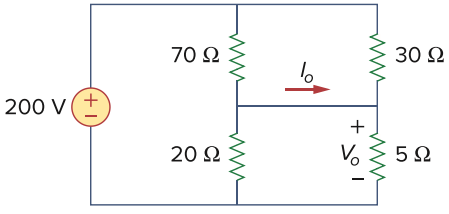



**Figure 2.99**
For Prob. 2.35.

**2.36** Find $i$ and $V_o$ in the circuit of Fig. 2.100.

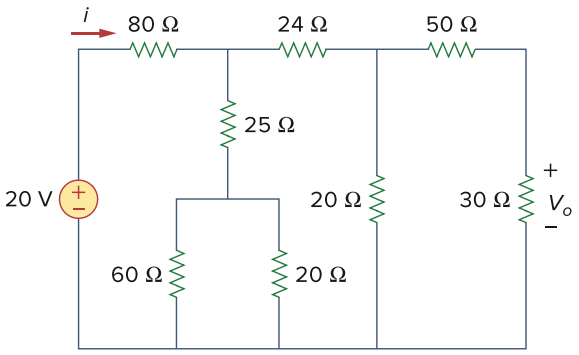



**Figure 2.100**
For Prob. 2.36.

**2.37** Given the circuit in Fig. 2.101 and that the resistance, $R_{eq}$, looking into the circuit from the left is equal to 100 Ω, determine the value of $R_1$.

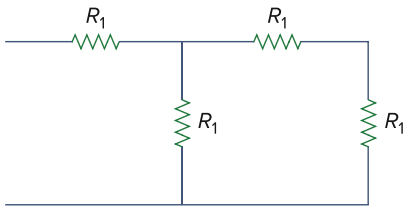



**Figure 2.101**
For Prob. 2.37.

**2.38** Find $R_{eq}$ and $i_o$ in the circuit of Fig. 2.102.

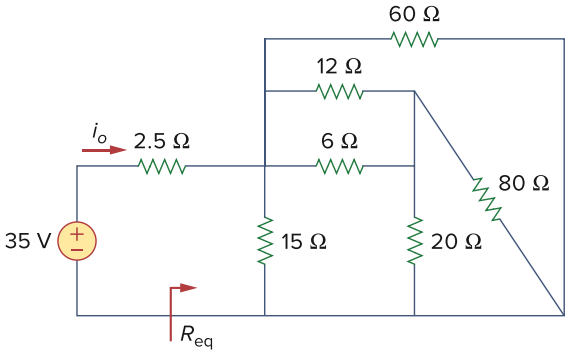



**Figure 2.102**
For Prob. 2.38.

**2.39** Evaluate $R_{eq}$ looking into each set of terminals for each of the circuits shown in Fig. 2.103.

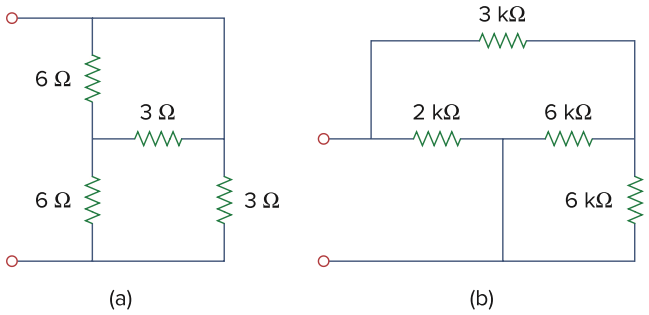



**Figure 2.103**
For Prob. 2.39.

**2.40** For the ladder network in Fig. 2.104, find $I$ and $R_{eq}$.

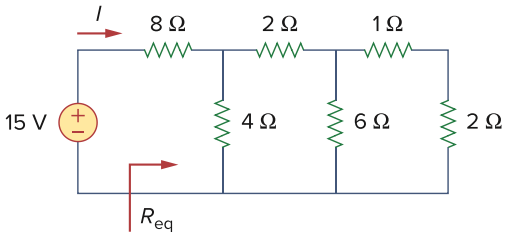



**Figure 2.104**
For Prob. 2.40.

**2.41** If $R_{eq} = 50\ \Omega$ in the circuit of Fig. 2.105, find $R$.

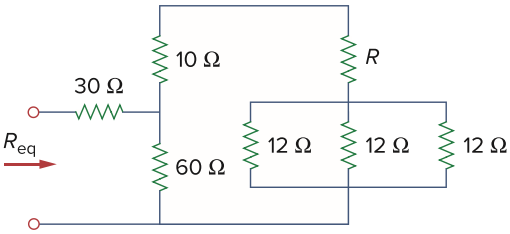
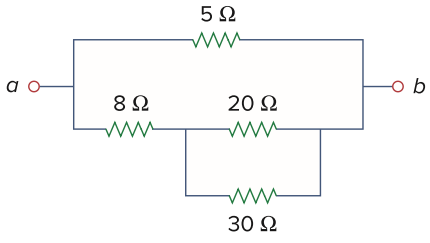



**Figure 2.105**
For Prob. 2.41.

**2.42** Reduce each of the circuits in Fig. 2.106 to a single resistor at terminals $a$-$b$.



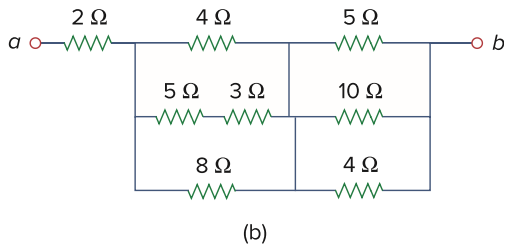



**Figure 2.106**
For Prob. 2.42.

**2.43** Calculate the equivalent resistance $R_{ab}$ at terminals *a-b* for each of the circuits in Fig. 2.107.

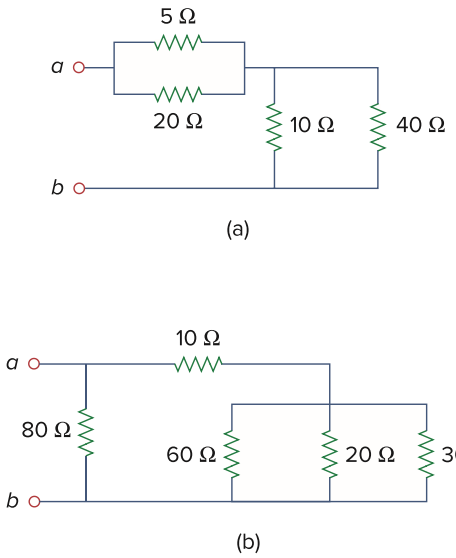



**Figure 2.107**
For Prob. 2.43.

**2.44** For the circuits in Fig. 2.108, obtain the equivalent resistance at terminals *a-b*.

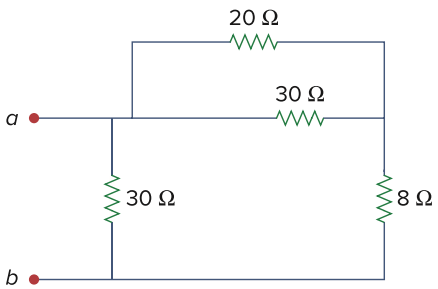
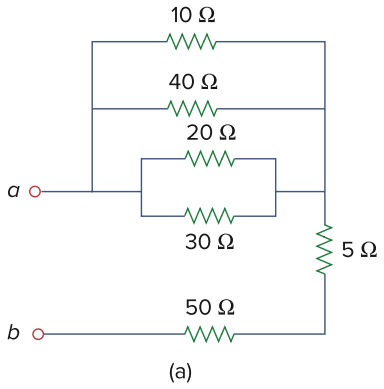



**Figure 2.108**
For Prob. 2.44.

**2.45** Find the equivalent resistance at terminals *a-b* of each circuit in Fig. 2.109.



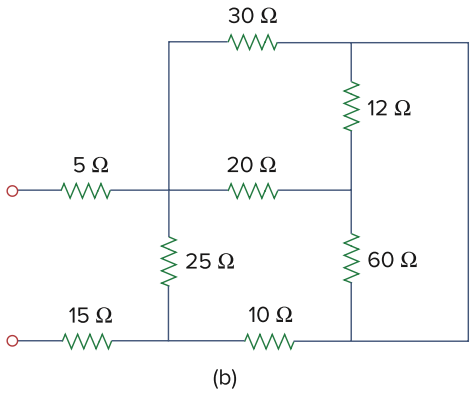




**Figure 2.109**
For Prob. 2.45.

**2.46** Find *I* in the circuit of Fig. 2.110.

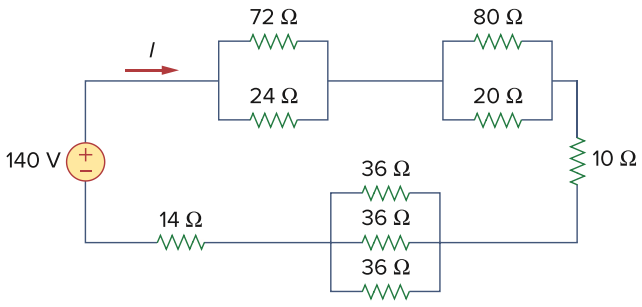



**Figure 2.110**
For Prob. 2.46.

**2.47** Find the equivalent resistance $R_{ab}$ in the circuit of Fig. 2.111.

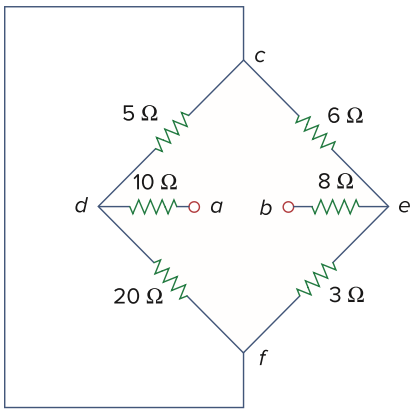



**Figure 2.111**
For Prob. 2.47.

## Section 2.7 Wye-Delta Transformations

**2.48** Convert the circuits in Fig. 2.112 from Y to Δ.

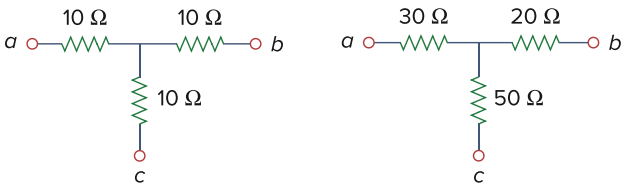



**Figure 2.112**
For Prob. 2.48.

**2.49** Transform the circuits in Fig. 2.113 from Δ to Y.

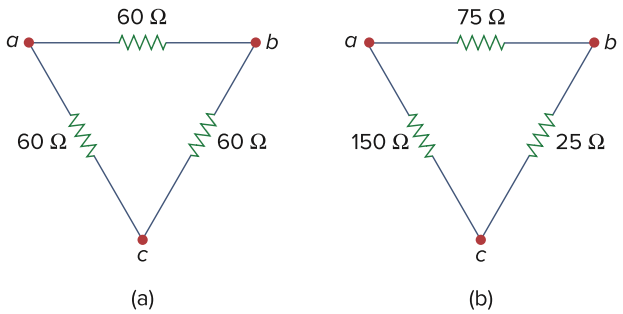



**Figure 2.113**
For Prob. 2.49.

**2.50** Design a problem to help other students better understand wye-delta transformations using Fig. 2.114.

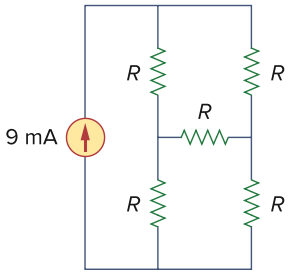



**Figure 2.114**
For Prob. 2.50.

**2.51** Obtain the equivalent resistance at the terminals *a*-*b* for each of the circuits in Fig. 2.115.

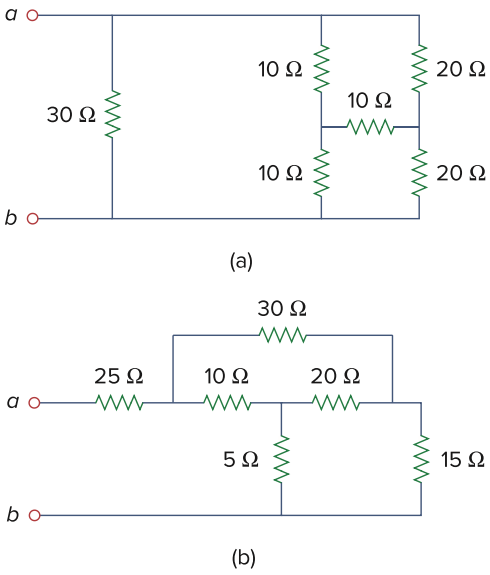



**Figure 2.115**
For Prob. 2.51.

***2.52** For the circuit shown in Fig. 2.116, find the equivalent resistance. All resistors are 3Ω.

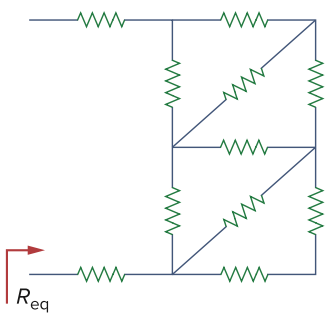



**Figure 2.116**
For Prob. 2.52.

* An asterisk indicates a challenging problem.

***2.53** Obtain the equivalent resistance $R_{ab}$ in each of the circuits of Fig. 2.117. In (b), all resistors have a value of 30 Ω.

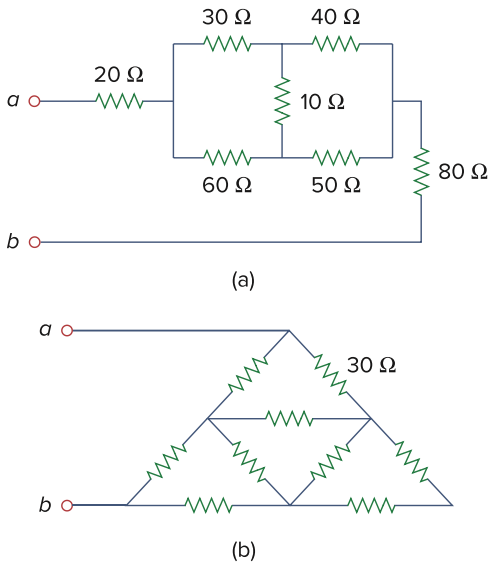



**Figure 2.117**
For Prob. 2.53.

**2.54** Consider the circuit in Fig. 2.118. Find the equivalent resistance at terminals: (a) *a*-*b*, (b) *c*-*d*.

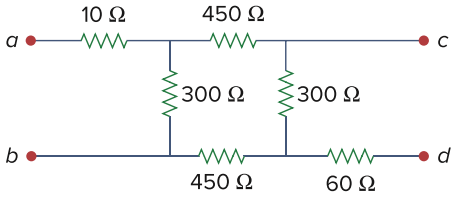



**Figure 2.118**
For Prob. 2.54.

**2.55** Calculate $I_o$ in the circuit of Fig. 2.119.

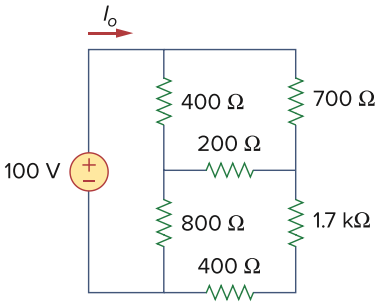



**Figure 2.119**
For Prob. 2.55.

**2.56** Determine $V$ in the circuit of Fig. 2.120.

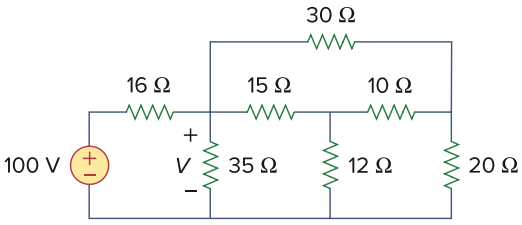



**Figure 2.120**
For Prob. 2.56.

***2.57** Find $R_{eq}$ and $I$ in the circuit of Fig. 2.121.

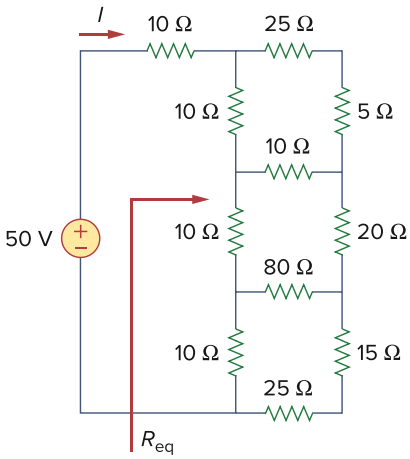



**Figure 2.121**
For Prob. 2.57.

## Section 2.8 Applications

**2.58** The 150 W light bulb in Fig. 2.122 is rated at 110 volts. Calculate the value of $V_s$ to make the light bulb operate at its rated conditions.

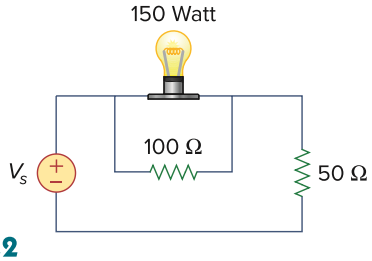



**Figure 2.122**
For Prob. 2.58.

**2.59** An enterprising young man travels to Europe carrying three light bulbs he had purchased in North America. The light bulbs he has are a 100-W light bulb, a 60-W light bulb, and a 40-W light bulb. Each light bulb is rated at 110 V. He wishes to connect these to a 220-V system that is found in Europe. For reasons we are not sure of, he connects the 40-W

light bulb in series with a parallel combination of the 60-W light bulb and the 100-W light bulb as shown in Fig. 2.123. How much power is actually being delivered to each light bulb? What does he see when he first turns on the light bulbs?

Is there a better way to connect these light bulbs in order to have them work more effectively?

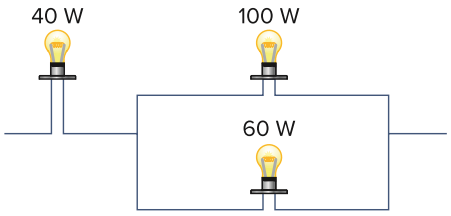



**Figure 2.123**
For Prob. 2.59.

**2.60** If the three bulbs of Prob. 2.59 are connected in parallel to the 120-V source, calculate the current through each bulb.

**2.61** As a design engineer, you are asked to design a lighting system consisting of a 70-W power supply and two light bulbs as shown in Fig. 2.124. You must select the two bulbs from the following three available bulbs.

$R_1 = 80\ \Omega$, cost = \$0.60 (standard size)
$R_2 = 90\ \Omega$, cost = \$0.90 (standard size)
$R_3 = 100\ \Omega$, cost = \$0.75 (nonstandard size)

The system should be designed for minimum cost such that $I$ lies within the range $I = 1.2\ \text{A} \pm 5$ percent.

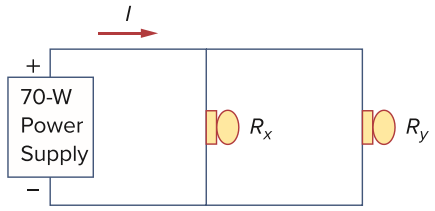



**Figure 2.124**
For Prob. 2.61.

**2.62** A three-wire system supplies two loads $A$ and $B$ as shown in Fig. 2.125. Load $A$ consists of a motor drawing a current of 8 A, while load $B$ is a PC drawing 2 A. Assuming 10 h/day of use for 365 days and 6 cents/kWh, calculate the annual energy cost of the system.

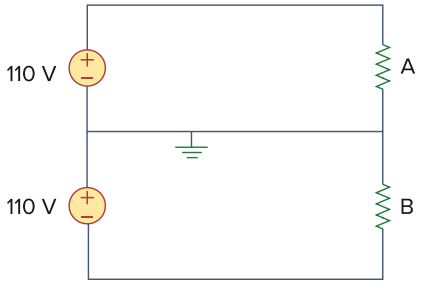



**Figure 2.125**

**2.63** If an ammeter with an internal resistance of 100 Ω and a current capacity of 2 mA is to measure 5 A, determine the value of the resistance needed. Calculate the power dissipated in the shunt resistor.

**2.64** The potentiometer (adjustable resistor) $R_x$ in Fig. 2.126 is to be designed to adjust current $i_x$ from 10 mA to 1 A. Calculate the values of $R$ and $R_x$ to achieve this.

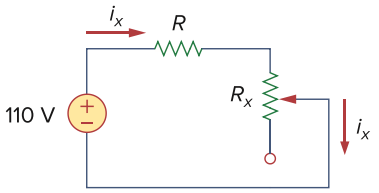



**Figure 2.126**
For Prob. 2.64.

**2.65** Design a circuit that uses a d'Arsonval meter (with an internal resistance of 2 kΩ that requires a current of 5 mA to cause the meter to deflect full scale) to build a voltmeter to read values of voltages up to 100 volts.

**2.66** A 20-kΩ/V voltmeter reads 10 V full scale.

(a) What series resistance is required to make the meter read 50 V full scale?

(b) What power will the series resistor dissipate when the meter reads full scale?

**2.67** (a) Obtain the voltage $V_o$ in the circuit of Fig. 2.127(a).

(b) Determine the voltage $V_o'$ measured when a voltmeter with 6-kΩ internal resistance is connected as shown in Fig. 2.127(b).

(c) The finite resistance of the meter introduces an error into the measurement. Calculate the percent error as

$$\left|\frac{V_o - V_o'}{V_o}\right| \times 100\ \%$$

(d) Find the percent error if the internal resistance were 36 kΩ.

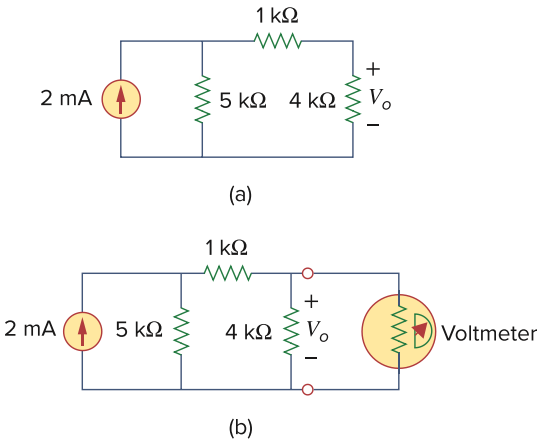



**Figure 2.127**
For Prob. 2.67.

**2.68** (a) Find the current $I$ in the circuit of Fig. 2.128(a).

(b) An ammeter with an internal resistance of 1 Ω is inserted in the network to measure $I'$ as shown in Fig. 2.128(b). What is $I'$?

(c) Calculate the percent error introduced by the meter as

$$\left|\frac{I - I'}{I}\right| \times 100\%$$

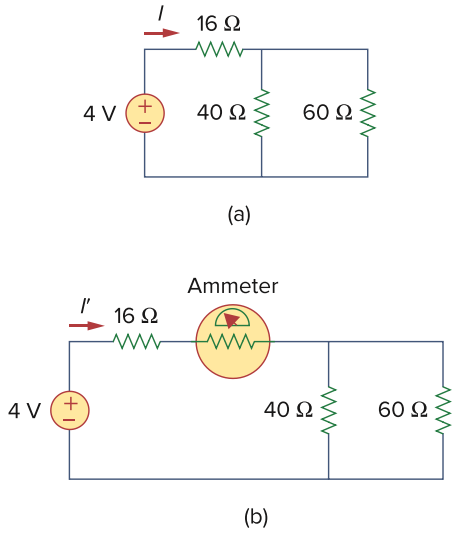



**Figure 2.128**
For Prob. 2.68.

**2.69** A voltmeter is used to measure $V_o$ in the circuit in Fig. 2.129. The voltmeter model consists of an ideal voltmeter in parallel with a 250-kΩ resistor. Let $V_s = 95$ V, $R_s = 25$ kΩ, and $R_1 = 40$ kΩ. Calculate $V_o$ with and without the voltmeter when

(a) $R_2 = 5$ kΩ (b) $R_2 = 25$ kΩ

(c) $R_2 = 250$ kΩ

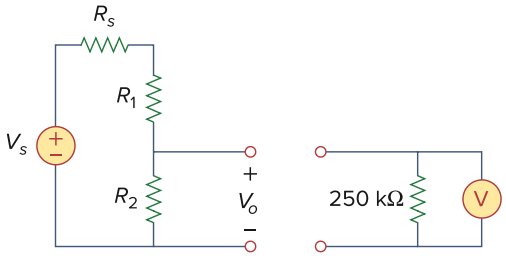



**Figure 2.129**
For Prob. 2.69.

**2.70** (a) Consider the Wheatstone bridge shown in Fig. 2.130. Calculate $v_a$, $v_b$, and $v_{ab}$.

(b) Rework part (a) if the ground is placed at $a$ instead of $o$.

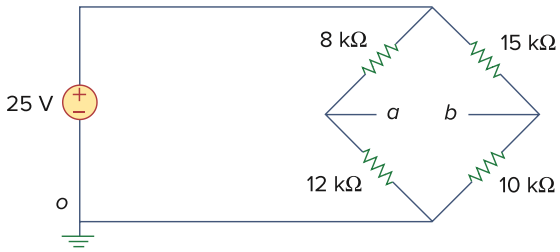



**Figure 2.130**
For Prob. 2.70.

**2.71** Figure 2.131 represents a model of a solar photovoltaic panel. Given that $V_s = 95$ V, $R_1 = 25$ Ω, and $i_L = 2$ A, find $R_L$.

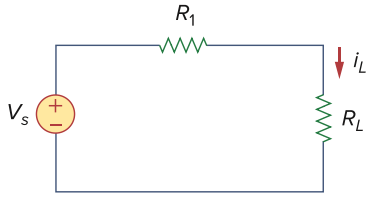



**Figure 2.131**
For Prob. 2.71.

**2.72** Find $V_o$ in the two-way power divider circuit in Fig. 2.132.

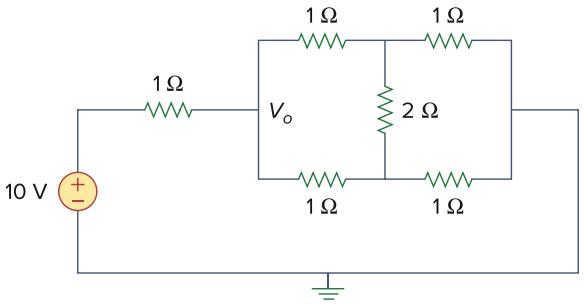



**Figure 2.132**
For Prob. 2.72.

**2.73** An ammeter model consists of an ideal ammeter in series with a 20-Ω resistor. It is connected with a current source and an unknown resistor $R_x$ as shown in Fig. 2.133. The ammeter reading is noted. When a potentiometer $R$ is added and adjusted until the ammeter reading drops to one half its previous reading, then $R = 65\ \Omega$. What is the value of $R_x$?

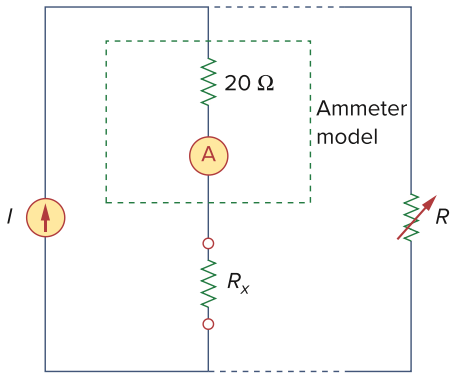



**Figure 2.133**
For Prob. 2.73.

**2.74** The circuit in Fig. 2.134 is to control the speed of a motor such that the motor draws currents 5 A, 3 A, and 1 A when the switch is at high, medium, and low positions, respectively. The motor can be modeled as a load resistance of 20 mΩ. Determine the series dropping resistances $R_1$, $R_2$, and $R_3$.

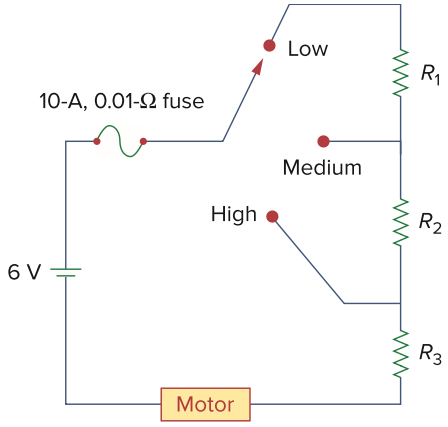



**Figure 2.134**
For Prob. 2.74.

**2.75** Find $R_{ab}$ in the four-way power divider circuit in Fig. 2.135. Assume each $R = 4\ \Omega$.

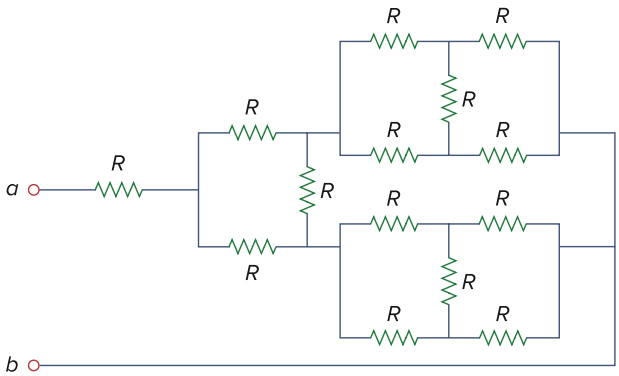



**Figure 2.135**
For Prob. 2.75.

## Comprehensive Problems

**2.76** Repeat Prob. 2.75 for the eight-way divider shown in Fig. 2.136.

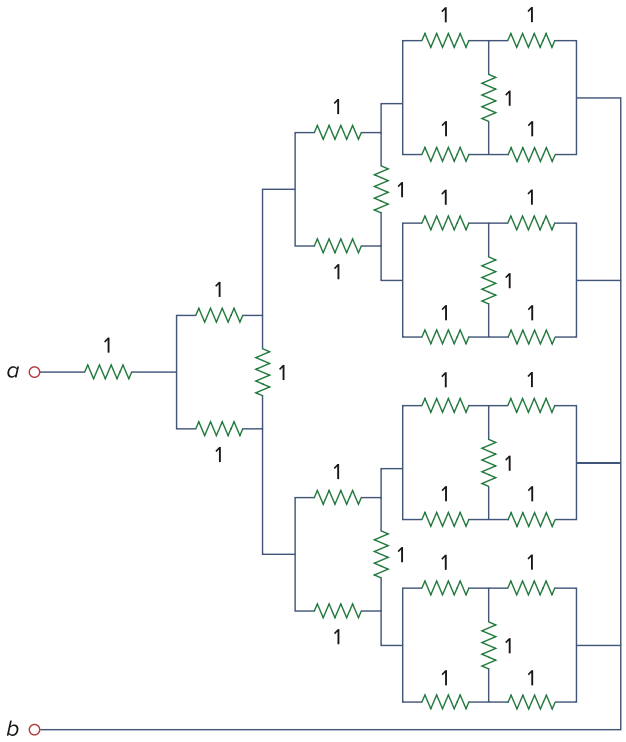



**Figure 2.136**
For Prob. 2.76.

**2.77** Suppose your circuit laboratory has the following standard commercially available resistors in large quantities:

1.8 Ω  20 Ω  300 Ω  24 kΩ  56 kΩ

Using series and parallel combinations and a minimum number of available resistors, how would you obtain the following resistances for an electronic circuit design?

(a) 5 Ω  (b) 311.8 Ω
(c) 40 kΩ  (d) 52.32 kΩ

**2.78** In the circuit in Fig. 2.137, the wiper divides the potentiometer resistance between $\alpha R$ and $(1 - \alpha)R$, $0 \le \alpha \le 1$. Find $v_o / v_s$.

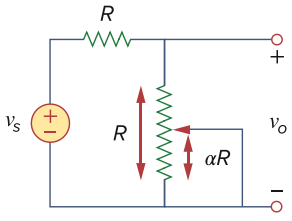



**Figure 2.137**
For Prob. 2.78.

**2.79** An electric pencil sharpener rated 240 mW, 6 V is connected to a 9-V battery as shown in Fig. 2.138. Calculate the value of the series-dropping resistor $R_x$ needed to power the sharpener.

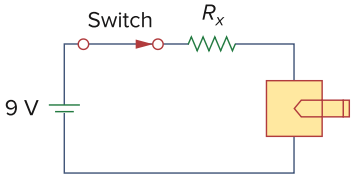



**Figure 2.138**
For Prob. 2.79.

**2.80** A loudspeaker is connected to an amplifier as shown in Fig. 2.139. If a 10-Ω loudspeaker draws the maximum power of 12 W from the amplifier, determine the maximum power a 4-Ω loudspeaker will draw.

**Figure 2.139**
For Prob. 2.80.

**2.81** For a specific application, the circuit shown in Fig. 2.140 was designed so that $I_L = 83.33$ mA and that $R_{in} = 5$ kΩ. What are the values of $R_1$ and $R_2$?

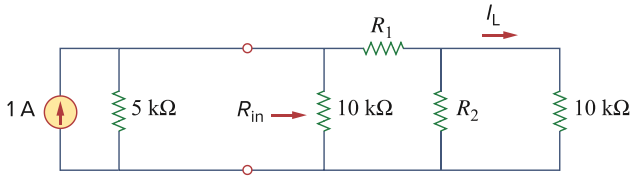



**Figure 2.140**
For Prob. 2.81.

**2.82** The pin diagram of a resistance array is shown in Fig. 2.141. Find the equivalent resistance between the following:

(a) 1 and 2
(b) 1 and 3
(c) 1 and 4

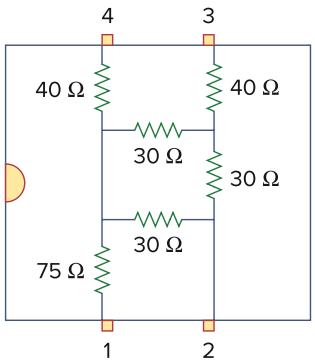



**Figure 2.141**
For Prob. 2.82.

**2.83** Two delicate devices are rated as shown in Fig. 2.142. Find the values of the resistors $R_1$ and $R_2$ needed to power the devices using a 36-V battery.

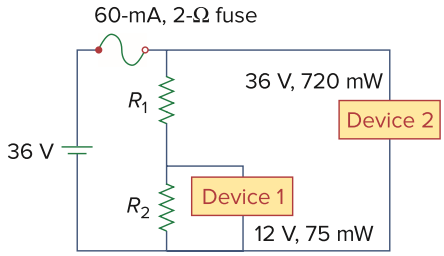



**Figure 2.142**
For Prob. 2.83.

chapter

# 3

# Methods of Analysis

*No great work is ever done in a hurry. To develop a great scientific discovery, to print a great picture, to write an immortal poem, to become a minister, or a famous general—to do anything great requires time, patience, and perseverance. These things are done by degrees, "little by little."*

—W. J. Wilmont Buxton

## Enhancing Your Career

### Career in Electronics

One area of application for electric circuit analysis is electronics. The term *electronics* was originally used to distinguish circuits of very low current levels. This distinction no longer holds, as power semiconductor devices operate at high le vels of current. Today, electronics is regarded as the science of the motion of char ges in a g as, vacuum, or semicon - ductor. Modern electronics in volves transistors and transistor circuits. The earlier electronic circuits were assembled from components. Man y electronic circuits are now produced as integrated circuits, fabricated in a semiconductor substrate or chip.

Electronic circuits find applications in many areas, such as automation, broadcasting, computers, and instrumentation. The range of devices that use electronic circuits is enormous and is limited only by our imagination. Radio, television, computers, and stereo systems are but a few.

An electrical engineer usually performs di verse functions and is likely to use, design, or construct systems that incorporate some form of electronic circuits. Therefore, an understanding of the operation and analysis of electronics is essential to the electrical engineer . Electronics has become a specialty distinct from other disciplines within electrical engi - neering. Because the field of electronics is ever advancing, an electronics engineer must update his/her knowledge from time to time. The best way to do this is by being a member of a professional organization such as the Institute of Electrical and Electronics Engineers (IEEE). With a membership of over 300,000, the IEEE is the largest professional organization in the world. Members benefit immensely from the numerous magazines, journals, transactions, and conference/symposium proceedings published yearly by IEEE. You should consider becoming an IEEE member.

Troubleshooting an electronic circuit board.
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## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Understand Kirchhoff's current law.
2. Understand Kirchhoff's voltage law.
3. Develop an understanding of how to use Kirchhoff's current law to write nodal equations and then to solve for unknown node voltages.
4. Develop an understanding of how to use Kirchhoff's voltage law to write mesh equations and then to solve for unknown loop currents.
5. Explain how to use *PSpice* to solve for unknown node voltages and currents.

## 3.1 Introduction

Having understood the fundamental laws of circuit theory (Ohm's law and Kirchhoff's laws), we are now prepared to apply these laws to develop two powerful techniques for circuit analysis: nodal analysis, which is based on a systematic application of Kirchhoff's current law (KCL), and mesh analysis, which is based on a systematic application of Kirchhoff's voltage law (KVL). The two techniques are so important that this chapter should be regarded as the most important in the book. Students are therefore encouraged to pay careful attention.

With the two techniques to be developed in this chapter, we can analyze any linear circuit by obtaining a set of simultaneous equations that are then solved to obtain the required values of current or voltage. One method of solving simultaneous equations involves Cramer's rule, which allows us to calculate circuit variables as a quotient of determinants. The examples in the chapter will illustrate this method; Appendix A also briefly summarizes the essentials the reader needs to know for applying Cramer's rule. Another method of solving simultaneous equations is to use *MATLAB*, a computer software discussed in Appendix E.

Also in this chapter, we introduce the use of *PSpice for Windows*, a circuit simulation computer software program that we will use throughout the text. Finally, we apply the techniques learned in this chapter to analyze transistor circuits.

## 3.2 Nodal Analysis

Nodal analysis is also known as the *node-voltage method.*

Nodal analysis provides a general procedure for analyzing circuits using node voltages as the circuit variables. Choosing node voltages instead of element voltages as circuit variables is convenient and reduces the number of equations one must solve simultaneously.

To simplify matters, we shall assume in this section that circuits do not contain voltage sources. Circuits that contain voltage sources will be analyzed in the next section.

In *nodal analysis*, we are interested in finding the node voltages. Given a circuit with n nodes without voltage sources, the nodal analysis of the circuit involves taking the following three steps.

### Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining $n - 1$ nodes. The voltages are referenced with respect to the reference node.
2. Apply KCL to each of the $n - 1$ nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

We shall now explain and apply these three steps.

The first step in nodal analysis is selecting a node as the *reference* or *datum node*. The reference node is commonly called the *ground* since it is assumed to have zero potential. A reference node is indicated by any of the three symbols in Fig. 3.1. The type of ground in Fig. 3.1(c) is called a *chassis ground* and is used in devices where the case, enclosure, or chassis acts as a reference point for all circuits. When the potential of the earth is used as reference, we use the *earth ground* in Fig. 3.1(a) or (b). We shall always use the symbol in Fig. 3.1(b).

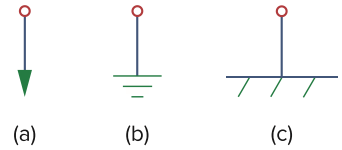



**Figure 3.1**
Common symbols for indicating a reference node, (a) common ground, (b) ground, (c) chassis ground.

Once we have selected a reference node, we assign voltage designations to nonreference nodes. Consider, for example, the circuit in Fig. 3.2(a). Node 0 is the reference node ($v = 0$), while nodes 1 and 2 are assigned voltages $v_1$ and $v_2$, respectively. Keep in mind that the node voltages are defined with respect to the reference node. As illustrated in Fig. 3.2(a), each node voltage is the voltage rise from the reference node to the corresponding nonreference node or simply the voltage of that node with respect to the reference node.

The number of nonreference nodes is equal to the number of independent equations that we will derive.

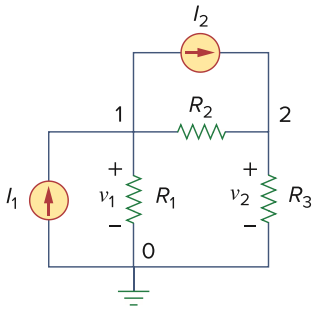
As the second step, we apply KCL to each nonreference node in the circuit. To avoid putting too much information on the same circuit, the circuit in Fig. 3.2(a) is redrawn in Fig. 3.2(b), where we now add $i_1$, $i_2$, and $i_3$ as the currents through resistors $R_1$, $R_2$, and $R_3$, respectively. At node 1, applying KCL gives

$$I_1 = I_2 + i_1 + i_2 \tag{3.1}$$

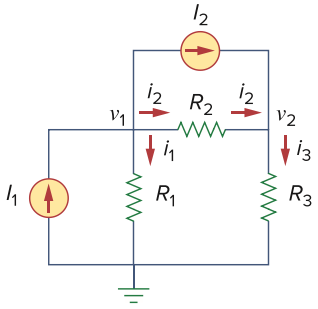
At node 2,

$$I_2 + i_2 = i_3 \tag{3.2}$$

We now apply Ohm's law to express the unknown currents $i_1$, $i_2$, and $i_3$ in terms of node voltages. The key idea to bear in mind is that, since resistance is a passive element, by the passive sign convention, current must always flow from a higher potential to a lower potential.




(a)




(b)

**Figure 3.2**
Typical circuit for nodal analysis.

Current flows from a higher potential to a lower potential in a resistor.

We can express this principle as

$$i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R} \tag{3.3}$$

Note that this principle is in agreement with the way we defined resistance in Chapter 2 (see Fig. 2.1). With this in mind, we obtain from Fig. 3.2(b),

$$i_1 = \frac{v_1 - 0}{R_1} \quad \text{or} \quad i_1 = G_1 v_1$$
$$i_2 = \frac{v_1 - v_2}{R_2} \quad \text{or} \quad i_2 = G_2(v_1 - v_2) \tag{3.4}$$
$$i_3 = \frac{v_2 - 0}{R_3} \quad \text{or} \quad i_3 = G_3 v_2$$

Substituting Eq. (3.4) in Eqs. (3.1) and (3.2) results, respectively, in

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} \tag{3.5}$$
$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3} \tag{3.6}$$

In terms of the conductances, Eqs. (3.5) and (3.6) become

$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2) \tag{3.7}$$
$$I_2 + G_2(v_1 - v_2) = G_3 v_2 \tag{3.8}$$

The third step in nodal analysis is to solve for the node voltages. If we apply KCL to $n - 1$ nonreference nodes, we obtain $n - 1$ simultaneous equations such as Eqs. (3.5) and (3.6) or (3.7) and (3.8). For the circuit of Fig. 3.2, we solve Eqs. (3.5) and (3.6) or (3.7) and (3.8) to obtain the node voltages $v_1$ and $v_2$ using any standard method, such as the substitution method, the elimination method, Cramer's rule, or matrix inversion. To use either of the last two methods, one must cast the simultaneous equations in matrix form. For example, Eqs. (3.7) and (3.8) can be cast in matrix form as

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix} \tag{3.9}$$

which can be solved to get $v_1$ and $v_2$. Equation 3.9 will be generalized in Section 3.6. The simultaneous equations may also be solved using calculators or with software packages such as *MATLAB, Mathcad, Maple,* and *Quattro Pro*.

Appendix A discusses how to use Cramer's rule.

## Example 3.1

Calculate the node voltages in the circuit shown in Fig. 3.3(a).

**Solution:**

Consider Fig. 3.3(b), where the circuit in Fig. 3.3(a) has been prepared for nodal analysis. Notice how the currents are selected for the application of KCL. Except for the branches with current sources, the labeling of the currents is arbitrary but consistent. (By consistent, we mean that if, for example, we assume that $i_2$ enters the 4-Ω resistor from the left-hand side, $i_2$ must leave the resistor from the right-hand side.) The reference node is selected, and the node voltages $v_1$ and $v_2$ are now to be determined.

At node 1, applying KCL and Ohm's law gives

$$i_1 = i_2 + i_3 \quad \Rightarrow \quad 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

Multiplying each term in the last equation by 4, we obtain

$$20 = v_1 - v_2 + 2v_1$$

or

$$3v_1 - v_2 = 20 \tag{3.1.1}$$

At node 2, we do the same thing and get

$$i_2 + i_4 = i_1 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

Multiplying each term by 12 results in

$$3v_1 - 3v_2 + 120 = 60 + 2v_2$$

or

$$-3v_1 + 5v_2 = 60 \tag{3.1.2}$$

Now we have two simultaneous Eqs. (3.1.1) and (3.1.2). We can solve the equations using any method and obtain the values of $v_1$ and $v_2$.

**METHOD 1** Using the elimination technique, we add Eqs. (3.1.1) and (3.1.2).

$$4v_2 = 80 \quad \Rightarrow \quad v_2 = 20 \text{ V}$$

Substituting $v_2 = 20$ in Eq. (3.1.1) gives

$$3v_1 - 20 = 20 \quad \Rightarrow \quad v_1 = \frac{40}{3} = 13.333 \text{ V}$$

**METHOD 2** To use Cramer's rule, we need to put Eqs. (3.1.1) and (3.1.2) in matrix form as

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix} \tag{3.1.3}$$

The determinant of the matrix is

$$\Delta = \begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} = 15 - 3 = 12$$

We now obtain $v_1$ and $v_2$ as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.333 \text{ V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\Delta} = \frac{180 + 60}{12} = 20 \text{ V}$$

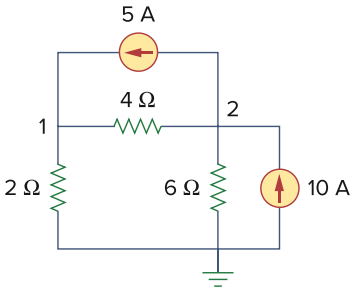
giving us the same result as did the elimination method.

If we need the currents, we can easily calculate them from the values of the nodal voltages.

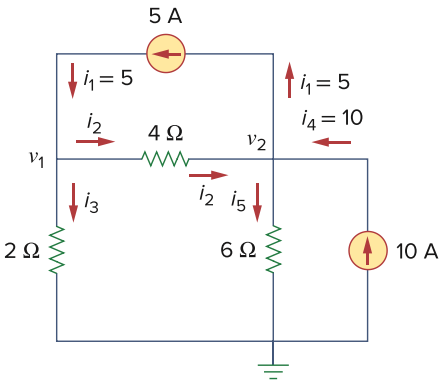
$$i_1 = 5 \text{ A}, \qquad i_2 = \frac{v_1 - v_2}{4} = -1.6668 \text{ A}, \qquad i_3 = \frac{v_1}{2} = 6.666 \text{ A}$$

$$i_4 = 10 \text{ A}, \qquad i_5 = \frac{v_2}{6} = 3.333 \text{ A}$$

The fact that $i_2$ is negative shows that the current flows in the direction opposite to the one assumed.




(a)




(b)

**Figure 3.3**
For Example 3.1: (a) original circuit, (b) circuit for analysis.

**Practice Problem 3.1**

Obtain the node voltages in the circuit of Fig. 3.4.

**Answer:** $v_1 = 30$ V, $v_2 = -2.5$ V.

## Example 3.2

Determine the voltages at the nodes in Fig. 3.5(a).

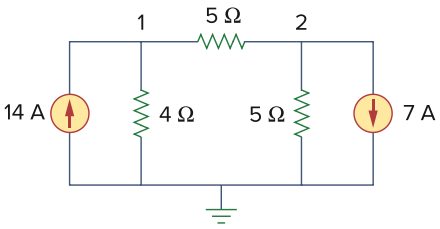



**Figure 3.4**
For Practice Prob. 3.1.

**Solution:**

The circuit in this example has three nonreference nodes, unlike the previous example which has two nonreference nodes. We assign voltages to the three nodes as shown in Fig. 3.5(b) and label the currents.

At node 1,

$$3 = i_1 + i_x \quad \Rightarrow \quad 3 = \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2}$$

Multiplying by 4 and rearranging terms, we get

$$3v_1 - 2v_2 - v_3 = 12 \tag{3.2.1}$$

At node 2,

$$i_x = i_2 + i_3 \quad \Rightarrow \quad \frac{v_1 - v_2}{2} = \frac{v_2 - v_3}{8} + \frac{v_2 - 0}{4}$$

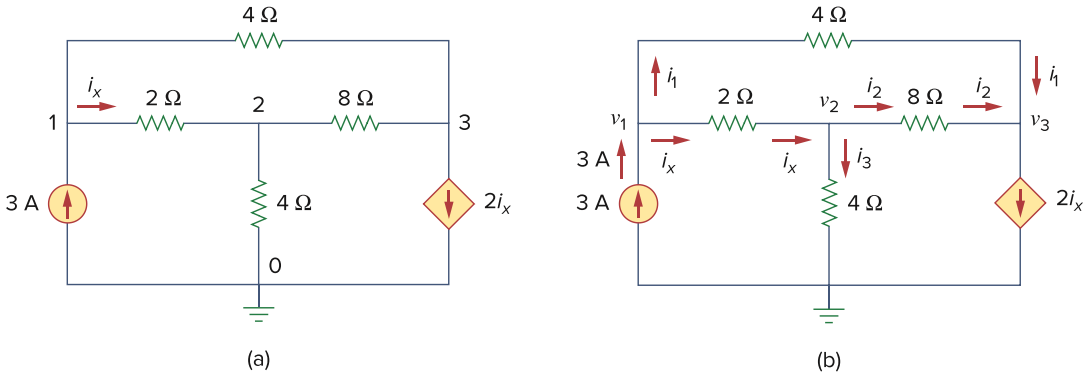



**Figure 3.5**
For Example 3.2: (a) original circuit, (b) circuit for analysis.

Multiplying by 8 and rearranging terms, we get

$$-4v_1 + 7v_2 - v_3 = 0 \tag{3.2.2}$$

At node 3,

$$i_1 + i_2 = 2i_x \quad \Rightarrow \quad \frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} = \frac{2(v_1 - v_2)}{2}$$

Multiplying by 8, rearranging terms, and dividing by 3, we get

$$2v_1 - 3v_2 + v_3 = 0 \tag{3.2.3}$$

We have three simultaneous equations to solve to get the node voltages $v_1$, $v_2$, and $v_3$. We shall solve the equations in three ways.

■ **METHOD 1** Using the elimination technique, we add Eqs. (3.2.1) and (3.2.3).

$$5v_1 - 5v_2 = 12$$

or

$$v_1 - v_2 = \frac{12}{5} = 2.4 \tag{3.2.4}$$

Adding Eqs. (3.2.2) and (3.2.3) gives

$$-2v_1 + 4v_2 = 0 \quad \Rightarrow \quad v_1 = 2v_2 \tag{3.2.5}$$

Substituting Eq. (3.2.5) into Eq. (3.2.4) yields

$$2v_2 - v_2 = 2.4 \quad \Rightarrow \quad v_2 = 2.4, \quad v_1 = 2v_2 = 4.8 \text{ V}$$

From Eq. (3.2.3), we get

$$v_3 = 3v_2 - 2v_1 = 3v_2 - 4v_2 = -v_2 = -2.4 \text{ V}$$

Thus,

$$v_1 = 4.8 \text{ V}, \quad v_2 = 2.4 \text{ V}, \quad v_3 = -2.4 \text{ V}$$

**METHOD 2** To use Cramer's rule, we put Eqs. (3.2.1) to (3.2.3) in matrix form.

$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} \tag{3.2.6}$$

From this, we obtain

$$v_1 = \frac{\Delta_1}{\Delta}, \quad v_2 = \frac{\Delta_2}{\Delta}, \quad v_3 = \frac{\Delta_3}{\Delta}$$

where $\Delta$, $\Delta_1$, $\Delta_2$, and $\Delta_3$ are the determinants to be calculated as follows. As explained in Appendix A, to calculate the determinant of a 3 by 3 matrix, we repeat the first two rows and cross multiply.

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{array}{ccc} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \\ - & - & - \\ 3 & -2 & -1 \\ - & - & - \\ -4 & 7 & -1 \\ - & - & - \end{array} \begin{array}{c} \\ \\ \\ + \\ \\ + \\ \\ + \end{array}$$

$$= 21 - 12 + 4 + 14 - 9 - 8 = 10$$

Similarly, we obtain

$$\Delta_1 = \begin{array}{ccc} 12 & -2 & -1 \\ 0 & 7 & -1 \\ 0 & -3 & 1 \\ - & - & - \\ 12 & -2 & -1 \\ - & - & - \\ 0 & 7 & -1 \\ - & - & - \end{array} \begin{array}{c} \\ \\ \\ + \\ \\ + \\ \\ + \end{array} = 84 + 0 + 0 - 0 - 36 - 0 = 48$$

$$\Delta_2 = \begin{array}{ccc} 3 & 12 & -1 \\ -4 & 0 & -1 \\ 2 & 0 & 1 \\ - & - & - \\ 3 & 12 & -1 \\ - & - & - \\ -4 & 0 & -1 \\ - & - & - \end{array} \begin{array}{c} \\ \\ \\ + \\ \\ + \\ \\ + \end{array} = 0 + 0 - 24 - 0 - 0 + 48 = 24$$

$$\Delta_3 = \begin{array}{ccc} 3 & -2 & 12 \\ -4 & 7 & 0 \\ 2 & -3 & 0 \\ - & - & - \\ 3 & -2 & 12 \\ - & - & - \\ -4 & 7 & 0 \\ - & - & - \end{array} \begin{array}{c} \\ \\ \\ + \\ \\ + \\ \\ + \end{array} = 0 + 144 + 0 - 168 - 0 - 0 = -24$$

Thus, we find

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{48}{10} = 4.8 \text{ V}, \qquad v_2 = \frac{\Delta_2}{\Delta} = \frac{24}{10} = 2.4 \text{ V}$$

$$v_3 = \frac{\Delta_3}{\Delta} = \frac{-24}{10} = -2.4 \text{ V}$$

as we obtained with Method 1.

■ **METHOD 3** We now use *MATLAB* to solve the matrix. Equation (3.2.6) can be written as

$$\mathbf{AV} = \mathbf{B} \quad \Rightarrow \quad \mathbf{V} = \mathbf{A}^{-1}\mathbf{B}$$

where $\mathbf{A}$ is the 3 by 3 square matrix, $\mathbf{B}$ is the column vector, and $\mathbf{V}$ is a column vector comprised of $v_1$, $v_2$, and $v_3$ that we want to determine. We use *MATLAB* to determine $\mathbf{V}$ as follows:

```
>>A = [3  -2  -1;  -4  7  -1;  2  -3  1];
>>B = [12  0  0]';
>>V = inv(A) * B
V =
    4.8000
    2.4000
   -2.4000
```

Thus, $v_1 = 4.8$ V, $v_2 = 2.4$ V, and $v_3 = -2.4$ V, as obtained previously.

## Practice Problem 3.2

Find the voltages at the three nonreference nodes in the circuit of Fig. 3.6.

**Answer:** $v_1 = 32$ V, $v_2 = -25.6$ V, $v_3 = 62.4$ V.

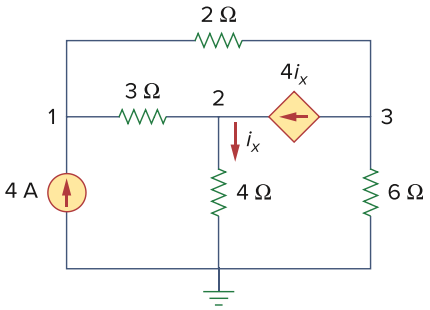



**Figure 3.6**
For Practice Prob. 3.2.

## 3.3 Nodal Analysis with Voltage Sources

We now consider how voltage sources affect nodal analysis. We use the circuit in Fig. 3.7 for illustration. Consider the following two possibilities.

■ **CASE 1** If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source. In Fig. 3.7, for example,

$$v_1 = 10 \text{ V} \tag{3.10}$$

Thus, our analysis is somewhat simplified by this knowledge of the voltage at this node.

■ **CASE 2** If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes

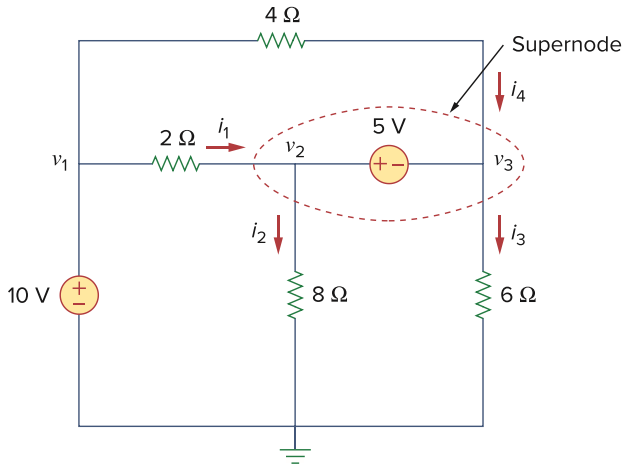



**Figure 3.7**
A circuit with a supernode.

form a *generalized node* or *supernode*; we apply both KCL and KVL to determine the node voltages.

A supernode may be regarded as a closed surface enclosing the voltage source and its two nodes.

A **supernode** is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

In Fig. 3.7, nodes 2 and 3 form a supernode. (We could have more than two nodes forming a single supernode. For example, see the circuit in Fig. 3.14.) We analyze a circuit with supernodes using the same three steps mentioned in the previous section except that the supernodes are treated differently. Why? Because an essential component of nodal analysis is applying KCL, which requires knowing the current through each element. There is no way of knowing the current through a voltage source in advance. However, KCL must be satisfied at a supernode like any other node. Hence, at the supernode in Fig. 3.7,

$$i_1 + i_4 = i_2 + i_3 \tag{3.11a}$$

or

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6} \tag{3.11b}$$

To apply Kirchhoff's voltage law to the supernode in Fig. 3.7, we redraw the circuit as shown in Fig. 3.8. Going around the loop in the clockwise direction gives

$$-v_2 + 5 + v_3 = 0 \quad \Rightarrow \quad v_2 - v_3 = 5 \tag{3.12}$$

From Eqs. (3.10), (3.11b), and (3.12), we obtain the node voltages.

Note the following properties of a supernode:

1. The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.
2. A supernode has no voltage of its own.
3. A supernode requires the application of both KCL and KVL.

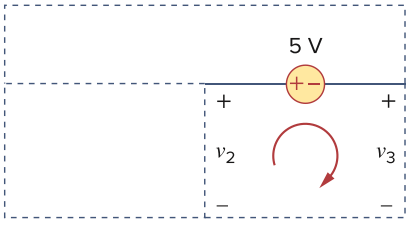



**Figure 3.8**
Applying KVL to a supernode.

## Example 3.3

For the circuit shown in Fig. 3.9, find the node voltages.

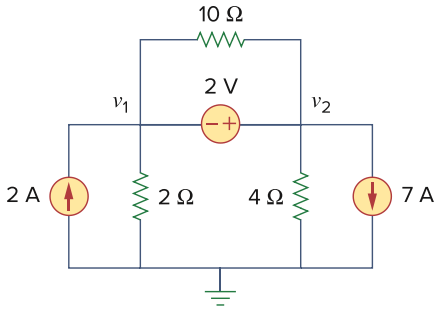



**Figure 3.9**
For Example 3.3.

**Solution:**
The supernode contains the 2-V source, nodes 1 and 2, and the 10-Ω resistor. Applying KCL to the supernode as shown in Fig. 3.10(a) gives

$$2 = i_1 + i_2 + 7$$

Expressing $i_1$ and $i_2$ in terms of the node voltages

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7 \qquad \Rightarrow \qquad 8 = 2v_1 + v_2 + 28$$

or

$$v_2 = -20 - 2v_1 \tag{3.3.1}$$

To get the relationship between $v_1$ and $v_2$, we apply KVL to the circuit in Fig. 3.10(b). Going around the loop, we obtain

$$-v_1 - 2 + v_2 = 0 \qquad \Rightarrow \qquad v_2 = v_1 + 2 \tag{3.3.2}$$

From Eqs. (3.3.1) and (3.3.2), we write

$$v_2 = v_1 + 2 = -20 - 2v_1$$

or

$$3v_1 = -22 \qquad \Rightarrow \qquad v_1 = -7.333 \text{ V}$$

and $v_2 = v_1 + 2 = -5.333$ V. Note that the 10-Ω resistor does not make any difference because it is connected across the supernode.

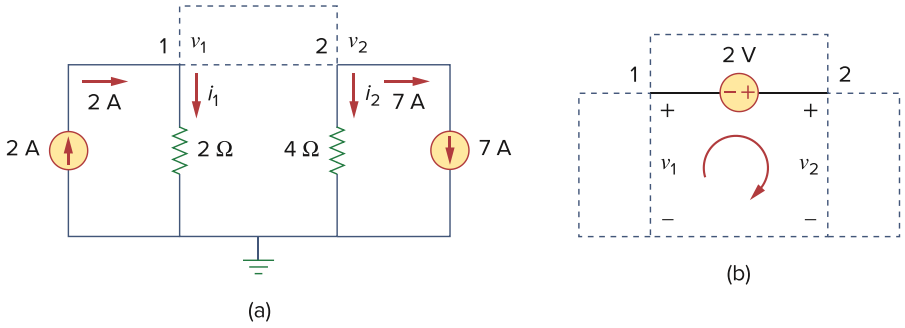



**Figure 3.10**
Applying: (a) KCL to the supernode, (b) KVL to the loop.

## Practice Problem 3.3

Find $v$ and $i$ in the circuit of Fig. 3.11.

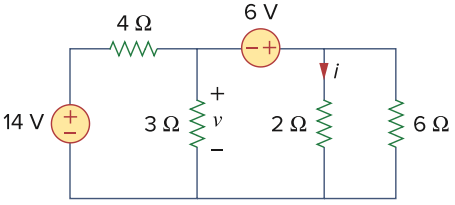



**Figure 3.11**
For Practice Prob. 3.3.

**Answer:** −400 mV, 2.8 A.

## Example 3.4

Find the node voltages in the circuit of Fig. 3.12.

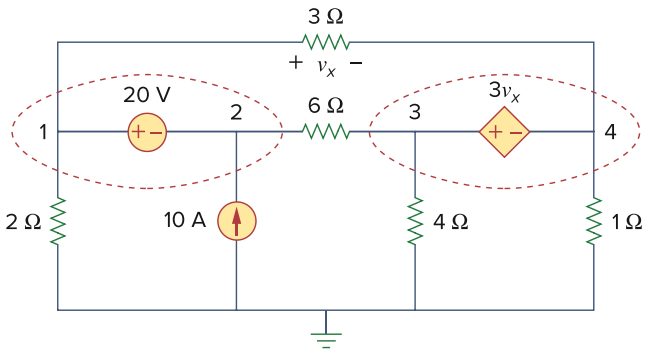



**Figure 3.12**
For Example 3.4.

**Solution:**
Nodes 1 and 2 form a supernode; so do nodes 3 and 4. We apply KCL to the two supernodes as in Fig. 3.13(a). At supernode 1-2,

$$i_3 + 10 = i_1 + i_2$$

Expressing this in terms of the node voltages,

$$\frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_1}{2}$$

or

$$5v_1 + v_2 - v_3 - 2v_4 = 60 \qquad \textbf{(3.4.1)}$$

At supernode 3-4,

$$i_1 = i_3 + i_4 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$$

or

$$4v_1 + 2v_2 - 5v_3 - 16v_4 = 0 \qquad \textbf{(3.4.2)}$$

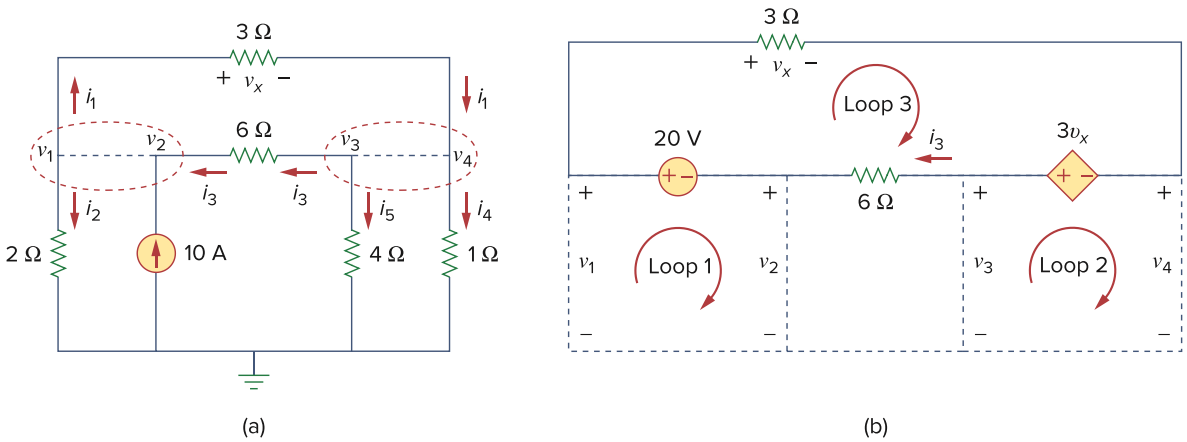



**Figure 3.13**
Applying: (a) KCL to the two supernodes, (b) KVL to the loops.

We now apply KVL to the branches involving the voltage sources as shown in Fig. 3.13(b). For loop 1,

$$-v_1 + 20 + v_2 = 0 \quad \Rightarrow \quad v_1 - v_2 = 20 \tag{3.4.3}$$

For loop 2,

$$-v_3 + 3v_x + v_4 = 0$$

But $v_x = v_1 - v_4$ so that

$$3v_1 - v_3 - 2v_4 = 0 \tag{3.4.4}$$

For loop 3,

$$v_x - 3v_x + 6i_3 - 20 = 0$$

But $6i_3 = v_3 - v_2$ and $v_x = v_1 - v_4$. Hence,

$$-2v_1 - v_2 + v_3 + 2v_4 = 20 \tag{3.4.5}$$

We need four node v oltages, $v_1$, $v_2$, $v_3$, and $v_4$, and it requires only four out of the five Eqs. (3.4.1) to (3.4.5) to find them. Although the fifth equation is redundant, it can be used to check results. We can solve Eqs. (3.4.1) to (3.4.4) directly using *MATLAB*. We can eliminate one node voltage so that we solv e three simultaneous equations instead of four . From Eq. (3.4.3), $v_2 = v_1 - 20$. Substituting this into Eqs. (3.4.1) and (3.4.2), respectively, gives

$$6v_1 - v_3 - 2v_4 = 80 \tag{3.4.6}$$

and

$$6v_1 - 5v_3 - 16v_4 = 40 \tag{3.4.7}$$

Equations (3.4.4), (3.4.6), and (3.4.7) can be cast in matrix form as

$$\begin{bmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 40 \end{bmatrix}$$

Using Cramer's rule gives

$$\Delta = \begin{vmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{vmatrix} = -18, \qquad \Delta_1 = \begin{vmatrix} 0 & -1 & -2 \\ 80 & -1 & -2 \\ 40 & -5 & -16 \end{vmatrix} = -480,$$

$$\Delta_3 = \begin{vmatrix} 3 & 0 & -2 \\ 6 & 80 & -2 \\ 6 & 40 & -16 \end{vmatrix} = -3120, \qquad \Delta_4 = \begin{vmatrix} 3 & -1 & 0 \\ 6 & -1 & 80 \\ 6 & -5 & 40 \end{vmatrix} = 840$$

Thus, we arrive at the node voltages as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{-480}{-18} = 26.67 \text{ V}, \qquad v_3 = \frac{\Delta_3}{\Delta} = \frac{-3120}{-18} = 173.33 \text{ V},$$

$$v_4 = \frac{\Delta_4}{\Delta} = \frac{840}{-18} = -46.67 \text{ V}$$

and $v_2 = v_1 - 20 = 6.667$ V. We have not used Eq. (3.4.5); it can be used to cross-check results.

## Practice Problem 3.4

Find $v_1$, $v_2$, and $v_3$ in the circuit of Fig. 3.14 using nodal analysis.

**Answer:** $v_1 = 7.608$ V, $v_2 = -17.39$ V, $v_3 = 1.6305$ V.

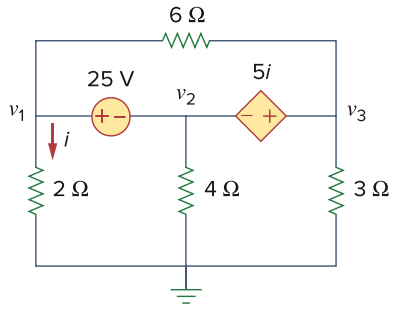


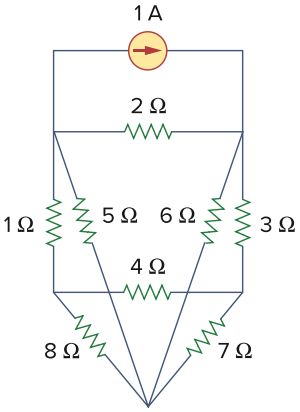

**Figure 3.14**
For Practice Prob. 3.4.

## 3.4 Mesh Analysis

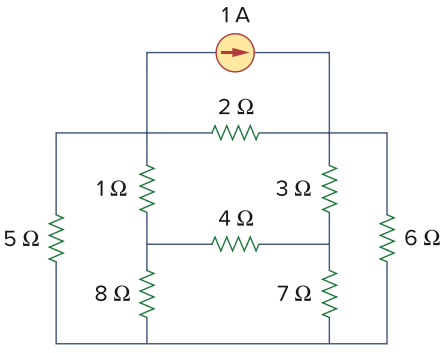
Mesh analysis provides another general procedure for analyzing circuits, using mesh currents as the circuit variables. Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously. Recall that a loop is a closed path with no node passed more than once. A mesh is a loop that does not contain any other loop within it.

Nodal analysis applies KCL to find unknown voltages in a given circuit, while mesh analysis applies KVL to find unknown currents. Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is *planar*. A planar circuit is one that can be drawn in a plane with no branches crossing one another; otherwise it is *nonplanar*. A circuit may have crossing branches and still be planar if it can be redrawn such that it has no crossing branches. For example, the circuit in Fig. 3.15(a) has two crossing branches, but it can be redrawn as in Fig. 3.15(b). Hence, the circuit in Fig. 3.15(a) is planar. However, the circuit in Fig. 3.16 is nonplanar, because there is no way to redraw it and avoid the branches crossing. Nonplanar circuits can be handled using nodal analysis, but they will not be considered in this text.

Mesh analysis is also known as *loop analysis* or the *mesh-current method*.




(a)




(b)

**Figure 3.15**
(a) A planar circuit with crossing branches, (b) the same circuit redrawn with no crossing branches.

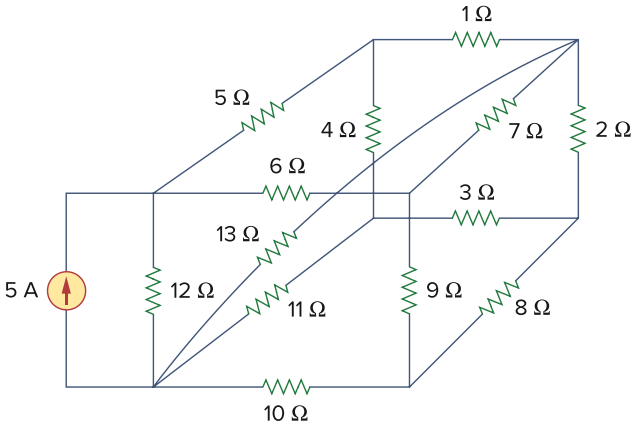



**Figure 3.16**
A nonplanar circuit.

To understand mesh analysis, we should first explain more about what we mean by a mesh.

A mesh is a loop that does not contain any other loops within it.

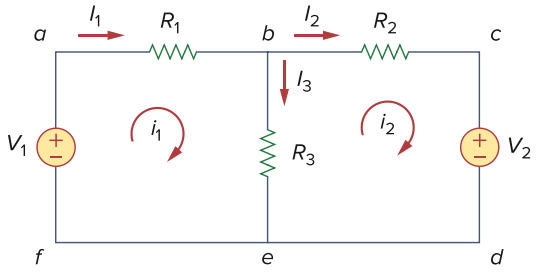



**Figure 3.17**
A circuit with two meshes.

In Fig. 3.17, for example, paths *abefa* and *bcdeb* are meshes, but path *abcdefa* is not a mesh. The current through a mesh is known as *mesh current*. In mesh analysis, we are interested in applying KVL to find the mesh currents in a given circuit.

Although path *abcdefa* is a loop and not a mesh, KVL still holds. This is the reason for loosely using the terms *loop analysis* and *mesh analysis* to mean the same thing.

In this section, we will apply mesh analysis to planar circuits that do not contain current sources. In the next section, we will consider circuits with current sources. In the mesh analysis of a circuit with $n$ meshes, we take the following three steps.

### Steps to Determine Mesh Currents:

1. Assign mesh currents $i_1, i_2, \ldots, i_n$ to the $n$ meshes.
2. Apply KVL to each of the $n$ meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the resulting $n$ simultaneous equations to get the mesh currents.

The direction of the mesh current is arbitrary—(clockwise or counterclockwise)—and does not affect the validity of the solution.

To illustrate the steps, consider the circuit in Fig. 3.17. The first step requires that mesh currents $i_1$ and $i_2$ are assigned to meshes 1 and 2. Although a mesh current may be assigned to each mesh in an arbitrary direction, it is conventional to assume that each mesh current flows clockwise.

As the second step, we apply KVL to each mesh. Applying KVL to mesh 1, we obtain

$$-V_1 + R_1 i_1 + R_3 (i_1 - i_2) = 0$$

or

$$(R_1 + R_3) i_1 - R_3 i_2 = V_1 \tag{3.13}$$

For mesh 2, applying KVL gives

$$R_2 i_2 + V_2 + R_3 (i_2 - i_1) = 0$$

or

$$-R_3 i_1 + (R_2 + R_3) i_2 = -V_2 \tag{3.14}$$

The shortcut way will not apply if one mesh current is assumed clockwise and the other assumed counterclockwise, although this is permissible.

Note in Eq. (3.13) that the coefficient of $i_1$ is the sum of the resistances in the first mesh, while the coefficient of $i_2$ is the negative of the resistance common to meshes 1 and 2. Now observe that the same is true in Eq. (3.14). This can serve as a shortcut way of writing the mesh equations. We will exploit this idea in Section 3.6.

The third step is to solve for the mesh currents. Putting Eqs. (3.13) and (3.14) in matrix form yields

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix} \tag{3.15}$$

which can be solved to obtain the mesh currents $i_1$ and $i_2$. We are at liberty to use any technique for solving the simultaneous equations. According to Eq. (2.12), if a circuit has $n$ nodes, $b$ branches, and $l$ independent loops or meshes, then $l = b - n + 1$. Hence, $l$ independent simultaneous equations are required to solve the circuit using mesh analysis.

Notice that the branch currents are different from the mesh currents unless the mesh is isolated. To distinguish between the two types of currents, we use $i$ for a mesh current and $I$ for a branch current. The current elements $I_1$, $I_2$, and $I_3$ are algebraic sums of the mesh currents. It is e vident from Fig. 3.17 that

$$I_1 = i_1, \qquad I_2 = i_2, \qquad I_3 = i_1 - i_2 \tag{3.16}$$

## Example 3.5

For the circuit in Fig. 3.18, find the branch currents $I_1$, $I_2$, and $I_3$ using mesh analysis.

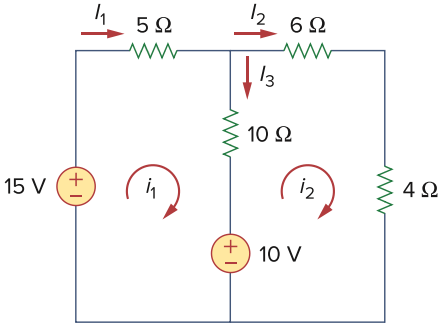



**Figure 3.18**
For Example 3.5.

**Solution:**
We first obtain the mesh currents using KVL. For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

or

$$3i_1 - 2i_2 = 1 \tag{3.5.1}$$

For mesh 2,

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

or

$$i_1 = 2i_2 - 1 \tag{3.5.2}$$

■ **METHOD 1** Using the substitution method, we substitute Eq. (3.5.2) into Eq. (3.5.1), and write

$$6i_2 - 3 - 2i_2 = 1 \qquad \Rightarrow \qquad i_2 = 1 \text{ A}$$

From Eq. (3.5.2), $i_1 = 2i_2 - 1 = 2 - 1 = 1$ A. Thus,

$$I_1 = i_1 = 1 \text{ A}, \qquad I_2 = i_2 = 1 \text{ A}, \qquad I_3 = i_1 - i_2 = 0$$

■ **METHOD 2** To use Cramer's rule, we cast Eqs. (3.5.1) and (3.5.2) in matrix form as

$$\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We obtain the determinants

$$\Delta = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 6 - 2 = 4$$

$$\Delta_1 = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = 2 + 2 = 4, \qquad \Delta_2 = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 3 + 1 = 4$$

Thus,

$$i_1 = \frac{\Delta_1}{\Delta} = 1 \text{ A}, \qquad i_2 = \frac{\Delta_2}{\Delta} = 1 \text{ A}$$

as before.

## Practice Problem 3.5

Calculate the mesh currents $i_1$ and $i_2$ of the circuit of Fig. 3.19.

**Answer:** $i_1 = 4.6$ A, $i_2 = 200$ mA.

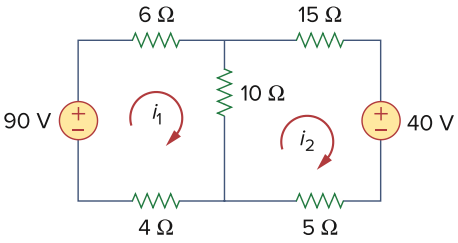



**Figure 3.19**
For Practice Prob. 3.5.

## Example 3.6

Use mesh analysis to find the current $I_o$ in the circuit of Fig. 3.20.

**Solution:**
We apply KVL to the three meshes in turn. For mesh 1,

$$-24 + 10(i_1 - i_2) + 12(i_1 - i_3) = 0$$

or

$$11i_1 - 5i_2 - 6i_3 = 12 \tag{3.6.1}$$

For mesh 2,

$$24i_2 + 4(i_2 - i_3) + 10(i_2 - i_1) = 0$$

or

$$-5i_1 + 19i_2 - 2i_3 = 0 \tag{3.6.2}$$

For mesh 3,

$$4I_o + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$

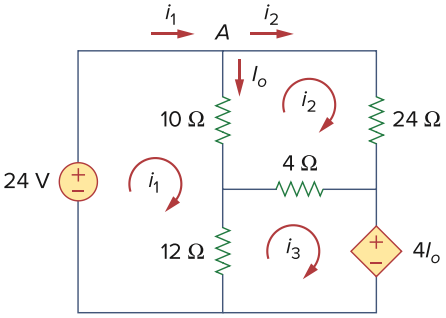



**Figure 3.20**
For Example 3.6.

But at node A, $I_o = i_1 - i_2$, so that

$$4(i_1 - i_2) + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$

or

$$-i_1 - i_2 + 2i_3 = 0 \qquad \textbf{(3.6.3)}$$

In matrix form, Eqs. (3.6.1) to (3.6.3) become

$$\begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

We obtain the determinants as

$$\Delta = \begin{vmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{vmatrix}$$

$$= 418 - 30 - 10 - 114 - 22 - 50 = 192$$

$$\Delta_1 = \begin{vmatrix} 12 & -5 & -6 \\ 0 & 19 & -2 \\ 0 & -1 & 2 \end{vmatrix} = 456 - 24 = 432$$

$$\Delta_2 = \begin{vmatrix} 11 & 12 & -6 \\ -5 & 0 & -2 \\ -1 & 0 & 2 \end{vmatrix} = 24 + 120 = 144$$

$$\Delta_3 = \begin{vmatrix} 11 & -5 & 12 \\ -5 & 19 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 60 + 228 = 288$$

We calculate the mesh currents using Cramer's rule as

$$i_1 = \frac{\Delta_1}{\Delta} = \frac{432}{192} = 2.25 \text{ A}, \qquad i_2 = \frac{\Delta_2}{\Delta} = \frac{144}{192} = 0.75 \text{ A},$$

$$i_3 = \frac{\Delta_3}{\Delta} = \frac{288}{192} = 1.5 \text{ A}$$

Thus, $I_o = i_1 - i_2 = 1.5$ A.

## Practice Problem 3.6

Using mesh analysis, find $I_o$ in the circuit of Fig. 3.21.

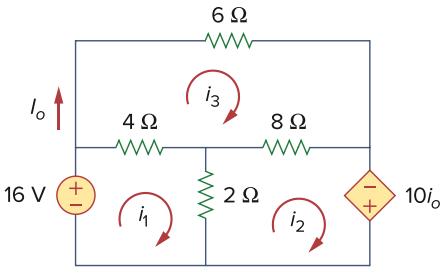



**Figure 3.21**
For Practice Prob. 3.6.

**Answer:** −4 A.

## 3.5 Mesh Analysis with Current Sources

Applying mesh analysis to circuits containing current sources (dependent or independent) may appear complicated. But it is actually much easier than what we encountered in the previous section, because the presence of the current sources reduces the number of equations. Consider the following two possible cases.

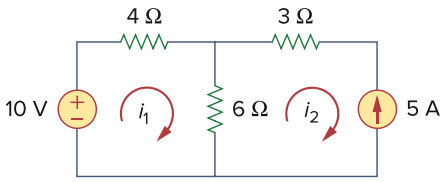



**Figure 3.22**
A circuit with a current source.

**CASE 1** When a current source exists only in one mesh: Consider the circuit in Fig. 3.22, for example. We set $i_2 = -5$ A and write a mesh equation for the other mesh in the usual way; that is,

$$-10 + 4i_1 + 6(i_1 - i_2) = 0 \quad \Rightarrow \quad i_1 = -2 \text{ A} \qquad (3.17)$$

**CASE 2** When a current source exists between two meshes: Consider the circuit in Fig. 3.23(a), for example. We create a *supermesh* by excluding the current source and any elements connected in series with it, as shown in Fig. 3.23(b). Thus,

> A **supermesh** results when two meshes have a (dependent or independent) current source in common.

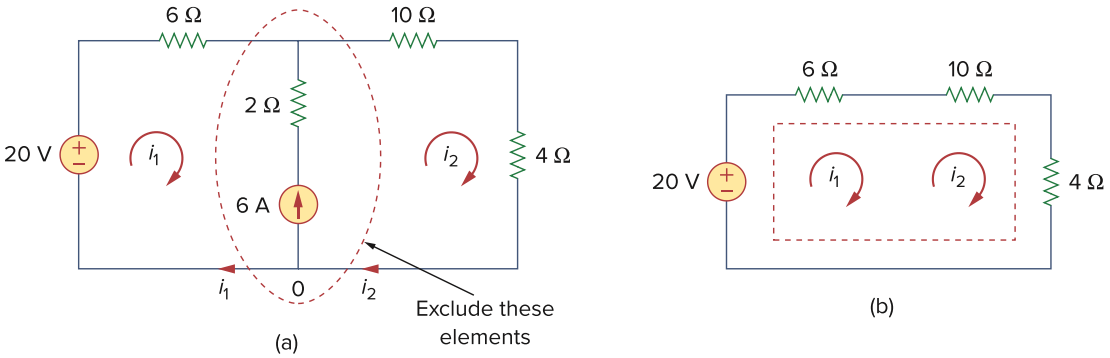



**Figure 3.23**
(a) Two meshes having a current source in common, (b) a supermesh, created by excluding the current source.

As shown in Fig. 3.23(b), we create a supermesh as the periphery of the two meshes and treat it differently. (If a circuit has two or more supermeshes that intersect, they should be combined to form a larger supermesh.) Why treat the supermesh differently? Because mesh analysis applies KVL—which requires that we know the voltage across each branch—and we do not know the voltage across a current source in advance. However, a supermesh must satisfy KVL like any other mesh. Therefore, applying KVL to the supermesh in Fig. 3.23(b) gives

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

or

$$6i_1 + 14i_2 = 20 \tag{3.18}$$

We apply KCL to a node in the branch where the two meshes intersect. Applying KCL to node 0 in Fig. 3.23(a) gives

$$i_2 = i_1 + 6 \tag{3.19}$$

Solving Eqs. (3.18) and (3.19), we get

$$i_1 = -3.2 \text{ A}, \qquad i_2 = 2.8 \text{ A} \tag{3.20}$$

Note the following properties of a supermesh:

1. The current source in the supermesh pro vides the constraint equation necessary to solve for the mesh currents.
2. A supermesh has no current of its own.
3. A supermesh requires the application of both KVL and KCL.

## Example 3.7

For the circuit in Fig. 3.24, find $i_1$ to $i_4$ using mesh analysis.

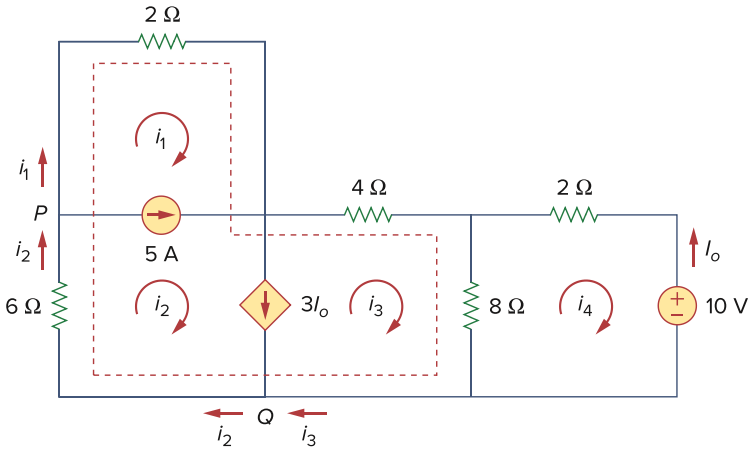



**Figure 3.24**
For Example 3.7.

**Solution:**

Note that meshes 1 and 2 form a supermesh because they have an independent current source in common. Also, meshes 2 and 3 form another supermesh because they have a dependent current source in common. The two supermeshes intersect and form a larger supermesh as shown. Applying KVL to the larger supermesh,

$$2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$$

or

$$i_1 + 3i_2 + 6i_3 - 4i_4 = 0 \tag{3.7.1}$$

For the independent current source, we apply KCL to node $P$:

$$i_2 = i_1 + 5 \tag{3.7.2}$$

For the dependent current source, we apply KCL to node $Q$:

$$i_2 = i_3 + 3I_o$$

But $I_o = -i_4$, hence,

$$i_2 = i_3 - 3i_4 \tag{3.7.3}$$

Applying KVL in mesh 4,

$$2i_4 + 8(i_4 - i_3) + 10 = 0$$

or

$$5i_4 - 4i_3 = -5 \tag{3.7.4}$$

From Eqs. (3.7.1) to (3.7.4),

$$i_1 = -7.5 \text{ A}, \quad i_2 = -2.5 \text{ A}, \quad i_3 = 3.93 \text{ A}, \quad i_4 = 2.143 \text{ A}$$

## Practice Problem 3.7

Use mesh analysis to determine $i_1$, $i_2$, and $i_3$ in Fig. 3.25.

**Answer:** $i_1 = 12.379$ A, $i_2 = 378.9$ mA, $i_3 = 3.284$ A.

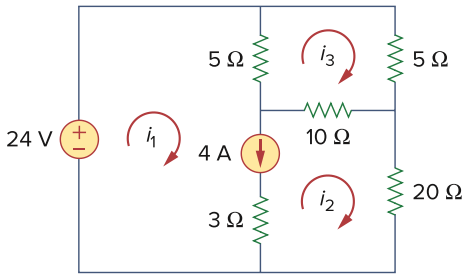



**Figure 3.25**
For Practice Prob. 3.7.

## 3.6 †Nodal and Mesh Analyses by Inspection

This section presents a generalized procedure for nodal or mesh analysis. It is a shortcut approach based on mere inspection of a circuit.

When all sources in a circuit are independent current sources, we do not need to apply KCL to each node to obtain the node-v oltage equations as we did in Section 3.2. We can obtain the equations by mere inspection of the circuit. As an e xample, let us ree xamine the circuit in Fig. 3.2, shown again in Fig. 3.26(a) for convenience. The circuit has two nonreference nodes and the node equations were derived in Section 3.2 as

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix} \tag{3.21}$$

Observe that each of the diagonal terms is the sum of the conductances connected directly to node 1 or 2, while the off-diagonal terms are the negatives of the conductances connected between the nodes. Also, each term on the right-hand side of Eq. (3.21) is the algebraic sum of the currents entering the node.

In general, if a circuit with independent current sources has $N$ nonreference nodes, the node-v oltage equations can be written in terms of the conductances as

$$\begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} \tag{3.22}$$

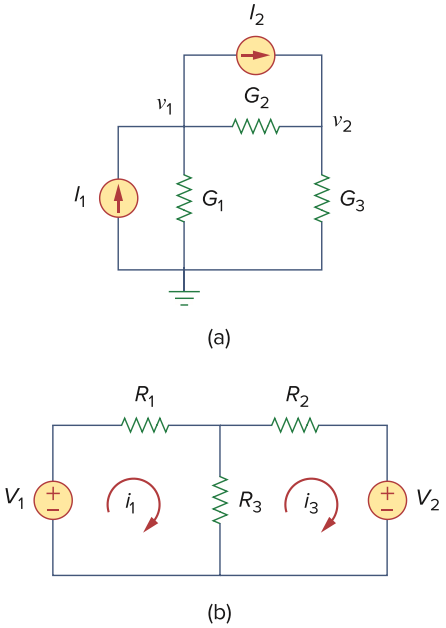



**Figure 3.26**
(a) The circuit in Fig. 3.2, (b) the circuit in Fig. 3.17.

or simply

$$\mathbf{Gv} = \mathbf{i} \tag{3.23}$$

where

$G_{kk}$ = Sum of the conductances connected to node $k$
$G_{kj} = G_{jk}$ = Negative of the sum of the conductances directly connecting nodes $k$ and $j$, $k \neq j$
$v_k$ = Unknown voltage at node $k$
$i_k$ = Sum of all independent current sources directly connected to node $k$, with currents entering the node treated as positive

**G** is called the *conductance matrix*; **v** is the output vector; and **i** is the input vector. Equation (3.22) can be solved to obtain the unknown node voltages. Keep in mind that this is valid for circuits with only independent current sources and linear resistors.

Similarly, we can obtain mesh-current equations by inspection when a linear resistive circuit has only independent voltage sources. Consider the circuit in Fig. 3.17, shown again in Fig. 3.26(b) for convenience. The circuit has two nonreference nodes and the node equations were derived in Section 3.4 as

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} \tag{3.24}$$

We notice that each of the diagonal terms is the sum of the resistances in the related mesh, while each of the off-diagonal terms is the negative of the resistance common to meshes 1 and 2. Each term on the right-hand side of Eq. (3.24) is the algebraic sum taken clockwise of all independent voltage sources in the related mesh.

In general, if the circuit has $N$ meshes, the mesh-current equations can be expressed in terms of the resistances as

$$\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1N} \\ R_{21} & R_{22} & \cdots & R_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ R_{N1} & R_{N2} & \cdots & R_{NN} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \tag{3.25}$$

or simply

$$\mathbf{Ri} = \mathbf{v} \tag{3.26}$$

where

$R_{kk}$ = Sum of the resistances in mesh $k$
$R_{kj} = R_{jk}$ = Negative of the sum of the resistances in common with meshes $k$ and $j$, $k \neq j$
$i_k$ = Unknown mesh current for mesh $k$ in the clockwise direction
$v_k$ = Sum taken clockwise of all independent voltage sources in mesh $k$, with voltage rise treated as positive

**R** is called the *resistance matrix*; **i** is the output vector; and **v** is the input vector. We can solve Eq. (3.25) to obtain the unknown mesh currents.

## Example 3.8

Write the node-voltage matrix equations for the circuit in Fig. 3.27 by inspection.

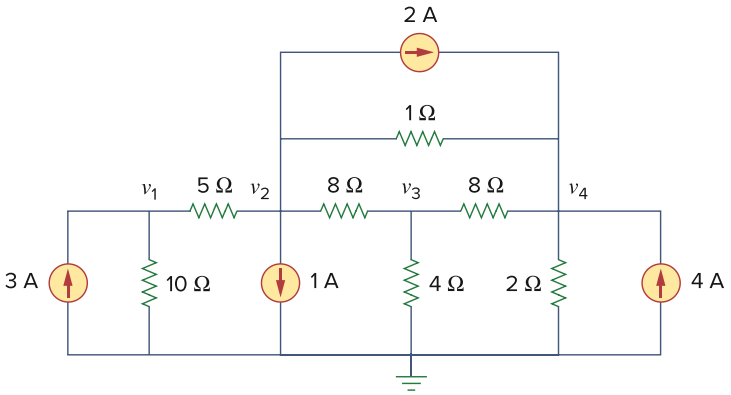



**Figure 3.27**
For Example 3.8.

**Solution:**
The circuit in Fig. 3.27 has four nonreference nodes, so we need four node equations. This implies that the size of the conductance matrix **G**, is 4 by 4. The diagonal terms of **G**, in siemens, are

$$G_{11} = \frac{1}{5} + \frac{1}{10} = 0.3, \qquad G_{22} = \frac{1}{5} + \frac{1}{8} + \frac{1}{1} = 1.325$$

$$G_{33} = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = 0.5, \qquad G_{44} = \frac{1}{8} + \frac{1}{2} + \frac{1}{1} = 1.625$$

The off-diagonal terms are

$$G_{12} = -\frac{1}{5} = -0.2, \qquad G_{13} = G_{14} = 0$$

$$G_{21} = -0.2, \qquad G_{23} = -\frac{1}{8} = -0.125, \qquad G_{24} = -\frac{1}{1} = -1$$

$$G_{31} = 0, \qquad G_{32} = -0.125, \qquad G_{34} = -\frac{1}{8} = -0.125$$

$$G_{41} = 0, \qquad G_{42} = -1, \qquad G_{43} = -0.125$$

The input current vector **i** has the following terms, in amperes:

$$i_1 = 3, \qquad i_2 = -1 - 2 = -3, \qquad i_3 = 0, \qquad i_4 = 2 + 4 = 6$$

Thus, the node-voltage equations are

$$\begin{bmatrix} 0.3 & -0.2 & 0 & 0 \\ -0.2 & 1.325 & -0.125 & -1 \\ 0 & -0.125 & 0.5 & -0.125 \\ 0 & -1 & -0.125 & 1.625 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 6 \end{bmatrix}$$

which can be solved using *MATLAB* to obtain the node voltages $v_1$, $v_2$, $v_3$, and $v_4$.

## Practice Problem 3.8

By inspection, obtain the node-voltage equations for the circuit in Fig. 3.28.

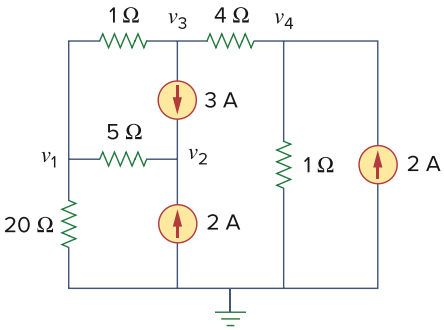



**Figure 3.28**
For Practice Prob. 3.8.

**Answer:**

$$\begin{bmatrix} 1.25 & -0.2 & -1 & 0 \\ -0.2 & 0.2 & 0 & 0 \\ -1 & 0 & 1.25 & -0.25 \\ 0 & 0 & -0.25 & 1.25 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -3 \\ 2 \end{bmatrix}$$

## Example 3.9

By inspection, write the mesh-current equations for the circuit in Fig. 3.29.

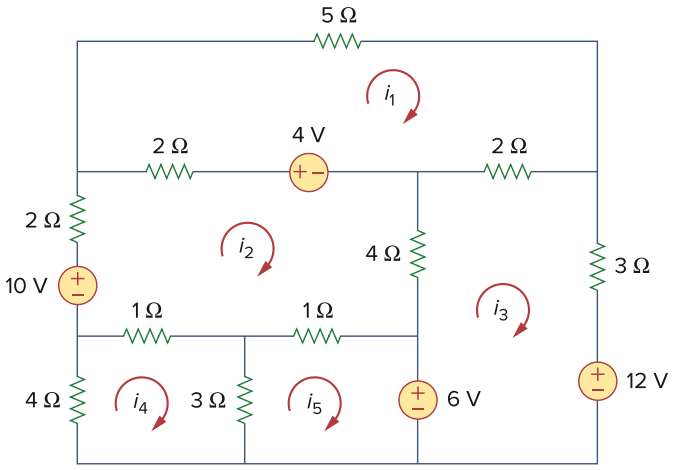



**Figure 3.29**
For Example 3.9.

**Solution:**

We have five meshes, so the resistance matrix is 5 by 5. The diagonal terms, in ohms, are:

$$R_{11} = 5 + 2 + 2 = 9, \quad R_{22} = 2 + 4 + 1 + 1 + 2 = 10,$$
$$R_{33} = 2 + 3 + 4 = 9, \quad R_{44} = 1 + 3 + 4 = 8, \quad R_{55} = 1 + 3 = 4$$

The off-diagonal terms are:

$$R_{12} = -2, \quad R_{13} = -2, \quad R_{14} = 0 = R_{15},$$
$$R_{21} = -2, \quad R_{23} = -4, \quad R_{24} = -1, \quad R_{25} = -1,$$
$$R_{31} = -2, \quad R_{32} = -4, \quad R_{34} = 0 = R_{35},$$
$$R_{41} = 0, \quad R_{42} = -1, \quad R_{43} = 0, \quad R_{45} = -3,$$
$$R_{51} = 0, \quad R_{52} = -1, \quad R_{53} = 0, \quad R_{54} = -3$$

The input voltage vector $\mathbf{v}$ has the following terms in volts:

$$v_1 = 4, \quad v_2 = 10 - 4 = 6,$$
$$v_3 = -12 + 6 = -6, \quad v_4 = 0, \quad v_5 = -6$$

Thus, the mesh-current equations are:

$$\begin{bmatrix} 9 & -2 & -2 & 0 & 0 \\ -2 & 10 & -4 & -1 & -1 \\ -2 & -4 & 9 & 0 & 0 \\ 0 & -1 & 0 & 8 & -3 \\ 0 & -1 & 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -6 \\ 0 \\ -6 \end{bmatrix}$$

From this, we can use *MATLAB* to obtain mesh currents $i_1$, $i_2$, $i_3$, $i_4$, and $i_5$.

## Practice Problem 3.9

By inspection, obtain the mesh-current equations for the circuit in Fig. 3.30.

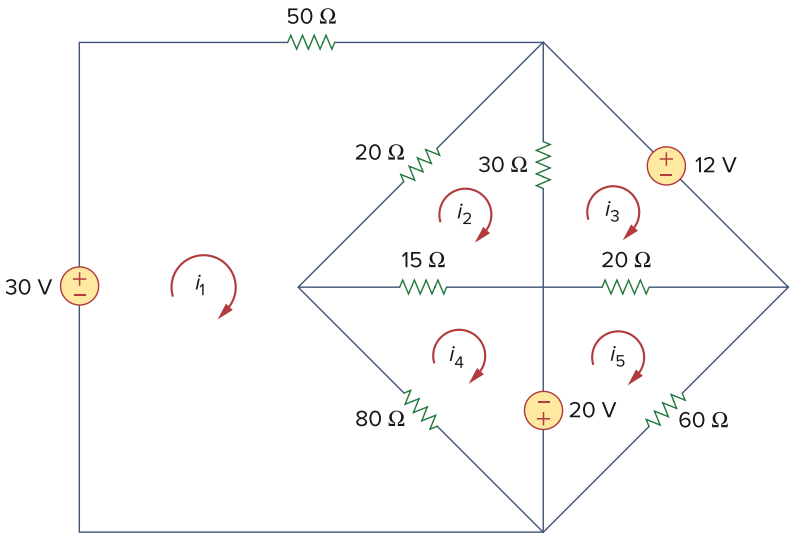



**Figure 3.30**
For Practice Prob. 3.9.

**Answer:**

$$\begin{bmatrix} 150 & -40 & 0 & -80 & 0 \\ -40 & 65 & -30 & -15 & 0 \\ -0 & -30 & 50 & 0 & -20 \\ 80 & -15 & 0 & 95 & 0 \\ 0 & 0 & -20 & 0 & 80 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ -12 \\ 20 \\ -20 \end{bmatrix}$$

## 3.7 Nodal Versus Mesh Analysis

Both nodal and mesh analyses provide a systematic way of analyzing a complex network. Someone may ask: Gi ven a netw ork to be analyzed, how do we know which method is better or more efficient? The choice of the better method is dictated by two factors.

The first factor is the nature of the particular network. Networks that contain many series-connected elements, voltage sources, or supermeshes are more suitable for mesh analysis, whereas networks with parallel-connected elements, current sources, or supernodes are more suitable for nodal analysis. Also, a circuit with fewer nodes than meshes is better analyzed using nodal analysis, while a circuit with fewer meshes than nodes is better analyzed using mesh analysis. The key is to select the method that results in the smaller number of equations.

The second factor is the information required. If node voltages are required, it may be expedient to apply nodal analysis. If branch or mesh currents are required, it may be better to use mesh analysis.

It is helpful to be familiar with both methods of analysis, for at least two reasons. First, one method can be used to check the results from the other method, if possible. Second, since each method has its limitations, only one method may be suitable for a particular problem. For example, mesh analysis is the only method to use in analyzing transistor circuits, as we shall see in Section 3.9. But mesh analysis cannot easily be used to solve an op amp circuit, as we shall see in Chapter 5, because there is no direct way to obtain the voltage across the op amp itself. For nonplanar networks, nodal analysis is the only option, because mesh analysis only applies to planar networks. Also, nodal analysis is more amenable to solution by computer, as it is easy to program. This allows one to analyze complicated circuits that defy hand calculation. A computer software package based on nodal analysis is introduced next.

## 3.8 Circuit Analysis with *PSpice*

*PSpice* is a computer software circuit analysis program that we will gradually learn to use throughout the course of this text. This section illustrates how to use *PSpice for Windows* to analyze the dc circuits we have studied so far.

A tutorial on using *PSpice for Windows* can be found in Connect.

The reader is expected to review the tutorial before proceeding in this section. It should be noted that *PSpice* is only helpful in determining branch voltages and currents when the numerical values of all the circuit components are known.

### Example 3.10

Use *PSpice* to find the node voltages in the circuit of Fig. 3.31.

**Solution:**

The first step is to draw the given circuit using Schematics. If one follows the instructions given in the *PSpice* tutorial found in Connect, the schematic in Fig. 3.32 is produced. Because this is a dc analysis, we use voltage source VDC and current source IDC. The pseudocomponent VIEWPOINTS are added to display the required node voltages. Once the circuit is drawn and saved as *exam310.sch*, we run *PSpice* by selecting **Analysis/Simulate**. The circuit is simulated and the results are displayed

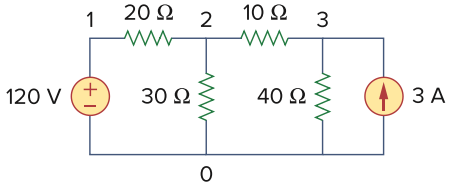



**Figure 3.31**
For Example 3.10.

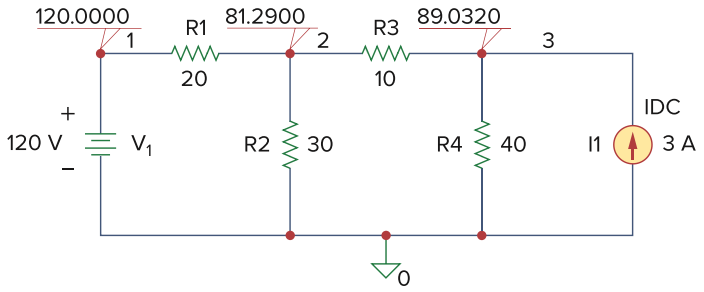



**Figure 3.32**
For Example 3.10; the schematic of the circuit in Fig. 3.31.

on VIEWPOINTS and also saved in output file *exam310.out*. The output file includes the following:

```
NODE   VOLTAGE    NODE   VOLTAGE    NODE   VOLTAGE
(1)    120.0000   (2)    81.2900    (3)    89.0320
```

indicating that $V_1 = 120$ V, $V_2 = 81.29$ V, $V_3 = 89.032$ V.

## Practice Problem 3.10

For the circuit in Fig. 3.33, use *PSpice* to find the node voltages.

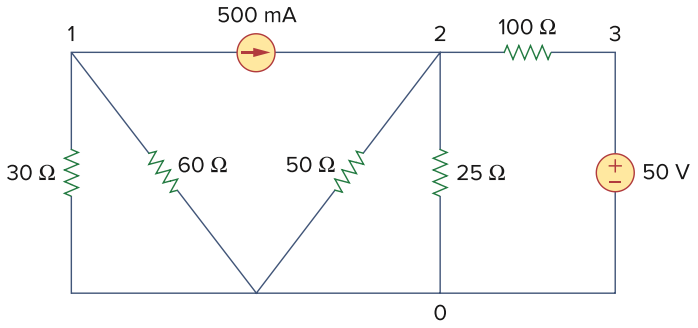



**Figure 3.33**
For Practice Prob. 3.10.

**Answer:** $V_1 = -10$ V, $V_2 = 14.286$ V, $V_3 = 50$ V.

## Example 3.11

In the circuit of Fig. 3.34, determine the currents $i_1$, $i_2$, and $i_3$.

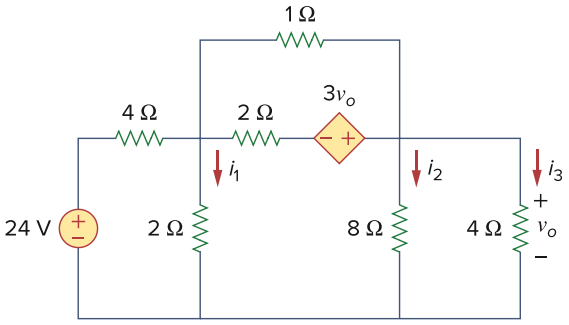



**Figure 3.34**
For Example 3.11.

**Solution:**
The schematic is shown in Fig. 3.35. (The schematic in Fig. 3.35 includes the output results, implying that it is the schematic displayed on the screen *after* the simulation.) Notice that the voltage-controlled voltage source E1 in Fig. 3.35 is connected so that its input is the voltage across the 4-Ω resistor; its gain is set equal to 3. In order to display the required currents, we insert pseudocomponent IPROBES in the appropriate branches. The schematic is saved as *exam311.sch* and simulated by selecting **Analysis/Simulate**. The results are displayed on IPROBES as shown in Fig. 3.35 and saved in output file *exam311.out*. From the output file or the IPROBES, we obtain $i_1 = i_2 = 1.333$ A and $i_3 = 2.667$ A.

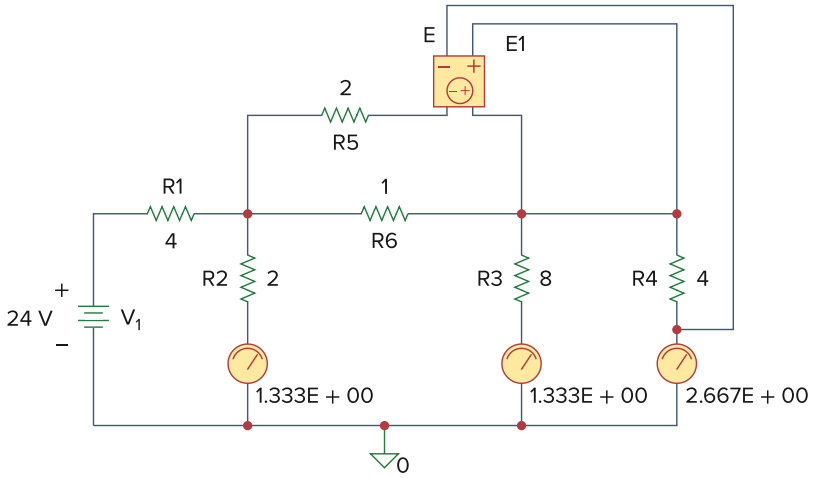



**Figure 3.35**
The schematic of the circuit in Fig. 3.34.

**Practice Problem 3.11**

Use *PSpice* to determine currents $i_1$, $i_2$, and $i_3$ in the circuit of Fig. 3.36.

**Answer:** $i_1 = -428.6$ mA, $i_2 = 2.286$ A, $i_3 = 2$ A.

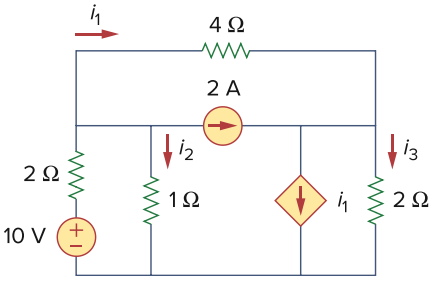



**Figure 3.36**
For Practice Prob. 3.11.

## 3.9 †Applications: DC Transistor Circuits

Most of us deal with electronic products on a routine basis and have some experience with personal computers. A basic component for the integrated circuits found in these electronics and computers is the active, three-terminal device known as the *transistor*. Understanding the transistor is essential before an engineer can start an electronic circuit design.

Figure 3.37 depicts various kinds of transistors commercially available. There are two basic types of transistors: *bipolar junction transistors* (BJTs) and *field-effect transistors* (FETs). Here, we consider only the BJTs, which were the first of the two and are still used today. Our objective is to present enough detail about the BJT to enable us to apply the techniques developed in this chapter to analyze dc transistor circuits.

## Historical

Courtesy of Lucent Technologies/Bell Labs

**William Shockley** (1910–1989), **John Bardeen** (1908–1991), and **Walter Brattain** (1902–1987) co-invented the transistor.

Nothing has had a greater impact on the transition from the "Industrial Age" to the "Age of the Engineer" than the transistor. I am sure that Dr. Shockley, Dr. Bardeen, and Dr. Brattain had no idea they would have this incredible effect on our history. While working at Bell Laboratories, they successfully demonstrated the point-contact transistor, invented by Bardeen and Brattain in 1947, and the junction transistor, which Shockley conceived in 1948 and successfully produced in 1951.

It is interesting to note that the idea of the field-effect transistor, the most commonly used one today, was first conceived in 1925–1928 by J. E. Lilienfeld, a German immigrant to the United States. This is evident from his patents of what appears to be a field-effect transistor. Unfortunately, the technology to realize this device had to wait until 1954 when Shockley's field-effect transistor became a reality. Just think what today would be like if we had this transistor 30 years earlier!

For their contributions to the creation of the transistor, Dr. Shockley, Dr. Bardeen, and Dr. Brattain received, in 1956, the Nobel Prize in physics. It should be noted that Dr. Bardeen is the only individual to win two Nobel prizes in physics; the second came later for work in superconductivity at the University of Illinois.

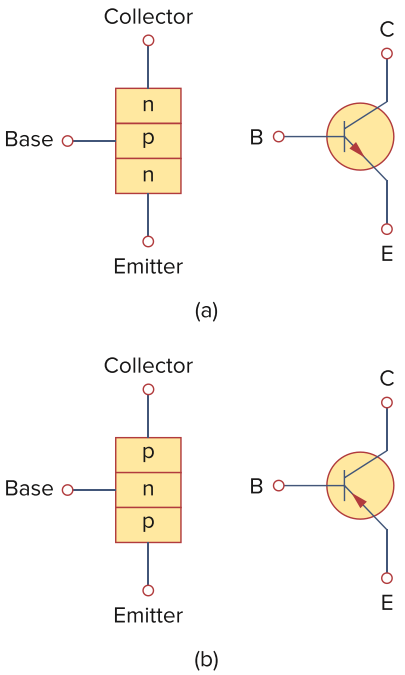



**Figure 3.38**
Two types of BJTs and their circuit symbols: (a) *npn*, (b) *pnp*.

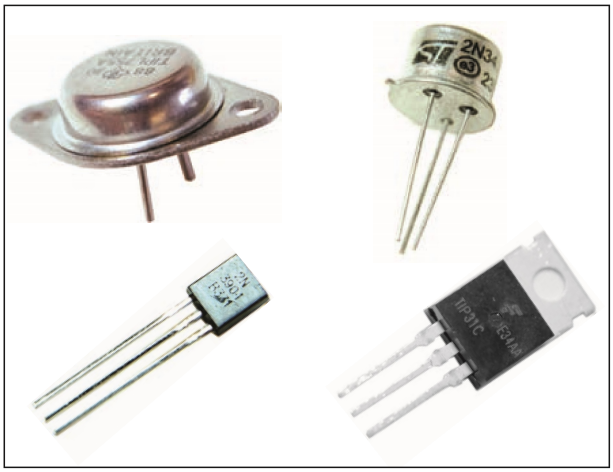


**Figure 3.37**
Various types of transistors.
*(© McGraw-Hill Education/Mark Dierker, photographer)*

There are two types of BJTs: *npn* and *pnp*, with their circuit symbols as shown in Fig. 3.38. Each type has three terminals, designated as emitter (E), base (B), and collector (C). For the *npn* transistor, the currents and voltages of the transistor are specified as in Fig. 3.39. Applying KCL to Fig. 3.39(a) gives

$$I_E = I_B + I_C \tag{3.27}$$

where $I_E$, $I_C$, and $I_B$ are emitter, collector, and base currents, respectively. Similarly, applying KVL to Fig. 3.39(b) gives

$$V_{CE} + V_{EB} + V_{BC} = 0 \tag{3.28}$$

where $V_{CE}$, $V_{EB}$, and $V_{BC}$ are collector -emitter, emitter-base, and base-collector voltages. The BJT can operate in one of three modes: acti ve, cutoff, and saturation. When transistors operate in the active mode, typically $V_{BE} \simeq 0.7$ V,

$$I_C = \alpha I_E \tag{3.29}$$

where $\alpha$ is called the *common-base curr ent gain*. In Eq. (3.29), $\alpha$ denotes the fraction of electrons injected by the emitter that are col - lected by the collector. Also,

$$\boxed{I_C = \beta I_B} \tag{3.30}$$

where $\beta$ is known as the *common-emitter current gain*. The $\alpha$ and $\beta$ are characteristic properties of a given transistor and assume constant values for that transistor. Typically, $\alpha$ takes values in the range of 0.98 to 0.999, while $\beta$ takes values in the range of 50 to 1000. From Eqs. (3.27) to (3.30), it is evident that

$$I_E = (1 + \beta)I_B \tag{3.31}$$

and

$$\beta = \frac{\alpha}{1 - \alpha} \tag{3.32}$$

These equations show that, in the active mode, the BJT can be modeled as a dependent current-controlled current source. Thus, in circuit analysis, the dc equi valent model in Fig. 3.40(b) may be used to replace the *npn* transistor in Fig. 3.40(a). Since $\beta$ in Eq. (3.32) is large, a small base current controls lar ge currents in the output circuit. Consequently , the bipolar transistor can serve as an amplifier, producing both current g ain and voltage gain. Such amplifiers can be used to furnish a considerable amount of power to transducers such as loudspeakers or control motors.

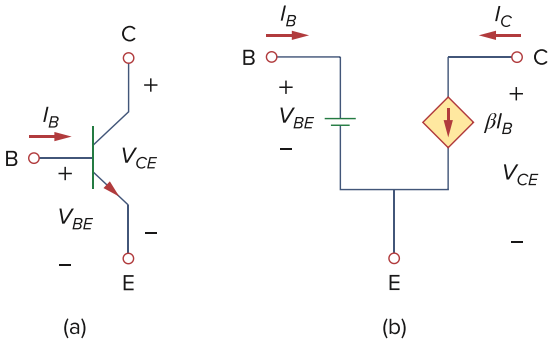



**Figure 3.40**
(a) An *npn* transistor, (b) its dc equivalent model.

It should be observ ed in the follo wing e xamples that one cannot directly analyze transistor circuits using nodal analysis because of the po- tential difference between the terminals of the transistor . Only when the transistor is replaced by its equivalent model can we apply nodalanalysis.

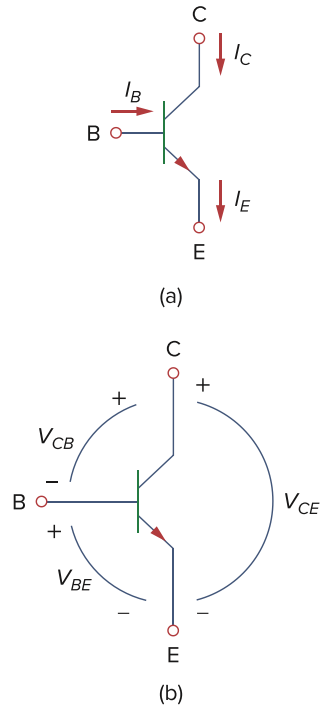



**Figure 3.39**
The terminal variables of an *npn* transistor: (a) currents, (b) voltages.

In fact, transistor circuits provide motivation to study dependent sources.

## Example 3.12

Find $I_B$, $I_C$, and $v_o$ in the transistor circuit of Fig. 3.41. Assume that the transistor operates in the active mode and that $\beta = 50$.

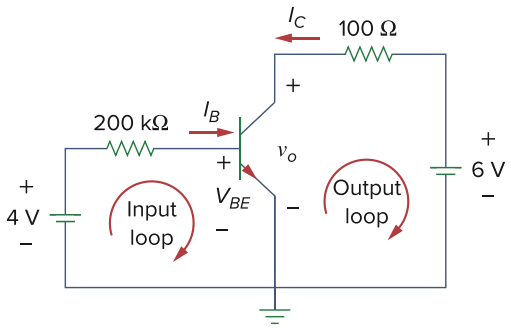



**Figure 3.41**
For Example 3.12.

**Solution:**
For the input loop, KVL gives

$$-4 + I_B(20 \times 10^3) + V_{BE} = 0$$

Since $V_{BE} = 0.7$ V in the active mode,

$$I_B = \frac{4 - 0.7}{200 \times 10^3} = 16.5\ \mu\text{A}$$

But

$$I_C = \beta I_B = 50 \times 16.5\ \mu\text{A} = 0.825\ \text{mA}$$

For the output loop, KVL gives

$$-v_o - 100I_C + 6 = 0$$

or

$$v_o = 6 - 100I_C = 6 - 0.0825 = 5.917\ \text{V}$$

Note that $v_o = V_{CE}$ in this case.

## Practice Problem 3.12

For the transistor circuit in Fig. 3.42, let $\beta = 100$ and $V_{BE} = 0.7$ V. Determine $v_o$ and $V_{CE}$.

**Answer:** 4.691 V, 888.5 mV.

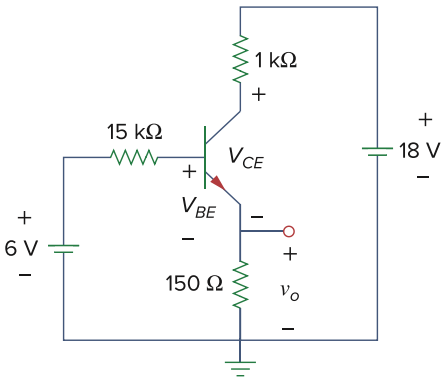



**Figure 3.42**
For Practice Prob. 3.12.

## Example 3.13

For the BJT circuit in Fig. 3.43, $\beta = 150$ and $V_{BE} = 0.7$ V. Find $v_o$.

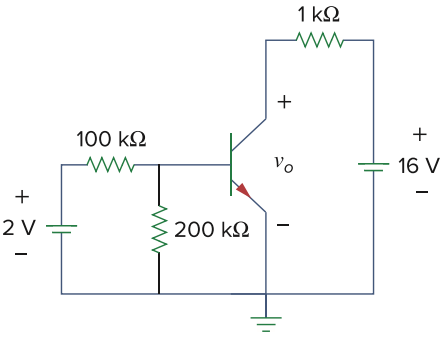


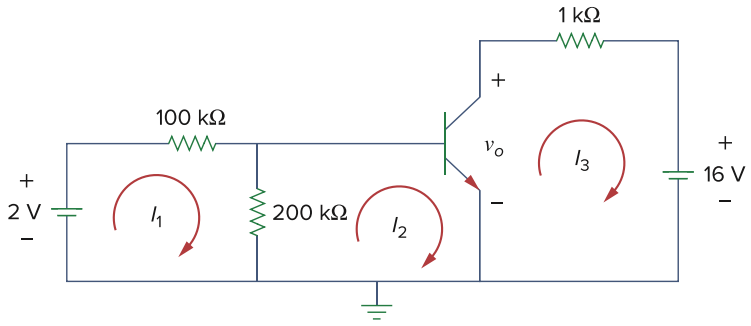

**Figure 3.43**
For Example 3.13.

**Solution:**

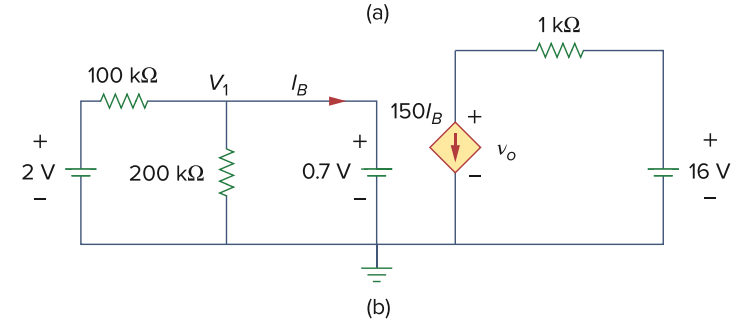
1. **Define.** The circuit is clearly defined and the problem is clearly stated. There appear to be no additional questions that need to be asked.
2. **Present.** We are to determine the output voltage of the circuit shown in Fig. 3.43. The circuit contains an ideal transistor with $\beta = 150$ and $V_{BE} = 0.7$ V.
3. **Alternative.** We can use mesh analysis to solve for $v_o$. We can replace the transistor with its equivalent circuit and use nodal analysis. We can try both approaches and use them to check each other. As a third check, we can use the equivalent circuit and solve it using *PSpice*.
4. **Attempt.**

■ **METHOD 1** Working with Fig. 3.44(a), we start with the first loop.

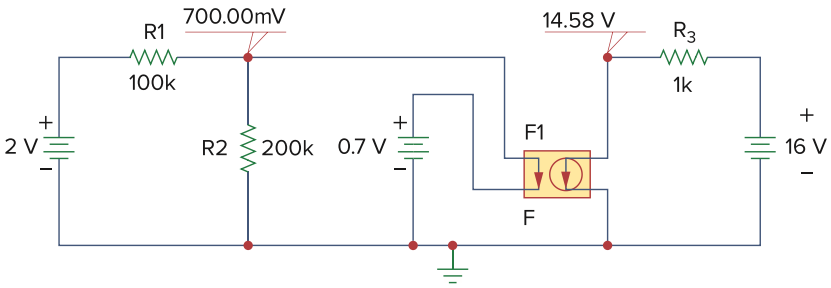
$$-2 + 100\text{k}I_1 + 200\text{k}(I_1 - I_2) = 0 \quad \text{or} \quad 3I_1 - 2I_2 = 2 \times 10^{-5} \tag{3.13.1}$$




(a)




(b)




(c)

**Figure 3.44**
Solution of the problem in Example 3.13: (a) Method 1, (b) Method 2, (c) Method 3.

Now for loop 2.

$$200\text{k}(I_2 - I_1) + V_{BE} = 0 \qquad \text{or} \qquad -2I_1 + 2I_2 = -0.7 \times 10^{-5} \tag{3.13.2}$$

Since we have two equations and two unknowns, we can solve for $I_1$ and $I_2$. Adding Eq. (3.13.1) to (3.13.2) we get;

$$I_1 = 1.3 \times 10^{-5}\text{A} \qquad \text{and} \qquad I_2 = (-0.7 + 2.6)10^{-5}/2 = 9.5\ \mu\text{A}$$

Since $I_3 = -150I_2 = -1.425$ mA, we can now solve for $v_o$ using loop 3:

$$-v_o + 1\text{ k}I_3 + 16 = 0 \qquad \text{or} \qquad v_o = -1.425 + 16 = \mathbf{14.575\ V}$$

■ **METHOD 2** Replacing the transistor with its equivalent circuit produces the circuit shown in Fig. 3.44(b). We can now use nodal analysis to solve for $v_o$.

At node number 1: $V_1 = 0.7$ V

$$(0.7 - 2)/100\text{k} + 0.7/200\text{k} + I_B = 0 \qquad \text{or} \qquad I_B = 9.5\ \mu\text{A}$$

At node number 2 we have:

$$150I_B + (v_o - 16)/1\text{k} = 0 \qquad \text{or}$$
$$v_o = 16 - 150 \times 10^3 \times 9.5 \times 10^{-6} = \mathbf{14.575\ V}$$

5. **Evaluate.** The answers check, b ut to further check we can use *PSpice* (Method 3), which gi ves us the solution sho wn in Fig. 3.44(c).
6. **Satisfactory?** Clearly, we have obtained the desired answer with a very high confidence level. We can now present our work as a solution to the problem.

## Practice Problem 3.13

The transistor circuit in Fig. 3.45 has $\beta = 80$ and $V_{BE} = 0.7$ V. Find $v_o$ and $I_o$.

**Answer:** 12 V, 600 $\mu$A.

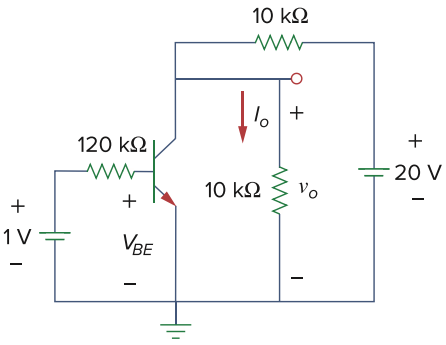



**Figure 3.45**
For Practice Prob. 3.13.

## 3.10 Summary

1. Nodal analysis is the application of Kirchhoff's current law at the nonreference nodes. (It is applicable to both planar and nonplanar circuits.) We express the result in terms of the node voltages. Solv - ing the simultaneous equations yields the node voltages.
2. A supernode consists of two nonreference nodes connected by a (dependent or independent) voltage source.
3. Mesh analysis is the application of Kirchhoff's voltage law around meshes in a planar circuit. We express the result in terms of mesh currents. Solving the simultaneous equations yields the mesh currents.

4. A supermesh consists of two meshes that have a (dependent or independent) current source in common.
5. Nodal analysis is normally used when a circuit has fewer node equations than mesh equations. Mesh analysis is normally used when a circuit has fewer mesh equations than node equations.
6. Circuit analysis can be carried out using *PSpice*.
7. DC transistor circuits can be analyzed using the techniques covered in this chapter.

## Review Questions

**3.1** At node 1 in the circuit of Fig. 3.46, applying KCL gives:

(a) $2 + \dfrac{12 - v_1}{3} = \dfrac{v_1}{6} + \dfrac{v_1 - v_2}{4}$

(b) $2 + \dfrac{v_1 - 12}{3} = \dfrac{v_1}{6} + \dfrac{v_2 - v_1}{4}$

(c) $2 + \dfrac{12 - v_1}{3} = \dfrac{0 - v_1}{6} + \dfrac{v_1 - v_2}{4}$

(d) $2 + \dfrac{v_1 - 12}{3} = \dfrac{0 - v_1}{6} + \dfrac{v_2 - v_1}{4}$

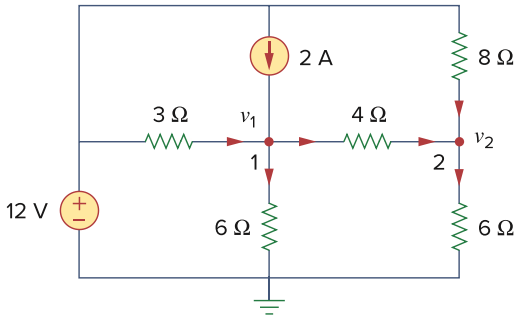



**Figure 3.46**
For Review Questions 3.1 and 3.2.

**3.2** In the circuit of Fig. 3.46, applying KCL at node 2 gives:

(a) $\dfrac{v_2 - v_1}{4} + \dfrac{v_2}{8} = \dfrac{v_2}{6}$

(b) $\dfrac{v_1 - v_2}{4} + \dfrac{v_2}{8} = \dfrac{v_2}{6}$

(c) $\dfrac{v_1 - v_2}{4} + \dfrac{12 - v_2}{8} = \dfrac{v_2}{6}$

(d) $\dfrac{v_2 - v_1}{4} + \dfrac{v_2 - 12}{8} = \dfrac{v_2}{6}$

**3.3** For the circuit in Fig. 3.47, $v_1$ and $v_2$ are related as:

(a) $v_1 = 6i + 8 + v_2$ (b) $v_1 = 6i - 8 + v_2$
(c) $v_1 = -6i + 8 + v_2$ (d) $v_1 = -6i - 8 + v_2$

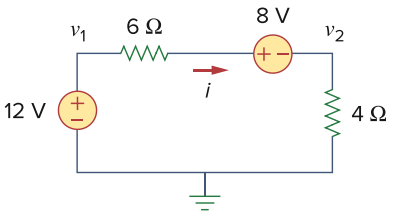



**Figure 3.47**
For Review Questions 3.3 and 3.4.

**3.4** In the circuit of Fig. 3.47, the voltage $v_2$ is:

(a) −8 V (b) −1.6 V
(c) 1.6 V (d) 8 V

**3.5** The current $i$ in the circuit of Fig. 3.48 is:

(a) −2.667 A (b) −0.667 A
(c) 0.667 A (d) 2.667 A

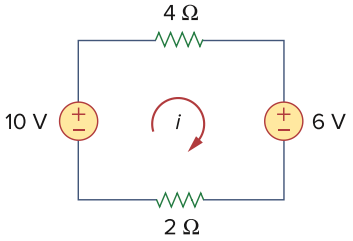



**Figure 3.48**
For Review Questions 3.5 and 3.6.

**3.6** The loop equation for the circuit in Fig. 3.48 is:

(a) $-10 + 4i + 6 + 2i = 0$
(b) $10 + 4i + 6 + 2i = 0$
(c) $10 + 4i - 6 + 2i = 0$
(d) $-10 + 4i - 6 + 2i = 0$

**3.7** In the circuit of Fig. 3.49, current $i_1$ is:

(a) 4 A (b) 3 A (c) 2 A (d) 1 A

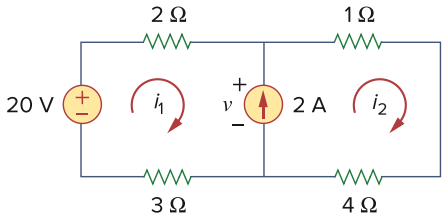



**Figure 3.49**
For Review Questions 3.7 and 3.8.

**3.8** The voltage $v$ across the current source in the circuit of Fig. 3.49 is:

(a) 20 V (b) 15 V (c) 10 V (d) 5 V

**3.9** The *PSpice* part name for a current-controlled voltage source is:

(a) EX (b) FX (c) HX (d) GX

**3.10** Which of the following statements are not true of the pseudocomponent IPROBE:

(a) It must be connected in series.
(b) It plots the branch current.
(c) It displays the current through the branch in which it is connected.
(d) It can be used to display voltage by connecting it in parallel.
(e) It is used only for dc analysis.
(f) It does not correspond to a particular circuit element.

*Answers: 3.1a, 3.2c, 3.3a, 3.4c, 3.5c, 3.6a, 3.7d, 3.8b, 3.9c, 3.10b,d.*

# Problems

## Sections 3.2 and 3.3 Nodal Analysis

**3.1** Using Fig. 3.50, design a problem to help other students better understand nodal analysis.

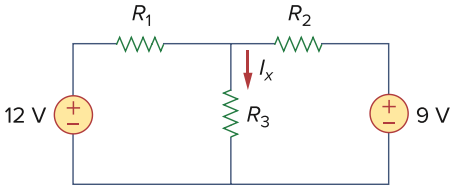



**Figure 3.50**
For Prob. 3.1 and Prob. 3.39.

**3.2** For the circuit in Fig. 3.51, obtain $v_1$ and $v_2$.

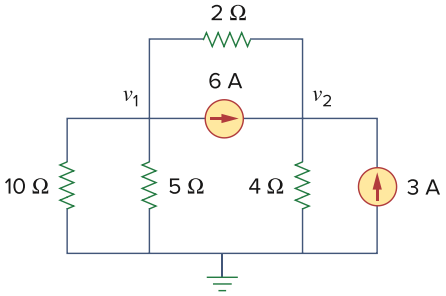



**Figure 3.51**
For Prob. 3.2.

**3.3** Find the currents $I_1$ through $I_4$ and the voltage $v_o$ in the circuit of Fig. 3.52.

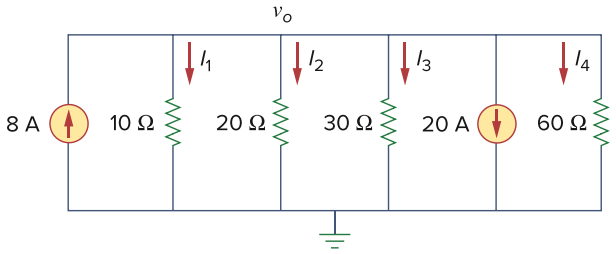



**Figure 3.52**
For Prob. 3.3.

**3.4** Given the circuit in Fig. 3.53, calculate the currents $i_1$ through $i_4$.

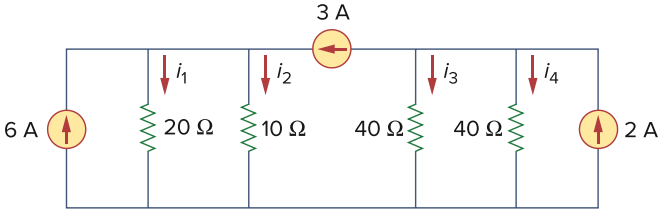



**Figure 3.53**
For Prob. 3.4.

**3.5** Obtain $v_o$ in the circuit of Fig. 3.54.

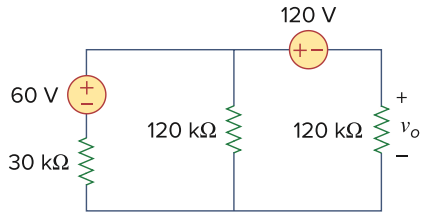



**Figure 3.54**
For Prob. 3.5.

**3.6** Solve for $V_1$ in the circuit of Fig. 3.55 using nodal analysis.

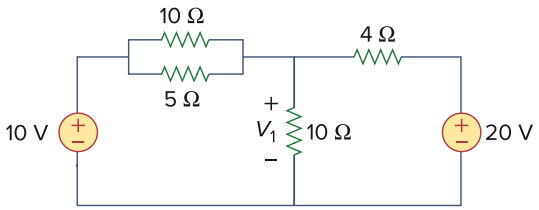



**Figure 3.55**
For Prob. 3.6.

**3.7** Apply nodal analysis to solve for $V_x$ in the circuit of Fig. 3.56.

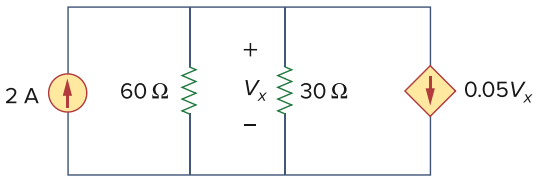



**Figure 3.56**
For Prob. 3.7.

**3.8** Using nodal analysis, find $v_o$ in the circuit of Fig. 3.57.

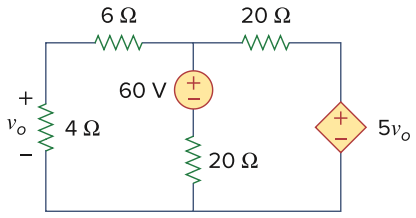



**Figure 3.57**
For Prob. 3.8 and Prob. 3.37.

**3.9** Determine $I_b$ in the circuit in Fig. 3.58 using nodal analysis.

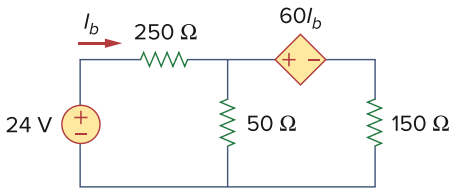



**Figure 3.58**
For Prob. 3.9.

**3.10** Find $I_o$ in the circuit of Fig. 3.59.

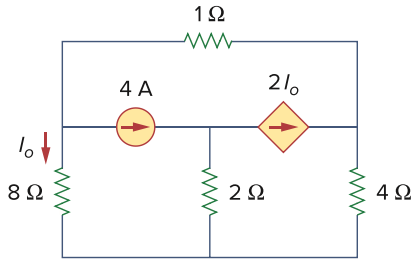



**Figure 3.59**
For Prob. 3.10.

**3.11** Find $V_o$ and the power dissipated in all the resistors in the circuit of Fig. 3.60.

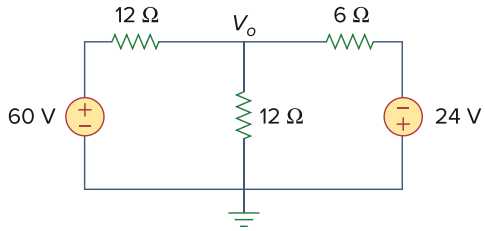



**Figure 3.60**
For Prob. 3.11.

**3.12** Using nodal analysis, determine $V_o$ in the circuit in Fig. 3.61.

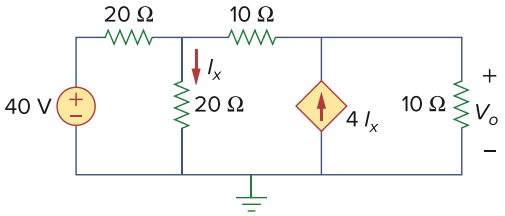



**Figure 3.61**
For Prob. 3.12.

**3.13** Calculate $v_1$ and $v_2$ in the circuit of Fig. 3.62 using nodal analysis.

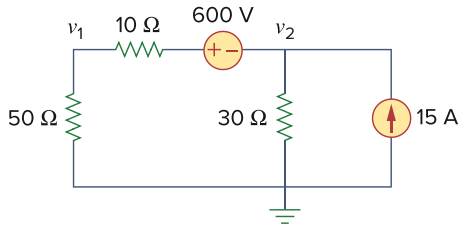



**Figure 3.62**
For Prob. 3.13.

**3.14** Using nodal analysis, find $v_o$ in the circuit of Fig. 3.63.

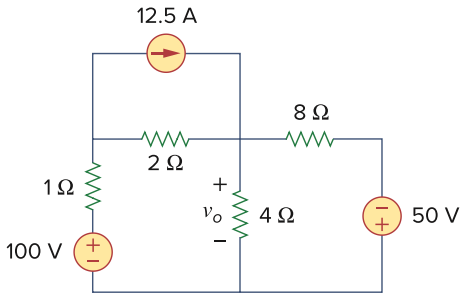



**Figure 3.63**
For Prob. 3.14.

**3.15** Apply nodal analysis to find $i_o$ and the power dissipated in each resistor in the circuit of Fig. 3.64.

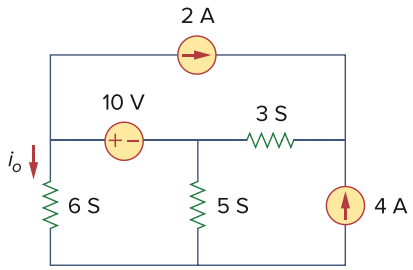



**Figure 3.64**
For Prob. 3.15.

**3.16** Determine voltages $v_1$ through $v_3$ in the circuit of Fig. 3.65 using nodal analysis.

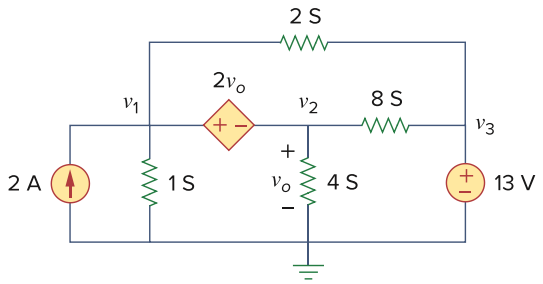



**Figure 3.65**
For Prob. 3.16.

**3.17** Using nodal analysis, find current $i_o$ in the circuit of Fig. 3.66.

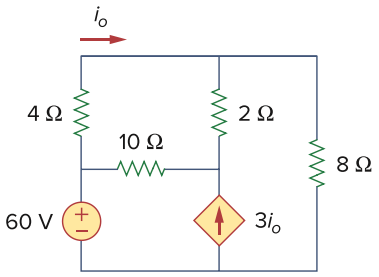



**Figure 3.66**
For Prob. 3.17.

**3.18** Determine the node voltages in the circuit in Fig. 3.67 using nodal analysis.

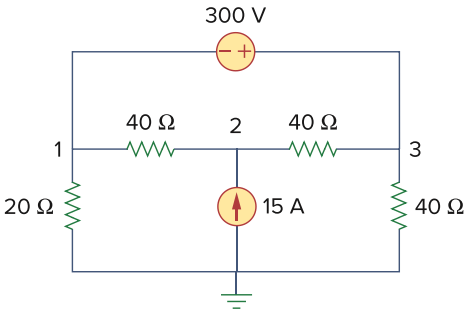



**Figure 3.67**
For Prob. 3.18.

**3.19** Use nodal analysis to find $v_1$, $v_2$, and $v_3$ in the circuit of Fig. 3.68.

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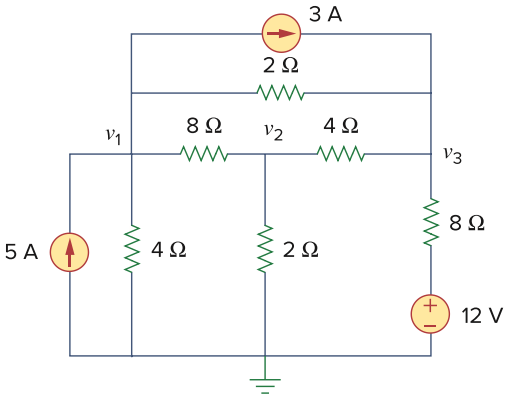


**Figure 3.68**
For Prob. 3.19.

**3.20** For the circuit in Fig. 3.69, find $v_1$, $v_2$, and $v_3$ using nodal analysis.

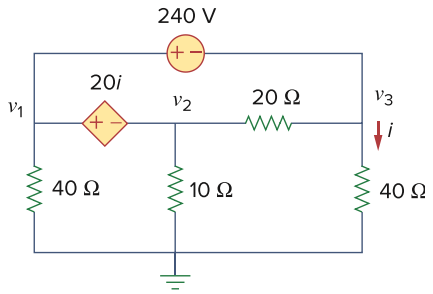


**Figure 3.69**
For Prob. 3.20.

**3.21** For the circuit in Fig. 3.70, find $v_1$ and $v_2$ using nodal analysis.

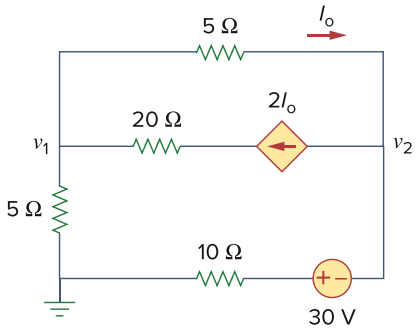


**Figure 3.70**
For Prob. 3.21.

**3.22** Determine $v_1$ and $v_2$ in the circuit of Fig. 3.71.

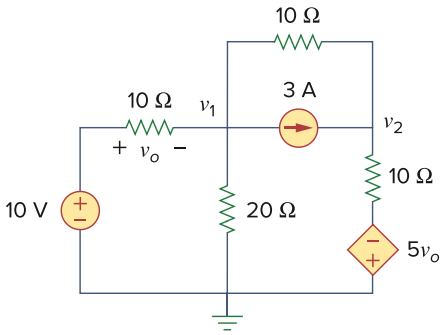


**Figure 3.71**
For Prob. 3.22.

**3.23** Use nodal analysis to find $V_o$ in the circuit of Fig. 3.72.

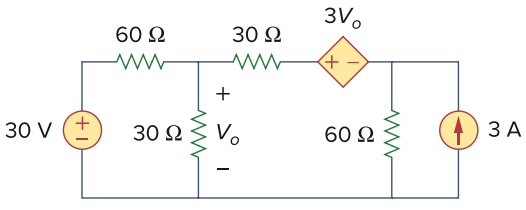


**Figure 3.72**
For Prob. 3.23.

**3.24** Use nodal analysis and *MATLAB* to find $V_o$ in the circuit of Fig. 3.73.

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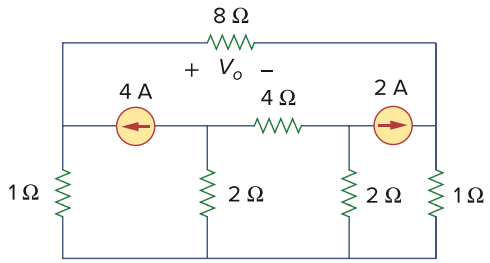



**Figure 3.73**
For Prob. 3.24.

**3.25** Use nodal analysis along with *MATLAB* to determine the node voltages in Fig. 3.74.

ML

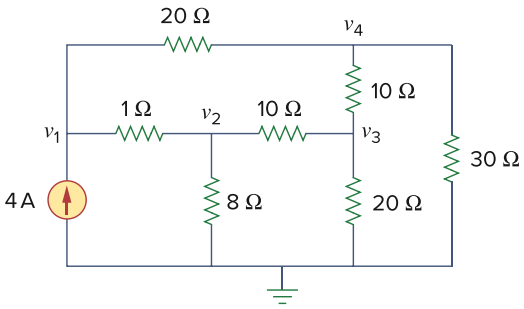



**Figure 3.74**
For Prob. 3.25.

**3.26** Calculate the node voltages $v_1$, $v_2$, and $v_3$ in the circuit of Fig. 3.75.

ML

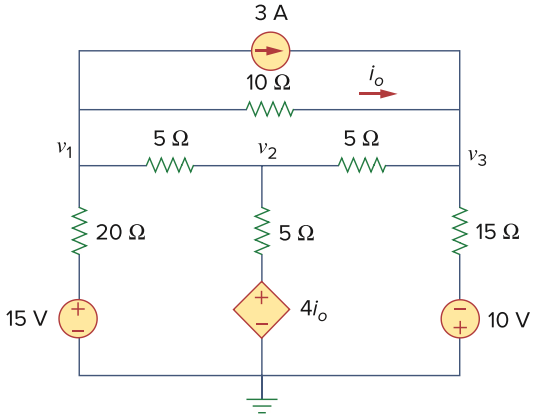



**Figure 3.75**
For Prob. 3.26.

*3.27 Use nodal analysis to determine voltages $v_1$, $v_2$, and $v_3$ in the circuit of Fig. 3.76.

ML

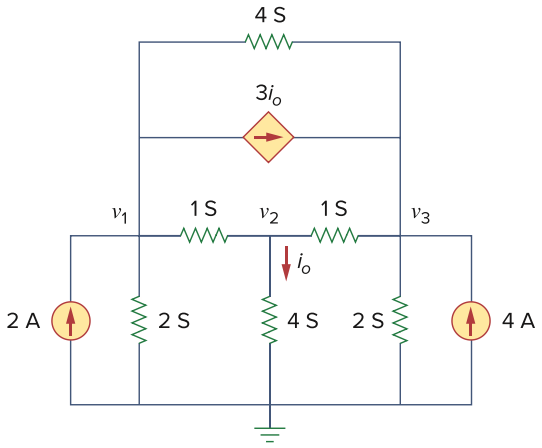



**Figure 3.76**
For Prob. 3.27.

*3.28 Use *MATLAB* to find the voltages at nodes *a*, *b*, *c*, and *d* in the circuit of Fig. 3.77.

ML

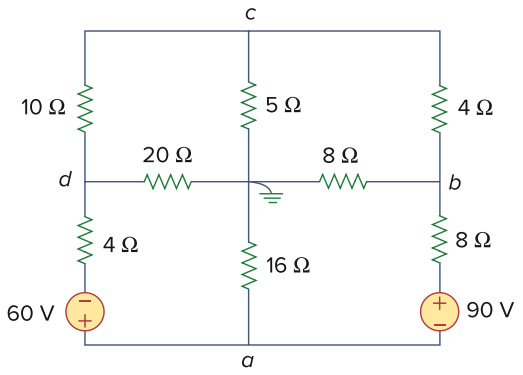



**Figure 3.77**
For Prob. 3.28.

3.29 Use *MATLAB* to solve for the node voltages in the circuit of Fig. 3.78.

ML

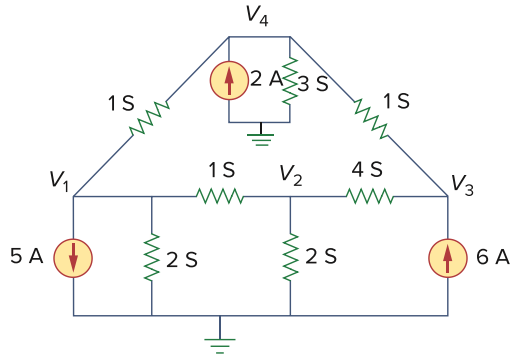



**Figure 3.78**
For Prob. 3.29.

* An asterisk indicates a challenging problem.

3.30 Using nodal analysis, find $v_o$ and $i_o$ in the circuit of Fig. 3.79.

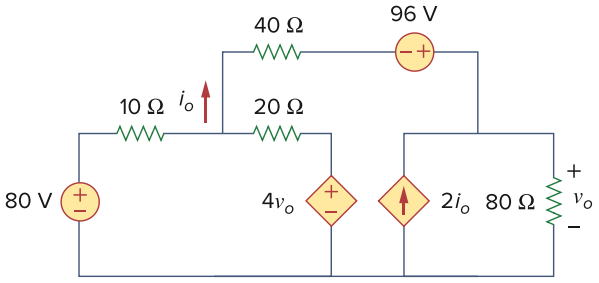



**Figure 3.79**
For Prob. 3.30.

3.31 Find the node voltages for the circuit in Fig. 3.80.

ML

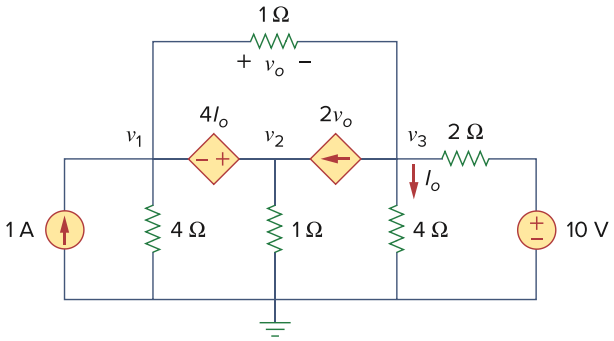



**Figure 3.80**
For Prob. 3.31.

3.32 Obtain the node voltages $v_1$, $v_2$, and $v_3$ in the circuit of Fig. 3.81.

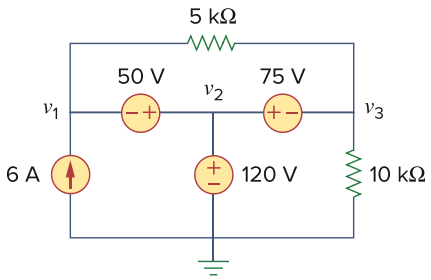
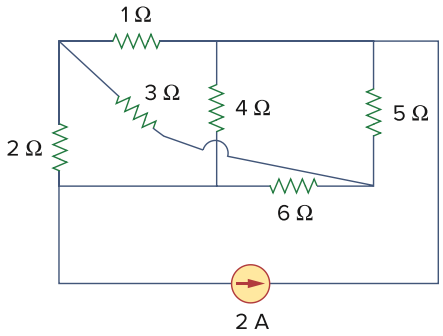


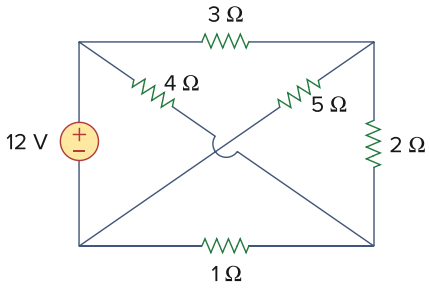

**Figure 3.81**
For Prob. 3.32.

## Sections 3.4 and 3.5 Mesh Analysis

**3.33** Which of the circuits in Fig. 3.82 is planar? For the planar circuit, redraw the circuits with no crossing branches.



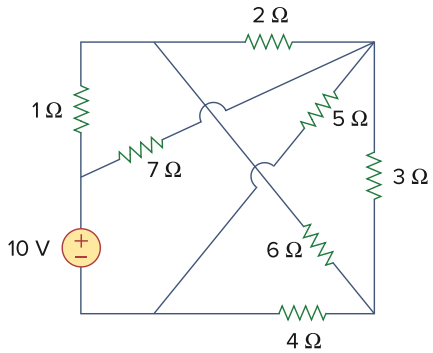

(a)



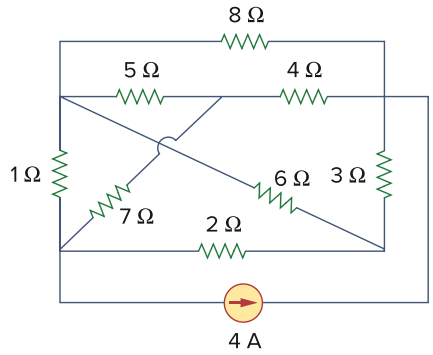

(b)

**Figure 3.82**
For Prob. 3.33.

**3.34** Determine which of the circuits in Fig. 3.83 is planar and redraw it with no crossing branches.




(a)




(b)

**Figure 3.83**
For Prob. 3.34.

**3.35** Rework Prob. 3.5 using mesh analysis.

**3.36** Use mesh analysis to obtain $i_a$, $i_b$, and $i_c$ in the circuit in Fig. 3.84.

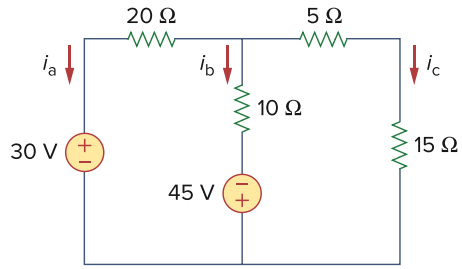



**Figure 3.84**
For Prob. 3.36.

**3.37** Solve Prob. 3.8 using mesh analysis.

**3.38** Apply mesh analysis to the circuit in Fig. 3.85 and obtain $I_o$.

ML

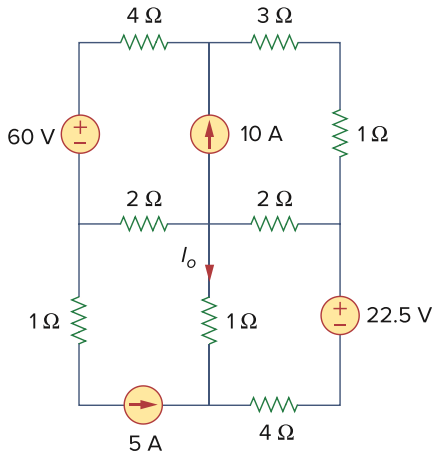



**Figure 3.85**
For Prob. 3.38.

**3.39** Using Fig. 3.50 from Prob. 3.1, design a problem to help other students better understand mesh analysis.

**3.40** For the bridge network in Fig. 3.86, find $i_o$ using mesh analysis.

ML

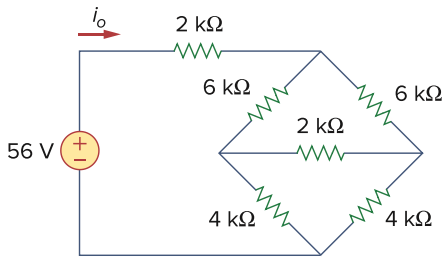



**Figure 3.86**
For Prob. 3.40.

**3.41** Apply mesh analysis to find $i$ in Fig. 3.87.

ML

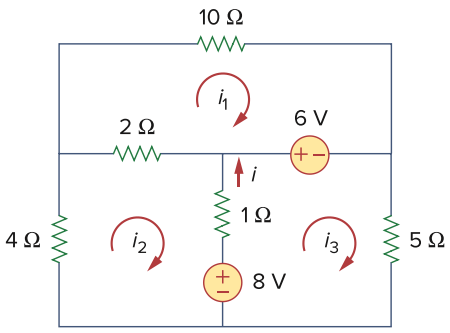



**Figure 3.87**
For Prob. 3.41.

**3.42** Using Fig. 3.88, design a problem to help students better understand mesh analysis using matrices.

edd

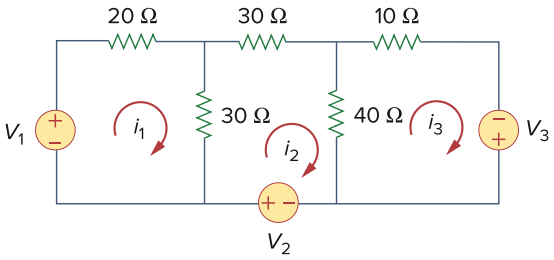



**Figure 3.88**
For Prob. 3.42.

**3.43** Use mesh analysis to find $v_{ab}$ and $i_o$ in the circuit of Fig. 3.89.

ML

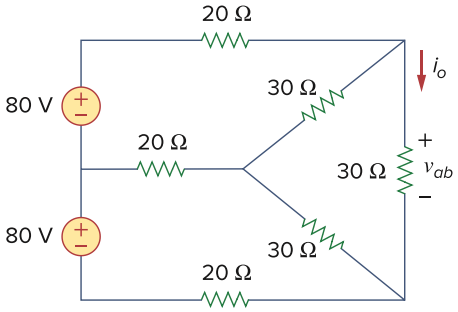



**Figure 3.89**
For Prob. 3.43.

**3.44** Use mesh analysis to obtain $i_o$ in the circuit of Fig. 3.90.

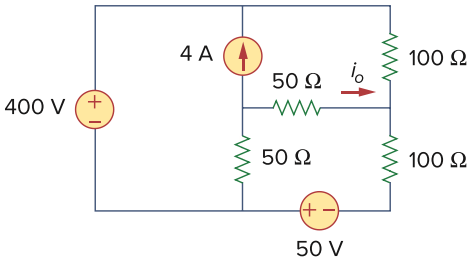



**Figure 3.90**
For Prob. 3.44.

**3.45** Find current $i$ in the circuit of Fig. 3.91.

ML

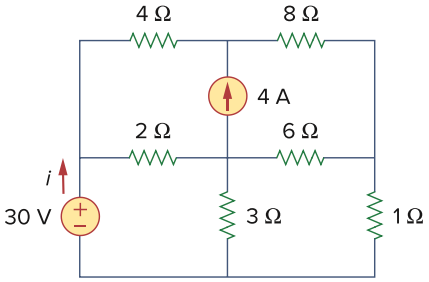



**Figure 3.91**
For Prob. 3.45.

**3.46** Calculate the mesh currents $i_1$ and $i_2$ in Fig. 3.92.

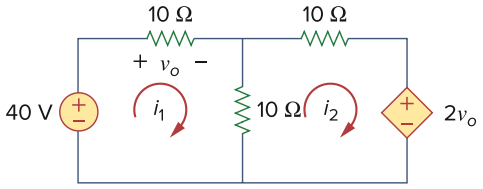



**Figure 3.92**
For Prob. 3.46.

**3.47** Rework Prob. 3.19 using mesh analysis.

ML

**3.48** Determine the current through the 10-kΩ resistor in the circuit of Fig. 3.93 using mesh analysis.

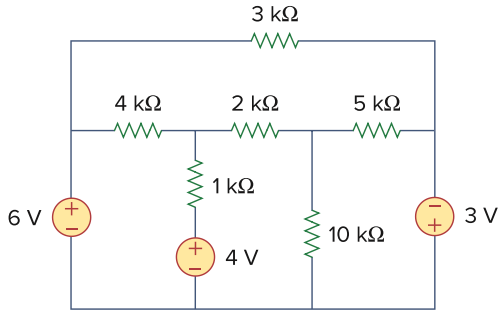



**Figure 3.93**
For Prob. 3.48.

**3.49** Find $v_o$ and $i_o$ in the circuit of Fig. 3.94.

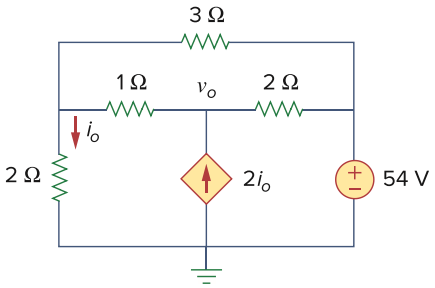



**Figure 3.94**
For Prob. 3.49.

**3.50** Use mesh analysis to find the current $i_o$ in the circuit of Fig. 3.95.

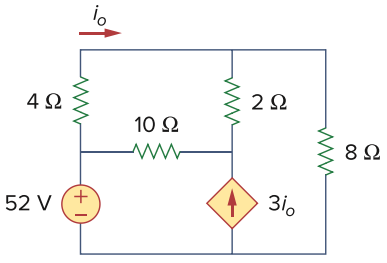



**Figure 3.95**
For Prob. 3.50.

**3.51** Apply mesh analysis to find $v_o$ in the circuit of Fig. 3.96.

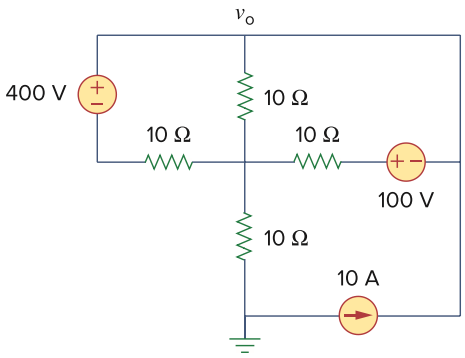



**Figure 3.96**
For Prob. 3.51.

**3.52** Use mesh analysis to find $i_1$, $i_2$, and $i_3$ in the circuit of Fig. 3.97.

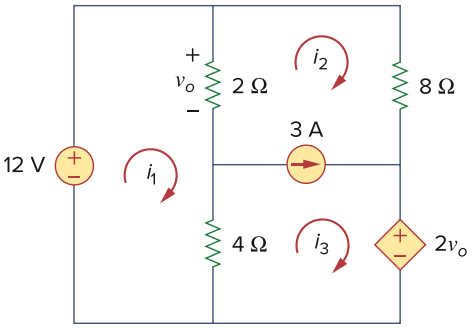



**Figure 3.97**
For Prob. 3.52.

**3.53** Find the mesh currents in the circuit of Fig. 3.98 using *MATLAB*.

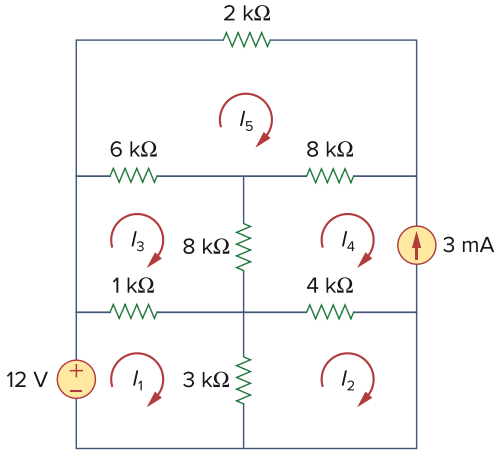



**Figure 3.98**
For Prob. 3.53.

**3.54** Find the mesh currents $i_1$, $i_2$, and $i_3$ in the circuit in Fig. 3.99.

ML

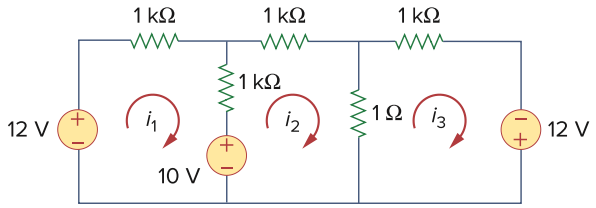



**Figure 3.99**
For Prob. 3.54.

*3.55 In the circuit of Fig. 3.100, solve for $I_1$, $I_2$, and $I_3$.

ML

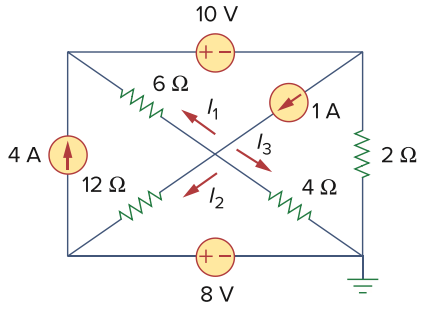



**Figure 3.100**
For Prob. 3.55.

**3.56** Determine $v_1$ and $v_2$ in the circuit of Fig. 3.101.

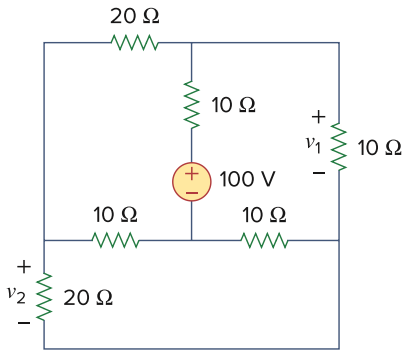



**Figure 3.101**
For Prob. 3.56.

**3.57** In the circuit of Fig. 3.102, find the values of $R$, $V_1$, and $V_2$ given that $i_o = 15$ mA.

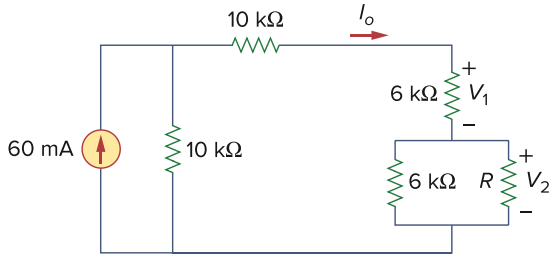



**Figure 3.102**
For Prob. 3.57.

**3.58** Find $i_1$, $i_2$, and $i_3$ in the circuit of Fig. 3.103.

ML

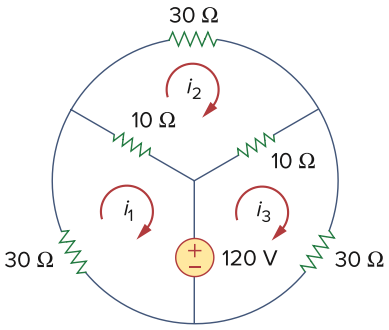



**Figure 3.103**
For Prob. 3.58.

**3.59** Rework Prob. 3.30 using mesh analysis.

ML

**3.60** Calculate the power dissipated in each resistor in the circuit of Fig. 3.104.

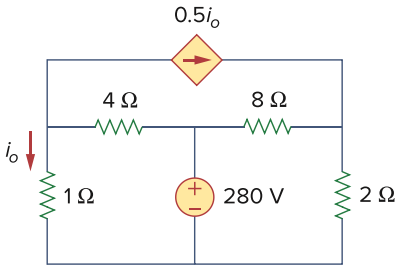



**Figure 3.104**
For Prob. 3.60.

**3.61** Calculate the current gain $i_o/i_s$ in the circuit of Fig. 3.105.

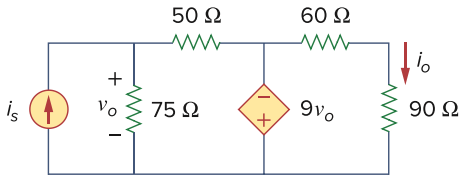



**Figure 3.105**
For Prob. 3.61.

**3.62** Find the mesh currents $i_1$, $i_2$, and $i_3$ in the network of Fig. 3.106.

ML

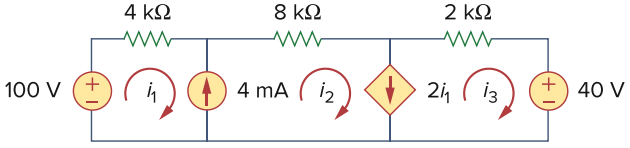



**Figure 3.106**
For Prob. 3.62.

**3.63** Find $v_x$ and $i_x$ in the circuit shown in Fig. 3.107.

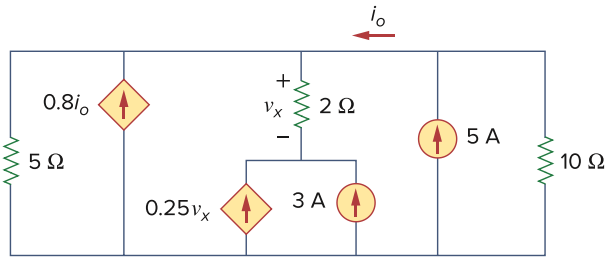



**Figure 3.107**
For Prob. 3.63.

**3.64** Find $v_o$ and $i_o$ in the circuit of Fig. 3.108.

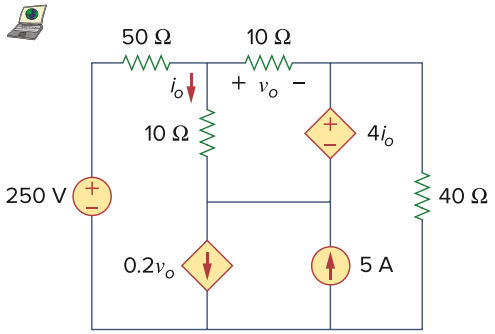



**Figure 3.108**
For Prob. 3.64.

**3.65** Use *MATLAB* to solve for the mesh currents in the circuit of Fig. 3.109.

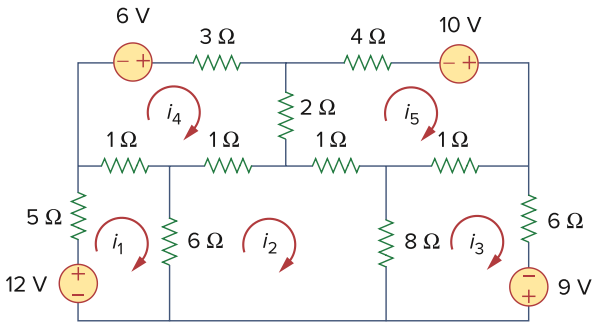



**Figure 3.109**
For Prob. 3.65.

**3.66** Write a set of mesh equations for the circuit in Fig. 3.110. Use *MATLAB* to determine the mesh currents.

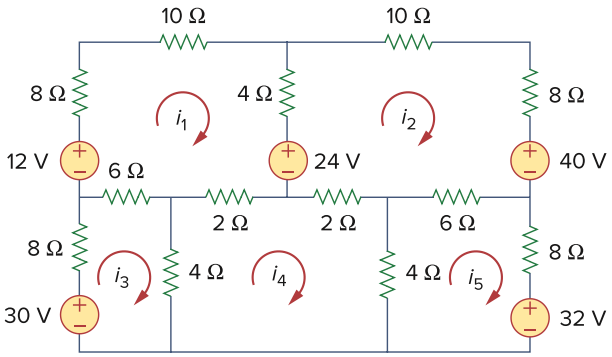



**Figure 3.110**
For Prob. 3.66.

## Section 3.6 Nodal and Mesh Analyses by Inspection

**3.67** Obtain the node-voltage equations for the circuit in Fig. 3.111 by inspection. Then solve for $V_o$.

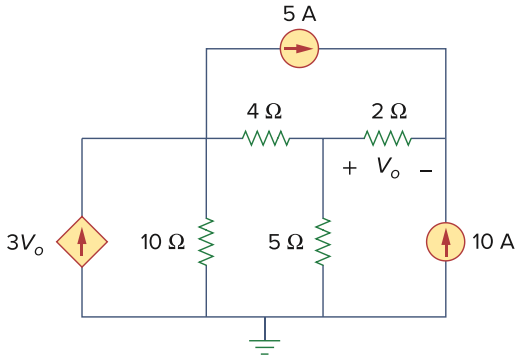



**Figure 3.111**
For Prob. 3.67.

**3.68** Using Fig. 3.112, design a problem, to solve for $V_o$, to help other students better understand nodal analysis. Try your best to come up with values to make the calculations easier.

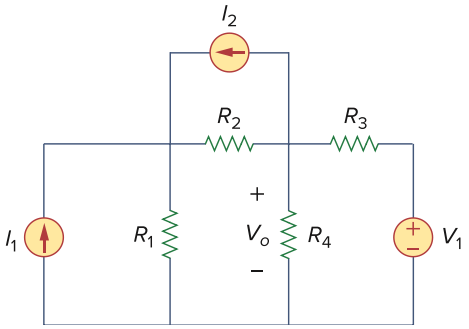



**Figure 3.112**
For Prob. 3.68.

**3.69** For the circuit shown in Fig. 3.113, write the node-voltage equations by inspection.

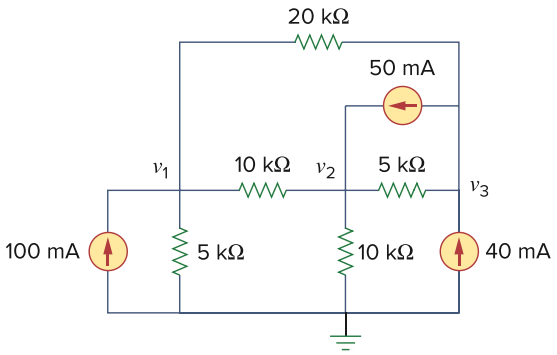



**Figure 3.113**
For Prob. 3.69.

**3.70** Write the node-voltage equations by inspection and then determine values of $V_1$ and $V_2$ in the circuit of Fig. 3.114.

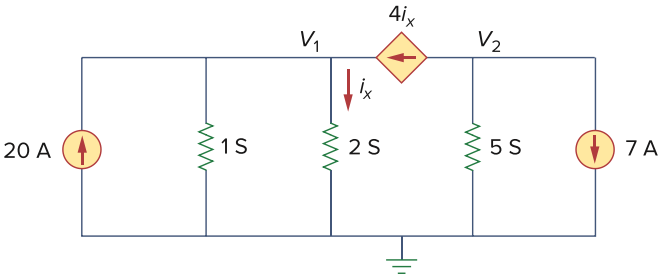



**Figure 3.114**
For Prob. 3.70.

**3.71** Write the mesh-current equations for the circuit in Fig. 3.115. Next, determine the values of $i_1$, $i_2$, and $i_3$.

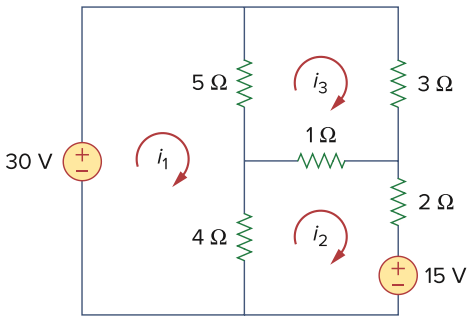



**Figure 3.115**
For Prob. 3.71.

**3.72** By inspection, write the mesh-current equations for the circuit in Fig. 3.116.

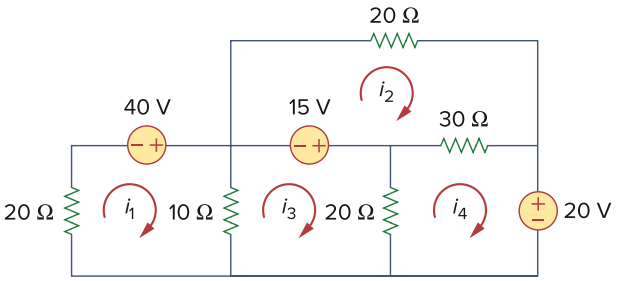



**Figure 3.116**
For Prob. 3.72.

**3.73** Write the mesh-current equations for the circuit in Fig. 3.117.

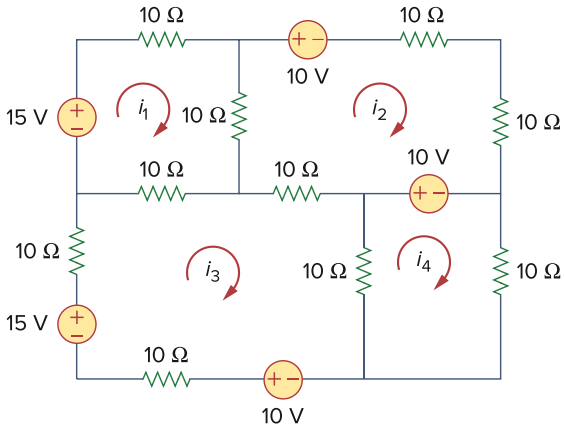



**Figure 3.117**
For Prob. 3.73.

**3.74** By inspection, obtain the mesh-current equations for the circuit in Fig. 3.118.

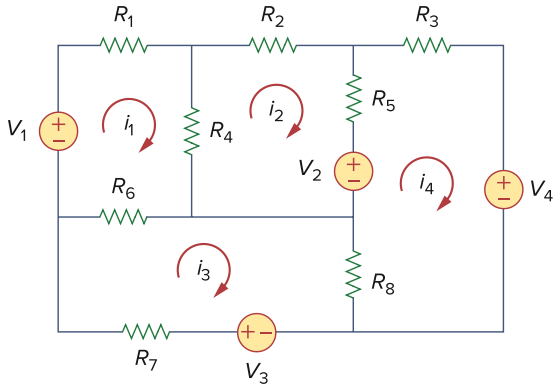



**Figure 3.118**
For Prob. 3.74.

## Section 3.8 Circuit Analysis with *PSpice* or *MultiSim*

**3.75** Use *PSpice* or *MultiSim* to solve Prob. 3.58.

**3.76** Use *PSpice* or *MultiSim* to solve Prob. 3.27.

**3.77** Solve for $V_1$ and $V_2$ in the circuit of Fig. 3.119 using *PSpice* or *MultiSim*.

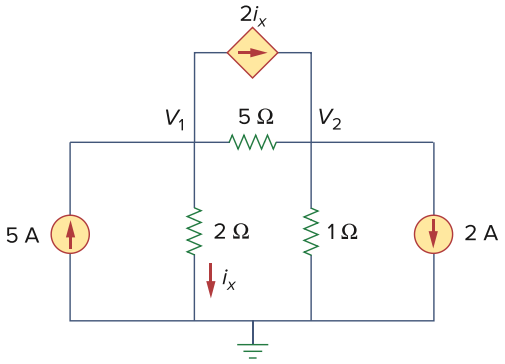



**Figure 3.119**
For Prob. 3.77.

**3.78** Solve Prob. 3.20 using *PSpice* or *MultiSim*.

**3.79** Rework Prob. 3.28 using *PSpice* or *MultiSim*.

**3.80** Find the nodal voltages $v_1$ through $v_4$ in the circuit of Fig. 3.120 using *PSpice* or *MultiSim*.

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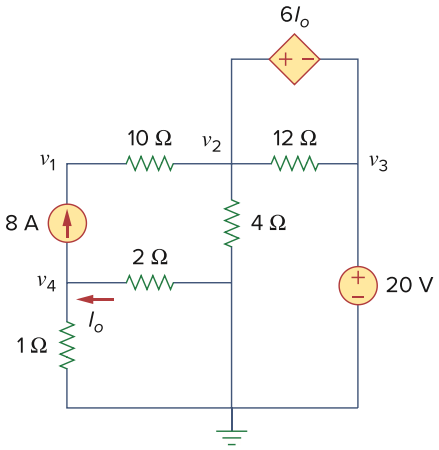


**Figure 3.120**
For Prob. 3.80.

**3.81** Use *PSpice* or *MultiSim* to solve the problem in Example 3.4.

**3.82** If the Schematics Netlist for a network is as follows, draw the network.

```
R_R1    1   2   2K
R_R2    2   0   4K
R_R3    3   0   8K
R_R4    3   4   6K
R_R5    1   3   3K
V_VS    4   0   DC      100
I_IS    0   1   DC      4
F_F1    1   3   VF_F1   2
VF_F1   5   0   0V
E_E1    3   2   1       3     3
```

**3.83** The following program is the Schematics Netlist of a particular circuit. Draw the circuit and determine the voltage at node 2.

```
R_R1    1   2   20
R_R2    2   0   50
R_R3    2   3   70
R_R4    3   0   30
V_VS    1   0   20V
I_IS    2   0   DC    2A
```

## Section 3.9 Applications

**3.84** Calculate $v_o$ and $I_o$ in the circuit of Fig. 3.121.

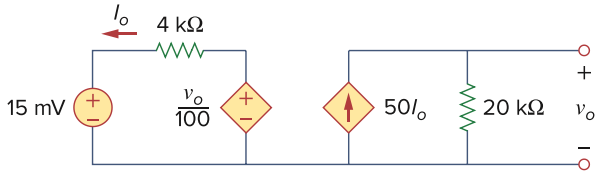


**Figure 3.121**
For Prob. 3.84.

**3.85** An audio amplifier with a resistance of 9 Ω supplies power to a speaker. What should be the resistance of the speaker for maximum power to be delivered?

**3.86** For the simplified transistor circuit of Fig. 3.122, calculate the voltage $v_o$.

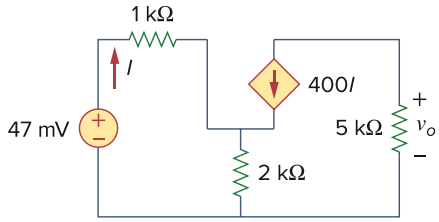


**Figure 3.122**
For Prob. 3.86.

**3.87** For the circuit in Fig. 3.123, find the gain $v_o/v_s$.

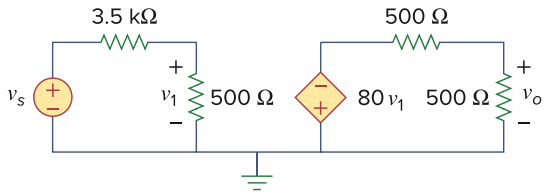


**Figure 3.123**
For Prob. 3.87.

*3.88 Determine the gain $v_o/v_s$ of the transistor amplifier circuit in Fig. 3.124.

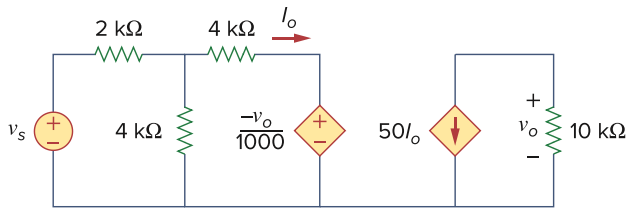


**Figure 3.124**
For Prob. 3.88.

**3.89** For the transistor circuit shown in Fig. 3.125, find $I_B$ and $V_{CE}$. Let $\beta = 100$, and $V_{BE} = 0.7$ V.

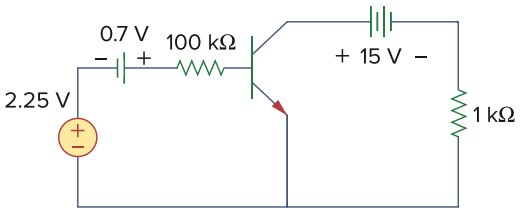


**Figure 3.125**
For Prob. 3.89.

**3.90** Calculate $v_s$ for the transistor in Fig. 3.126 given that $v_o = 6$ V, $\beta = 90$, $V_{BE} = 0.7$ V.

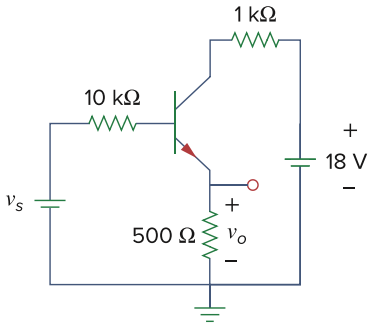


**Figure 3.126**
For Prob. 3.90.

**3.91** For the transistor circuit of Fig. 3.127, find $I_B$, $V_{CE}$, and $v_o$. Take $\beta = 150$, $V_{BE} = 0.7$ V.

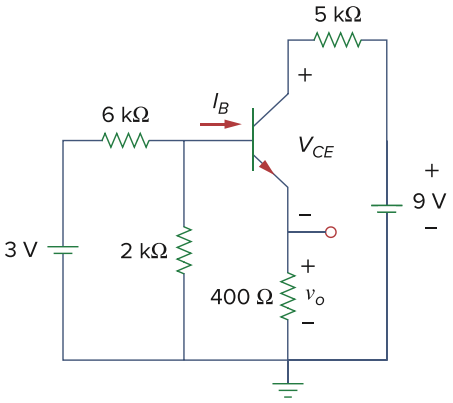


**Figure 3.127**
For Prob. 3.91.

**3.92** Using Fig. 3.128, design a problem to help other students better understand transistors. Make sure you use reasonable numbers!

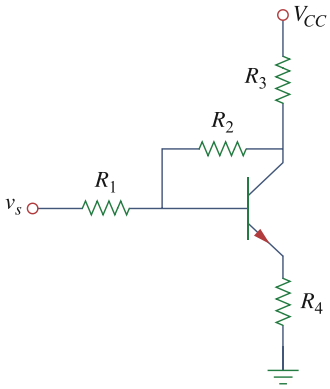


**Figure 3.128**
For Prob. 3.92.

## Comprehensive Problem

*3.93 Rework Example 3.11 with hand calculation.

chapter

# 4

# Circuit Theorems

*Your success as an engineer will be directly proportional to your ability to communicate!*

—Charles K. Alexander

## Enhancing Your Skills and Your Career

### Enhancing Your Communication Skills

Taking a course in circuit analysis is one step in preparing yourself for a career in electrical engineering. Enhancing your communication skills while in school should also be part of that preparation, as a large part of your time will be spent communicating.

People in industry have complained again and again that graduating engineers are ill-prepared in written and oral communication. An engineer who communicates effectively becomes a valuable asset.

You can probably speak or write easily and quickly. But how *effectively* do you communicate? The art of effective communication is of the utmost importance to your success as an engineer.

For engineers in industry, communication is key to promotability. Consider the result of a survey of U.S. corporations that asked what factors influence managerial promotion. The survey includes a listing of 22 personal qualities and their importance in advancement. You may be surprised to note that "technical skill based on experience" placed fourth from the bottom. Attributes such as self-confidence, ambition, flexibility, maturity, ability to make sound decisions, getting things done with and through people, and capacity for hard work all ranked higher. At the top of the list was "ability to communicate." The higher your professional career progresses, the more you will need to communicate. Therefore, you should regard effective communication as an important tool in your engineering tool chest.

Learning to communicate effectively is a lifelong task you should always work toward. The best time to begin is while still in school. Continually look for opportunities to develop and strengthen your reading, writing, listening, and speaking skills. You can do this through classroom presentations, team projects, active participation in student organizations, and enrollment in communication courses. The risks are less now than later in the workplace.

*Ability to communicate effectively is regarded by many as the most important step to an executive promotion.*
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## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Develop and enhance your skills in using nodal analysis and mesh analysis to analyze basic circuits.
2. Understand how linearity works with basic circuits.
3. Explain the principle of superposition and how it can be used to help analyze circuits.
4. Understand the value of source transformation and how it can be used to simplify circuits.
5. Recognize Thevenin's and Norton's theorems and know how they can lead to greatly simplified circuits.
6. Explain the maximum power transfer concept.

## 4.1 Introduction

A major advantage of analyzing circuits using Kirchhoff's laws as we did in Chapter 3 is that we can analyze a circuit without tampering with its original configuration. A major disadvantage of this approach is that, for a large, complex circuit, tedious computation is involved.

The growth in areas of application of electric circuits has led to an evolution from simple to complex circuits. To handle the complexity, engineers over the years have developed some theorems to simplify circuit analysis. Such theorems include Thevenin's and Norton's theorems. Since these theorems are applicable to *linear* circuits, we first discuss the concept of circuit linearity. In addition to circuit theorems, we discuss the concepts of superposition, source transformation, and maximum power transfer in this chapter. The concepts we develop are applied in the last section to source modeling and resistance measurement.

## 4.2 Linearity Property

Linearity is the property of an element describing a linear relationship between cause and effect. Although the property applies to many circuit elements, we shall limit its applicability to resistors in this chapter. The property is a combination of both the homogeneity (scaling) property and the additivity property.

The homogeneity property requires that if the input (also called the *excitation*) is multiplied by a constant, then the output (also called the *response*) is multiplied by the same constant. For a resistor, for example, Ohm's law relates the input $i$ to the output $v$,

$$v = iR \tag{4.1}$$

If the current is increased by a constant $k$, then the voltage increases correspondingly by $k$; that is,

$$kiR = kv \tag{4.2}$$

The additivity property requires that the response to a sum of inputs is the sum of the responses to each input applied separately . Using the voltage-current relationship of a resistor, if

$$v_1 = i_1 R \tag{4.3a}$$

and

$$v_2 = i_2 R \tag{4.3b}$$

then applying $(i_1 + i_2)$ gives

$$\boxed{v = (i_1 + i_2)R = i_1 R + i_2 R = v_1 + v_2} \tag{4.4}$$

We say that a resistor is a linear element because the voltage-current relationship satisfies both the homogeneity and the additivity properties.

In general, a circuit is linear if it is both additive and homogeneous. A linear circuit consists of only linear elements, linear dependent sources, and independent sources.

A **linear circuit** is one whose output is linearly related (or directly proportional) to its input.

Throughout this book we consider only linear circuits. Note that since $p = i^2R = v^2/R$ (making it a quadratic function rather than a linear one), the relationship between power and voltage (or current) is nonlinear. Therefore, the theorems covered in this chapter are not applicable to power.

To illustrate the linearity principle, consider the linear circuit shown in Fig. 4.1. The linear circuit has no independent sources inside it. It is excited by a voltage source $v_s$, which serves as the input. The circuit is terminated by a load $R$. We may take the current $i$ through $R$ as the output. Suppose $v_s = 10$ V gives $i = 2$ A. According to the linearity principle, $v_s = 1$ V will give $i = 0.2$ A. By the same token, $i = 1$ mA must be due to $v_s = 5$ mV.

For example, when current $i_1$ flows through resistor $R$, the power is $p_1 = Ri_1^2$, and when current $i_2$ flows through $R$, the power is $p_2 = Ri_2^2$. If current $i_1 + i_2$ flows through $R$, the power absorbed is $p_3 = R(i_1 + i_2)^2 = Ri_1^2 + Ri_2^2 + 2Ri_1 i_2 \neq p_1 + p_2$. Thus, the power relation is nonlinear.

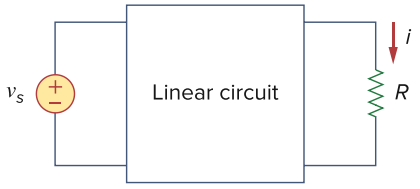


**Figure 4.1**
A linear circuit with input $v_s$ and output $i$.

## Example 4.1

For the circuit in Fig. 4.2, find $I_o$ when $v_s = 12$ V and $v_s = 24$ V.

**Solution:**
Applying KVL to the two loops, we obtain

$$12i_1 - 4i_2 + v_s = 0 \tag{4.1.1}$$

$$-4i_1 + 16i_2 - 3v_x - v_s = 0 \tag{4.1.2}$$

But $v_x = 2i_1$. Equation (4.1.2) becomes

$$-10i_1 + 16i_2 - v_s = 0 \tag{4.1.3}$$

Adding Eqs. (4.1.1) and (4.1.3) yields

$$2i_1 + 12i_2 = 0 \quad \Rightarrow \quad i_1 = -6i_2$$

Substituting this in Eq. (4.1.1), we get

$$-76i_2 + v_s = 0 \quad \Rightarrow \quad i_2 = \frac{v_s}{76}$$

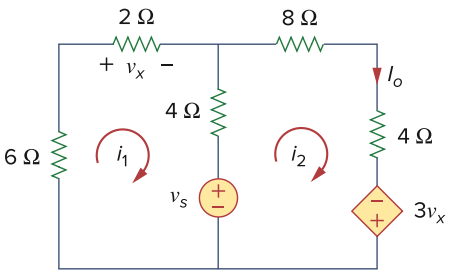


**Figure 4.2**
For Example 4.1.

When $v_s = 12$ V,

$$I_o = i_2 = \frac{12}{76}\text{ A}$$

When $v_s = 24$ V,

$$I_o = i_2 = \frac{24}{76}\text{ A}$$

showing that when the source value is doubled, $I_o$ doubles.

## Practice Problem 4.1

For the circuit in Fig. 4.3, find $v_o$ when $i_s = 30$ and $i_s = 45$ A.

**Answer:** 40 V, 60 V.

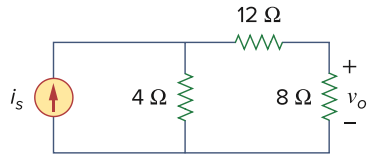


**Figure 4.3**
For Practice Prob. 4.1.

## Example 4.2

Assume $I_o = 1$ A and use linearity to find the actual value of $I_o$ in the circuit of Fig. 4.4.

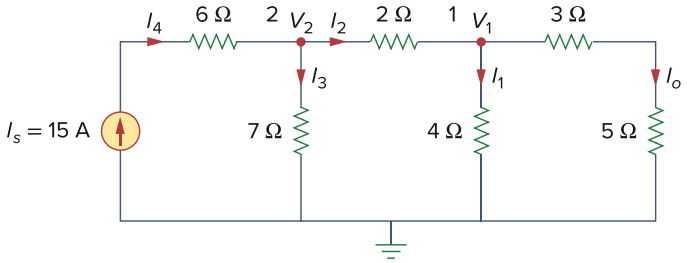


**Figure 4.4**
For Example 4.2.

**Solution:**
If $I_o = 1$ A, then $V_1 = (3 + 5)I_o = 8$ V and $I_1 = V_1/4 = 2$ A. Applying KCL at node 1 gives

$$I_2 = I_1 + I_o = 3\text{ A}$$

$$V_2 = V_1 + 2I_2 = 8 + 6 = 14\text{ V}, \qquad I_3 = \frac{V_2}{7} = 2\text{ A}$$

Applying KCL at node 2 gives

$$I_4 = I_3 + I_2 = 5\text{ A}$$

Therefore, $I_s = 5$ A. This shows that assuming $I_o = 1$ gives $I_s = 5$ A, the actual source current of 15 A will give $I_o = 3$ A as the actual value.

## Practice Problem 4.2

Assume that $V_o = 1$ V and use linearity to calculate the actual value of $V_o$ in the circuit of Fig. 4.5.

**Answer:** 16 V.

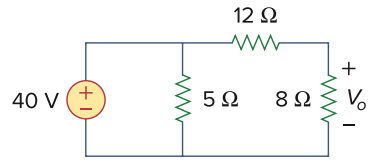


**Figure 4.5**
For Practice Prob. 4.2.

## 4.3 Superposition

If a circuit has two or more independent sources, one way to determine the value of a specific variable (voltage or current) is to use nodal or mesh analysis as in Chapter 3. Another way is to determine the contribution of each independent source to the variable and then add them up. The latter approach is known as the *superposition principle*.

The idea of superposition rests on the linearity property.

Superposition is not limited to circuit analysis but is applicable in many fields where cause and effect bear a linear relationship to one another.

The **superposition** principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately. However, to apply the superposition principle, we must keep two things in mind:

Other terms such as *killed, made inactive, deadened, or set equal to zero* are often used to convey the same idea.

1. We consider one independent source at a time while all other independent sources are *turned off*. This implies that we replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit). This way we obtain a simpler and more manageable circuit.
2. Dependent sources are left intact because they are controlled by circuit variables.

With these in mind, we apply the superposition principle in three steps:

### Steps to Apply Superposition Principle:

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using the techniques covered in Chapters 2 and 3.
2. Repeat step 1 for each of the other independent sources.
3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

Analyzing a circuit using superposition has one major disadvantage: It may very likely involve more work. If the circuit has three independent sources, we may have to analyze three simpler circuits each providing the contribution due to the respective individual source. However, superposition does help reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and of current sources by open circuits.

Keep in mind that superposition is based on linearity. For this reason, it is not applicable to the effect on power due to each source, because the power absorbed by a resistor depends on the square of the voltage or current. If the power value is needed, the current through (or voltage across) the element must be calculated first using superposition.

## Example 4.3

Use the superposition theorem to find $v$ in the circuit of Fig. 4.6.

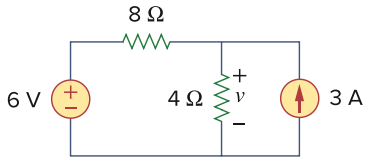


**Figure 4.6**
For Example 4.3.

**Solution:**
Since there are two sources, let

$$v = v_1 + v_2$$

where $v_1$ and $v_2$ are the contributions due to the 6-V voltage source and the 3-A current source, respectively. To obtain $v_1$, we set the current source to zero, as shown in Fig. 4.7(a). Applying KVL to the loop in Fig. 4.7(a) gives

$$12i_1 - 6 = 0 \quad \Rightarrow \quad i_1 = 0.5 \text{ A}$$

Thus,

$$v_1 = 4i_1 = 2 \text{ V}$$

We may also use voltage division to get $v_1$ by writing

$$v_1 = \frac{4}{4+8}(6) = 2 \text{ V}$$

To get $v_2$, we set the voltage source to zero, as in Fig. 4.7(b). Using current division,

$$i_3 = \frac{8}{4+8}(3) = 2 \text{ A}$$

Hence,

$$v_2 = 4i_3 = 8 \text{ V}$$

And we find

$$v = v_1 + v_2 = 2 + 8 = 10 \text{ V}$$

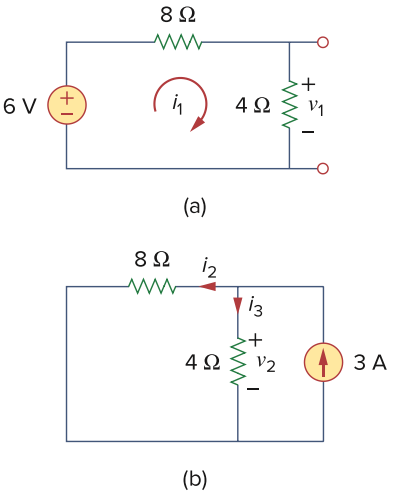


**Figure 4.7**
For Example 4.3: (a) calculating $v_1$, (b) calculating $v_2$.

## Practice Problem 4.3

Using the superposition theorem, find $v_o$ in the circuit of Fig. 4.8.

**Answer:** 16 V.

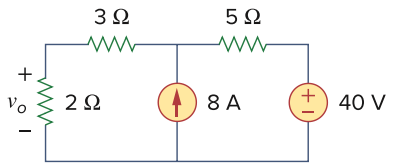


**Figure 4.8**
For Practice Prob. 4.3.

## Example 4.4

Find $i_o$ in the circuit of Fig. 4.9 using superposition.

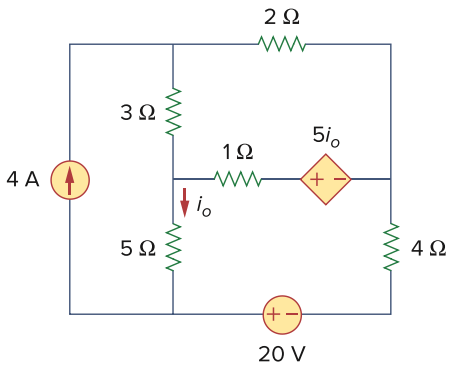


**Figure 4.9**
For Example 4.4.

**Solution:**
The circuit in Fig. 4.9 involves a dependent source, which must be left intact. We let

$$i_o = i_o' + i_o'' \tag{4.4.1}$$

where $i_o'$ and $i_o''$ are due to the 4-A current source and 20-V voltage source respectively. To obtain $i_o'$, we turn off the 20-V source so that we have the circuit in Fig. 4.10(a).We apply mesh analysis in order to obtain $i_o'$. For loop 1,

$$i_1 = 4 \text{ A} \tag{4.4.2}$$

For loop 2,

$$-3i_1 + 6i_2 - 1i_3 - 5i_o' = 0 \tag{4.4.3}$$

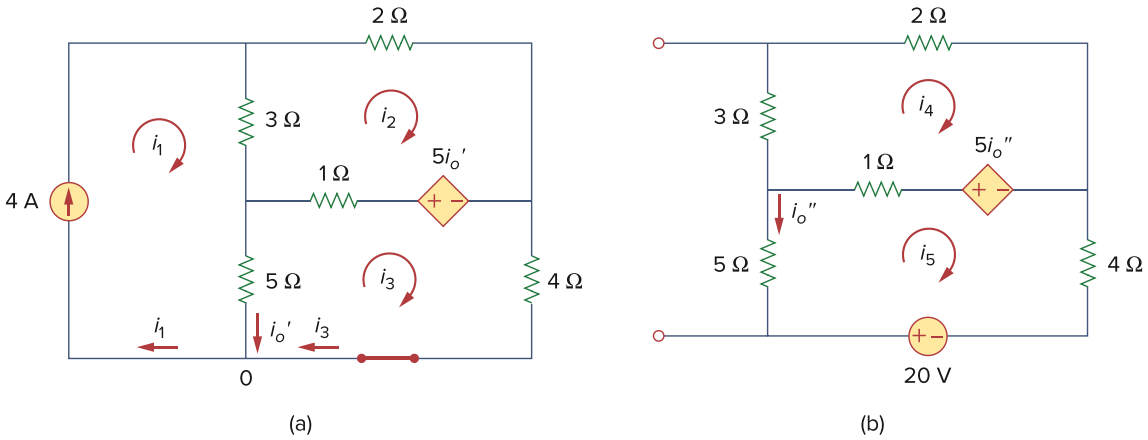


**Figure 4.10**
For Example 4.4: Applying superposition to (a) obtain $i_o'$, (b) obtain $i_o''$.

For loop 3,

$$-5i_1 - 1i_2 + 10i_3 + 5i_o' = 0 \qquad \textbf{(4.4.4)}$$

But at node 0,

$$i_3 = i_1 - i_o' = 4 - i_o' \qquad \textbf{(4.4.5)}$$

Substituting Eqs. (4.4.2) and (4.4.5) into Eqs. (4.4.3) and (4.4.4) gives two simultaneous equations

$$3i_2 - 2i_o' = 8 \qquad \textbf{(4.4.6)}$$

$$i_2 + 5i_o' = 20 \qquad \textbf{(4.4.7)}$$

which can be solved to get

$$i_o' = \frac{52}{17}\,\text{A} \qquad \textbf{(4.4.8)}$$

To obtain $i_o''$, we turn off the 4-A current source so that the circuit becomes that shown in Fig. 4.10(b). For loop 4, KVL gives

$$6i_4 - i_5 - 5i_o'' = 0 \qquad \textbf{(4.4.9)}$$

and for loop 5,

$$-i_4 + 10i_5 - 20 + 5i_o'' = 0 \qquad \textbf{(4.4.10)}$$

But $i_5 = -i_o''$. Substituting this in Eqs. (4.4.9) and (4.4.10) gives

$$6i_4 - 4i_o'' = 0 \qquad \textbf{(4.4.11)}$$

$$i_4 + 5i_o'' = -20 \qquad \textbf{(4.4.12)}$$

which we solve to get

$$i_o'' = -\frac{60}{17}\,\text{A} \qquad \textbf{(4.4.13)}$$

Now substituting Eqs. (4.4.8) and (4.4.13) into Eq. (4.4.1) gives

$$i_o = -\frac{8}{17} = -0.4706\ \text{A}$$

## Practice Problem 4.4

Use superposition to find $v_x$ in the circuit of Fig. 4.11.

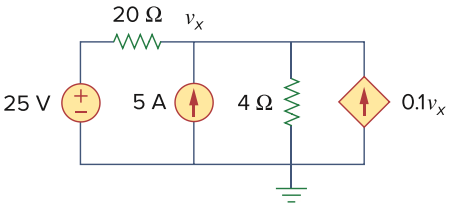


**Figure 4.11**
For Practice Prob. 4.4.

**Answer:** $v_x = 31.25$ V.

## Example 4.5

For the circuit in Fig. 4.12, use the superposition principle to find $i$.

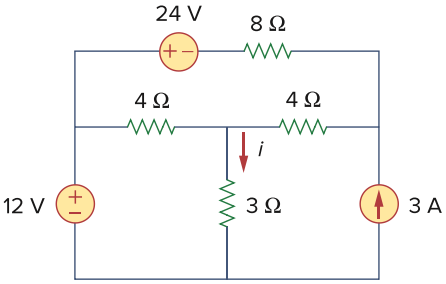


**Figure 4.12**
For Example 4.5.

**Solution:**
In this case, we have three sources. Let

$$i = i_1 + i_2 + i_3$$

where $i_1$, $i_2$, and $i_3$ are due to the 12-V , 24-V, and 3-A sources respec - tively. To get $i_1$, consider the circuit in Fig. 4.13(a). Combining 4 Ω (on the right-hand side) in series with 8 Ω gives 12 Ω. The 12 Ω in parallel with 4 Ω gives $12 \times 4/16 = 3$ Ω. Thus,

$$i_1 = \frac{12}{6} = 2 \text{ A}$$

To get $i_2$, consider the circuit in Fig. 4.13(b). Applying mesh analysis gives

$$16i_a - 4i_b + 24 = 0 \quad \Rightarrow \quad 4i_a - i_b = -6 \qquad \textbf{(4.5.1)}$$

$$7i_b - 4i_a = 0 \quad \Rightarrow \quad i_a = \frac{7}{4} i_b \qquad \textbf{(4.5.2)}$$

Substituting Eq. (4.5.2) into Eq. (4.5.1) gives

$$i_2 = i_b = -1$$

To get $i_3$, consider the circuit in Fig. 4.13(c). Using nodal analysis gives

$$3 = \frac{v_2}{8} + \frac{v_2 - v_1}{4} \quad \Rightarrow \quad 24 = 3v_2 - 2v_1 \qquad \textbf{(4.5.3)}$$

$$\frac{v_2 - v_1}{4} = \frac{v_1}{4} + \frac{v_1}{3} \quad \Rightarrow \quad v_2 = \frac{10}{3} v_1 \qquad \textbf{(4.5.4)}$$

Substituting Eq. (4.5.4) into Eq. (4.5.3) leads to $v_1 = 3$ and

$$i_3 = \frac{v_1}{3} = 1 \text{ A}$$

Thus,

$$i = i_1 + i_2 + i_3 = 2 - 1 + 1 = 2 \text{ A}$$

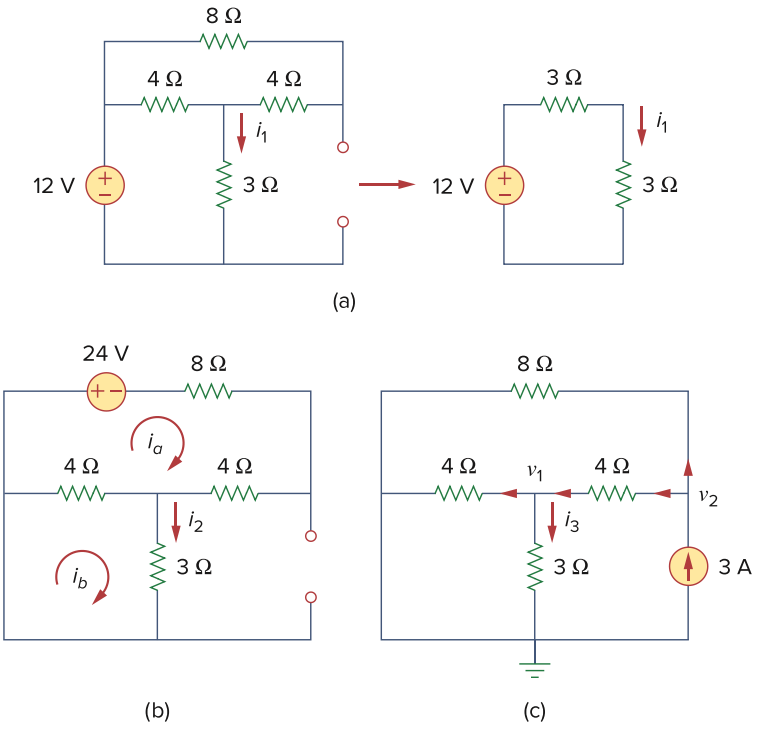


**Figure 4.13**
For Example 4.5.

## Practice Problem 4.5

Find $I$ in the circuit of Fig. 4.14 using the superposition principle.

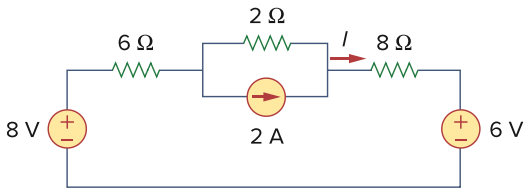


**Figure 4.14**
For Practice Prob. 4.5.

**Answer:** 375 mA.

## 4.4 Source Transformation

We have noticed that series-parallel combination and wye-delta transformation help simplify circuits. *Source transformation* is another tool for simplifying circuits. Basic to these tools is the concept of *equivalence*. We recall that an equivalent circuit is one whose $v$-$i$ characteristics are identical with the original circuit.

In Section 3.6, we saw that node-voltage (or mesh-current) equations can be obtained by mere inspection of a circuit when the sources are all independent current (or all independent voltage) sources. It is therefore expedient in circuit analysis to be able to substitute a voltage source in series with a resistor for a current source in parallel with a

resistor, or vice versa, as shown in Fig. 4.15. Either substitution is known as a *source transformation*.

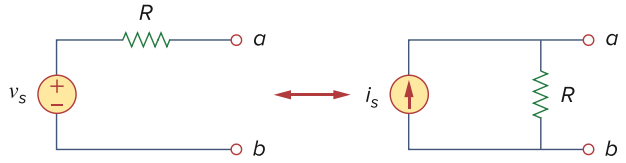


**Figure 4.15**
Transformation of independent sources.

A source transformation is the process of replacing a voltage source $v_s$ in series with a resistor $R$ by a current source $i_s$ in parallel with a resistor $R$, or vice versa.

The two circuits in Fig. 4.15 are equivalent—provided they have the same voltage-current relation at terminals $a$-$b$. It is easy to show that they are indeed equivalent. If the sources are turned off, the equivalent resistance at terminals $a$-$b$ in both circuits is $R$. Also, when terminals $a$-$b$ are short-circuited, the short-circuit current flowing from $a$ to $b$ is $i_{sc} = v_s/R$ in the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the right-hand side. Thus, $v_s/R = i_s$ in order for the two circuits to be equivalent. Hence, source transformation requires that

$$v_s = i_s R \qquad \text{or} \qquad i_s = \frac{v_s}{R} \tag{4.5}$$

Source transformation also applies to dependent sources, provided we carefully handle the dependent variable. As shown in Fig. 4.16, a dependent voltage source in series with a resistor can be transformed to a dependent current source in parallel with the resistor or vice versa where we make sure that Eq. (4.5) is satisfied.

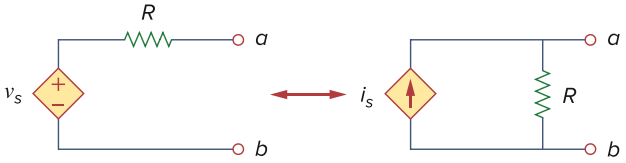


**Figure 4.16**
Transformation of dependent sources.

Like the wye-delta transformation we studied in Chapter 2, a source transformation does not affect the remaining part of the circuit. When applicable, source transformation is a powerful tool that allows circuit manipulations to ease circuit analysis. However, we should keep the following points in mind when dealing with source transformation.

1. Note from Fig. 4.15 (or Fig. 4.16) that the arrow of the current source is directed toward the positive terminal of the voltage source.
2. Note from Eq. (4.5) that source transformation is not possible when $R = 0$, which is the case with an ideal voltage source. However, for a practical, nonideal voltage source, $R \neq 0$. Similarly, an ideal current source with $R = \infty$ cannot be replaced by a finite voltage source. More will be said on ideal and nonideal sources in Section 4.10.1.

## Example 4.6

Use source transformation to find $v_o$ in the circuit of Fig. 4.17.

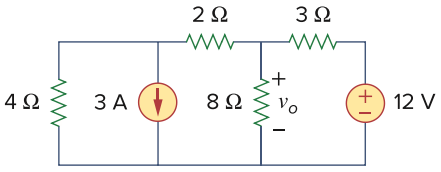


**Figure 4.17**
For Example 4.6.

**Solution:**
We first transform the current and voltage sources to obtain the cir - cuit in Fig. 4.18(a). Combining the 4-Ω and 2-Ω resistors in series and transforming the 12-V v oltage source gi ves us Fig. 4.18(b). We now combine the 3-Ω and 6-Ω resistors in parallel to get 2-Ω. We also combine the 2-A and 4-A current sources to get a 2-A source. Thus, by repeatedly applying source transformations, we obtain the circuit in Fig. 4.18(c).

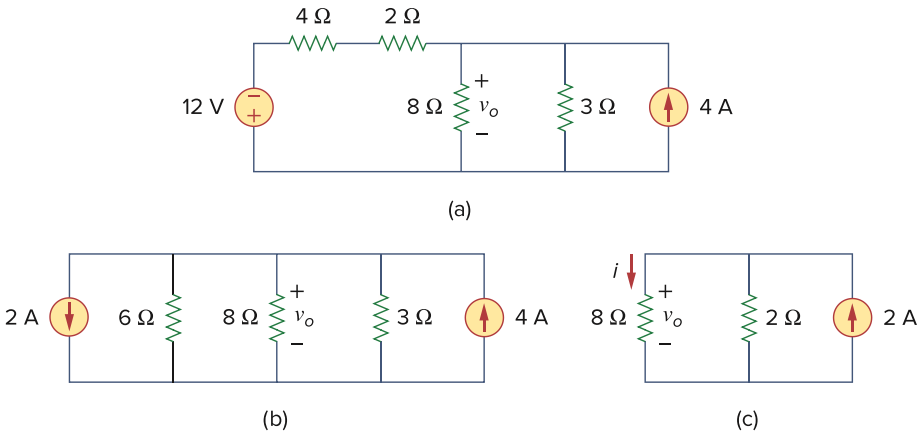


**Figure 4.18**
For Example 4.6.

We use current division in Fig. 4.18(c) to get

$$i = \frac{2}{2+8}(2) = 0.4 \text{ A}$$

and

$$v_o = 8i = 8(0.4) = 3.2 \text{ V}$$

Alternatively, since the 8-Ω and 2-Ω resistors in Fig. 4.18(c) are in parallel, they have the same voltage $v_o$ across them. Hence,

$$v_o = (8 \parallel 2)(2 \text{ A}) = \frac{8 \times 2}{10}(2) = 3.2 \text{ V}$$

## Practice Problem 4.6

Find $i_o$ in the circuit of Fig. 4.19 using source transformation.

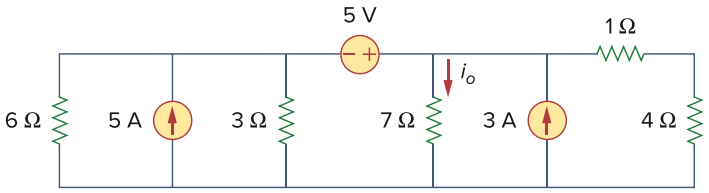


**Figure 4.19**
For Practice Prob. 4.6.

**Answer:** 1.78 A.

## Example 4.7

Find $v_x$ in Fig. 4.20 using source transformation.

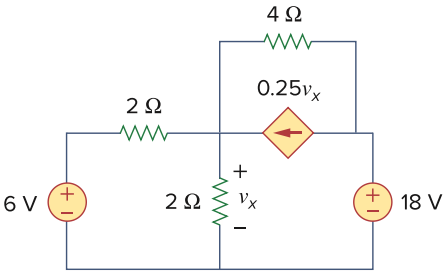


**Figure 4.20**
For Example 4.7.

**Solution:**

The circuit in Fig. 4.20 involves a voltage-controlled dependent current source. We transform this dependent current source as well as the 6-V independent voltage source as shown in Fig. 4.21(a). The 18-V voltage source is not transformed because it is not connected in series with any resistor. The two 2-Ω resistors in parallel combine to give a 1-Ω resistor, which is in parallel with the 3-A current source. The current source is transformed to a voltage source as shown in Fig. 4.21(b). Notice that the terminals for $v_x$ are intact. Applying KVL around the loop in Fig. 4.21(b) gives

$$-3 + 5i + v_x + 18 = 0 \qquad \textbf{(4.7.1)}$$

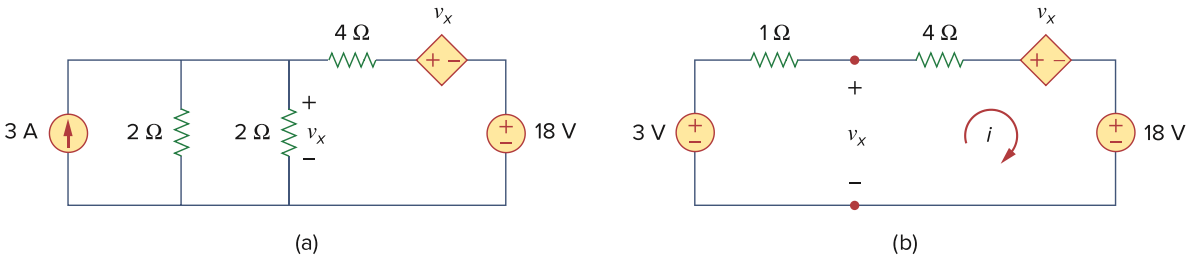


**Figure 4.21**
For Example 4.7: Applying source transformation to the circuit in Fig. 4.20.

Applying KVL to the loop containing only the 3-V voltage source, the 1-Ω resistor, and $v_x$ yields

$$-3 + 1i + v_x = 0 \quad \Rightarrow \quad v_x = 3 - i \qquad \textbf{(4.7.2)}$$

Substituting this into Eq. (4.7.1), we obtain

$$15 + 5i + 3 - i = 0 \quad \Rightarrow \quad i = -4.5 \text{ A}$$

Alternatively, we may apply KVL to the loop containing $v_x$, the 4-Ω resistor, the voltage-controlled dependent voltage source, and the 18-V voltage source in Fig. 4.21(b). We obtain

$$-v_x + 4i + v_x + 18 = 0 \quad \Rightarrow \quad i = -4.5 \text{ A}$$

Thus, $v_x = 3 - i = 7.5$ V.

## Practice Problem 4.7

Use source transformation to find $i_x$ in the circuit shown in Fig. 4.22.

**Answer:** 7.059 mA.

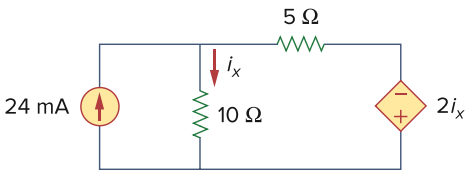


**Figure 4.22**
For Practice Prob. 4.7.

## 4.5 Thevenin's Theorem

It often occurs in practice that a particular element in a circuit is variable (usually called the *load*) while other elements are fixed. As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load. Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenin's theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

According to Thevenin's theorem, the linear circuit in Fig. 4.23(a) can be replaced by that in Fig. 4.23(b). (The load in Fig. 4.23 may be a single resistor or another circuit.) The circuit to the left of the terminals *a-b* in Fig. 4.23(b) is known as the *Thevenin equivalent circuit*; it was developed in 1883 by M. Leon Thevenin (1857–1926), a French telegraph engineer.

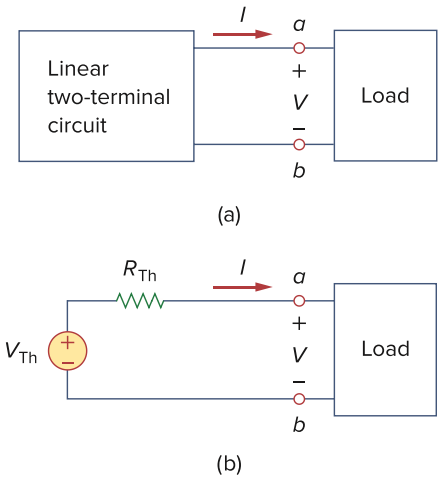


**Figure 4.23**
Replacing a linear two-terminal circuit by its Thevenin equivalent: (a) original circuit, (b) the Thevenin equivalent circuit.

**Thevenin's theorem** states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source $V_{Th}$ in series with a resistor $R_{Th}$, where $V_{Th}$ is the open-circuit voltage at the terminals and $R_{Th}$ is the input or equivalent resistance at the terminals when the independent sources are turned off.

The proof of the theorem will be given later, in Section 4.7. Our major concern right now is how to find the Thevenin equivalent voltage $V_{Th}$ and resistance $R_{Th}$. To do so, suppose the two circuits in Fig. 4.23 are equivalent. Two circuits are said to be *equivalent* if they have the same voltage-current relation at their terminals. Let us find out what will make the two circuits in Fig. 4.23 equivalent. If the terminals *a-b* are made open-circuited (by removing the load), no current flows, so that the open-circuit voltage across the terminals *a-b* in Fig. 4.23(a) must be equal to the voltage source $V_{Th}$ in Fig. 4.23(b), since the two circuits are equivalent. Thus, $V_{Th}$ is the open-circuit voltage across the terminals as shown in Fig. 4.24(a); that is,

$$V_{Th} = v_{oc} \tag{4.6}$$

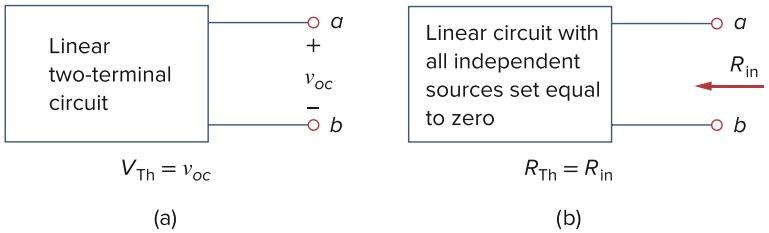


**Figure 4.24**
Finding $V_{Th}$ and $R_{Th}$.

Again, with the load disconnected and terminals *a-b* open-circuited, we turn off all independent sources. The input resistance (or equivalent resistance) of the dead circuit at the terminals *a-b* in Fig. 4.23(a) must be equal to $R_{Th}$ in Fig. 4.23(b) because the two circuits are equivalent. Thus, $R_{Th}$ is the input resistance at the terminals when the independent sources are turned off, as shown in Fig. 4.24(b); that is,

$$R_{Th} = R_{in} \tag{4.7}$$

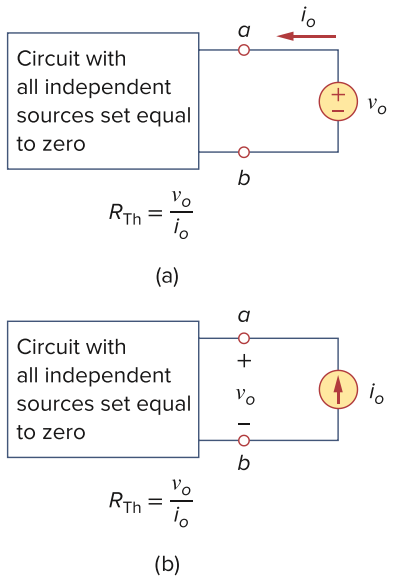


**Figure 4.25**
Finding $R_{Th}$ when circuit has dependent sources.

Later we will see that an alternative way of finding $R_{Th}$ is $R_{Th} = v_{oc}/i_{sc}$.

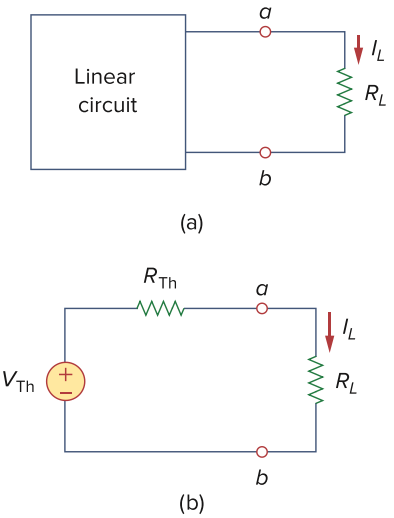


**Figure 4.26**
A circuit with a load: (a) original circuit, (b) Thevenin equivalent.

To apply this idea in finding the Thevenin resistance $R_{Th}$, we need to consider two cases.

■ **CASE 1** If the network has no dependent sources, we turn off all independent sources. $R_{Th}$ is the input resistance of the network looking between terminals $a$ and $b$, as shown in Fig. 4.24(b).

■ **CASE 2** If the network has dependent sources, we turn off all independent sources. As with superposition, dependent sources are not to be turned off because they are controlled by circuit variables. We apply a voltage source $v_o$ at terminals $a$ and $b$ and determine the resulting current $i_o$. Then $R_{Th} = v_o/i_o$, as shown in Fig. 4.25(a). Alternatively, we may insert a current source $i_o$ at terminals $a$-$b$ as shown in Fig. 4.25(b) and find the terminal voltage $v_o$. Again $R_{Th} = v_o/i_o$. Either of the two approaches will give the same result. In either approach we may assume any value of $v_o$ and $i_o$. For example, we may use $v_o = 1$ V or $i_o = 1$ A, or even use unspecified values of $v_o$ or $i_o$.

It often occurs that $R_{Th}$ takes a negative value. In this case, the negative resistance ($v = -iR$) implies that the circuit is supplying power. This is possible in a circuit with dependent sources; Example 4.10 will illustrate this.

Thevenin's theorem is very important in circuit analysis. It helps simplify a circuit. A large circuit may be replaced by a single independent voltage source and a single resistor. This replacement technique is a powerful tool in circuit design.

As mentioned earlier, a linear circuit with a variable load can be replaced by the Thevenin equivalent, exclusive of the load. The equivalent network behaves the same way externally as the original circuit. Consider a linear circuit terminated by a load $R_L$, as shown in Fig. 4.26(a). The current $I_L$ through the load and the voltage $V_L$ across the load are easily determined once the Thevenin equivalent of the circuit at the load's terminals is obtained, as shown in Fig. 4.26(b). From Fig. 4.26(b), we obtain

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} \tag{4.8a}$$

$$V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th} \tag{4.8b}$$

Note from Fig. 4.26(b) that the Thevenin equivalent is a simple voltage divider, yielding $V_L$ by mere inspection.

## Example 4.8

Find the Thevenin equivalent circuit of the circuit shown in Fig. 4.27, to the left of the terminals $a$-$b$. Then find the current through $R_L = 6$, 16, and 36 Ω.

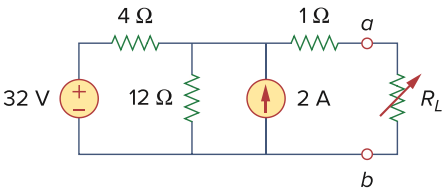


**Figure 4.27**
For Example 4.8.

**Solution:**
We find $R_{Th}$ by turning off the 32-V voltage source (replacing it with a short circuit) and the 2-A current source (replacing it with an

open circuit). The circuit becomes what is shown in Fig. 4.28(a). Thus,

$$R_{\text{Th}} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

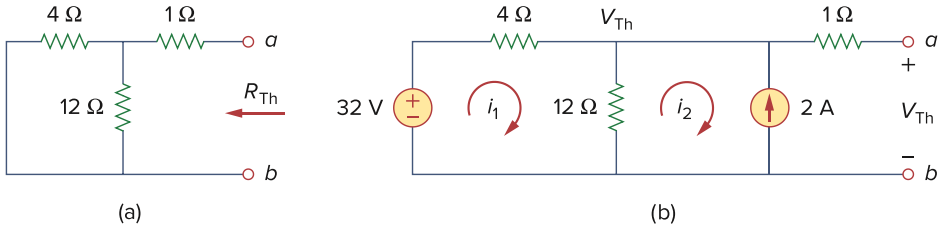


**Figure 4.28**
For Example 4.8: (a) finding $R_{\text{Th}}$, (b) finding $V_{\text{Th}}$.

To find $V_{\text{Th}}$, consider the circuit in Fig. 4.28(b).Applying mesh analysis to the two loops, we obtain

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \qquad i_2 = -2\ \text{A}$$

Solving for $i_1$, we get $i_1 = 0.5$ A. Thus,

$$V_{\text{Th}} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\ \text{V}$$

Alternatively, it is even easier to use nodal analysis. We ignore the 1-Ω resistor since no current flows through it. At the top node, KCL gives

$$\frac{32 - V_{\text{Th}}}{4} + 2 = \frac{V_{\text{Th}}}{12}$$

or

$$96 - 3V_{\text{Th}} + 24 = V_{\text{Th}} \qquad \Rightarrow \qquad V_{\text{Th}} = 30\ \text{V}$$

as obtained before. We could also use source transformation to find $V_{\text{Th}}$.

The Thevenin equivalent circuit is shown in Fig. 4.29. The current through $R_L$ is

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L} = \frac{30}{4 + R_L}$$

When $R_L = 6$,

$$I_L = \frac{30}{10} = 3\ \text{A}$$

When $R_L = 16$,

$$I_L = \frac{30}{20} = 1.5\ \text{A}$$

When $R_L = 36$,

$$I_L = \frac{30}{40} = 0.75\ \text{A}$$

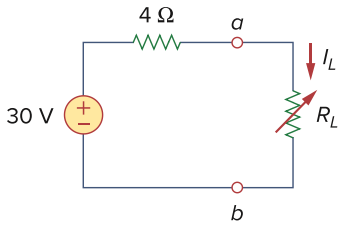


**Figure 4.29**
The Thevenin equivalent circuit for Example 4.8.

## Practice Problem 4.8

Using Thevenin's theorem, find the equivalent circuit to the left of the terminals in the circuit of Fig. 4.30. Then find $I$.

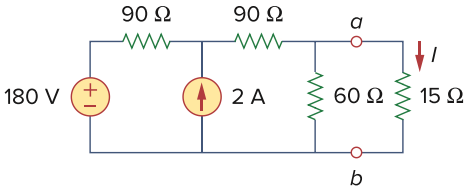


**Figure 4.30**
For Practice Prob. 4.8.

**Answer:** $V_{Th} = 90$ V, $R_{Th} = 45\ \Omega$, $I = 1.5$ A.

## Example 4.9

Find the Thevenin equivalent of the circuit in Fig. 4.31 at terminals $a$-$b$.

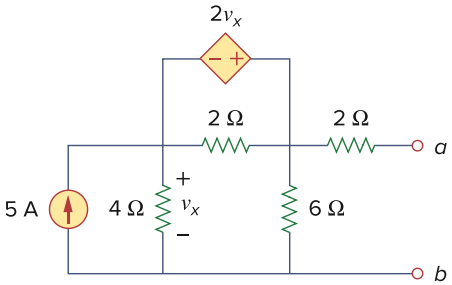


**Figure 4.31**
For Example 4.9.

**Solution:**
This circuit contains a dependent source, unlik e the circuit in the pre - vious e xample. To find $R_{Th}$, we set the independent source equal to zero b ut lea ve the dependent source alone. Because of the presence of the dependent source, ho wever, we e xcite the netw ork with a v olt- age source $v_o$ connected to the terminals as indicated in Fig. 4.32(a). We may set $v_o = 1$ V to ease calculation, since the circuit is linear . Our goal is to find the current $i_o$ through the terminals, and then obtain $R_{Th} = 1/i_o$. (Alternatively, we may insert a 1-A current source, find the corresponding voltage $v_o$, and obtain $R_{Th} = v_o/1$.)

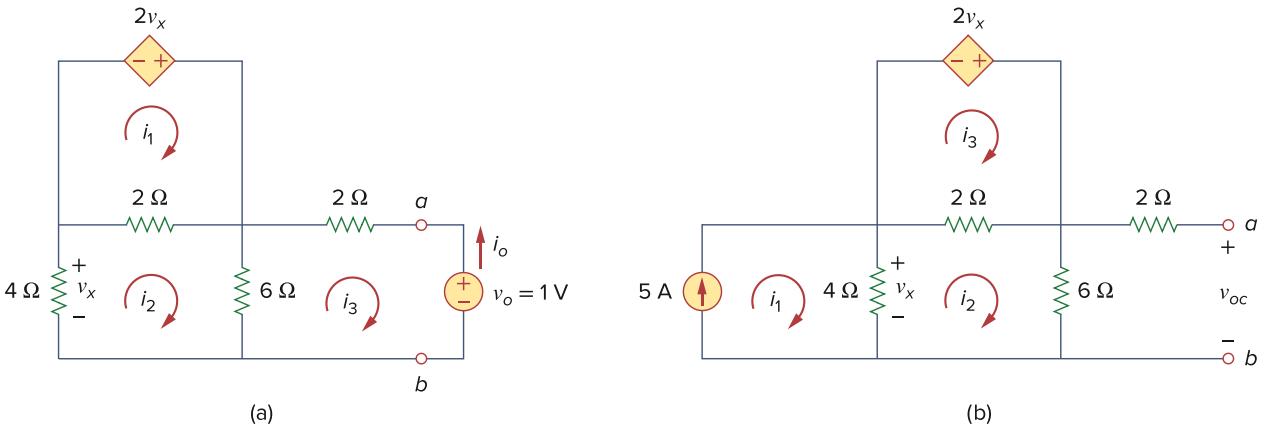


**Figure 4.32**
Finding $R_{Th}$ and $V_{Th}$ for Example 4.9.

Applying mesh analysis to loop 1 in the circuit of Fig. 4.32(a) results in

$$-2v_x + 2(i_1 - i_2) = 0 \qquad \text{or} \qquad v_x = i_1 - i_2$$

But $-4i_2 = v_x = i_1 - i_2$; hence,

$$i_1 = -3i_2 \tag{4.9.1}$$

For loops 2 and 3, applying KVL produces

$$4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0 \tag{4.9.2}$$

$$6(i_3 - i_2) + 2i_3 + 1 = 0 \tag{4.9.3}$$

Solving these equations gives

$$i_3 = -\frac{1}{6}\,\text{A}$$

But $i_o = -i_3 = 1/6$ A. Hence,

$$R_{\text{Th}} = \frac{1\ \text{V}}{i_o} = 6\ \Omega$$

To get $V_{\text{Th}}$, we find $v_{oc}$ in the circuit of Fig. 4.32(b). Applying mesh analysis, we get

$$i_1 = 5 \tag{4.9.4}$$

$$-2v_x + 2(i_3 - i_2) = 0 \quad \Rightarrow \quad v_x = i_3 - i_2 \tag{4.9.5}$$

$$4(i_2 - i_1) + 2(i_2 - i_3) + 6i_2 = 0$$

or

$$12i_2 - 4i_1 - 2i_3 = 0 \tag{4.9.6}$$

But $4(i_1 - i_2) = v_x$. Solving these equations leads to $i_2 = 10/3$. Hence,

$$V_{\text{Th}} = v_{oc} = 6i_2 = 20\ \text{V}$$

The Thevenin equivalent is as shown in Fig. 4.33.

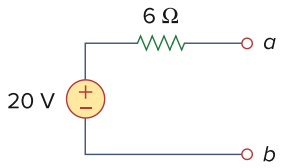


**Figure 4.33**
The Thevenin equivalent of the circuit in Fig. 4.31.

## Practice Problem 4.9

Find the Thevenin equivalent circuit of the circuit in Fig. 4.34 to the left of the terminals.

**Answer:** $V_{\text{Th}} = 5.333$ V, $R_{\text{Th}} = 444.4$ mΩ.

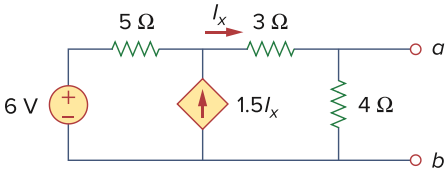


**Figure 4.34**
For Practice Prob. 4.9.

## Example 4.10

Determine the Thevenin equi valent of the circuit in Fig. 4.35(a) at terminals *a*-*b*.

**Solution:**

1. **Define.** The problem is clearly defined; we are to determine the Thevenin equivalent of the circuit shown in Fig. 4.35(a).
2. **Present.** The circuit contains a 2-Ω resistor in parallel with a 4-Ω resistor. These are, in turn, in parallel with a dependent current source. It is important to note that there are no independent sources.
3. **Alternative.** The first thing to consider is that, since we have no independent sources in this circuit, we must excite the circuit externally. In addition, when you have no independent sources the value for $V_{\text{Th}}$ will be equal to zero so you will only have to find $R_{\text{Th}}$.

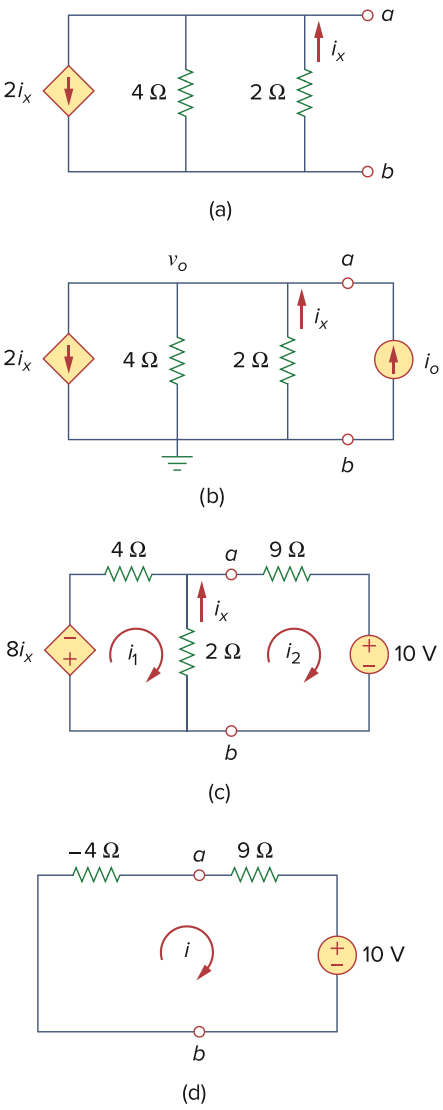


**Figure 4.35**
For Example 4.10.

The simplest approach is to excite the circuit with either a 1-V voltage source or a 1-A current source. Because we will end up with an equivalent resistance (either positive or negative), I prefer to use the current source and nodal analysis which will yield a voltage at the output terminals equal to the resistance (with 1 A flowing in, $v_o$ is equal to 1 times the equivalent resistance).

As an alternative, the circuit could also be excited by a 1-V voltage source and mesh analysis could be used to find the equivalent resistance.

4. **Attempt.** We start by writing the nodal equation at $a$ in Fig. 4.35(b) assuming $i_o = 1$ A.

$$2i_x + (v_o - 0)/4 + (v_o - 0)/2 + (-1) = 0 \qquad \textbf{(4.10.1)}$$

Given that we have two unknowns and only one equation, we will need a constraint equation.

$$i_x = (0 - v_o)/2 = -v_o/2 \qquad \textbf{(4.10.2)}$$

Substituting Eq. (4.10.2) into Eq. (4.10.1) yields

$$2(-v_o/2) + (v_o - 0)/4 + (v_o - 0)/2 + (-1) = 0$$

$$= (-1 + \tfrac{1}{4} + \tfrac{1}{2})v_o - 1 \qquad \text{or} \qquad v_o = -4 \text{ V}$$

Since $v_o = 1 \times R_{\text{Th}}$, then $R_{\text{Th}} = v_o/1 = -4\,\Omega$.

The negative value of the resistance tells us that, according to the passive sign convention, the circuit in Fig. 4.35(a) is supplying power. Of course, the resistors in Fig. 4.35(a) cannot supply power (they absorb power); it is the dependent source that supplies the power. This is an example of how a dependent source and resistors could be used to simulate negative resistance.

5. **Evaluate.** First of all, we note that the answer has a negative value. We know this is not possible in a passive circuit, but in this circuit we do have an active device (the dependent current source). Thus, the equivalent circuit is essentially an active circuit that can supply power.

Now we must evaluate the solution. The best way to do this is to perform a check, using a different approach, and see if we obtain the same solution. Let us try connecting a 9-Ω resistor in series with a 10-V voltage source across the output terminals of the original circuit and then the Thevenin equivalent. To make the circuit easier to solve, we can take and change the parallel current source and 4-Ω resistor to a series voltage source and 4-Ω resistor by using source transformation. This, with the new load, gives us the circuit shown in Fig. 4.35(c).

We can now write two mesh equations.

$$8i_x + 4i_1 + 2(i_1 - i_2) = 0$$

$$2(i_2 - i_1) + 9i_2 + 10 = 0$$

Note, we only have two equations but have three unknowns, so we need a constraint equation. We can use

$$i_x = i_2 - i_1$$

This leads to a new equation for loop 1. Simplifying leads to

$$(4 + 2 - 8)i_1 + (-2 + 8)i_2 = 0$$

or

$$-2i_1 + 6i_2 = 0 \quad \text{or} \quad i_1 = 3i_2$$

$$-2i_1 + 11i_2 = -10$$

Substituting the first equation into the second gives

$$-6i_2 + 11i_2 = -10 \quad \text{or} \quad i_2 = -10/5 = \mathbf{-2\ A}$$

Using the Thevenin equivalent is quite easy since we have only one loop, as shown in Fig. 4.35(d).

$$-4i + 9i + 10 = 0 \quad \text{or} \quad i = -10/5 = \mathbf{-2\ A}$$

5. **Satisfactory?** Clearly we have found the value of the equivalent circuit as required by the problem statement. Checking does validate that solution (we compared the answer we obtained by using the equivalent circuit with one obtained by using the load with the original circuit). We can present all this as a solution to the problem.

## Practice Problem 4.10

Obtain the Thevenin equivalent of the circuit in Fig. 4.36.

**Answer:** $V_{Th} = 0$ V, $R_{Th} = -7.5\ \Omega$.

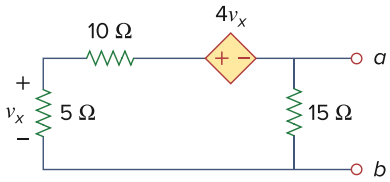


**Figure 4.36**
For Practice Prob. 4.10.

## 4.6 Norton's Theorem

In 1926, about 43 years afterThevenin published his theorem, E. L. Norton, an American engineer at Bell Telephone Laboratories, proposed a similar theorem.

Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source $I_N$ in parallel with a resistor $R_N$, where $I_N$ is the short-circuit current through the terminals and $R_N$ is the input or equivalent resistance at the terminals when the independent sources are turned off.

Thus, the circuit in Fig. 4.37(a) can be replaced by the one in Fig. 4.37(b).

The proof of Norton' s theorem will be gi ven in the ne xt section. For now, we are mainly concerned with ho w to get $R_N$ and $I_N$. We find $R_N$ in the same w ay we find $R_{Th}$. In f act, from what we kno w about source transformation, the Thevenin and Norton resistances are equal; that is,

$$R_N = R_{Th} \tag{4.9}$$

To find the Norton current $I_N$, we determine the short-circuit current flowing from terminal $a$ to $b$ in both circuits in Fig. 4.37. It is e vident

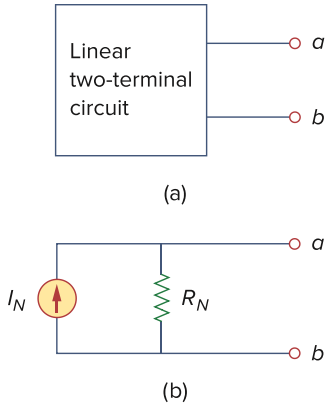


**Figure 4.37**
(a) Original circuit, (b) Norton equivalent circuit.

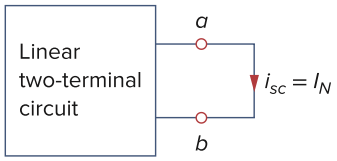


**Figure 4.38**
Finding Norton current $I_N$.

that the short-circuit current in Fig. 4.37(b) is $I_N$. This must be the same short-circuit current from terminal $a$ to $b$ in Fig. 4.37(a), since the two circuits are equivalent. Thus,

$$I_N = i_{sc} \tag{4.10}$$

shown in Fig. 4.38. Dependent and independent sources are treated the same way as in Thevenin's theorem.

Observe the close relationship between Norton's and Thevenin's theorems: $R_N = R_{\text{Th}}$ as in Eq. (4.9), and

$$\boxed{I_N = \frac{V_{\text{Th}}}{R_{\text{Th}}}} \tag{4.11}$$

The Thevenin and Norton equivalent circuits are related by a source transformation.

This is essentially source transformation. For this reason, source transformation is often called Thevenin-Norton transformation.

Since $V_{\text{Th}}$, $I_N$, and $R_{\text{Th}}$ are related according to Eq. (4.11), to determine the Thevenin or Norton equivalent circuit requires that we find:

- The open-circuit voltage $v_{oc}$ across terminals $a$ and $b$.
- The short-circuit current $i_{sc}$ at terminals $a$ and $b$.
- The equivalent or input resistance $R_{\text{in}}$ at terminals $a$ and $b$ when all independent sources are turned off.

We can calculate any two of the three using the method that takes the least effort and use them to get the third using Ohm's law. Example 4.11 will illustrate this. Also, since

$$V_{\text{Th}} = v_{oc} \tag{4.12a}$$

$$I_N = i_{sc} \tag{4.12b}$$

$$R_{\text{Th}} = \frac{v_{oc}}{i_{sc}} = R_N \tag{4.12c}$$

the open-circuit and short-circuit tests are sufficient to find any Thevenin or Norton equivalent, of a circuit which contains at least one independent source.

## Example 4.11

Find the Norton equivalent circuit of the circuit in Fig. 4.39 at terminals $a$-$b$.

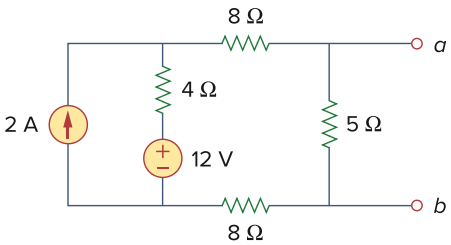


**Figure 4.39**
For Example 4.11.

**Solution:**

We find $R_N$ in the same way we find $R_{\text{Th}}$ in the Thevenin equivalent circuit. Set the independent sources equal to zero. This leads to the circuit in Fig. 4.40(a), from which we find $R_N$. Thus,

$$R_N = 5 \parallel (8 + 4 + 8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

To find $I_N$, we short-circuit terminals $a$ and $b$, as shown in Fig. 4.40(b). We ignore the 5-Ω resistor because it has been short-circuited. Applying mesh analysis, we obtain

$$i_1 = 2\ \text{A}, \qquad 20i_2 - 4i_1 - 12 = 0$$

From these equations, we obtain

$$i_2 = 1\ \text{A} = i_{sc} = I_N$$

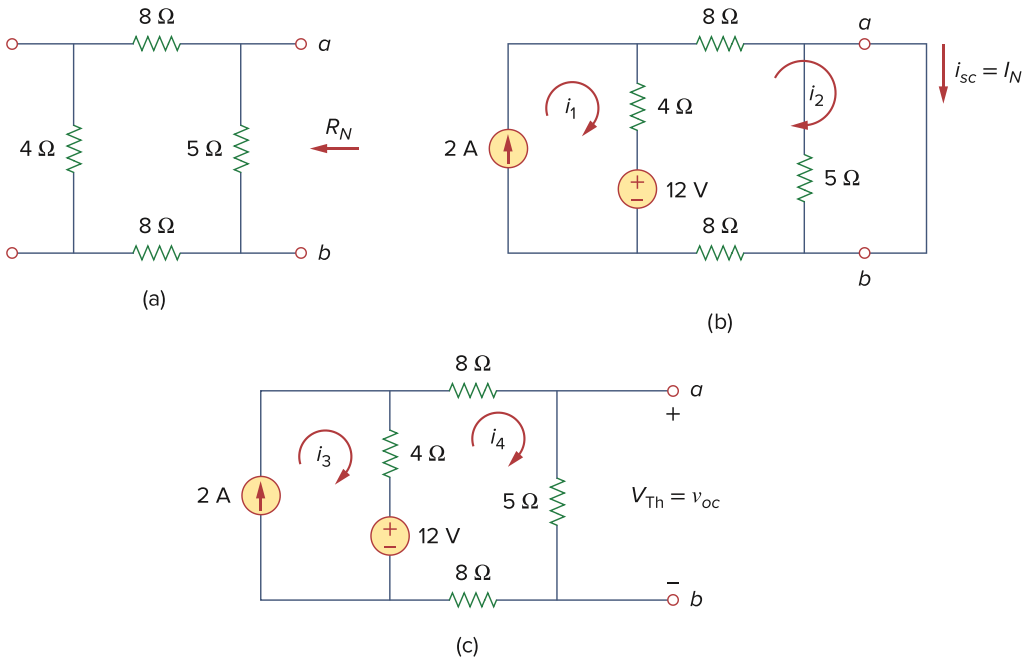


**Figure 4.40**
For Example 4.11; finding: (a) $R_N$, (b) $I_N = i_{sc}$, (c) $V_{\text{Th}} = v_{oc}$.

Alternatively, we may determine $I_N$ from $V_{\text{Th}}/R_{\text{Th}}$. We obtain $V_{\text{Th}}$ as the open-circuit voltage across terminals $a$ and $b$ in Fig. 4.40(c). Using mesh analysis, we obtain

$$i_3 = 2 \text{ A}$$

$$25i_4 - 4i_3 - 12 = 0 \quad \Rightarrow \quad i_4 = 0.8 \text{ A}$$

and

$$v_{oc} = V_{\text{Th}} = 5i_4 = 4 \text{ V}$$

Hence,

$$I_N = \frac{V_{\text{Th}}}{R_{\text{Th}}} = \frac{4}{4} = 1 \text{ A}$$

as obtained previously. This also serves to confirm Eq. (4.12c) that $R_{\text{Th}} = v_{oc}/i_{sc} = 4/1 = 4\ \Omega$. Thus, the Norton equivalent circuit is as shown in Fig. 4.41.

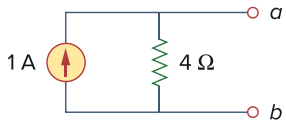


**Figure 4.41**
Norton equivalent of the circuit in Fig. 4.39.

## Practice Problem 4.11

Find the Norton equivalent circuit for the circuit in Fig. 4.42, at terminals $a$-$b$.

**Answer:** $R_N = 90\ \Omega$, $I_N = 4.5$ A.

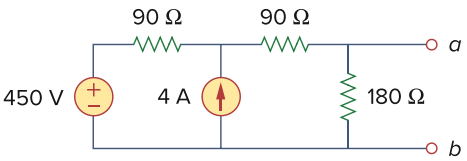


**Figure 4.42**
For Practice Prob. 4.11.

## Example 4.12

Using Norton's theorem, find $R_N$ and $I_N$ of the circuit in Fig. 4.43 at terminals $a$-$b$.

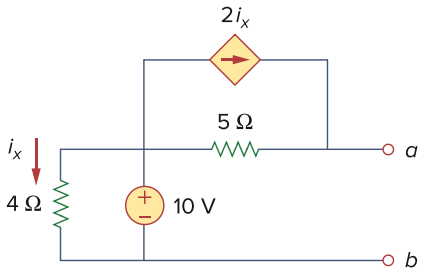


**Figure 4.43**
For Example 4.12.

**Solution:**

To find $R_N$, we set the independent voltage source equal to zero and connect a voltage source of $v_o = 1$ V (or any unspecified voltage $v_o$) to the terminals. We obtain the circuit in Fig. 4.44(a).We ignore the 4-Ω resistor because it is short-circuited. Also due to the short circuit, the 5- Ω resistor, the voltage source, and the dependent current source are all in parallel. Hence, $i_x = 0$. At node $a$, $i_o = \frac{1\text{ V}}{5\ \Omega} = 0.2$ A, and

$$R_N = \frac{v_o}{i_o} = \frac{1}{0.2} = 5\ \Omega$$

To find $I_N$, we short-circuit terminals $a$ and $b$ and find the current $i_{sc}$, as indicated in Fig. 4.44(b). Note from this figure that the 4 Ω resistor, the 10-V voltage source, the 5-Ω resistor, and the dependent current source are all in parallel. Hence,

$$i_x = \frac{10}{4} = 2.5\text{ A}$$

At node $a$, KCL gives

$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\text{ A}$$

Thus,

$$I_N = 7\text{ A}$$

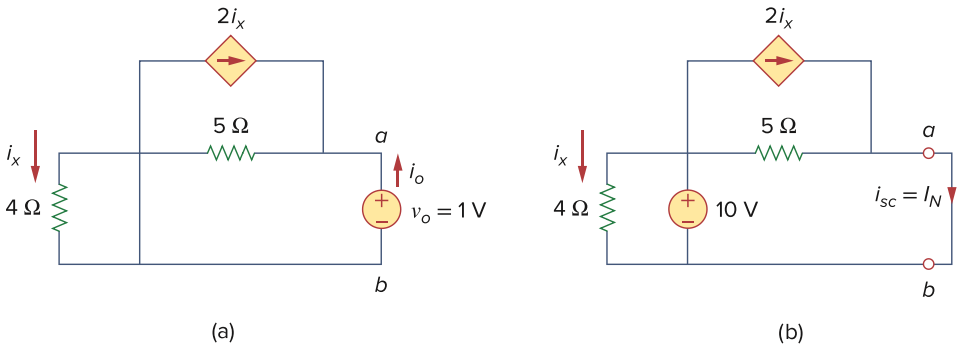


**Figure 4.44**
For Example 4.12: (a) finding $R_N$, (b) finding $I_N$.

## Practice Problem 4.12

Find the Norton equivalent circuit of the circuit in Fig. 4.45 at terminals $a$-$b$.

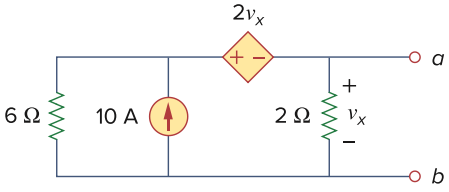


**Figure 4.45**
For Practice Prob. 4.12.

**Answer:** $R_N = 1\ \Omega$, $I_N = 10$ A.

## 4.7 †Derivations of Thevenin's and Norton's Theorems

In this section, we will prove Thevenin's and Norton's theorems using the superposition principle.

Consider the linear circuit in Fig. 4.46(a). It is assumed that the circuit contains resistors and dependent and independent sources. We have access to the circuit via terminals $a$ and $b$, through which current from an external source is applied. Our objective is to ensure that the voltage-current relation at terminals $a$ and $b$ is identical to that of the Thevenin equivalent in Fig. 4.46(b). For the sake of simplicity, suppose the linear circuit in Fig. 4.46(a) contains two independent voltage sources $v_{s1}$ and $v_{s2}$ and two independent current sources $i_{s1}$ and $i_{s2}$. We may obtain any circuit variable, such as the terminal voltage $v$, by applying superposition. That is, we consider the contribution due to each independent source including the external source $i$. By superposition, the terminal voltage $v$ is

$$v = A_0 i + A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2} \tag{4.13}$$

where $A_0$, $A_1$, $A_2$, $A_3$, and $A_4$ are constants. Each term on the right-hand side of Eq. (4.13) is the contribution of the related independent source; that is, $A_0 i$ is the contribution to $v$ due to the external current source $i$, $A_1 v_{s1}$ is the contribution due to the voltage source $v_{s1}$, and so on. We may collect terms for the internal independent sources together as $B_0$, so that Eq. (4.13) becomes

$$v = A_0 i + B_0 \tag{4.14}$$

where $B_0 = A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}$. We now want to evaluate the values of constants $A_0$ and $B_0$. When the terminals $a$ and $b$ are open-circuited, $i = 0$ and $v = B_0$. Thus, $B_0$ is the open-circuit voltage $v_{oc}$, which is the same as $V_{\text{Th}}$, so

$$B_0 = V_{\text{Th}} \tag{4.15}$$

When all the internal sources are turned off, $B_0 = 0$. The circuit can then be replaced by an equivalent resistance $R_{\text{eq}}$, which is the same as $R_{\text{Th}}$, and Eq. (4.14) becomes

$$v = A_0 i = R_{\text{Th}} i \quad \Rightarrow \quad A_0 = R_{\text{Th}} \tag{4.16}$$

Substituting the values of $A_0$ and $B_0$ in Eq. (4.14) gives

$$v = R_{\text{Th}} i + V_{\text{Th}} \tag{4.17}$$

which expresses the voltage-current relation at terminals $a$ and $b$ of the circuit in Fig. 4.46(b). Thus, the two circuits in Fig. 4.46(a) and 4.46(b) are equivalent.

When the same linear circuit is driven by a voltage source $v$ as shown in Fig. 4.47(a), the current flowing into the circuit can be obtained by superposition as

$$i = C_0 v + D_0 \tag{4.18}$$

where $C_0 v$ is the contribution to $i$ due to the external voltage source $v$ and $D_0$ contains the contributions to $i$ due to all internal independent sources. When the terminals $a$-$b$ are short-circuited, $v = 0$ so that

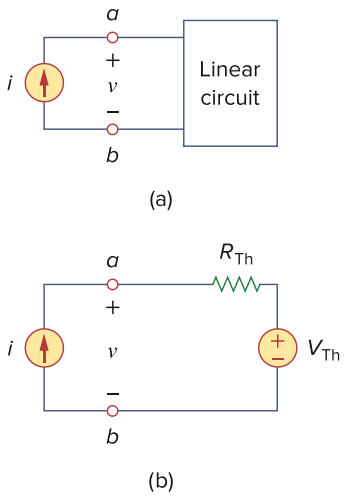


**Figure 4.46**
Derivation of Thevenin equivalent: (a) a current-driven circuit, (b) its Thevenin equivalent.

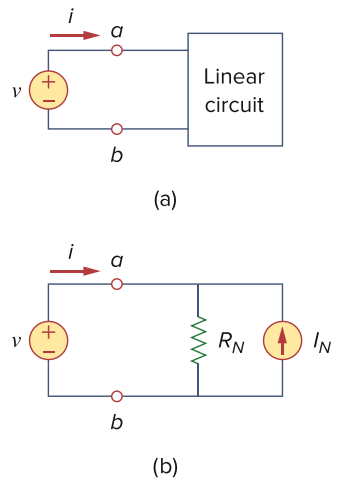


**Figure 4.47**
Derivation of Norton equivalent: (a) a voltage-driven circuit, (b) its Norton equivalent.

$i = D_0 = -i_{sc}$, where $i_{sc}$ is the short-circuit current flowing out of terminal $a$, which is the same as the Norton current $I_N$, i.e.,

$$D_0 = -I_N \tag{4.19}$$

When all the internal independent sources are turned off, $D_0 = 0$ and the circuit can be replaced by an equivalent resistance $R_{eq}$ (or an equivalent conductance $G_{eq} = 1/R_{eq}$), which is the same as $R_{Th}$ or $R_N$. Thus, Eq. (4.19) becomes

$$i = \frac{v}{R_{Th}} - I_N \tag{4.20}$$

This expresses the voltage-current relation at terminals $a$-$b$ of the circuit in Fig. 4.47(b), confirming that the two circuits in Fig. 4.47(a) and 4.47(b) are equivalent.

## 4.8 Maximum Power Transfer

In many practical situations, a circuit is designed to provide power to a load. There are applications in areas such as communications where it is desirable to maximize the power delivered to a load. We now address the problem of delivering the maximum power to a load when given a system with known internal losses. It should be noted that this will result in significant internal losses greater than or equal to the power delivered to the load.

The Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load. We assume that we can adjust the load resistance $R_L$. If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown in Fig. 4.48, the power delivered to the load is

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L \tag{4.21}$$

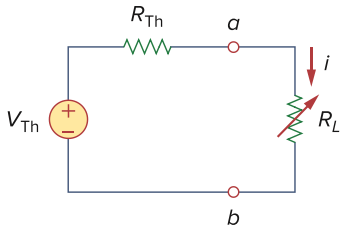


**Figure 4.48**
The circuit used for maximum power transfer.

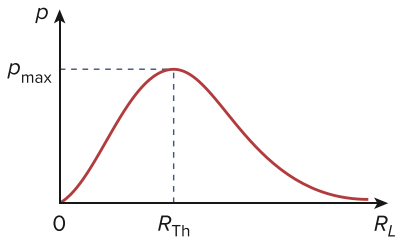


**Figure 4.49**
Power delivered to the load as a function of $R_L$.

For a given circuit, $V_{Th}$ and $R_{Th}$ are fixed. By varying the load resistance $R_L$, the power delivered to the load varies as sketched in Fig. 4.49. We notice from Fig. 4.49 that the power is small for small or large values of $R_L$ but maximum for some value of $R_L$ between 0 and $\infty$. We now want to show that this maximum power occurs when $R_L$ is equal to $R_{Th}$. This is known as the *maximum power theorem.*

> **Maximum power** is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).

To prove the maximum power transfer theorem, we differentiate $p$ in Eq. (4.21) with respect to $R_L$ and set the result equal to zero. We obtain

$$\frac{dp}{dR_L} = V_{Th}^2 \left[\frac{(R_{Th} + R_L)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4}\right]$$

$$= V_{Th}^2 \left[\frac{(R_{Th} + R_L - 2R_L)}{(R_{Th} + R_L)^3}\right] = 0$$

This implies that

$$0 = (R_{\text{Th}} + R_L - 2R_L) = (R_{\text{Th}} - R_L) \tag{4.22}$$

which yields

$$\boxed{R_L = R_{\text{Th}}} \tag{4.23}$$

showing that the maximum power transfer takes place when the load resistance $R_L$ equals the Thevenin resistance $R_{\text{Th}}$. We can readily confirm that Eq. (4.23) gives the maximum power by showing that $d^2p/dR_L^2 < 0$.

The source and load are said to be *matched* when $R_L = R_{\text{Th}}$.

The maximum power transferred is obtained by substituting Eq. (4.23) into Eq. (4.21), for

$$\boxed{p_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}}} \tag{4.24}$$

Equation (4.24) applies only when $R_L = R_{\text{Th}}$. When $R_L \neq R_{\text{Th}}$, we compute the power delivered to the load using Eq. (4.21).

## Example 4.13

Find the value of $R_L$ for maximum power transfer in the circuit of Fig. 4.50. Find the maximum power.

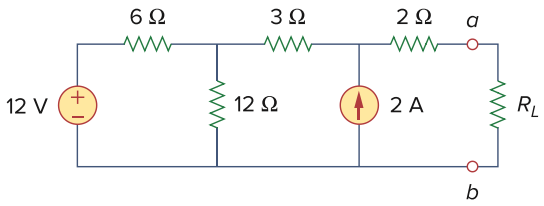


**Figure 4.50**
For Example 4.13.

**Solution:**

We need to find the Thevenin resistance $R_{\text{Th}}$ and the Thevenin voltage $V_{\text{Th}}$ across the terminals $a$-$b$. To get $R_{\text{Th}}$, we use the circuit in Fig. 4.51(a) and obtain

$$R_{\text{Th}} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6 \times 12}{18} = 9\ \Omega$$

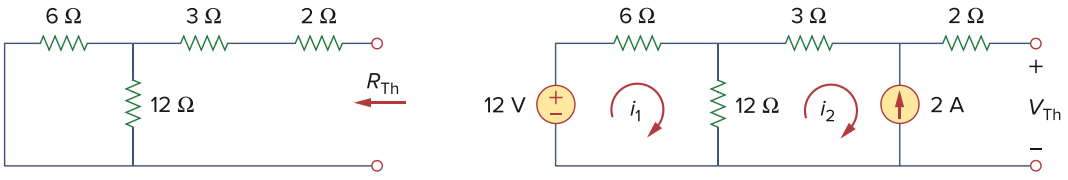


**Figure 4.51**
For Example 4.13: (a) finding $R_{\text{Th}}$, (b) finding $V_{\text{Th}}$.

To get $V_{Th}$, we consider the circuit in Fig. 4.51(b). Applying mesh analysis gives

$$-12 + 18i_1 - 12i_2 = 0, \qquad i_2 = -2 \text{ A}$$

Solving for $i_1$, we get $i_1 = -2/3$. Applying KVL around the outer loop to get $V_{Th}$ across terminals *a*-*b*, we obtain

$$-12 + 6i_1 + 3i_2 + 2(0) + V_{Th} = 0 \quad \Rightarrow \quad V_{Th} = 22 \text{ V}$$

For maximum power transfer,

$$R_L = R_{Th} = 9\ \Omega$$

and the maximum power is

$$p_{max} = \frac{V_{Th}^2}{4R_L} = \frac{22^2}{4 \times 9} = 13.44 \text{ W}$$

## Practice Problem 4.13

Determine the value of $R_L$ that will draw the maximum power from the rest of the circuit in Fig. 4.52. Calculate the maximum power.

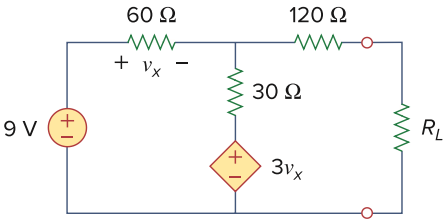


**Figure 4.52**
For Practice Prob. 4.13.

**Answer:** 126.67 Ω, 96.71 mW.

## 4.9 Verifying Circuit Theorems with *PSpice*

In this section, we learn how to use *PSpice* to verify the theorems covered in this chapter. Specifically, we will consider using DC Sweep analysis to find the Thevenin or Norton equivalent at any pair of nodes in a circuit and the maximum power transfer to a load. The reader is advised to read Section D.3 of Appendix D in preparation for this section.

To find the Thevenin equivalent of a circuit at a pair of open terminals using *PSpice*, we use the schematic editor to draw the circuit and insert an independent probing current source, say, Ip, at the terminals. The probing current source must have a part name ISRC. We then perform a DC Sweep on Ip, as discussed in Section D.3. Typically, we may let the current through Ip vary from 0 to 1 A in 0.1-A increments. After saving and simulating the circuit, we use Probe to display a plot of the voltage across Ip versus the current through Ip. The zero intercept of the plot gives us the Thevenin equivalent voltage, while the slope of the plot is equal to the Thevenin resistance.

To find the Norton equivalent involves similar steps except that we insert a probing independent voltage source (with a part name VSRC), say, Vp, at the terminals. We perform a DC Sweep on Vp and let Vp vary from 0 to 1 V in 0.1-V increments. A plot of the current through Vp versus the voltage across Vp is obtained using the Probe menu after simulation. The zero intercept is equal to the Norton current, while the slope of the plot is equal to the Norton conductance.

To find the maximum power transfer to a load using *PSpice* involves performing a DC parametric Sweep on the component value of $R_L$ in Fig. 4.48 and plotting the power delivered to the load as a function of $R_L$. According to Fig. 4.49, the maximum power occurs

when $R_L = R_{Th}$. This is best illustrated with an example, and Example 4.15 provides one.

We use VSRC and ISRC as part names for the independent voltage and current sources, respectively.

## Example 4.14

Consider the circuit in Fig. 4.31 (see Example 4.9). Use *PSpice* to find the Thevenin and Norton equivalent circuits.

**Solution:**

(a) To find the Thevenin resistance $R_{Th}$ and Thevenin voltage $V_{Th}$ at the terminals *a-b* in the circuit in Fig. 4.31, we first use Schematics to draw the circuit as shown in Fig. 4.53(a). Notice that a probing current source I2 is inserted at the terminals. Under **Analysis/Setput**, we select DC Sweep. In the DC Sweep dialog box, we select Linear for the *Sweep Type* and Current Source for the *Sweep Var. Type*. We enter I2 under the *Name* box, 0 as *Start Value*, 1 as *End Value*, and 0.1 as *Increment*. After simulation, we add trace V(I2:–) from the *PSpice* A/D window and obtain the plot shown in Fig. 4.53(b). From the plot, we obtain

$$V_{Th} = \text{Zero intercept} = 20\ \text{V}, \qquad R_{Th} = \text{Slope} = \frac{26 - 20}{1} = 6\ \Omega$$

These agree with what we got analytically in Example 4.9.

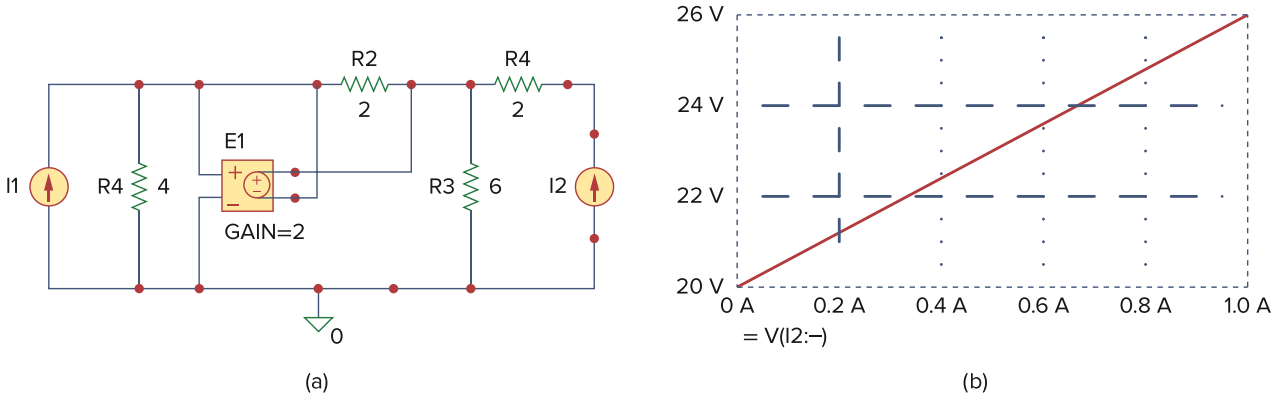


**Figure 4.53**
For Example 4.14: (a) schematic and (b) plot for finding $R_{Th}$ and $V_{Th}$.

(b) To find the Norton equivalent, we modify the schematic in Fig. 4.53(a) by replaying the probing current source with a probing voltage source V1. The result is the schematic in Fig. 4.54(a). Again, in the DC Sweep dialog box, we select Linear for the *Sweep Type* and Voltage Source for the *Sweep Var. Type*. We enter V1 under *Name* box, 0 as *Start Value*, 1 as *End Value*, and 0.1 as *Increment*. Under the *PSpice* A/D Window, we add trace I (V1) and obtain the plot in Fig. 4.54(b). From the plot, we obtain

$$I_N = \text{Zero intercept} = 3.335\ \text{A}$$

$$G_N = \text{Slope} = \frac{3.335 - 3.165}{1} = 0.17\ \text{S}$$

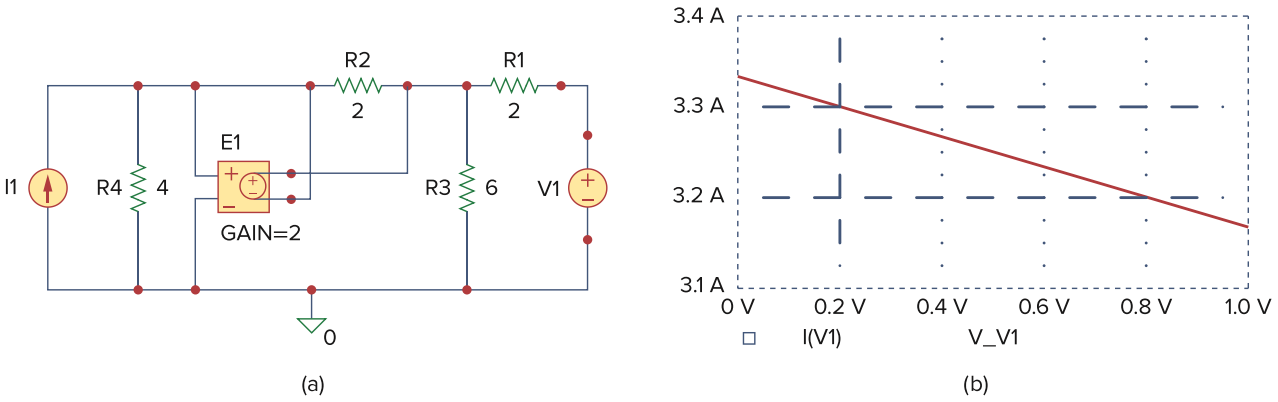


**Figure 4.54**
For Example 4.14: (a) schematic and (b) plot for finding $G_N$ and $I_N$.

## Practice Problem 4.14

Rework Practice Prob. 4.9 using *PSpice*.

**Answer:** $V_{Th} = 5.333$ V, $R_{Th} = 444.4$ mΩ.

## Example 4.15

Refer to the circuit in Fig. 4.55. Use *PSpice* to find the maximum power transfer to $R_L$.

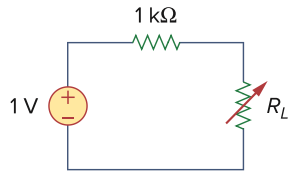


**Figure 4.55**
For Example 4.15.

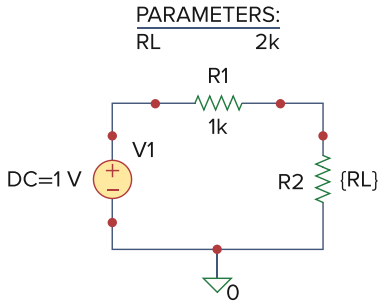


**Figure 4.56**
Schematic for the circuit in Fig. 4.55.

**Solution:**

We need to perform a DC Sweep on $R_L$ to determine when the power across it is maximum. We first draw the circuit using Schematics as shown in Fig. 4.56. Once the circuit is drawn, we take the following three steps to further prepare the circuit for a DC Sweep.

The first step involves defining the value of $R_L$ as a parameter, since we want to vary it. To do this:

1. **DCLICKL** the value 1k of R2 (representing $R_L$) to open up the *Set Attribute Value* dialog box.
2. Replace 1k with {RL} and click **OK** to accept the change.

Note that the curly brackets are necessary.

The second step is to define parameter. To achieve this:

1. Select **Draw/Get New Part/Libraries ···/special.slb**.
2. Type PARAM in the *PartName* box and click **OK**.
3. **DRAG** the box to any position near the circuit.
4. **CLICKL** to end placement mode.
5. **DCLICKL** to open up the *PartName: PARAM* dialog box.
6. **CLICKL** on *NAME1* = and enter RL (with no curly brackets) in the *Value* box, and **CLICKL Save Attr** to accept change.
7. **CLICKL** on *VALUE1* = and enter 2k in the *Value* box, and **CLICKL Save Attr** to accept change.
8. Click **OK**.

The value 2k in item 7 is necessary for a bias point calculation; it cannot be left blank.

The third step is to set up the DC Sweep to sweep the parameter. To do this:

1. Select **Analysis/Setput** to bring up the DC Sweep dialog box.
2. For the *Sweep Type*, select Linear (or Octave for a wide range of $R_L$).
3. For the *Sweep Var. Type*, select Global Parameter.
4. Under the *Name* box, enter RL.
5. In the *Start Value* box, enter 100.
6. In the *End Value* box, enter 5k.
7. In the *Increment* box, enter 100.
8. Click **OK** and **Close** to accept the parameters.

After taking these steps and saving the circuit, we are ready to simulate. Select **Analysis/Simulate**. If there are no errors, we select **Add Trace** in the *PSpice* A/D window and type −V(R2:2)*I(R2) in the *Trace Command* box. [The negative sign is needed since I(R2) is negative.] This gives the plot of the power delivered to $R_L$ as $R_L$ varies from 100 Ω to 5 kΩ. We can also obtain the power absorbed by $R_L$ by typing V(R2:2)* V(R2:2)/RL in the *Trace Command* box. Either way, we obtain the plot in Fig. 4.57. It is evident from the plot that the maximum power is 250 μW. Notice that the maximum occurs when $R_L = 1$ kΩ, as expected analytically.

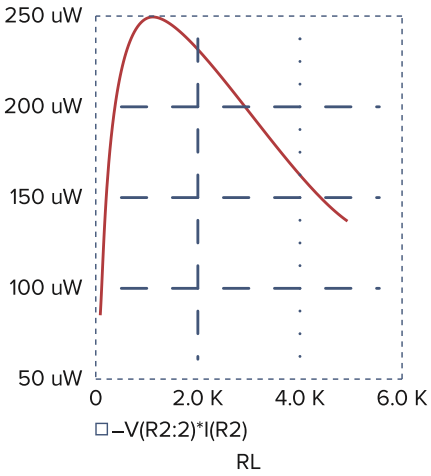


**Figure 4.57**
For Example 4.15: the plot of power across $R_L$.

## Practice Problem 4.15

Find the maximum power transferred to $R_L$ if the 1-kΩ resistor in Fig. 4.55 is replaced by a 2-kΩ resistor.

**Answer:** 125 μW.

## 4.10 †Applications

In this section we will discuss two important practical applications of the concepts covered in this chapter: source modeling and resistance measurement.

### 4.10.1 Source Modeling

Source modeling provides an example of the usefulness of the Thevenin or the Norton equivalent. An active source such as a battery is often characterized by its Thevenin or Norton equivalent circuit. An ideal voltage source provides a constant voltage irrespective of the current drawn by the load, while an ideal current source supplies a constant current regardless of the load voltage. As Fig. 4.58 shows, practical voltage and current sources are not ideal, due to their *internal resistances* or *source resistances* $R_s$ and $R_p$. They become ideal as $R_s \to 0$ and $R_p \to \infty$. To show that this is the case, consider the effect of the load on voltage sources,

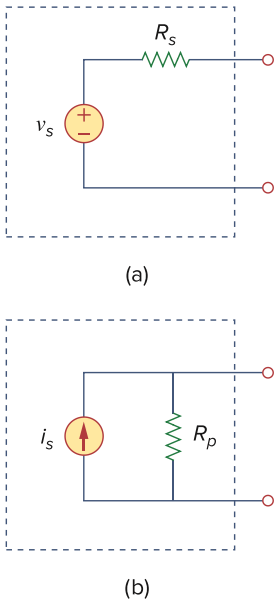


**Figure 4.58**
(a) Practical voltage source, (b) practical current source.

as shown in Fig. 4.59(a). By the voltage division principle, the load voltage is

$$v_L = \frac{R_L}{R_s + R_L} v_s \tag{4.25}$$

As $R_L$ increases, the load voltage approaches a source voltage $v_s$, as illustrated in Fig. 4.59(b). From Eq. (4.25), we should note that:

1. The load voltage will be constant if the internal resistance $R_s$ of the source is zero or, at least, $R_s \ll R_L$. In other words, the smaller $R_s$ is compared with $R_L$, the closer the voltage source is to being ideal.

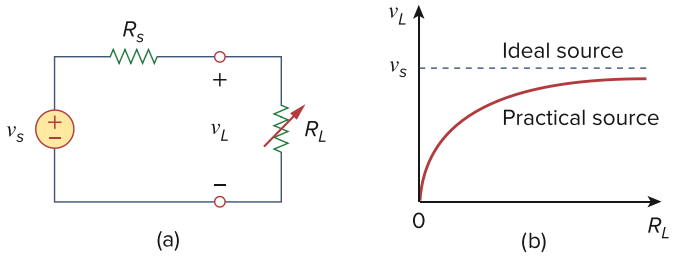


**Figure 4.59**
(a) Practical voltage source connected to a load $R_L$, (b) load voltage decreases as $R_L$ decreases.

2. When the load is disconnected (i.e., the source is open-circuited so that $R_L \to \infty$), $v_{oc} = v_s$. Thus, $v_s$ may be regarded as the *unloaded source* voltage. The connection of the load causes the terminal voltage to drop in magnitude; this is known as the *loading effect*.

The same argument can be made for a practical current source when connected to a load as shown in Fig. 4.60(a). By the current division principle,

$$i_L = \frac{R_p}{R_p + R_L} i_s \tag{4.26}$$

Figure 4.60(b) shows the variation in the load current as the load resistance increases. Again, we notice a drop in current due to the load (loading effect), and load current is constant (ideal current source) when the internal resistance is very large (i.e., $R_p \to \infty$ or, at least, $R_p \gg R_L$).

Sometimes, we need to know the unloaded source voltage $v_s$ and the internal resistance $R_s$ of a voltage source. To find $v_s$ and $R_s$, we follow the procedure illustrated in Fig. 4.61. First, we measure the open-circuit voltage $v_{oc}$ as in Fig. 4.61(a) and set

$$v_s = v_{oc} \tag{4.27}$$

Then, we connect a variable load $R_L$ across the terminals as in Fig. 4.61(b). We adjust the resistance $R_L$ until we measure a load voltage of exactly one-half of the open-circuit voltage, $v_L = v_{oc}/2$, because now $R_L = R_{\text{Th}} = R_s$. At that point, we disconnect $R_L$ and measure it. We set

$$R_s = R_L \tag{4.28}$$

For example, a car battery may have $v_s = 12$ V and $R_s = 0.05\ \Omega$.

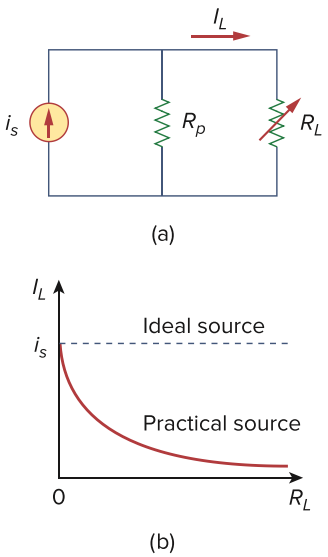


**Figure 4.60**
(a) Practical current source connected to a load $R_L$, (b) load current decreases as $R_L$ increases.

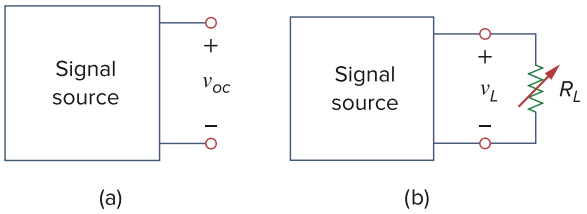


**Figure 4.61**
(a) Measuring $v_{oc}$, (b) measuring $v_L$.

## Example 4.16

The terminal voltage of a voltage source is 12 V when connected to a 2-W load. When the load is disconnected, the terminal voltage rises to 12.4 V. (a) Calculate the source voltage $v_s$ and internal resistance $R_s$. (b) Determine the voltage when an 8-Ω load is connected to the source.

**Solution:**
(a) We replace the source by its Thevenin equivalent. The terminal voltage when the load is disconnected is the open-circuit voltage,

$$v_s = v_{oc} = 12.4 \text{ V}$$

When the load is connected, as shown in Fig. 4.62(a), $v_L = 12$ V and $p_L = 2$ W. Hence,

$$p_L = \frac{v_L^2}{R_L} \quad \Rightarrow \quad R_L = \frac{v_L^2}{p_L} = \frac{12^2}{2} = 72\ \Omega$$

The load current is

$$i_L = \frac{v_L}{R_L} = \frac{12}{72} = \frac{1}{6}\text{ A}$$

The voltage across $R_s$ is the difference between the source voltage $v_s$ and the load voltage $v_L$, or

$$12.4 - 12 = 0.4 = R_s i_L, \qquad R_s = \frac{0.4}{I_L} = 2.4\ \Omega$$

(b) Now that we have the Thevenin equivalent of the source, we connect the 8-Ω load across the Thevenin equivalent as shown in Fig. 4.62(b). Using voltage division, we obtain

$$v = \frac{8}{8 + 2.4}(12.4) = 9.538 \text{ V}$$

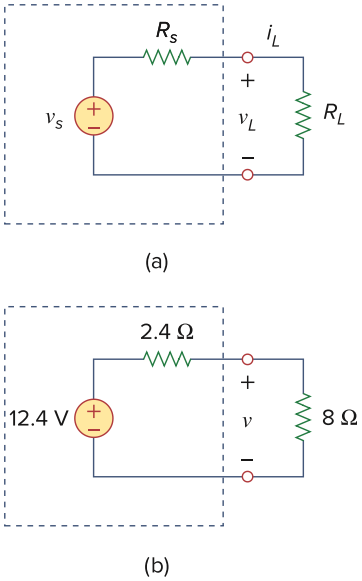


**Figure 4.62**
For Example 4.16.

## Practice Problem 4.16

The measured open-circuit voltage across a certain amplifier is 16 V. The voltage drops to 8 V when a 8-Ω loudspeaker is connected to the amplifier. Calculate the voltage when a 24-Ω loudspeaker is used instead.

**Answer:** 12 V.

### 4.10.2 Resistance Measurement

Although the ohmmeter method provides the simplest way to measure resistance, more accurate measurement may be obtained using the Wheatstone bridge. While ohmmeters are designed to measure resistance in low, mid, or high range, a Wheatstone bridge is used to measure resistance in the mid range, say, between 1 Ω and 1 MΩ. Very low values of resistances are measured with a *milliohmmeter,* while very high values are measured with a *Megger tester*.

*Historical note:* The bridge was invented by Charles Wheatstone (1802–1875), a British professor who also invented the telegraph, as Samuel Morse did independently in the United States.

The Wheatstone bridge (or resistance bridge) circuit is used in a number of applications. Here we will use it to measure an unknown resistance. The unknown resistance $R_x$ is connected to the bridge as shown in Fig. 4.63. The variable resistance is adjusted until no current flows through the galvanometer, which is essentially a d'Arsonval movement operating as a sensitive current-indicating device like an ammeter in the microamp range. Under this condition $v_1 = v_2$, and the bridge is said to be *balanced*. Since no current flows through the galvanometer, $R_1$ and $R_2$ behave as though they were in series; so do $R_3$ and $R_x$. The fact that no current flows through the galvanometer also implies that $v_1 = v_2$. Applying the voltage division principle,

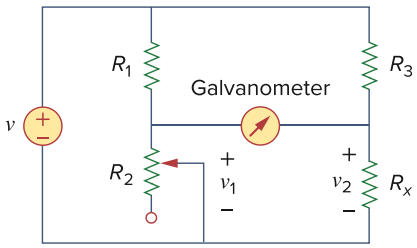


**Figure 4.63**
The Wheatstone bridge; $R_x$ is the resistance to be measured.

$$v_1 = \frac{R_2}{R_1 + R_2} v = v_2 = \frac{R_x}{R_3 + R_x} v \tag{4.29}$$

Hence, no current flows through the galvanometer when

$$\frac{R_2}{R_1 + R_2} = \frac{R_x}{R_3 + R_x} \quad \Rightarrow \quad R_2 R_3 = R_1 R_x$$

or

$$\boxed{R_x = \frac{R_3}{R_1} R_2} \tag{4.30}$$

If $R_1 = R_3$, and $R_2$ is adjusted until no current flows through the galvanometer, then $R_x = R_2$.

How do we find the current through the galvanometer when the Wheatstone bridge is *unbalanced*? We find the Thevenin equivalent ($V_{\text{Th}}$ and $R_{\text{Th}}$) with respect to the galvanometer terminals. If $R_m$ is the resistance of the galvanometer, the current through it under the unbalanced condition is

$$I = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_m} \tag{4.31}$$

Example 4.18 will illustrate this.

## Example 4.17

In Fig. 4.63, $R_1 = 500\,\Omega$ and $R_3 = 200\,\Omega$. The bridge is balanced when $R_2$ is adjusted to be 125 Ω. Determine the unknown resistance $R_x$.

**Solution:**
Using Eq. (4.30) gives

$$R_x = \frac{R_3}{R_1} R_2 = \frac{200}{500} 125 = 50\,\Omega$$

## Practice Problem 4.17

A Wheatstone bridge has $R_1 = R_3 = 2\ \text{k}\Omega$. $R_2$ is adjusted until no current flows through the galvanometer. At that point, $R_2 = 6.3\ \text{k}\Omega$. What is the value of the unknown resistance?

**Answer:** 6.3 kΩ.

## Example 4.18

The circuit in Fig. 4.64 represents an unbalanced bridge. If the galvanometer has a resistance of 40 Ω, find the current through the galvanometer.

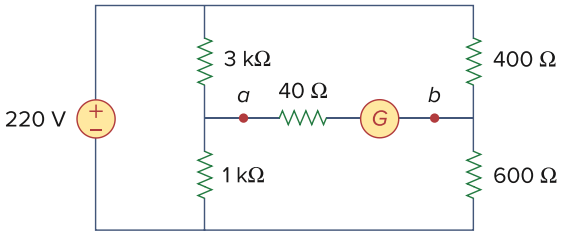


**Figure 4.64**
Unbalanced bridge of Example 4.18.

**Solution:**

We first need to replace the circuit by its Thevenin equivalent at terminals $a$ and $b$. The Thevenin resistance is found using the circuit in Fig. 4.65($a$). Notice that the 3-kΩ and 1-KΩ resistors are in parallel; so are the 400 and 600- Ω resistors. The two parallel combinations form a series combination with respect to terminals $a$ and $b$. Hence,

$$R_{\text{Th}} = 3000 \parallel 1000 + 400 \parallel 600$$

$$= \frac{3000 \times 1000}{3000 + 1000} + \frac{400 \times 600}{400 + 600} = 750 + 240 = 990\ \Omega$$

To find the Thevenin voltage, we consider the circuit in Fig. 4.65(b). Using the voltage division principle gives

$$v_1 = \frac{1000}{1000 + 3000}(220) = 55\ \text{V}, \qquad v_2 = \frac{600}{600 + 400}(220) = 132\ \text{V}$$

Applying KVL around loop $ab$ gives

$$-v_1 + V_{\text{Th}} + v_2 = 0 \qquad \text{or} \qquad V_{\text{Th}} = v_1 - v_2 = 55 - 132 = -77\ \text{V}$$

Having determined the Thevenin equivalent, we find the current through the galvanometer using Fig. 4.65(c).

$$I_G = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_m} = \frac{-77}{990 + 40} = -74.76\ \text{mA}$$

The negative sign indicates that the current flows in the direction opposite to the one assumed, that is, from terminal $b$ to terminal $a$.

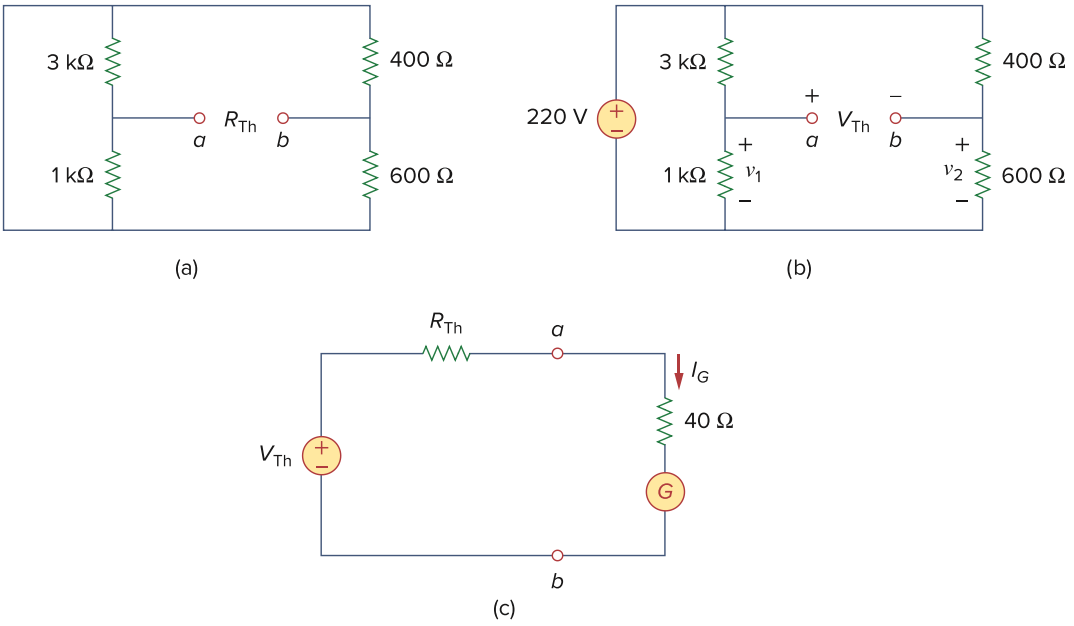


**Figure 4.65**
For Example 4.18: (a) Finding $R_{Th}$, (b) finding $V_{Th}$, (c) determining the current through the galvanometer.

## Practice Problem 4.18

Obtain the current through the galvanometer, having a resistance of 14 Ω, in the Wheatstone bridge shown in Fig. 4.66.

**Answer:** 64 mA.

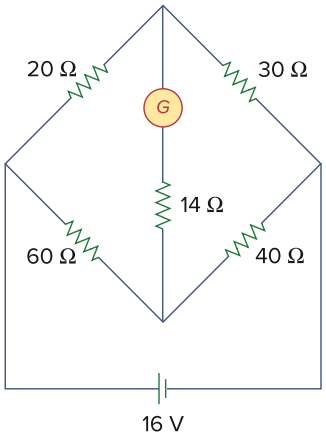


**Figure 4.66**
For Practice Prob. 4.18.

## 4.11 Summary

1. A linear network consists of linear elements, linear dependent sources, and linear independent sources.
2. Network theorems are used to reduce a complex circuit to a simpler one, thereby making circuit analysis much simpler.
3. The superposition principle states that for a circuit having multiple independent sources, the voltage across (or current through) an element is equal to the algebraic sum of all the individual voltages (or currents) due to each independent source acting one at a time.
4. Source transformation is a procedure for transforming a voltage source in series with a resistor to a current source in parallel with a resistor, or vice versa.
5. Thevenin's and Norton's theorems allow us to isolate a portion of a network while the remaining portion of the network is replaced by an equivalent network. The Thevenin equivalent consists of a voltage source $V_{Th}$ in series with a resistor $R_{Th}$, while the Norton equivalent consists of a current source $I_N$ in parallel with a resistor $R_N$. The two theorems are related by source transformation.

$$R_N = R_{Th}, \qquad I_N = \frac{V_{Th}}{R_{Th}}$$

6. For a given Thevenin equivalent circuit, maximum power transfer occurs when $R_L = R_{Th}$; that is, when the load resistance is equal to the Thevenin resistance.
7. The maximum power transfer theorem states that the maximum power is delivered by a source to the load $R_L$ when $R_L$ is equal to $R_{Th}$, the Thevenin resistance at the terminals of the load.
8. *PSpice* can be used to verify the circuit theorems covered in this chapter.
9. Source modeling and resistance measurement using the Wheatstone bridge provide applications for Thevenin's theorem.

## Review Questions

**4.1** The current through a branch in a linear network is 2 A when the input source voltage is 10 V. If the voltage is reduced to 1 V and the polarity is reversed, the current through the branch is:

(a) – 2 A (b) – 0.2 A (c) 0.2 A
(d) 2 A (e) 20 A

**4.2** For superposition, it is not required that only one independent source be considered at a time; any number of independent sources may be considered simultaneously.

(a) True (b) False

**4.3** The superposition principle applies to power calculation.

(a) True (b) False

**4.4** Refer to Fig. 4.67. The Thevenin resistance at terminals *a* and *b* is:

(a) 25 Ω (b) 20 Ω
(c) 5 Ω (d) 4 Ω

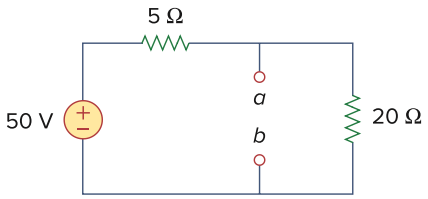


**Figure 4.67**
For Review Questions 4.4 to 4.6.

**4.5** The Thevenin voltage across terminals *a* and *b* of the circuit in Fig. 4.67 is:

(a) 50 V (b) 40 V
(c) 20 V (d) 10 V

**4.6** The Norton current at terminals *a* and *b* of the circuit in Fig. 4.67 is:

(a) 10 A (b) 2.5 A
(c) 2 A (d) 0 A

**4.7** The Norton resistance $R_N$ is exactly equal to the Thevenin resistance $R_{Th}$.

(a) True (b) False

**4.8** Which pair of circuits in Fig. 4.68 are equivalent?

(a) a and b (b) b and d
(c) a and c (d) c and d

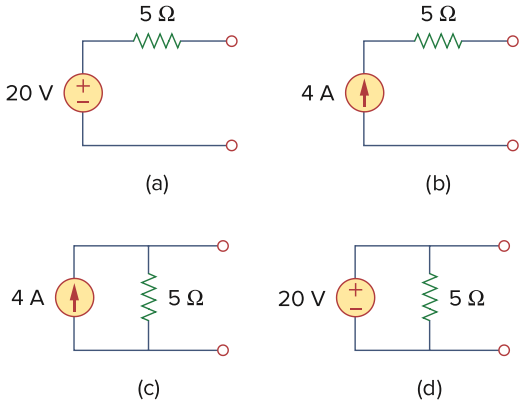


**Figure 4.68**
For Review Question 4.8.

**4.9** A load is connected to a network. At the terminals to which the load is connected, $R_{Th} = 10\ \Omega$ and $V_{Th} = 40$ V. The maximum possible power supplied to the load is:

(a) 160 W (b) 80 W
(c) 40 W (d) 1 W

**4.10** The source is supplying the maximum power to the load when the load resistance equals the source resistance.

(a) True (b) False

*Answers: 4.1b, 4.2a, 4.3b, 4.4d, 4.5b, 4.6a, 4.7a, 4.8c, 4.9c, 4.10a.*

# Problems

## Section 4.2 Linearity Property

**4.1** Calculate the current $i_o$ in the circuit of Fig. 4.69. What value of input voltage is necessary to make $i_o$ equal to 5 amps?

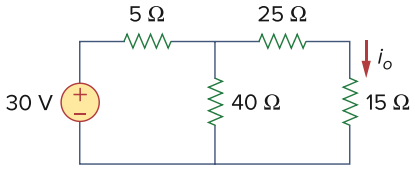


**Figure 4.69**
For Prob. 4.1.

**4.2** Using Fig. 4.70, design a problem to help other students better understand linearity.

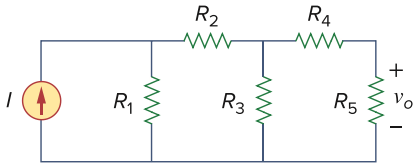


**Figure 4.70**
For Prob. 4.2.

**4.3** (a) In the circuit of Fig. 4.71, calculate $v_o$ and $i_o$ when $v_s = 1$ V.

(b) Find $v_o$ and $i_o$ when $v_s = 10$ V.

(c) What are $v_o$ and $i_o$ when each of the 1-Ω resistors is replaced by a 10-Ω resistor and $v_s = 10$ V?

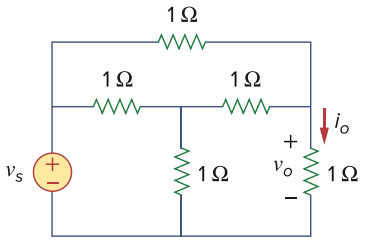


**Figure 4.71**
For Prob. 4.3.

**4.4** Use linearity to determine $i_o$ in the circuit of Fig. 4.72.

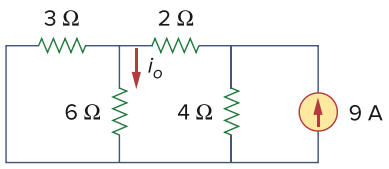


**Figure 4.72**
For Prob. 4.4.

**4.5** For the circuit in Fig. 4.73, assume $v_o = 1$ V, and use linearity to find the actual value of $v_o$.

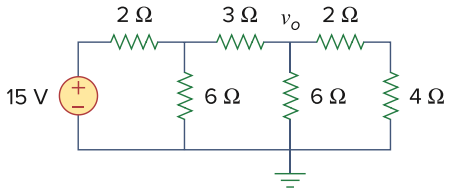


**Figure 4.73**
For Prob. 4.5.

**4.6** For the linear circuit shown in Fig. 4.74, use linearity to complete the following table.

| Experiment | $V_s$ | $V_o$ |
|---|---|---|
| 1 | 12 V | 4 V |
| 2 | | 16 V |
| 3 | 1 V | |
| 4 | | –2 V |

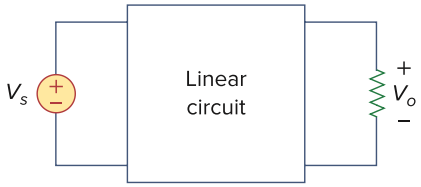


**Figure 4.74**
For Prob. 4.6.

**4.7** Use linearity and the assumption that $V_o = 1$ V to find the actual value of $V_o$ in Fig. 4.75.

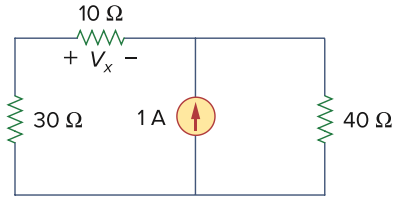


**Figure 4.75**
For Prob. 4.7.

## Section 4.3 Superposition

**4.8** Using superposition, find $V_o$ in the circuit of Fig. 4.76. Check with *PSpice or MultiSim*.

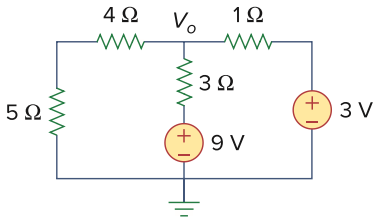


**Figure 4.76**
For Prob. 4.8.

**4.9** Given that $I = 6$ amps when $V_s = 160$ volts and $I_s = -10$ amps and $I = 5$ amp when $V_s = 200$ volts and $I_s = 0$, use superposition and linearity to determine the value of $I$ when $V_s = 120$ volts and $I_s = 5$ amps.

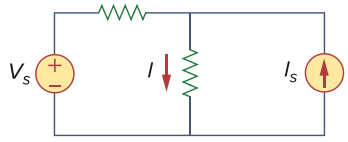


**Figure 4.77**
For Prob. 4.9.

**4.10** Using Fig. 4.78, design a problem to help other students better understand superposition. Note, the letter $k$ is a gain you can specify to make the problem easier to solve but must not be zero.

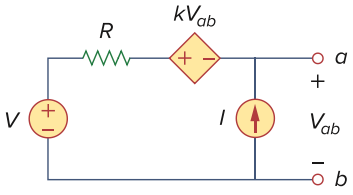


**Figure 4.78**
For Prob. 4.10.

**4.11** Use the superposition principle to find $i_o$ and $v_o$ in the circuit of Fig. 4.79.

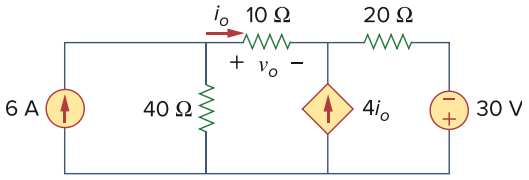


**Figure 4.79**
For Prob. 4.11.

**4.12** Determine $v_o$ in the circuit of Fig. 4.80 using the superposition principle.

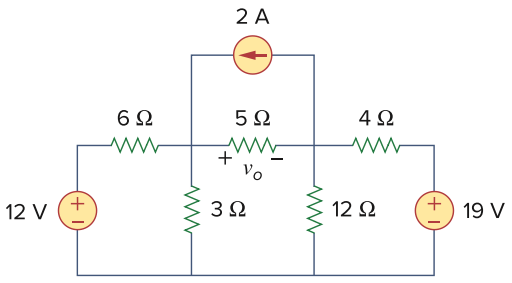


**Figure 4.80**
For Prob. 4.12.

**4.13** Use superposition to find $v_o$ in the circuit of Fig. 4.81.

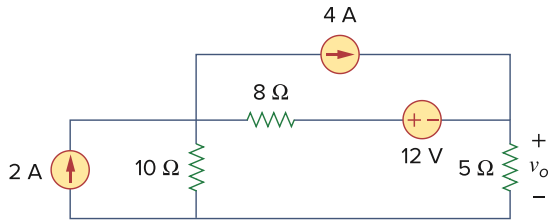


**Figure 4.81**
For Prob. 4.13.

**4.14** Apply the superposition principle to find $v_o$ in the circuit of Fig. 4.82.

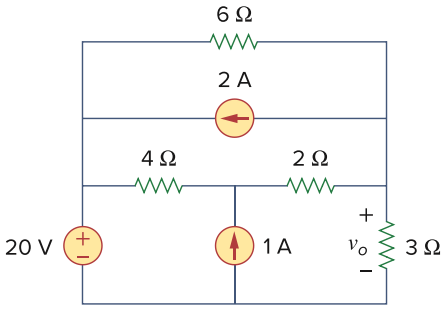


**Figure 4.82**
For Prob. 4.14.

**4.15** For the circuit in Fig. 4.83, use superposition to find $i$. Calculate the power delivered to the 3-Ω resistor.

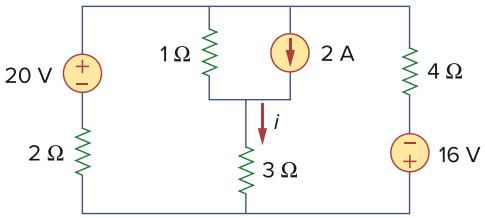


**Figure 4.83**
For Probs. 4.15 and 4.56.

**4.16** Given the circuit in Fig. 4.84, use superposition to obtain $i_o$.

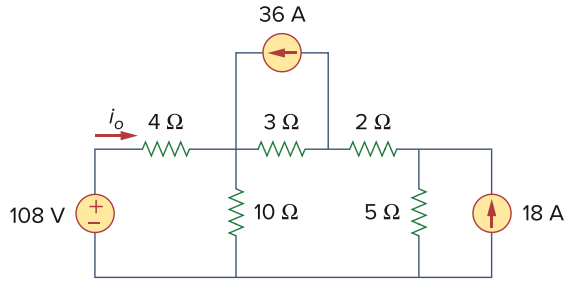


**Figure 4.84**
For Prob. 4.16.

**4.17** Use superposition to obtain $v_x$ in the circuit of Fig. 4.85. Check your result using *PSpice or MultiSim*.

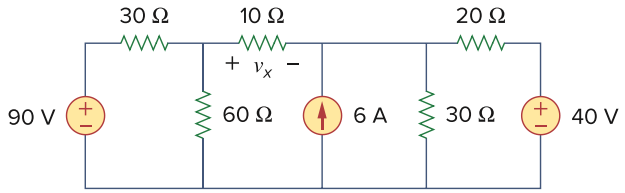


**Figure 4.85**
For Prob. 4.17.

**4.18** Use superposition to find $V_o$ in the circuit of Fig. 4.86.

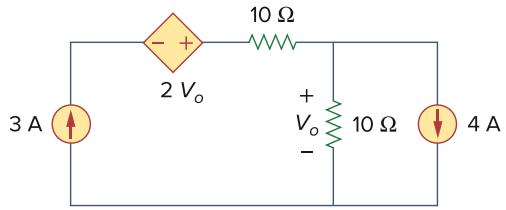


**Figure 4.86**
For Prob. 4.18.

**4.19** Use superposition to solve for $v_x$ in the circuit of Fig. 4.87.

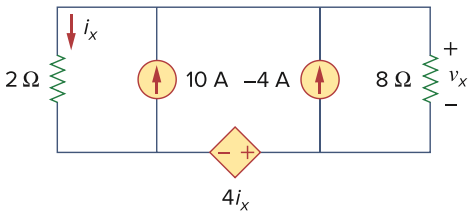


**Figure 4.87**
For Prob. 4.19.

## Section 4.4 Source Transformation

**4.20** Use source transformation to reduce the circuit between terminals *a* and *b* shown in Fig. 4.88 to a single voltage source in series with a single resistor.

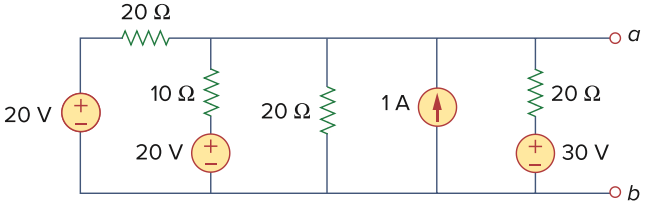


**Figure 4.88**
For Prob. 4.20.

**4.21** Using Fig. 4.89, design a problem to help other students better understand source transformation.

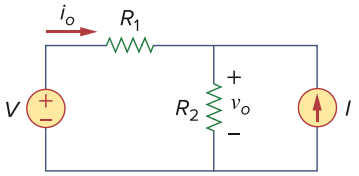


**Figure 4.89**
For Prob. 4.21.

**4.22** For the circuit in Fig. 4.90, use source transformation to find $i$.

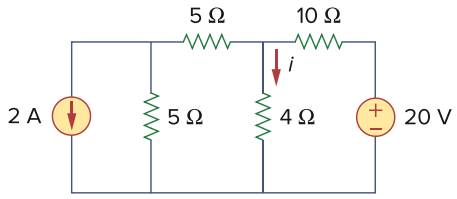


**Figure 4.90**
For Prob. 4.22.

**4.23** Referring to Fig. 4.91, use source transformation to determine the current and power absorbed by the 8-Ω resistor.

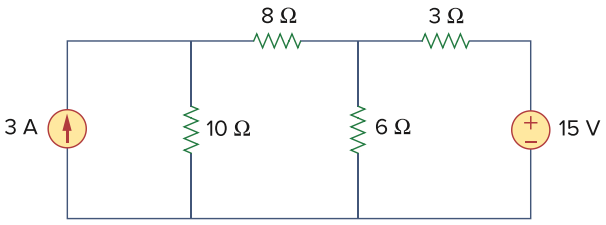


**Figure 4.91**
For Prob. 4.23.

**4.24** Use source transformation to find the voltage $V_x$ in the circuit of Fig. 4.92.

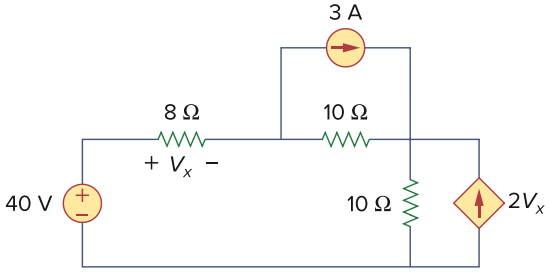


**Figure 4.92**
For Prob. 4.24.

**4.25** Obtain $v_o$ in the circuit of Fig. 4.93 using source transformation. Check your result using *PSpice or MultiSim*.

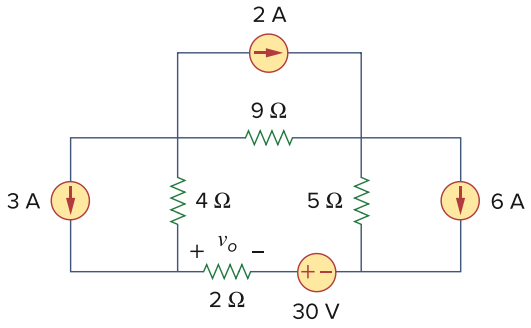


**Figure 4.93**
For Prob. 4.25.

**4.26** Use source transformation to find $i_o$ in the circuit of Fig. 4.94.

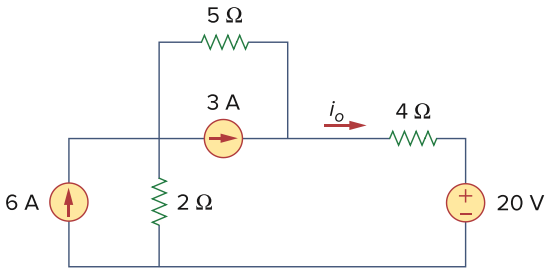


**Figure 4.94**
For Prob. 4.26.

**4.27** Apply source transformation to find $v_x$ in the circuit of Fig. 4.95.

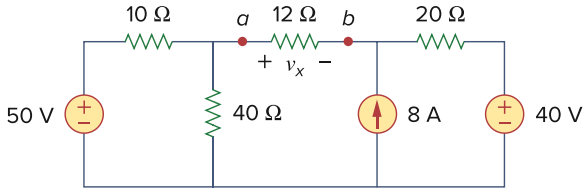


**Figure 4.95**
For Probs. 4.27 and 4.40.

**4.28** Use source transformation to find $I_o$ in Fig. 4.96.

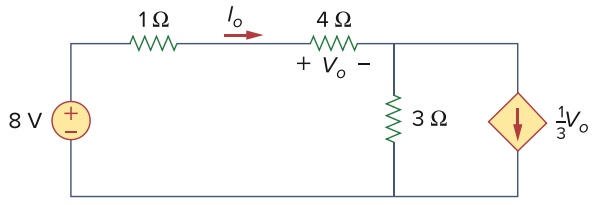


**Figure 4.96**
For Prob. 4.28.

**4.29** Use source transformation to find $v_o$ in the circuit of Fig. 4.97.

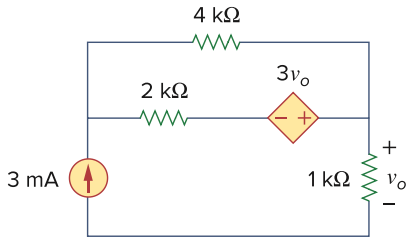


**Figure 4.97**
For Prob. 4.29.

**4.30** Use source transformation on the circuit shown in Fig 4.98 to find $i_x$.

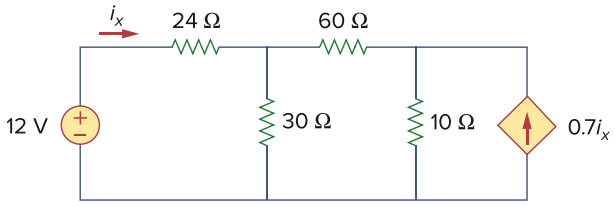


**Figure 4.98**
For Prob. 4.30.

**4.31** Determine $v_x$ in the circuit of Fig. 4.99 using source transformation.

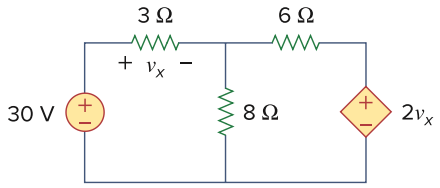


**Figure 4.99**
For Prob. 4.31.

**4.32** Use source transformation to find $i_x$ in the circuit of Fig. 4.100.

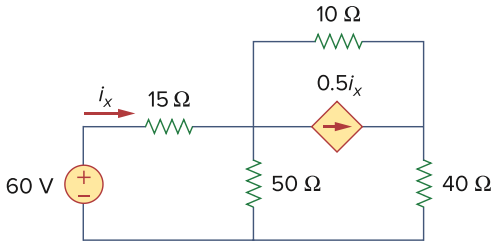


**Figure 4.100**
For Prob. 4.32.

## Sections 4.5 and 4.6 Thevenin's and Norton's Theorems

**4.33** Determine the Thevenin equivalent circuit, shown in Fig. 4.101, as seen by the 7-ohm resistor.

Then calculate the current flowing through the 7-ohm resistor.

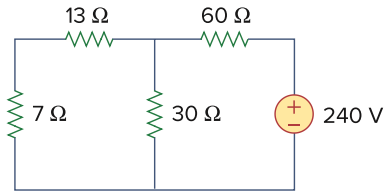


**Figure 4.101**
For Prob. 4.33.

**4.34** Using Fig. 4.102, design a problem that will help other students better understand Thevenin equivalent circuits.

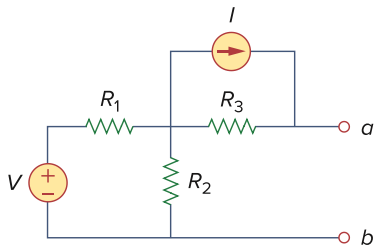


**Figure 4.102**
For Probs. 4.34 and 4.49.

**4.35** Use Thevenin's theorem to find $v_o$ in Prob. 4.12.

**4.36** Solve for the current $i$ in the circuit of Fig. 4.103 using Thevenin's theorem. (*Hint:* Find the Thevenin equivalent seen by the 12-Ω resistor.)

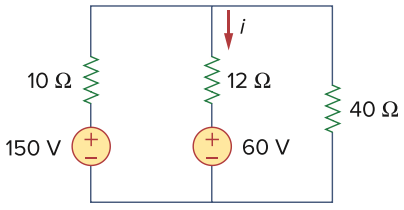


**Figure 4.103**
For Prob. 4.36.

**4.37** Find the Norton equivalent with respect to terminals $a$-$b$ in the circuit shown in Fig. 4.104.

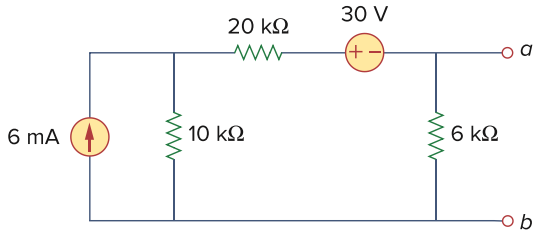


**Figure 4.104**
For Prob. 4.37.

**4.38** Apply Thevenin's theorem to find $V_o$ in the circuit of Fig. 4.105.

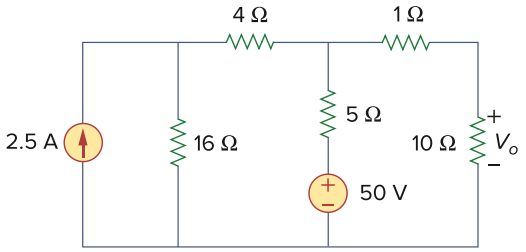


**Figure 4.105**
For Prob. 4.38.

**4.39** Obtain the Thevenin equivalent at terminals $a$-$b$ of the circuit shown in Fig. 4.106.

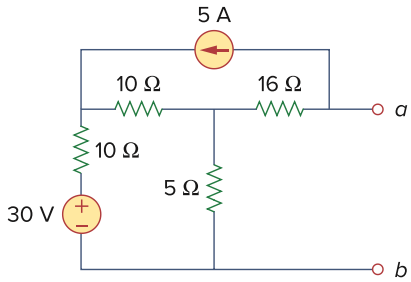


**Figure 4.106**
For Prob. 4.39.

**4.40** Find the Thevenin equivalent at terminals $a$-$b$ of the circuit in Fig. 4.107.

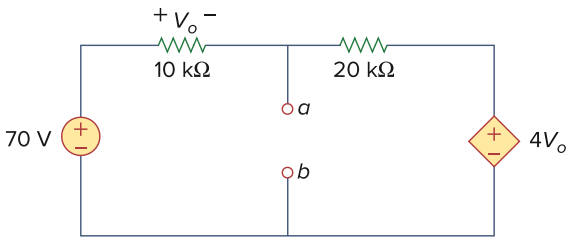


**Figure 4.107**
For Prob. 4.40.

**4.41** Find the Thevenin and Norton equivalents at terminals *a-b* of the circuit shown in Fig. 4.108.

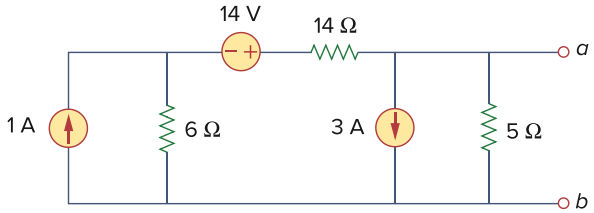


**Figure 4.108**
For Prob. 4.41.

***4.42** For the circuit in Fig. 4.109, find the Thevenin equivalent between terminals *a* and *b*.

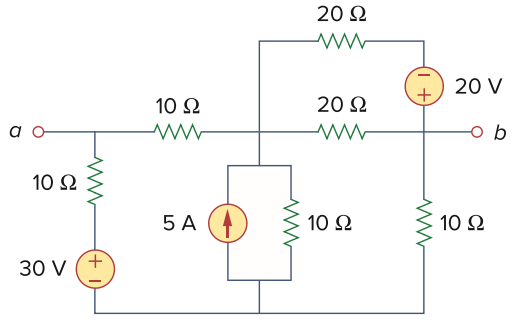


**Figure 4.109**
For Prob. 4.42.

**4.43** Find the Thevenin equivalent looking into terminals *a-b* of the circuit in Fig. 4.110 and solve for $i_x$.

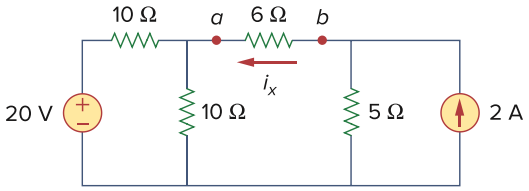


**Figure 4.110**
For Prob. 4.43.

**4.44** For the circuit in Fig. 4.111, obtain the Thevenin equivalent as seen from terminals:

(a) *a-b* (b) *b-c*

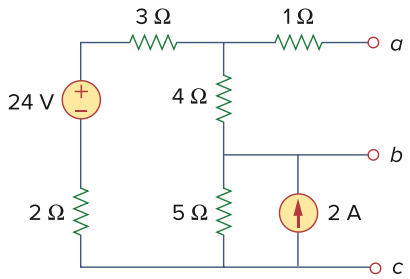


**Figure 4.111**
For Prob. 4.44.

* An asterisk indicates a challenging problem.

**4.45** Find the Thevenin equivalent of the circuit in Fig. 4.112 as seen by looking into terminals *a* and *b*.

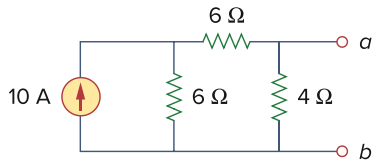


**Figure 4.112**
For Prob. 4.45.

**4.46** Using Fig. 4.113, design a problem to help other students better understand Norton equivalent circuits.

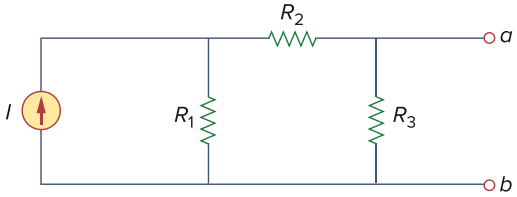


**Figure 4.113**
For Prob. 4.46.

**4.47** Obtain the Thevenin and Norton equivalent circuits of the circuit in Fig. 4.114 with respect to terminals *a* and *b*.

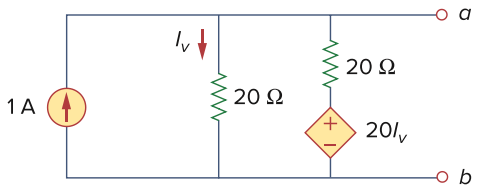


**Figure 4.114**
For Prob. 4.47.

**4.48** Determine the Norton equivalent at terminals *a-b* for the circuit in Fig. 4.115.

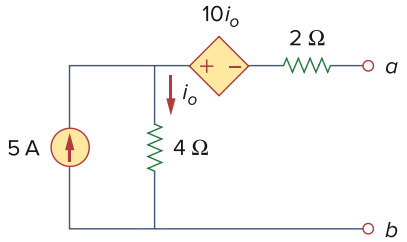


**Figure 4.115**
For Prob. 4.48.

**4.49** Find the Norton equivalent looking into terminals *a-b* of the circuit in Fig. 4.102. Let $V = 40$ V, $I = 3$ A, $R_1 = 10\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 20\ \Omega$.

**4.50** Obtain the Norton equivalent of the circuit in Fig. 4.116 to the left of terminals *a-b*. Use the result to find current *i*.

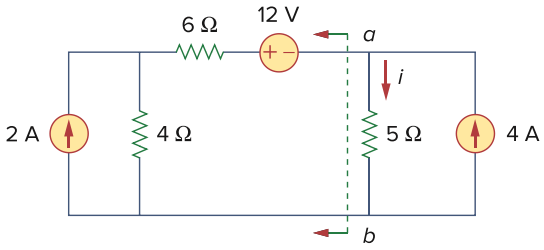


**Figure 4.116**
For Prob. 4.50.

**4.51** Given the circuit in Fig. 4.117, obtain the Norton equivalent as viewed from terminals:

(a) *a-b* (b) *c-d*

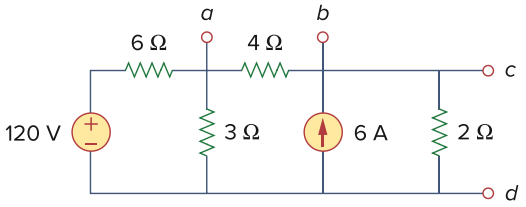


**Figure 4.117**
For Prob. 4.51.

**4.52** For the transistor model in Fig. 4.118, obtain the Thevenin equivalent at terminals *a-b*.

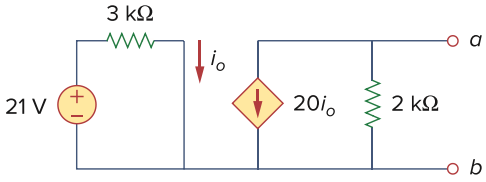


**Figure 4.118**
For Prob. 4.52.

**4.53** Find the Norton equivalent at terminals *a-b* of the circuit in Fig. 4.119.

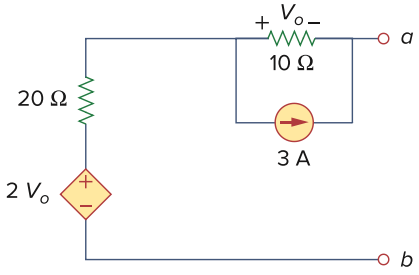


**Figure 4.119**
For Prob. 4.53.

**4.54** Find the Thevenin equivalent between terminals *a-b* of the circuit in Fig. 4.120.

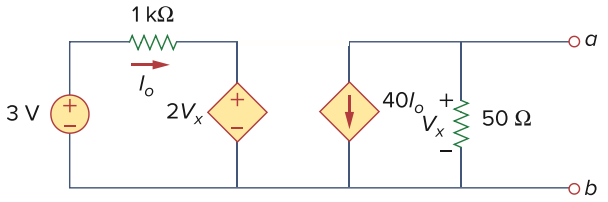


**Figure 4.120**
For Prob. 4.54.

***4.55** Obtain the Norton equivalent at terminals *a-b* of the circuit in Fig. 4.121.

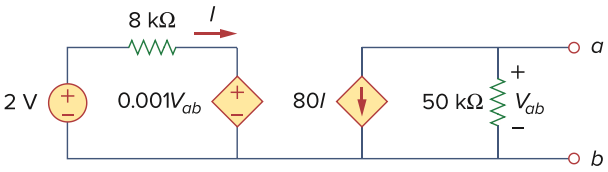


**Figure 4.121**
For Prob. 4.55.

**4.56** Use Norton's theorem to find $V_o$ in the circuit of Fig. 4.122.

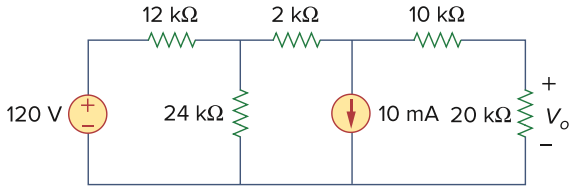


**Figure 4.122**
For Prob. 4.56.

**4.57** Obtain the Thevenin and Norton equivalent circuits at terminals *a-b* for the circuit in Fig. 4.123.

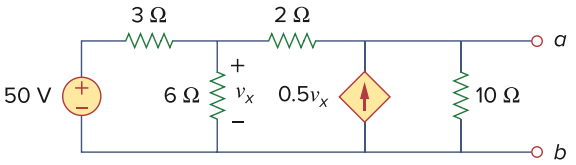


**Figure 4.123**
For Probs. 4.57 and 4.79.

**4.58** The network in Fig. 4.124 models a bipolar transistor common-emitter amplifier connected to a load. Find the Thevenin resistance seen by the load.

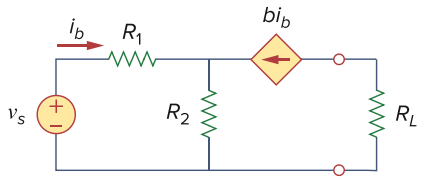


**Figure 4.124**
For Prob. 4.58.

**4.59** Determine the Thevenin and Norton equivalents at terminals *a-b* of the circuit in Fig. 4.125.

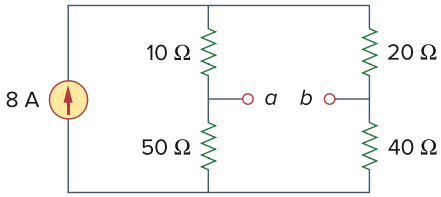


**Figure 4.125**
For Probs. 4.59 and 4.80.

***4.60** For the circuit in Fig. 4.126, find the Thevenin and Norton equivalent circuits at terminals *a-b*.

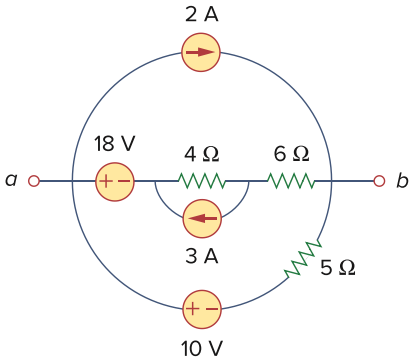


**Figure 4.126**
For Probs. 4.60 and 4.81.

***4.61** Obtain the Thevenin and Norton equivalent circuits at terminals *a-b* of the circuit in Fig. 4.127.

ML

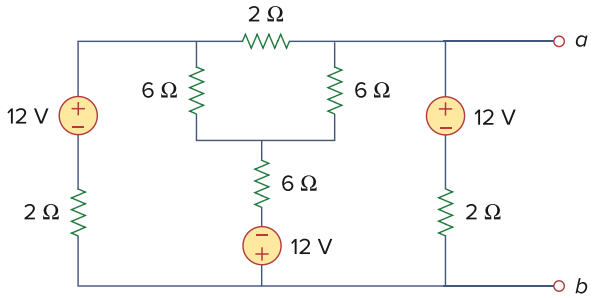


**Figure 4.127**
For Prob. 4.61.

***4.62** Find the Thevenin equivalent of the circuit in Fig. 4.128.

ML

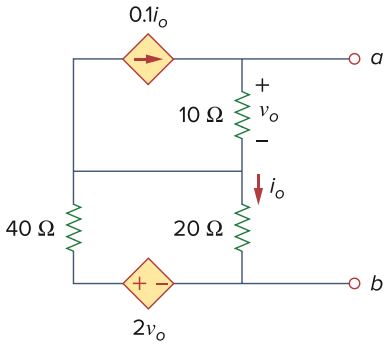


**Figure 4.128**
For Prob. 4.62.

**4.63** Find the Norton equivalent for the circuit in Fig. 4.129.

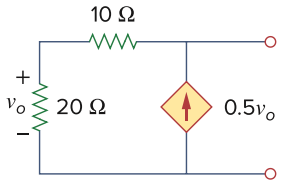


**Figure 4.129**
For Prob. 4.63.

**4.64** Obtain the Thevenin equivalent seen at terminals *a-b* of the circuit in Fig. 4.130.

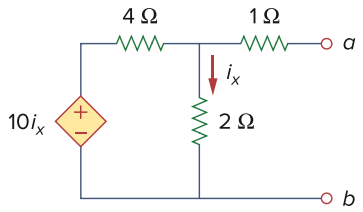


**Figure 4.130**
For Prob. 4.64.

**4.65** For the circuit shown in Fig. 4.131, determine the relationship between $V_o$ and $I_o$.

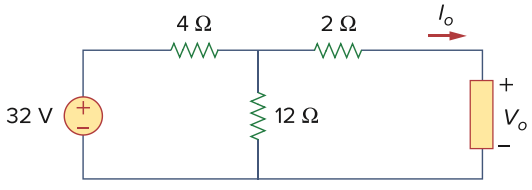


**Figure 4.131**
For Prob. 4.65.

## Section 4.8 Maximum Power Transfer

**4.66** Find the maximum power that can be delivered to the resistor $R$ in the circuit of Fig. 4.132.

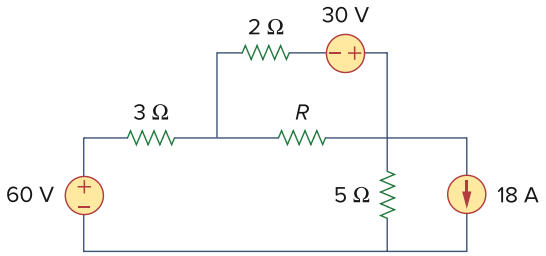


**Figure 4.132**
For Prob. 4.66.

**4.67** The variable resistor $R$ in Fig. 4.133 is adjusted until it absorbs the maximum power from the circuit.
(a) Calculate the value of $R$ for maximum power.
(b) Determine the maximum power absorbed by $R$.

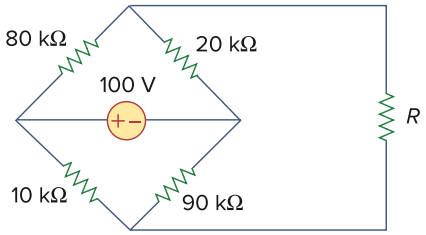


**Figure 4.133**
For Prob. 4.67.

***4.68** Consider the 30-Ω resistor in Fig. 4.134. First compute the Thevenin equivalent circuit as seen by the 30-Ω resistor. Compute the value of $R$ that results in Thevenin equivalent resistance equal to the 30-Ω resistance and then calculate power delivered to the 30-Ω resistor. Now let $R = 0\ \Omega$, 110 Ω, and ∞, calculate the power delivered to the 30-Ω resistor in each case. What can you say about the value of $R$ that will result in the maximum power that can be delivered to the 30-Ω resistor?

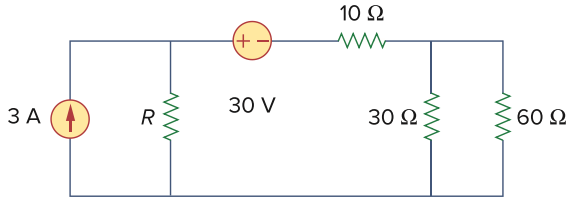


**Figure 4.134**
For Prob. 4.68.

**4.69** Find the maximum power transferred to resistor $R$ in the circuit of Fig. 4.135.

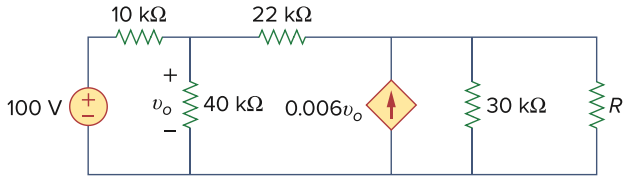


**Figure 4.135**
For Prob. 4.69.

**4.70** Determine the maximum power delivered to the variable resistor $R$ shown in the circuit of Fig. 4.136.

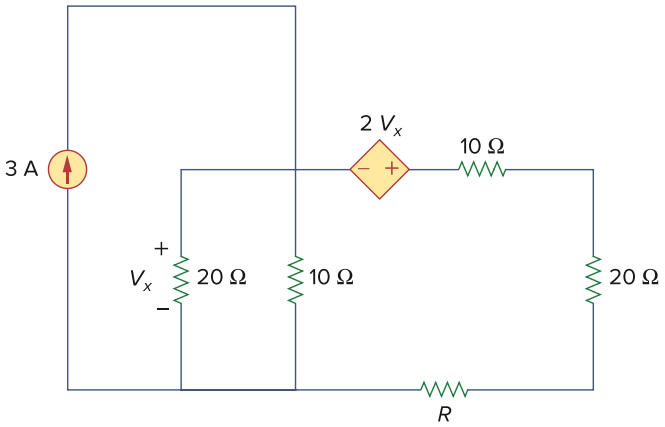


**Figure 4.136**
For Prob. 4.70.

**4.71** For the circuit in Fig. 4.137, what resistor connected across terminals $a$-$b$ will absorb maximum power from the circuit? What is that power?

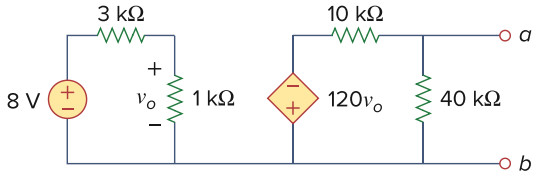


**Figure 4.137**
For Prob. 4.71.

**4.72** (a) For the circuit in Fig. 4.138, obtain the Thevenin equivalent at terminals $a$-$b$.
(b) Calculate the current in $R_L = 13\ \Omega$.
(c) Find $R_L$ for maximum power deliverable to $R_L$.
(d) Determine that maximum power.

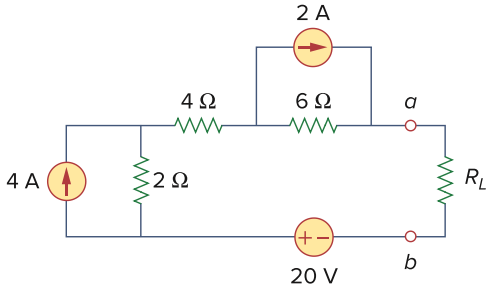


**Figure 4.138**
For Prob. 4.72.

**4.73** Determine the maximum power that can be delivered to the variable resistor $R$ in the circuit of Fig. 4.139.

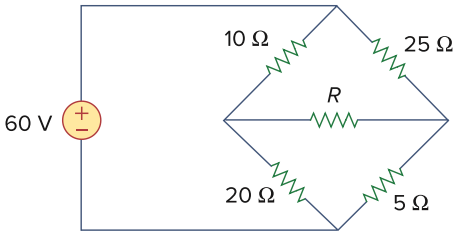


**Figure 4.139**
For Prob. 4.73.

**4.74** For the bridge circuit shown in Fig. 4.140, find the load $R_L$ for maximum power transfer and the maximum power absorbed by the load.

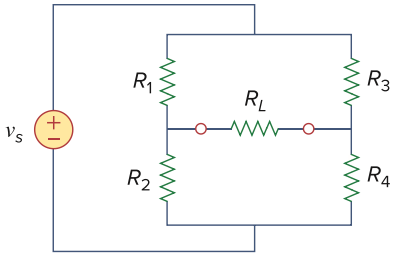


**Figure 4.140**
For Prob. 4.74.

***4.75** For the circuit in Fig. 4.141, determine the value of $R$ such that the maximum power delivered to the load is 12 mW.

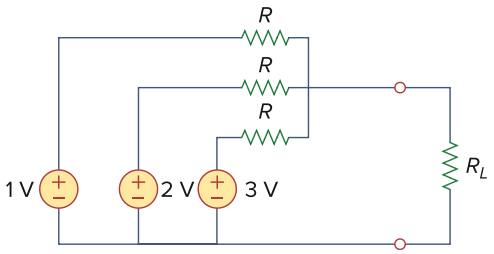


**Figure 4.141**
For Prob. 4.75.

## Section 4.9 Verifying Circuit Theorems with *PSpice*

**4.76** Solve Prob. 4.34 using *PSpice or MultiSim*. Let $V = 40$ V, $I = 3$ A, $R_1 = 10\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 20\ \Omega$.

**4.77** Use *PSpice or MultiSim* to solve Prob. 4.44.

**4.78** Use *PSpice or MultiSim* to solve Prob. 4.52.

**4.79** Obtain the Thevenin equivalent of the circuit in Fig. 4.123 using *PSpice or MultiSim*.

**4.80** Use *PSpice or MultiSim* to find the Thevenin equivalent circuit at terminals $a$-$b$ of the circuit in Fig. 4.125.

**4.81** For the circuit in Fig. 4.126, use *PSpice or MultiSim* to find the Thevenin equivalent at terminals $a$-$b$.

## Section 4.10 Applications

**4.82** An automobile battery has an open circuit voltage of 14.7 V which drops to 12 V when connected to two 65-W headlights. What is the resistance of the headlights and the value of the internal resistance of the battery?

**4.83** The following results were obtained from measurements taken between the two terminals of a resistive network.

| | | |
|---|---|---|
| Terminal Voltage | 72 V | 0 V |
| Terminal Current | 0 A | 9 A |

Find the Thevenin equivalent of the network.

**4.84** When connected to a 4- Ω resistor, a battery has a terminal voltage of 10.8 V but produces 12 V on an open circuit. Determine the Thevenin equivalent circuit for the battery.

**4.85** The Thevenin equivalent at terminals $a$-$b$ of the linear network shown in Fig. 4.142 is to be determined by measurement. When a 10-kΩ resistor is connected to terminals $a$-$b$, the voltage $V_{ab}$ is measured as 20 V. When a 30-kΩ resistor is connected to the terminals, $V_{ab}$ is measured as 40 V. Determine: (a) the Thevenin equivalent at terminals $a$-$b$, (b) $V_{ab}$ when a 20-kΩ resistor is connected to terminals $a$-$b$.

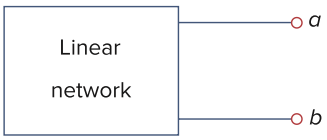


**Figure 4.142**
For Prob. 4.85.

**4.86** A black box with a circuit in it is connected to a variable resistor. An ideal ammeter (with zero resistance) and an ideal voltmeter (with infinite resistance) are used to measure current and voltage as shown in Fig. 4.143. The results are shown in the table on the next page.

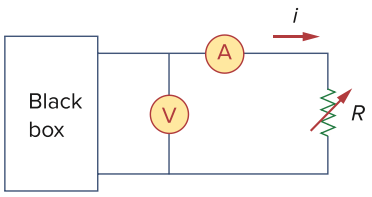


**Figure 4.143**
For Prob. 4.86.

(a) Find $i$ when $R = 12\ \Omega$.

(b) Determine the maximum power from the box.

| $R(\Omega)$ | $V(\text{V})$ | $i(\text{A})$ |
|---|---|---|
| 2 | 6 | 3 |
| 8 | 16 | 2 |
| 14 | 21 | 1.5 |

**4.87** A transducer is modeled with a current source $I_s$ and a parallel resistance $R_s$. The current at the terminals of the source is measured to be 9.975 mA when an ammeter with an internal resistance of 20 Ω is used.

(a) If adding a 2-kΩ resistor across the source terminals causes the ammeter reading to fall to 9.876 mA, calculate $I_s$ and $R_s$.

(b) What will the ammeter reading be if the resistance between the source terminals is changed to 4 kΩ?

**4.88** Consider the circuit in Fig. 4.144. An ammeter with internal resistance $R_i$ is inserted between $A$ and $B$ to measure $I_o$. Determine the reading of the ammeter if: (a) $R_i = 500\ \Omega$, (b) $R_i = 0\ \Omega$. (*Hint:* Find the Thevenin equivalent circuit at terminals $a$-$b$.)

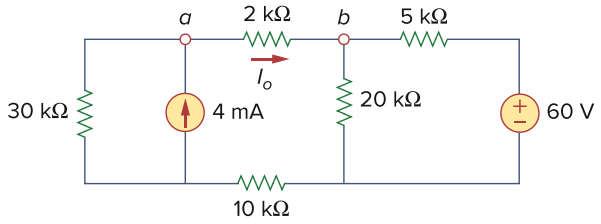


**Figure 4.144**
For Prob. 4.88.

**4.89** Consider the circuit in Fig. 4.145. (a) Replace the resistor $R_L$ by a zero resistance ammeter and determine the ammeter reading. (b) To verify the reciprocity theorem, interchange the ammeter and the 12-V source and determine the ammeter reading again.

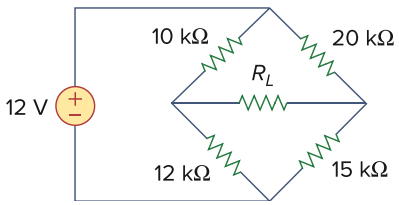


**Figure 4.145**
For Prob. 4.89.

**4.90** The Wheatstone bridge circuit shown in Fig. 4.146 is used to measure the resistance of a strain gauge. The adjustable resistor has a linear taper with a maximum value of 100 Ω. If the resistance of the strain gauge is found to be 42.6 Ω, what fraction of the full slider travel is the slider when the bridge is balanced?

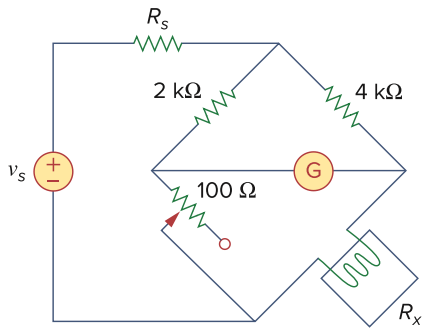


**Figure 4.146**
For Prob. 4.90.

**4.91** (a) In the Wheatstone bridge circuit of Fig. 4.147 select the values of $R_a$ and $R_b$ such that the bridge can measure $R_x$ in the range of 0–25 Ω.

(b) Repeat for the range of 0–250 Ω.

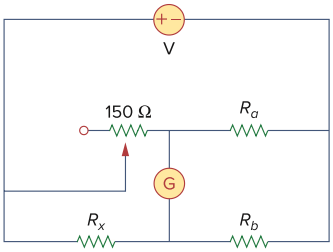


**Figure 4.147**
For Prob. 4.91.

***4.92** Consider the bridge circuit of Fig. 4.148. Is the bridge balanced? If the 10-kΩ resistor is replaced by an 18-kΩ resistor, what resistor connected between terminals $a$-$b$ absorbs the maximum power? What is this power?

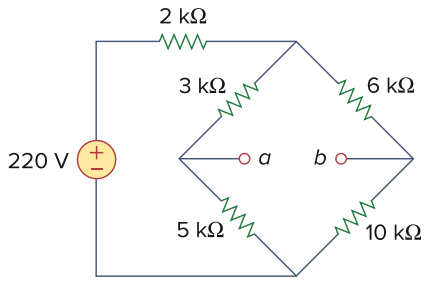


**Figure 4.148**
For Prob. 4.92.

# Comprehensive Problems

**4.93** The circuit in Fig. 4.149 models a common-emitter transistor amplifier. Find $i_x$ using source transformation.

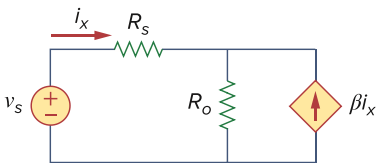


**Figure 4.149**
For Prob. 4.93.

**4.94** An attenuator is an interface circuit that reduces the voltage level without changing the output resistance.

(a) By specifying $R_s$ and $R_p$ of the interface circuit in Fig. 4.150, design an attenuator that will meet the following requirements:

$$\frac{V_o}{V_g} = 0.125, \qquad R_{eq} = R_{Th} = R_g = 100\ \Omega$$

(b) Using the interface designed in part (a), calculate the current through a load of $R_L = 50\ \Omega$ when $V_g = 12$ V.

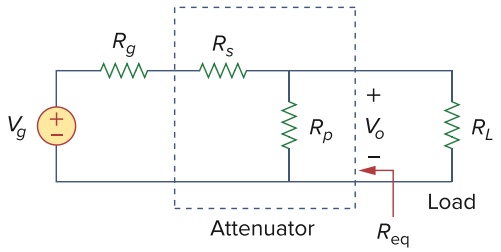


**Figure 4.150**
For Prob. 4.94.

***4.95** A dc voltmeter with a sensitivity of 10 kΩ/V is used to find the Thevenin equivalent of a linear network. Readings on two scales are as follows:

(a) 0–10 V scale: 8 V (b) 0–50 V scale: 10 V

Obtain the Thevenin voltage and the Thevenin resistance of the network.

***4.96** A resistance array is connected to a load resistor $R$ and a 9-V battery as shown in Fig. 4.151.

(a) Find the value of $R$ such that $V_o = 1.8$ V.

(b) Calculate the value of $R$ that will draw the maximum current. What is the maximum current?

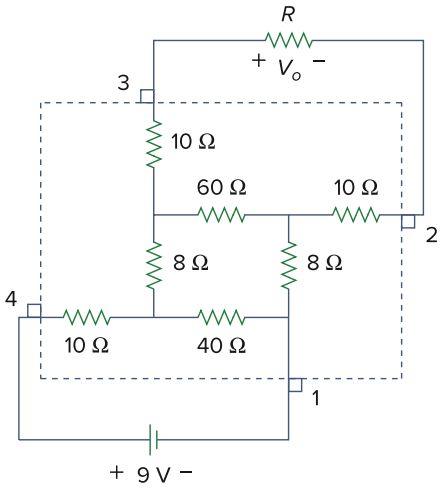


**Figure 4.151**
For Prob. 4.96.

**4.97** A common-emitter amplifier circuit is shown in Fig. 4.152. Obtain the Thevenin equivalent to the left of points $B$ and $E$.

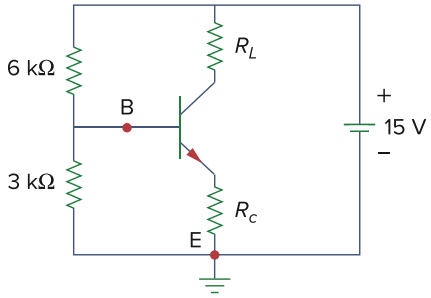


**Figure 4.152**
For Prob. 4.97.

***4.98** For Practice Prob. 4.18, determine the current through the 40-Ω resistor and the power dissipated by the resistor.

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chapter

# 6

# Capacitors and Inductors

*But in science the credit goes to the man who convinces the world, not to the man to whom the idea first occurs.*

—Francis Darwin

## Enhancing Your Skills and Your Career

### ABET EC 2000 criteria (3.c), *"an ability to design a system, component, or process to meet desired needs."*

The "ability to design a system, component, or process to meet desired needs" is why engineers are hired. That is why this is the most important *technical* skill that an engineer has. Interestingly your success as an engineer is directly proportional to your ability to communicate but your being able to design is wh y you will be hired in the first place.

Design takes place when you ha ve what is termed an open-ended problem that e ventually is defined by the solution. Within the conte xt of this course or te xtbook, we can only e xplore some of the elements of design. Pursuing all of the steps of our problem-solving technique teaches you se veral of the most important elements of the design process.

Probably the most important part of design is clearly defining what the system, component, process, or, in our case, problem is. Rarely is an engineer given a perfectly clear assignment. Therefore, as a student, you can develop and enhance this skill by asking yourself, your colleagues, or your professors questions designed to clarify the problem statement.

Exploring alternati ve solutions is another important part of the design process. Again, as a student, you can practice this part of the design process on almost every problem you work.

Evaluating your solutions is critical to an y engineering assignment. Again, this is a skill that you as a student can practice on e very problem you work.

Photo by Charles Alexander

## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Fully understand the volt-amp characteristics of capacitors and inductors and their use in basic circuits.
2. Explain how capacitors behave when combined in parallel and in series.
3. Understand how inductors behave when combined in parallel and in series.
4. Know how to create integrators using capacitors and op amps.
5. Learn how to create differentiators and their limitations.
6. Learn how to create analog computers and to understand how they can be used to solve linear differential equations.

In contrast to a resistor, which spends or dissipates energy irreversibly, an inductor or capacitor stores or releases energy (i.e., has a memory).

## 6.1 Introduction

So far we have limited our study to resistive circuits. In this chapter, we shall introduce two new and important passive linear circuit elements: the capacitor and the inductor. Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements.

The application of resistive circuits is quite limited. With the introduction of capacitors and inductors in this chapter, we will be able to analyze more important and practical circuits. Be assured that the circuit analysis techniques covered in Chapters 3 and 4 are equally applicable to circuits with capacitors and inductors.

We begin by introducing capacitors and describing how to combine them in series or in parallel. Later, we do the same for inductors. As typical applications, we explore how capacitors are combined with op amps to form integrators, differentiators, and analog computers.

## 6.2 Capacitors

A capacitor is a passive element designed to store energy in its electric field. Besides resistors, capacitors are the most common electrical components. Capacitors are used extensively in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.

A capacitor is typically constructed as depicted in Fig. 6.1.

A capacitor consists of two conducting plates separated by an insulator (or dielectric).

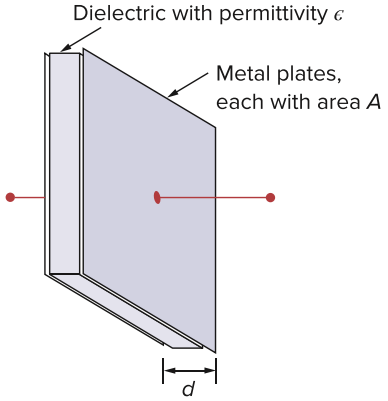


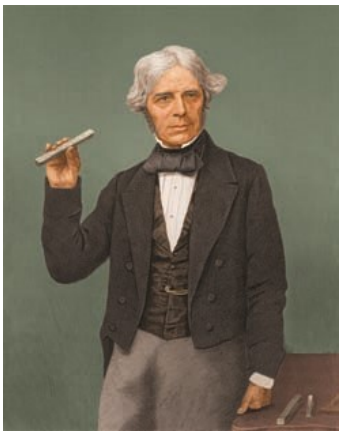

**Figure 6.1**
A typical capacitor.

In many practical applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica.

## Historical

**Michael Faraday** (1791–1867), an English chemist and physicist, was probably the greatest experimentalist who ever lived.

Born near London, Faraday realized his boyhood dream by working with the great chemist Sir Humphry Davy at the Royal Institution, where he worked for 54 years. He made several contributions in all areas of physical science and coined such words as electrolysis, anode, and cathode. His discovery of electromagnetic induction in 1831 was a major breakthrough in engineering because it provided a way of generating electricity. The electric motor and generator operate on this principle. The unit of capacitance, the farad, was named in his honor.



© Stock Montage/Getty Images

When a voltage source $v$ is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge $q$ on one plate and a negative charge $-q$ on the other. The capacitor is said to store the electric charge. The amount of charge stored, represented by $q$, is directly proportional to the applied voltage $v$ so that

$$q = Cv \tag{6.1}$$

where $C$, the constant of proportionality, is known as the *capacitance* of the capacitor. The unit of capacitance is the farad (F), in honor of the English physicist Michael Faraday (1791–1867). From Eq. (6.1), we may derive the following definition.

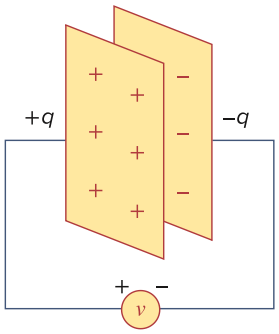



**Figure 6.2**
A capacitor with applied voltage $v$.

**Capacitance** is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).

Alternatively, capacitance is the amount of charge stored per plate for a unit voltage difference in a capacitor.

Note from Eq. (6.1) that 1 farad = 1 coulomb/volt.

Although the capacitance $C$ of a capacitor is the ratio of the charge $q$ per plate to the applied voltage $v$, it does not depend on $q$ or $v$. It depends on the physical dimensions of the capacitor. For example, for the parallel-plate capacitor shown in Fig. 6.1, the capacitance is given by

$$C = \frac{\epsilon A}{d} \tag{6.2}$$

where $A$ is the surface area of each plate, $d$ is the distance between the plates, and $\epsilon$ is the permittivity of the dielectric material between the plates. Although Eq. (6.2) applies to only parallel-plate capacitors, we may infer from it that, in general, three factors determine the value of the capacitance:

1. The surface area of the plates—the larger the area, the greater the capacitance.
2. The spacing between the plates—the smaller the spacing, the greater the capacitance.

Capacitor voltage rating and capacitance are typically inversely rated due to the relationships in Eqs. (6.1) and (6.2). Arcing occurs if $d$ is small and $V$ is high.

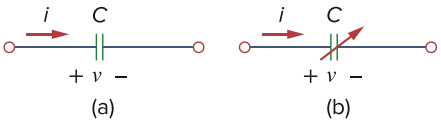



**Figure 6.3**
Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

3. The permittivity of the material—the higher the permittivity, the greater the capacitance.

Capacitors are commercially available in different values and types. Typically, capacitors have values in the picofarad (pF) to microfarad ($\mu$F) range. They are described by the dielectric material they are made of and by whether they are of fixed or variable type. Figure 6.3 shows the circuit symbols for fixed and variable capacitors. Note that according to the passive sign convention, if $v > 0$ and $i > 0$ or if $v < 0$ and $i < 0$, the capacitor is being charged, and if $v \cdot i < 0$, the capacitor is discharging.

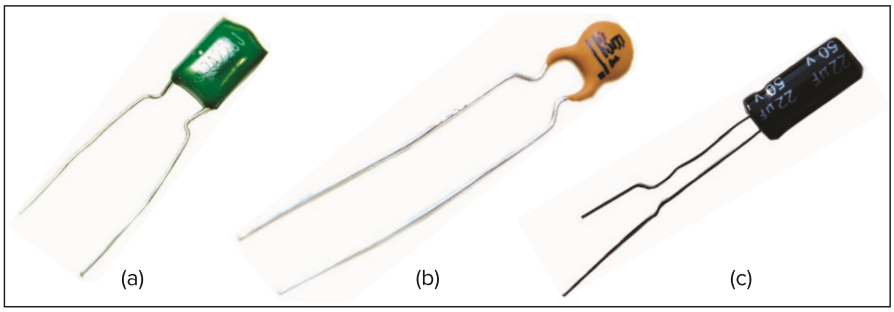



**Figure 6.4**
Fixed capacitors: (a) polyester capacitor, (b) ceramic capacitor, (c) electrolytic capacitor.
© McGraw-Hill Education/Mark Dierker, photographer

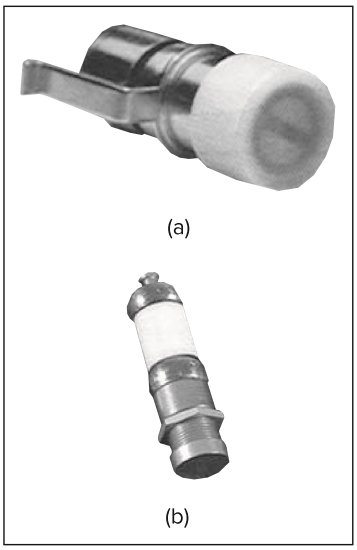



**Figure 6.5**
Variable capacitors: (a) trimmer capacitor, (b) filmtrim capacitor.
Courtesy of Johanson.

Figure 6.4 shows common types of fixed-value capacitors. Polyester capacitors are light in weight, stable, and their change with temperature is predictable. Instead of polyester, other dielectric materials such as mica and polystyrene may be used. Film capacitors are rolled and housed in metal or plastic films. Electrolytic capacitors produce very high capacitance. Figure 6.5 shows the most common types of variable capacitors. The capacitance of a trimmer (or padder) capacitor is often placed in parallel with another capacitor so that the equivalent capacitance can be varied slightly. The capacitance of the variable air capacitor (meshed plates) is varied by turning the shaft. Variable capacitors are used in radio receivers allowing one to tune to various stations. In addition, capacitors are used to block dc, pass ac, shift phase, store energy, start motors, and suppress noise.

To obtain the current-voltage relationship of the capacitor, we take the derivative of both sides of Eq. (6.1). Since

$$i = \frac{dq}{dt} \tag{6.3}$$

differentiating both sides of Eq. (6.1) gives

$$\boxed{i = C\frac{dv}{dt}} \tag{6.4}$$

This is the current-voltage relationship for a capacitor, assuming the passive sign convention. The relationship is illustrated in Fig. 6.6 for a capacitor whose capacitance is independent of voltage. Capacitors that satisfy Eq. (6.4) are said to be *linear*. For a *nonlinear capacitor*, the plot of the current-voltage relationship is not a straight line. Although some

According to Eq. (6.4), for a capacitor to carry current, its voltage must vary with time. Hence, for constant voltage, $i = 0$.

capacitors are nonlinear, most are linear. We will assume linear capacitors in this book.

The voltage-current relation of the capacitor can be obtained by integrating both sides of Eq. (6.4). We get

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau \tag{6.5}$$

or

$$\boxed{v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\, d\tau + v(t_0)} \tag{6.6}$$

where $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at time $t_0$. Equation (6.6) shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.

The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt} \tag{6.7}$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p(\tau)\, d\tau = C\int_{-\infty}^{t} v\frac{dv}{d\tau}\, d\tau = C\int_{v(-\infty)}^{v(t)} v\, dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)} \tag{6.8}$$

We note that $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$. Thus,

$$\boxed{w = \frac{1}{2}Cv^2} \tag{6.9}$$

Using Eq. (6.1), we may rewrite Eq. (6.9) as

$$w = \frac{q^2}{2C} \tag{6.10}$$

Equation (6.9) or (6.10) represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy. In fact, the word *capacitor* is derived from this element's capacity to store energy in an electric field.

We should note the following important properties of a capacitor:

1. Note from Eq. (6.4) that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus,

   A capacitor is an open circuit to dc.

   However, if a battery (dc voltage) is connected across a capacitor, the capacitor charges.

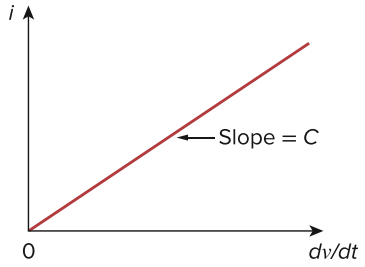



**Figure 6.6**
Current-voltage relationship of a capacitor.

An alternative way of looking at this is using Eq. (6.9), which indicates that energy is proportional to voltage squared. Since injecting or extracting energy can only be done over some finite time, voltage cannot change instantaneously across a capacitor.

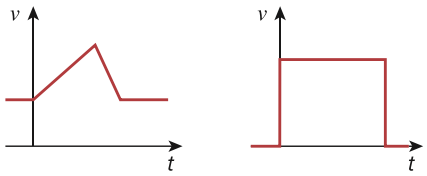



**Figure 6.7**
Voltage across a capacitor: (a) allowed, (b) not allowable; an abrupt change is not possible.

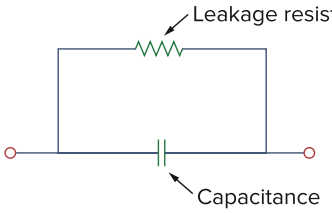



**Figure 6.8**
Circuit model of a nonideal capacitor.

2. The voltage on the capacitor must be continuous.

> The voltage on a capacitor cannot change abruptly.

The capacitor resists an abrupt change in the voltage across it. According to Eq. (6.4), a discontinuous change in voltage requires an infinite current, which is physically impossible. For example, the voltage across a capacitor may take the form shown in Fig. 6.7(a), whereas it is not physically possible for the capacitor voltage to take the form shown in Fig. 6.7(b) because of the abrupt changes. Conversely, the current through a capacitor can change instantaneously.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
4. A real, nonideal capacitor has a parallel-model leakage resistance, as shown in Fig. 6.8. The leakage resistance may be as high as 100 MΩ and can be neglected for most practical applications. For this reason, we will assume ideal capacitors in this book.

## Example 6.1

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it. (b) Find the energy stored in the capacitor.

**Solution:**

(a) Since $q = Cv$,

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

(b) The energy stored is

$$w = \frac{1}{2} Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

## Practice Problem 6.1

What is the voltage across a 4.5-$\mu$F capacitor if the charge on one plate is 0.12 mC? How much energy is stored?

**Answer:** 26.67 V, 1.6 mJ.

## Example 6.2

The voltage across a 5-$\mu$F capacitor is

$$v(t) = 10 \cos 6000t \text{ V}$$

Calculate the current through it.

**Solution:**
By definition, the current is

$$i(t) = C \frac{dv}{dt} = 5 \times 10^{-6} \frac{d}{dt}(10 \cos 6000t)$$

$$= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A}$$

## Practice Problem 6.2

If a 10-$\mu$F capacitor is connected to a voltage source with

$$v(t) = 20\cos(200t)\ \text{V}$$

determine the current through the capacitor.

**Answer:** $-40\sin(200t)$ mA.

## Example 6.3

Determine the voltage across a 2-$\mu$F capacitor if the current through it is

$$i(t) = 6e^{-3000t}\ \text{mA}$$

Assume that the initial capacitor voltage is zero.

**Solution:**

Since $v = \frac{1}{C}\int_0^t i\,d\tau + v(0)$ and $v(0) = 0$,

$$v = \frac{1}{2\times 10^{-6}}\int_0^t 6e^{-3000\tau}\,d\tau \cdot 10^{-3}$$

$$= \frac{3\times 10^{3}}{-3000}e^{-3000\tau}\Big|_0^t = (1 - e^{-3000t})\ \text{V}$$

## Practice Problem 6.3

The current through a 100-$\mu$F capacitor is $i(t) = 50\sin 120\pi t$ mA. Calculate the voltage across it at $t = 1$ ms and $t = 5$ ms. Take $v(0) = 0$.

**Answer:** 93.14 mV, 1.736 V.

## Example 6.4

Determine the current through a 200- $\mu$F capacitor whose voltage is shown in Fig. 6.9.

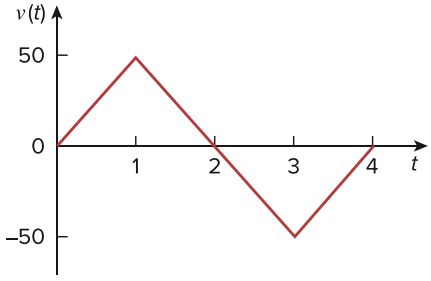



**Figure 6.9**
For Example 6.4.

**Solution:**
The voltage waveform can be described mathematically as

$$v(t) = \begin{cases} 50t\ \text{V} & 0 < t < 1 \\ 100 - 50t\ \text{V} & 1 < t < 3 \\ -200 + 50t\ \text{V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Since $i = C\,dv/dt$ and $C = 200\ \mu$F, we take the derivative of $v$ to obtain

$$i(t) = 200\times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 10\ \text{mA} & 0 < t < 1 \\ -10\ \text{mA} & 1 < t < 3 \\ 10\ \text{mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the current waveform is as shown in Fig. 6.10.

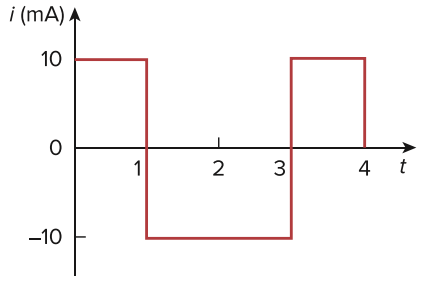



**Figure 6.10**
For Example 6.4.

## Practice Problem 6.4

An initially uncharged 1-mF capacitor has the current shown in Fig. 6.11 across it. Calculate the voltage across it at $t = 2$ ms and $t = 5$ ms.

**Answer:** 100 mV, 400 mV.

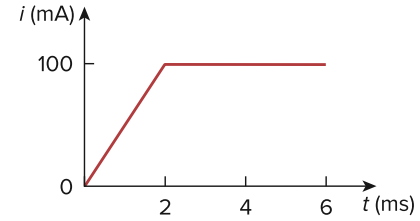



**Figure 6.11**
For Practice Prob. 6.4.

## Example 6.5

Obtain the energy stored in each capacitor in Fig. 6.12(a) under dc conditions.

**Solution:**
Under dc conditions, we replace each capacitor with an open circuit, as shown in Fig. 6.12(b). The current through the series combination of the 2-kΩ and 4-kΩ resistors is obtained by current division as

$$i = \frac{3}{3+2+4}(6\text{ mA}) = 2\text{ mA}$$

Hence, the voltages $v_1$ and $v_2$ across the capacitors are

$$v_1 = 2000i = 4\text{ V} \qquad v_2 = 4000i = 8\text{ V}$$

and the energies stored in them are

$$w_1 = \frac{1}{2}C_1v_1^2 = \frac{1}{2}(2\times10^{-3})(4)^2 = 16\text{ mJ}$$

$$w_2 = \frac{1}{2}C_2v_2^2 = \frac{1}{2}(4\times10^{-3})(8)^2 = 128\text{ mJ}$$

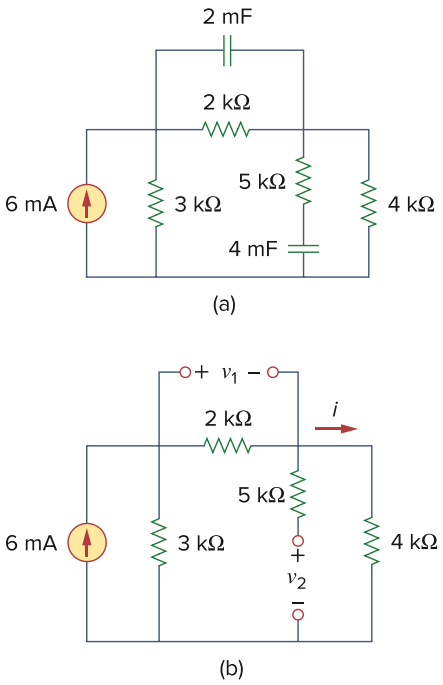



**Figure 6.12**
For Example 6.5.

## Practice Problem 6.5

Under dc conditions, find the energy stored in the capacitors in Fig. 6.13.

**Answer:** 20.25 mJ, 3.375 mJ.

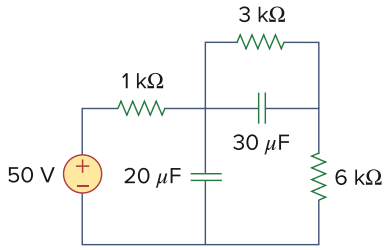



**Figure 6.13**
For Practice Prob. 6.5.

## 6.3 Series and Parallel Capacitors

We know from resistive circuits that the series-parallel combination is a powerful tool for reducing circuits. This technique can be extended to series-parallel connections of capacitors, which are sometimes encountered. We desire to replace these capacitors by a single equivalent capacitor $C_{eq}$.

In order to obtain the equivalent capacitor $C_{eq}$ of $N$ capacitors in parallel, consider the circuit in Fig. 6.14(a). The equivalent circuit is

in Fig. 6.14(b). Note that the capacitors have the same voltage $v$ across them. Applying KCL to Fig. 6.14(a),

$$i = i_1 + i_2 + i_3 + \cdots + i_N \tag{6.11}$$

But $i_k = C_k\, dv/dt$. Hence,

$$\begin{aligned} i &= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt} \\ &= \left( \sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = C_{\text{eq}} \frac{dv}{dt} \end{aligned} \tag{6.12}$$

where

$$C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots + C_N \tag{6.13}$$

The **equivalent capacitance** of $N$ parallel-connected capacitors is the sum of the individual capacitances.

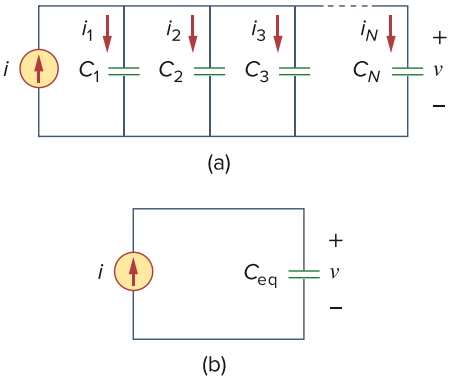



**Figure 6.14**
(a) Parallel-connected $N$ capacitors, (b) equivalent circuit for the parallel capacitors.

We observe that capacitors in parallel combine in the same manner as resistors in series.

We now obtain $C_{\text{eq}}$ of $N$ capacitors connected in series by comparing the circuit in Fig. 6.15(a) with the equivalent circuit in Fig. 6.15(b). Note that the same current $i$ flows (and consequently the same charge) through the capacitors. Applying KVL to the loop in Fig. 6.15(a),

$$v = v_1 + v_2 + v_3 + \cdots + v_N \tag{6.14}$$

But $v_k = \dfrac{1}{C_k} \displaystyle\int_{t_0}^{t} i(\tau)\, d\tau + v_k(t_0)$. Therefore,

$$\begin{aligned} v &= \frac{1}{C_1} \int_{t_0}^{t} i(\tau)\, d\tau + v_1(t_0) + \frac{1}{C_2} \int_{t_0}^{t} i(\tau)\, d\tau + v_2(t_0) \\ &\quad + \cdots + \frac{1}{C_N} \int_{t_0}^{t} i(\tau)\, d\tau + v_N(t_0) \\ &= \left( \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} \right) \int_{t_0}^{t} i(\tau)\, d\tau + v_1(t_0) + v_2(t_0) \\ &\quad + \cdots + v_N(t_0) \\ &= \frac{1}{C_{\text{eq}}} \int_{t_0}^{t} i(\tau)\, d\tau + v(t_0) \end{aligned} \tag{6.15}$$

where

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N} \tag{6.16}$$

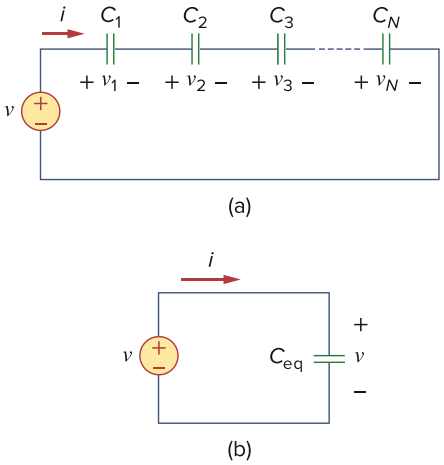



**Figure 6.15**
(a) Series-connected $N$ capacitors, (b) equivalent circuit for the series capacitor.

The initial voltage $v(t_0)$ across $C_{eq}$ is required by KVL to be the sum of the capacitor voltages at $t_0$. Or according to Eq. (6.15),

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

Thus, according to Eq. (6.16),

> The **equivalent capacitance** of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

Note that capacitors in series combine in the same manner as resistors in parallel. For $N = 2$ (i.e., two capacitors in series), Eq. (6.16) becomes

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$\boxed{C_{eq} = \frac{C_1 C_2}{C_1 + C_2}} \tag{6.17}$$

## Example 6.6

Find the equi valent capacitance seen between terminals $a$ and $b$ of the circuit in Fig. 6.16.

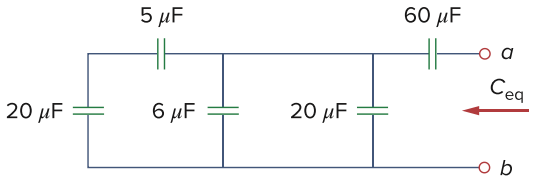



**Figure 6.16**
For Example 6.6.

**Solution:**
The 20-$\mu$F and 5-$\mu$F capacitors are in series; their equivalent capaci - tance is

$$\frac{20 \times 5}{20 + 5} = 4\ \mu\text{F}$$

This 4-$\mu$F capacitor is in parallel with the 6- $\mu$F and 20-$\mu$F capacitors; their combined capacitance is

$$4 + 6 + 20 = 30\ \mu\text{F}$$

This 30-$\mu$F capacitor is in series with the 60- $\mu$F capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{eq} = \frac{30 \times 60}{30 + 60} = 20\ \mu\text{F}$$

## Practice Problem 6.6

Find the equivalent capacitance seen at the terminals of the circuit in Fig. 6.17.

**Answer:** 40 $\mu$F.

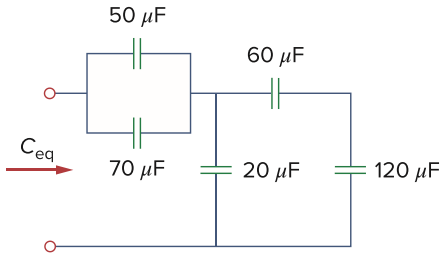



**Figure 6.17**
For Practice Prob. 6.6.

## Example 6.7

For the circuit in Fig. 6.18, find the voltage across each capacitor.

**Solution:**
We first find the equivalent capacitance $C_{eq}$, shown in Fig. 6.19. The two parallel capacitors in Fig. 6.18 can be combined to get 40 + 20 = 60 mF. This 60-mF capacitor is in series with the 20-mF and 30-mF capacitors. Thus,

$$C_{eq} = \frac{1}{\frac{1}{60} + \frac{1}{30} + \frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

The total charge is

$$q = C_{eq} v = 10 \times 10^{-3} \times 30 = 0.3 \text{ C}$$

This is the charge on the 20-mF and 30-mF capacitors, because they are in series with the 30-V source. (A crude way to see this is to imagine that charge acts like current, since $i = dq/dt$.) Therefore,

$$v_1 = \frac{q}{C_1} = \frac{0.3}{20 \times 10^{-3}} = 15 \text{ V} \qquad v_2 = \frac{q}{C_2} = \frac{0.3}{30 \times 10^{-3}} = 10 \text{ V}$$

Having determined $v_1$ and $v_2$, we now use KVL to determine $v_3$ by

$$v_3 = 30 - v_1 - v_2 = 5 \text{ V}$$

Alternatively, since the 40-mF and 20-mF capacitors are in parallel, they have the same voltage $v_3$ and their combined capacitance is 40 + 20 = 60 mF. This combined capacitance is in series with the 20-mF and 30-mF capacitors and consequently has the same charge on it. Hence,

$$v_3 = \frac{q}{60 \text{ mF}} = \frac{0.3}{60 \times 10^{-3}} = 5 \text{ V}$$

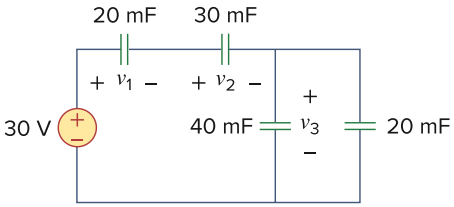



**Figure 6.18**
For Example 6.7.

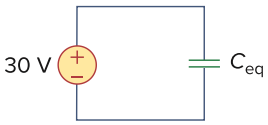



**Figure 6.19**
Equivalent circuit for Fig. 6.18.

## Practice Problem 6.7

Find the voltage across each of the capacitors in Fig. 6.20.

**Answer:** $v_1 = 75$ V, $v_2 = 75$ V, $v_3 = 25$ V, $v_4 = 50$ V.

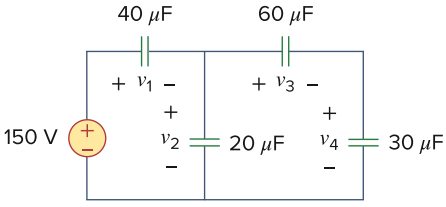



**Figure 6.20**
For Practice Prob. 6.7.

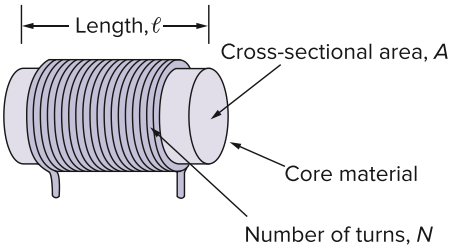



**Figure 6.21**
Typical form of an inductor.

## 6.4 Inductors

An inductor is a passive element designed to store energy in its magnetic field. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

Any conductor of electric current has inductive properties and may be regarded as an inductor. But in order to enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown in Fig. 6.21.

An **inductor** consists of a coil of conducting wire.

If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention,

$$v = L\frac{di}{dt} \tag{6.18}$$

In view of Eq. (6.18), for an inductor to have voltage across its terminals, its current must vary with time. Hence, $v = 0$ for constant current through the inductor.

where $L$ is the constant of proportionality called the *inductance* of the inductor. The unit of inductance is the henry (H), named in honor of the American inventor Joseph Henry (1797–1878). It is clear from Eq. (6.18) that 1 henry equals 1 volt-second per ampere.

**Inductance** is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).

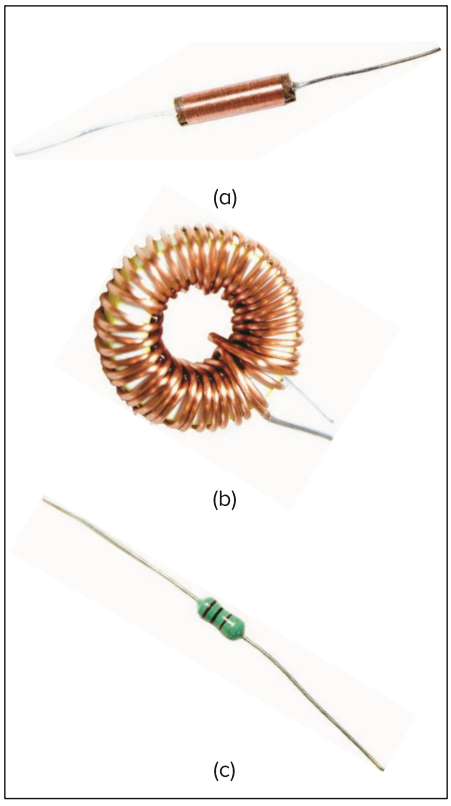



**Figure 6.22**
Various types of inductors: (a) solenoidal wound inductor, (b) toroidal inductor, (c) axial lead inductor.
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The inductance of an inductor depends on its physical dimension and construction. Formulas for calculating the inductance of inductors of different shapes are derived from electromagnetic theory and can be found in standard electrical engineering handbooks. For example, for the inductor, (solenoid) shown in Fig. 6.21,

$$L = \frac{N^2 \mu A}{\ell} \tag{6.19}$$

where $N$ is the number of turns, $\ell$ is the length, A is the cross-sectional area, and $\mu$ is the permeability of the core. We can see from Eq. (6.19) that inductance can be increased by increasing the number of turns of coil, using material with higher permeability as the core, increasing the cross-sectional area, or reducing the length of the coil.

Like capacitors, commercially available inductors come in different values and types. Typical practical inductors have inductance values ranging from a few microhenrys ($\mu$H), as in communication systems, to tens of henrys (H) as in power systems. Inductors may be fixed or variable. The core may be made of iron, steel, plastic, or air. The terms *coil* and *choke* are also used for inductors. Common inductors are shown in Fig. 6.22. The circuit symbols for inductors are shown in Fig. 6.23, following the passive sign convention.

Equation (6.18) is the voltage-current relationship for an inductor. Figure 6.24 shows this relationship graphically for an inductor whose

## Historical

**Joseph Henry** (1797–1878), an American physicist, discovered inductance and constructed an electric motor.

Born in Albany, New York, Henry graduated from Albany Academy and taught philosophy at Princeton University from 1832 to 1846. He was the first secretary of the Smithsonian Institution. He conducted several experiments on electromagnetism and developed powerful electromagnets that could lift objects weighing thousands of pounds. Interestingly, Joseph Henry discovered electromagnetic induction before Faraday but failed to publish his findings. The unit of inductance, the henry, was named after him.

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inductance is independent of current. Such an inductor is known as a *linear inductor*. For a *nonlinear inductor,* the plot of Eq. (6.18) will not be a straight line because its inductance varies with current. We will assume linear inductors in this textbook unless stated otherwise.

The current-voltage relationship is obtained from Eq. (6.18) as

$$di = \frac{1}{L} v\, dt$$

Integrating gives

$$i = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\, d\tau \tag{6.20}$$

or

$$\boxed{i = \frac{1}{L}\int_{t_0}^{t} v(\tau) d\tau + i(t_0)} \tag{6.21}$$

where $i(t_0)$ is the total current for $-\infty < t < t_0$ and $i(-\infty) = 0$. The idea of making $i(-\infty) = 0$ is practical and reasonable, because there must be a time in the past when there was no current in the inductor.

The inductor is designed to store energy in its magnetic field. The energy stored can be obtained from Eq. (6.18). The power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right) i \tag{6.22}$$

The energy stored is

$$\begin{aligned} w &= \int_{-\infty}^{t} p(\tau)\, d\tau = L\int_{-\infty}^{t} \frac{di}{d\tau}\, i\, d\tau \\ &= L\int_{-\infty}^{t} i\, di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty) \end{aligned} \tag{6.23}$$

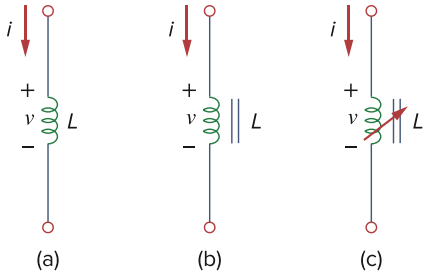



**Figure 6.23**
Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.

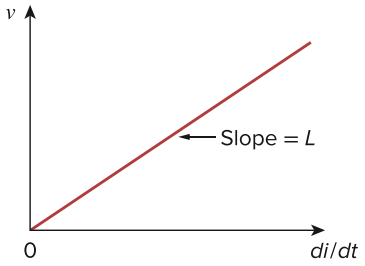



**Figure 6.24**
Voltage-current relationship of an inductor.

Since $i(-\infty) = 0$,

$$w = \frac{1}{2}Li^2 \tag{6.24}$$

We should note the following important properties of an inductor.

1. Note from Eq. (6.18) that the voltage across an inductor is zero when the current is constant. Thus,

An inductor acts like a short circuit to dc.

2. An important property of the inductor is its opposition to the change in current flowing through it.

The current through an inductor cannot change instantaneously.

According to Eq. (6.18), a discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible. Thus, an inductor opposes an abrupt change in the current through it. For example, the current through an inductor may take the form shown in Fig. 6.25(a), whereas the inductor current cannot take the form shown in Fig. 6.25(b) in real-life situations due to the discontinuities. However, the voltage across an inductor can change abruptly.

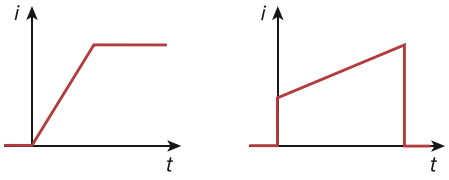



**Figure 6.25**
Current through an inductor: (a) allowed, (b) not allowable; an abrupt change is not possible.

3. Like the ideal capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.

Since an inductor is often made of a highly conducting wire, it has a very small resistance.

4. A practical, nonideal inductor has a significant resistive component, as shown in Fig. 6.26. This is due to the fact that the inductor is made of a conducting material such as copper, which has some resistance. This resistance is called the *winding resistance* $R_w$, and it appears in series with the inductance of the inductor. The presence of $R_w$ makes it both an energy storage device and an energy dissipation device. Since $R_w$ is usually very small, it is ignored in most cases. The nonideal inductor also has a *winding capacitance* $C_w$ due to the capacitive coupling between the conducting coils. $C_w$ is very small and can be ignored in most cases, except at high frequencies. We will assume ideal inductors in this book.

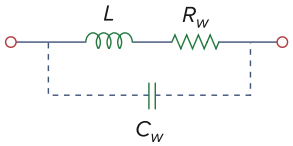



**Figure 6.26**
Circuit model for a practical inductor.

## Example 6.8

The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

**Solution:**
Since $v = L\,di/dt$ and $L = 0.1$ H,

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1 - 5t)\ \text{V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}\ \text{J}$$

**Practice Problem 6.8**

If the current through a 1-mH inductor is $i(t) = 90\sin(200t)$ mA, find the terminal voltage and the energy stored.

**Answer:** $18\cos(200t)$ mV, $4.05\sin^2(200t)\ \mu$J.

**Example 6.9**

Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also, find the energy stored at $t = 5$ s. Assume $i(v) > 0$.

**Solution:**

Since $i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)$ and $L = 5$ H,

$$i = \frac{1}{5}\int_0^t 30\tau^2\,d\tau + 0 = 6 \times \frac{t^3}{3} = 2t^3\ \text{A}$$

The power $p = vi = 60t^5$, and the energy stored is then

$$w = \int p\,dt = \int_0^5 60t^5\,dt = 60\frac{t^6}{6}\Big|_0^5 = 156.25\ \text{kJ}$$

Alternatively, we can obtain the energy stored using Eq. (6.24), by writing

$$w\Big|_0^5 = \frac{1}{2}Li^2(5) - \frac{1}{2}Li(0) = \frac{1}{2}(5)(2 \times 5^3)^2 - 0 = 156.25\ \text{kJ}$$

as obtained before.

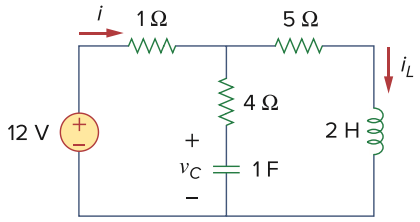
**Practice Problem 6.9**

The terminal voltage of a 2-H inductor is $v = 10(1 - t)$ V. Find the current flowing through it at $t = 4$ s and the energy stored in it at $t = 4$ s. Assume $i(0) = 2$ A.

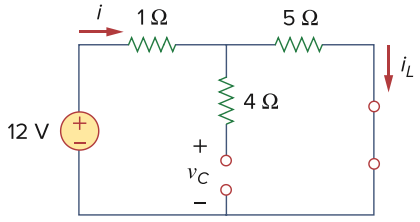
**Answer:** $-18$ A, 324 J.

## Example 6.10

Consider the circuit in Fig. 6.27(a). Under dc conditions, find: (a) $i$, $v_C$, and $i_L$, (b) the energy stored in the capacitor and inductor.




(a)




(b)

**Figure 6.27**
For Example 6.10.

**Solution:**

(a) Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit, as in Fig. 6.27(b). It is evident from Fig. 6.27(b) that

$$i = i_L = \frac{12}{1+5} = 2 \text{ A}$$

The voltage $v_C$ is the same as the voltage across the 5-Ω resistor. Hence,

$$v_C = 5i = 10 \text{ V}$$

(b) The energy in the capacitor is

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50 \text{ J}$$

and that in the inductor is

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4 \text{ J}$$

## Practice Problem 6.10

Determine $v_C$, $i_L$, and the energy stored in the capacitor and inductor in the circuit of Fig. 6.28 under dc conditions.

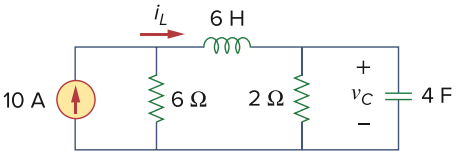



**Figure 6.28**
For Practice Prob. 6.10.

**Answer:** 15 V, 7.5 A, 450 J, 168.75 J.

## 6.5 Series and Parallel Inductors

Now that the inductor has been added to our list of passive elements, it is necessary to extend the powerful tool of series-parallel combination. We need to know how to find the equivalent inductance of a series-connected or parallel-connected set of inductors found in practical circuits.

Consider a series connection of $N$ inductors, as shown in Fig. 6.29(a), with the equivalent circuit shown in Fig. 6.29(b). The inductors have the same current through them. Applying KVL to the loop,

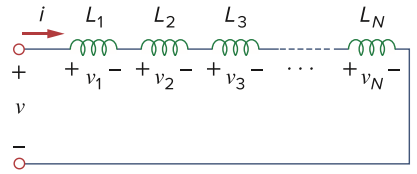
$$v = v_1 + v_2 + v_3 + \cdots + v_N \tag{6.25}$$

Substituting $v_k = L_k \, di/dt$ results in

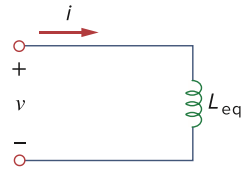
$$\begin{aligned} v &= L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} + \cdots + L_N\frac{di}{dt} \\ &= (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt} \\ &= \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{\text{eq}}\frac{di}{dt} \end{aligned} \tag{6.26}$$

where

$$L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N \tag{6.27}$$




(a)




(b)

**Figure 6.29**
(a) A series connection of $N$ inductors, (b) equivalent circuit for the series inductors.

Thus,

The **equivalent inductance** of series-connected inductors is the sum of the individual inductances.

Inductors in series are combined in exactly the same way as resistors in series.

We now consider a parallel connection of $N$ inductors, as shown in Fig. 6.30(a), with the equivalent circuit in Fig. 6.30(b). The inductors have the same voltage across them. Using KCL,

$$i = i_1 + i_2 + i_3 + \cdots + i_N \tag{6.28}$$

But $i_k = \frac{1}{L_k}\int_{t_0}^{t} v\,dt + i_k(t_0)$; hence,

$$\begin{aligned} i &= \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) \\ &\quad + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0) \\ &= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0) \\ &\quad + \cdots + i_N(t_0) \\ &= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{\text{eq}}}\int_{t_0}^{t} v\,dt + i(t_0) \end{aligned} \tag{6.29}$$

where

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N} \tag{6.30}$$

The initial current $i(t_0)$ through $L_{\text{eq}}$ at $t = t_0$ is expected by KCL to be the sum of the inductor currents at $t_0$. Thus, according to Eq. (6.29),

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

According to Eq. (6.30),

The **equivalent inductance** of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances.

Note that the inductors in parallel are combined in the same way as resistors in parallel.

For two inductors in parallel ($N = 2$), Eq. (6.30) becomes

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} \quad \text{or} \quad L_{\text{eq}} = \frac{L_1 L_2}{L_1 + L_2} \tag{6.31}$$

As long as all the elements are of the same type, the Δ-Y transformations for resistors discussed in Section 2.7 can be extended to capacitors and inductors.

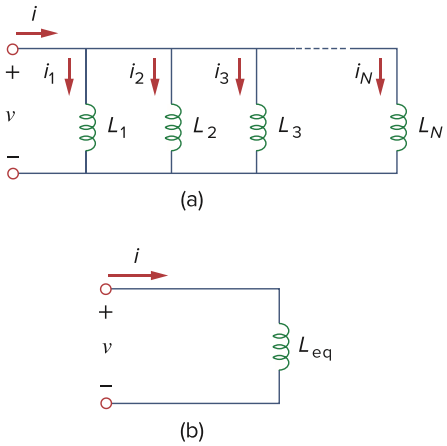



**Figure 6.30**
(a) A parallel connection of $N$ inductors, (b) equivalent circuit for the parallel inductors.

**TABLE 6.1**

Important characteristics of the basic elements.†

| Relation | Resistor ($R$) | Capacitor ($C$) | Inductor ($L$) |
|---|---|---|---|
| $v$-$i$: | $v = iR$ | $v = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$ | $v = L\frac{di}{dt}$ |
| $i$-$v$: | $i = v/R$ | $i = C\frac{dv}{dt}$ | $i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)$ |
| $p$ or $w$: | $p = i^2R = \frac{v^2}{R}$ | $w = \frac{1}{2}Cv^2$ | $w = \frac{1}{2}Li^2$ |
| Series: | $R_{eq} = R_1 + R_2$ | $C_{eq} = \frac{C_1C_2}{C_1 + C_2}$ | $L_{eq} = L_1 + L_2$ |
| Parallel: | $R_{eq} = \frac{R_1R_2}{R_1 + R_2}$ | $C_{eq} = C_1 + C_2$ | $L_{eq} = \frac{L_1L_2}{L_1 + L_2}$ |
| At dc: | Same | Open circuit | Short circuit |
| Circuit variable that cannot change abruptly: | Not applicable | $v$ | $i$ |

†*Passive sign convention is assumed.*

It is appropriate at this point to summarize the most important characteristics of the three basic circuit elements we have studied. The summary is given in Table 6.1.

## Example 6.11

Find the equivalent inductance of the circuit shown in Fig. 6.31.

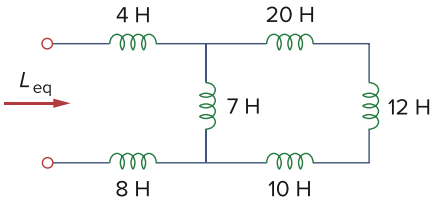



**Figure 6.31**
For Example 6.11.

**Solution:**
The 10-H, 12-H, and 20-H inductors are in series; thus, combining them gives a 42-H inductance. This 42-H inductor is in parallel with the 7-H inductor so that they are combined, to give

$$\frac{7 \times 42}{7 + 42} = 6 \text{ H}$$

This 6-H inductor is in series with the 4-H and 8-H inductors. Hence,

$$L_{eq} = 4 + 6 + 8 = 18 \text{ H}$$

## Practice Problem 6.11

Calculate the equivalent inductance for the inductive ladder network in Fig. 6.32.

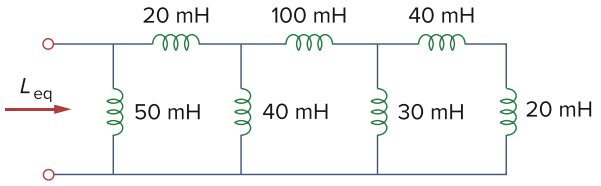



**Figure 6.32**
For Practice Prob. 6.11.

**Answer:** 25 mH.

## Example 6.12

For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.

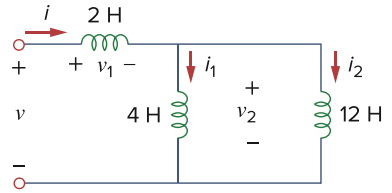



**Figure 6.33**
For Example 6.12.

**Solution:**

(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \| 12 = 2 + 3 = 5 \text{ H}$$

Thus,

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

and

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current $i_1$ is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2\, dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\, dt + 5 \text{ mA}$$

$$= -3e^{-10t}\Big|_0^t + 5 \text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA}$$

Similarly,

$$i_2(t) = \frac{1}{12}\int_0^t v_2\, dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\, dt - 1 \text{ mA}$$

$$= -e^{-10t}\Big|_0^t - 1 \text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA}$$

Note that $i_1(t) + i_2(t) = i(t)$.

## Practice Problem 6.12

In the circuit of Fig. 6.34, $i_1(t) = 3e^{-2t}$ A. If $i(0) = 7$ A, find: (a) $i_2(0)$; (b) $i_2(t)$ and $i(t)$; (c) $v_1(t)$, $v_2(t)$, and $v(t)$.

**Answer:** (a) 4 A, (b) $(-2 + 6e^{-2t})$ A, $(-2 + 9e^{-2t})$ A, (c) $-36e^{-2t}$ V, $-144e^{-2t}$ V, $-180e^{-2t}$ V.

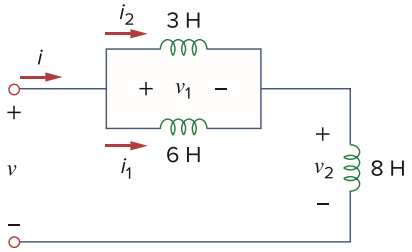



**Figure 6.34**
For Practice Prob. 6.12.

## 6.6 † Applications

Circuit elements such as resistors and capacitors are commercially available in either discrete form or integrated-circuit (IC) form. Unlike capacitors and resistors, inductors with appreciable inductance are difficult to produce on IC substrates. Therefore, inductors (coils) usually

come in discrete form and tend to be more bulky and expensive. For this reason, inductors are not as versatile as capacitors and resistors, and they are more limited in applications. However, there are several applications in which inductors have no practical substitute. They are routinely used in relays, delays, sensing devices, pick-up heads, telephone circuits, radio and TV receivers, power supplies, electric motors, microphones, and loudspeakers, to mention a few.

Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:

1. The capacity to store energy makes them useful as temporary voltage or current sources. Thus, they can be used for generating a large amount of current or voltage for a short period of time.
2. Capacitors oppose any abrupt change in voltage, while inductors oppose any abrupt change in current. This property makes inductors useful for spark or arc suppression and for converting pulsating dc voltage into relatively smooth dc voltage.
3. Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination.

The first two properties are put to use in dc circuits, while the third one is taken advantage of in ac circuits. We will see how useful these properties are in later chapters. For now, consider three applications involving capacitors and op amps: integrator, differentiator, and analog computer.

### 6.6.1 Integrator

Important op amp circuits that use energy-storage elements include integrators and differentiators. These op amp circuits often involve resistors and capacitors; inductors (coils) tend to be more bulky and expensive.

The op amp integrator is used in numerous applications, especially in analog computers, to be discussed in Section 6.6.3.

An **integrator** is an op amp circuit whose output is proportional to the integral of the input signal.

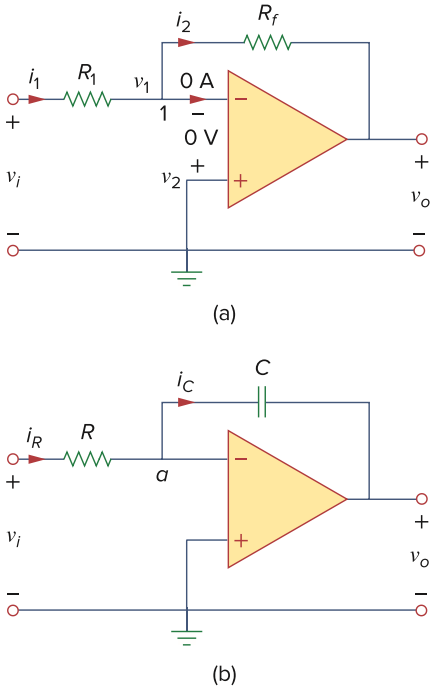



**Figure 6.35**
Replacing the feedback resistor in the inverting amplifier in (a) produces an integrator in (b).

If the feedback resistor $R_f$ in the familiar inverting amplifier of Fig. 6.35(a) is replaced by a capacitor, we obtain an ideal integrator, as shown in Fig. 6.35(b). It is interesting that we can obtain a mathematical representation of integration this way. At node $a$ in Fig. 6.35(b),

$$i_R = i_C \tag{6.32}$$

But

$$i_R = \frac{v_i}{R}, \qquad i_C = -C\frac{dv_o}{dt}$$

Substituting these in Eq. (6.32), we obtain

$$\frac{v_i}{R} = -C\frac{dv_o}{dt} \tag{6.33a}$$

$$dv_o = -\frac{1}{RC}v_i\,dt \tag{6.33b}$$

Integrating both sides gives

$$v_o(t) - v_o(0) = -\frac{1}{RC}\int_0^t v_i(\tau)\,d\tau \tag{6.34}$$

To ensure that $v_o(0) = 0$, it is always necessary to discharge the integrator's capacitor prior to the application of a signal. Assuming $v_o(0) = 0$,

$$\boxed{v_o = -\frac{1}{RC}\int_0^t v_i(\tau)\,d\tau} \tag{6.35}$$

which shows that the circuit in Fig. 6.35(b) provides an output voltage proportional to the integral of the input. In practice, the op amp integrator requires a feedback resistor to reduce dc gain and prevent saturation. Care must be taken that the op amp operates within the linear range so that it does not saturate.

## Example 6.13

If $v_1 = 10\cos 2t$ mV and $v_2 = 0.5t$ mV, find $v_o$ in the op amp circuit in Fig. 6.36. Assume that the voltage across the capacitor is initially zero.

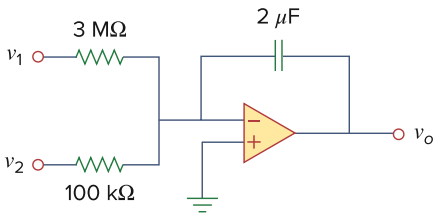



**Figure 6.36**
For Example 6.13.

**Solution:**
This is a summing integrator, and

$$v_o = -\frac{1}{R_1C}\int v_1\,dt - \frac{1}{R_2C}\int v_2\,dt$$

$$= -\frac{1}{3\times 10^6 \times 2\times 10^{-6}}\int_0^t 10\cos(2\tau)\,d\tau - \frac{1}{100\times 10^3\times 2\times 10^{-6}}\int_0^t 0.5\tau\,d\tau$$

$$= -\frac{1}{6}\frac{10}{2}\sin 2t - \frac{1}{0.2}\frac{0.5t^2}{2} = -0.833\sin 2t - 1.25t^2 \text{ mV}$$

## Practice Problem 6.13

The integrator in Fig. 6.35(b) has $R = 100$ kΩ, $C = 20\,\mu$F. Determine the output voltage when a dc voltage of 2.5 mV is applied at $t = 0$. Assume that the op amp is initially nulled.

**Answer:** $-1.25t$ mV.

### 6.6.2 Differentiator

A differentiator is an op amp circuit whose output is proportional to the rate of change of the input signal.

In Fig. 6.35(a), if the input resistor is replaced by a capacitor , the resulting circuit is a differentiator, shown in Fig. 6.37. Applying KCL at node $a$,

$$i_R = i_C \tag{6.36}$$

But

$$i_R = -\frac{v_o}{R}, \qquad i_C = C\frac{dv_i}{dt}$$

Substituting these in Eq. (6.36) yields

$$\boxed{v_o = -RC\frac{dv_i}{dt}} \tag{6.37}$$

showing that the output is the deri vative of the input. Dif ferentiator circuits are electronically unstable because an y electrical noise within the circuit is exaggerated by the differentiator. For this reason, the differentiator circuit in Fig. 6.37 is not as useful and popular as the integrator. It is seldom used in practice.

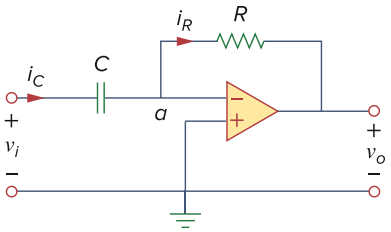
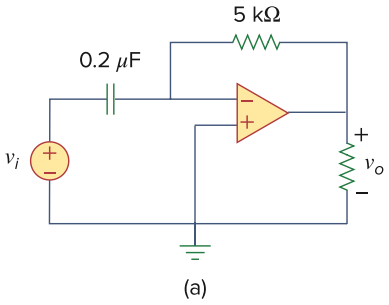


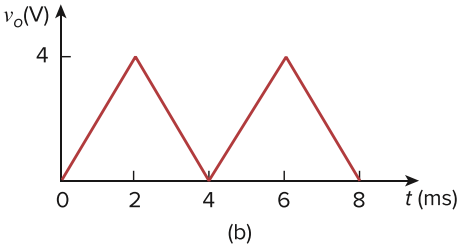

**Figure 6.37**
An op amp differentiator.

## Example 6.14

Sketch the output voltage for the circuit in Fig. 6.38(a), given the input voltage in Fig. 6.38(b). Take $v_o = 0$ at $t = 0$.




(a)




(b)

**Figure 6.38**
For Example 6.14.

**Solution:**
This is a differentiator with

$$RC = 5 \times 10^3 \times 0.2 \times 10^{-6} = 10^{-3} \text{ s}$$

For $0 < t < 4$ ms, we can express the input voltage in Fig. 6.38(b) as

$$v_i = \begin{cases} 2000t & 0 < t < 2 \text{ ms} \\ 8 - 2000t & 2 < t < 4 \text{ ms} \end{cases}$$

This is repeated for $4 < t < 8$ ms. Using Eq. (6.37), the output is obtained as

$$v_o = -RC\frac{dv_i}{dt} = \begin{cases} -2 \text{ V} & 0 < t < 2 \text{ ms} \\ 2 \text{ V} & 2 < t < 4 \text{ ms} \end{cases}$$

Thus, the output is as sketched in Fig. 6.39.

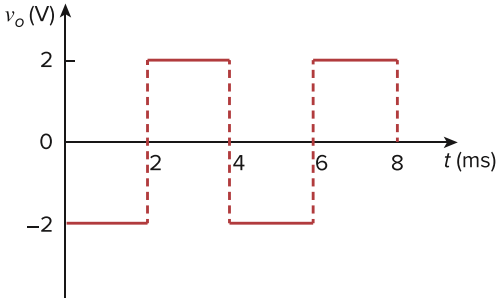



**Figure 6.39**
Output of the circuit in Fig. 6.38(a).

## Practice Problem 6.14

The differentiator in Fig. 6.37 has $R = 100\ \text{k}\Omega$ and $C = 0.1\ \mu\text{F}$. Given that $v_i = 5t$ V, determine the output $v_o$.

**Answer:** −50 mV.

### 6.6.3 Analog Computer

Op amps were initially developed for electronic analog computers. Analog computers can be programmed to solve mathematical models of mechanical or electrical systems. These models are usually expressed in terms of differential equations.

To solve simple differential equations using the analog computer requires cascading three types of op amp circuits: integrator circuits, summing amplifiers, and inverting/noninverting amplifiers for negative/positive scaling. The best way to illustrate how an analog computer solves a differential equation is with an example.

Suppose we desire the solution $x(t)$ of the equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \qquad t > 0 \tag{6.38}$$

where $a$, $b$, and $c$ are constants, and $f(t)$ is an arbitrary forcing function. The solution is obtained by first solving the highest-order derivative term. Solving for $d^2x/dt^2$ yields

$$\frac{d^2x}{dt^2} = \frac{f(t)}{a} - \frac{b}{a}\frac{dx}{dt} - \frac{c}{a}x \tag{6.39}$$

To obtain $dx/dt$, the $d^2x/dt^2$ term is integrated and inverted. Finally, to obtain $x$, the $dx/dt$ term is integrated and inverted. The forcing function is injected at the proper point. Thus, the analog computer for solving Eq. (6.38) is implemented by connecting the necessary summers, inverters, and integrators. A plotter or oscilloscope may be used to view the output $x$, or $dx/dt$, or $d^2x/dt^2$, depending on where it is connected in the system.

Although the above example is on a second-order differential equation, any differential equation can be simulated by an analog computer comprising integrators, inverters, and inverting summers. But care must be exercised in selecting the values of the resistors and capacitors, to ensure that the op amps do not saturate during the solution time interval.

The analog computers with vacuum tubes were built in the 1950s and 1960s. Recently their use has declined. They have been superseded by modern digital computers. However, we still study analog computers for two reasons. First, the availability of integrated op amps has made it possible to build analog computers easily and cheaply. Second, understanding analog computers helps with the appreciation of the digital computers.

---

**Example 6.15**

Design an analog computer circuit to solve the differential equation:

$$\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + v_o = 10\sin 4t, \qquad t > 0$$

subject to $v_o(0) = -4$, $v_o'(0) = 1$, where the prime refers to the time derivative.

**Solution:**

1. **Define.** We have a clearly defined problem and expected solution. I might remind the student that many times the problem is not so well defined and this portion of the problem-solving process could

require much more effort. If this is so, then you should always keep in mind that time spent here will result in much less effort later and most likely save you a lot of frustration in the process.

2. **Present.** Clearly, using the devices developed in Section 6.6.3 will allow us to create the desired analog computer circuit. We will need the integrator circuits (possibly combined with a summing capability) and one or more inverter circuits.
3. **Alternative.** The approach for solving this problem is straightforward. We will need to pick the correct values of resistances and capacitors to allow us to realize the equation we are representing. The final output of the circuit will give the desired result.
4. **Attempt.** There are an infinite number of possibilities for picking the resistors and capacitors, many of which will result in correct solutions. Extreme values of resistors and capacitors will result in incorrect outputs. For example, low values of resistors will overload the electronics. Picking values of resistors that are too large will cause the op amps to stop functioning as ideal devices. The limits can be determined from the characteristics of the real op amp.

We first solve for the second derivative as

$$\frac{d^2v_o}{dt^2} = 10 \sin 4t - 2\frac{dv_o}{dt} - v_o \qquad \textbf{(6.15.1)}$$

Solving this requires some mathematical operations, including summing, scaling, and integration. Integrating both sides of Eq. (6.15.1) gives

$$\frac{dv_o}{dt} = -\int_0^t \left(-10 \sin(4\tau) + 2\frac{dv_o(\tau)}{d\tau} + v_o(\tau)\right) d\tau + v_o'(0) \qquad \textbf{(6.15.2)}$$

where $v_o'(0) = 1$. We implement Eq. (6.15.2) using the summing integrator shown in Fig. 6.40(a). The values of the resistors and capacitors have been chosen so that $RC = 1$ for the term

$$-\frac{1}{RC}\int_0^t v_o(\tau)\, d\tau$$

Other terms in the summing integrator of Eq. (6.15.2) are implemented accordingly. The initial condition $dv_o(0)/dt = 1$ is implemented by connecting a 1-V battery with a switch across the capacitor as shown in Fig. 6.40(a).

The next step is to obtain $v_o$ by integrating $dv_o/dt$ and inverting the result,

$$v_o = -\int_0^t -\frac{dv_o(\tau)}{d\tau}\, d\tau + v(0) \qquad \textbf{(6.15.3)}$$

This is implemented with the circuit in Fig. 6.40(b) with the battery giving the initial condition of −4 V. We now combine the two circuits in Fig. 6.40(a) and (b) to obtain the complete circuit shown in Fig. 6.40(c). When the input signal 10 sin 4$t$ is applied, we open the switches at $t = 0$ to obtain the output waveform $v_o$, which may be viewed on an oscilloscope.

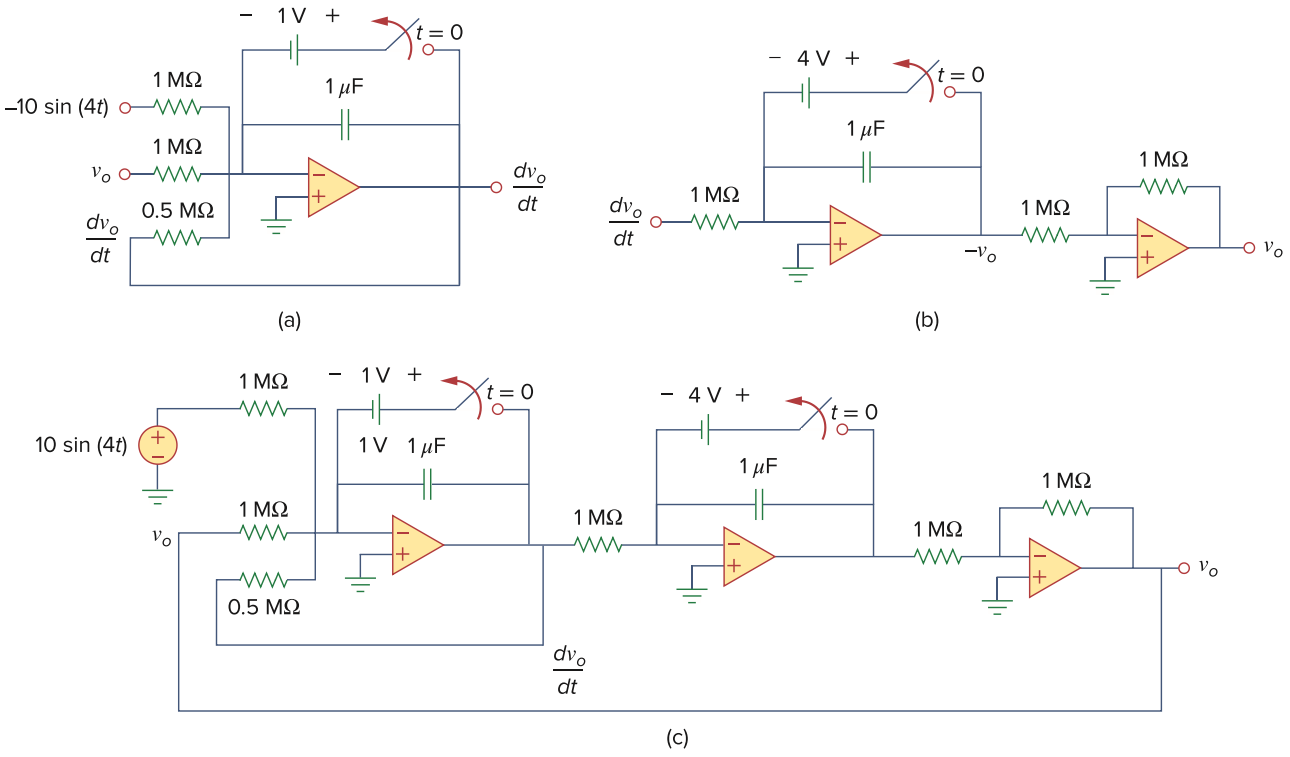



**Figure 6.40**
For Example 6.15.

5. **Evaluate.** The answer looks correct, but is it? If an actual solution for $v_o$ is desired, then a good check would be to first find the solution by realizing the circuit in *PSpice*. This result could then be compared with a solution using the differential solution capability of *MATLAB*.

   Since all we need to do is check the circuit and confirm that it represents the equation, we have an easier technique to use. We just go through the circuit and see if it generates the desired equation.

   However, we still have choices to make. We could go through the circuit from left to right but that would involve differentiating the result to obtain the original equation. An easier approach would be to go from right to left. This is the approach we will use to check the answer.

   Starting with the output, $v_o$, we see that the right-hand op amp is nothing more than an inverter with a gain of one. This means that the output of the middle circuit is $-v_o$. The following represents the action of the middle circuit.

$$-v_o = -\left(\int_0^t \frac{dv_o}{d\tau}\,d\tau + v_o(0)\right) = -\left(v_o\Big|_0^t + v_o(0)\right)$$

$$= -(v_o(t) - v_o(0) + v_o(0))$$

   where $v_o(0) = -4$ V is the initial voltage across the capacitor.

   We check the circuit on the left the same way.

$$\frac{dv_o}{dt} = -\left(\int_0^t -\frac{d^2v_o}{d\tau^2}\,d\tau - v_o'(0)\right) = -\left(-\frac{dv_o}{dt} + v_o'(0) - v_o'(0)\right)$$

Now, all we need to verify is that the input to the first op amp is $-d^2v_o/dt^2$.

Looking at the input we see that it is equal to

$$-10\sin(4t) + v_o + \frac{1/10^{-6}}{0.5\ \text{M}\Omega}\frac{dv_o}{dt} = -10\sin(4t) + v_o + 2\frac{dv_o}{dt}$$

which does produce $-d^2v_o/dt^2$ from the original equation.

6. **Satisfactory?** The solution we have obtained is satisfactory. We can now present this work as a solution to the problem.

## Practice Problem 6.15

Design an analog computer circuit to solve the differential equation:

$$\frac{d^2v_o}{dt^2} + 3\frac{dv_o}{dt} + 2v_o = 4\cos 10t, \qquad t > 0$$

subject to $v_o(0) = 2$, $v_o'(0) = 0$.

**Answer:** See Fig. 6.41, where $RC = 1$ s.

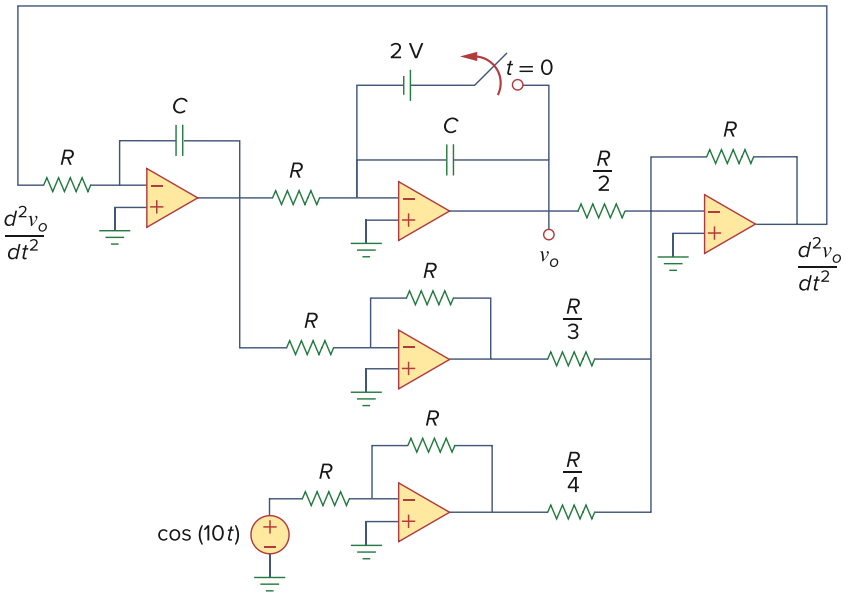



**Figure 6.41**
For Practice Prob. 6.15.

## 6.7 Summary

1. The current through a capacitor is directly proportional to the time rate of change of the voltage across it.

$$i = C\frac{dv}{dt}$$

The current through a capacitor is zero unless the v oltage is changing. Thus, a capacitor acts like an open circuit to a dc source.

2. The voltage across a capacitor is directly proportional to the time integral of the current through it.

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,d\tau = \frac{1}{C}\int_{t_0}^{t} i\,d\tau + v(t_0)$$

The voltage across a capacitor cannot change instantly.

3. Capacitors in series and in parallel are combined in the same way as conductances.
4. The voltage across an inductor is directly proportional to the time rate of change of the current through it.

$$v = L\frac{di}{dt}$$

The voltage across the inductor is zero unless the current is changing. Thus, an inductor acts like a short circuit to a dc source.

5. The current through an inductor is directly proportional to the time integral of the voltage across it.

$$i = \frac{1}{L}\int_{-\infty}^{t} v\,d\tau = \frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0)$$

The current through an inductor cannot change instantly.

6. Inductors in series and in parallel are combined in the same way resistors in series and in parallel are combined.
7. At any given time $t$, the energy stored in a capacitor is $\frac{1}{2}Cv^2$, while the energy stored in an inductor is $\frac{1}{2}Li^2$.
8. Three application circuits, the integrator, the differentiator, and the analog computer, can be realized using resistors, capacitors, and op amps.

## Review Questions

**6.1** What charge is on a 5-F capacitor when it is connected across a 120-V source?

(a) 600 C (b) 300 C
(c) 24 C (d) 12 C

**6.2** Capacitance is measured in:

(a) coulombs (b) joules
(c) henrys (d) farads

**6.3** When the total charge in a capacitor is doubled, the energy stored:

(a) remains the same (b) is halved
(c) is doubled (d) is quadrupled

**6.4** Can the voltage waveform in Fig. 6.42 be associated with a real capacitor?

(a) Yes (b) No

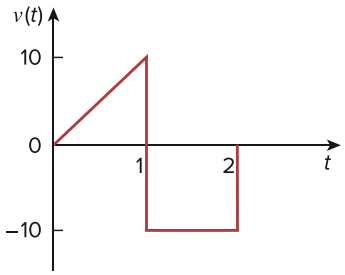



**Figure 6.42**
For Review Question 6.4.

**6.5** The total capacitance of two 40-mF series-connected capacitors in parallel with a 4-mF capacitor is:

(a) 3.8 mF (b) 5 mF (c) 24 mF
(d) 44 mF (e) 84 mF

**6.6** In Fig. 6.43, if $i = \cos 4t$ and $v = \sin 4t$, the element is:

(a) a resistor (b) a capacitor (c) an inductor

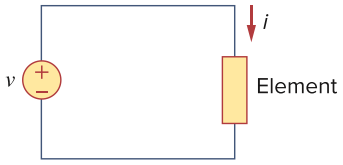



**Figure 6.43**
For Review Question 6.6.

**6.7** A 5-H inductor changes its current by 3 A in 0.2 s. The voltage produced at the terminals of the inductor is:

(a) 75 V (b) 8.888 V
(c) 3 V (d) 1.2 V

**6.8** If the current through a 10-mH inductor increases from zero to 2 A, how much energy is stored in the inductor?

(a) 40 mJ (b) 20 mJ
(c) 10 mJ (d) 5 mJ

**6.9** Inductors in parallel can be combined just like resistors in parallel.

(a) True (b) False

**6.10** For the circuit in Fig. 6.44, the voltage divider formula is:

(a) $v_1 = \dfrac{L_1 + L_2}{L_1} v_s$ (b) $v_1 = \dfrac{L_1 + L_2}{L_2} v_s$

(c) $v_1 = \dfrac{L_2}{L_1 + L_2} v_s$ (d) $v_1 = \dfrac{L_1}{L_1 + L_2} v_s$

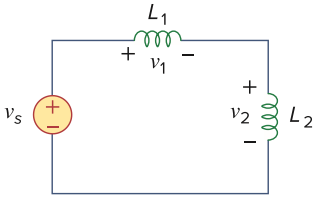



**Figure 6.44**
For Review Question 6.10.

*Answers: 6.1a, 6.2d, 6.3d, 6.4b, 6.5c, 6.6b, 6.7a, 6.8b, 6.9a, 6.10d.*

# Problems

## Section 6.2 Capacitors

**6.1** If the voltage across a 7.5-F capacitor is $2te^{-3t}$ V, find the current and the power.

**6.2** A 50-$\mu$F capacitor has energy $w(t) = 10\cos^2 377t$ J. Determine the current through the capacitor.

**6.3** Design a problem to help other students better understand how capacitors work.

**6.4** A voltage across a capacitor is equal to $[2 - 2\cos(4t)]$ V and the current flowing through it is equal to $2\sin(4t)$ $\mu$A. Determine the value of the capacitance. Calculate the power being stored by the capacitor.

**6.5** The voltage across a 4-$\mu$F capacitor is shown in Fig. 6.45. Find the current waveform.

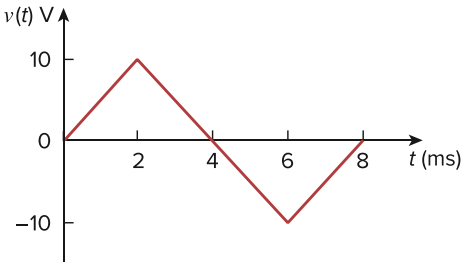



**Figure 6.45**
For Prob. 6.5.

**6.6** The voltage waveform in Fig. 6.46 is applied across a 55-$\mu$F capacitor. Draw the current waveform through it.

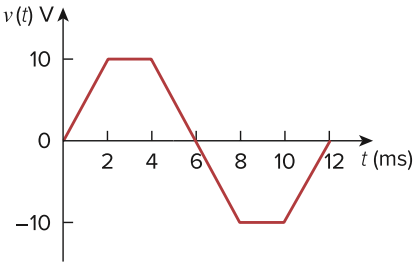



**Figure 6.46**
For Prob. 6.6.

**6.7** At $t = 0$, the voltage across a 25-mF capacitor is 10 V. Calculate the voltage across the capacitor for $t > 0$ when current $5t$ mA flows through it.

**6.8** A 4-mF capacitor has the terminal voltage

$$v = \begin{cases} 50 \text{ V}, & t \le 0 \\ Ae^{-100t} + Be^{-600t} \text{ V}, & t \ge 0 \end{cases}$$

If the capacitor has an initial current of 2 A, find:

(a) the constants $A$ and $B$,
(b) the energy stored in the capacitor at $t = 0$,
(c) the capacitor current for $t > 0$.

**6.9** The current through a 0.5-F capacitor is $6(1 - e^{-t})$ A. Determine the voltage and power at $t = 2$ s. Assume $v(0) = 0$.

**6.10** The voltage across a 5-mF capacitor is shown in Fig. 6.47. Determine the current through the capacitor.

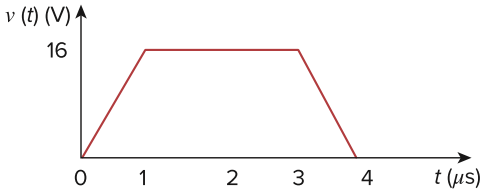



**Figure 6.47**
For Prob. 6.10.

**6.11** A 4-mF capacitor has the current waveform shown in Fig. 6.48. Assuming that $v(0) = 10$ V, sketch the voltage waveform $v(t)$.

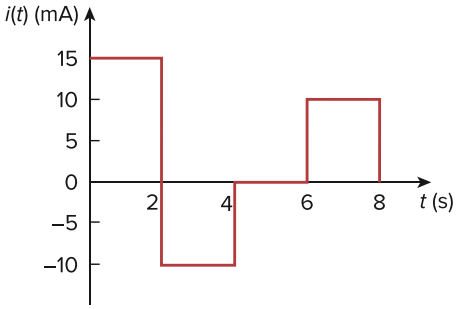



**Figure 6.48**
For Prob. 6.11.

**6.12** A voltage of $45e^{-2000t}$ V appears across a parallel combination of a 100-mF capacitor and a 12-Ω resistor. Calculate the power absorbed by the parallel combination.

**6.13** Find the voltage across the capacitors in the circuit of Fig. 6.49 under dc conditions.

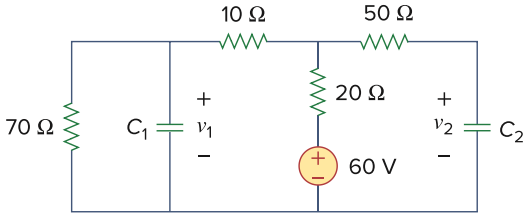



**Figure 6.49**
For Prob. 6.13.

## Section 6.3 Series and Parallel Capacitors

**6.14** Series-connected 20- and 60-pF capacitors are placed in parallel with series-connected 30- and 70-pF capacitors. Determine the equivalent capacitance.

**6.15** Two capacitors (25 and 75 $\mu$F) are connected to a 100-V source. Find the energy stored in each capacitor if they are connected in:

(a) parallel (b) series

**6.16** The equivalent capacitance at terminals $a$-$b$ in the circuit of Fig. 6.50 is 20 $\mu$F. Calculate the value of $C$.

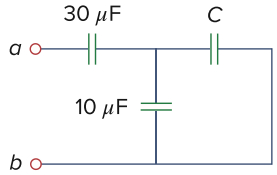



**Figure 6.50**
For Prob. 6.16.

**6.17** Determine the equivalent capacitance for each of the circuits of Fig. 6.51.

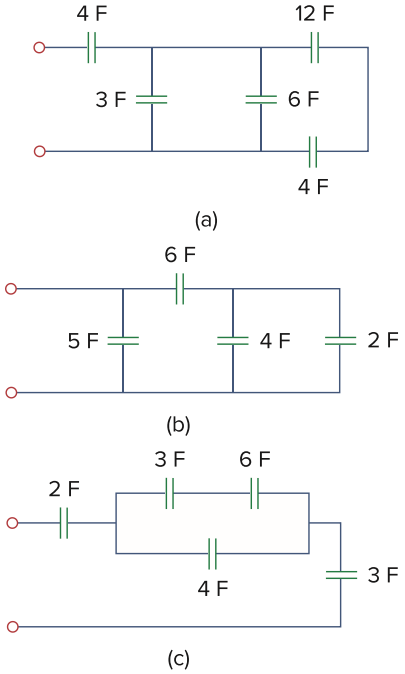



**Figure 6.51**
For Prob. 6.17.

**6.18** Find $C_{eq}$ in the circuit of Fig. 6.52 if all capacitors are 4 $\mu$F.

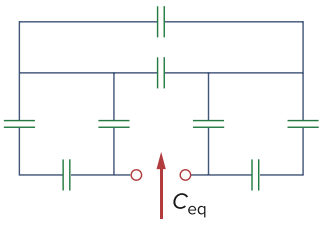



**Figure 6.52**
For Prob. 6.18.

**6.19** Find the equivalent capacitance between terminals *a* and *b* in the circuit of Fig. 6.53. All capacitances are in μF.

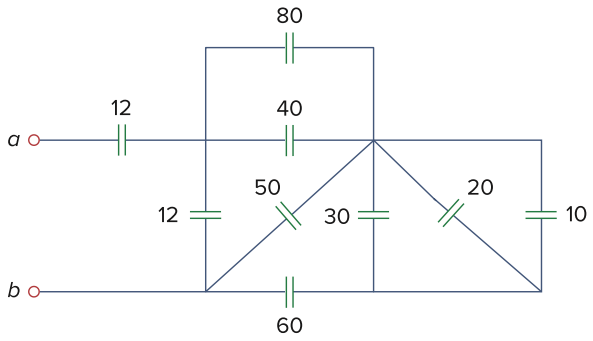



**Figure 6.53**
For Prob. 6.19.

**6.20** Find the equivalent capacitance at terminals *a*-*b* of the circuit in Fig. 6.54.

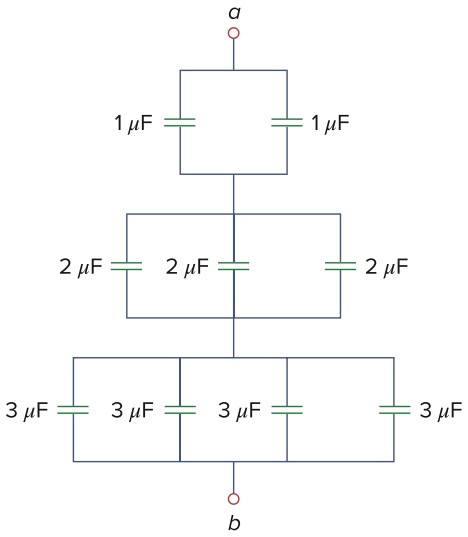



**Figure 6.54**
For Prob. 6.20.

**6.21** Determine the equivalent capacitance at terminals *a*-*b* of the circuit in Fig. 6.55.

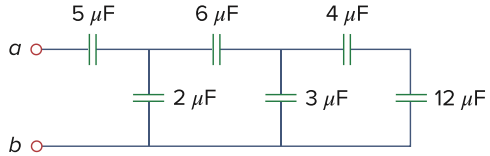



**Figure 6.55**
For Prob. 6.21.

**6.22** Obtain the equivalent capacitance of the circuit in Fig. 6.56.

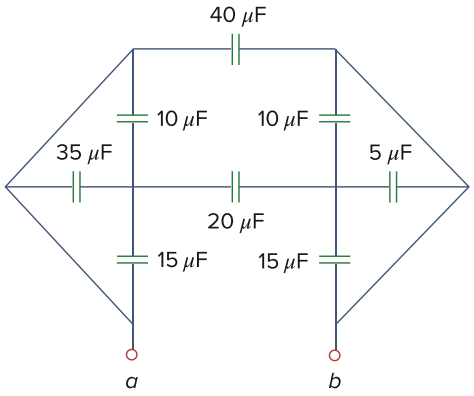



**Figure 6.56**
For Prob. 6.22.

**6.23** Using Fig. 6.57, design a problem that will help other students better understand how capacitors work together when connected in series and in parallel.

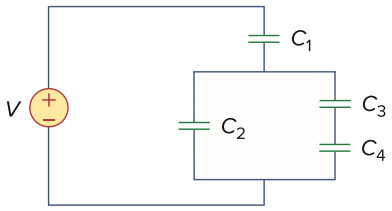



**Figure 6.57**
For Prob. 6.23.

**6.24** In the circuit shown in Fig. 6.58 assume that the capacitors were initially uncharged and that the current source has been connected to the circuit long enough for all the capacitors to reach steady state (no current flowing through the capacitors). Determine the voltage across each capacitor and the energy stored in each.

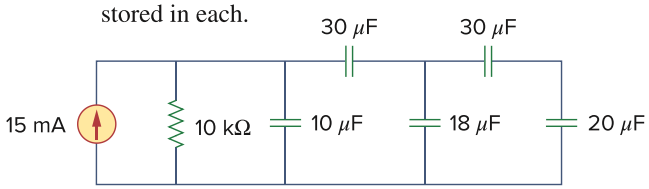



**Figure 6.58**
For Prob. 6.24.

**6.25** (a) Show that the voltage-division rule for two capacitors in series as in Fig. 6.59(a) is

$$v_1 = \frac{C_2}{C_1 + C_2} v_s, \qquad v_2 = \frac{C_1}{C_1 + C_2} v_s$$

assuming that the initial conditions are zero.

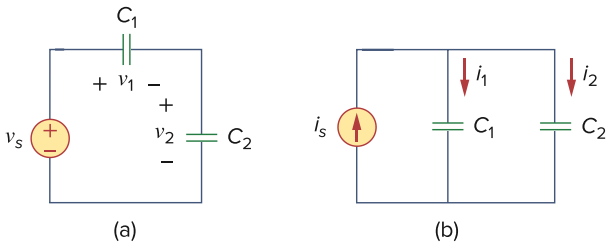



**Figure 6.59**
For Prob. 6.25.

(b) For two capacitors in parallel as in Fig. 6.59(b), show that the current-division rule is

$$i_1 = \frac{C_1}{C_1 + C_2} i_s, \qquad i_2 = \frac{C_2}{C_1 + C_2} i_s$$

assuming that the initial conditions are zero.

**6.26** Three capacitors, $C_1 = 5\ \mu$F, $C_2 = 10\ \mu$F, and $C_3 = 20\ \mu$F, are connected in parallel across a 200-V source. Determine:

(a) the total capacitance,
(b) the charge on each capacitor,
(c) the total energy stored in the parallel combination.

**6.27** Given that four 10-$\mu$F capacitors can be connected in series and in parallel, find the minimum and maximum values that can be obtained by such series/parallel combinations.

***6.28** Obtain the equivalent capacitance of the network shown in Fig. 6.60.

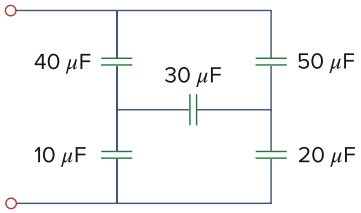



**Figure 6.60**
For Prob. 6.28.

**6.29** Determine $C_{eq}$ for each circuit in Fig. 6.61.

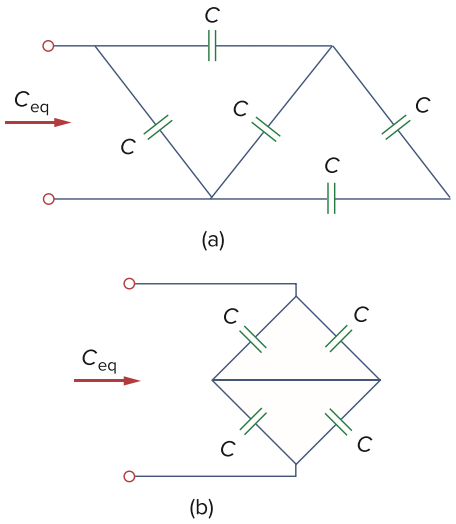



**Figure 6.61**
For Prob. 6.29.

* An asterisk indicates a challenging problem.

**6.30** Assuming that the capacitors are initially uncharged, find $v_o(t)$ in the circuit of Fig. 6.62.

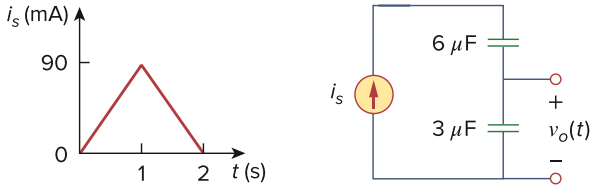



**Figure 6.62**
For Prob. 6.30.

**6.31** If $v(0) = 0$, find $v(t)$, $i_1(t)$, and $i_2(t)$ in the circuit of Fig. 6.63.

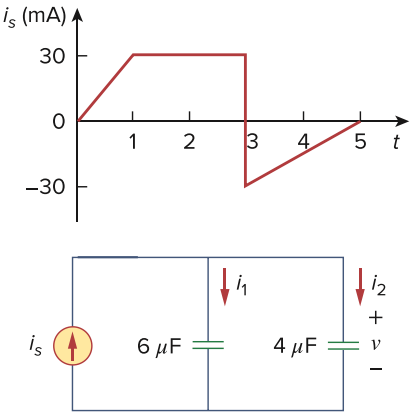



**Figure 6.63**
For Prob. 6.31.

**6.32** In the circuit in Fig. 6.64, let $i_s = 4.5e^{-2t}$ mA and the voltage across each capacitor is equal to zero at $t = 0$. Determine $v_1$ and $v_2$ and the energy stored in each capacitor for all $t > 0$.

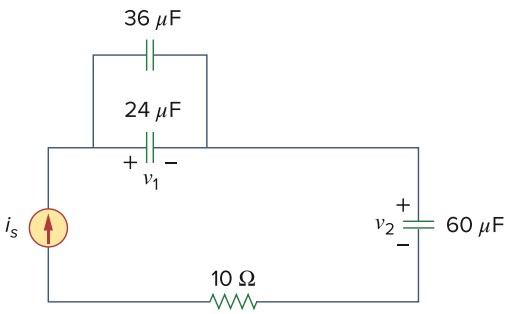



**Figure 6.64**
For Prob. 6.32.

**6.33** Obtain the Thevenin equivalent at the terminals, *a-b*, of the circuit shown in Fig. 6.65. Please note that Thevenin equivalent circuits do not generally exist for circuits involving capacitors and resistors. This is a special case where the Thevenin equivalent circuit does exist.

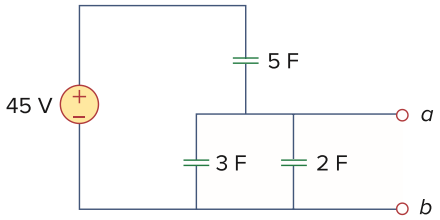



**Figure 6.65**
For Prob. 6.33.

## Section 6.4 Inductors

**6.34** The current through a 25-mH inductor is $10e^{-t/2}$ A. Find the voltage and the power at $t = 3$ s.

**6.35** An inductor has a linear change in current from 100 mA to 200 mA in 2 ms and induces a voltage of 160 mV. Calculate the value of the inductor.

**6.36** Design a problem to help other students better understand how inductors work.

**6.37** The current through a 12-mH inductor is 4 sin 100$t$ A. Find the voltage, and the energy stored at $t = \frac{\pi}{200}$ s.

**6.38** The current through a 40-mH inductor is

$$i(t) = \begin{cases} 0, & t < 0 \\ te^{-2t}\ \text{A}, & t > 0 \end{cases}$$

Find the voltage $v(t)$.

**6.39** The voltage across a 50-mH inductor is given by

$$v(t) = [5e^{-2t} + 2t + 4]\ \text{V} \qquad \text{for } t > 0.$$

Determine the current $i(t)$ through the inductor. Assume that $i(0) = 0$ A.

**6.40** The current through a 5-mH inductor is shown in Fig. 6.66. Determine the voltage across the inductor at $t = 1$, 3, and 5 ms.

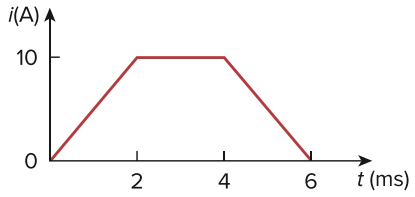



**Figure 6.66**
For Prob. 6.40.

**6.41** The voltage across a 2-H inductor is $20(1 - e^{-2t})$ V. If the initial current through the inductor is 0.3 A, find the current and the energy stored in the inductor at $t = 1$ s.

**6.42** If the voltage waveform in Fig. 6.67 is applied across the terminals of a 5-H inductor, calculate the current through the inductor. Assume $i(0) = -1$ A.

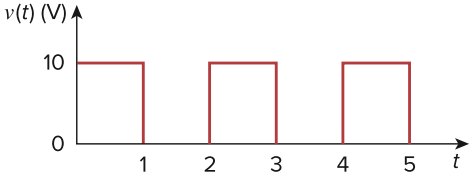



**Figure 6.67**
For Prob. 6.42.

**6.43** The current in a 150-mH inductor increases from 0 to 60 mA (steady state). How much energy is stored in the inductor?

***6.44** A 100-mH inductor is connected in parallel with a 2-kΩ resistor. The current through the inductor is $i(t) = 35e^{-400t}$ mA. (a) Find the voltage $v_L$ across the inductor. (b) Find the voltage $v_R$ across the resistor. (c) Does $v_R(t) + v_L(t) = 0$? (d) Calculate the energy stored in the inductor at $t = 0$.

**6.45** If the voltage waveform in Fig. 6.68 is applied to a 25-mH inductor, find the inductor current $i(t)$ for $0 < t < 2$ seconds. Assume $i(0) = 0$.

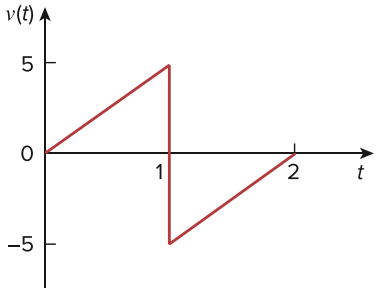



**Figure 6.68**
For Prob. 6.45.

**6.46** Find $v_C$, $i_L$, and the energy stored in the capacitor and inductor in the circuit of Fig. 6.69 under dc conditions.

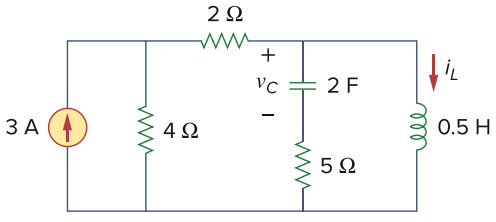



**Figure 6.69**
For Prob. 6.46.

**6.47** For the circuit in Fig. 6.70, calculate the value of $R$ that will make the energy stored in the capacitor the same as that stored in the inductor under dc conditions.

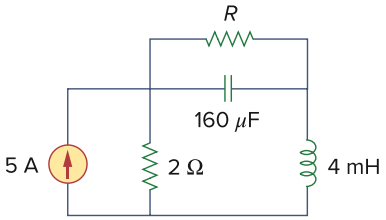



**Figure 6.70**
For Prob. 6.47.

**6.48** Under steady-state dc conditions, find $i$ and $v$ in the circuit in Fig. 6.71.

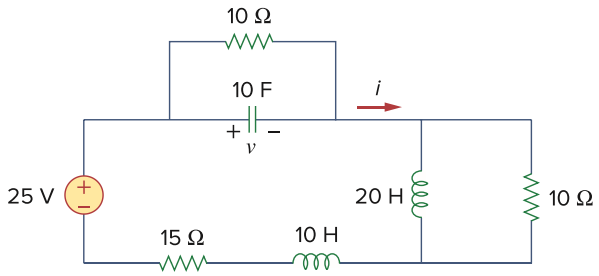



**Figure 6.71**
For Prob. 6.48.

## Section 6.5 Series and Parallel Inductors

**6.49** Find the equivalent inductance of the circuit in Fig. 6.72. Assume all inductors are 40 mH.

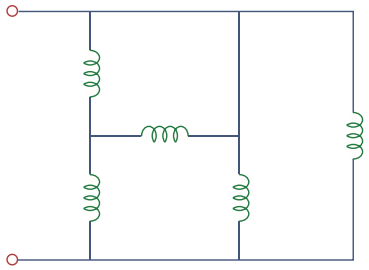


**Figure 6.72**
For Prob. 6.49.

**6.50** An energy-storage network consists of series-connected 16- and 14-mH inductors in parallel with series-connected 24- and 36-mH inductors. Calculate the equivalent inductance.

**6.51** Determine $L_{eq}$ at terminals $a$-$b$ of the circuit in Fig. 6.73.

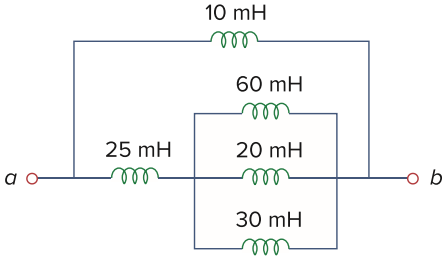



**Figure 6.73**
For Prob. 6.51.

**6.52** Using Fig. 6.74, design a problem to help other students better understand how inductors behave when connected in series and when connected in parallel.

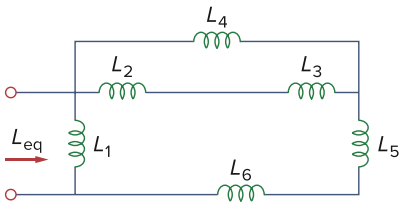



**Figure 6.74**
For Prob. 6.52.

**6.53** Find $L_{eq}$ at the terminals of the circuit in Fig. 6.75.

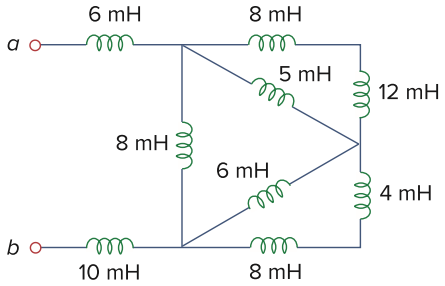



**Figure 6.75**
For Prob. 6.53.

**6.54** Find the equivalent inductance looking into the terminals of the circuit in Fig. 6.76.

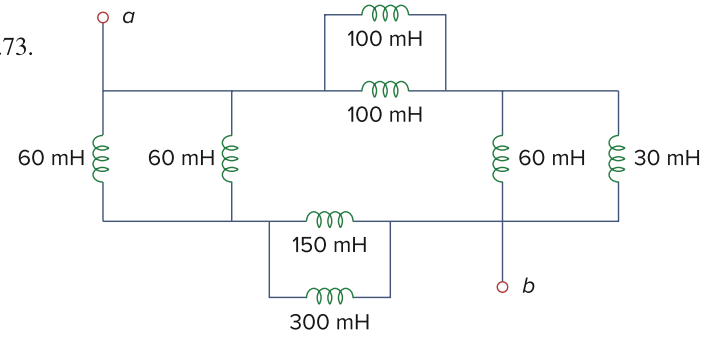



**Figure 6.76**
For Prob. 6.54.

**6.55** Find $L_{eq}$ in each of the circuits in Fig. 6.77.

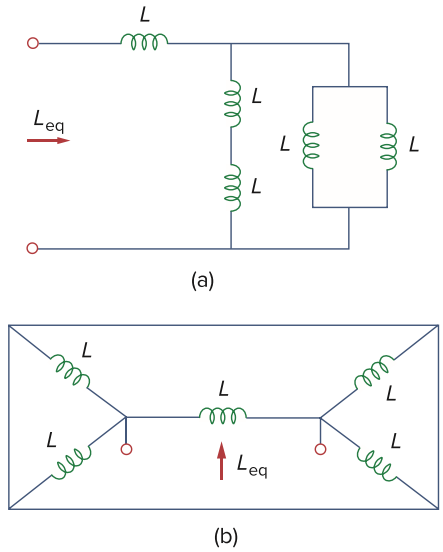



**Figure 6.77**
For Prob. 6.55.

**6.56** Find $L_{eq}$ in the circuit of Fig. 6.78.

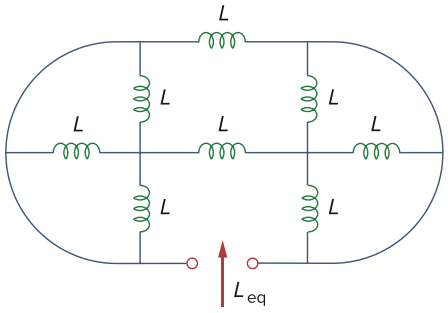



**Figure 6.78**
For Prob. 6.56.

***6.57** Determine $L_{eq}$ that may be used to represent the inductive network of Fig. 6.79 at the terminals.

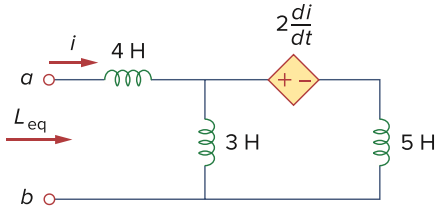



**Figure 6.79**
For Prob. 6.57.

**6.58** The current waveform in Fig. 6.80 flows through a 3-H inductor. Sketch the voltage across the inductor over the interval $0 < t < 6$ s.

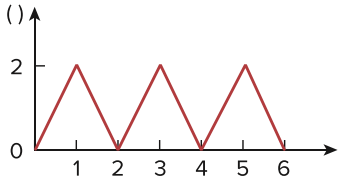



**Figure 6.80**
For Prob. 6.58.

**6.59** (a) For two inductors in series as in Fig. 6.81(a), show that the voltage division principle is

$$v_1 = \frac{L_1}{L_1 + L_2} v_s, \qquad v_2 = \frac{L_2}{L_1 + L_2} v_s$$

assuming that the initial conditions are zero.

(b) For two inductors in parallel as in Fig. 6.81(b), show that the current-division principle is

$$i_1 = \frac{L_2}{L_1 + L_2} i_s, \qquad i_2 = \frac{L_1}{L_1 + L_2} i_s$$

assuming that the initial conditions are zero.

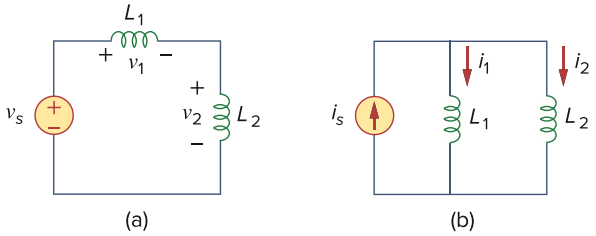



**Figure 6.81**
For Prob. 6.59.

**6.60** In the circuit of Fig. 6.82, $i_o(0) = 2$ A. Determine $i_o(t)$ and $v_o(t)$ for $t > 0$.

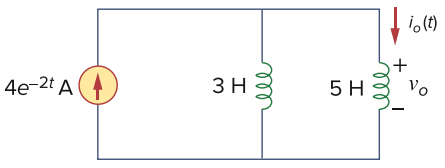



**Figure 6.82**
For Prob. 6.60.

**6.61** Consider the circuit in Fig. 6.83. Find: (a) $L_{eq}$, $i_1(t)$, and $i_2(t)$ if $i_s = 3e^{-t}$ mA, (b) $v_o(t)$, (c) energy stored in the 20-mH inductor at $t = 1$ s.

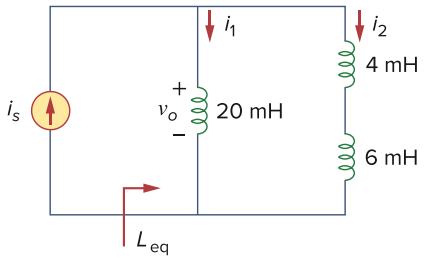



**Figure 6.83**
For Prob. 6.61.

**6.62** Consider the circuit in Fig. 6.84. Given that $v(t) = 12e^{-3t}$ mV for $t > 0$ and $i_1(0) = -30$ mA, find: (a) $i_2(0)$, (b) $i_1(t)$ and $i_2(t)$.

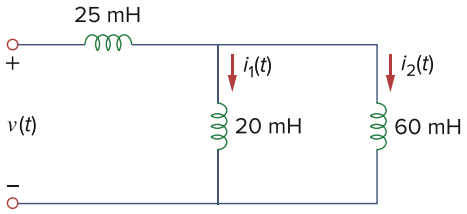



**Figure 6.84**
For Prob. 6.62.

**6.63** In the circuit of Fig. 6.85, sketch $v_o$.

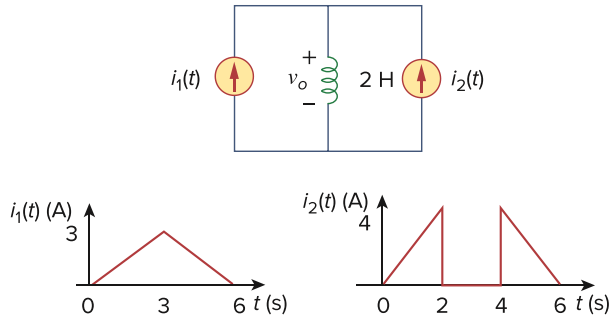



**Figure 6.85**
For Prob. 6.63.

**6.64** The switch in Fig. 6.86 has been in position $A$ for a long time. At $t = 0$, the switch moves from position $A$ to $B$. The switch is a make-before-break type so that there is no interruption in the inductor current. Find:

(a) $i(t)$ for $t > 0$,
(b) $v$ *just after* the switch has been moved to position $B$,
(c) $v(t)$ long after the switch is in position $B$.

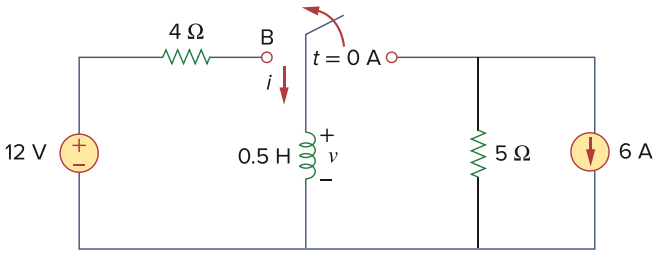



**Figure 6.86**
For Prob. 6.64.

**6.65** The inductors in Fig. 6.87 are initially charged and are connected to the black box at $t = 0$. If $i_1(0) = 4$ A, $i_2(0) = -2$ A, and $v(t) = 50e^{-200t}$ mV, $t \geq 0$, find:

(a) the energy initially stored in each inductor,
(b) the total energy delivered to the black box from $t = 0$ to $t = \infty$,
(c) $i_1(t)$ and $i_2(t)$, $t \geq 0$,
(d) $i(t)$, $t \geq 0$.

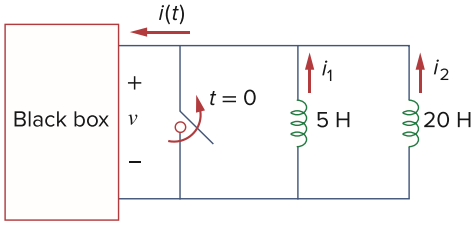



**Figure 6.87**
For Prob. 6.65.

**6.66** The current $i(t)$ through a 20-mH inductor is equal, in magnitude, to the voltage across it for all values of time. If $i(0) = 2$ A, find $i(t)$.

## Section 6.6 Applications

**6.67** An op amp integrator has $R = 50$ kΩ and $C = 0.04$ μF. If the input voltage is $v_i = 10 \sin 50t$ mV, obtain the output voltage. Assume that at $t$ equal to zero, the output is equal to zero.

**6.68** A 6-V dc voltage is applied to an integrator with $R = 50\ \text{k}\Omega$, $C = 100\ \mu\text{F}$ at $t = 0$. How long will it take for the op amp to saturate if the saturation voltages are +12 V and −12 V? Assume that the initial capacitor voltage was zero.

**6.69** An op amp integrator with $R = 4\ \text{M}\Omega$ and $C = 1\ \mu\text{F}$ has the input waveform shown in Fig. 6.88. Plot the output waveform.

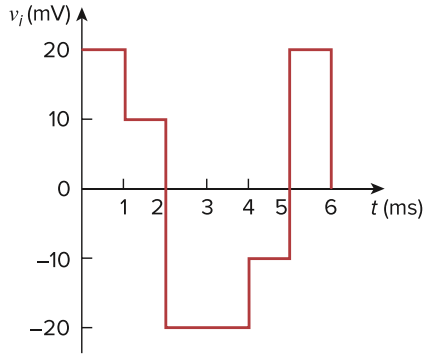



**Figure 6.88**
For Prob. 6.69.

**6.70** Using a single op amp, a capacitor, and resistors of 100 kΩ or less, design a circuit to implement

$$v_o = -2\int_0^t v_i(\tau)\,d\tau$$

Assume $v_o = 0$ at $t = 0$.

**6.71** Show how you would use a single op amp to generate

$$v_o = -\int_0^t (v_1 + 4v_2 + 10v_3)\,d\tau$$

If the inte grating capacitor is $C = 5\ \mu\text{F}$, obtain the other component values.

**6.72** At $t = 1.5$ ms, calculate $v_o$ due to the cascaded integrators in Fig. 6.89. Assume that the integrators are reset to 0 V at $t = 0$.

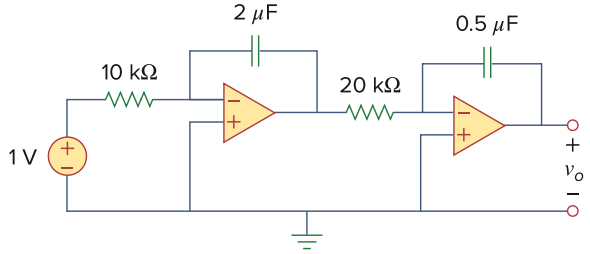



**Figure 6.89**
For Prob. 6.72.

**6.73** Show that the circuit in Fig. 6.90 is a noninverting integrator.

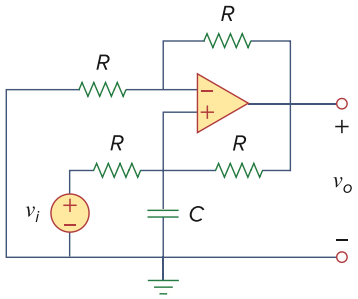



**Figure 6.90**
For Prob. 6.73.

**6.74** The triangular waveform in Fig. 6.91(a) is applied to the input of the op amp differentiator in Fig. 6.91(b). Plot the output.

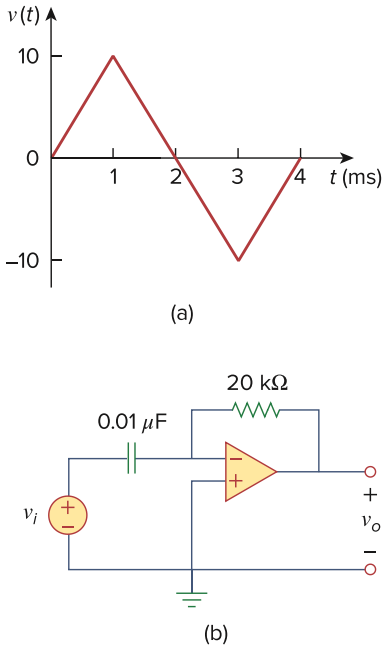



**Figure 6.91**
For Prob. 6.74.

**6.75** An op amp differentiator has $R = 250\ \text{k}\Omega$ and $C = 10\ \mu\text{F}$. The input voltage is a ramp $r(t) = 7t$ mV. Find the output voltage.

**6.76** A voltage waveform has the following characteristics: a positive slope of 20 V/s for 5 ms followed by a negative slope of 10 V/s for 10 ms. If the waveform is applied to a differentiator with $R = 50\ \text{k}\Omega$, $C = 10\ \mu\text{F}$, sketch the output voltage waveform.

***6.77** The output $v_o$ of the op amp circuit in Fig. 6.92(a) is shown in Fig. 6.92(b). Let $R_i = R_f = 1$ MΩ and $C = 1$ μF. Determine the input voltage waveform and sketch it.

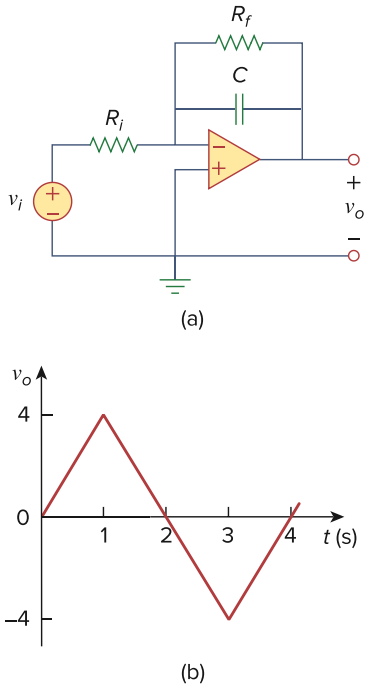



**Figure 6.92**
For Prob. 6.77.

**6.78** Design an analog computer to simulate

$$\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + v_o = 10 \sin 2t$$

where $v_0(0) = -6$ V and $v_0'(0) = 0$.

**6.79** Design an analog computer circuit to solve for $v(t)$, given the following equation and a value for $f(t)$ and that $v(0) = 0$ V.

$$(dv(t)/dt) + 3vt = ftdt$$

**6.80** Figure 6.93 presents an analog computer designed to solve a differential equation. Assuming $f(t)$ is known, set up the equation for $f(t)$.

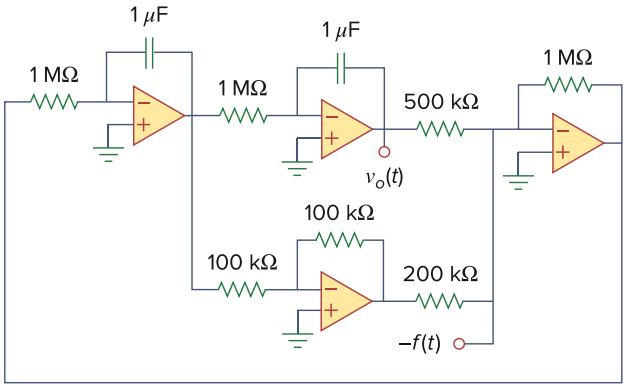



**Figure 6.93**
For Prob. 6.80.

**6.81** Design an analog computer to simulate the following equation to solve for $v(t)$ (assume the initial conditions are zero):

$$(d^3v(t)/dt^3) + 3(dv(t)/dt) = 4f(t)$$

**6.82** Design an op amp circuit such that

$$v_o = 10v_s + 2\int v_s dt$$

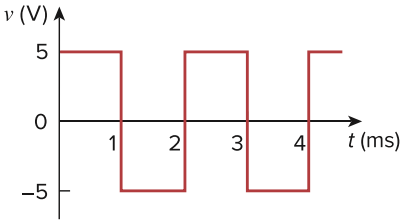
where $v_s$ and $v_o$ are the input voltage and output voltage, respectively.

## Comprehensive Problems

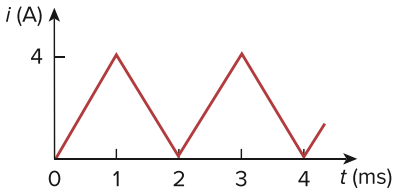
**6.83** Your laboratory has available a large number of 5-μF capacitors rated at 150 V. To design a capacitor bank of 10 μF rated at 600 V, how many 5-μF capacitors are needed and how would you connect them?

**6.84** An 8-mH inductor is used in a fusion power experiment. If the current through the inductor is $i(t) = 10\cos^2(\pi t)$ mA, for all $t > 0$ s, find the power being delivered to the inductor and the energy stored in it at $t = 0.5$ s.

**6.85** A square-wave generator produces the voltage waveform shown in Fig. 6.94(a). What kind of a circuit component is needed to convert the voltage waveform to the triangular current waveform shown in Fig. 6.94(b)? Calculate the value of the component, assuming that it is initially uncharged.




(a)




(b)

**Figure 6.94**
For Prob. 6.85.

**6.86** An electric motor can be modeled as a series combination of a 12-Ω resistor and 200-mH inductor. If a current $i(t) = 2te^{-10t}$ A flows through the series combination, find the voltage across the combination.

chapter

# 7

# First-Order Circuits

*We live in deeds, not years; in thoughts, not breaths; in feelings, not in figures on a dial. We should count time in heart-throbs. He most lives who thinks most, feels the noblest, acts the best.*

—P. J. Bailey

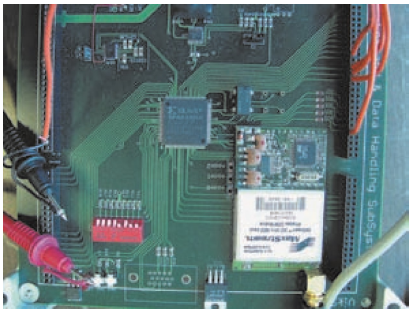
## Enhancing Your Career

### Careers in Computer Engineering

Electrical engineering education has gone through drastic changes in recent decades. Most departments have come to be known as Department of Electrical and Computer Engineering, emphasizing the rapid changes due to computers. Computers occupy a prominent place in modern society and education. They have become commonplace and are helping to change the face of research, development, production, business, and entertainment. The scientist, engineer, doctor, attorney, teacher, airline pilot, businessperson—almost anyone benefits from a computer's abilities to store large amounts of information and to process that information in very short periods of time. The internet, a computer communication network, is essential in business, education, and library science. Computer usage continues to grow by leaps and bounds.

An education in computer engineering should provide breadth in software, hardware design, and basic modeling techniques. It should include courses in data structures, digital systems, computer architecture, microprocessors, interfacing, software engineering, and operating systems.

Electrical engineers who specialize in computer engineering find jobs in computer industries and in numerous fields where computers are being used. Companies that produce software are growing rapidly in number and size and providing employment for those who are skilled in programming. An excellent way to advance one's knowledge of computers is to join the IEEE Computer Society, which sponsors diverse magazines, journals, and conferences.



Computer design of very large scale integrated (VLSI) circuits.
Courtesy Brian Fast, Cleveland State University

## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Understand solutions to unforced, first-order linear differential equations.
2. Comprehend singularity equations and know their importance in solving linear differential equations.
3. Understand the effect of unit step sources on first-order linear differential equations.
4. Explain how dependent sources and op amps influence simple first-order linear differential equations.
5. Use *PSpice* to solve simple transient circuits with an inductor or a capacitor.

## 7.1 Introduction

Now that we have considered the three passive elements (resistors, capacitors, and inductors) and one active element (the op amp) individually, we are prepared to consider circuits that contain various combinations of two or three of the passive elements. In this chapter, we shall examine two types of simple circuits: a circuit comprising a resistor and capacitor and a circuit comprising a resistor and an inductor. These are called *RC* and *RL* circuits, respectively. As simple as these circuits are, they find continual applications in electronics, communications, and control systems, as we shall see.

We carry out the analysis of *RC* and *RL* circuits by applying Kirchhoff's laws, as we did for resistive circuits. The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to *RC* and *RL* circuits produces differential equations, which are more difficult to solve than algebraic equations. The differential equations resulting from analyzing *RC* and *RL* circuits are of the first order. Hence, the circuits are collectively known as *first-order* circuits.

A **first-order** circuit is characterized by a first-order differential equation.

In addition to there being two types of first-order circuits *(RC* and *RL)*, there are two ways to excite the circuits. The first way is by initial conditions of the storage elements in the circuits. In these so-called *source-free circuits*, we assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors. Although source-free circuits are by definition free of independent sources, they may have dependent sources. The second way of exciting first-order circuits is by independent sources. In this chapter, the independent sources we will consider are dc sources. (In later chapters, we shall consider sinusoidal and exponential sources.) The two types of first-order circuits and the two ways of exciting them add up to the four possible situations we will study in this chapter.

Finally, we consider four typical applications of *RC* and *RL* circuits: delay and relay circuits, a photoflash unit, and an automobile ignition circuit.

## 7.2 The Source-Free *RC* Circuit

A source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 7.1. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Our objective is to determine the circuit response, which, for pedagogic reasons, we assume to be the voltage $v(t)$ across the capacitor. Since the capacitor is initially charged, we can assume that at time $t = 0$, the initial voltage is

$$v(0) = V_0 \tag{7.1}$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2} CV_0^2 \tag{7.2}$$

Applying KCL at the top node of the circuit in Fig. 7.1 yields

$$i_C + i_R = 0 \tag{7.3}$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \tag{7.4a}$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0 \tag{7.4b}$$

This is a *first-order differential equation,* since only the first derivative of $v$ is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt \tag{7.5}$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

where $\ln A$ is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC} \tag{7.6}$$

Taking powers of $e$ produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(0) = A = V_0$. Hence,

$$v(t) = V_0 e^{-t/RC} \tag{7.7}$$

This shows that the voltage response of the *RC* circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the *natural response* of the circuit.

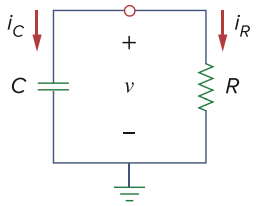



**Figure 7.1**
A source-free *RC* circuit.

A circuit response is the manner in which the circuit reacts to an excitation.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response depends on the nature of the circuit alone, with no external sources. In fact, the circuit has a response only because of the energy initially stored in the capacitor.

The natural response is illustrated graphically in Fig. 7.2. Note that at $t = 0$, we have the correct initial condition as in Eq. (7.1). As $t$ increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the *time constant*, denoted by $\tau$, the lowercase Greek letter tau.

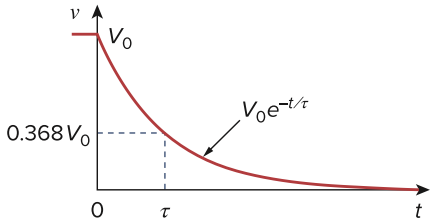



**Figure 7.2**
The voltage response of the *RC* circuit.

The time constant of a circuit is the time required for the response to decay to a factor of $1/e$ or 36.8 percent of its initial value.[1]

This implies that at $t = \tau$, Eq. (7.7) becomes

$$V_0e^{-\tau/RC} = V_0e^{-1} = 0.368V_0$$

or

$$\boxed{\tau = RC} \tag{7.8}$$

In terms of the time constant, Eq. (7.7) can be written as

$$\boxed{v(t) = V_0e^{-t/\tau}} \tag{7.9}$$

**TABLE 7.1**

Values of $v(t)/V_0 = e^{-t/\tau}$.

| $t$ | $v(t)/V_0$ |
|---|---|
| $\tau$ | 0.36788 |
| $2\tau$ | 0.13534 |
| $3\tau$ | 0.04979 |
| $4\tau$ | 0.01832 |
| $5\tau$ | 0.00674 |

With a calculator it is easy to show that the value of $v(t)/V_0$ is as shown in Table 7.1. It is evident from Table 7.1 that the voltage $v(t)$ is less than 1 percent of $V_0$ after $5\tau$ (five time constants). Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants. In other words, it takes $5\tau$ for the circuit to reach its final state or steady state when no changes take place with time. Notice that for every time interval of $\tau$, the voltage is reduced by 36.8 percent of its previous value, $v(t + \tau) = v(t)/e = 0.368v(t)$, regardless of the value of $t$.

Observe from Eq. (7.8) that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated in Fig. 7.4. A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state. At any rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.

With the voltage $v(t)$ in Eq. (7.9), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R}e^{-t/\tau} \tag{7.10}$$

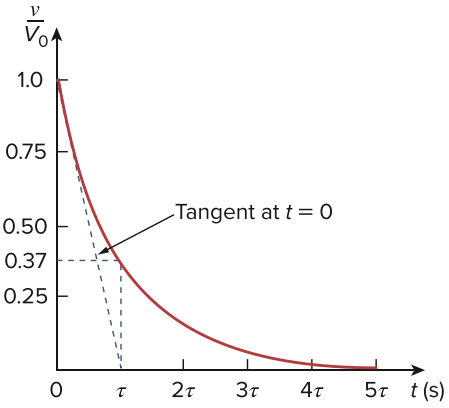



**Figure 7.3**
Graphical determination of the time constant $\tau$ from the response curve.

[1] The time constant may be viewed from another perspective. Evaluating the derivative of $v(t)$ in Eq. (7.7) at $t = 0$, we obtain

$$\frac{d}{dt}\left(\frac{v}{V_0}\right)\Big|_{t=0} = -\frac{1}{\tau}e^{-t/\tau}\Big|_{t=0} = -\frac{1}{\tau}$$

Thus, the time constant is the initial rate of decay, or the time taken for $v/V_0$ to decay from unity to zero, assuming a constant rate of decay. This initial slope interpretation of the time constant is often used in the laboratory to find $\tau$ graphically from the response curve displayed on an oscilloscope. To find $\tau$ from the response curve, draw the tangent to the curve at $t = 0$, as shown in Fig. 7.3. The tangent intercepts with the time axis at $t = \tau$.

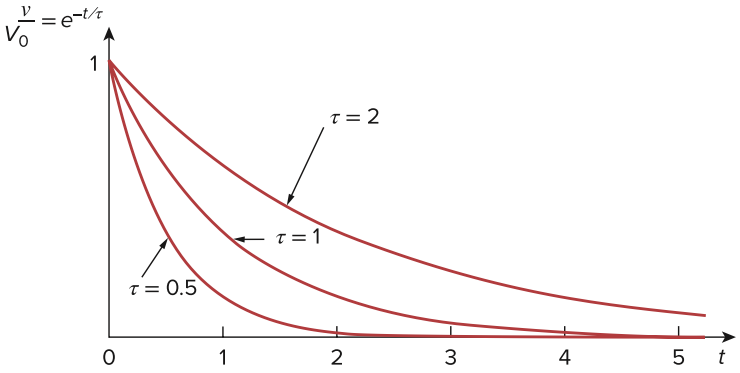



**Figure 7.4**
Plot of $v/V_0 = e^{-t/\tau}$ for various values of the time constant.

The power dissipated in the resistor is

$$p(t) = vi_R = \frac{V_0^2}{R} e^{-2t/\tau} \tag{7.11}$$

The energy absorbed by the resistor up to time $t$ is

$$\begin{aligned} w_R(t) &= \int_0^t p(\lambda)d\lambda = \int_0^t \frac{V_0^2}{R} e^{-2\lambda/\tau} d\lambda \\ &= -\frac{\tau V_0^2}{2R} e^{-2\lambda/\tau}\Big|_0^t = \frac{1}{2} CV_0^2(1 - e^{-2t/\tau}), \qquad \tau = RC \end{aligned} \tag{7.12}$$

Notice that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2}CV_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

In summary:

## The Key to Working with a Source-Free *RC* Circuit Is Finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant $\tau$.

The time constant is the same regardless of what the output is defined to be.

With these tw o items, we obtain the response as the capacitor v oltage $v_C(t) = v(t) = v(0)e^{-t/\tau}$. Once the capacitor voltage is first obtained, other variables (capacitor current $i_C$, resistor voltage $v_R$, and resistor current $i_R$) can be determined. In finding the time constant $\tau = RC$, $R$ is often the Thevenin equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor $C$ and find $R = R_{\text{Th}}$ at its terminals.

When a circuit contains a single capacitor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the capacitor to form a simple *RC* circuit. Also, one can use Thevenin's theorem when several capacitors can be combined to form a single equivalent capacitor.

## Example 7.1

In Fig. 7.5, let $v_C(0) = 15$ V. Find $v_C$, $v_x$, and $i_x$ for $t > 0$.

**Solution:**
We first need to make the circuit in Fig. 7.5 conform with the standard *RC* circuit in Fig. 7.1. We find the equivalent resistance or the Thevenin

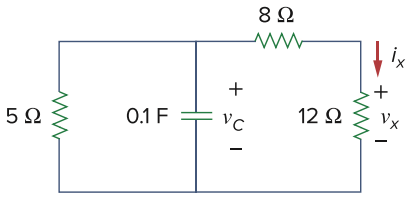



**Figure 7.5**
For Example 7.1.

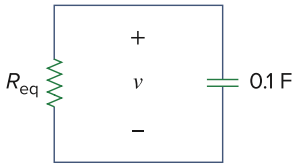



**Figure 7.6**
Equivalent circuit for the circuit in Fig. 7.5.

resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage $v_C$. From this, we can determine $v_x$ and $i_x$.

The 8-Ω and 12-Ω resistors in series can be combined to give a 20-Ω resistor. This 20-Ω resistor in parallel with the 5-Ω resistor can be combined so that the equivalent resistance is

$$R_{\text{eq}} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

Hence, the equivalent circuit is as shown in Fig. 7.6, which is analogous to Fig. 7.1. The time constant is

$$\tau = R_{\text{eq}}C = 4(0.1) = 0.4\ \text{s}$$

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}, \qquad v_C = v = 15e^{-2.5t}\ \text{V}$$

From Fig. 7.5, we can use voltage division to get $v_x$; so

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

## Practice Problem 7.1

Refer to the circuit in Fig. 7.7. Let $v_C(0) = 60$ V. Determine $v_C$, $v_x$, and $i_o$ for $t \geq 0$.

**Answer:** $60e^{-0.25t}$ V, $20e^{-0.25t}$ V, $-5e^{-0.25t}$ A.

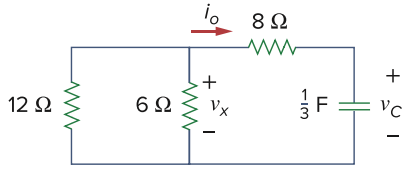



**Figure 7.7**
For Practice Prob. 7.1.

## Example 7.2

The switch in the circuit in Fig. 7.8 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

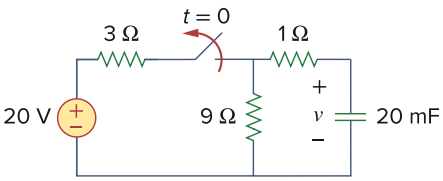



**Figure 7.8**
For Example 7.2.

**Solution:**

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in Fig. 7.9(a). Using voltage division

$$v_C(t) = \frac{9}{9 + 3}(20) = 15\ \text{V}, \qquad t < 0$$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15\ \text{V}$$

For $t > 0$, the switch is opened, and we have the *RC* circuit shown in Fig. 7.9(b). [Notice that the *RC* circuit in Fig. 7.9(b) is source free; the independent source in Fig. 7.8 is needed to provide $V_0$ or the initial energy in the capacitor.] The 1-Ω and 9-Ω resistors in series give

$$R_{eq} = 1 + 9 = 10\ \Omega$$

The time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2\ \text{s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2}\ \text{V}$$

or

$$v(t) = 15e^{-5t}\ \text{V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25\ \text{J}$$

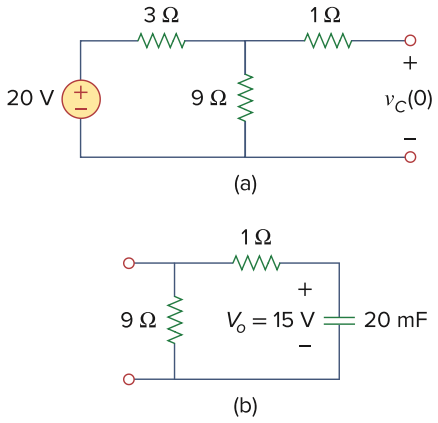



**Figure 7.9**
For Example 7.2: (a) $t < 0$, (b) $t > 0$.

## Practice Problem 7.2

If the switch in Fig. 7.10 opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

**Answer:** $8e^{-2t}$ V, 5.333 J.

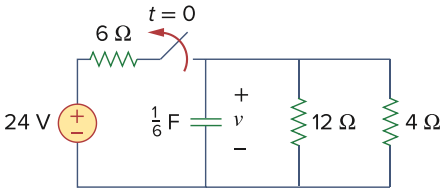



**Figure 7.10**
For Practice Prob. 7.2.

## 7.3 The Source-Free *RL* Circuit

Consider the series connection of a resistor and an inductor, as shown in Fig. 7.11. Our goal is to determine the circuit response, which we will assume to be the current $i(t)$ through the inductor. We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At $t = 0$, we assume that the inductor has an initial current $I_0$, or

$$i(0) = I_0 \tag{7.13}$$

with the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2}LI_0^2 \tag{7.14}$$

Applying KVL around the loop in Fig. 7.11,

$$v_L + v_R = 0 \tag{7.15}$$

But $v_L = L\,di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

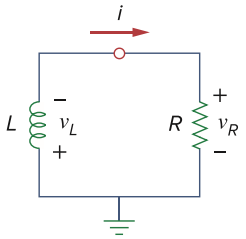



**Figure 7.11**
A source-free *RL* circuit.

or

$$\frac{di}{dt} + \frac{R}{L} i = 0 \tag{7.16}$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}\, dt$$

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Rightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

or

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \tag{7.17}$$

Taking the powers of $e$, we have

$$i(t) = I_0 e^{-Rt/L} \tag{7.18}$$

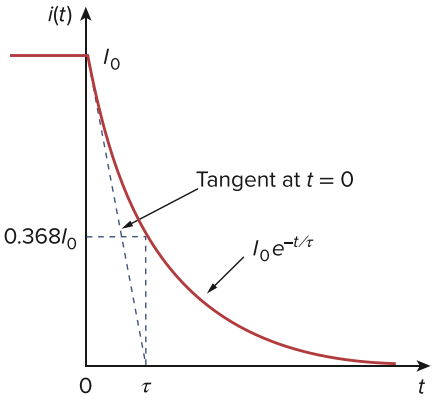



**Figure 7.12**
The current response of the *RL* circuit.

This shows that the natural response of the *RL* circuit is an exponential decay of the initial current. The current response is shown in Fig. 7.12. It is evident from Eq. (7.18) that the time constant for the *RL* circuit is

$$\boxed{\tau = \frac{L}{R}} \tag{7.19}$$

with $\tau$ again having the unit of seconds. Thus, Eq. (7.18) may be written as

$$\boxed{i(t) = I_0 e^{-t/\tau}} \tag{7.20}$$

With the current in Eq. (7.20), we can find the voltage across the resistor as

$$v_R(t) = iR = I_0 R e^{-t/\tau} \tag{7.21}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \tag{7.22}$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p(\lambda)d\lambda = \int_0^t I_0^2 R e^{-2\lambda/\tau}\, d\lambda = -\frac{\tau}{2} I_0^2 R e^{-2\lambda/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

or

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}) \tag{7.23}$$

Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} L I_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor as in Eq. (7.14). Again, the energy initially stored in the inductor is eventually dissipated in the resistor.

The smaller the time constant $\tau$ of a circuit, the faster the rate of decay of the response. The larger the time constant, the slower the rate of decay of the response. At any rate, the response decays to less than 1 percent of its initial value (i.e., reaches steady state) after $5\tau$.

Figure 7.12 shows an initial slope interpretation may be given to $\tau$.

In summary:

### The Key to Working with a Source-Free *RL* Circuit Is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant $\tau$ of the circuit.

With the tw o items, we obtain the response as the inductor current $i_L(t) = i(t) = i(0)e^{-t/\tau}$. Once we determine the inductor current $i_L$, other variables (inductor voltage $v_L$, resistor voltage $v_R$, and resistor current $i_R$) can be obtained. Note that in general $R$ in Eq. (7.19) is the Thevenin resistance at the terminals of the inductor.

When a circuit has a single inductor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the inductor to form a simple *RL* circuit. Also, one can use Thevenin's theorem when several inductors can be combined to form a single equivalent inductor.

## Example 7.3

Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit of Fig. 7.13.

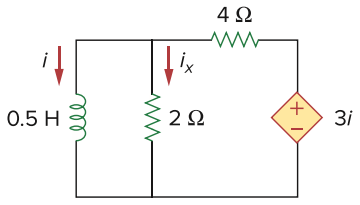



**Figure 7.13**
For Example 7.3.

**Solution:**
There are two ways we can solve this problem. One way is to obtain the equivalent resistance at the inductor terminals and then use Eq. (7.20). The other way is to start from scratch by using Kirchhoff's voltage law. Whichever approach is taken, it is always better to first obtain the inductor current.

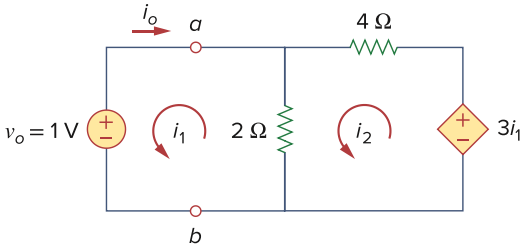
■ **METHOD 1** The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1$ V at the inductor terminals $a$-$b$, as in Fig. 7.14(a). (We could also insert a 1-A current source at the terminals.) Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \quad \Rightarrow \quad i_1 - i_2 = -\frac{1}{2} \tag{7.3.1}$$

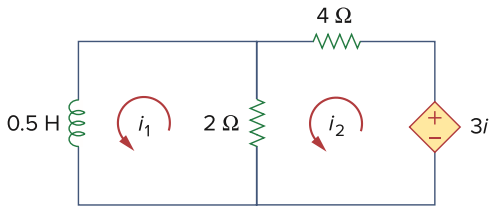
$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1 \tag{7.3.2}$$

Substituting Eq. (7.3.2) into Eq. (7.3.1) gives

$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A}$$




(a)




(b)

**Figure 7.14**
Solving the circuit in Fig. 7.13.

Hence,

$$R_{\text{eq}} = R_{\text{Th}} = \frac{v_o}{i_o} = \frac{1}{3}\,\Omega$$

The time constant is

$$\tau = \frac{L}{R_{\text{eq}}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{s}$$

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}\text{ A}, \qquad t > 0$$

■ **METHOD 2** We may directly apply KVL to the circuit as in Fig. 7.14(b). For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

or

$$\frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \tag{7.3.3}$$

For loop 2,

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1 \tag{7.3.4}$$

Substituting Eq. (7.3.4) into Eq. (7.3.3) gives

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

Rearranging terms,

$$\frac{di_1}{i_1} = -\frac{2}{3}dt$$

Since $i_1 = i$, we may replace $i_1$ with $i$ and integrate:

$$\ln i\Big|_{i(0)}^{i(t)} = -\frac{2}{3}t\Big|_0^t$$

or

$$\ln\frac{i(t)}{i(0)} = -\frac{2}{3}t$$

Taking the powers of $e$, we finally obtain

$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t}\text{ A}, \qquad t > 0$$

which is the same as by Method 1.

The voltage across the inductor is

$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t}\text{ V}$$

Since the inductor and the 2-Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.6667e^{-(2/3)t}\,\text{A}, \qquad t > 0$$

## Practice Problem 7.3

Find $i$ and $v_x$ in the circuit of Fig. 7.15. Let $i(0) = 7$ A.

**Answer:** $7e^{-2t}$ A, $-7e^{-2t}$ V, $t > 0$.

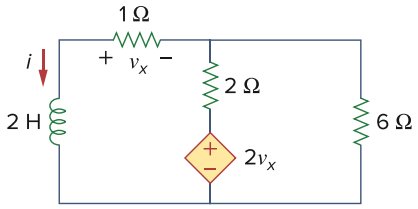



**Figure 7.15**
For Practice Prob. 7.3.

## Example 7.4

The switch in the circuit of Fig. 7.16 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

**Solution:**
When $t < 0$, the switch is closed, and the inductor acts as a short circuit to dc. The 16-Ω resistor is short-circuited; the resulting circuit is shown in Fig. 7.17(a). To get $i_1$ in Fig. 7.17(a), we combine the 4- Ω and 12-Ω resistors in parallel to get

$$\frac{4 \times 12}{4 + 12} = 3\ \Omega$$

Hence,

$$i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

We obtain $i(t)$ from $i_1$ in Fig. 7.17(a) using current division, by writing

$$i(t) = \frac{12}{12 + 4}\, i_1 = 6\ \text{A}, \qquad t < 0$$

Since the current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = 6\,\text{A}$$

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the source-free *RL* circuit in Fig. 7.17(b). Combining the resistors, we have

$$R_{\text{eq}} = (12 + 4) \parallel 16 = 8\ \Omega$$

The time constant is

$$\tau = \frac{L}{R_{\text{eq}}} = \frac{2}{8} = \frac{1}{4}\,\text{s}$$

Thus,

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\ \text{A}$$

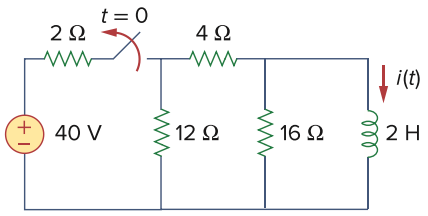



**Figure 7.16**
For Example 7.4.

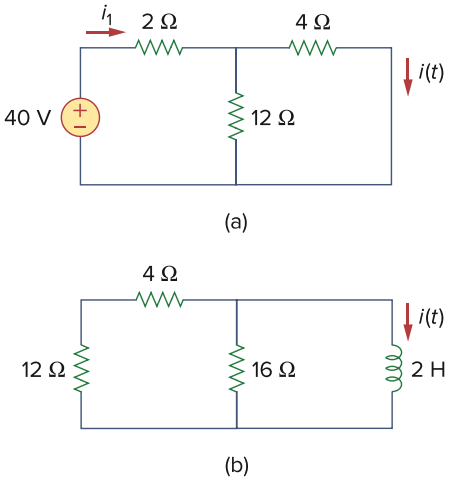



**Figure 7.17**
Solving the circuit of Fig. 7.16: (a) for $t < 0$, (b) for $t > 0$.

## Practice Problem 7.4

For the circuit in Fig. 7.18, find $i(t)$ for $t > 0$.

**Answer:** $2e^{-2t}$ A, $t > 0$.

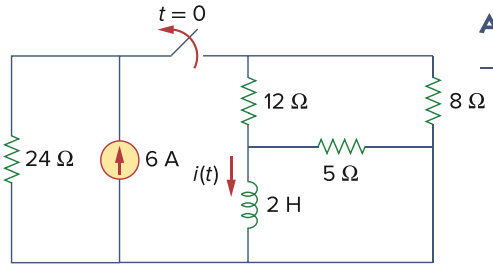



**Figure 7.18**
For Practice Prob. 7.4.

## Example 7.5

In the circuit shown in Fig. 7.19, find $i_o$, $v_o$, and $i$ for all time, assuming that the switch was open for a long time.

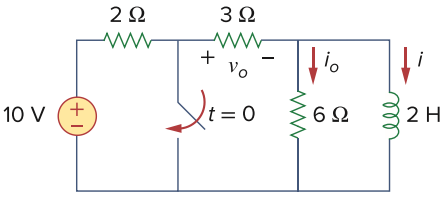



**Figure 7.19**
For Example 7.5.

**Solution:**
It is better to first find the inductor current $i$ and then obtain other quantities from it.

For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, so that we ha ve the circuit shown in Fig. 7.20(a). Hence, $i_o = 0$, and

$$i(t) = \frac{10}{2+3} = 2 \text{ A}, \qquad t < 0$$

$$v_o(t) = 3i(t) = 6 \text{ V}, \qquad t < 0$$

Thus, $i(0) = 2$.

For $t > 0$, the switch is closed, so that the v oltage source is short-circuited. We now have a source-free $RL$ circuit as shown in Fig. 7.20(b). At the inductor terminals,

$$R_{\text{Th}} = 3 \parallel 6 = 2\ \Omega$$

so that the time constant is

$$\tau = \frac{L}{R_{\text{Th}}} = 1 \text{ s}$$

Hence,

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\text{A}, \qquad t > 0$$

Because the inductor is in parallel with the 6- and 3-Ω resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t} \text{ V}, \qquad t > 0$$

and

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\text{A}, \qquad t > 0$$

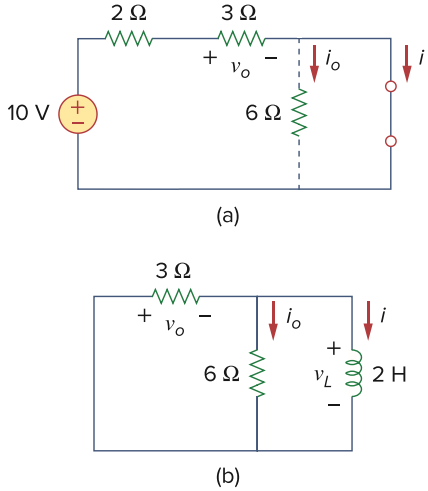



**Figure 7.20**
The circuit in Fig. 7.19 for: (a) $t < 0$, (b) $t > 0$.

Thus, for all time,

$$i_o(t) = \begin{cases} 0\text{ A}, & t < 0 \\ -\frac{2}{3}e^{-t}\text{ A}, & t > 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6\text{ V}, & t < 0 \\ 4e^{-t}\text{ V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\text{ A}, & t < 0 \\ 2e^{-t}\text{ A}, & t \geq 0 \end{cases}$$

We notice that the inductor current is continuous at $t = 0$, while the current through the 6-Ω resistor drops from 0 to $-2/3$ at $t = 0$, and the voltage across the 3-Ω resistor drops from 6 to 4 at $t = 0$. We also notice that the time constant is the same regardless of what the output is defined to be. Figure 7.21 plots $i$ and $i_o$.

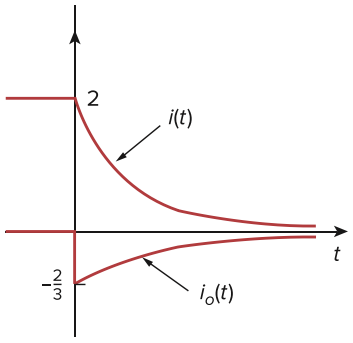



**Figure 7.21**
A plot of $i$ and $i_o$.

## Practice Problem 7.5

Determine $i$, $i_o$, and $v_o$ for all $t$ in the circuit shown in Fig. 7.22. Assume that the switch was closed for a long time. It should be noted that opening a switch in series with an ideal current source creates an infinite voltage at the current source terminals. Clearly this is impossible. For the purposes of problem solving, we can place a shunt resistor in parallel with the source (which now makes it a voltage source in series with a resistor). In more practical circuits, devices that act like current sources are, for the most part, electronic circuits. These circuits will allow the source to act like an ideal current source over its operating range but voltage-limit it when the load resistor becomes too large (as in an open circuit).

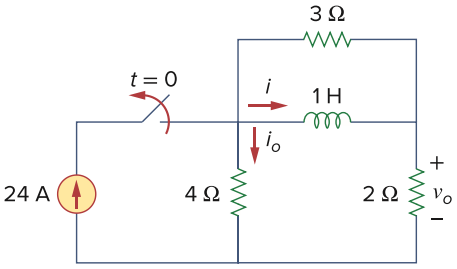



**Figure 7.22**
For Practice Prob. 7.5.

**Answer:**

$$i = \begin{cases} 16\text{ A}, & t < 0 \\ 16e^{-2t}\text{ A}, & t \geq 0 \end{cases}, \qquad i_o = \begin{cases} 8\text{ A}, & t < 0 \\ -5.333e^{-2t}\text{ A}, & t > 0 \end{cases}$$

$$v_o = \begin{cases} 32\text{ V}, & t < 0 \\ 10.667e^{-2t}\text{ V}, & t > 0 \end{cases}$$

## 7.4 Singularity Functions

Before going on with the second half of this chapter, we need to digress and consider some mathematical concepts that will aid our understanding of transient analysis. A basic understanding of singularity functions will help us make sense of the response of first-order circuits to a sudden application of an independent dc voltage or current source.

Singularity functions (also called *switching functions*) are very useful in circuit analysis. They serve as good approximations to the switching signals that arise in circuits with switching operations. They are helpful in the neat, compact description of some circuit phenomena, especially the step response of *RC* or *RL* circuits to be discussed in the next sections. By definition,

**Singularity functions** are functions that either are discontinuous or have discontinuous derivatives.

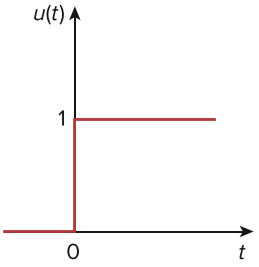



**Figure 7.23**
The unit step function.

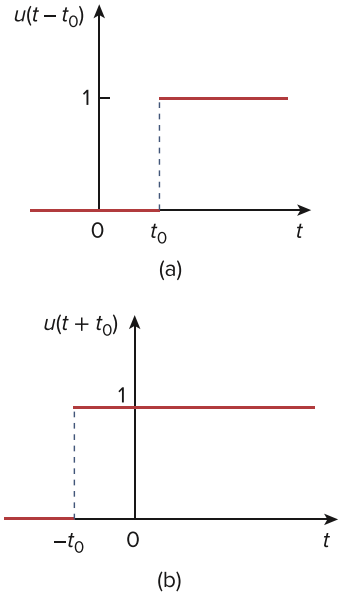



**Figure 7.24**
(a) The unit step function delayed by $t_0$, (b) the unit step advanced by $t_0$.

The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.

The unit step function $u(t)$ is 0 for negative values of $t$ and 1 for positive values of $t$.

In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{7.24}$$

The unit step function is undefined at $t = 0$, where it changes abruptly from 0 to 1. It is dimensionless, like other mathematical functions such as sine and cosine. Figure 7.23 depicts the unit step function. If the abrupt change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, the unit step function becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} \tag{7.25}$$

which is the same as saying that $u(t)$ is delayed by $t_0$ seconds, as shown in Fig. 7.24(a). To get Eq. (7.25) from Eq. (7.24), we simply replace every $t$ by $t - t_0$. If the change is at $t = -t_0$, the unit step function becomes

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} \tag{7.26}$$

meaning that $u(t)$ is advanced by $t_0$ seconds, as shown in Fig. 7.24(b).

We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \tag{7.27}$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0) \tag{7.28}$$

If we let $t_0 = 0$, then $v(t)$ is simply the step voltage $V_0 u(t)$. A voltage source of $V_0 u(t)$ is shown in Fig. 7.25(a); its equivalent circuit is shown in Fig. 7.25(b). It is evident in Fig. 7.25(b) that terminals $a$-$b$ are short-circuited ($v = 0$) for $t < 0$ and that $v = V_0$ appears at the terminals for

Alternatively, we may derive Eqs. (7.25) and (7.26) from Eq. (7.24) by writing $u[f(t)] = 1$, $f(t) > 0$, where $f(t)$ may be $t - t_0$ or $t + t_0$.

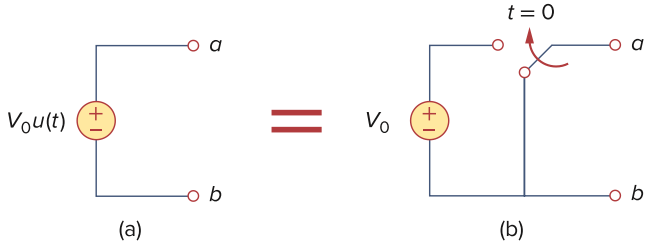



**Figure 7.25**
(a) Voltage source of $V_0 u(t)$, (b) its equivalent circuit.

$t > 0$. Similarly, a current source of $I_0u(t)$ is shown in Fig. 7.26(a), while its equivalent circuit is in Fig. 7.26(b). Notice that for $t < 0$, there is an open circuit ($i = 0$), and that $i = I_0$ flows for $t > 0$.

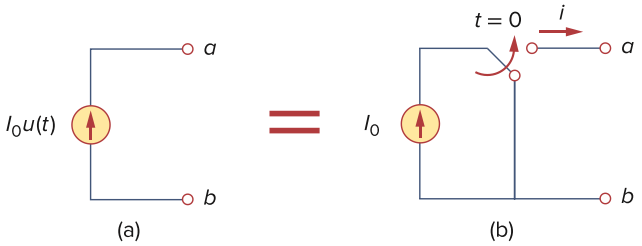



**Figure 7.26**
(a) Current source of $I_0u(t)$, (b) its equivalent circuit.

The derivative of the unit step function $u(t)$ is the *unit impulse function* $\delta(t)$, which we write as

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases} \tag{7.29}$$

The unit impulse function—also known as the *delta* function—is shown in Fig. 7.27.

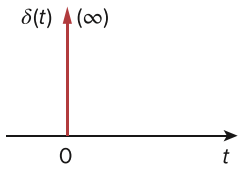



**Figure 7.27**
The unit impulse function.

The unit impulse function $\delta(t)$ is zero everywhere except at $t = 0$, where it is undefined.

Impulsive currents and voltages occur in electric circuits as a result of switching operations or impulsive sources. Although the unit impulse function is not physically realizable (just like ideal sources, ideal resistors, etc.), it is a very useful mathematical tool.

The unit impulse may be regarded as an applied or resulting shock. It may be visualized as a very short duration pulse of unit area. This may be expressed mathematically as

$$\int_{0^-}^{0^+} \delta(t)\,dt = 1 \tag{7.30}$$

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ is the time just after $t = 0$. For this reason, it is customary to write 1 (denoting unit area) beside the arrow that is used to symbolize the unit impulse function, as in Fig. 7.27. The unit area is known as the *strength* of the impulse function. When an impulse function has a strength other than unity, the area of the impulse is equal to its strength. For example, an impulse function $10\delta(t)$ has an area of 10. Figure 7.28 shows the impulse functions $5\delta(t + 2)$, $10\delta(t)$, and $-4\delta(t - 3)$.

To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t - t_0)dt \tag{7.31}$$

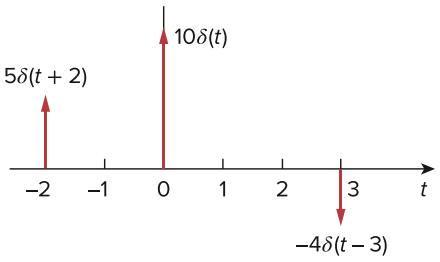



**Figure 7.28**
Three impulse functions.

where $a < t_0 < b$. Since $\delta(t - t_0) = 0$ except at $t = t_0$, the integrand is zero except at $t_0$. Thus,

$$\int_a^b f(t)\delta(t - t_0)\,dt = \int_a^b f(t_0)\delta(t - t_0)\,dt$$
$$= f(t_0)\int_a^b \delta(t - t_0)dt = f(t_0)$$

or

$$\boxed{\int_a^b f(t)\delta(t - t_0)\,dt = f(t_0)} \tag{7.32}$$

This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs. This is a highly useful property of the impulse function known as the *sampling* or *sifting* property. The special case of Eq. (7.31) is for $t_0 = 0$. Then Eq. (7.32) becomes

$$\int_{0^-}^{0^+} f(t)\delta(t)\,dt = f(0) \tag{7.33}$$

Integrating the unit step function $u(t)$ results in the *unit ramp function* $r(t)$; we write

$$r(t) = \int_{-\infty}^{t} u(\lambda)d\lambda = tu(t) \tag{7.34}$$

or

$$\boxed{r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}} \tag{7.35}$$

The unit ramp function is zero for negative values of $t$ and has a unit slope for positive values of $t$.

Figure 7.29 shows the unit ramp function. In general, a ramp is a function that changes at a constant rate.

The unit ramp function may be delayed or advanced as shown in Fig. 7.30. For the delayed unit ramp function,

$$r(t - t_0) = \begin{cases} 0, & t \le t_0 \\ t - t_0, & t \ge t_0 \end{cases} \tag{7.36}$$

and for the advanced unit ramp function,

$$r(t + t_0) = \begin{cases} 0, & t \le -t_0 \\ t + t_0, & t \ge -t_0 \end{cases} \tag{7.37}$$

We should keep in mind that the three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt} \tag{7.38}$$

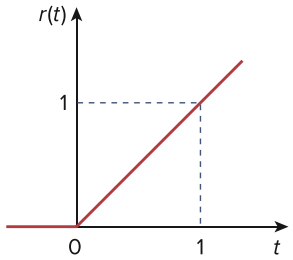
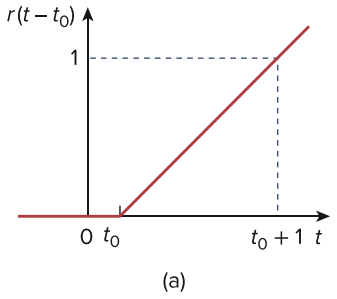
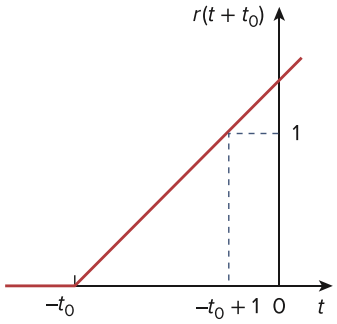



**Figure 7.29**
The unit ramp function.




(a)




(b)

**Figure 7.30**
The unit ramp function: (a) delayed by $t_0$, (b) advanced by $t_0$.

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda, \quad r(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda \tag{7.39}$$

Although there are many more singularity functions, we are only interested in these three (the impulse function, the unit step function, and the ramp function) at this point.

## Example 7.6

Express the voltage pulse in Fig. 7.31 in terms of the unit step. Calculate its derivative and sketch it.

Gate functions are used along with switches to pass or block another signal.

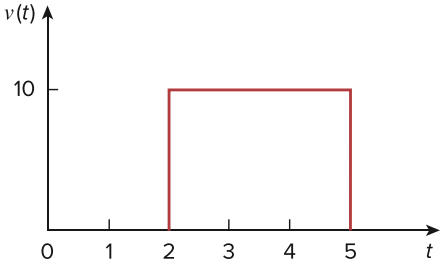



**Figure 7.31**
For Example 7.6.

**Solution:**

The type of pulse in Fig. 7.31 is called the *gate function*. It may be regarded as a step function that switches on at one value of $t$ and switches off at another value of $t$. The gate function shown in Fig. 7.31 switches on at $t = 2$ s and switches off at $t = 5$ s. It consists of the sum of two unit step functions as shown in Fig. 7.32(a). From the figure, it is evident that

$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

Taking the derivative of this gives

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$

which is shown in Fig. 7.32(b). We can obtain Fig. 7.32(b) directly from Fig. 7.31 by simply observing that there is a sudden increase by 10 V at $t = 2$ s leading to $10\,\delta(t-2)$. At $t = 5$ s, there is a sudden decrease by 10 V leading to $-10\text{V}\ \delta(t-5)$.

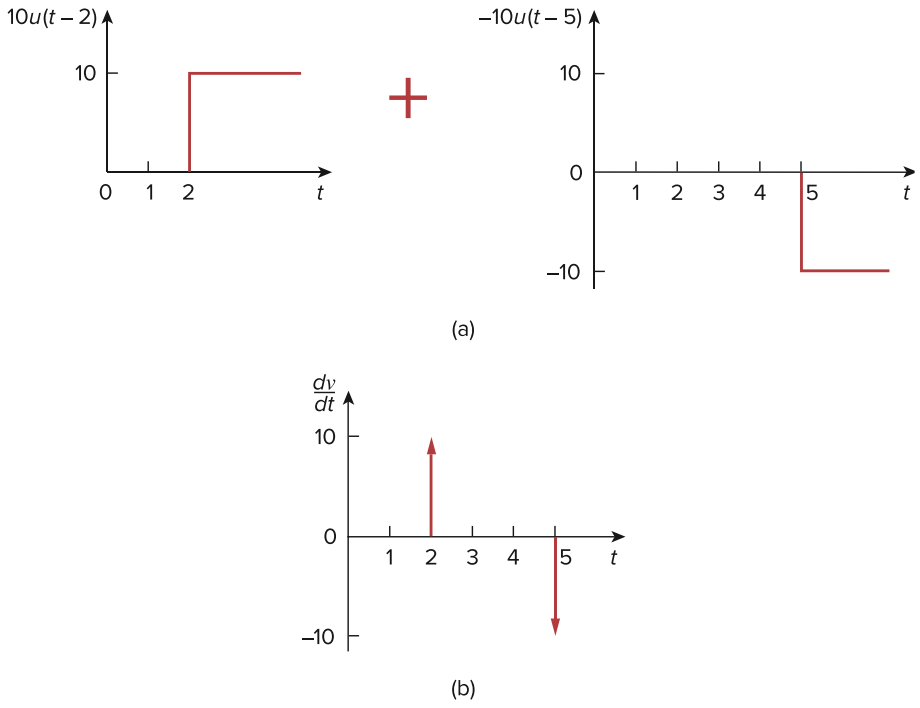



**Figure 7.32**
(a) Decomposition of the pulse in Fig. 7.31, (b) derivative of the pulse in Fig. 7.31.

## Practice Problem 7.6

Express the current pulse in Fig. 7.33 in terms of the unit step. Find its integral and sketch it.

**Answer:** $10[u(t) - 2u(t-2) + u(t-4)]$ A, $10[r(t) - 2r(t-2) + r(t-4)]$ amp-sec. See Fig. 7.34.

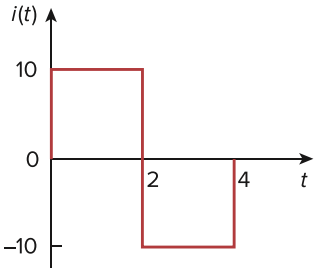



**Figure 7.33**
For Practice Prob. 7.6.

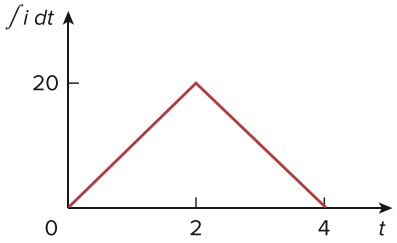



**Figure 7.34**
Integral of $i(t)$ in Fig. 7.33.

## Example 7.7

Express the *sawtooth* function shown in Fig. 7.35 in terms of singularity functions.

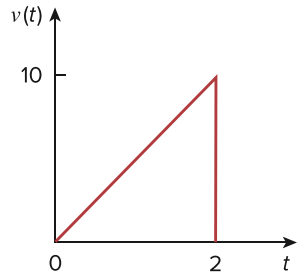



**Figure 7.35**
For Example 7.7.

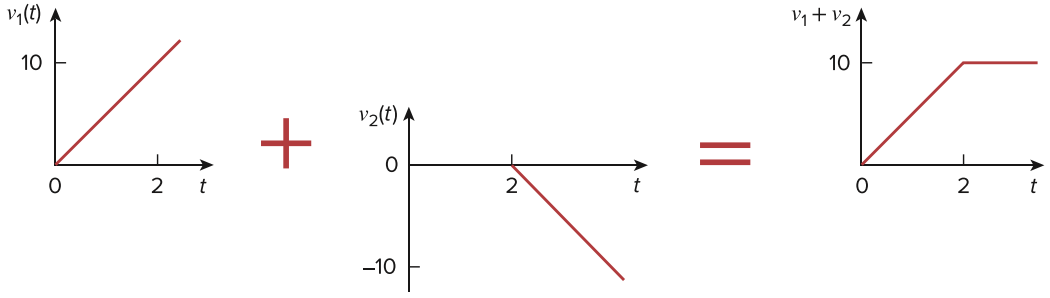
**Solution:**
There are three ways of solving this problem. The first method is by mere observation of the given function, while the other methods involve some graphical manipulations of the function.

■ **METHOD 1** By looking at the sketch of $v(t)$ in Fig. 7.35, it is not hard to notice that the given function $v(t)$ is a combination of singularity functions. So we let

$$v(t) = v_1(t) + v_2(t) + \cdots \tag{7.7.1}$$

The function $v_1(t)$ is the ramp function of slope 5, shown in Fig. 7.36(a); that is,

$$v_1(t) = 5r(t) \tag{7.7.2}$$




(a) (b) (c)

**Figure 7.36**
Partial decomposition of $v(t)$ in Fig. 7.35.

Since $v_1(t)$ goes to infinity, we need another function at $t = 2$s in order to get $v(t)$. We let this function be $v_2$, which is a ramp function of slope $-5$, as shown in Fig. 7.36(b); that is,

$$v_2(t) = -5r(t-2) \tag{7.7.3}$$

Adding $v_1$ and $v_2$ gives us the signal in Fig. 7.36(c). Obviously, this is not the same as $v(t)$ in Fig. 7.35. But the difference is simply a constant 10 units for $t > 2$ s. By adding a third signal $v_3$, where

$$v_3 = -10u(t-2) \tag{7.7.4}$$

we get $v(t)$, as shown in Fig. 7.37. Substituting Eqs. (7.7.2) through (7.7.4) into Eq. (7.7.1) gives

$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$$

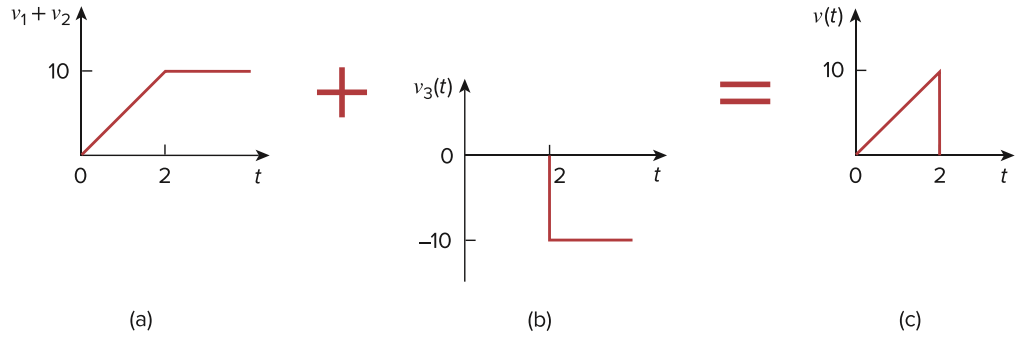



**Figure 7.37**
Complete decomposition of $v(t)$ in Fig. 7.35.

**METHOD 2** A close observation of Fig. 7.35 reveals that $v(t)$ is a multiplication of two functions: a ramp function and a gate function. Thus,

$$\begin{aligned} v(t) &= 5t[u(t) - u(t-2)] \\ &= 5tu(t) - 5tu(t-2) \\ &= 5r(t) - 5(t-2+2)u(t-2) \\ &= 5r(t) - 5(t-2)u(t-2) - 10u(t-2) \\ &= 5r(t) - 5r(t-2) - 10u(t-2) \end{aligned}$$

the same as before.

**METHOD 3** This method is similar to Method 2. We observe from Fig. 7.35 that $v(t)$ is a multiplication of a ramp function and a unit step function, as shown in Fig. 7.38. Thus,

$$v(t) = 5r(t)u(-t+2)$$

If we replace $u(-t)$ by $1 - u(t)$, then we can replace $u(-t+2)$ by $1 - u(t-2)$. Hence,

$$v(t) = 5r(t)[1 - u(t-2)]$$

which can be simplified as in Method 2 to get the same result.

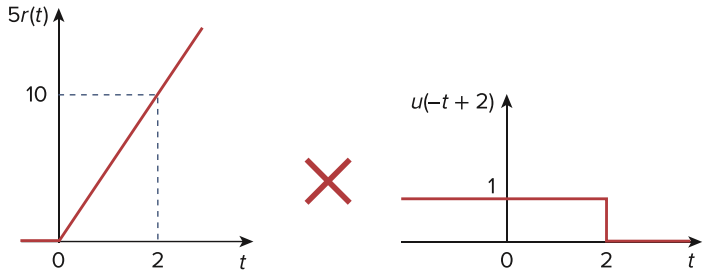



**Figure 7.38**
Decomposition of $v(t)$ in Fig. 7.35.

## Practice Problem 7.7

Refer to Fig. 7.39. Express $i(t)$ in terms of singularity functions.

**Answer:** $2u(t) - 2r(t) + 4r(t-2) - 2r(t-3)$ A.

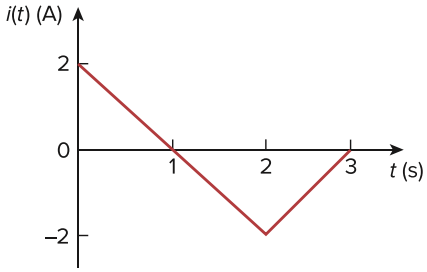



**Figure 7.39**
For Practice Prob. 7.7.

## Example 7.8

Given the signal

$$g(t) = \begin{cases} 3, & t < 0 \\ -2, & 0 < t < 1 \\ 2t - 4, & t > 1 \end{cases}$$

express $g(t)$ in terms of step and ramp functions.

**Solution:**
The signal $g(t)$ may be regarded as the sum of three functions specified within the three intervals $t < 0$, $0 < t < 1$, and $t > 1$.

For $t < 0$, $g(t)$ may be regarded as 3 multiplied by $u(-t)$, where $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Within the time interval $0 < t < 1$, the function may be considered as $-2$ multiplied by a gated function $[u(t) - u(t-1)]$. For $t > 1$, the function may be regarded as $2t - 4$ multiplied by the unit step function $u(t-1)$. Thus,

$$\begin{aligned} g(t) &= 3u(-t) - 2[u(t) - u(t-1)] + (2t-4)u(t-1) \\ &= 3u(-t) - 2u(t) + (2t - 4 + 2)u(t-1) \\ &= 3u(-t) - 2u(t) + 2(t-1)u(t-1) \\ &= 3u(-t) - 2u(t) + 2r(t-1) \end{aligned}$$

One may avoid the trouble of using $u(-t)$ by replacing it with $1 - u(t)$. Then

$$g(t) = 3[1 - u(t)] - 2u(t) + 2r(t-1) = 3 - 5u(t) + 2r(t-1)$$

Alternatively, we may plot $g(t)$ and apply Method 1 from Example 7.7.

## Practice Problem 7.8

If

$$h(t) = \begin{cases} 0, & t < 0 \\ -4, & 0 < t < 2 \\ 3t - 8, & 2 < t < 6 \\ 0, & t > 6 \end{cases}$$

express $h(t)$ in terms of the singularity functions.

**Answer:** $-4u(t) + 2u(t-2) + 3r(t-2) - 10u(t-6) - 3r(t-6)$.

## Example 7.9

Evaluate the following integrals involving the impulse function:

$$\int_0^{10} (t^2 + 4t - 2)\delta(t-2)\, dt$$

$$\int_{-\infty}^{\infty} [\delta(t-1)e^{-t}\cos t + \delta(t+1)e^{-t}\sin t]\, dt$$

**Solution:**

For the first integral, we apply the sifting property in Eq. (7.32).

$$\int_0^{10} (t^2 + 4t - 2)\delta(t-2)\, dt = (t^2 + 4t - 2)|_{t=2} = 4 + 8 - 2 = 10$$

Similarly, for the second integral,

$$\begin{aligned} &\int_{-\infty}^{\infty} [\delta(t-1)e^{-t}\cos t + \delta(t+1)e^{-t}\sin t]\, dt \\ &= e^{-t}\cos t|_{t=1} + e^{-t}\sin t|_{t=-1} \\ &= e^{-1}\cos 1 + e^{1}\sin(-1) = 0.1988 - 2.2873 = -2.0885 \end{aligned}$$

## Practice Problem 7.9

Evaluate the following integrals:

$$\int_{-\infty}^{\infty} (t^3 + 5t^2 + 10)\delta(t+3)\, dt, \qquad \int_0^{10} \delta(t-\pi)\cos 3t\, dt$$

**Answer:** 28, $-1$.

# 7.5 Step Response of an *RC* Circuit

When the dc source of an *RC* circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a *step response*.

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

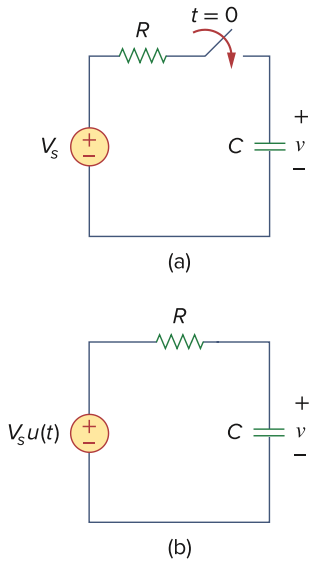



**Figure 7.40**
An *RC* circuit with voltage step input.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the *RC* circuit in Fig. 7.40(a) which can be replaced by the circuit in Fig. 7.40(b), where $V_s$ is a constant dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined. We assume an initial voltage $V_0$ on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \tag{7.40}$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \tag{7.41}$$

where $v$ is the voltage across the capacitor. For $t > 0$, Eq. (7.41) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \tag{7.42}$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

or

$$\frac{dv}{v - V_s} = -\frac{dt}{RC} \tag{7.43}$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \tag{7.44}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0 \tag{7.45}$$

Thus,

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases} \tag{7.46}$$

This is known as the *complete response* (or total response) of the *RC* circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later . Assuming that $V_s > V_0$, a plot of $v(t)$ is sho wn in Fig. 7.41.

If we assume that the capacitor is uncharged initially, we set $V_0 = 0$ in Eq. (7.46) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases} \tag{7.47}$$

which can be written alternatively as

$$v(t) = V_s(1 - e^{-t/\tau})u(t) \tag{7.48}$$

This is the complete step response of the *RC* circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq. (7.47) using $i(t) = C\,dv/dt$. We get

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad \tau = RC, \qquad t > 0$$

or

$$i(t) = \frac{V_s}{R}e^{-t/\tau}\,u(t) \tag{7.49}$$

Figure 7.42 shows the plots of capacitor v oltage $v(t)$ and capacitor current $i(t)$.

Rather than going through the derivations above, there is a systematic approach—or rather, a shortcut method—for finding the step response of an *RC* or *RL* circuit. Let us reexamine Eq. (7.45), which is more general than Eq. (7.48). It is evident that $v(t)$ has two components. Classically there are two ways of decomposing this into tw o components. The first is to break it into a "natural response and a forced response" and the second is to break it into a "transient response and a steady-state response." Starting with the natural response and forced response, we write the total or complete response as

Complete response = natural response + forced response
(stored energy; independent source)

or

$$v = v_n + v_f \tag{7.50}$$

where

$$v_n = V_o e^{-t/\tau}$$

and

$$v_f = V_s(1 - e^{-t/\tau})$$

We are familiar with the natural response $v_n$ of the circuit, as discussed in Section 7.2. $v_f$ is known as the *forced* response because it is produced by the circuit when an external "force" (a voltage source in this case) is applied. It represents what the circuit is forced to do by the input excitation. The natural response e ventually dies out along with the transient component of the forced response, leaving only the steady-state component of the forced response.

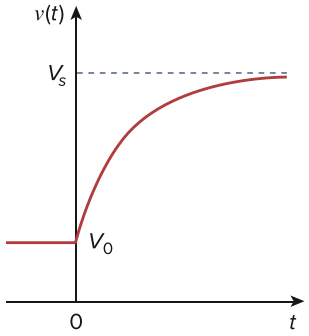



**Figure 7.41**
Response of an *RC* circuit with initially charged capacitor.

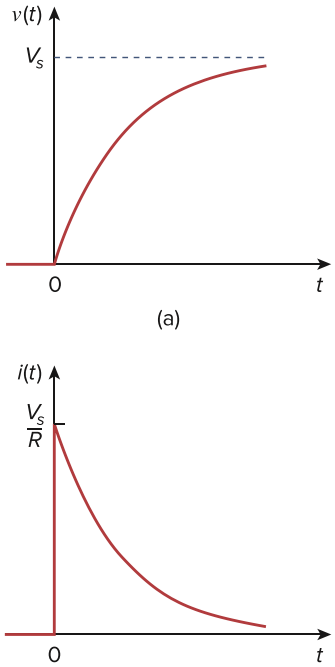



**Figure 7.42**
Step response of an *RC* circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Another way of looking at the complete response is to break into two components—one temporary and the other permanent, that is

$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}$$

or

$$v = v_t + v_{ss} \tag{7.51}$$

where

$$v_t = (V_o - V_s)e^{-t/\tau} \tag{7.52a}$$

and

$$v_{ss} = V_s \tag{7.52b}$$

The *transient response* $v_t$ is temporary; it is the portion of the complete response that decays to zero as time approaches infinity. Thus,

The **transient response** is the circuit's temporary response that will die out with time.

The *steady-state response* $v_{ss}$ is the portion of the complete response that remains after the transient reponse has died out. Thus,

The **steady-state response** is the behavior of the circuit a long time after an external excitation is applied.

The first decomposition of the complete response is in terms of the source of the responses, while the second decomposition is in terms of the permanency of the responses. Under certain conditions, the natural response and transient response are the same.The same can be said about the forced response and steady-state response.

Whichever way we look at it, the complete response in Eq. (7.45) may be written as

This is the same as saying that the complete response is the sum of the transient response and the steady-state response.

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \tag{7.53}$$

where $v(0)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady-state value. Thus, to find the step response of an *RC* circuit requires three things:

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant $\tau$.

Once we know $x(0)$, $x(\infty)$, and $\tau$, almost all the circuit problems in this chapter can be solved using the formula

$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}$$

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the

response using Eq. (7.53). This technique equally applies to *RL* circuits, as we shall see in the next section.

Note that if the switch changes position at time $t = t_0$ instead of at $t = 0$, there is a time delay in the response so that Eq. (7.53) becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t - t_0)/\tau} \tag{7.54}$$

where $v(t_0)$ is the initial value at $t = t_0^+$. Keep in mind that Eq. (7.53) or (7.54) applies only to step responses, that is, when the input e xcitation is constant.

## Example 7.10

The switch in Fig. 7.43 has been in position *A* for a long time. At $t = 0$, the switch moves to *B*. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ and 4 s.

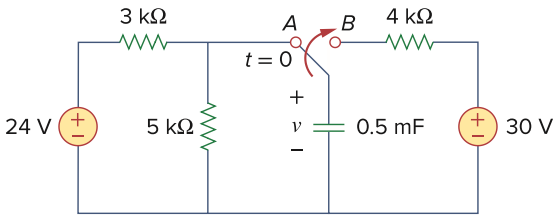



**Figure 7.43**
For Example 7.10.

**Solution:**

For $t < 0$, the switch is at position *A*. The capacitor acts like an open circuit to dc, but $v$ is the same as the voltage across the 5-k Ω resistor. Hence, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

$$v(0^-) = \frac{5}{5 + 3}(24) = 15 \text{ V}$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$

For $t > 0$, the switch is in position *B*. The Thevenin resistance connected to the capacitor is $R_{\text{Th}} = 4$ kΩ, and the time constant is

$$\tau = R_{\text{Th}} C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V}$$

At $t = 1$,

$$v(1) = 30 - 15e^{-0.5} = 20.9 \text{ V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97 \text{ V}$$

## Practice Problem 7.10

Find $v(t)$ for $t > 0$ in the circuit of Fig. 7.44. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$.

**Answer:** $(9.375 + 5.625e^{-2t})$ V for all $t > 0$, 11.444 V.

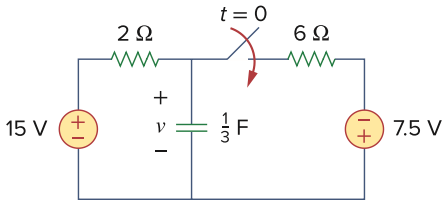



**Figure 7.44**
For Practice Prob. 7.10.

## Example 7.11

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find $i$ and $v$ for all time.

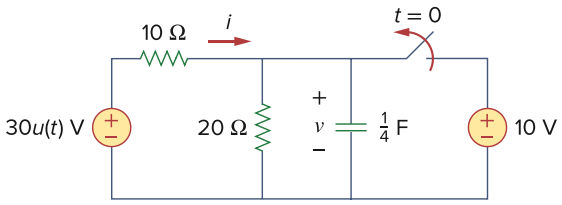



**Figure 7.45**
For Example 7.11.

**Solution:**

The resistor current $i$ can be discontinuous at $t = 0$, while the capacitor voltage $v$ cannot. Hence, it is always better to find $v$ and then obtain $i$ from $v$.

By definition of the unit step function,

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$, the switch is closed and $30\,u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit and should be regarded as contributing nothing to $v$. Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit. Hence, the circuit becomes that shown in Fig. 7.46(a) for $t < 0$. From this circuit we obtain

$$v = 10 \text{ V}, \qquad i = -\frac{v}{10} = -1 \text{ A}$$

Since the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = 10 \text{ V}$$

For $t > 0$, the switch is opened and the 10-V voltage source is disconnected from the circuit. The $30\,u(t)$ voltage source is now operative, so the circuit becomes that shown in Fig. 7.46(b). After a long time, the circuit reaches steady state and the capacitor acts like an open circuit again. We obtain $v(\infty)$ by using voltage division, writing

$$v(\infty) = \frac{20}{20 + 10}(30) = 20 \text{ V}$$

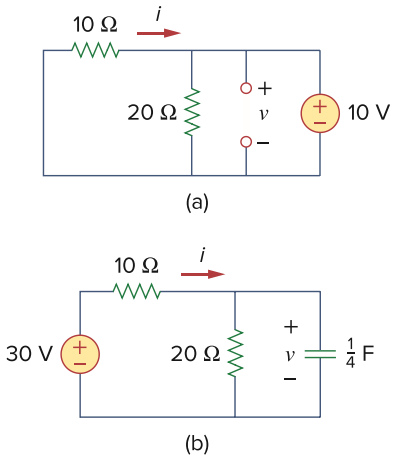



**Figure 7.46**
Solution of Example 7.11: (a) for $t < 0$, (b) for $t > 0$.

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \| 20 = \frac{10 \times 20}{30} = \frac{20}{3}\,\Omega$$

and the time constant is

$$\tau = R_{\text{Th}}\, C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3}\,\text{s}$$

Thus,

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t})\ \text{V} \end{aligned}$$

To obtain $i$, we notice from Fig. 7.46(b) that $i$ is the sum of the currents through the 20-Ω resistor and the capacitor; that is,

$$\begin{aligned} i &= \frac{v}{20} + C\frac{dv}{dt} \\ &= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})\ \text{A} \end{aligned}$$

Notice from Fig. 7.46(b) that $v + 10i = 30$ is satisfied, as expected. Hence,

$$v = \begin{cases} 10\ \text{V}, & t < 0 \\ (20 - 10e^{-0.6t})\ \text{V}, & t \geq 0 \end{cases}$$

$$i = \begin{cases} -1\ \text{A}, & t < 0 \\ (1 + e^{-0.6t})\ \text{A}, & t > 0 \end{cases}$$

Notice that the capacitor voltage is continuous while the resistor current is not.

## Practice Problem 7.11

The switch in Fig. 7.47 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

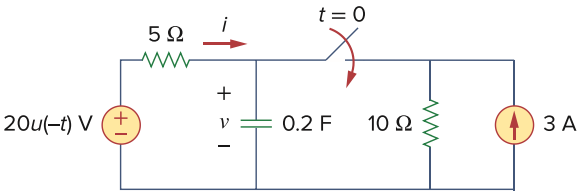



**Figure 7.47**
For Practice Prob. 7.11.

**Answer:** $i(t) = \begin{cases} 0, & t < 0 \\ -2(1 + e^{-1.5t})\ \text{A}, & t > 0, \end{cases}$

$$v = \begin{cases} 20\ \text{V}, & t < 0 \\ 10(1 + e^{-1.5t})\ \text{V}, & t > 0 \end{cases}$$

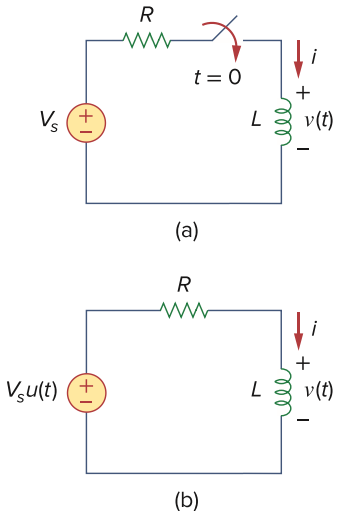



**Figure 7.48**
An *RL* circuit with a step input voltage.

## 7.6 Step Response of an *RL* Circuit

Consider the *RL* circuit in Fig. 7.48(a), which may be replaced by the circuit in Fig. 7.48(b). Again, our goal is to find the inductor current $i$ as the circuit response. Rather than apply Kirchhoff's laws, we will use the simple technique in Eqs. (7.50) through (7.53). Let the response be the sum of the transient response and the steady-state response,

$$i = i_t + i_{ss} \tag{7.55}$$

We know that the transient response is always a decaying exponential, that is,

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R} \tag{7.56}$$

where $A$ is a constant to be determined.

The steady-state response is the value of the current a long time after the switch in Fig. 7.48(a) is closed. We know that the transient response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage $V_s$ appears across $R$. Thus, the steady-state response is

$$i_{ss} = \frac{V_s}{R} \tag{7.57}$$

Substituting Eqs. (7.56) and (7.57) into Eq. (7.55) gives

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \tag{7.58}$$

We now determine the constant $A$ from the initial value of $i$. Let $I_0$ be the initial current through the inductor, which may come from a source other than $V_s$. Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \tag{7.59}$$

Thus, at $t = 0$, Eq. (7.58) becomes

$$I_0 = A + \frac{V_s}{R}$$

From this, we obtain $A$ as

$$A = I_0 - \frac{V_s}{R}$$

Substituting for $A$ in Eq. (7.58), we get

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau} \tag{7.60}$$

This is the complete response of the *RL* circuit. It is illustrated in Fig. 7.49. The response in Eq. (7.60) may be written as

$$\boxed{i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}} \tag{7.61}$$

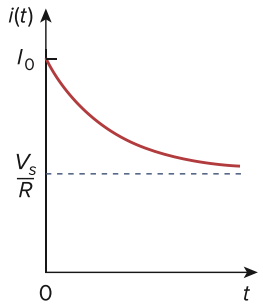



**Figure 7.49**
Total response of the *RL* circuit with initial inductor current $I_0$.

where $i(0)$ and $i(\infty)$ are the initial and final values of $i$, respectively. Thus, to find the step response of an *RL* circuit requires three things:

1. The initial inductor current $i(0)$ at $t = 0$.
2. The final inductor current $i(\infty)$.
3. The time constant $\tau$.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (7.61). Keep in mind that this technique applies only for step responses.

Again, if the switching takes place at time $t = t_0$ instead of $t = 0$, Eq. (7.61) becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau} \tag{7.62}$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \tag{7.63a}$$

or

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t) \tag{7.63b}$$

This is the step response of the *RL* circuit with no initial inductor current. The voltage across the inductor is obtained from Eq. (7.63) using $v = L\, di/dt$. We get

$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \qquad t > 0$$

or

$$v(t) = V_s e^{-t/\tau}u(t) \tag{7.64}$$

Figure 7.50 shows the step responses in Eqs. (7.63) and (7.64).

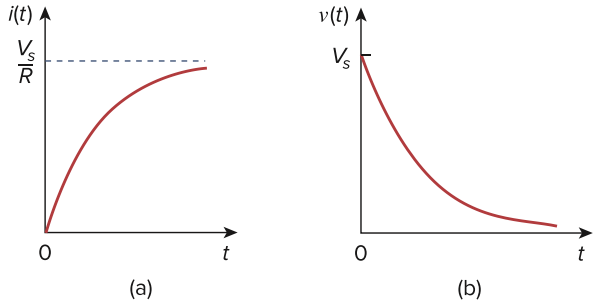



**Figure 7.50**
Step responses of an *RL* circuit with no initial inductor current: (a) current response, (b) voltage response.

## Example 7.12

Find $i(t)$ in the circuit of Fig. 7.51 for $t > 0$. Assume that the switch has been closed for a long time.

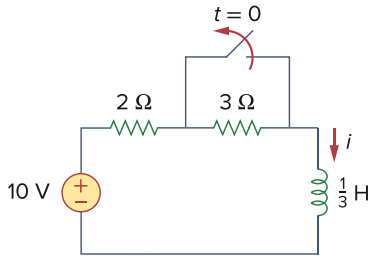



**Figure 7.51**
For Example 7.12.

**Solution:**

When $t < 0$, the 3-Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$

Because the inductor current cannot change instantaneously,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, the switch is open. The 2- and 3- Ω resistors are in series, so that

$$i(\infty) = \frac{10}{2+3} = 2 \text{ A}$$

The Thevenin resistance across the inductor terminals is

$$R_{\text{Th}} = 2 + 3 = 5 \ \Omega$$

For the time constant,

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{s}$$

Thus,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$= 2 + (5 - 2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0$$

*Check:* In Fig. 7.51, for $t > 0$, KVL must be satisfied; that is,

$$10 = 5i + L\frac{di}{dt}$$

$$5i + L\frac{di}{dt} = [10 + 15e^{-15t}] + \left[\frac{1}{3}(3)(-15)e^{-15t}\right] = 10$$

This confirms the result.

## Practice Problem 7.12

The switch in Fig. 7.52 has been closed for a long time. It opens at $t = 0$. Find $i(t)$ for $t > 0$.

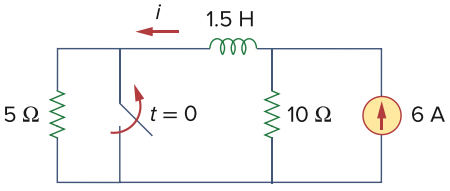



**Figure 7.52**
For Practice Prob. 7.12.

**Answer:** $(4 + 2e^{-10t})$ A for all $t > 0$.

## Example 7.13

At $t = 0$, switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate $i$ for $t = 2$ s and $t = 5$ s.

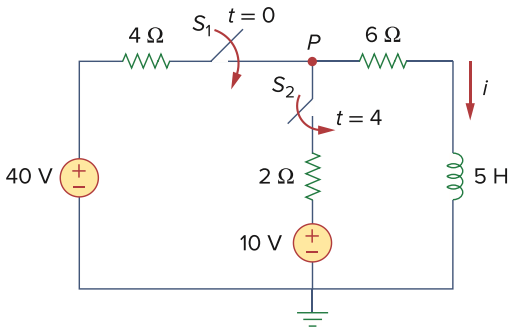



**Figure 7.53**
For Example 7.13.

**Solution:**
We need to consider the three time intervals $t \le 0$, $0 \le t \le 4$, and $t \ge 4$ separately. For $t < 0$, switches $S_1$ and $S_2$ are open so that $i = 0$. Since the inductor current cannot change instantly,

$$i(0^-) = i(0) = i(0^+) = 0$$

For $0 \le t \le 4$, $S_1$ is closed so that the 4- and 6- Ω resistors are in series. (Remember, at this time, $S_2$ is still open.) Hence, assuming for now that $S_1$ is closed forever,

$$i(\infty) = \frac{40}{4+6} = 4 \text{ A}, \qquad R_{\text{Th}} = 4 + 6 = 10\ \Omega$$

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{5}{10} = \frac{1}{2} \text{ s}$$

Thus,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$= 4 + (0 - 4)e^{-2t} = 4(1 - e^{-2t}) \text{ A}, \qquad 0 \le t \le 4$$

For $t \ge 4$, $S_2$ is closed; the 10-V voltage source is connected, and the circuit changes. This sudden change does not affect the inductor current because the current cannot change abruptly. Thus, the initial current is

$$i(4) = i(4^-) = 4(1 - e^{-8}) \simeq 4 \text{ A}$$

To find $i(\infty)$, let $v$ be the voltage at node $P$ in Fig. 7.53. Using KCL,

$$\frac{40 - v}{4} + \frac{10 - v}{2} = \frac{v}{6} \qquad \Rightarrow \qquad v = \frac{180}{11} \text{ V}$$

$$i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727 \text{ A}$$

The Thevenin resistance at the inductor terminals is

$$R_{\text{Th}} = 4 \| 2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3}\ \Omega$$

and

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{5}{\frac{22}{3}} = \frac{15}{22} \text{ s}$$

Hence,

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, \qquad t \geq 4$$

We need $(t-4)$ in the exponential because of the time delay. Thus,

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \qquad \tau = \frac{15}{22}$$

$$= 2.727 + 1.273e^{-1.4667(t-4)}, \qquad t \geq 4$$

Putting all this together,

$$i(t) = \begin{cases} 0, & t \leq 0 \\ 4(1 - e^{-2t}), & 0 \leq t \leq 4 \\ 2.727 + 1.273e^{-1.4667(t-4)}, & t \geq 4 \end{cases}$$

At $t = 2$,

$$i(2) = 4(1 - e^{-4}) = 3.93 \text{ A}$$

At $t = 5$,

$$i(5) = 2.727 + 1.273e^{-1.4667} = 3.02 \text{ A}$$

## Practice Problem 7.13

Switch $S_1$ in Fig. 7.54 is closed at $t = 0$, and switch $S_2$ is closed at $t = 2$s. Calculate $i(t)$ for all $t$. Find $i(1)$ and $i(3)$.

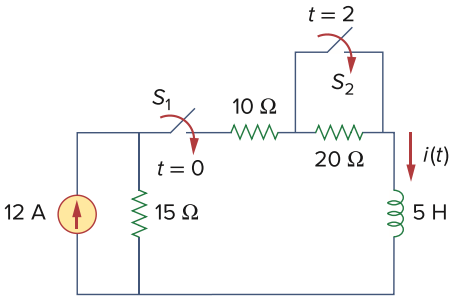



**Figure 7.54**
For Practice Prob. 7.13.

**Answer:**

$$i(t) = \begin{cases} 0, & t < 0 \\ 4(1 - e^{-9t}), & 0 < t < 2 \\ 7.2 - 3.2e^{-5(t-2)}, & t > 2 \end{cases}$$

$i(1) = 4$ A, $i(3) = 7.178$ A.

## 7.7 † First-Order Op Amp Circuits

An op amp circuit containing a storage element will exhibit first-order behavior. Differentiators and integrators treated in Section 6.6 are examples of first-order op amp circuits. Again, for practical reasons, inductors are hardly ever used in op amp circuits; therefore, the op amp circuits we consider here are of the *RC* type.

As usual, we analyze op amp circuits using nodal analysis. Sometimes, the Thevenin equivalent circuit is used to reduce the op amp circuit to one that we can easily handle. The following three examples illustrate the concepts. The first one deals with a source-free op amp circuit, while the other two involve step responses. The three examples have been carefully selected to cover all possible *RC* types of op amp circuits, depending on the location of the capacitor with respect to the op amp; that is, the capacitor can be located in the input, the output, or the feedback loop.

## Example 7.14

For the op amp circuit in Fig. 7.55(a), find $v_o$ for $t > 0$, gi ven that $v(0) = 3$ V. Let $R_f = 80$ kΩ, $R_1 = 20$ kΩ, and $C = 5$ μF.

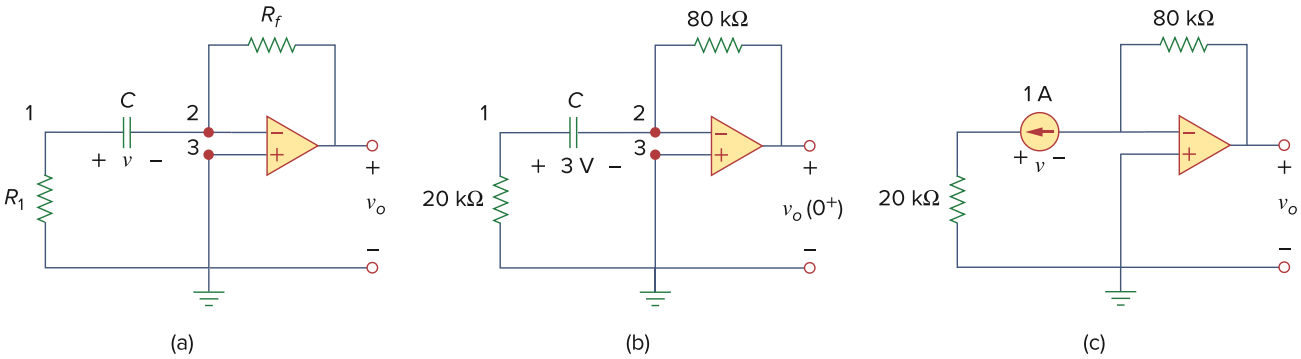



**Figure 7.55**
For Example 7.14.

**Solution:**
This problem can be solved in two ways:

■ **METHOD 1** Consider the circuit in Fig. 7.55(a). Let us derive the appropriate differential equation using nodal analysis. If $v_1$ is the voltage at node 1, at that node, KCL gives

$$\frac{0 - v_1}{R_1} = C\frac{dv}{dt} \tag{7.14.1}$$

Because nodes 2 and 3 must be at the same potential, the potential at node 2 is zero. Thus, $v_1 - 0 = v$ or $v_1 = v$ and Eq. (7.14.1) becomes

$$\frac{dv}{dt} + \frac{v}{CR_1} = 0 \tag{7.14.2}$$

This is similar to Eq. (7.4b) so that the solution is obtained the same way as in Section 7.2, i.e.,

$$v(t) = V_0 e^{-t/\tau}, \qquad \tau = R_1 C \tag{7.14.3}$$

where $V_0$ is the initial voltage across the capacitor. But$v(0) = 3 = V_0$ and $\tau = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1$. Hence,

$$v(t) = 3e^{-10t} \tag{7.14.4}$$

Applying KCL at node 2 gives

$$C\frac{dv}{dt} = \frac{0 - v_o}{R_f}$$

or

$$v_o = -R_f C\frac{dv}{dt} \tag{7.14.5}$$

Now we can find $v_0$ as

$$v_o = -80 \times 10^3 \times 5 \times 10^{-6}(-30e^{-10t}) = 12e^{-10t}\ \text{V}, \qquad t > 0$$

**METHOD 2** Let us apply the shortcut method from Eq. (7.53). We need to find $v_o(0^+)$, $v_o(\infty)$, and $\tau$. Since $v(0^+) = v(0^-) = 3$ V, we apply KCL at node 2 in the circuit of Fig. 7.55(b) to obtain

$$\frac{3}{20{,}000} + \frac{0 - v_o(0^+)}{80{,}000} = 0$$

or $v_o(0^+) = 12$ V. Since the circuit is source free, $v(\infty) = 0$ V. To find $\tau$, we need the equivalent resistance $R_{eq}$ across the capacitor terminals. If we remove the capacitor and replace it by a 1-A current source, we have the circuit shown in Fig. 7.55(c). Applying KVL to the input loop yields

$$20{,}000(1) - v = 0 \quad \Rightarrow \quad v = 20 \text{ kV}$$

Then

$$R_{\text{eq}} = \frac{v}{1} = 20 \text{ k}\Omega$$

and $\tau = R_{\text{eq}}C = 0.1$. Thus,

$$v_o(t) = v_o(\infty) + [v_o(0) - v_o(\infty)]e^{-t/\tau}$$

$$= 0 + (12 - 0)e^{-10t} = 12e^{-10t} \text{ V}, \qquad t > 0$$

as before.

## Practice Problem 7.14

For the op amp circuit in Fig. 7.56, find $v_o$ for $t > 0$ if $v(0) = 4$ V. Assume that $R_f = 50$ kΩ, $R_1 = 10$ kΩ, and $C = 10$ μF.

**Answer:** $-4e^{-2t}$ V, $t > 0$.

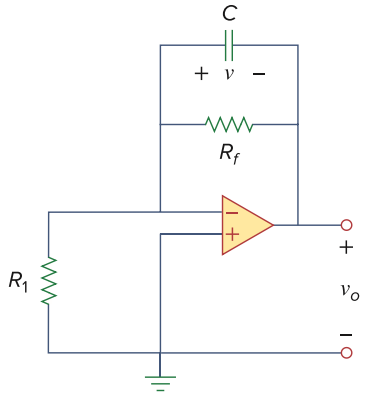



**Figure 7.56**
For Practice Prob. 7.14.

## Example 7.15

Determine $v(t)$ and $v_o(t)$ in the circuit of Fig. 7.57.

**Solution:**
This problem can be solved in two ways, just like the previous example. However, we will apply only the second method. Since what we are looking for is the step response, we can apply Eq. (7.53) and write

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}, \qquad t > 0 \qquad \textbf{(7.15.1)}$$

where we need only find the time constant $\tau$, the initial value $v(0)$, and the final value $v(\infty)$. Notice that this applies strictly to the capacitor voltage due a step input. Since no current enters the input terminals of the op amp, the elements on the feedback loop of the op amp constitute an *RC* circuit, with

$$\tau = RC = 50 \times 10^3 \times 10^{-6} = 0.05 \qquad \textbf{(7.15.2)}$$

For $t < 0$, the switch is open and there is no voltage across the capacitor. Hence, $v(0) = 0$. For $t > 0$, we obtain the voltage at node 1 by voltage division as

$$v_1 = \frac{20}{20 + 10}3 = 2 \text{ V} \qquad \textbf{(7.15.3)}$$

Since there is no storage element in the input loop, $v_1$ remains constant for all $t$. At steady state, the capacitor acts like an open circuit so that the op amp circuit is a noninverting amplifier. Thus,

$$v_o(\infty) = \left(1 + \frac{50}{20}\right)v_1 = 3.5 \times 2 = 7 \text{ V} \qquad \textbf{(7.15.4)}$$

But

$$v_1 - v_o = v \qquad \textbf{(7.15.5)}$$

so that

$$v(\infty) = 2 - 7 = -5 \text{ V}$$

Substituting $\tau$, $v(0)$, and $v(\infty)$ into Eq. (7.15.1) gives

$$v(t) = -5 + [0 - (-5)]e^{-20t} = 5(e^{-20t} - 1) \text{ V}, \qquad t > 0 \qquad \textbf{(7.15.6)}$$

From Eqs. (7.15.3), (7.15.5), and (7.15.6), we obtain

$$v_o(t) = v_1(t) - v(t) = 7 - 5e^{-20t} \text{ V}, \qquad t > 0 \qquad \textbf{(7.15.7)}$$

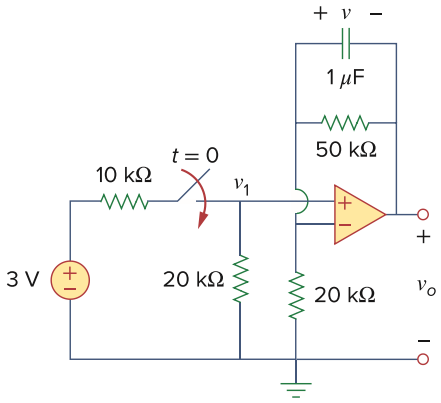



**Figure 7.57**
For Example 7.15.

## Practice Problem 7.15

Find $v(t)$ and $v_o(t)$ in the op amp circuit of Fig. 7.58.

**Answer:** (Note, the voltage across the capacitor and the output voltage must be both equal to zero, for $t < 0$, since the input was zero for all $t < 0$.) $40(1 - e^{-10t})\,u(t)$ mV, $40(e^{-10t} - 1)\,u(t)$ mV.

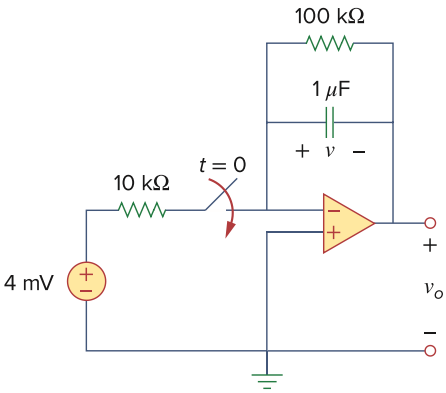



**Figure 7.58**
For Practice Prob. 7.15.

## Example 7.16

Find the step response $v_o(t)$ for $t > 0$ in the op amp circuit of Fig. 7.59. Let $v_i = 2u(t)$ V, $R_1 = 20$ kΩ, $R_f = 50$ kΩ, $R_2 = R_3 = 10$ kΩ, $C = 2$ μF.

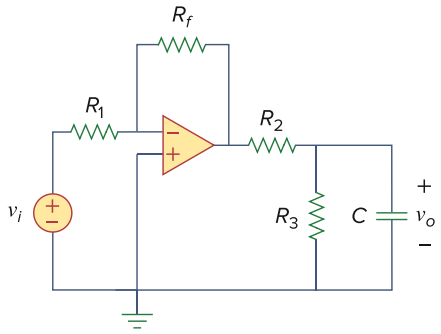



**Figure 7.59**
For Example 7.16.

**Solution:**

Notice that the capacitor in Example 7.14 is located in the input loop, while the capacitor in Example 7.15 is located in the feedback loop. In this example, the capacitor is located in the output of the op amp. Again, we can solve this problem directly using nodal analysis. However, using the Thevenin equivalent circuit may simplify the problem.

We temporarily remove the capacitor and find the Thevenin equivalent at its terminals. To obtain $V_{\text{Th}}$, consider the circuit in Fig. 7.60(a). Since the circuit is an inverting amplifier,

$$V_{ab} = -\frac{R_f}{R_1} v_i$$

By voltage division,

$$V_{\text{Th}} = \frac{R_3}{R_2 + R_3} V_{ab} = -\frac{R_3}{R_2 + R_3} \frac{R_f}{R_1} v_i$$

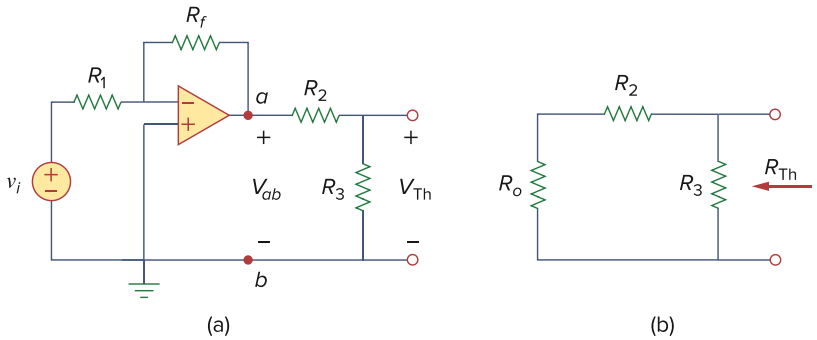



**Figure 7.60**
Obtaining $V_{\text{Th}}$ and $R_{\text{Th}}$ across the capacitor in Fig. 7.59.

To obtain $R_{\text{Th}}$, consider the circuit in Fig. 7.60(b), where $R_o$ is the output resistance of the op amp. Since we are assuming an ideal op amp, $R_o = 0$, and

$$R_{\text{Th}} = R_2 \parallel R_3 = \frac{R_2 R_3}{R_2 + R_3}$$

Substituting the given numerical values,

$$V_{\text{Th}} = -\frac{R_3}{R_2 + R_3} \frac{R_f}{R_1} v_i = -\frac{10}{20} \frac{50}{20} 2u(t) = -2.5u(t)$$

$$R_{\text{Th}} = \frac{R_2 R_3}{R_2 + R_3} = 5\text{k}\Omega$$

The Thevenin equivalent circuit is shown in Fig. 7.61, which is similar to Fig. 7.40. Hence, the solution is similar to that in Eq. (7.48); that is,

$$v_o(t) = -2.5(1 - e^{-t/\tau})u(t)$$

where $\tau = R_{\text{Th}} C = 5 \times 10^3 \times 2 \times 10^{-6} = 0.01$. Thus, the step response for $t > 0$ is

$$v_o(t) = 2.5(e^{-100t} - 1)u(t) \text{ V}$$

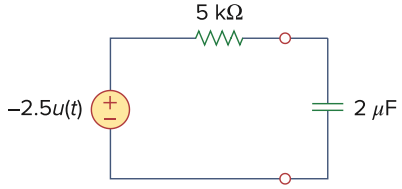



**Figure 7.61**
Thevenin equivalent circuit of the circuit in Fig. 7.59.

## Practice Problem 7.16

Obtain the step response $v_o(t)$ for the circuit in Fig. 7.62. Let $v_i = 9u(t)$ V, $R_1 = 20\ \text{k}\Omega$, $R_f = 40\ \text{k}\Omega$, $R_2 = R_3 = 10\ \text{k}\Omega$, $C = 2\ \mu\text{F}$.

**Answer:** $27(1 - e^{-50t})u(t)$ V.

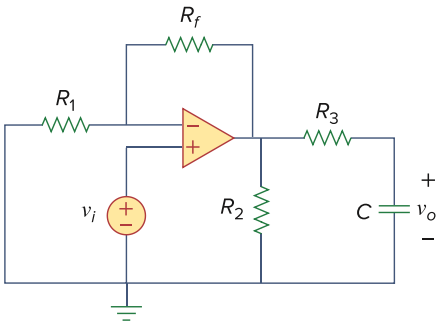



**Figure 7.62**
For Practice Prob. 7.16.

## 7.8 Transient Analysis with *PSpice*

As we discussed in Section 7.5, the transient response is the temporary response of the circuit that soon disappears. *PSpice* can be used to obtain the transient response of a circuit with storage elements. Section D.4 in Appendix D provides a review of transient analysis using *PSpice for Windows*. It is recommended that you read Section D.4 before continuing with this section.

If necessary, dc *PSpice* analysis is first carried out to determine the initial conditions. Then the initial conditions are used in the transient *PSpice* analysis to obtain the transient responses. It is recommended but not necessary that during this dc analysis, all capacitors should be open-circuited while all inductors should be short-circuited.

*PSpice* uses "transient" to mean "function of time." Therefore, the transient response in *PSpice* may not actually die out as expected.

## Example 7.17

Use *PSpice* to find the response $i(t)$ for $t > 0$ in the circuit of Fig. 7.63.

**Solution:**

Solving this problem by hand gives $i(0) = 0$, $i(\infty) = 2\text{A}$, $R_{\text{Th}} = 6$, $\tau = 3/6 = 0.5$ s, so that

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = 2(1 - e^{-2t}), \qquad t > 0$$

To use *PSpice,* we first draw the schematic as shown in Fig. 7.64. We recall from Appendix D that the part name for a closed switch is Sw_tclose. We do not need to specify the initial condition of the inductor because *PSpice* will determine that from the circuit. By selecting **Analysis/Setup/Transient**, we set *Print Step* to 25 ms and *Final Step* to $5\tau = 2.5$ s. After saving the circuit, we simulate by selecting **Analysis/Simulate**. In the *PSpice* A/D window, we select **Trace/Add** and display –I(L1) as the current through the inductor. Figure 7.65 shows the plot of $i(t)$, which agrees with that obtained by hand calculation.

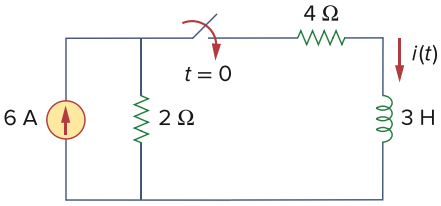



**Figure 7.63**
For Example 7.17.

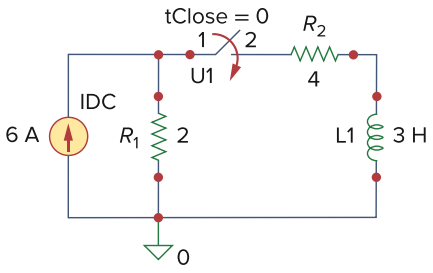



**Figure 7.64**
The schematic of the circuit in Fig. 7.63.

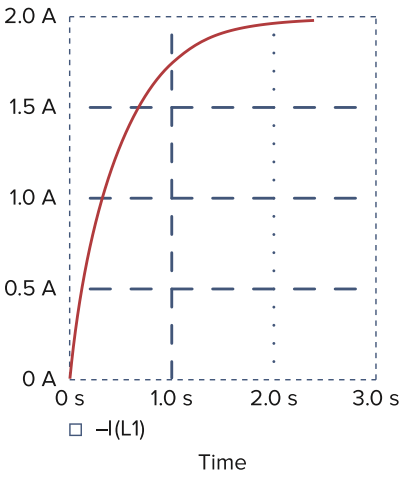



**Figure 7.65**
For Example 7.17; the response of the circuit in Fig. 7.63.

Note that the negative sign on I(L1) is needed because the current enters through the upper terminal of the inductor, which happens to be the negative terminal after one counterclock wise rotation. A way to avoid the negative sign is to ensure that current enters pin 1 of the inductor. To obtain this desired direction of positive current flow, the initially horizontal inductor symbol should be rotated counterclockwise 270° and placed in the desired location.

## Practice Problem 7.17

For the circuit in Fig. 7.66, use *Pspice* to find $v(t)$ for $t > 0$.

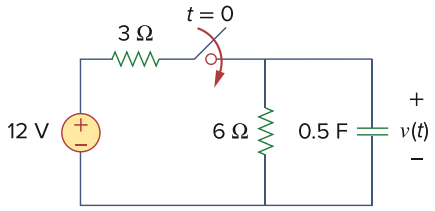



**Figure 7.66**
For Practice Prob. 7.17.

**Answer:** $v(t) = 8(1 - e^{-t})$ V, $t > 0$. The response is similar in shape to that in Fig. 7.65.

## Example 7.18

In the circuit of Fig. 7.67(a), determine the response $v(t)$.

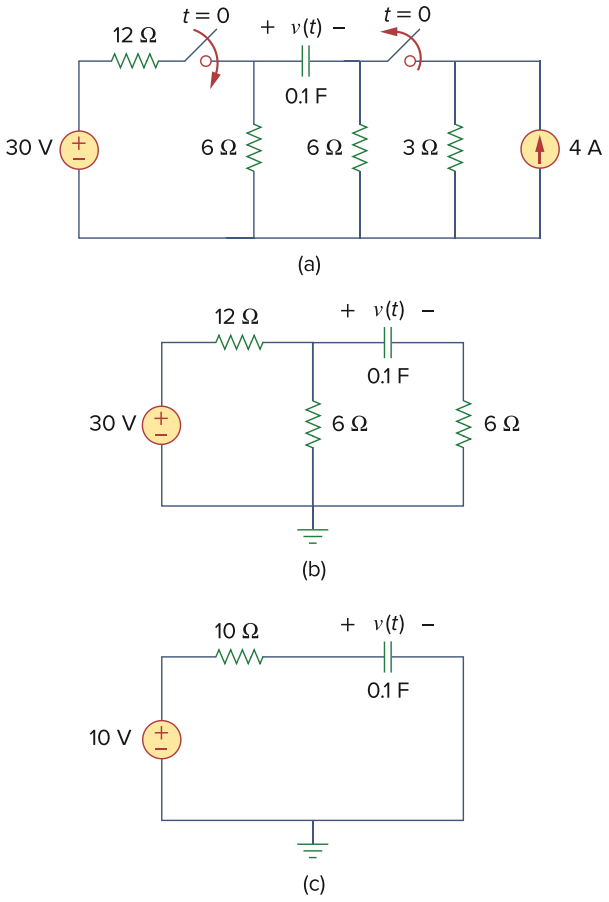



**Figure 7.67**
For Example 7.18. Original circuit (a), circuit *for* $t > 0$ (b), and reduced circuit for $t > 0$ (c).

**Solution:**

1. **Define.** The problem is clearly stated and the circuit is clearly labeled.
2. **Present.** Given the circuit shown in Fig. 7.67(a), determine the response $v(t)$.
3. **Alternative.** We can solve this circuit using circuit analysis techniques, nodal analysis, mesh analysis, or *PSpice*. Let us solve the problem using circuit analysis techniques (this time Thevenin equivalent circuits) and then check the answer using two methods of *PSpice*.
4. **Attempt.** For time $< 0$, the switch on the left is open and the switch on the right is closed. Assume that the switch on the right has been closed long enough for the circuit to reach steady state; then the capacitor acts like an open circuit and the current from the 4-A source flows through the parallel combination of the 6-Ω and 3-Ω resistors ($6 \parallel 3 = 18/9 = 2$), producing a voltage equal to $2 \times 4 = 8\ \text{V} = -v(0)$.

   At $t = 0$, the switch on the left closes and the switch on the right opens, producing the circuit shown in Fig. 7.67(b).

   The easiest way to complete the solution is to find the Thevenin equivalent circuit as seen by the capacitor. The open-circuit voltage (with the capacitor removed) is equal to the voltage drop across the 6-Ω resistor on the left, or 10 V (the voltage drops uniformly across the 12-Ω resistor, 20 V, and across the 6-Ω resistor, 10 V). This is $V_{\text{Th}}$. The resistance looking in where the capacitor was is equal to $12 \parallel 6 + 6 = 72/18 + 6 = 10\ \Omega$, which is $R_{\text{eq}}$. This produces the Thevenin equivalent circuit shown in Fig. 7.67(c). Matching up the boundary conditions ($v(0) = -8$ V and $v(\infty) = 10$ V) and $\tau = RC = 1$, we get

$$v(t) = 10 - 18\text{e}^{-\text{t}}\ \text{V}$$

5. **Evaluate.** There are two ways of solving the problem using *PSpice*.

■ **METHOD 1** One way is to first do the dc *PSpice* analysis to determine the initial capacitor voltage. The schematic of the revelant circuit is in Fig. 7.68(a). Two pseudocomponent VIEWPOINTs are inserted to measure the voltages at nodes 1 and 2. When the circuit is simulated, we obtain the displayed values in Fig. 7.68(a) as $V_1 = 0$ V and $V_2 = 8$ V. Thus, the initial capacitor voltage is $v(0) = V_1 - V_2 = -8$ V. The *PSpice* transient analysis uses this value along with the schematic in Fig. 7.68(b). Once the circuit in Fig. 7.68(b) is drawn, we insert the capacitor initial voltage as IC $= -8$. We select **Analysis/Setup/Transient** and set *Print Step* to 0.1 s and *Final Step* to $4\tau = 4$ s. After saving the circuit, we select **Analysis/Simulate** to simulate the circuit. In the *PSpice* A/D window, we select **Trace/Add** and display V(R2:2) – V(R3:2) or V(C1:1) − V(C1:2) as the capacitor voltage $v(t)$. The plot of $v(t)$ is shown in Fig. 7.69. This agrees with the result obtained by hand calculation, $v(t) = 10 - 18e^{-t}$ V.

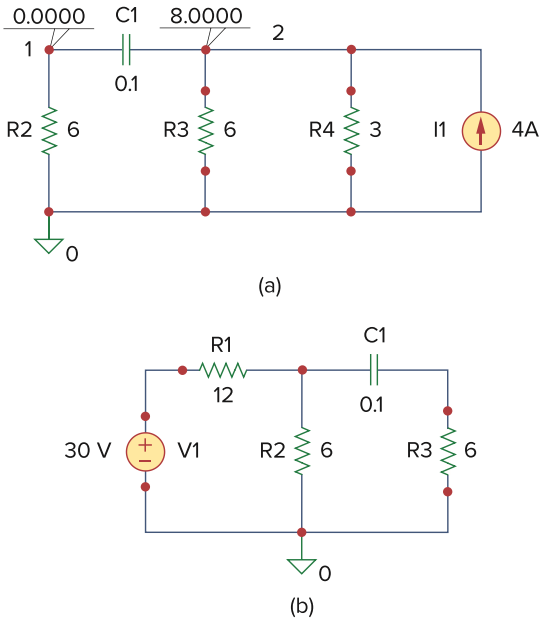



**Figure 7.68**
(a) Schematic for dc analysis to get $v(0)$, (b) schematic for transient analysis used in getting the response $v(t)$.

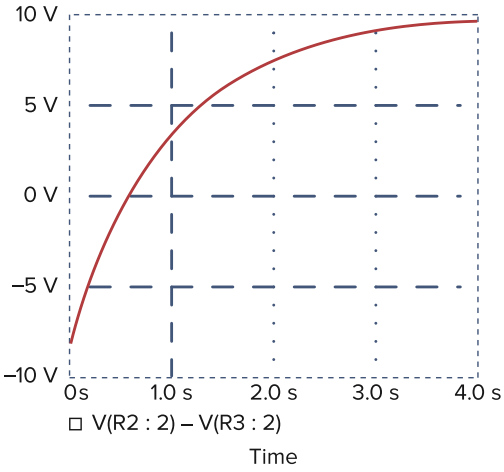



**Figure 7.69**
Response $v(t)$ for the circuit in Fig. 7.67.

■ **METHOD 2** We can simulate the circuit in Fig. 7.67 directly, since *PSpice* can handle the open and closed switches and determine the initial conditions automatically. Using this approach, the schematic is drawn as shown in Fig. 7.70. After drawing the circuit, we select **Analysis/Setup/Transient** and set *Print Step* to 0.1 s and *Final Step* to $4\tau = 4$ s. We save the circuit, then select **Analysis/Simulate** to simulate the circuit. In the *PSpice* A/D window, we select **Trace/Add** and display V(R2:2) − V(R3:2) as the capacitor voltage $v(t)$. The plot of $v(t)$ is the same as that shown in Fig. 7.69.

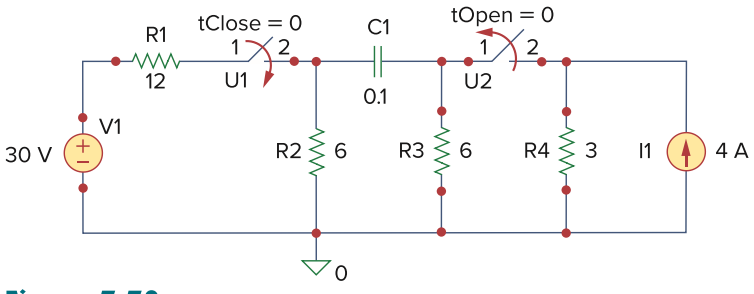



**Figure 7.70**
For Example 7.18.

6. **Satisfactory?** Clearly, we have found the value of the output response $v(t)$, as required by the problem statement. Checking does validate that solution. We can present all this as a complete solution to the problem.

### Practice Problem 7.18

The switch in Fig. 7.71 was open for a long time but closed at $t = 0$. If $i(0) = 10\text{A}$, find $i(t)$ for $t > 0$ by hand and also by *PSpice*.

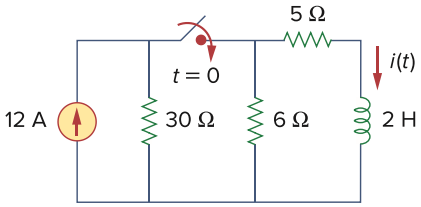



**Figure 7.71**
For Practice Prob. 7.18.

**Answer:** $i(t) = 6 + 4e^{-5t}$ A. The plot of $i(t)$ obtained by *PSpice* analysis is shown in Fig. 7.72.

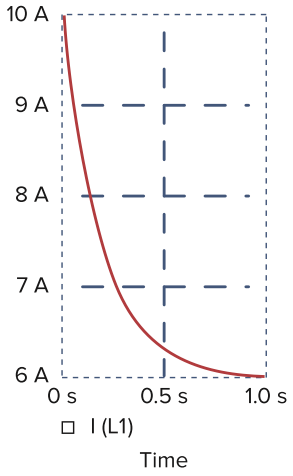



**Figure 7.72**
For Practice Prob. 7.18.

## 7.9 Applications

The various devices in which *RC* and *RL* circuits find applications include filtering in dc power supplies, smoothing circuits in digital communications, differentiators, integrators, delay circuits, and relay circuits. Some of these applications take advantage of the short or long time constants of the *RC* or *RL* circuits. We will consider four simple applications here. The first two are *RC* circuits, the last two are *RL* circuits.

### 7.9.1 Delay Circuits

An *RC* circuit can be used to provide various time delays. Figure 7.73 shows such a circuit. It basically consists of an *RC* circuit with the capacitor connected in parallel with a neon lamp. The voltage source can provide enough voltage to fire the lamp. When the switch is closed, the capacitor voltage increases gradually toward 110 V at a rate determined by the circuit's time constant, $(R_1 + R_2)C$. The lamp will act as an open

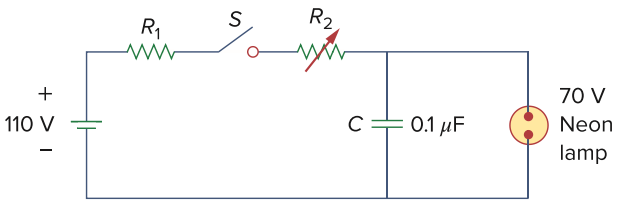



**Figure 7.73**
An *RC* delay circuit.

circuit and not emit light until the voltage across it exceeds a particular level, say 70 V. When the voltage level is reached, the lamp fires (goes on), and the capacitor discharges through it. Due to the low resistance of the lamp when on, the capacitor voltage drops fast and the lamp turns off. The lamp acts again as an open circuit and the capacitor recharges. By adjusting $R_2$, we can introduce either short or long time delays into the circuit and make the lamp fire, recharge, and fire repeatedly every time constant $\tau = (R_1 + R_2)C$, because it takes a time period $\tau$ to get the capacitor voltage high enough to fire or low enough to turn off.

The warning blinkers commonly found on road construction sites are one example of the usefulness of such an $RC$ delay circuit.

## Example 7.19

Consider the circuit in Fig. 7.73, and assume that $R_1 = 1.5\ \text{M}\Omega$, $0 < R_2 < 2.5\ \text{M}\Omega$. (a) Calculate the extreme limits of the time constant of the circuit. (b) How long does it take for the lamp to glow for the first time after the switch is closed? Let $R_2$ assume its largest value.

**Solution:**

(a) The smallest value for $R_2$ is $0\ \Omega$, and the corresponding time constant for the circuit is

$$\tau = (R_1 + R_2)C = (1.5 \times 10^6 + 0) \times 0.1 \times 10^{-6} = 0.15\ \text{s}$$

The largest value for $R_2$ is $2.5\ \text{M}\Omega$, and the corresponding time constant for the circuit is

$$\tau = (R_1 + R_2)C = (1.5 + 2.5) \times 10^6 \times 0.1 \times 10^{-6} = 0.4\ \text{s}$$

Thus, by proper circuit design, the time constant can be adjusted to introduce a proper time delay in the circuit.

(b) Assuming that the capacitor is initially uncharged, $v_C(0) = 0$, while $v_C(\infty) = 110$. But

$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)]e^{-t/\tau} = 110[1 - e^{-t/\tau}]$$

where $\tau = 0.4$ s, as calculated in part (a). The lamp glows when $v_C = 70$ V. If $v_C(t) = 70$ V at $t = t_0$, then

$$70 = 110[1 - e^{-t_0/\tau}] \quad \Rightarrow \quad \frac{7}{11} = 1 - e^{-t_0/\tau}$$

or

$$e^{-t_0/\tau} = \frac{4}{11} \quad \Rightarrow \quad e^{t_0/\tau} = \frac{11}{4}$$

Taking the natural logarithm of both sides gives

$$t_0 = \tau \ln \frac{11}{4} = 0.4 \ln 2.75 = 0.4046\ \text{s}$$

A more general formula for finding $t_0$ is

$$t_0 = \tau \ln \frac{-v(\infty)}{v(t_0) - v(\infty)}$$

The lamp will fire repeatedly every $t_0$ seconds if and only if $v(t_0) < v(\infty)$.

## Practice Problem 7.19

The *RC* circuit in Fig. 7.74 is designed to operate an alarm which activates when the current through it exceeds 90 $\mu$A. If $0 \le R \le 6$ k$\Omega$, find the range of the time delay that the variable resistor can create.

**Answer:** Between 34.47 and 89.26 ms.

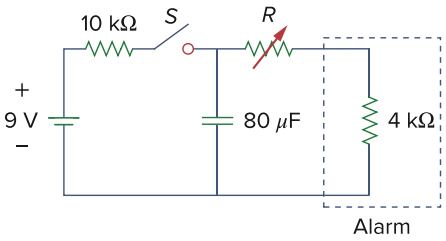



**Figure 7.74**
For Practice Prob. 7.19.

### 7.9.2 Photoflash Unit

An electronic flash unit provides a common example of an *RC* circuit. This application exploits the ability of the capacitor to oppose any abrupt change in voltage. Figure 7.75 shows a simplified circuit. It consists essentially of a high-voltage dc supply, a current-limiting large resistor $R_1$, and a capacitor $C$ in parallel with the flashlamp of low resistance $R_2$. When the switch is in position 1, the capacitor charges slowly due to the large time constant ($\tau_1 = R_1C$). As shown in Fig. 7.76(a), the capacitor voltage rises gradually from zero to $V_s$, while its current decreases gradually from $I_1 = V_s/R_1$ to zero. The charging time is approximately five times the time constant,

$$t_{\text{charge}} = 5R_1C \tag{7.65}$$

With the switch in position 2, the capacitor voltage is discharged. The low resistance $R_2$ of the photolamp permits a high discharge current with peak $I_2 = V_s/R_2$ in a short duration, as depicted in Fig. 7.76(b). Discharging takes place in approximately five times the time constant,

$$t_{\text{discharge}} = 5R_2C \tag{7.66}$$

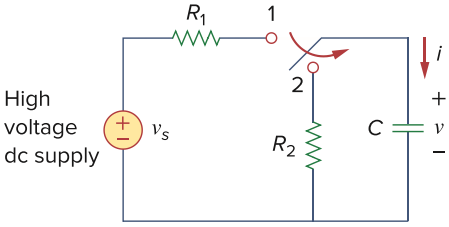



**Figure 7.75**
Circuit for a flash unit providing slow charge in position 1 and fast discharge in position 2.

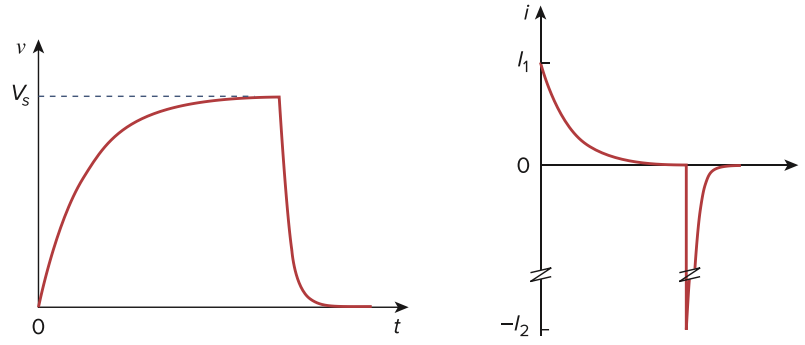



**Figure 7.76**
(a) Capacitor voltage showing slow charge and fast discharge, (b) capacitor current showing low charging current $I_1 = V_s/R_1$ and high discharge current $I_2 = V_s/R_2$.

Thus, the simple *RC* circuit of Fig. 7.75 provides a short-duration, high-current pulse. Such a circuit also finds applications in electric spot welding and the radar transmitter tube.

## Example 7.20

An electronic flashgun has a current-limiting 6-kΩ resistor and 2000-$\mu$F electrolytic capacitor charged to 240 V. If the lamp resistance is 12 Ω, find: (a) the peak charging current, (b) the time required for the capacitor to fully charge, (c) the peak discharging current, (d) the total energy stored in the capacitor, and the average power dissipated by the lamp.

**Solution:**

(a) The peak charging current is

$$I_1 = \frac{V_s}{R_1} = \frac{240}{6 \times 10^3} = 40 \text{ mA}$$

(b) From Eq. (7.65),

$$t_{\text{charge}} = 5R_1C = 5 \times 6 \times 10^3 \times 2000 \times 10^{-6} = 60 \text{ s} = 1 \text{ minute}$$

(c) The peak discharging current is

$$I_2 = \frac{V_s}{R_2} = \frac{240}{12} = 20 \text{ A}$$

(d) The energy stored is

$$W = \frac{1}{2}CV_s^2 = \frac{1}{2} \times 2000 \times 10^{-6} \times 240^2 = 57.6 \text{ J}$$

The energy stored in the capacitor is dissipated across the lamp during the discharging period. From Eq. (7.66),

$$t_{\text{discharge}} = 5R_2C = 5 \times 12 \times 2000 \times 10^{-6} = 0.12 \text{ s}$$

Thus, the average power dissipated is

$$p = \frac{W}{t_{\text{discharge}}} = \frac{57.6}{0.12} = 480 \text{ watts}$$

## Practice Problem 7.20

The flash unit of a camera has a 2-mF capacitor charged to 40 V.

(a) How much charge is on the capacitor?
(b) What is the energy stored in the capacitor?
(c) If the flash fires in 0.8 ms, what is the average current through the flashtube?
(d) How much power is delivered to the flashtube?
After a picture has been taken, the capacitor needs to be recharged by a power unit that supplies a maximum of 5 mA. How much time does it take to charge the capacitor?

**Answer:** (a) 80 mC, (b) 1.6 J, (c) 100 A, (d) 2 kW, 16 s.

### 7.9.3 Relay Circuits

A magnetically controlled switch is called a *relay*. A relay is essentially an electromagnetic device used to open or close a switch that controls another circuit. Figure 7.77(a) shows a typical relay circuit. The coil

circuit is an $RL$ circuit like that in Fig. 7.77(b), where $R$ and $L$ are the resistance and inductance of the coil. When switch $S_1$ in Fig. 7.77(a) is closed, the coil circuit is energized. The coil current gradually increases and produces a magnetic field. Eventually the magnetic field is sufficiently strong to pull the mo vable contact in the other circuit and close switch $S_2$. At this point, the relay is said to be *pulled in*. The time interval $t_d$ between the closure of switches $S_1$ and $S_2$ is called the *relay delay time*.

Relays were used in the earliest digital circuits and are still used for switching high-power circuits.

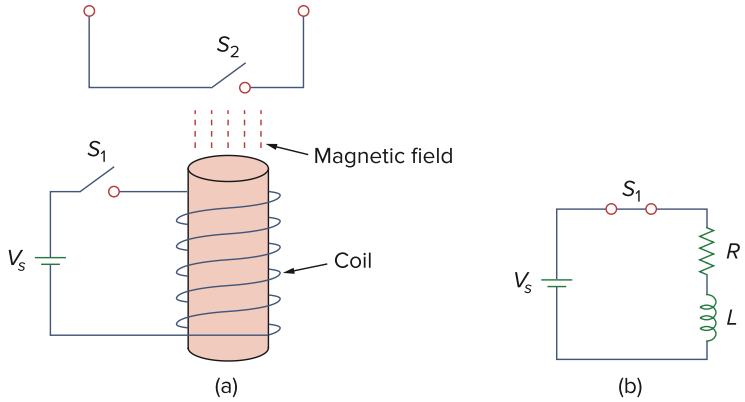



**Figure 7.77**
A relay circuit.

## Example 7.21

The coil of a certain relay is operated by a 12-V battery. If the coil has a resistance of 150 Ω and an inductance of 30 mH and the current needed to pull in is 50 mA, calculate the relay delay time.

**Solution:**
The current through the coil is given by

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

where

$$i(0) = 0, \qquad i(\infty) = \frac{12}{150} = 80 \text{ mA}$$

$$\tau = \frac{L}{R} = \frac{30 \times 10^{-3}}{150} = 0.2 \text{ ms}$$

Thus,

$$i(t) = 80[1 - e^{-t/\tau}] \text{ mA}$$

If $i(t_d) = 50$ mA, then

$$50 = 80[1 - e^{-t_d/\tau}] \quad \Rightarrow \quad \frac{5}{8} = 1 - e^{-t_d/\tau}$$

or

$$e^{-t_d/\tau} = \frac{3}{8} \quad \Rightarrow \quad e^{t_d/\tau} = \frac{8}{3}$$

By taking the natural logarithm of both sides, we get

$$t_d = \tau \ln\frac{8}{3} = 0.2 \ln\frac{8}{3}\ \text{ms} = 0.1962\ \text{ms}$$

Alternatively, we may find $t_d$ using

$$t_d = \tau \ln\frac{i(0) - i(\infty)}{i(t_d) - i(\infty)}$$

## Practice Problem 7.21

A relay has a resistance of 200 Ω and an inductance of 500 mH. The relay contacts close when the current through the coil reaches 175 μA. What time elapses between the application of 110 V to the coil and contact closure?

**Answer:** 957.5 μs.

### 7.9.4 Automobile Ignition Circuit

The ability of inductors to oppose rapid change in current makes them useful for arc or spark generation. An automobile ignition system takes advantage of this feature.

The gasoline engine of an automobile requires that the fuel-air mixture in each cylinder be ignited at proper times. This is achieved by means of a spark plug (Fig. 7.78), which essentially consists of a pair of electrodes separated by an air gap. By creating a large voltage (thousands of volts) between the electrodes, a spark is formed across the air gap, thereby igniting the fuel. But how can such a large voltage be obtained from the car battery, which supplies only 12 V? This is achieved by means of an inductor (the spark coil) $L$. Since the voltage across the inductor is $v = L\,di/dt$, we can make $di/dt$ large by creating a large change in current in a very short time. When the ignition switch in Fig. 7.78 is closed, the current through the inductor increases gradually and reaches the final value of $i = V_s/R$, where $V_s = 12$ V. Again, the time taken for the inductor to charge is five times the *time constant* of the circuit ($\tau = L/R$),

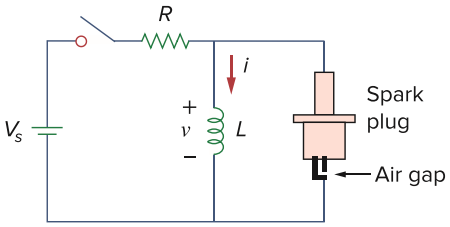



**Figure 7.78**
Circuit for an automobile ignition system.

$$t_{\text{charge}} = 5\frac{L}{R} \tag{7.67}$$

Since at steady state, $i$ is constant, $di/dt = 0$, and the inductor voltage $v = 0$. When the switch suddenly opens, a large voltage is developed across the inductor (due to the rapidly collapsing field) causing a spark or arc in the air gap. The spark continues until the energy stored in the inductor is dissipated in the spark discharge. In laboratories, when one is working with inductive circuits, this same effect causes a very nasty shock, and one must exercise caution.

### Example 7.22

A solenoid with resistance 4 Ω and inductance 6 mH is used in an automobile ignition circuit similar to that in Fig. 7.78. If the battery supplies 12 V, determine: the final current through the solenoid when the switch is closed, the energy stored in the coil, and the voltage across the air gap, assuming that the switch takes 1 $\mu$s to open.

**Solution:**
The final current through the coil is

$$I = \frac{V_s}{R} = \frac{12}{4} = 3 \text{ A}$$

The energy stored in the coil is

$$W = \frac{1}{2} L I^2 = \frac{1}{2} \times 6 \times 10^{-3} \times 3^2 = 27 \text{ mJ}$$

The voltage across the gap is

$$V = L\frac{\Delta I}{\Delta t} = 6 \times 10^{-3} \times \frac{3}{1 \times 10^{-6}} = 18 \text{ kV}$$

### Practice Problem 7.22

The spark coil of an automobile ignition system has a 20-mH inductance and a 5-Ω resistance. With a supply voltage of 12 V, calculate: the time needed for the coil to fully charge, the energy stored in the coil, and the voltage developed at the spark gap if the switch opens in 2 $\mu$s.

**Answer:** 20 ms, 57.6 mJ, and 24 kV.

## 7.10 Summary

1. The analysis in this chapter is applicable to any circuit that can be reduced to an equivalent circuit comprising a resistor and a single energy-storage element (inductor or capacitor). Such a circuit is first-order because its behavior is described by a first-order differential equation. When analyzing *RC* and *RL* circuits, one must always keep in mind that the capacitor is an open circuit to steady-state dc conditions while the inductor is a short circuit to steady-state dc conditions.
2. The natural response is obtained when no independent source is present. It has the general form

$$x(t) = x(0)e^{-t/\tau}$$

   where $x$ represents current through (or voltage across) a resistor, a capacitor, or an inductor, and $x(0)$ is the initial value of $x$. Because most practical resistors, capacitors, and inductors always have losses, the natural response is a transient response, i.e. it dies out with time.
3. The time constant $\tau$ is the time required for a response to decay to 1/e of its initial value. For *RC* circuits, $\tau = RC$ and for *RL* circuits, $\tau = L/R$.

4. The singularity functions include the unit step, the unit ramp function, and the unit impulse functions. The unit step function $u(t)$ is

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

The unit impulse function is

$$\delta(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

The unit ramp function is

$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$

5. The steady-state response is the behavior of the circuit after an independent source has been applied for a long time. The transient response is the component of the complete response that dies out with time.
6. The total or complete response consists of the steady-state response and the transient response.
7. The step response is the response of the circuit to a sudden application of a dc current or voltage. Finding the step response of a first-order circuit requires the initial value $x(0^+)$, the final value $x(\infty)$, and the time constant $\tau$. With these three items, we obtain the step response as

$$x(t) = x(\infty) + [x(0^+) - x(\infty)]e^{-t/\tau}$$

A more general form of this equation is

$$x(t) = x(\infty) + [x(t_0^+) - x(\infty)]e^{-(t-t_0)/\tau}$$

Or we may write it as

$$\text{Instantaneous value} = \text{Final} + [\text{Initial} - \text{Final}]e^{-(t-t_0)/\tau}$$

8. *PSpice* is very useful for obtaining the transient response of a circuit.
9. Four practical applications of *RC* and *RL* circuits are: a delay circuit, a photoflash unit, a relay circuit, and an automobile ignition circuit.

## Review Questions

**7.1** An *RC* circuit has $R = 2\ \Omega$ and $C = 4$ F. The time constant is:

(a) 0.5 s (b) 2 s (c) 4 s
(d) 8 s 15 s

**7.2** The time constant for an *RL* circuit with $R = 2\ \Omega$ and $L = 4$ H is:

(a) 0.5 s (b) 2 s (c) 4 s
(d) 8 s 15 s

**7.3** A capacitor in an *RC* circuit with $R = 2\ \Omega$ and $C = 4$ F is being charged. The time required for the capacitor voltage to reach 63.2 percent of its steady-state value is:

(a) 2 s (b) 4 s (c) 8 s
(d) 16 s none of the above

**7.4** An *RL* circuit has $R = 2\ \Omega$ and $L = 4$ H. The time needed for the inductor current to reach 40 percent of its steady-state value is:

(a) 0.5 s (b) 1 s (c) 2 s
(d) 4 s none of the above

**7.5** In the circuit of Fig. 7.79, the capacitor voltage just before $t = 0$ is:

(a) 10 V (b) 7 V (c) 6 V
(d) 4 V 0 V

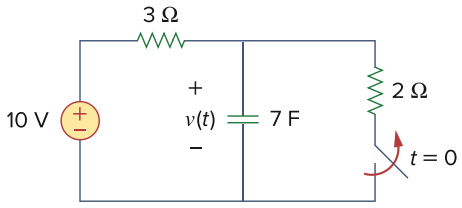



**Figure 7.79**
For Review Questions 7.5 and 7.6.

**7.6** In the circuit in Fig. 7.79, $v(\infty)$ is:

(a) 10 V (b) 7 V (c) 6 V
(d) 4 V 0 V

**7.7** For the circuit in Fig. 7.80, the inductor current just before $t = 0$ is:

(a) 8 A (b) 6 A (c) 4 A
(d) 2 A 0 A

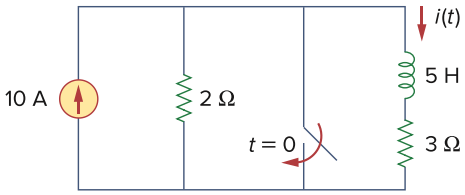



**Figure 7.80**
For Review Questions 7.7 and 7.8.

**7.8** In the circuit of Fig. 7.80, $i(\infty)$ is:

(a) 10 A (b) 6 A (c) 4 A
(d) 2 A 0 A

**7.9** If $v_s$ changes from 2 V to 4 V at $t = 0$, we may express $v_s$ as:

(a) $\delta(t)$ V (b) $2u(t)$ V
(c) $2u(-t) + 4u(t)$ V (d) $2 + 2u(t)$ V
$4u(t) - 2$ V

**7.10** The pulse in Fig. 7.116(a) can be expressed in terms of singularity functions as:

(a) $2u(t) + 2u(t-1)$ V (b) $2u(t) - 2u(t-1)$ V
(c) $2u(t) - 4u(t-1)$ V (d) $2u(t) + 4u(t-1)$ V

*Answers: 7.1d, 7.2b, 7.3c, 7.4b, 7.5d, 7.6a, 7.7c, 7.8e , 7.9c,d, 7.10b.*

# Problems

## Section 7.2 The Source-Free *RC* Circuit

**7.1** In the circuit shown in Fig. 7.81

$$v(t) = 56e^{-200t}\,\text{V}, \quad t > 0$$

$$i(t) = 8e^{-200t}\,\text{mA}, \quad t > 0$$

(a) Find the values of $R$ and $C$.

(b) Calculate the time constant $\tau$.

(c) Determine the time required for the voltage to decay half its initial value at $t = 0$.

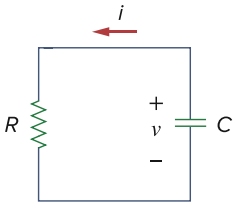



**Figure 7.81**
For Prob. 7.1.

**7.2** Find the time constant for the *RC* circuit in Fig. 7.82.

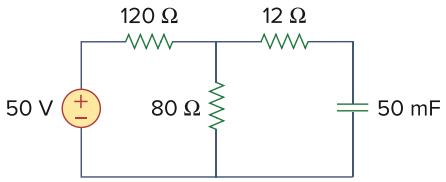



**Figure 7.82**
For Prob. 7.2.

**7.3** Determine the time constant for the circuit in Fig. 7.83.

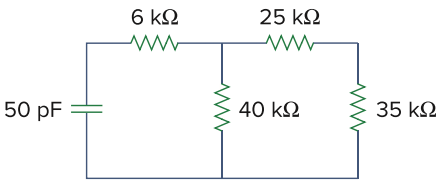



**Figure 7.83**
For Prob. 7.3.

**7.4** The switch in Fig. 7.84 has been in position *A* for a long time. Assume the switch moves instantaneously from *A* to *B* at $t = 0$. Find $v$ for $t > 0$.

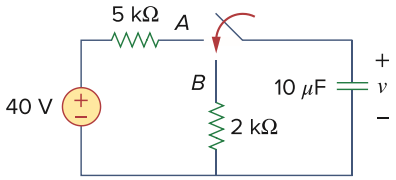



**Figure 7.84**
For Prob. 7.4.

**7.5** Using Fig. 7.85, design a problem to help other students better understand source-free *RC* circuits.

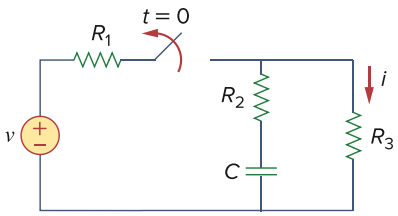



**Figure 7.85**
For Prob. 7.5.

**7.6** The switch in Fig. 7.86 has been closed for a long time, and it opens at $t = 0$. Find $v(t)$ for $t \geq 0$.

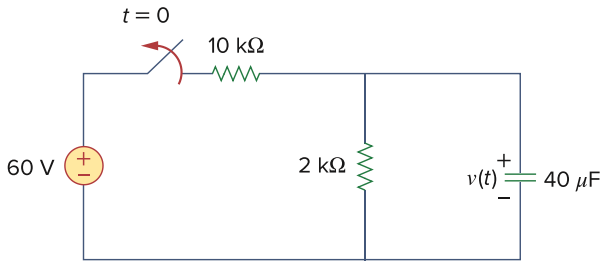



**Figure 7.86**
For Prob. 7.6.

**7.7** Assuming that the switch in Fig. 7.87 has been in position *A* for a long time and is moved to position *B* at $t = 0$, Then at $t = 1$ second, the switch moves from *B* to *C*. Find $v_C(t)$ for $t \geq 0$.

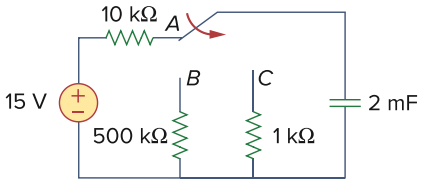



**Figure 7.87**
For Prob. 7.7.

**7.8** For the circuit in Fig. 7.88, if

$$v = 10e^{-4t}\ \text{V} \quad \text{and} \quad i = 0.2e^{-4t}\ \text{A}, \quad t > 0$$

(a) Find *R* and *C*.
(b) Determine the time constant.
(c) Calculate the initial energy in the capacitor.
(d) Obtain the time it takes to dissipate 50 percent of the initial energy.

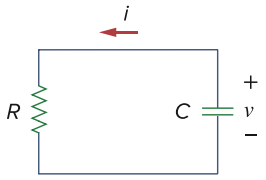



**Figure 7.88**
For Prob. 7.8.

**7.9** The switch in Fig. 7.89 opens at $t = 0$. Find $v_o$ for $t > 0$.

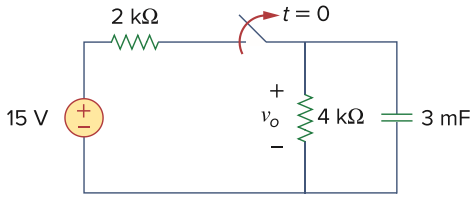



**Figure 7.89**
For Prob. 7.9.

**7.10** For the circuit in Fig. 7.90, find $v_o(t)$ for $t > 0$. Determine the time necessary for the capacitor voltage to decay to one-third of its value at $t = 0$.

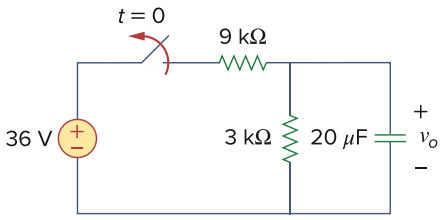



**Figure 7.90**
For Prob. 7.10.

## Section 7.3 The Source-Free *RL* Circuit

**7.11** For the circuit in Fig. 7.91, find $i_o$ for $t > 0$.

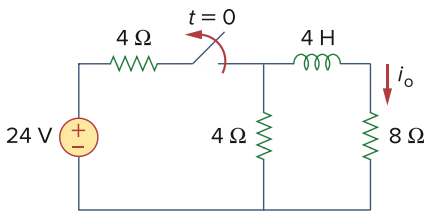



**Figure 7.91**
For Prob. 7.11.

**7.12** Using Fig. 7.92, design a problem to help other students better understand source-free *RL* circuits.

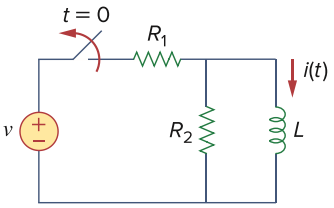



**Figure 7.92**
For Prob. 7.12.

**7.13** In the circuit of Fig. 7.93,

$$v(t) = 80e^{-10^3 t}\text{V}, \quad t > 0$$
$$i(t) = 5e^{-10^3 t}\text{mA}, \quad t > 0$$

(a) Find $R$, $L$, and $\tau$.

(b) Calculate the energy dissipated in the resistance for $0 < t < 0.5$ ms.

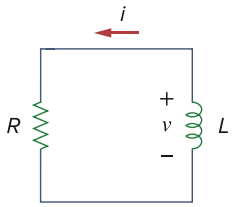



**Figure 7.93**
For Prob. 7.13.

**7.14** Calculate the time constant of the circuit in Fig. 7.94.

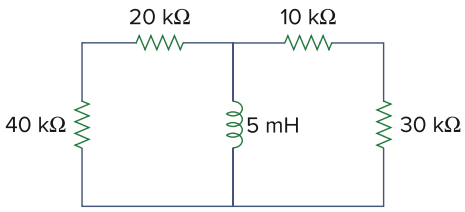



**Figure 7.94**
For Prob. 7.14.

**7.15** Find the time constant for each of the circuits in Fig. 7.95.

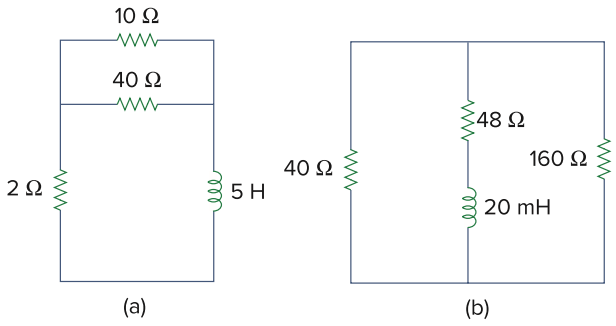



**Figure 7.95**
For Prob. 7.15.

**7.16** Determine the time constant for each of the circuits in Fig. 7.96.

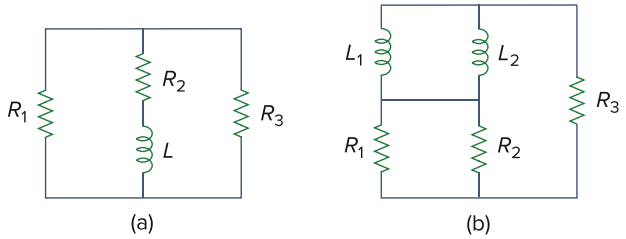



**Figure 7.96**
For Prob. 7.16.

**7.17** Consider the circuit of Fig. 7.97. Find $v_o(t)$ if $i(0) = 15$ A and $v(t) = 0$.

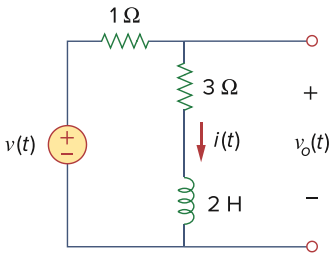



**Figure 7.97**
For Prob. 7.17.

**7.18** For the circuit in Fig. 7.98, determine $v_o(t)$ when $i(0) = 5$ A and $v(t) = 0$.

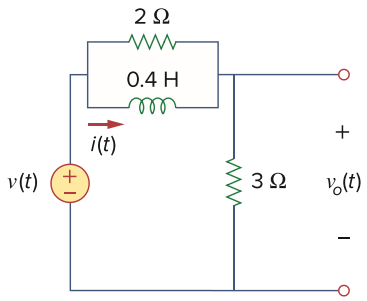



**Figure 7.98**
For Prob. 7.18.

**7.19** In the circuit of Fig. 7.99, find $i(t)$ for $t > 0$ if $i(0) = 5$ A.

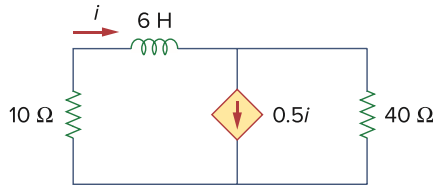



**Figure 7.99**
For Prob. 7.19.

**7.20** For the circuit in Fig. 7.100,

$$v = 90e^{-50t}\,\text{V}$$

and

$$i = 30e^{-50t}\,\text{A}, \qquad t > 0$$

(a) Find $L$ and $R$.
(b) Determine the time constant.
(c) Calculate the initial energy in the inductor.
(d) What fraction of the initial energy is dissipated in 10 ms?

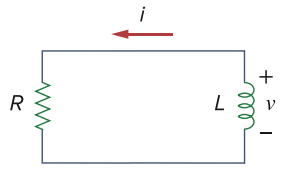



**Figure 7.100**
For Prob. 7.20.

**7.21** In the circuit of Fig. 7.101, find the value of $R$ for which the steady-state energy stored in the inductor will be 2 J.

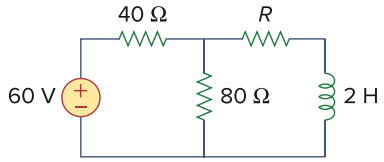



**Figure 7.101**
For Prob. 7.21.

**7.22** Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit of Fig. 7.102 if $i(0) = 10$ A.

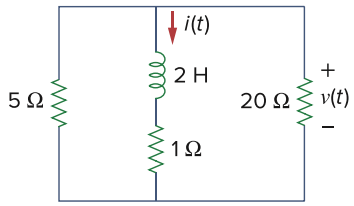



**Figure 7.102**
For Prob. 7.22.

**7.23** Consider the circuit in Fig. 7.103. Given that $v_o(0) = 10$ V, find $v_o$ and $v_x$ for $t > 0$.

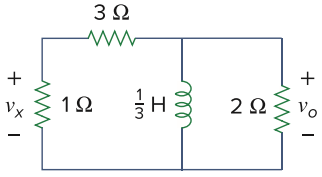



**Figure 7.103**
For Prob. 7.23.

## Section 7.4 Singularity Functions

**7.24** Express the following signals in terms of singularity functions.

(a) $v(t) = \begin{cases} 0, & t < 0 \\ -5, & t > 0 \end{cases}$

(b) $i(t) = \begin{cases} 0, & t < 1 \\ -10, & 1 < t < 3 \\ 10, & 3 < t < 5 \\ 0, & t > 5 \end{cases}$

(c) $x(t) = \begin{cases} t - 1, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ 4 - t, & 3 < t < 4 \\ 0, & \text{Otherwise} \end{cases}$

(d) $y(t) = \begin{cases} 2, & t < 0 \\ -5, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

**7.25** Design a problem to help other students better understand singularity functions.

**7.26** Express the signals in Fig. 7.104 in terms of singularity functions.

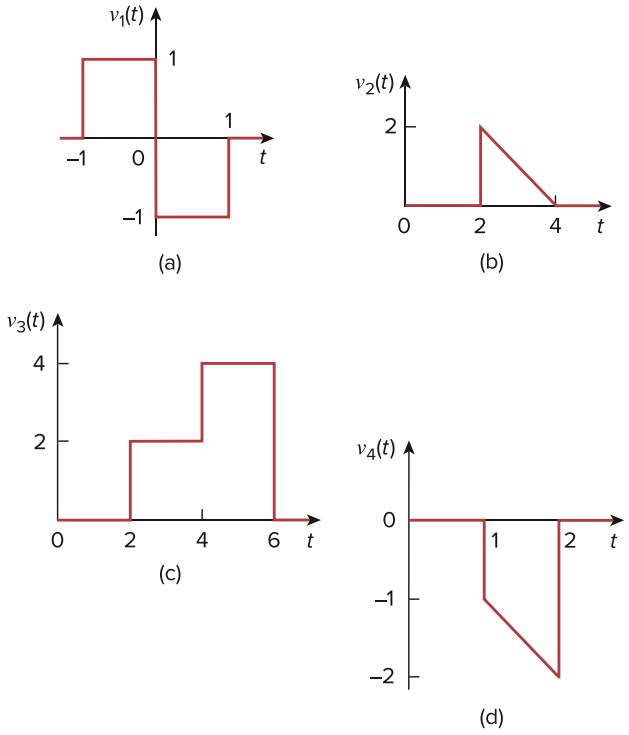



**Figure 7.104**
For Prob. 7.26.

**7.27** Express $v(t)$ in Fig. 7.105 in terms of step functions.

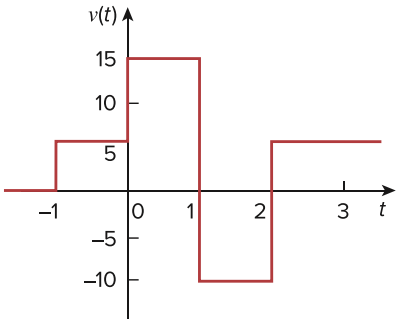



**Figure 7.105**
For Prob. 7.27.

**7.28** Sketch the waveform represented by

$$i(t) = [r(t) - r(t-1) - u(t-2) - r(t-2) + r(t-3) + u(t)(t-4)]\ \text{A}$$

**7.29** Sketch the following functions:

(a) $x(t) = 10e^{-t}u(t-1)$,

(b) $y(t) = 10e^{-(t-1)}u(t)$,

(c) $z(t) = \cos 4t\delta(t-1)$

**7.30** Evaluate the following integrals involving the impulse functions:

(a) $\int_{-\infty}^{\infty} 4t^2\delta(t-1)dt$

(b) $\int_{-\infty}^{\infty} 4t^2\cos 2\pi t\delta(t-0.5)dt$

**7.31** Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} e^{-4t^2}\delta(t-2)dt$

(b) $\int_{-\infty}^{\infty} [5\delta(t) + e^{-t}\delta(t) + \cos 2\pi t\delta(t)]dt$

**7.32** Evaluate the following integrals:

(a) $\int_{1}^{t} u(\lambda)d\lambda$

(b) $\int_{0}^{4} r(t-1)dt$

(c) $\int_{1}^{5} (t-6)^2\delta(t-2)dt$

**7.33** The voltage across a 10-mH inductor is $45\delta(t-2)$mV. Find the inductor current, assuming that the inductor is initially uncharged.

**7.34** Evaluate the following derivatives:

(a) $\frac{d}{dt}[u(t-1)u(t+1)]$

(b) $\frac{d}{dt}[r(t-6)u(t-2)]$

(c) $\frac{d}{dt}[\sin 4tu(t-3)]$

**7.35** Find the solution to the following differential equations:

(a) $\frac{dv}{dt} + 2v = 0, \qquad v(0) = -1\ \text{V}$

(b) $2\frac{di}{dt} - 3i = 0, \qquad i(0) = 2$

**7.36** Solve for $v$ in the following differential equations, subject to the stated initial condition.

(a) $dv/dt + v = u(t), \qquad v(0) = 0$

(b) $2\,dv/dt - v = 3u(t), \qquad v(0) = -6$

**7.37** A circuit is described by

$$4\frac{dv}{dt} + v = 10$$

(a) What is the time constant of the circuit?

(b) What is $v(\infty)$, the final value of $v$?

(c) If $v(0) = 2$, find $v(t)$ for $t \geq 0$.

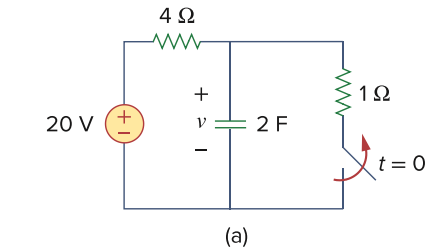
**7.38** A circuit is described by

$$\frac{di}{dt} + 3i = 2u(t)$$

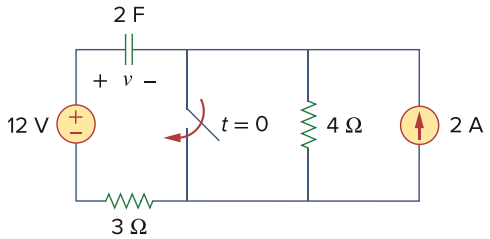
Find $i(t)$ for $t > 0$ given that $i(0) = 0$.

## Section 7.5 Step Response of an *RC* Circuit

**7.39** Calculate the capacitor voltage for $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.106.




(a)




(b)

**Figure 7.106**
For Prob. 7.39.

**7.40** Find the capacitor voltage for $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.107.

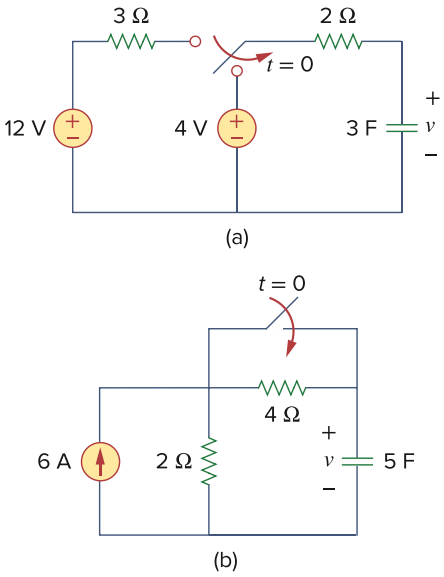



**Figure 7.107**
For Prob. 7.40.

**7.41** Using Fig. 7.108, design a problem to help other students better understand the step response of an *RC* circuit.

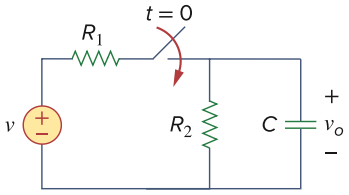



**Figure 7.108**
For Prob. 7.41.

**7.42** (a) If the switch in Fig. 7.109 has been open for a long time and is closed at $t = 0$, find $v_o(t)$.

(b) Suppose that the switch has been closed for a long time and is opened at $t = 0$. Find $v_o(t)$.

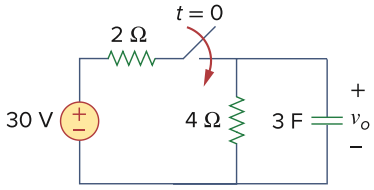



**Figure 7.109**
For Prob. 7.42.

**7.43** Consider the circuit in Fig. 7.110. Find $i(t)$ for $t < 0$ and $t > 0$.

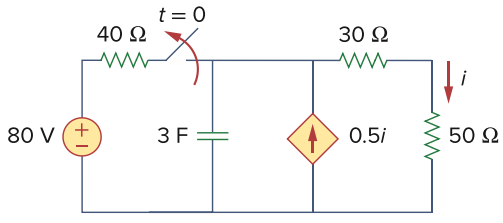



**Figure 7.110**
For Prob. 7.43.

**7.44** The switch in Fig. 7.111 has been in position *a* for a long time. At $t = 0$, it moves to position *b*. Calculate $i(t)$ for all $t > 0$.

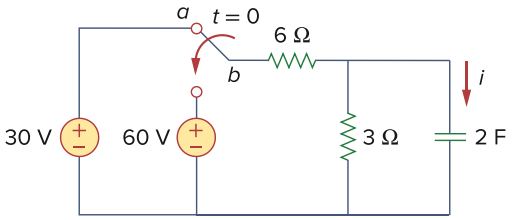



**Figure 7.111**
For Prob. 7.44.

**7.45** Find $v_o$ in the circuit of Fig. 7.112 when $v_s = 30u(t)$ V. Assume that $v_o(0) = 5$ V.

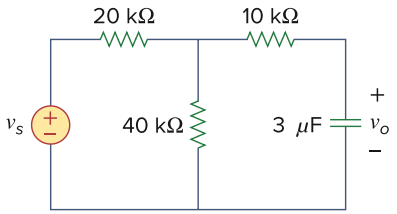



**Figure 7.112**
For Prob. 7.45.

**7.46** For the circuit in Fig. 7.113, $i_s(t) = 5u(t)$. Find $v(t)$.

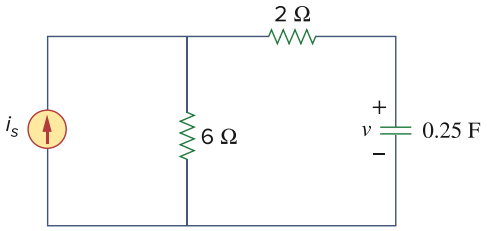



**Figure 7.113**
For Prob. 7.46.

**7.47** Determine $v(t)$ for $t > 0$ in the circuit of Fig. 7.114 if $v(0) = 0$.

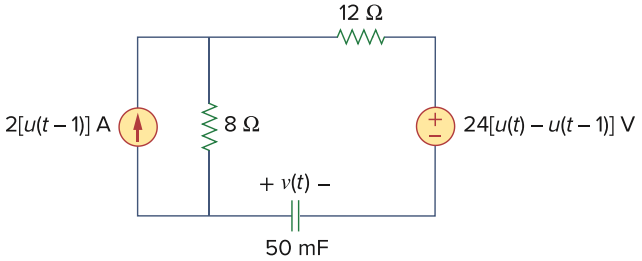



**Figure 7.114**
For Prob. 7.47.

**7.48** Find $v(t)$ and $i(t)$ in the circuit of Fig. 7.115.

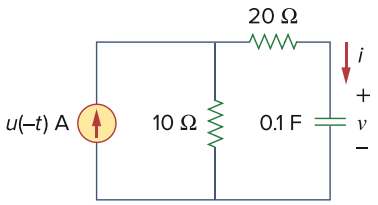



**Figure 7.115**
For Prob. 7.48.

**7.49** If the waveform in Fig. 7.116(a) is applied to the circuit of Fig. 7.116(b), find $v(t)$. Assume $v(0) = 0$.

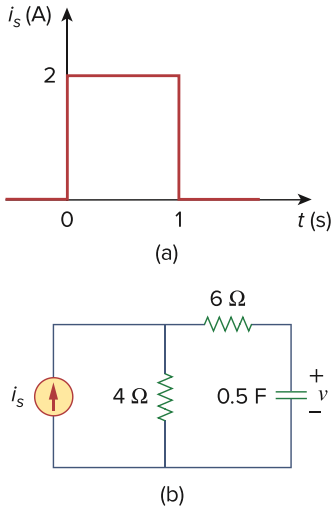



**Figure 7.116**
For Prob. 7.49 and Review Question 7.10.

***7.50** In the circuit of Fig. 7.117, find $i_x$ for $t > 0$. Let $R_1 = R_2 = 1\ \text{k}\Omega$, $R_3 = 2\ \text{k}\Omega$, and $C = 0.25\ \text{mF}$.

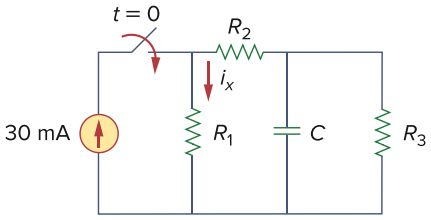



**Figure 7.117**
For Prob. 7.50.

## Section 7.6 Step Response of an *RL* Circuit

**7.51** Rather than applying the shortcut technique used in Section 7.6, use KVL to obtain Eq. (7.60).

**7.52** Using Fig. 7.118, design a problem to help other students better understand the step response of an *RL* circuit.

* An asterisk indicates a challenging problem.

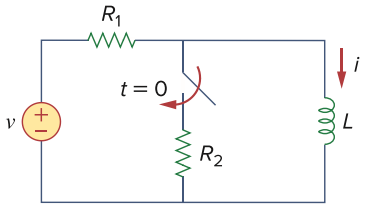



**Figure 7.118**
For Prob. 7.52.

**7.53** Determine the inductor current $i(t)$ for both $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.119.

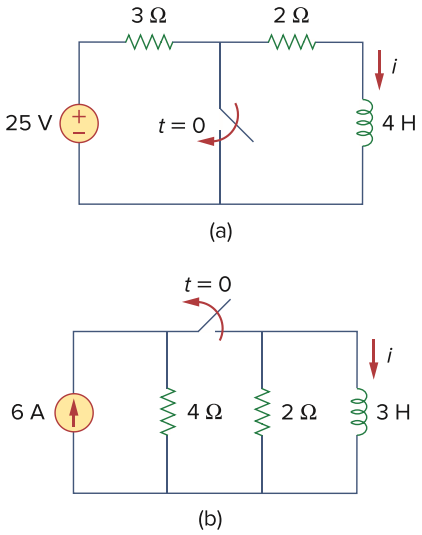



**Figure 7.119**
For Prob. 7.53.

**7.54** Obtain the inductor current for both $t < 0$ and $t > 0$ in each of the circuits in Fig. 7.120.

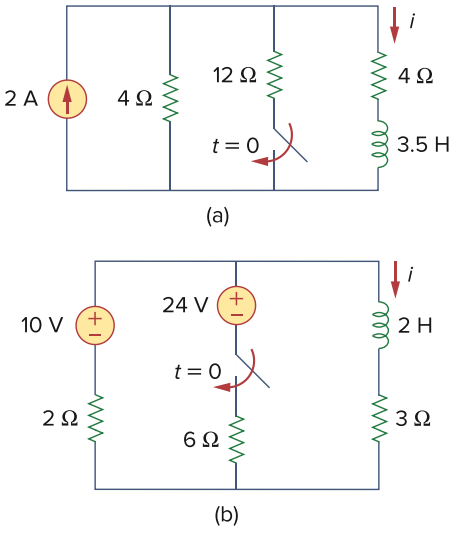



**Figure 7.120**
For Prob. 7.54.

**7.55** Find $v(t)$ for $t < 0$ and $t > 0$ in the circuit of Fig. 7.121.

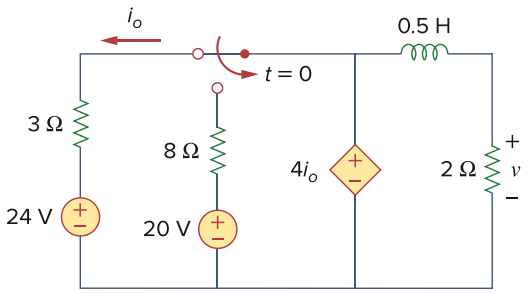



**Figure 7.121**
For Prob. 7.55.

**7.56** For the network shown in Fig. 7.122, find $v(t)$ for $t > 0$.

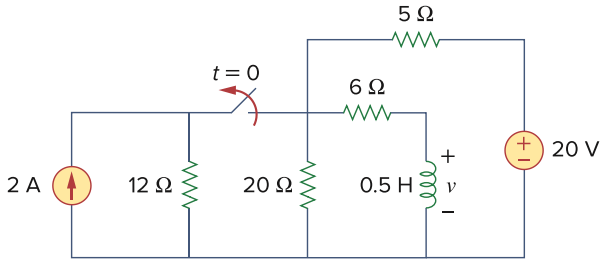



**Figure 7.122**
For Prob. 7.56.

***7.57** Find $i_1(t)$ and $i_2(t)$ for $t > 0$ in the circuit of Fig. 7.123.

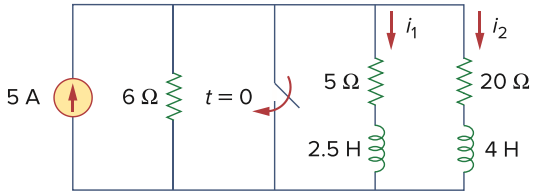



**Figure 7.123**
For Prob. 7.57.

**7.58** Rework Prob. 7.17 if $i(0) = 10$ A and $v(t) = 20u(t)$ V.

**7.59** Determine the step response $v_o(t)$ to $i_s = 6u(t)$ A in the circuit of Fig. 7.124.

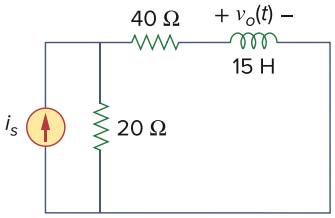



**Figure 7.124**
For Prob. 7.59.

**7.60** Find $v(t)$ for $t > 0$ in the circuit of Fig. 7.125 if the initial current in the inductor is zero.

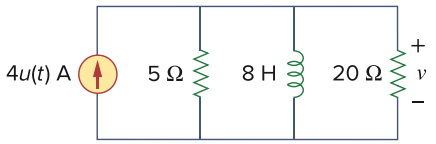



**Figure 7.125**
For Prob. 7.60.

**7.61** In the circuit in Fig. 7.126, $i_s$ changes from 5 A to 10 A at $t = 0$; that is, $i_s = 5u(-t) + 10u(t)$. Find $v$ and $i$.

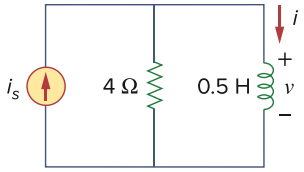



**Figure 7.126**
For Prob. 7.61.

**7.62** For the circuit in Fig. 7.127, calculate $i(t)$ if $i(0) = 0$.

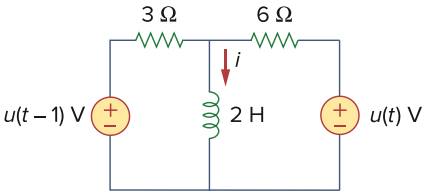



**Figure 7.127**
For Prob. 7.62.

**7.63** Obtain $v(t)$ and $i(t)$ in the circuit of Fig. 7.128.

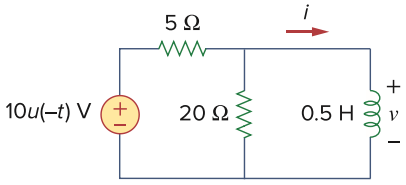



**Figure 7.128**
For Prob. 7.63.

**7.64** Determine the value of $i_L(t)$ and the total energy dissipated by the circuit from $t = 0$ sec to $t = \infty$ sec. The value of $i_{in}(t)$ is equal to $[6 - 6u(t)]$ A.

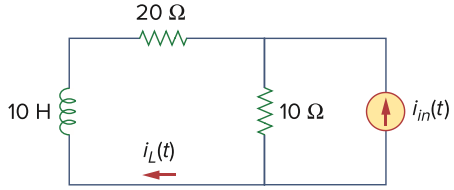



**Figure 7.129**
For Prob. 7.64.

**7.65** If the input pulse in Fig. 7.130(a) is applied to the circuit in Fig. 7.130(b), determine the response $i(t)$.

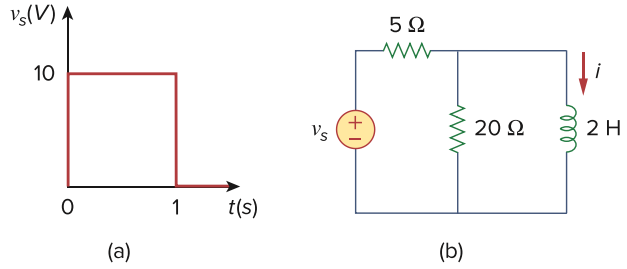



**Figure 7.130**
For Prob. 7.65.

## Section 7.7 First-order Op Amp Circuits

**7.66** Using Fig. 7.131, design a problem to help other students better understand first-order op amp circuits.

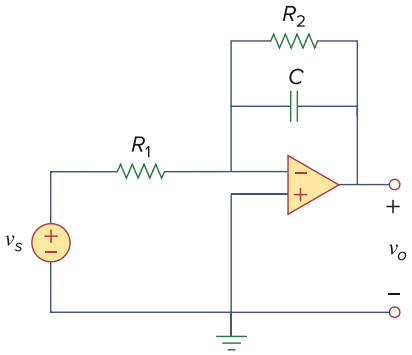



**Figure 7.131**
For Prob. 7.66.

**7.67** If $v(0) = 10$ V, find $v_o(t)$ for $t > 0$ in the op amp circuit in Fig. 7.132. Let $R = 100$ kΩ and $C = 20$ μF.

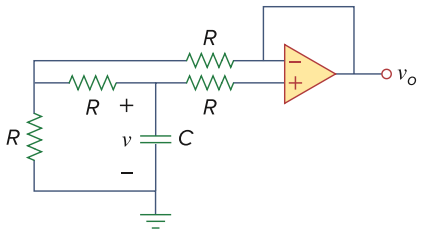



**Figure 7.132**
For Prob. 7.67.

**7.68** Obtain $v_o$ for $t > 0$ in the circuit of Fig. 7.133.

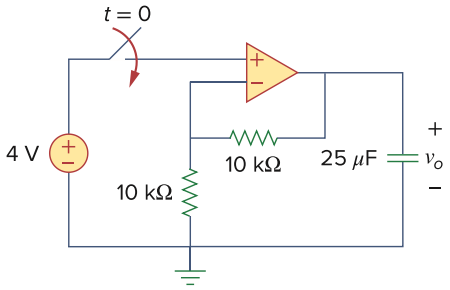



**Figure 7.133**
For Prob. 7.68.

**7.69** For the op amp circuit in Fig. 7.134, find $v_o(t)$ for $t > 0$.

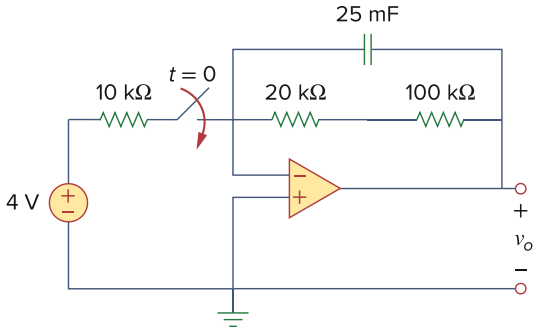



**Figure 7.134**
For Prob. 7.69.

**7.70** Determine $v_o$ for $t > 0$ when $v_s = 20$ mV in the op amp circuit of Fig. 7.135.

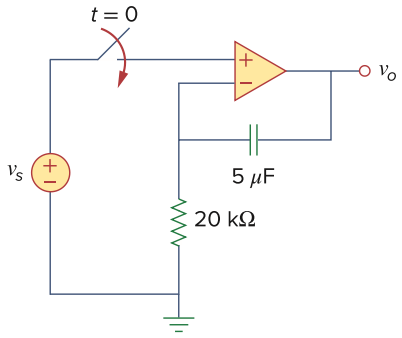



**Figure 7.135**
For Prob. 7.70.

**7.71** For the op amp circuit in Fig. 7.136, suppose $v_s = 10u(t)$ V. Find $v(t)$ for $t > 0$.

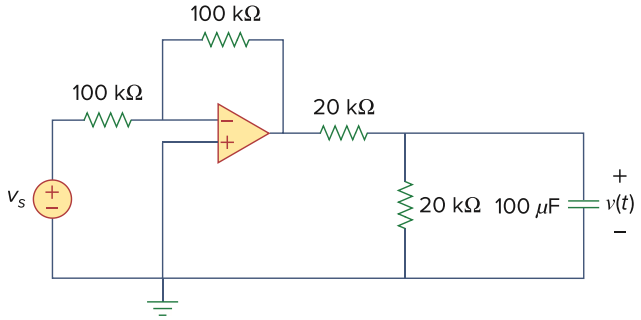



**Figure 7.136**
For Prob. 7.71.

**7.72** Find $i_o$ in the op amp circuit in Fig. 7.137. Assume that $v(0) = -2$ V, $R = 10$ kΩ, and $C = 10$ μF.

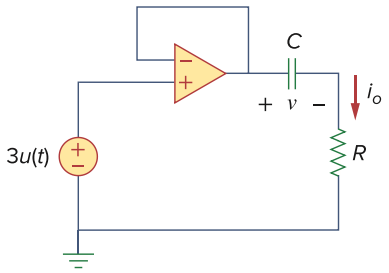



**Figure 7.137**
For Prob. 7.72.

**7.73** For the op amp circuit of Fig. 7.138, let $R_1 = 10$ kΩ, $R_f = 30$ kΩ, $C = 20$ μF, and $v(0) = 1$ V. Find $v_0$.

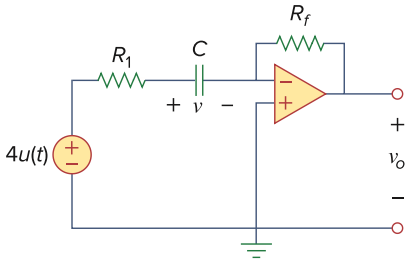



**Figure 7.138**
For Prob. 7.73.

**7.74** Determine $v_o(t)$ for $t > 0$ in the circuit of Fig. 7.139. Let $i_s = 10u(t)$ μA and assume that the capacitor is initially uncharged.

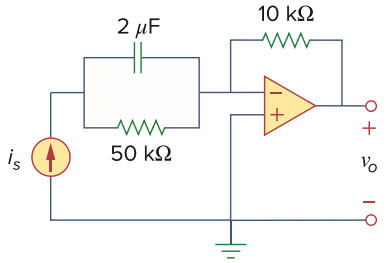



**Figure 7.139**
For Prob. 7.74.

**7.75** In the circuit of Fig. 7.140, find $v_o$ and $i_o$, given that $v_s = 10[1 - e^{-t}]u(t)$ V.

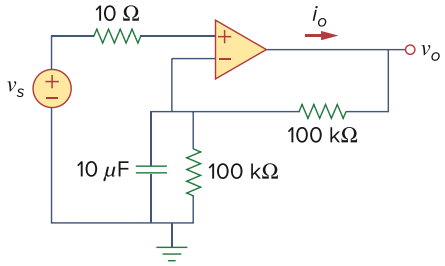



**Figure 7.140**
For Prob. 7.75.

## Section 7.8 Transient Analysis with *PSpice*

**7.76** Repeat Prob. 7.49 using *PSpice or MultiSim*.

**7.77** The switch in Fig. 7.141 opens at $t = 0$. Use *PSpice or MultiSim* to determine $v(t)$ for $t > 0$.

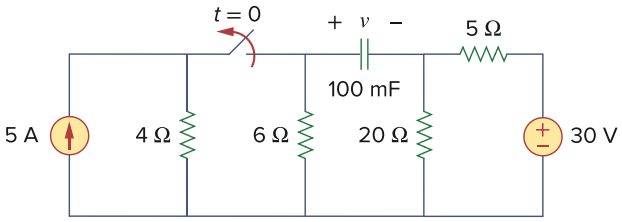



**Figure 7.141**
For Prob. 7.77.

**7.78** The switch in Fig. 7.142 moves from position $a$ to $b$ at $t = 0$. Use *PSpice or MultiSim* to find $i(t)$ for $t > 0$.

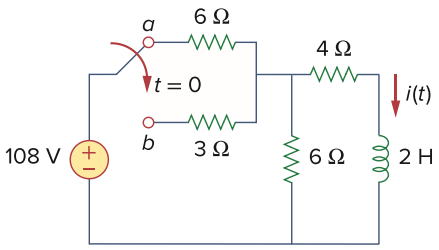



**Figure 7.142**
For Prob. 7.78.

**7.79** In the circuit of Fig. 7.143, determine $i_o(t)$.

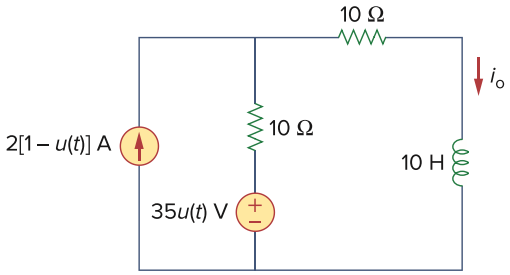



**Figure 7.143**
For Prob. 7.79.

**7.80** In the circuit of Fig. 7.144, find the value of $i_o$ for all values of $0 < t$.

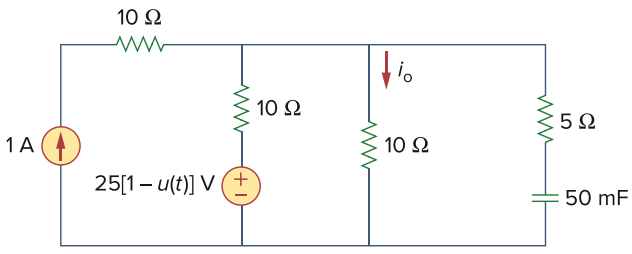



**Figure 7.144**
For Prob. 7.80.

**7.81** Repeat Prob. 7.65 using *PSpice or MultiSim*.

## Section 7.9 Applications

**7.82** In designing a signal-switching circuit, it was found that a 100-$\mu$F capacitor was needed for a time constant of 3 ms. What value resistor is necessary for the circuit?

**7.83** An *RC* circuit consists of a series connection of a 120-V source, a switch, a 34-MΩ resistor, and a 15-$\mu$F capacitor. The circuit is used in estimating the speed of a horse running a 4-km racetrack. The switch closes when the horse begins and opens when the horse crosses the finish line. Assuming that the capacitor charges to 85.6 V, calculate the speed of the horse.

**7.84** A capacitor with a value of 10 mF has a leakage resistance of 2 MΩ. How long does it take the voltage across the capacitor to decay to 40% of the initial voltage to which the capacitor is charged? Assume that the capacitor is charged and then set aside by itself.

**7.85** A simple relaxation oscillator circuit is shown in Fig. 7.145. The neon lamp fires when its voltage reaches 75 V and turns off when its voltage drops to 30 V. Its resistance is 120 Ω when on and infinitely high when off.

(a) For how long is the lamp on each time the capacitor discharges?
(b) What is the time interval between light flashes?

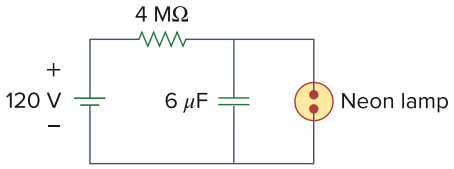



**Figure 7.145**
For Prob. 7.85.

**7.86** Figure 7.146 shows a circuit for setting the length of time voltage is applied to the electrodes of a welding machine. The time is taken as how long it takes the capacitor to charge from 0 to 8 V. What is the time range covered by the variable resistor?

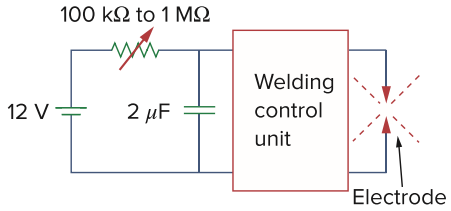



**Figure 7.146**
For Prob. 7.86.

**7.87** A 120-V dc generator energizes a motor whose coil has an inductance of 50 H and a resistance of 100 Ω. A field discharge resistor of 400 Ω is connected in parallel with the motor to avoid damage to the motor, as shown in Fig. 7.147. The system is at steady state. Find the current through the discharge resistor 100 ms after the breaker is tripped.

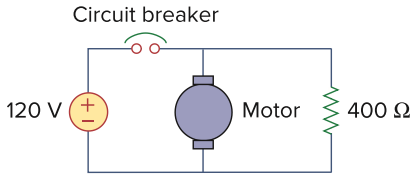



**Figure 7.147**
For Prob. 7.87.

# Comprehensive Problems

**7.88** The circuit in Fig. 7.148(a) can be designed as an approximate differentiator or an integrator, depending on whether the output is taken across the resistor or the capacitor, and also on the time constant $\tau = RC$ of the circuit and the width $T$ of the input pulse in Fig. 7.148(b). The circuit is a differentiator if $\tau \ll T$, say $\tau < 0.1T$, or an integrator if $\tau \gg T$, say $\tau > 10T$.

(a) What is the minimum pulse width that will allow a differentiator output to appear across the capacitor?
(b) If the output is to be an integrated form of the input, what is the maximum value the pulse width can assume?

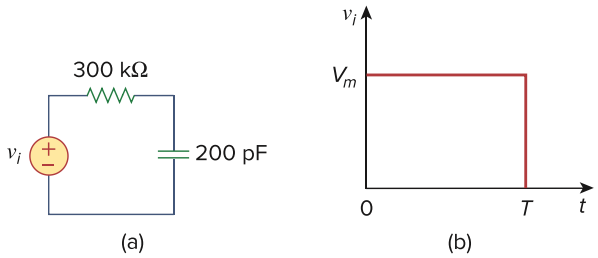



**Figure 7.148**
For Prob. 7.88.

**7.89** An *RL* circuit may be used as a differentiator if the output is taken across the inductor and $\tau \ll T$ (say $\tau < 0.1T$), where $T$ is the width of the input pulse. If $R$ is fixed at 200 kΩ, determine the maximum value of $L$ required to differentiate a pulse with $T = 10\ \mu$s.

**7.90** An attenuator probe employed with oscilloscopes was designed to reduce the magnitude of the input voltage $v_i$ by a factor of 10. As shown in Fig. 7.149, the oscilloscope has internal resistance $R_s$ and capacitance $C_s$, while the probe has an internal resistance $R_p$. If $R_p$ is fixed at 6 MΩ, find $R_s$ and $C_s$ for the circuit to have a time constant of 15 $\mu$s.

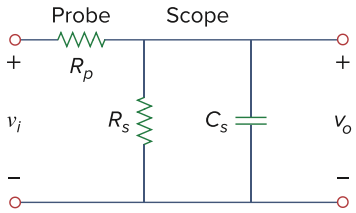



**Figure 7.149**
For Prob. 7.90.

**7.91** The circuit in Fig. 7.150 is used by a biology student to study "frog kick." She noticed that the frog kicked a little when the switch was closed but kicked violently for 5 s when the switch was opened. Model the frog as a resistor and calculate its resistance. Assume that it takes 10 mA for the frog to kick violently.

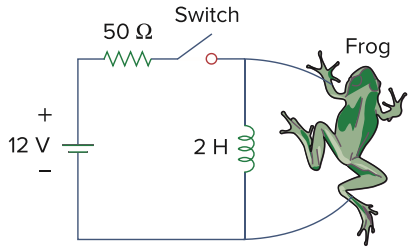



**Figure 7.150**
For Prob. 7.91.

**7.92** To move a spot of a cathode-ray tube across the screen requires a linear increase in the voltage across the deflection plates, as shown in Fig. 7.151. Given that the capacitance of the plates is 4 nF, sketch the current flowing through the plates.

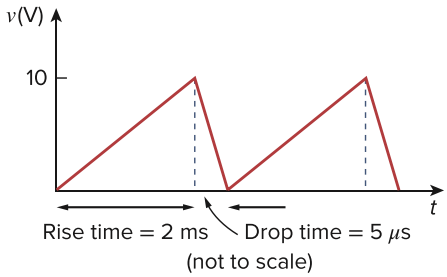



**Figure 7.151**
For Prob. 7.92.

chapter

# 8

# Second-Order Circuits

*Everyone who can earn a masters degree in engineering must earn a masters degree in engineering in order to maximize the success of their career! If you want to do research, state-of-the-art engineering, teach in a university, or start your own business, you really need to earn a doctoral degree!*

—Charles K. Alexander

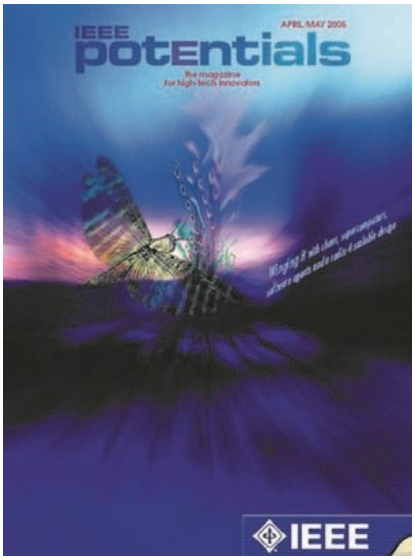
## Enhancing Your Career

To increase your engineering career opportunities after graduation, develop a strong fundamental understanding in a broad set of engineering areas. When possible, this might best be accomplished by working toward a graduate degree immediately upon receiving your undergraduate degree.

Each degree in engineering represents certain skills the student acquires. At the Bachelor degree level, you learn the language of engineering and the fundamentals of engineering and design. At the Master's level, you acquire the ability to do advanced engineering projects and to communicate your work effectively both orally and in writing. The Ph.D. represents a thorough understanding of the fundamentals of electrical engineering and a mastery of the skills necessary both for working at the frontiers of an engineering area and for communicating one's effort to others.

If you have no idea what career you should pursue after graduation, a graduate degree program will enhance your ability to explore career options. Since your undergraduate degree will only provide you with the fundamentals of engineering, a Master's degree in engineering supplemented by business courses benefits more engineering students than does getting a Master's of Business Administration (MBA). The best time to get your MBA is after you have been a practicing engineer for some years and decide your career path would be enhanced by strengthening your business skills.

Engineers should constantly educate themselves, formally and informally, taking advantage of all means of education. Perhaps there is no better way to enhance your career than to join a professional society such as IEEE and be an active member.




Enhancing your career involves understanding your goals, adapting to changes, anticipating opportunities, and planning your own niche.
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## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Develop a better understanding of the solution of general-second order differential equations.
2. Learn how to determine initial and final values.
3. Understand the response in source-free series *RLC* circuits.
4. Understand the response in source-free parallel *RLC* circuits.
5. Understand the step response of series *RLC* circuits.
6. Understand the step response of parallel *RLC* circuits.
7. Understand general second-order circuits.
8. Understand general second-order circuits with op amps.

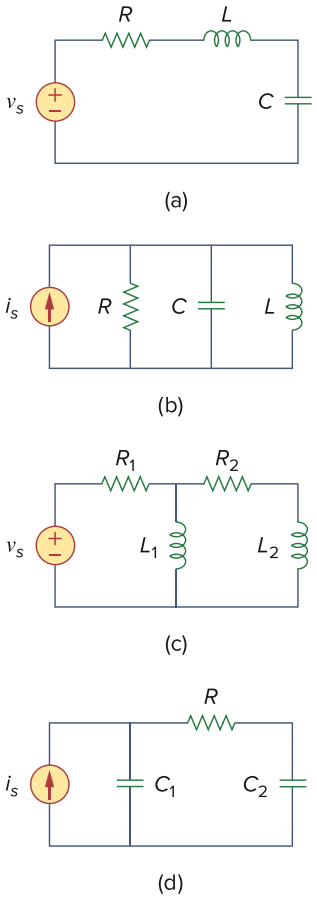



**Figure 8.1**
Typical examples of second-order circuits: (a) series *RLC* circuit, (b) parallel *RLC* circuit, (c) *RL* circuit, (d) *RC* circuit.

## 8.1 Introduction

In the previous chapter we considered circuits with a single storage element (a capacitor or an inductor). Such circuits are first-order because the differential equations describing them are first-order. In this chapter we will consider circuits containing two storage elements. These are known as *second-order* circuits because their responses are described by differential equations that contain second derivatives.

Typical examples of second-order circuits are *RLC* circuits, in which the three kinds of passive elements are present. Examples of such circuits are shown in Fig. 8.1(a) and (b). Other examples are *RL* and *RC* circuits, as shown in Fig. 8.1(c) and (d). It is apparent from Fig. 8.1 that a second-order circuit may have two storage elements of different type or the same type (provided elements of the same type cannot be represented by an equivalent single element). An op amp circuit with two storage elements may also be a second-order circuit. As with first-order circuits, a second-order circuit may contain several resistors and dependent and independent sources.

A **second-order circuit** is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.

Our analysis of second-order circuits will be similar to that used for first-order. We will first consider circuits that are excited by the initial conditions of the storage elements. Although these circuits may contain dependent sources, they are free of independent sources. These source-free circuits will give natural responses as expected. Later we will consider circuits that are excited by independent sources. These circuits will give both the transient response and the steady-state response. We consider only dc independent sources in this chapter. The case of sinusoidal and exponential sources is deferred to later chapters.

We begin by learning how to obtain the initial conditions for the circuit variables and their derivatives, as this is crucial to analyzing second-order circuits. Then we consider series and parallel *RLC* circuits such as shown in Fig. 8.1 for the two cases of excitation: by initial

conditions of the energy storage elements and by step inputs. Later we examine other types of second-order circuits, including op amp circuits. We will consider *PSpice* analysis of second-order circuits. Finally, we will consider the automobile ignition system and smoothing circuits as typical applications of the circuits treated in this chapter. Other applications such as resonant circuits and filters will be covered in Chapter 14.

## 8.2 Finding Initial and Final Values

Perhaps the major problem students face in handling second-order circuits is finding the initial and final conditions on circuit variables. Students are usually comfortable getting the initial and final values of $v$ and $i$ but often have difficulty finding the initial values of their derivatives: $dv/dt$ and $di/dt$. For this reason, this section is explicitly devoted to the subtleties of getting $v(0)$, $i(0)$, $dv(0)/dt$, $di(0)/dt$, $i(\infty)$, and $v(\infty)$. Unless otherwise stated in this chapter, $v$ denotes capacitor voltage, while $i$ is the inductor current.

There are two key points to keep in mind in determining the initial conditions.

First—as always in circuit analysis—we must carefully handle the polarity of voltage $v(t)$ across the capacitor and the direction of the current $i(t)$ through the inductor. Keep in mind that $v$ and $i$ are defined strictly according to the passive sign convention (see Figs. 6.3 and 6.23). One should carefully observe how these are defined and apply them accordingly.

Second, keep in mind that the capacitor voltage is always continuous so that

$$v(0^+) = v(0^-) \tag{8.1a}$$

and the inductor current is always continuous so that

$$i(0^+) = i(0^-) \tag{8.1b}$$

where $t = 0^-$ denotes the time just before a switching event and $t = 0^+$ is the time just after the switching event, assuming that the switching event takes place at $t = 0$.

Thus, in finding initial conditions, we first focus on those variables that cannot change abruptly, capacitor voltage and inductor current, by applying Eq. (8.1). The following examples illustrate these ideas.

### Example 8.1

The switch in Fig. 8.2 has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

**Solution:**

(a) If the switch is closed a long time before $t = 0$, it means that the circuit has reached dc steady state at $t = 0$. At dc steady state, the inductor acts like a short circuit, while the capacitor acts like an open circuit, so we have the circuit in Fig. 8.3(a) at $t = 0^-$. Thus,

$$i(0^-) = \frac{12}{4+2} = 2\ \text{A}, \qquad v(0^-) = 2i(0^-) = 4\ \text{V}$$

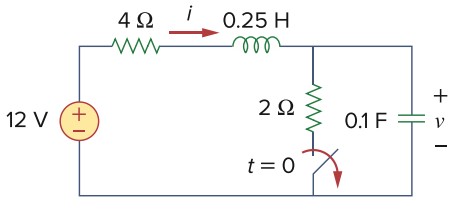



**Figure 8.2**
For Example 8.1.

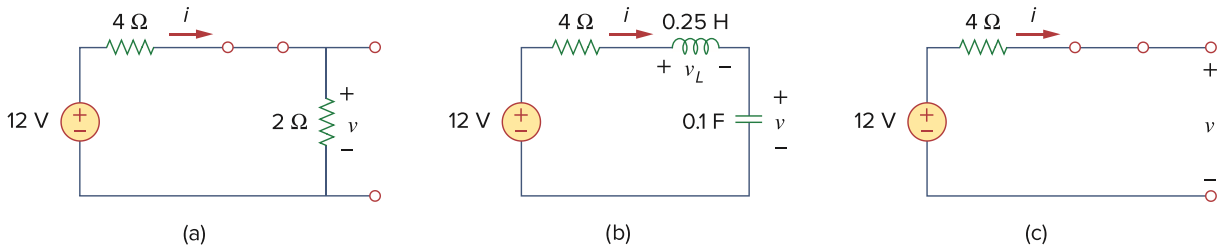



**Figure 8.3**
Equivalent circuit of that in Fig. 8.2 for: (a) $t = 0^-$, (b) $t = 0^+$, (c) $t \to \infty$.

As the inductor current and the capacitor voltage cannot change abruptly,

$$i(0^+) = i(0^-) = 2 \text{ A}, \qquad v(0^+) = v(0^-) = 4 \text{ V}$$

(b) At $t = 0^+$, the switch is open; the equivalent circuit is as shown in Fig. 8.3(b). The same current flows through both the inductor and capacitor. Hence,

$$i_C(0^+) = i(0^+) = 2 \text{ A}$$

Since $C\, dv/dt = i_C$, $dv/dt = i_C/C$, and

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20 \text{ V/s}$$

Similarly, since $L\, di/dt = v_L$, $di/dt = v_L/L$. We now obtain $v_L$ by applying KVL to the loop in Fig. 8.3(b). The result is

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

or

$$v_L(0^+) = 12 - 8 - 4 = 0$$

Thus,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0 \text{ A/s}$$

(c) For $t > 0$, the circuit undergoes transience. But as $t \to \infty$, the circuit reaches steady state again. The inductor acts like a short circuit and the capacitor like an open circuit, so that the circuit in Fig. 8.3(b) becomes that shown in Fig. 8.3(c), from which we have

$$i(\infty) = 0 \text{ A}, \qquad v(\infty) = 12 \text{ V}$$

## Practice Problem 8.1

The switch in Fig. 8.4 was open for a long time but closed at $t = 0$. Determine: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

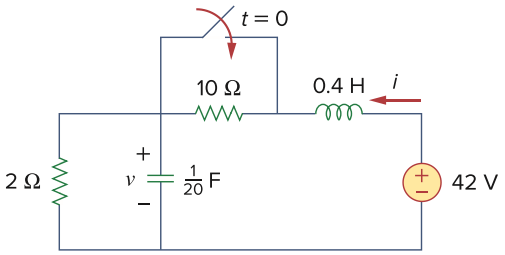



**Figure 8.4**
For Practice Prob. 8.1.

**Answer:** (a) 3.5 A, 7 V, (b) 87.5 A/s, 0 V/s, (c) 21 A, 42 V.

## Example 8.2

In the circuit of Fig. 8.5, calculate: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

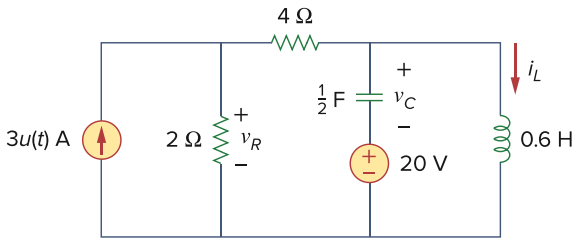



**Figure 8.5**
For Example 8.2.

### Solution:

(a) For $t < 0$, $3u(t) = 0$. At $t = 0^-$, since the circuit has reached steady state, the inductor can be replaced by a short circuit, while the capacitor is replaced by an open circuit as shown in Fig. 8.6(a). From this figure we obtain

$$i_L(0^-) = 0, \qquad v_R(0^-) = 0, \qquad v_C(0^-) = -20 \text{ V} \qquad \textbf{(8.2.1)}$$

Although the derivatives of these quantities at $t = 0^-$ are not required, it is evident that they are all zero, since the circuit has reached steady state and nothing changes.

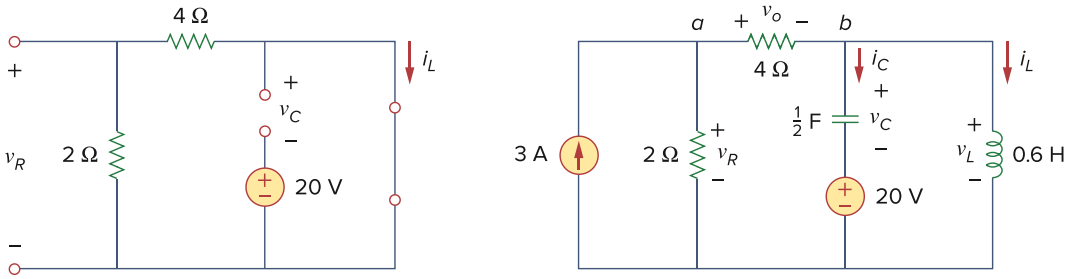



**Figure 8.6**
The circuit in Fig. 8.5 for: (a) $t = 0^-$, (b) $t = 0^+$.

For $t > 0$, $3u(t) = 3$, so that the circuit is now equivalent to that in Fig. 8.6(b). Since the inductor current and capacitor voltage cannot change abruptly,

$$i_L(0^+) = i_L(0^-) = 0, \qquad v_C(0^+) = v_C(0^-) = -20 \text{ V} \qquad \textbf{(8.2.2)}$$

Although the voltage across the 4-Ω resistor is not required, we will use it to apply KVL and KCL; let it be called $v_o$. Applying KCL at node $a$ in Fig. 8.6(b) gives

$$3 = \frac{v_R(0^+)}{2} + \frac{v_o(0^+)}{4} \qquad \textbf{(8.2.3)}$$

Applying KVL to the middle mesh in Fig. 8.6(b) yields

$$-v_R(0^+) + v_o(0^+) + v_C(0^+) + 20 = 0 \qquad \textbf{(8.2.4)}$$

Since $v_C(0^+) = -20$ V from Eq. (8.2.2), Eq. (8.2.4) implies that

$$v_R(0^+) = v_o(0^+) \tag{8.2.5}$$

From Eqs. (8.2.3) and (8.2.5), we obtain

$$v_R(0^+) = v_o(0^+) = 4 \text{ V} \tag{8.2.6}$$

(b) Since $L\, di_L/dt = v_L$,

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

But applying KVL to the right mesh in Fig. 8.6(b) gives

$$v_L(0^+) = v_C(0^+) + 20 = 0$$

Hence,

$$\frac{di_L(0^+)}{dt} = 0 \tag{8.2.7}$$

Similarly, since $C\, dv_C/dt = i_C$, then $dv_C/dt = i_C/C$. We apply KCL at node $b$ in Fig. 8.6(b) to get $i_C$:

$$\frac{v_o(0^+)}{4} = i_C(0^+) + i_L(0^+) \tag{8.2.8}$$

Since $v_o(0^+) = 4$ and $i_L(0^+) = 0$, $i_C(0^+) = 4/4 = 1$ A. Then

$$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{1}{0.5} = 2 \text{ V/s} \tag{8.2.9}$$

To get $dv_R(0^+)/dt$, we apply KCL to node $a$ and obtain

$$3 = \frac{v_R}{2} + \frac{v_o}{4}$$

Taking the derivative of each term and setting $t = 0^+$ gives

$$0 = 2\frac{dv_R(0^+)}{dt} + \frac{dv_o(0^+)}{dt} \tag{8.2.10}$$

We also apply KVL to the middle mesh in Fig. 8.6(b) and obtain

$$-v_R + v_C + 20 + v_o = 0$$

Again, taking the derivative of each term and setting $t = 0^+$ yields

$$-\frac{dv_R(0^+)}{dt} + \frac{dv_C(0^+)}{dt} + \frac{dv_o(0^+)}{dt} = 0$$

Substituting for $dv_C(0^+)/dt = 2$ gives

$$\frac{dv_R(0^+)}{dt} = 2 + \frac{dv_o(0^+)}{dt} \tag{8.2.11}$$

From Eqs. (8.2.10) and (8.2.11), we get

$$\frac{dv_R(0^+)}{dt} = \frac{2}{3} \text{ V/s}$$

We can find $di_R(0^+)/dt$ although it is not required. Since $v_R = 2i_R$,

$$\frac{di_R(0^+)}{dt} = \frac{1}{2}\frac{dv_R(0^+)}{dt} = \frac{1}{2}\frac{2}{3} = \frac{1}{3}\text{ A/s}$$

(c) As $t \to \infty$, the circuit reaches steady state. We have the equivalent circuit in Fig. 8.6(a) except that the 3-A current source is now operative. By current division principle,

$$i_L(\infty) = \frac{2}{2+4}3\text{ A} = 1\text{ A}$$
$$v_R(\infty) = \frac{4}{2+4}3\text{ A} \times 2 = 4\text{ V}, \qquad v_C(\infty) = -20\text{ V} \tag{8.2.12}$$

## Practice Problem 8.2

For the circuit in Fig. 8.7, find: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

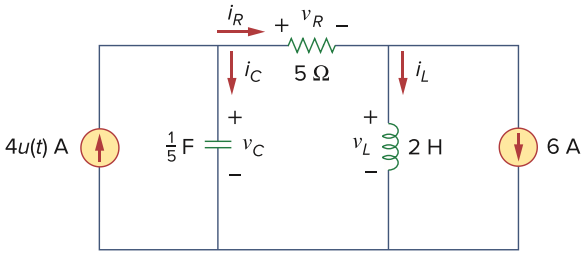



**Figure 8.7**
For Practice Prob. 8.2.

**Answer:** (a) −6 A, 0, 0, (b) 0, 20 V/s, 0, (c) −2 A, 20 V, 20 V.

## 8.3 The Source-Free Series *RLC* Circuit

An understanding of the natural response of the series *RLC* circuit is a necessary background for future studies in filter design and communications networks.

Consider the series *RLC* circuit sho wn in Fig. 8.8. The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor v oltage $V_0$ and initial inductor current $I_0$. Thus, at $t = 0$,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \tag{8.2a}$$
$$i(0) = I_0 \tag{8.2b}$$

Applying KVL around the loop in Fig. 8.8,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau = 0 \tag{8.3}$$

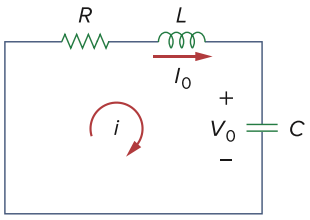



**Figure 8.8**
A source-free series *RLC* circuit.

To eliminate the integral, we differentiate with respect to $t$ and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{8.4}$$

This is a *second-order differential equation* and is the reason for calling the *RLC* circuits in this chapter second-order circuits. Our goal is to solve Eq. (8.4). To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of $i$ and its first derivative or initial values of some $i$ and $v$. The initial value of $i$ is given in Eq. (8.2b). We get the initial value of the derivative of $i$ from Eqs. (8.2a) and (8.3); that is,

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

or

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) \tag{8.5}$$

With the two initial conditions in Eqs. (8.2b) and (8.5), we can now solve Eq. (8.4). Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we let

$$i = Ae^{st} \tag{8.6}$$

where $A$ and $s$ are constants to be determined. Substituting Eq. (8.6) into Eq. (8.4) and carrying out the necessary differentiations, we obtain

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0 \tag{8.7}$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{8.8}$$

See Appendix C.1 for the formula to find the roots of a quadratic equation.

This quadratic equation is known as the *characteristic equation* of the differential Eq. (8.4), since the roots of the equation dictate the character of $i$. The two roots of Eq. (8.8) are

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{8.9a}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{8.9b}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}} \tag{8.10}$$

where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{8.11}$$

The roots $s_1$ and $s_2$ are called *natural frequencies*, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit; $\omega_0$ is known as the *resonant frequency* or strictly as the *undamped natural frequency*, expressed in radians per second (rad/s); and $\alpha$ is the *neper frequency* expressed in nepers per second. In terms of $\alpha$ and $\omega_0$, Eq. (8.8) can be written as

The *neper* (Np) is a dimensionless unit named after John Napier (1550–1617), a Scottish mathematician.

$$s^2 + 2\alpha s + \omega_0^2 = 0 \tag{8.8a}$$

The variables $s$ and $\omega_0$ are important quantities we will be discussing throughout the rest of the text.

The ratio $\alpha/\omega_0$ is known as the *damping ratio* $\zeta$.

The two values of $s$ in Eq. (8.10) indicate that there are two possible solutions for $i$, each of which is of the form of the assumed solution in Eq. (8.6); that is,

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t} \tag{8.12}$$

Since Eq. (8.4) is a linear equation, any linear combination of the two distinct solutions $i_1$ and $i_2$ is also a solution of Eq. (8.4). A complete or total solution of Eq. (8.4) would therefore require a linear combination of $i_1$ and $i_2$. Thus, the natural response of the series *RLC* circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{8.13}$$

where the constants $A_1$ and $A_2$ are determined from the initial values $i(0)$ and $di(0)/dt$ in Eqs. (8.2b) and (8.5).

From Eq. (8.10), we can infer that there are three types of solutions:

The response is *overdamped* when the roots of the circuit's characteristic equation are unequal and real, *critically damped* when the roots are equal and real, and *underdamped* when the roots are complex.

1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

We will consider each of these cases separately.

## Overdamped Case ($\alpha > \omega_0$)

From Eqs. (8.9) and (8.10), $\alpha > \omega_0$ implies $C > 4L/R^2$. When this happens, both roots $s_1$ and $s_2$ are negative and real. The response is

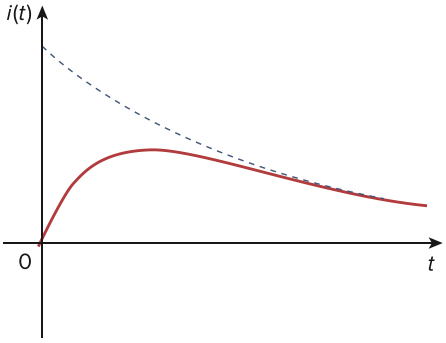
$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{8.14}$$

which decays and approaches zero as $t$ increases. Figure 8.9(a) illustrates a typical overdamped response.

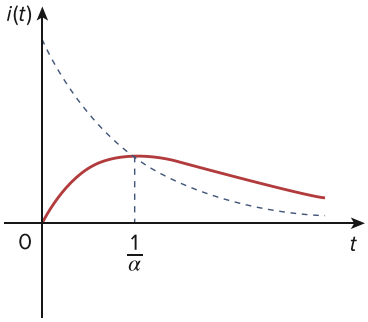
## Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$, $C = 4L/R^2$ and

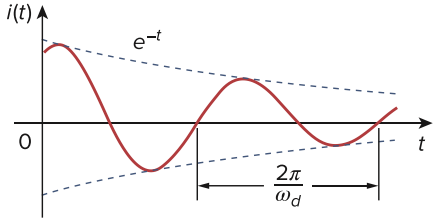
$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \tag{8.15}$$




(a)




(b)




(c)

**Figure 8.9**
(a) Overdamped response, (b) critically damped response, (c) underdamped response.

For this case, Eq. (8.13) yields

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

where $A_3 = A_1 + A_2$. This cannot be the solution, because the two initial conditions cannot be satisfied with the single constant $A_3$. What then could be wrong? Our assumption of an exponential solution is incorrect for the special case of critical damping. Let us go back to Eq. (8.4). When $\alpha = \omega_0 = R/2L$, Eq. (8.4) becomes

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \alpha^2 i = 0$$

or

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0 \tag{8.16}$$

If we let

$$f = \frac{di}{dt} + \alpha i \tag{8.17}$$

then Eq. (8.16) becomes

$$\frac{df}{dt} + \alpha f = 0$$

which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$, where $A_1$ is a constant. Equation (8.17) then becomes

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

or

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1 \tag{8.18}$$

This can be written as

$$\frac{d}{dt}(e^{\alpha t} i) = A_1 \tag{8.19}$$

Integrating both sides yields

$$e^{\alpha t} i = A_1 t + A_2$$

or

$$i = (A_1 t + A_2) e^{-\alpha t} \tag{8.20}$$

where $A_2$ is another constant. Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term, or

$$\boxed{i(t) = (A_2 + A_1 t) e^{-\alpha t}} \tag{8.21}$$

A typical critically damped response is shown in Fig. 8.9(b). In fact, Fig. 8.9(b) is a sketch of $i(t) = te^{-\alpha t}$, which reaches a maximum value of $e^{-1}/\alpha$ at $t = 1/\alpha$, one time constant, and then decays all the way to zero.

### Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0$, $C < 4L/R^2$. The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \tag{8.22a}$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \tag{8.22b}$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damped frequency*. Both $\omega_0$ and $\omega_d$ are natural frequencies because they help determine the natural response; while $\omega_0$ is often called the *undamped natural frequency*, $\omega_d$ is called the *damped natural frequency*. The natural response is

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 e^{-j\omega_d t} + A_2 e^{-j\omega_d t}) \end{aligned} \tag{8.23}$$

Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta \tag{8.24}$$

we get

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned} \tag{8.25}$$

Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants $B_1$ and $B_2$, we write

$$\boxed{i(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)} \tag{8.26}$$

With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature. The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$. Figure 8.9(c) depicts a typical underdamped response. Part (a) and (b) of Fig. 8.9 assume for each case that $i(0) = 0$.

Once the inductor current $i(t)$ is found for the *RLC* series circuit as shown above, other circuit quantities such as individual element voltages can easily be found. For example, the resistor voltage is $v_R = Ri$, and the inductor voltage is $v_L = L\,di/dt$. The inductor current $i(t)$ is selected as the key variable to be determined first in order to take advantage of Eq. (8.1b).

We conclude this section by noting the following interesting, peculiar properties of an *RLC* network:

1. The behavior of such a network is captured by the idea of *damping*, which is the gradual loss of the initial stored energy, as evidenced by the continuous decrease in the amplitude of the response. The damping effect is due to the presence of resistance $R$. The neper frequency $\alpha$ determines the rate at which the response is damped. If $R = 0$, then $\alpha = 0$, and we have an *LC* circuit with $1/\sqrt{LC}$ as the undamped natural frequency. Since $\alpha < \omega_0$ in this case, the response is not only undamped but also oscillatory. The circuit is said to be *loss-less*, because the dissipating or damping element ($R$) is absent. By adjusting the value of $R$, the response may be made undamped, overdamped, critically damped, or underdamped.
2. Oscillatory response is possible due to the presence of the two types of storage elements. Having both $L$ and $C$ allows the flow

$R = 0$ produces a perfectly sinusoidal response. This response cannot be practically accomplished with $L$ and $C$ because of the inherent losses in them. See Figs 6.8 and 6.26. An electronic device called an *oscillator* can produce a perfectly sinusoidal response.

Examples 8.5 and 8.7 demonstrate the effect of varying $R$.

The response of a second-order circuit with two storage elements of the same type, as in Fig. 8.1(c) and (d), cannot be oscillatory.

of energy back and forth between the two. The damped oscillation exhibited by the underdamped response is known as *ringing*. It stems from the ability of the storage elements $L$ and $C$ to transfer energy back and forth between them.

3. Observe from Fig. 8.9 that the waveforms of the responses differ. In general, it is difficult to tell from the waveforms the difference between the overdamped and critically damped responses. The critically damped case is the borderline between the underdamped and overdamped cases and it decays the fastest. With the same initial conditions, the overdamped case has the longest settling time, because it takes the longest time to dissipate the initial stored energy. If we desire the response that approaches the final value most rapidly without oscillation or ringing, the critically damped circuit is the right choice.

What this means in most practical circuits is that we seek an overdamped circuit that is as close as possible to a critically damped circuit.

## Example 8.3

In Fig. 8.8, $R = 40\ \Omega$, $L = 4$ H, and $C = 1/4$ F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

**Solution:**

We first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

or

$$s_1 = -0.101, \qquad s_2 = -9.899$$

Since $\alpha > \omega_0$, we conclude that the response is overdamped. This is also evident from the fact that the roots are real and negative.

## Practice Problem 8.3

If $R = 10\ \Omega$, $L = 5$ H, and $C = 2$ mF in Fig. 8.8, find $\alpha$, $\omega_0$, $s_1$, and $s_2$. What type of natural response will the circuit have?

**Answer:** 1, 10, $-1 \pm j9.95$, underdamped.

## Example 8.4

Find $i(t)$ in the circuit of Fig. 8.10. Assume that the circuit has reached steady state at $t = 0^-$.

**Solution:**

For $t < 0$, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit. The equivalent circuit is shown in Fig. 8.11(a). Thus, at $t = 0$,

$$i(0) = \frac{10}{4 + 6} = 1\ \text{A}, \qquad v(0) = 6i(0) = 6\ \text{V}$$

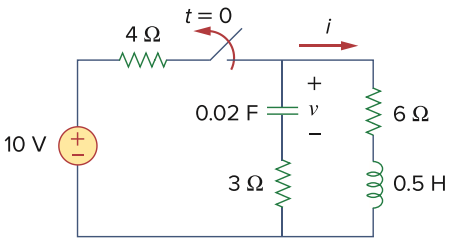



**Figure 8.10**
For Example 8.4.

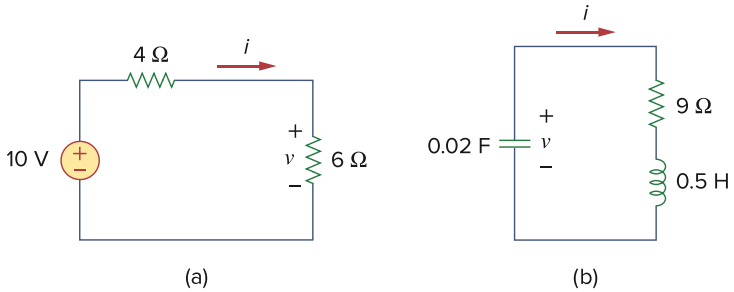



**Figure 8.11**
The circuit in Fig. 8.10: (a) for $t < 0$, (b) for $t > 0$.

where $i(0)$ is the initial current through the inductor and $v(0)$ is the initial voltage across the capacitor.

For $t > 0$, the switch is opened and the voltage source is disconnected. The equivalent circuit is shown in Fig. 8.11(b), which is a source-free series *RLC* circuit. Notice that the 3-Ω and 6-Ω resistors, which are in series in Fig. 8.10 when the switch is opened, have been combined to give $R = 9\ \Omega$ in Fig. 8.11(b). The roots are calculated as follows:

$$\alpha = \frac{R}{2L} = \frac{9}{2\left(\frac{1}{2}\right)} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100}$$

or

$$s_{1,2} = -9 \pm j4.359$$

Hence, the response is underdamped ($\alpha < \omega$); that is,

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359\,t) \tag{8.4.1}$$

We now obtain $A_1$ and $A_2$ using the initial conditions. At $t = 0$,

$$i(0) = 1 = A_1 \tag{8.4.2}$$

From Eq. (8.5),

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6\ \text{A/s} \tag{8.4.3}$$

Note that $v(0) = V_0 = -6$ V is used, because the polarity of $v_0$ in Fig. 8.11(b) is opposite that in Fig. 8.8. Taking the derivative of $i(t)$ in Eq. (8.4.1),

$$\frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359\,t) + e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359\,t)$$

Imposing the condition in Eq. (8.4.3) at $t = 0$ gives

$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2)$$

But $A_1 = 1$ from Eq. (8.4.2). Then

$$-6 = -9 + 4.359A_2 \qquad \Rightarrow \qquad A_2 = 0.6882$$

Substituting the values of $A_1$ and $A_2$ in Eq. (8.4.1) yields the complete solution as

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359\,t)\ \text{A}$$

## Practice Problem 8.4

The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position $b$ at $t = 0$, calculate $i(t)$ for $t > 0$.

**Answer:** $e^{-2.5t}(10\cos 1.6583t - 15.076\sin 1.6583t)$ A.

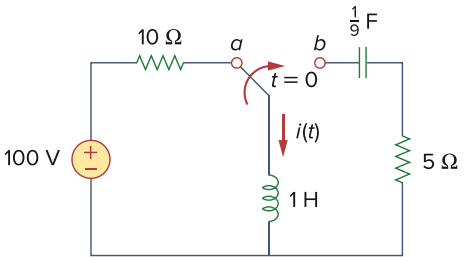



**Figure 8.12**
For Practice Prob. 8.4.

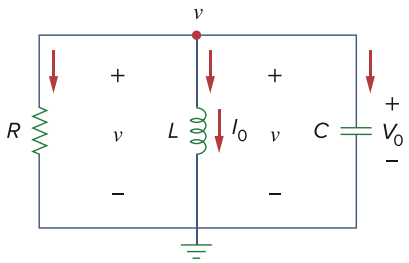



**Figure 8.13**
A source-free parallel *RLC* circuit.

## 8.4 The Source-Free Parallel *RLC* Circuit

Parallel *RLC* circuits find many practical applications, notably in communications networks and filter designs.

Consider the parallel *RLC* circuit shown in Fig. 8.13. Assume initial inductor current $I_0$ and initial capacitor voltage $V_0$,

$$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)\,dt \tag{8.27a}$$

$$v(0) = V_0 \tag{8.27b}$$

Because the three elements are in parallel, they have the same voltage $v$ across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node. Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau + C\frac{dv}{dt} = 0 \tag{8.28}$$

Taking the derivative with respect to $t$ and dividing by $C$ results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{8.29}$$

We obtain the characteristic equation by replacing the first derivative by $s$ and the second derivative by $s^2$. By following the same reasoning used in establishing Eqs. (8.4) through (8.8), the characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \tag{8.30}$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \tag{8.31}$$

where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{8.32}$$

The names of these terms remain the same as in the preceding section, as they play the same role in the solution. Again, there are three possible solutions, depending on whether $\alpha > \omega_0$, $\alpha = \omega_0$, or $\alpha < \omega_0$. Let us consider these cases separately.

### Overdamped Case ($\alpha > \omega_0$)

From Eq. (8.32), $\alpha > \omega_0$ when $L > 4R^2C$. The roots of the characteristic equation are real and negative. The response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{8.33}$$

### Critically Damped Case ($\alpha = \omega_0$)

For $\alpha = \omega_0$, $L = 4R^2C$. The roots are real and equal so that the response is

$$v(t) = (A_1 + A_2 t)e^{-\alpha t} \tag{8.34}$$

### Underdamped Case ($\alpha < \omega_0$)

When $\alpha < \omega_0$, $L < 4R^2C$. In this case the roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d \tag{8.35}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{8.36}$$

The response is

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) \tag{8.37}$$

The constants $A_1$ and $A_2$ in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$. The first term is known from Eq. (8.27b). We find the second term by combining Eqs. (8.27) and (8.28), as

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \tag{8.38}$$

The voltage waveforms are similar to those shown in Fig. 8.9 and will depend on whether the circuit is overdamped, underdamped, or critically damped.

Having found the capacitor voltage $v(t)$ for the parallel *RLC* circuit as shown above, we can readily obtain other circuit quantities such as individual element currents. For example, the resistor current is $i_R = v/R$ and the capacitor current is $i_C = C\,dv/dt$. We have selected the capacitor voltage $v(t)$ as the key variable to be determined first in order to take advantage of Eq. (8.1a). Notice that we first found the inductor current $i(t)$ for the *RLC* series circuit, whereas we first found the capacitor voltage $v(t)$ for the parallel *RLC* circuit.

## Example 8.5

In the parallel circuit of Fig. 8.13, find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF. Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.

**Solution:**

**CASE 1** If $R = 1.923\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is overdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

and the corresponding response is

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t} \tag{8.5.1}$$

We now apply the initial conditions to get $A_1$ and $A_2$.

$$v(0) = 5 = A_1 + A_2 \tag{8.5.2}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = -260$$

But differentiating Eq. (8.5.1),

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At $t = 0$,

$$-260 = -2A_1 - 50A_2 \tag{8.5.3}$$

From Eqs. (8.5.2) and (8.5.3), we obtain $A_1 = -0.2083$ and $A_2 = 5.208$. Substituting $A_1$ and $A_2$ in Eq. (8.5.1) yields

$$v(t) = -0.2083e^{-2t} + 5.208e^{-50t} \tag{8.5.4}$$

**CASE 2** When $R = 5\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10$$

while $\omega_0 = 10$ remains the same. Since $\alpha = \omega_0 = 10$, the response is critically damped. Hence, $s_1 = s_2 = -10$, and

$$v(t) = (A_1 + A_2 t)e^{-10t} \tag{8.5.5}$$

To get $A_1$ and $A_2$, we apply the initial conditions

$$v(0) = 5 = A_1 \tag{8.5.6}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{5 \times 10 \times 10^{-3}} = -100$$

But differentiating Eq. (8.5.5),

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t = 0$,

$$-100 = -10A_1 + A_2 \tag{8.5.7}$$

From Eqs. (8.5.6) and (8.5.7), $A_1 = 5$ and $A_2 = -50$. Thus,

$$v(t) = (5 - 50t)e^{-10t}\ \text{V} \tag{8.5.8}$$

**CASE 3** When $R = 6.25\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8$$

while $\omega_0 = 10$ remains the same. As $\alpha < \omega_0$ in this case, the response is underdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

Hence,

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t} \tag{8.5.9}$$

We now obtain $A_1$ and $A_2$, as

$$v(0) = 5 = A_1 \tag{8.5.10}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{6.25 \times 10 \times 10^{-3}} = -80$$

But differentiating Eq. (8.5.9),

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$,

$$-80 = -8A_1 + 6A_2 \tag{8.5.11}$$

From Eqs. (8.5.10) and (8.5.11), $A_1 = 5$ and $A_2 = -6.667$. Thus,

$$v(t) = (5 \cos 6t - 6.667 \sin 6t)e^{-8t} \tag{8.5.12}$$

Notice that by increasing the value of $R$, the degree of damping decreases and the responses differ. Figure 8.14 plots the three cases.

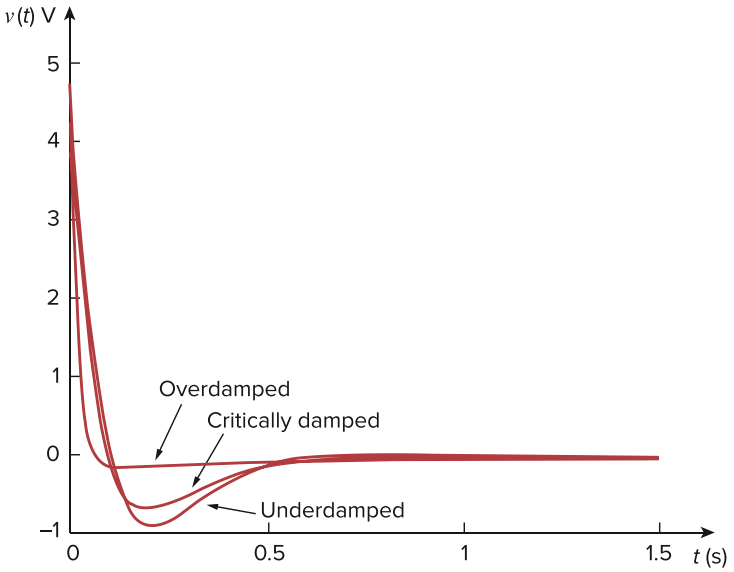



**Figure 8.14**
For Example 8.5: responses for three degrees of damping.

## Practice Problem 8.5

In Fig. 8.13, let $R = 2\ \Omega$, $L = 0.4$ H, $C = 25$ mF, $v(0) = 0$, $i(0) = 50$ mA. Find $v(t)$ for $t > 0$.

**Answer:** $-2te^{-10t}$ V.

## Example 8.6

Find $v(t)$ for $t > 0$ in the *RLC* circuit of Fig. 8.15.

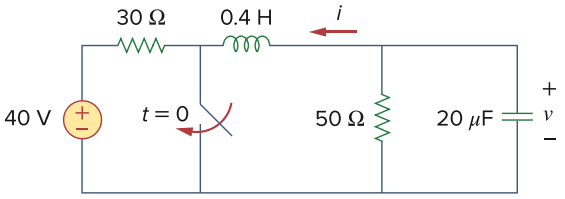



**Figure 8.15**
For Example 8.6.

**Solution:**

When $t < 0$, the switch is open; the inductor acts like a short circuit while the capacitor behaves like an open circuit. The initial voltage across the capacitor is the same as the voltage across the 50-Ω resistor; that is,

$$v(0) = \frac{50}{30 + 50}(40) = \frac{5}{8} \times 40 = 25 \text{ V} \tag{8.6.1}$$

The initial current through the inductor is

$$i(0) = -\frac{40}{30 + 50} = -0.5 \text{ A}$$

The direction of $i$ is as indicated in Fig. 8.15 to conform with the direction of $I_0$ in Fig. 8.13, which is in agreement with the convention that current flows into the positive terminal of an inductor (see Fig. 6.23). We need to express this in terms of $dv/dt$, since we are looking for $v$.

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{25 - 50 \times 0.5}{50 \times 20 \times 10^{-6}} = 0 \tag{8.6.2}$$

When $t > 0$, the switch is closed. The voltage source along with the 30-Ω resistor is separated from the rest of the circuit. The parallel *RLC* circuit acts independently of the voltage source, as illustrated in Fig. 8.16. Next, we determine that the roots of the characteristic equation are

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 50 \times 20 \times 10^{-6}} = 500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{0.4 \times 20 \times 10^{-6}} = 354$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -500 \pm \sqrt{250{,}000 - 124{,}997.6} = -500 \pm 354$$

or

$$s_1 = -854, \qquad s_2 = -146$$

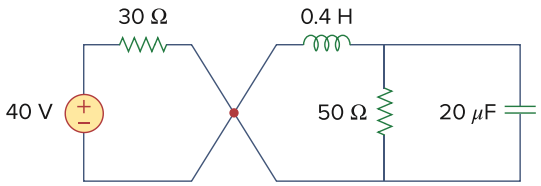



**Figure 8.16**
The circuit in Fig. 8.15 when $t > 0$. The parallel *RLC* circuit on the right-hand side acts independently of the circuit on the left-hand side of the junction.

Since $\alpha > \omega_0$, we have the overdamped response

$$v(t) = A_1 e^{-854t} + A_2 e^{-146t} \tag{8.6.3}$$

At $t = 0$, we impose the condition in Eq. (8.6.1),

$$v(0) = 25 = A_1 + A_2 \quad \Rightarrow \quad A_2 = 25 - A_1 \tag{8.6.4}$$

Taking the derivative of $v(t)$ in Eq. (8.6.3),

$$\frac{dv}{dt} = -854A_1 e^{-854t} - 146A_2 e^{-146t}$$

Imposing the condition in Eq. (8.6.2),

$$\frac{dv(0)}{dt} = 0 = -854A_1 - 146A_2$$

or

$$0 = 854A_1 + 146A_2 \tag{8.6.5}$$

Solving Eqs. (8.6.4) and (8.6.5) gives

$$A_1 = -5.156, \qquad A_2 = 30.16$$

Thus, the complete solution in Eq. (8.6.3) becomes

$$v(t) = -5.156e^{-854t} + 30.16e^{-146t} \text{ V}$$

## Practice Problem 8.6

Refer to the circuit in Fig. 8.17. Find $v(t)$ for $t > 0$.

**Answer:** $50(e^{-10t} - e^{-2.5t})$ V.

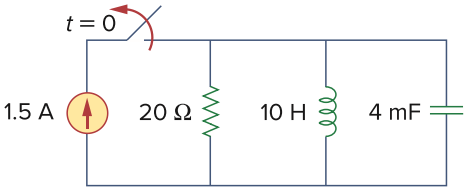



**Figure 8.17**
For Practice Prob. 8.6.

## 8.5 Step Response of a Series *RLC* Circuit

As we learned in the preceding chapter, the step response is obtained by the sudden application of a dc source. Consider the series *RLC* circuit shown in Fig. 8.18. Applying KVL around the loop for $t > 0$,

$$L\frac{di}{dt} + Ri + v = V_s \tag{8.39}$$

But

$$i = C\frac{dv}{dt}$$

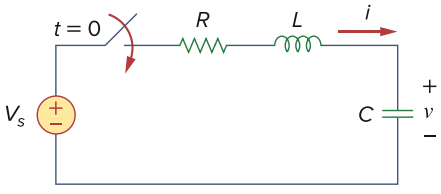



**Figure 8.18**
Step voltage applied to a series *RLC* circuit.

Substituting for $i$ in Eq. (8.39) and rearranging terms,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \tag{8.40}$$

which has the same form as Eq. (8.4). More specifically, the coefficients are the same (and that is important in determining the frequency parameters) but the variable is different. (Likewise, see Eq. (8.47).) Hence, the characteristic equation for the series *RLC* circuit is not af fected by the presence of the dc source.

The solution to Eq. (8.40) has two components: the transient response $v_t(t)$ and the steady-state response $v_{ss}(t)$; that is,

$$v(t) = v_t(t) + v_{ss}(t) \tag{8.41}$$

The transient response $v_t(t)$ is the component of the total response that dies out with time. The form of the transient response is the same as the form of the solution obtained in Section 8.3 for the source-free circuit, given by Eqs. (8.14), (8.21), and (8.26). Therefore, the transient response $v_t(t)$ for the overdamped, underdamped, and critically damped cases are:

$$v_t(t) = A_1e^{s_1t} + A_2e^{s_2t} \qquad \text{(Overdamped)} \tag{8.42a}$$

$$v_t(t) = (A_1 + A_2t)e^{-\alpha t} \qquad \text{(Critically damped)} \tag{8.42b}$$

$$v_t(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)} \tag{8.42c}$$

The steady-state response is the final value of $v(t)$. In the circuit in Fig. 8.18, the final value of the capacitor voltage is the same as the source voltage $V_s$. Hence,

$$v_{ss}(t) = v(\infty) = V_s \tag{8.43}$$

Thus, the complete solutions for the o verdamped, underdamped, and critically damped cases are:

$$v(t) = V_s + A_1e^{s_1t} + A_2e^{s_2t} \qquad \text{(Overdamped)} \tag{8.44a}$$

$$v(t) = V_s + (A_1 + A_2t)e^{-\alpha t} \qquad \text{(Critically damped)} \tag{8.44b}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)} \tag{8.44c}$$

The values of the constants $A_1$ and $A_2$ are obtained from the initial conditions: $v(0)$ and $dv(0)/dt$. Keep in mind that $v$ and $i$ are, respectively, the voltage across the capacitor and the current through the inductor. Therefore, Eq. (8.44) only applies for finding $v$. But once the capacitor voltage $v_C = v$ is known, we can determine $i = C\,dv/dt$, which is the same current through the capacitor, inductor, and resistor. Hence, the v oltage across the resistor is $v_R = iR$, while the inductor voltage is $v_L = L\,di/dt$.

Alternatively, the complete response for any v ariable $x(t)$ can be found directly, because it has the general form

$$x(t) = x_{ss}(t) + x_t(t) \tag{8.45}$$

where the $x_{ss} = x(\infty)$ is the final value and $x_t(t)$ is the transient response. The final value is found as in Section 8.2. The transient response has the same form as in Eq. (8.42), and the associated constants are determined from Eq. (8.44) based on the values of $x(0)$ and $dx(0)/dt$.

## Example 8.7

For the circuit in Fig. 8.19, find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, and $R = 1\ \Omega$.

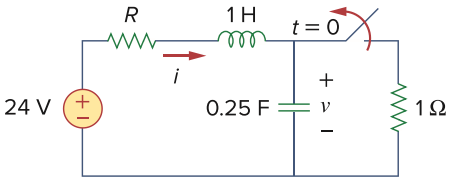



**Figure 8.19**
For Example 8.7.

**Solution:**

**CASE 1** When $R = 5\ \Omega$. For $t < 0$, the switch is closed for a long time. The capacitor behaves like an open circuit while the inductor acts like a short circuit. The initial current through the inductor is

$$i(0) = \frac{24}{5+1} = 4\ \text{A}$$

and the initial voltage across the capacitor is the same as the voltage across the 1-Ω resistor; that is,

$$v(0) = 1i(0) = 4\ \text{V}$$

For $t > 0$, the switch is opened, so that we ha ve the 1- Ω resistor disconnected. What remains is the series *RLC* circuit with the v oltage source. The characteristic roots are determined as follows:

$$\alpha = \frac{R}{2L} = \frac{5}{2 \times 1} = 2.5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, -4$$

Since $\alpha > \omega_0$, we have the overdamped natural response. The total response is therefore

$$v(t) = v_{ss} + (A_1 e^{-t} + A_2 e^{-4t})$$

where $v_{ss}$ is the steady-state response. It is the final value of the capacitor voltage. In Fig. 8.19, $v_f = 24$ V. Thus,

$$v(t) = 24 + (A_1 e^{-t} + A_2 e^{-4t}) \tag{8.7.1}$$

We now need to find $A_1$ and $A_2$ using the initial conditions.

$$v(0) = 4 = 24 + A_1 + A_2$$

or

$$-20 = A_1 + A_2 \tag{8.7.2}$$

The current through the inductor cannot change abruptly and is the same current through the capacitor at $t = 0^+$ because the inductor and capacitor are now in series. Hence,

$$i(0) = C\frac{dv(0)}{dt} = 4 \qquad \Rightarrow \qquad \frac{dv(0)}{dt} = \frac{4}{C} = \frac{4}{0.25} = 16$$

Before we use this condition, we need to take the derivative of $v$ in Eq. (8.7.1).

$$\frac{dv}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t} \tag{8.7.3}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 16 = -A_1 - 4A_2 \tag{8.7.4}$$

From Eqs. (8.7.2) and (8.7.4), $A_1 = -64/3$ and $A_2 = 4/3$. Substituting $A_1$ and $A_2$ in Eq. (8.7.1), we get

$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t}) \text{ V} \tag{8.7.5}$$

Since the inductor and capacitor are in series for $t > 0$, the inductor current is the same as the capacitor current. Hence,

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.3) by $C = 0.25$ and substituting the values of $A_1$ and $A_2$ gives

$$i(t) = \frac{4}{3}(4e^{-t} - e^{-4t}) \text{ A} \tag{8.7.6}$$

Note that $i(0) = 4$ A, as expected.

**CASE 2** When $R = 4\ \Omega$. Again, the initial current through the inductor is

$$i(0) = \frac{24}{4+1} = 4.8 \text{ A}$$

and the initial capacitor voltage is

$$v(0) = 1i(0) = 4.8 \text{ V}$$

For the characteristic roots,

$$\alpha = \frac{R}{2L} = \frac{4}{2 \times 1} = 2$$

while $\omega_0 = 2$ remains the same. In this case, $s_1 = s_2 = -\alpha = -2$, and we have the critically damped natural response. The total response is therefore

$$v(t) = v_{ss} + (A_1 + A_2 t)e^{-2t}$$

and, as before $v_{ss} = 24$ V,

$$v(t) = 24 + (A_1 + A_2 t)e^{-2t} \tag{8.7.7}$$

To find $A_1$ and $A_2$, we use the initial conditions. We write

$$v(0) = 4.8 = 24 + A_1 \quad \Rightarrow \quad A_1 = -19.2 \tag{8.7.8}$$

Since $i(0) = C\,dv(0)/dt = 4.8$ or

$$\frac{dv(0)}{dt} = \frac{4.8}{C} = 19.2$$

From Eq. (8.7.7),

$$\frac{dv}{dt} = (-2A_1 - 2tA_2 + A_2)e^{-2t} \tag{8.7.9}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 19.2 = -2A_1 + A_2 \tag{8.7.10}$$

From Eqs. (8.7.8) and (8.7.10), $A_1 = -19.2$ and $A_2 = -19.2$. Thus, Eq. (8.7.7) becomes

$$v(t) = 24 - 19.2(1 + t)e^{-2t}\ \text{V} \tag{8.7.11}$$

The inductor current is the same as the capacitor current; that is,

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.9) by $C = 0.25$ and substituting the values of $A_1$ and $A_2$ gives

$$i(t) = (4.8 + 9.6t)e^{-2t}\ \text{A} \tag{8.7.12}$$

Note that $i(0) = 4.8$ A, as expected.

**CASE 3** When $R = 1\ \Omega$. The initial inductor current is

$$i(0) = \frac{24}{1 + 1} = 12\ \text{A}$$

and the initial voltage across the capacitor is the same as the voltage across the 1-Ω resistor,

$$v(0) = 1i(0) = 12\ \text{V}$$

$$\alpha = \frac{R}{2L} = \frac{1}{2 \times 1} = 0.5$$

Since $\alpha = 0.5 < \omega_0 = 2$, we have the underdamped response

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0.5 \pm j1.936$$

The total response is therefore

$$v(t) = 24 + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t} \tag{8.7.13}$$

We now determine $A_1$ and $A_2$. We write

$$v(0) = 12 = 24 + A_1 \quad \Rightarrow \quad A_1 = -12 \tag{8.7.14}$$

Since $i(0) = C\,dv(0)/dt = 12$,

$$\frac{dv(0)}{dt} = \frac{12}{C} = 48 \tag{8.7.15}$$

But

$$\frac{dv}{dt} = e^{-0.5t}(-1.936A_1 \sin 1.936t + 1.936A_2 \cos 1.936t) - 0.5e^{-0.5t}(A_1 \cos 1.936t + A_2 \sin 1.936t) \tag{8.7.16}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 48 = (-0 + 1.936A_2) - 0.5(A_1 + 0)$$

Substituting $A_1 = -12$ gives $A_2 = 21.694$, and Eq. (8.7.13) becomes

$$v(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t}\ \text{V} \tag{8.7.17}$$

The inductor current is

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.16) by $C = 0.25$ and substituting the values of $A_1$ and $A_2$ gives

$$i(t) = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t} \text{ A} \qquad \textbf{(8.7.18)}$$

Note that $i(0) = 12$ A, as expected.

Figure 8.20 plots the responses for the three cases. From this figure, we observe that the critically damped response approaches the step input of 24 V the fastest.

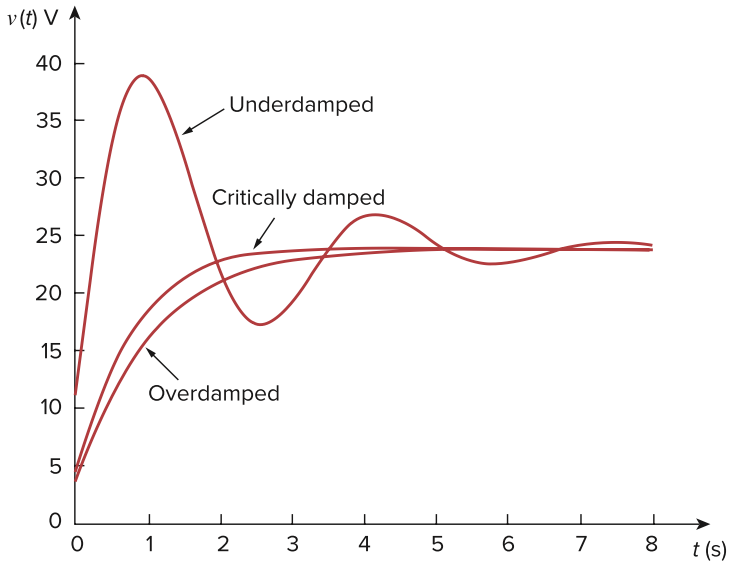



**Figure 8.20**
For Example 8.7: response for three degrees of damping.

## Practice Problem 8.7

Having been in position $a$ for a long time, the switch in Fig. 8.21 is moved to position $b$ at $t = 0$. Find $v(t)$ and $v_R(t)$ for $t > 0$.

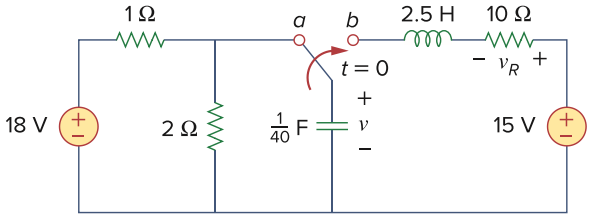



**Figure 8.21**
For Practice Prob. 8.7.

**Answer:** $15 - (1.7321 \sin 3.464t + 3 \cos 3.464t)e^{-2t}$ V, $3.464e^{-2t} \sin 3.464t$ V.

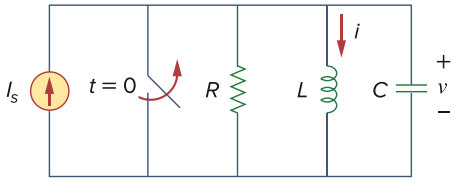



**Figure 8.22**
Parallel *RLC* circuit with an applied current.

## 8.6 Step Response of a Parallel *RLC* Circuit

Consider the parallel *RLC* circuit shown in Fig. 8.22. We want to find $i$ due to a sudden application of a dc current. Applying KCL at the top node for $t > 0$,

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \qquad \textbf{(8.46)}$$

But

$$v = L\frac{di}{dt}$$

Substituting for $v$ in Eq. (8.46) and dividing by $LC$, we get

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC} \tag{8.47}$$

which has the same characteristic equation as Eq. (8.29).

The complete solution to Eq. (8.47) consists of the transient response $i_t(t)$ and the steady-state response $i_{ss}$; that is,

$$i(t) = i_t(t) + i_{ss}(t) \tag{8.48}$$

The transient response is the same as what we had in Section 8.4. The steady-state response is the final value of $i$. In the circuit in Fig. 8.22, the final value of the current through the inductor is the same as the source current $I_s$. Thus,

$$\begin{aligned} i(t) &= I_s + A_1e^{s_1t} + A_2e^{s_2t} \quad \text{(Overdamped)} \\ i(t) &= I_s + (A_1 + A_2t)e^{-\alpha t} \quad \text{(Critically damped)} \\ i(t) &= I_s + (A_1\cos\omega_d t + A_2\sin\omega_d t)e^{-\alpha t} \quad \text{(Underdamped)} \end{aligned} \tag{8.49}$$

The constants $A_1$ and $A_2$ in each case can be determined from the initial conditions for $i$ and $di/dt$. Again, we should keep in mind that Eq. (8.49) only applies for finding the inductor current $i$. But once the inductor current $i_L = i$ is known, we can find $v = L\,di/dt$, which is the same voltage across inductor, capacitor, and resistor. Hence, the current through the resistor is $i_R = v/R$, while the capacitor current is $i_C = C\,dv/dt$. Alternatively, the complete response for any variable $x(t)$ may be found directly, using

$$x(t) = x_{ss}(t) + x_t(t) \tag{8.50}$$

where $x_{ss}$ and $x_t$ are its final value and transient response, respectively.

## Example 8.8

In the circuit of Fig. 8.23, find $i(t)$ and $i_R(t)$ for $t > 0$.

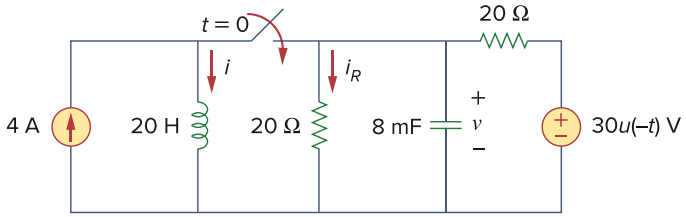



**Figure 8.23**
For Example 8.8.

**Solution:**

For $t < 0$, the switch is open, and the circuit is partitioned into two independent subcircuits. The 4-A current flows through the inductor, so that

$$i(0) = 4\text{ A}$$

Since $30u(-t) = 30$ when $t < 0$ and 0 when $t > 0$, the voltage source is operative for $t < 0$. The capacitor acts like an open circuit and the voltage across it is the same as the voltage across the 20-Ω resistor connected in parallel with it. By voltage division, the initial capacitor voltage is

$$v(0) = \frac{20}{20 + 20}(30) = 15 \text{ V}$$

For $t > 0$, the switch is closed, and we have a parallel $RLC$ circuit with a current source. The voltage source is zero which means it acts like a short-circuit. The two 20-Ω resistors are now in parallel. They are combined to give $R = 20 \parallel 20 = 10\ \Omega$. The characteristic roots are determined as follows:

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 10 \times 8 \times 10^{-3}} = 6.25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20 \times 8 \times 10^{-3}}} = 2.5$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -6.25 \pm \sqrt{39.0625 - 6.25}$$
$$= -6.25 \pm 5.7282$$

or

$$s_1 = -11.978, \qquad s_2 = -0.5218$$

Since $\alpha > \omega_0$, we have the overdamped case. Hence,

$$i(t) = I_s + A_1 e^{-11.978t} + A_2 e^{-0.5218t} \tag{8.8.1}$$

where $I_s = 4$ is the final value of $i(t)$. We now use the initial conditions to determine $A_1$ and $A_2$. At $t = 0$,

$$i(0) = 4 = 4 + A_1 + A_2 \qquad \Rightarrow \qquad A_2 = -A_1 \tag{8.8.2}$$

Taking the derivative of $i(t)$ in Eq. (8.8.1),

$$\frac{di}{dt} = -11.978A_1 e^{-11.978t} - 0.5218A_2 e^{-0.5218t}$$

so that at $t = 0$,

$$\frac{di(0)}{dt} = -11.978A_1 - 0.5218A_2 \tag{8.8.3}$$

But

$$L\frac{di(0)}{dt} = v(0) = 15 \qquad \Rightarrow \qquad \frac{di(0)}{dt} = \frac{15}{L} = \frac{15}{20} = 0.75$$

Substituting this into Eq. (8.8.3) and incorporating Eq. (8.8.2), we get

$$0.75 = (11.978 - 0.5218)A_2 \qquad \Rightarrow \qquad A_2 = 0.0655$$

Thus, $A_1 = -0.0655$ and $A_2 = 0.0655$. Inserting $A_1$ and $A_2$ in Eq. (8.8.1) gives the complete solution as

$$i(t) = 4 + 0.0655(e^{-0.5218t} - e^{-11.978t}) \text{ A}$$

From $i(t)$, we obtain $v(t) = L\, di/dt$ and

$$i_R(t) = \frac{v(t)}{20} = \frac{L}{20}\frac{di}{dt} = 0.785e^{-11.978t} - 0.0342e^{-0.5218t} \text{ A}$$

**Practice Problem 8.8**

Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit of Fig. 8.24.

**Answer:** $10(1-\cos(0.5t))$ A, $100\sin(0.5t)$ V.

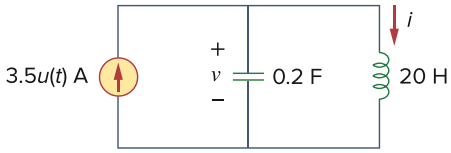



**Figure 8.24**
For Practice Prob. 8.8.

## 8.7 General Second-Order Circuits

Now that we have mastered series and parallel *RLC* circuits, we are prepared to apply the ideas to any second-order circuit having one or more independent sources with constant values. Although the series and parallel *RLC* circuits are the second-order circuits of greatest interest, other second-order circuits including op amps are also useful. Given a second-order circuit, we determine its step response $x(t)$ (which may be voltage or current) by taking the following four steps:

1. We first determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$, as discussed in Section 8.2.
2. We turn off the independent sources and find the form of the transient response $x_t(t)$ by applying KCL and KVL. Once a second-order differential equation is obtained, we determine its characteristic roots. Depending on whether the response is overdamped, critically damped, or underdamped, we obtain $x_t(t)$ with two unknown constants as we did in the previous sections.
3. We obtain the steady-state response as

$$x_{ss}(t) = x(\infty) \tag{8.51}$$

   where $x(\infty)$ is the final value of $x$, obtained in step 1.
4. The total response is now found as the sum of the transient response and steady-state response

$$x(t) = x_t(t) + x_{ss}(t) \tag{8.52}$$

   We finally determine the constants associated with the transient response by imposing the initial conditions $x(0)$ and $dx(0)/dt$, determined in step 1.

We can apply this general procedure to find the step response of any second-order circuit, including those with op amps. The following examples illustrate the four steps.

A circuit may look complicated at first. But once the sources are turned off in an attempt to find the form of the transient response, it may be reducible to a first-order circuit, when the storage elements can be combined, or to a parallel/series *RLC* circuit. If it is reducible to a first-order circuit, the solution becomes simply what we had in Chapter 7. If it is reducible to a parallel or series *RLC* circuit, we apply the techniques of previous sections in this chapter.

Problems in this chapter can also be solved by using Laplace transforms, which are covered in Chapters 15 and 16.

**Example 8.9**

Find the complete response $v$ and then $i$ for $t > 0$ in the circuit of Fig. 8.25.

**Solution:**
We first find the initial and final values. At $t = 0^-$, the circuit is at steady state. The switch is open; the equivalent circuit is shown in Fig. 8.26(a). It is evident from the figure that

$$v(0^-) = 12\text{ V}, \qquad i(0^-) = 0$$

At $t = 0^+$, the switch is closed; the equivalent circuit is in Fig. 8.26(b). By the continuity of capacitor voltage and inductor current, we know that

$$v(0^+) = v(0^-) = 12\text{ V}, \qquad i(0^+) = i(0^-) = 0 \tag{8.9.1}$$

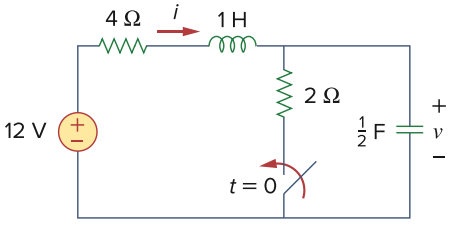



**Figure 8.25**
For Example 8.9.

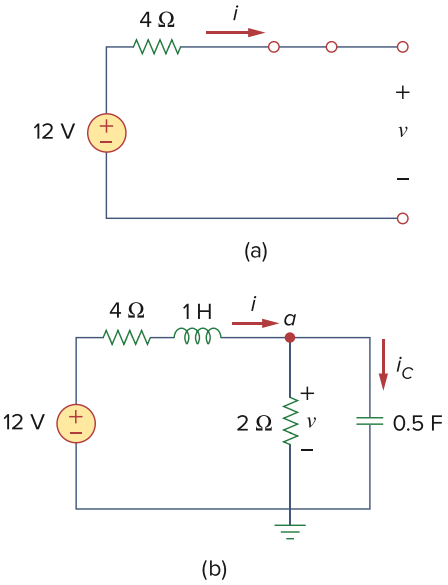



**Figure 8.26**
Equivalent circuit of the circuit in Fig. 8.25 for: (a) $t < 0$, (b) $t > 0$.

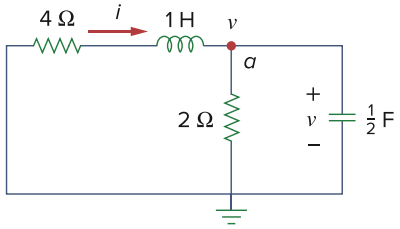



**Figure 8.27**
Obtaining the form of the transient response for Example 8.9.

To get $dv(0^+)/dt$, we use $C\,dv/dt = i_C$ or $dv/dt = i_C/C$. Applying KCL at node $a$ in Fig. 8.26(b),

$$i(0^+) = i_C(0^+) + \frac{v(0^+)}{2}$$

$$0 = i_C(0^+) + \frac{12}{2} \quad \Rightarrow \quad i_C(0^+) = -6 \text{ A}$$

Hence,

$$\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12 \text{ V/s} \tag{8.9.2}$$

The final values are obtained when the inductor is replaced by a short circuit and the capacitor by an open circuit in Fig. 8.26(b), giving

$$i(\infty) = \frac{12}{4+2} = 2 \text{ A}, \qquad v(\infty) = 2i(\infty) = 4 \text{ V} \tag{8.9.3}$$

Next, we obtain the form of the transient response for $t > 0$. By turning off the 12-V voltage source, we have the circuit in Fig. 8.27. Applying KCL at node $a$ in Fig. 8.27 gives

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} \tag{8.9.4}$$

Applying KVL to the left mesh results in

$$4i + 1\frac{di}{dt} + v = 0 \tag{8.9.5}$$

Since we are interested in $v$ for the moment, we substitute $i$ from Eq. (8.9.4) into Eq. (8.9.5). We obtain

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0$$

or

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0$$

From this, we obtain the characteristic equation as

$$s^2 + 5s + 6 = 0$$

with roots $s = -2$ and $s = -3$. Thus, the natural response is

$$v_n(t) = Ae^{-2t} + Be^{-3t} \tag{8.9.6}$$

where $A$ and $B$ are unknown constantsto be determined later. The steady-state response is

$$v_{ss}(t) = v(\infty) = 4 \tag{8.9.7}$$

The complete response is

$$v(t) = v_t + v_{ss} = 4 + Ae^{-2t} + Be^{-3t} \tag{8.9.8}$$

We now determine $A$ and $B$ using the initial values. From Eq. (8.9.1), $v(0) = 12$. Substituting this into Eq. (8.9.8) at $t = 0$ gives

$$12 = 4 + A + B \quad \Rightarrow \quad A + B = 8 \tag{8.9.9}$$

Taking the derivative of $v$ in Eq. (8.9.8),

$$\frac{dv}{dt} = -2Ae^{-2t} - 3Be^{-3t} \tag{8.9.10}$$

Substituting Eq. (8.9.2) into Eq. (8.9.10) at $t = 0$ gives

$$-12 = -2A - 3B \quad \Rightarrow \quad 2A + 3B = 12 \tag{8.9.11}$$

From Eqs. (8.9.9) and (8.9.11), we obtain

$$A = 12, \qquad B = -4$$

so that Eq. (8.9.8) becomes

$$v(t) = 4 + 12e^{-2t} - 4e^{-3t}\ \text{V}, \qquad t > 0 \tag{8.9.12}$$

From $v$, we can obtain other quantities of interest by referring to Fig. 8.26(b). To obtain $i$, for example,

$$\begin{aligned} i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} &= 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t} \\ &= 2 - 6e^{-2t} + 4e^{-3t}\ \text{A}, \qquad t > 0 \end{aligned} \tag{8.9.13}$$

Notice that $i(0) = 0$, in agreement with Eq. (8.9.1).

## Practice Problem 8.9

Determine $v$ and $i$ for $t > 0$ in the circuit of Fig. 8.28. (See comments about current sources in Practice Prob. 7.5.)

**Answer:** $20(1 - e^{-5t})$ V, $5(1 - e^{-5t})$ A.

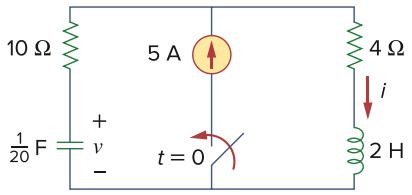



**Figure 8.28**
For Practice Prob. 8.9.

## Example 8.10

Find $v_o(t)$ for $t > 0$ in the circuit of Fig. 8.29.

**Solution:**
This is an example of a second-order circuit with two inductors. We first obtain the mesh currents $i_1$ and $i_2$, which happen to be the currents through the inductors. We need to obtain the initial and final values of these currents.

For $t < 0$, $7u(t) = 0$, so that $i_1(0^-) = 0 = i_2(0^-)$. For $t > 0$, $7u(t) = 7$, so that the equivalent circuit is as shown in Fig. 8.30(a). Due to the continuity of inductor current,

$$i_1(0^+) = i_1(0^-) = 0, \qquad i_2(0^+) = i_2(0^-) = 0 \tag{8.10.1}$$

$$v_{L_2}(0^+) = v_o(0^+) = 1[(i_1(0^+) - i_2(0^+)] = 0 \tag{8.10.2}$$

Applying KVL to the left loop in Fig. 8.30(a) at $t = 0^+$,

$$7 = 3i_1(0^+) + v_{L_1}(0^+) + v_o(0^+)$$

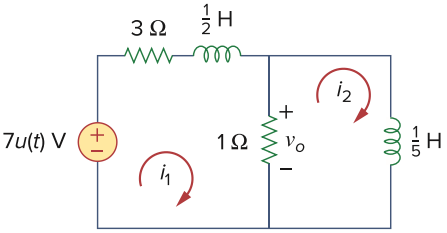



**Figure 8.29**
For Example 8.10.

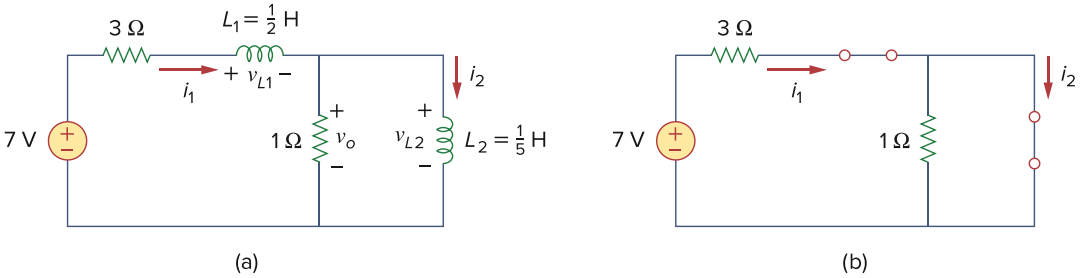



**Figure 8.30**
Equivalent circuit of that in Fig. 8.29 for: (a) $t > 0$, (b) $t \to \infty$.

or

$$v_{L_1}(0^+) = 7 \text{ V}$$

Since $L_1\, di_1/dt = v_{L_1}$,

$$\frac{di_1(0^+)}{dt} = \frac{v_{L1}}{L_1} = \frac{7}{\frac{1}{2}} = 14 \text{ A/s} \tag{8.10.3}$$

Similarly, since $L_2\, di_2/dt = v_{L_2}$,

$$\frac{di_2(0^+)}{dt} = \frac{v_{L2}}{L_2} = 0 \tag{8.10.4}$$

As $t \to \infty$, the circuit reaches steady state, and the inductors can be replaced by short circuits, as shown in Fig. 8.30(b). From this figure,

$$i_1(\infty) = i_2(\infty) = \frac{7}{3} \text{ A} \tag{8.10.5}$$

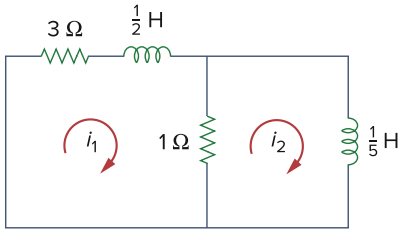



**Figure 8.31**
Obtaining the form of the transient response for Example 8.10.

Next, we obtain the form of the transient responses by removing the voltage source, as shown in Fig. 8.31. Applying KVL to the two meshes yields

$$4i_1 - i_2 + \frac{1}{2}\frac{di_1}{dt} = 0 \tag{8.10.6}$$

and

$$i_2 + \frac{1}{5}\frac{di_2}{dt} - i_1 = 0 \tag{8.10.7}$$

From Eq. (8.10.6),

$$i_2 = 4i_1 + \frac{1}{2}\frac{di_1}{dt} \tag{8.10.8}$$

Substituting Eq. (8.10.8) into Eq. (8.10.7) gives

$$4i_1 + \frac{1}{2}\frac{di_1}{dt} + \frac{4}{5}\frac{di_1}{dt} + \frac{1}{10}\frac{d^2i_1}{dt^2} - i_1 = 0$$

$$\frac{d^2i_1}{dt^2} + 13\frac{di_1}{dt} + 30i_1 = 0$$

From this we obtain the characteristic equation as

$$s^2 + 13s + 30 = 0$$

which has roots $s = -3$ and $s = -10$. Hence, the form of the transient response is

$$i_{1n} = Ae^{-3t} + Be^{-10t} \tag{8.10.9}$$

where $A$ and $B$ are constants. The steady-state response is

$$i_{1ss} = i_1(\infty) = \frac{7}{3}\,\text{A} \tag{8.10.10}$$

From Eqs. (8.10.9) and (8.10.10), we obtain the complete response as

$$i_1(t) = \frac{7}{3} + Ae^{-3t} + Be^{-10t} \tag{8.10.11}$$

We finally obtain $A$ and $B$ from the initial values. From Eqs. (8.10.1) and (8.10.11),

$$0 = \frac{7}{3} + A + B \tag{8.10.12}$$

Taking the derivative of Eq. (8.10.11), setting$t = 0$ in the derivative, and enforcing Eq. (8.10.3), we obtain

$$14 = -3A - 10B \tag{8.10.13}$$

From Eqs. (8.10.12) and (8.10.13), $A = -4/3$ and $B = -1$. Thus,

$$i_1(t) = \frac{7}{3} - \frac{4}{3}e^{-3t} - e^{-10t} \tag{8.10.14}$$

We no w obtain $i_2$ from $i_1$. Applying KVL to the left loop in Fig. 8.30(a) gives

$$7 = 4i_1 - i_2 + \frac{1}{2}\frac{di_1}{dt} \quad \Rightarrow \quad i_2 = -7 + 4i_1 + \frac{1}{2}\frac{di_1}{dt}$$

Substituting for $i_1$ in Eq. (8.10.14) gives

$$\begin{aligned} i_2(t) &= -7 + \frac{28}{3} - \frac{16}{3}e^{-3t} - 4e^{-10t} + 2e^{-3t} + 5e^{-10t} \\ &= \frac{7}{3} - \frac{10}{3}e^{-3t} + e^{-10t} \end{aligned} \tag{8.10.15}$$

From Fig. 8.29,

$$v_o(t) = 1[i_1(t) - i_2(t)] \tag{8.10.16}$$

Substituting Eqs. (8.10.14) and (8.10.15) into Eq. (8.10.16) yields

$$v_o(t) = 2(e^{-3t} - e^{-10t}) \tag{8.10.17}$$

Note that $v_o(0) = 0$, as expected from Eq. (8.10.2).

## Practice Problem 8.10

For $t > 0$, obtain $v_o(t)$ in the circuit of Fig. 8.32. (*Hint:* First find $v_1$ and $v_2$.)

**Answer:** $14(e^{-t} - e^{-6t})$ V, $t > 0$.

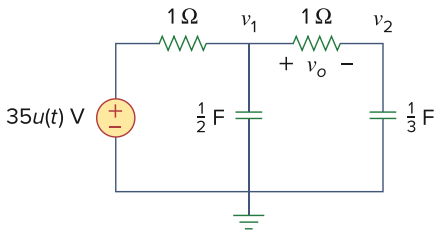



**Figure 8.32**
For Practice Prob. 8.10.

The use of op amps in second-order circuits avoids the use of inductors, which are undesirable in some applications.

## 8.8 Second-Order Op Amp Circuits

An op amp circuit with two storage elements that cannot be combined into a single equivalent element is second-order. Because inductors are bulky and heavy, they are rarely used in practical op amp circuits. For this reason, we will only consider *RC* second-order op amp circuits here. Such circuits find a wide range of applications in devices such as filters and oscillators.

The analysis of a second-order op amp circuit follows the same four steps given and demonstrated in the previous section.

### Example 8.11

In the op amp circuit of Fig. 8.33, find $v_o(t)$ for $t > 0$ when $v_s = 10u(t)$ mV. Let $R_1 = R_2 = 10$ kΩ, $C_1 = 20$ μF, and $C_2 = 100$ μF.

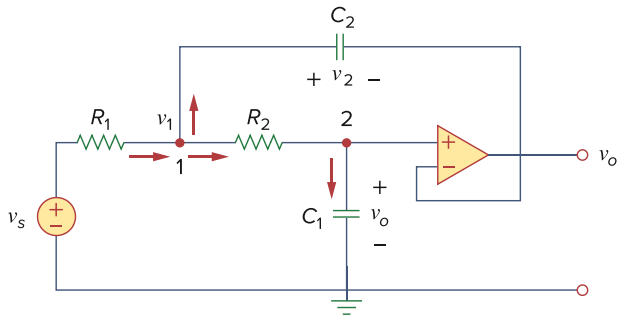



**Figure 8.33**
For Example 8.11.

**Solution:**
Although we could follow the same four steps given in the previous section to solve this problem, we will solve it a little differently. Due to the voltage follower configuration, the voltage across $C_1$ is $v_o$. Applying KCL at node 1,

$$\frac{v_s - v_1}{R_1} = C_2 \frac{dv_2}{dt} + \frac{v_1 - v_o}{R_2} \tag{8.11.1}$$

At node 2, KCL gives

$$\frac{v_1 - v_o}{R_2} = C_1 \frac{dv_o}{dt} \tag{8.11.2}$$

But

$$v_2 = v_1 - v_o \tag{8.11.3}$$

We now try to eliminate $v_1$ and $v_2$ in Eqs. (8.11.1) to (8.11.3). Substituting Eqs. (8.11.2) and (8.11.3) into Eq. (8.11.1) yields

$$\frac{v_s - v_1}{R_1} = C_2 \frac{dv_1}{dt} - C_2 \frac{dv_o}{dt} + C_1 \frac{dv_o}{dt} \tag{8.11.4}$$

From Eq. (8.11.2),

$$v_1 = v_o + R_2 C_1 \frac{dv_o}{dt} \tag{8.11.5}$$

Substituting Eq. (8.11.5) into Eq. (8.11.4), we obtain

$$\frac{v_s}{R_1} = \frac{v_o}{R_1} + \frac{R_2C_1}{R_1}\frac{dv_o}{dt} + C_2\frac{dv_o}{dt} + R_2C_1C_2\frac{d^2v_o}{dt^2} - C_2\frac{dv_o}{dt} + C_1\frac{dv_o}{dt}$$

or

$$\frac{d^2v_o}{dt^2} + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\frac{dv_o}{dt} + \frac{v_o}{R_1R_2C_1C_2} = \frac{v_s}{R_1R_2C_1C_2} \qquad \textbf{(8.11.6)}$$

With the given values of $R_1$, $R_2$, $C_1$, and $C_2$, Eq. (8.11.6) becomes

$$\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + 5v_o = 5v_s \qquad \textbf{(8.11.7)}$$

To obtain the form of the transient response, set $v_s = 0$ in Eq. (8.11.7), which is the same as turning off the source. The characteristic equation is

$$s^2 + 2s + 5 = 0$$

which has complex roots $s_{1,2} = -1 \pm j2$. Hence, the form of the transient response is

$$v_{ot} = e^{-t}(A\cos 2t + B\sin 2t) \qquad \textbf{(8.11.8)}$$

where $A$ and $B$ are unknown constants to be determined.

As $t \to \infty$, the circuit reaches the steady-state condition, and the capacitors can be replaced by open circuits. Since no current flows through $C_1$ and $C_2$ under steady-state conditions and no current can enter the input terminals of the ideal op amp, current does not flow through $R_1$ and $R_2$.

Thus,

$$v_o(\infty) = v_1(\infty) = v_s$$

The steady-state response is then

$$v_{oss} = v_o(\infty) = v_s = 10\text{ mV}, \qquad t > 0 \qquad \textbf{(8.11.9)}$$

The complete response is

$$v_o(t) = v_{ot} + v_{oss} = 10 + e^{-t}(A\cos 2t + B\sin 2t)\text{ mV} \qquad \textbf{(8.11.10)}$$

To determine $A$ and $B$, we need the initial conditions. For $t < 0$, $v_s = 0$, so that

$$v_o(0^-) = v_2(0^-) = 0$$

For $t > 0$, the source is operative. However, due to capacitor voltage continuity,

$$v_o(0^+) = v_2(0^+) = 0 \qquad \textbf{(8.11.11)}$$

From Eq. (8.11.3),

$$v_1(0^+) = v_2(0^+) + v_o(0^+) = 0$$

and, hence, from Eq. (8.11.2),

$$\frac{dv_o(0^+)}{dt} = \frac{v_1 - v_o}{R_2C_1} = 0 \qquad \textbf{(8.11.12)}$$

We now impose Eq. (8.11.11) on the complete response in Eq. (8.11.10) at $t = 0$, for

$$0 = 10 + A \qquad \Rightarrow \qquad A = -10 \qquad \textbf{(8.11.13)}$$

Taking the derivative of Eq. (8.11.10),

$$\frac{dv_o}{dt} = e^{-t}(-A\cos 2t - B\sin 2t - 2A\sin 2t + 2B\cos 2t)$$

Setting $t = 0$ and incorporating Eq. (8.11.12), we obtain

$$0 = -A + 2B \tag{8.11.14}$$

From Eqs. (8.11.13) and (8.11.14), $A = -10$ and $B = -5$. Thus, the step response becomes

$$v_o(t) = 10 - e^{-t}(10\cos 2t + 5\sin 2t)\text{ mV}, \qquad t > 0$$

## Practice Problem 8.11

In the op amp circuit shown in Fig. 8.34, $v_s = 25u(t)$ V, find $v_o(t)$ for $t > 0$. Assume that $R_1 = R_2 = 10\text{ k}\Omega$, $C_1 = 20\ \mu\text{F}$, and $C_2 = 100\ \mu\text{F}$.

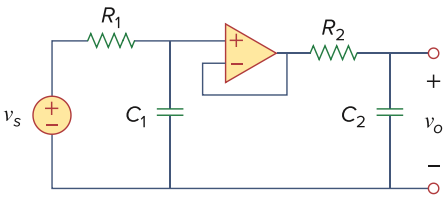



**Figure 8.34**
For Practice Prob. 8.11.

**Answer:** $(25 - 31.25e^{-t} + 6.25e^{-5t})$ V, $t > 0$.

## 8.9 PSpice Analysis of RLC Circuits

*RLC* circuits can be analyzed with great ease using *PSpice*, just like the *RC* or *RL* circuits of Chapter 7. The following two examples will illustrate this. The reader may review Section D.4 in Appendix D on *PSpice* for transient analysis.

## Example 8.12

The input voltage in Fig. 8.35(a) is applied to the circuit in Fig. 8.35(b). Use *PSpice* to plot $v(t)$ for $0 < t < 4$ s.

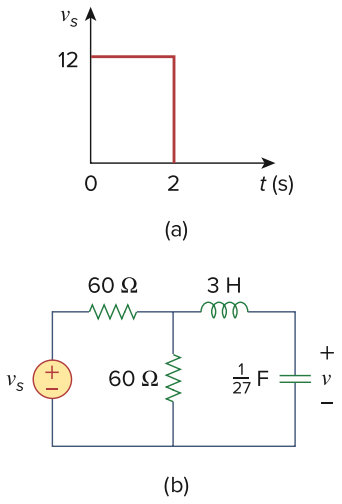



**Figure 8.35**
For Example 8.12.

**Solution:**

1. **Define.** As true with most textbook problems, the problem is clearly defined.
2. **Present.** The input is equal to a single square wave of amplitude 12 V with a period of 2 s. We are asked to plot the output, using *PSpice*.
3. **Alternative.** Since we are required to use *PSpice*, that is the only alternative for a solution. However, we can check it using the technique illustrated in Section 8.5 (a step response for a series *RLC* circuit).
4. **Attempt.** The given circuit is drawn using Schematics as in Fig. 8.36. The pulse is specified using VPWL voltage source, but VPULSE could be used instead. Using the piecewise linear function, we set the attributes of VPWL as T1 = 0, V1 = 0, T2 = 0.001, V2 = 12, and so forth, as shown in Fig. 8.36. Two voltage markers are inserted to plot the input and output voltages. Once the circuit is drawn and the attributes are set, we select **Analysis/Setup/Transient** to open up the *Transient Analysis* dialog box. As a parallel *RLC* circuit, the roots of the characteristic equation are $-1$ and $-9$. Thus, we may set *Final Time* as 4 s (four times the magnitude of the lower root). When the schematic

is saved, we select **Analysis/Simulate** and obtain the plots for the input and output voltages under the *PSpice* A/D window as shown in Fig. 8.37.

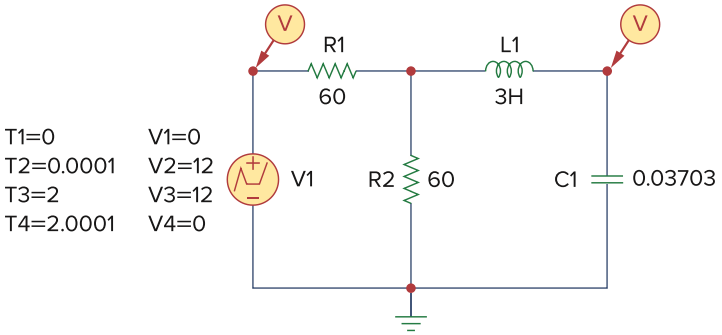



**Figure 8.36**
Schematic for the circuit in Fig. 8.35(b).

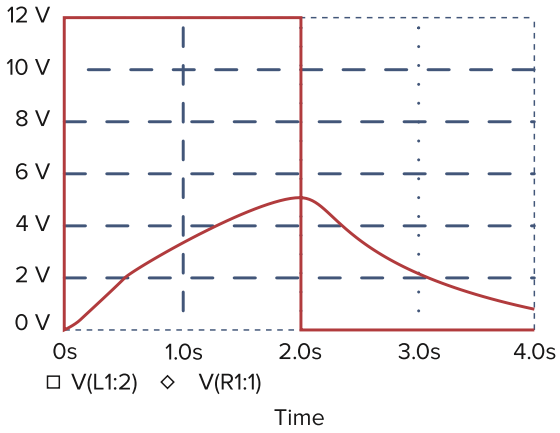



**Figure 8.37**
For Example 8.12: input and output.

Now we check using the technique from Section 8.5. We can start by realizing the Thevenin equivalent for the resistor-source combination is $V_{\text{Th}} = 12/2$ (the open circuit voltage divides equally across both resistors) = 6 V. The equivalent resistance is 30 Ω (60 || 60). Thus, we can now solve for the response using $R = 30\ \Omega$, $L = 3$ H, and $C = (1/27)$ F.

We first need to solve for $\alpha$ and $\omega_0$:

$$\alpha = R/(2L) = 30/6 = 5 \qquad \text{and} \qquad \omega_0 = \frac{1}{\sqrt{3\frac{1}{27}}} = 3$$

Since 5 is greater than 3, we have the overdamped case

$$s_{1,2} = -5 \pm \sqrt{5^2 - 9} = -1,\ -9, \qquad v(0) = 0, \quad v(\infty) = 6\ \text{V}, \qquad i(0) = 0$$

$$i(t) = C\frac{dv(t)}{dt},$$

where

$$v(t) = A_1 e^{-t} + A_2 e^{-9t} + 6$$
$$v(0) = 0 = A_1 + A_2 + 6$$
$$i(0) = 0 = C(-A_1 - 9A_2)$$

which yields $A_1 = -9A_2$. Substituting this into the above, we get $0 = 9A_2 - A_2 + 6$, or $A_2 = 0.75$ and $A_1 = -6.75$.

$$v(t) = (\mathbf{-6.75}e^{-t} + \mathbf{0.75}e^{-9t} + \mathbf{6})\boldsymbol{u}(\boldsymbol{t})\ \boldsymbol{V} \text{ for all } 0 < t < 2 \text{ s}.$$

At $t = 1$ s, $v(1) = -6.75e^{-1} + 0.75e^{-9} + 6 = -2.483 + 0.0001 + 6 = 3.552$ V. At $t = 2$ s, $v(2) = -6.75e^{-2} + 0 + 6 = 5.086$ V.

Note that from $2 < t < 4$ s, $V_{\text{Th}} = 0$, which implies that $v(\infty) = 0$. Therefore, $v(t) = (A_3 e^{-(t-2)} + A_4 e^{-9(t-2)})u(t-2)$ V. At $t = 2$ s, $A_3 + A_4 = 5.086$.

$$i(t) = \frac{(-A_3 e^{-(t-2)} - 9A_4 e^{-9(t-2)})}{27}$$

and

$$i(2) = \frac{(6.75e^{-2} - 6.75e^{-18})}{27} = 33.83 \text{ mA}$$

Therefore, $-A_3 - 9A_4 = 0.9135$.

Combining the two equations, we get $-A_3 - 9(5.086 - A_3) = 0.9135$, which leads to $A_3 = 5.835$ and $A_4 = -0.749$.

$$v(t) = (\mathbf{5.835}e^{-(t-2)} - \mathbf{0.749}e^{-9(t-2)})\, \boldsymbol{u}(t-2) \text{ V}$$

At $t = 3$ s, $v(3) = (2.147 - 0) = 2.147$ V. At $t = 4$ s, $v(4) = 0.7897$ V.

5. **Evaluate.** A check between the values calculated above and the plot shown in Figure 8.37 shows good agreement within the obvious level of accuracy.
6. **Satisfactory?** Yes, we have agreement and the results can be presented as a solution to the problem.

## Practice Problem 8.12

Find $i(t)$ using *PSpice* for $0 < t < 4$ s if the pulse voltage in Fig. 8.35(a) is applied to the circuit in Fig. 8.38.

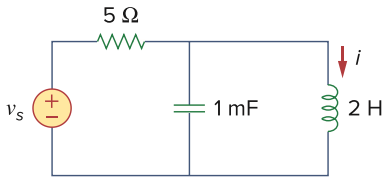



**Figure 8.38**
For Practice Prob. 8.12.

**Answer:** See Fig. 8.39.

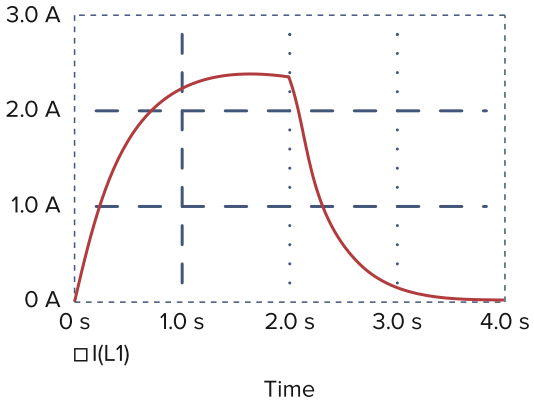



**Figure 8.39**
Plot of $i(t)$ for Practice Prob. 8.12.

## Example 8.13

For the circuit in Fig. 8.40, use *PSpice* to obtain $i(t)$ for $0 < t < 3$ s.

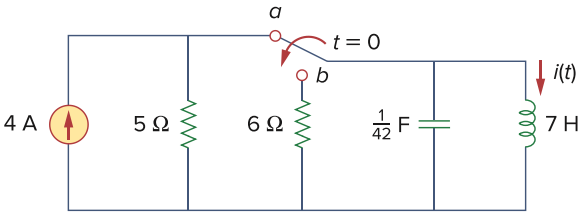



**Figure 8.40**
For Example 8.13.

**Solution:**

When the switch is in position $a$, the 6- Ω resistor is redundant. The schematic for this case is shown in Fig. 8.41(a). To ensure that current

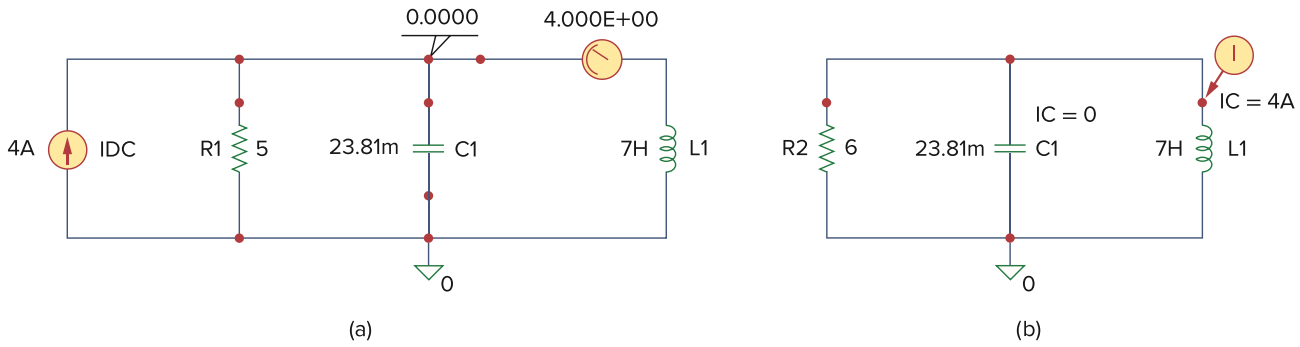



**Figure 8.41**
For Example 8.13: (a) for dc analysis, (b) for transient analysis.

$i(t)$ enters pin 1, the inductor is rotated three times before it is placed in the circuit. The same applies for the capacitor. We insert pseudo -components VIEWPOINT and IPROBE to determine the initial capacitor voltage and initial inductor current. We carry out a dc *PSpice* analysis by selecting **Analysis/Simulate.** As shown in Fig. 8.41(a), we obtain the initial capacitor voltage as 0 V and the initial inductor current $i(0)$ as 4 A from the dc analysis. These initial values will be used in the transient analysis.

When the switch is moved to position *b*, the circuit becomes a source-free parallel *RLC* circuit with the schematic in Fig. 8.41(b). We set the initial condition IC = 0 for the capacitor and IC = 4 A for the inductor. A current marker is inserted at pin 1 of the inductor. We select **Analysis/Setup/Transient** to open up the *Transient Analysis* dialog box and set *Final Time* to 3 s. After saving the schematic, we select **Analysis/Transient**. Figure 8.42 shows the plot of $i(t)$. The plot agrees with $i(t) = 4.8e^{-t} - 0.8e^{-6t}$ A, which is the solution by hand calculation.

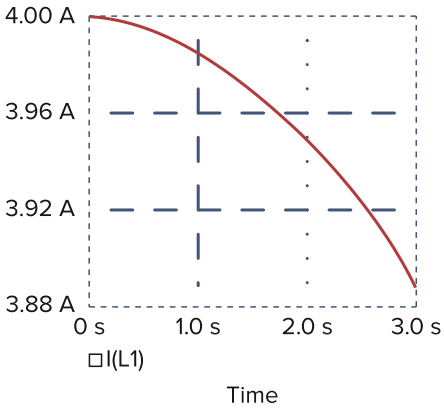



**Figure 8.42**
Plot of $i(t)$ for Example 8.13.

## Practice Problem 8.13

Refer to the circuit in Fig. 8.21 (see Practice Prob. 8.7). Use *PSpice* to obtain $v(t)$ for $0 < t < 2$.

**Answer:** See Fig. 8.43.

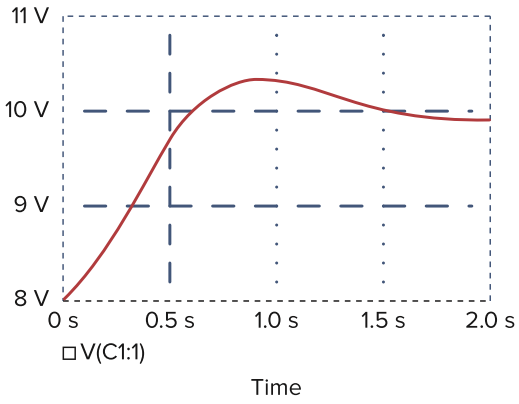



**Figure 8.43**
Plot of $v(t)$ for Practice Prob. 8.13.

## 8.10 †Duality

The concept of duality is a time-saving, effort-effective measure of solving circuit problems. Consider the similarity between Eqs. (8.4) and (8.29). The two equations are the same, except that we must interchange the following quantities: (1) voltage and current, (2) resistance and conductance, (3) capacitance and inductance. Thus, it sometimes occurs in circuit analysis that two different circuits have the same equations and solutions, except that the roles of certain complementary elements are interchanged. This interchangeability is known as the principle of *duality*.

The duality principle asserts a parallelism between pairs of characterizing equations and theorems of electric circuits.

Dual pairs are shown in Table 8.1. Note that power does not appear in Table 8.1, because power has no dual. The reason for this is the principle of linearity; since power is not linear, duality does not apply. Also notice from Table 8.1 that the principle of duality extends to circuit elements, configurations, and theorems.

Two circuits that are described by equations of the same form, but in which the variables are interchanged, are said to be dual to each other.

Two circuits are said to be duals of one another if they are described by the same characterizing equations with dual quantities interchanged.

**TABLE 8.1**

Dual pairs.

| | |
|---|---|
| Resistance $R$ | Conductance $G$ |
| Inductance $L$ | Capacitance $C$ |
| Voltage $v$ | Current $i$ |
| Voltage source | Current source |
| Node | Mesh |
| Series path | Parallel path |
| Open circuit | Short circuit |
| KVL | KCL |
| Thevenin | Norton |

Even when the principle of linearity applies, a circuit element or variable may not have a dual. For example, mutual inductance (to be covered in Chapter 13) has no dual.

The usefulness of the duality principle is self-evident. Once we know the solution to one circuit, we automatically have the solution for the dual circuit. It is obvious that the circuits in Figs. 8.8 and 8.13 are dual. Consequently, the result in Eq. (8.32) is the dual of that in Eq. (8.11). We must keep in mind that the method described here for finding a dual is limited to planar circuits. Finding a dual for a nonplanar circuit is beyond the scope of this textbook because nonplanar circuits cannot be described by a system of mesh equations.

To find the dual of a given circuit, we do not need to write down the mesh or node equations. We can use a graphical technique. Given a planar circuit, we construct the dual circuit by taking the following three steps:

1. Place a node at the center of each mesh of the given circuit. Place the reference node (the ground) of the dual circuit outside the given circuit.
2. Draw lines between the nodes such that each line crosses an element. Replace that element by its dual (see Table 8.1).
3. To determine the polarity of voltage sources and direction of current sources, follow this rule: A voltage source that produces a positive (clockwise) mesh current has as its dual a current source whose reference direction is from the ground to the nonreference node.

In case of doubt, one may verify the dual circuit by writing the nodal or mesh equations. The mesh (or nodal) equations of the original circuit are similar to the nodal (or mesh) equations of the dual circuit. The duality principle is illustrated with the following two examples.

## Example 8.14

Construct the dual of the circuit in Fig. 8.44.

**Solution:**
As shown in Fig. 8.45(a), we first locate nodes 1 and 2 in the two meshes and also the ground node 0 for the dual circuit. We draw a line between one node and another crossing an element. We replace the line joining the nodes by the duals of the elements which it crosses. For example, a line between nodes 1 and 2 crosses a 2-H inductor, and we place a 2-F capacitor (an inductor's dual) on the line. A line between nodes 1 and 0 crossing the 6-V voltage source will contain a 6-A current source. By drawing lines crossing all the elements, we construct the dual circuit on the given circuit as in Fig. 8.45(a). The dual circuit is redrawn in Fig. 8.45(b) for clarity.

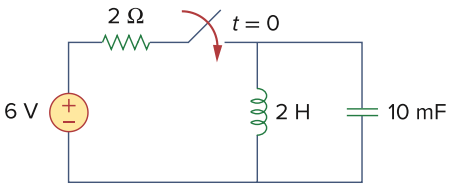
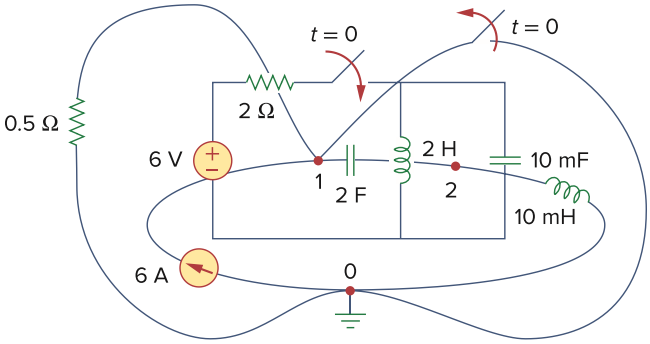
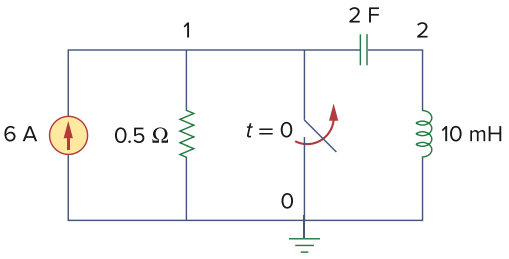



**Figure 8.44**
For Example 8.14.




(a)




(b)

**Figure 8.45**
(a) Construction of the dual circuit of Fig. 8.44, (b) dual circuit redrawn.

## Practice Problem 8.14

Draw the dual circuit of the one in Fig. 8.46.

**Answer:** See Fig. 8.47.

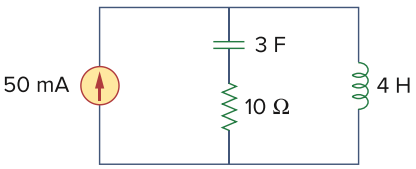



**Figure 8.46**
For Practice Prob. 8.14.

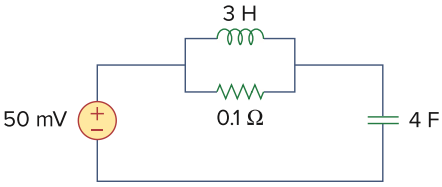



**Figure 8.47**
Dual of the circuit in Fig. 8.46.

## Example 8.15

Obtain the dual of the circuit in Fig. 8.48.

**Solution:**
The dual circuit is constructed on the original circuit as in Fig. 8.49(a). We first locate nodes 1 to 3 and the reference node 0. Joining nodes 1 and 2, we cross the 2-F capacitor, which is replaced by a 2-H inductor.

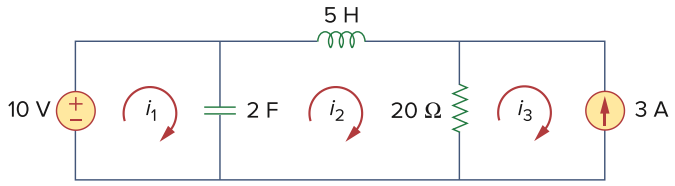



**Figure 8.48**
For Example 8.15.

Joining nodes 2 and 3, we cross the 20- Ω resistor, which is replaced by a $\frac{1}{20}$-Ω resistor. We keep doing this until all the elements are crossed. The result is in Fig. 8.49(a). The dual circuit is redrawn in Fig. 8.49(b).

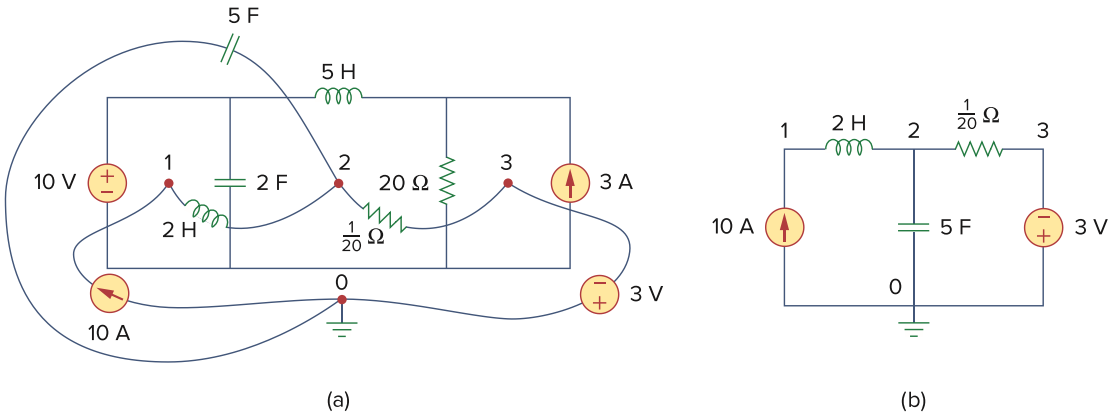



**Figure 8.49**
For Example 8.15: (a) construction of the dual circuit of Fig. 8.48, (b) dual circuit redrawn.

To verify the polarity of the voltage source and the direction of the current source, we may apply mesh currents $i_1$, $i_2$, and $i_3$ (all in the clockwise direction) in the original circuit in Fig. 8.48.The 10-V voltage source produces positive mesh current $i_1$, so that its dual is a 10-A current source directed from 0 to 1. Also, $i_3 = -3$ A in Fig. 8.48 has as its dual $v_3 = -3$ V in Fig. 8.49(b).

## Practice Problem 8.15

For the circuit in Fig. 8.50, obtain the dual circuit.

**Answer:** See Fig. 8.51.

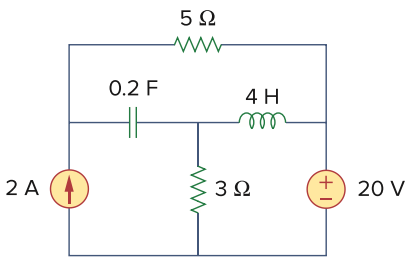



**Figure 8.50**
For Practice Prob. 8.15.

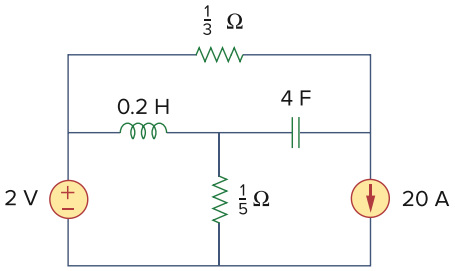



**Figure 8.51**
Dual of the circuit in Fig. 8.50.

## 8.11 †Applications

Practical applications of *RLC* circuits are found in control and communications circuits such as ringing circuits, peaking circuits, resonant circuits, smoothing circuits, and filters. Most of these circuits cannot be covered until we treat ac sources. For now, we will limit ourselves to two simple applications: automobile ignition and smoothing circuits.

### 8.11.1 Automobile Ignition System

In Section 7.9.4, we considered the automobile ignition system as a charging system. That was only a part of the system. Here, we consider another part—the voltage generating system. The system is modeled by the circuit shown in Fig. 8.52. The 12-V source is due to the battery and alternator. The 4-Ω resistor represents the resistance of the wiring. The ignition coil is modeled by the 8-mH inductor . The 1-$\mu$F capacitor (known as the *condenser* to automechanics) is in parallel with the switch (known as the *breaking points* or *electronic ignition*). In the follo wing example, we determine how the *RLC* circuit in Fig. 8.52 is used in generating high voltage.

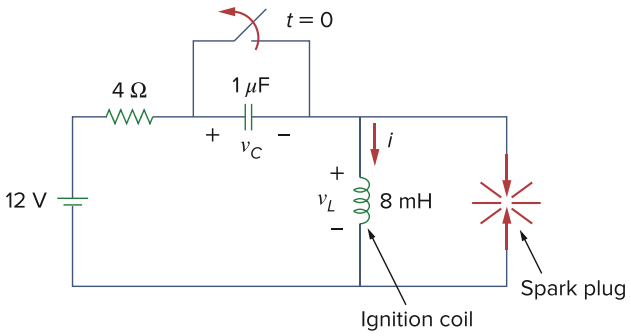



**Figure 8.52**
Automobile ignition circuit.

**Example 8.16**

Assuming that the switch in Fig. 8.52 is closed prior to $t = 0^-$, find the inductor voltage $v_L$ for $t > 0$.

**Solution:**
If the switch is closed prior to $t = 0^-$ and the circuit is in steady state, then

$$i(0^-) = \frac{12}{4} = 3 \text{ A}, \qquad v_C(0^-) = 0$$

At $t = 0^+$, the switch is opened. The continuity conditions require that

$$i(0^+) = 3 \text{ A}, \qquad v_C(0^+) = 0 \tag{8.16.1}$$

We obtain $di(0^+)/dt$ from $v_L(0^+)$. Applying KVL to the mesh at $t = 0^+$ yields

$$-12 + 4i(0^+) + v_L(0^+) + v_C(0^+) = 0$$
$$-12 + 4 \times 3 + v_L(0^+) + 0 = 0 \quad \Rightarrow \quad v_L(0^+) = 0$$

Hence,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = 0 \tag{8.16.2}$$

As $t \to \infty$, the system reaches steady state, so that the capacitor acts like an open circuit. Then

$$i(\infty) = 0 \tag{8.16.3}$$

If we apply KVL to the mesh for $t > 0$, we obtain

$$12 = Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + v_C(0)$$

Taking the derivative of each term yields

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{8.16.4}$$

We obtain the form of the transient response by following the procedure in Section 8.3. Substituting $R = 4\ \Omega$, $L = 8$ mH, and $C = 1\ \mu$F, we get

$$\alpha = \frac{R}{2L} = 250, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 1.118 \times 10^4$$

Since $\alpha < \omega_0$, the response is underdamped. The damped natural frequency is

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \simeq \omega_0 = 1.118 \times 10^4$$

The form of the transient response is

$$i_t(t) = e^{-\alpha}\,(A\cos\omega_d t + B\sin\omega_d t) \tag{8.16.5}$$

where $A$ and $B$ are constants. The steady-state response is

$$i_{ss}(t) = i(\infty) = 0 \tag{8.16.6}$$

so that the complete response is

$$i(t) = i_t(t) + i_{ss}(t) = e^{-250t}(A\cos 11{,}180t + B\sin 11{,}180t) \tag{8.16.7}$$

We now determine $A$ and $B$.

$$i(0) = 3 = A + 0 \qquad \Rightarrow \qquad A = 3$$

Taking the derivative of Eq. (8.16.7),

$$\frac{di}{dt} = -250e^{-250t}(A\cos 11{,}180t + B\sin 11{,}180t) + e^{-250t}(-11{,}180A\sin 11{,}180t + 11{,}180B\cos 11{,}180t)$$

Setting $t = 0$ and incorporating Eq. (8.16.2),

$$0 = -250A + 11{,}180B \qquad \Rightarrow \qquad B = 0.0671$$

Thus,

$$i(t) = e^{-250t}(3\cos 11{,}180t + 0.0671\sin 11{,}180t) \tag{8.16.8}$$

The voltage across the inductor is then

$$v_L(t) = L\frac{di}{dt} = -268e^{-250t}\sin 11{,}180t \tag{8.16.9}$$

This has a maximum value when sine is unity, that is, at $11{,}180t_0 = \pi/2$ or $t_0 = 140.5\ \mu$s. At time $= t_0$, the inductor voltage reaches its peak, which is

$$v_L(t_0) = -268e^{-250t_0} = -259\ \text{V} \qquad \textbf{(8.16.10)}$$

Although this is far less than the voltage range of 6000 to 10,000 V required to fire the spark plug in a typical automobile, a device known as a *transformer* (to be discussed in Chapter 13) is used to step up the inductor voltage to the required level.

## Practice Problem 8.16

In Fig. 8.52, find the capacitor voltage $v_C$ for $t > 0$.

**Answer:** $12 - 12e^{-250t}\cos 11{,}180t + 267.7e^{-250t}\sin 11{,}180t\,\text{V}$.

### 8.11.2 Smoothing Circuits

In a typical digital communication system, the signal to be transmitted is first sampled. Sampling refers to the procedure of selecting samples of a signal for processing, as opposed to processing the entire signal. Each sample is con verted into a binary number represented by a series of pulses. The pulses are transmitted by a transmission line such as a coaxial cable, twisted pair, or optical fiber. At the receiving end, the signal is applied to a digital-to-analog (D/A) con verter whose output is a "staircase" function, that is, constant at each time interv al. In order to recover the transmitted analog signal, the output is smoothed by letting it pass through a "smoothing" circuit, as illustrated in Fig. 8.53. An *RLC* circuit may be used as the smoothing circuit.

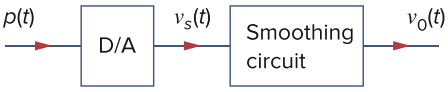
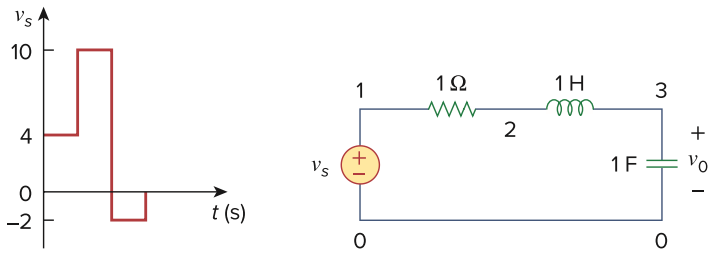



**Figure 8.53**
A series of pulses is applied to the digital-to-analog (D/A) converter, whose output is applied to the smoothing circuit.

## Example 8.17

The output of a D/A converter is shown in Fig. 8.54(a). If the *RLC* circuit in Fig. 8.54(b) is used as the smoothing circuit, determine the output voltage $v_o(t)$.




(a) (b)

**Figure 8.54**
For Example 8.17: (a) output of a D/A converter, (b) an *RLC* smoothing circuit.

**Solution:**
This problem is best solved using *PSpice*. The schematic is shown in Fig. 8.55(a). The pulse in Fig. 8.54(a) is specified using the piecewise

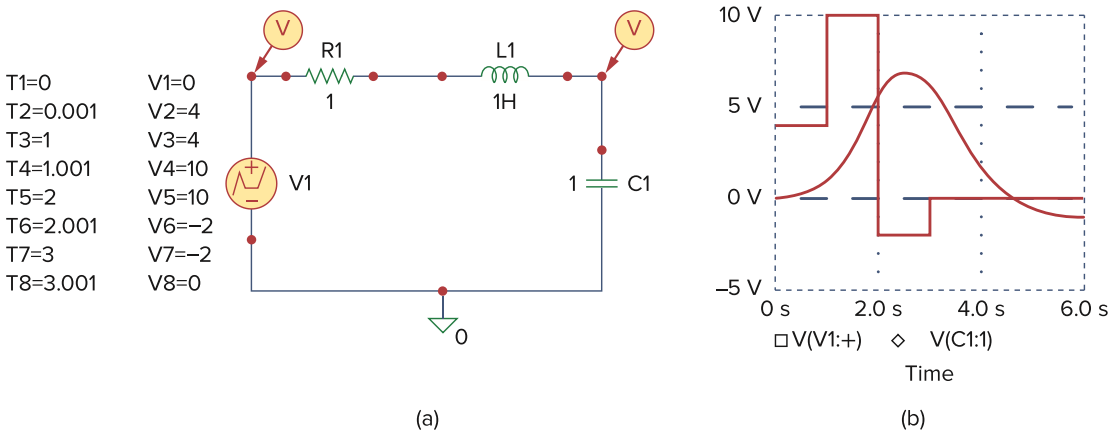



**Figure 8.55**
For Example 8.17: (a) schematic, (b) input and output voltages.

linear function. The attributes of V1 are set as T1 = 0, V1 = 0, T2 = 0.001, V2 = 4, T3 = 1, V3 = 4, and so on. To be able to plot both input and output voltages, we insert two voltage markers as shown. We select **Analysis/Setup/Transient** to open up the *Transient Analysis* dialog box and set *Final Time* as 6 s. Once the schematic is saved, we select **Analysis/Simulate** to run and obtain the plots shown in Fig. 8.55(b).

## Practice Problem 8.17

Rework Example 8.17 if the output of the D/A converter is as shown in Fig. 8.56.

**Answer:** See Fig. 8.57.

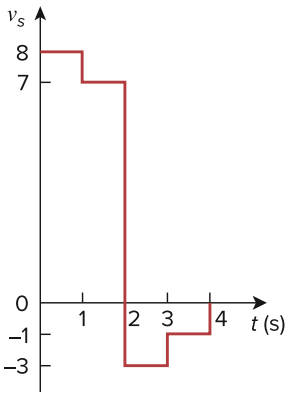



**Figure 8.56**
For Practice Prob. 8.17.

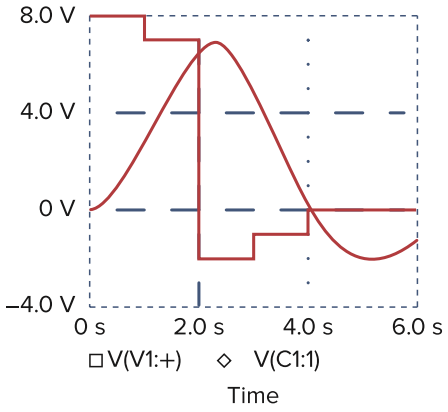



**Figure 8.57**
Result of Practice Prob. 8.17.

## 8.12 Summary

1. The determination of the initial values $x(0)$ and $dx(0)/dt$ and final value $x(\infty)$ is crucial to analyzing second-order circuits.
2. The *RLC* circuit is second-order because it is described by a second-order differential equation. Its characteristic equation is

$s^2 + 2\alpha s + \omega_0^2 = 0$, where $\alpha$ is the neper frequenc y and $\omega_0$ is the undamped natural frequency. For a series circuit, $\alpha = R/2L$, for a parallel circuit $\alpha = 1/2RC$, and for both cases $\omega_0 = 1/\sqrt{LC}$.

3. If there are no independent sources in the circuit after switching (or sudden change), we regard the circuit as source-free. The complete solution is the natural response.
4. The natural response of an *RLC* circuit is o verdamped, under - damped, or critically damped, depending on the roots of the characteristic equation. The response is critically damped when the roots are equal ($s_1 = s_2$ or $\alpha = \omega_0$), overdamped when the roots are real and unequal ($s_1 \neq s_2$ or $\alpha > \omega_0$), or underdamped when the roots are complex conjugate ($s_1 = s_2^*$ or $\alpha < \omega_0$).
5. If independent sources are present in the circuit after switching, the complete response is the sum of the transient response and the steady-state response.
6. *PSpice* is used to analyze *RLC* circuits in the same w ay as for *RC* or *RL* circuits.
7. Two circuits are dual if the mesh equations that describe one circuit have the same form as the nodal equations that describe the otherThe analysis of one circuit gives the analysis of its dual circuit.
8. The automobile ignition circuit and the smoothing circuit are typical applications of the material covered in this chapter.

## Review Questions

**8.1** For the circuit in Fig. 8.58, the capacitor voltage at $t = 0^-$ (just before the switch is closed) is:

(a) 0 V (b) 4 V (c) 8 V (d) 12 V

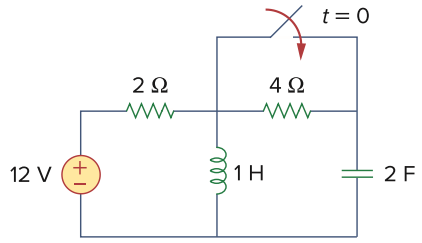



**Figure 8.58**
For Review Questions 8.1 and 8.2.

**8.2** For the circuit in Fig. 8.58, the initial inductor current (at $t = 0$) is:

(a) 0 A (b) 2 A (c) 6 A (d) 12 A

**8.3** When a step input is applied to a second-order circuit, the final values of the circuit variables are found by:

(a) Replacing capacitors with closed circuits and inductors with open circuits.
(b) Replacing capacitors with open circuits and inductors with closed circuits.
(c) Doing neither of the above.

**8.4** If the roots of the characteristic equation of an *RLC* circuit are $-2$ and $-3$, the response is:

(a) $(A \cos 2t + B \sin 2t)e^{-3t}$
(b) $(A + 2Bt)e^{-3t}$
(c) $Ae^{-2t} + Bte^{-3t}$
(d) $Ae^{-2t} + Be^{-3t}$

where $A$ and $B$ are constants.

**8.5** In a series *RLC* circuit, setting $R = 0$ will produce:

(a) an overdamped response
(b) a critically damped response
(c) an underdamped response
(d) an undamped response
(e) none of the above

**8.6** A parallel *RLC* circuit has $L = 2$ H and $C = 0.25$ F. The value of $R$ that will produce a unity neper frequency is:

(a) 0.5 Ω (b) 1 Ω (c) 2 Ω (d) 4 Ω

**8.7** Refer to the series *RLC* circuit in Fig. 8.59. What kind of response will it produce?

(a) overdamped
(b) underdamped
(c) critically damped
(d) none of the above

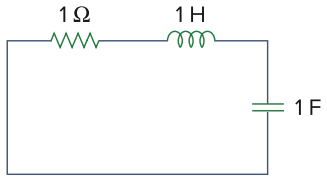



**Figure 8.59**
For Review Question 8.7.

**8.8** Consider the parallel *RLC* circuit in Fig. 8.60. What type of response will it produce?

(a) overdamped
(b) underdamped
(c) critically damped
(d) none of the above

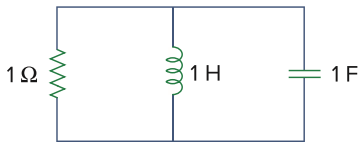



**Figure 8.60**
For Review Question 8.8.

**8.9** Match the circuits in Fig. 8.61 with the following items:

(i) first-order circuit
(ii) second-order series circuit
(iii) second-order parallel circuit
(iv) none of the above

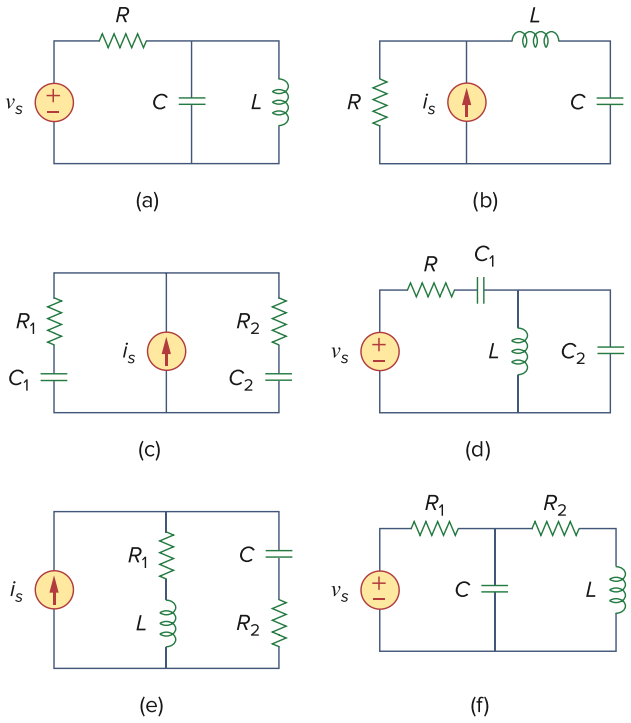



**Figure 8.61**
For Review Question 8.9.

**8.10** In an electric circuit, the dual of resistance is:

(a) conductance (b) inductance
(c) capacitance (d) open circuit
(e) short circuit

*Answers: 8.1a, 8.2c, 8.3b, 8.4d, 8.5d, 8.6c, 8.7b, 8.8b, 8.9 (i)-c, (ii)-b, e, (iii)-a, (iv)-d, f, 8.10a.*

# Problems

## Section 8.2 Finding Initial and Final Values

**8.1** For the circuit in Fig. 8.62, find:

(a) $i(0^+)$ and $v(0^+)$,
(b) $di(0^+)/dt$ and $dv(0^+)/dt$,
(c) $i(\infty)$ and $v(\infty)$.

**8.2** Using Fig. 8.63, design a problem to help other students better understand finding initial and final values.

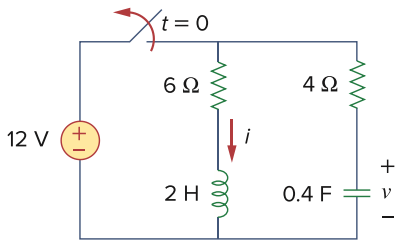



**Figure 8.62**
For Prob. 8.1.

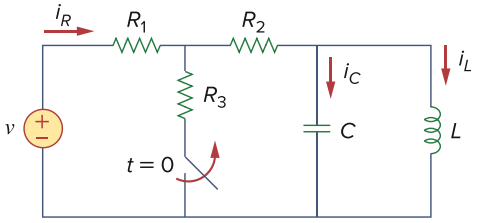



**Figure 8.63**
For Prob. 8.2.

**8.3** Refer to the circuit shown in Fig. 8.64. Calculate:

(a) $i_L(0^+)$, $v_C(0^+)$, and $v_R(0^+)$,
(b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, and $dv_R(0^+)/dt$,
(c) $i_L(\infty)$, $v_C(\infty)$, and $v_R(\infty)$.

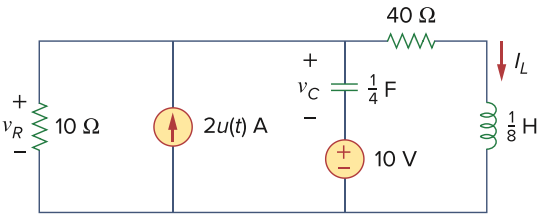



**Figure 8.64**
For Prob. 8.3.

**8.4** In the circuit of Fig. 8.65, find:

(a) $v(0^+)$ and $i(0^+)$,
(b) $dv(0^+)/dt$ and $di(0^+)/dt$,
(c) $v(\infty)$ and $i(\infty)$.

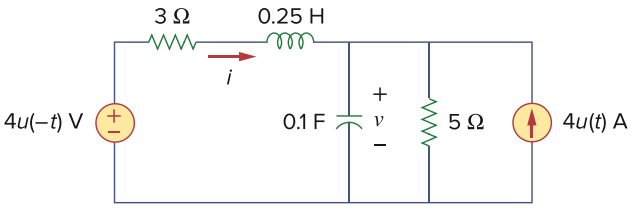



**Figure 8.65**
For Prob. 8.4.

**8.5** Refer to the circuit in Fig. 8.66. Determine:

(a) $i(0^+)$ and $v(0^+)$,
(b) $di/(0^+)dt$ and $dv(0^+)/dt$,
(c) $i(\infty)$ and $v(\infty)$.

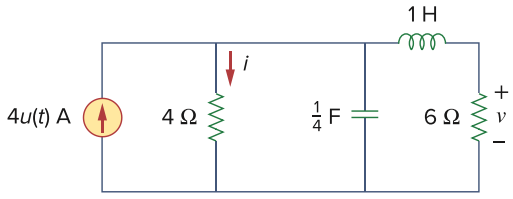



**Figure 8.66**
For Prob. 8.5.

**8.6** In the circuit of Fig. 8.67, find:

(a) $v_R(0^+)$ and $v_L(0^+)$,
(b) $dv_R(0^+)/dt$ and $dv_L(0^+)/dt$,
(c) $v_R(\infty)$ and $v_L(\infty)$.

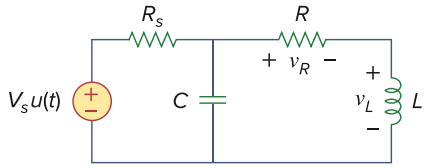



**Figure 8.67**
For Prob. 8.6.

## Section 8.3 Source-Free Series *RLC* Circuit

**8.7** A series *RLC* circuit has $R = 20$ kΩ, $L = 0.2$ mH, and $C = 5$ μF. What type of damping is exhibited by the circuit?

**8.8** Design a problem to help other students better understand source-free *RLC* circuits.

**8.9** The current in an *RLC* circuit is described by

$$\frac{d^2i}{dt^2} + 10\frac{di}{dt} + 25i = 0$$

If $i(0) = 10$ A and $di(0)/dt = 0$, find $i(t)$ for $t > 0$.

**8.10** The differential equation that describes the current in an *RLC* network is

$$3\frac{d^2i}{dt^2} + 15\frac{di}{dt} + 12i = 0$$

Given that $i(0) = 0$, $di(0)/dt = 6$ mA/s, obtain $i(t)$.

**8.11** The natural response of an *RLC* circuit is described by the differential equation

$$\frac{d^2v}{dt^2} + 2\frac{dv}{dt} + v = 0$$

for which the initial conditions are $v(0) = 10$ V and $dv(0)/dt = 0$. Solve for $v(t)$.

**8.12** If $R = 50$ Ω, $L = 1.5$ H, what value of $C$ will make an *RLC* series circuit:

(a) overdamped,
(b) critically damped,
(c) underdamped?

**8.13** For the circuit in Fig. 8.68, calculate the value of $R$ needed to have a critically damped response.

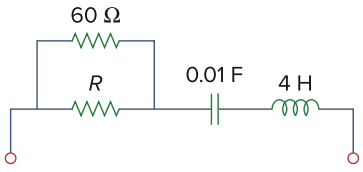



**Figure 8.68**
For Prob. 8.13.

**8.14** The switch in Fig. 8.69 moves from position *A* to position *B* at $t = 0$ (please note that the switch must

connect to point $B$ before it breaks the connection at $A$, a make-before-break switch). Let $v(0) = 0$, find $v(t)$ for $t > 0$.

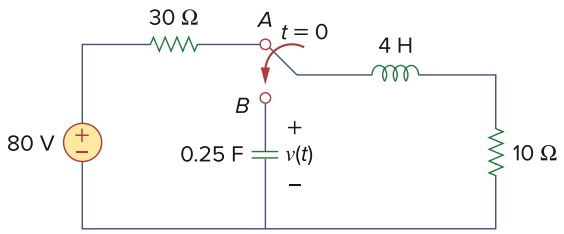



**Figure 8.69**
For Prob. 8.14.

**8.15** The responses of a series *RLC* circuit are

$$v_C(t) = 30 - 10e^{-20t} + 30e^{-10t} \text{ V}$$
$$i_L(t) = 40e^{-20t} - 60e^{-10t} \text{ mA}$$

where $v_C$ and $i_L$ are the capacitor voltage and inductor current, respectively. Determine the values of $R$, $L$, and $C$.

**8.16** Find $i(t)$ for $t > 0$ in the circuit of Fig. 8.70.

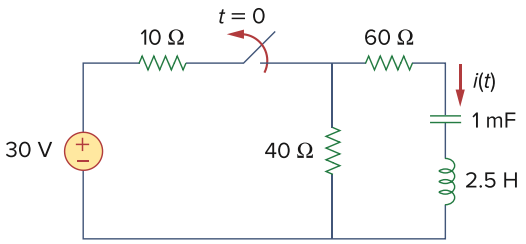



**Figure 8.70**
For Prob. 8.16.

**8.17** In the circuit of Fig. 8.71, the switch instantaneously moves from position $A$ to $B$ at $t = 0$. Find $v(t)$ for all $t \geq 0$.

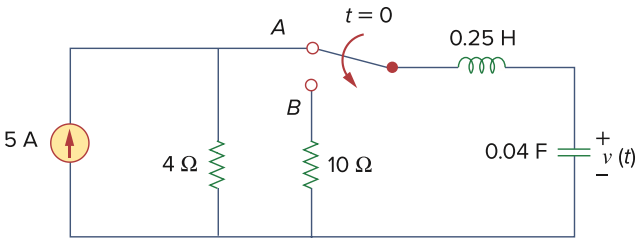



**Figure 8.71**
For Prob. 8.17.

**8.18** Find the voltage across the capacitor as a function of time for $t > 0$ for the circuit in Fig. 8.72. Assume steady-state conditions exist at $t = 0^-$.

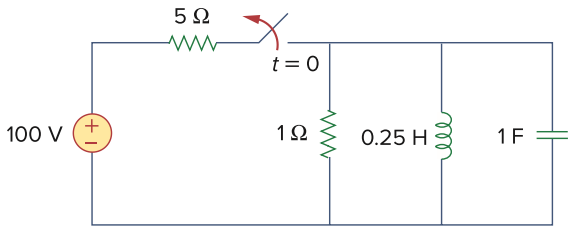



**Figure 8.72**
For Prob. 8.18.

**8.19** Obtain $v(t)$ for $t > 0$ in the circuit of Fig. 8.73.

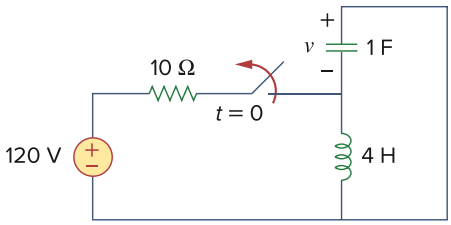



**Figure 8.73**
For Prob. 8.19.

**8.20** The switch in the circuit of Fig. 8.74 has been closed for a long time but is opened at $t = 0$. Determine $i(t)$ for $t > 0$.

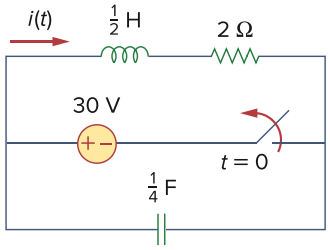



**Figure 8.74**
For Prob. 8.20.

*8.21 Calculate $v(t)$ for $t > 0$ in the circuit of Fig. 8.75.

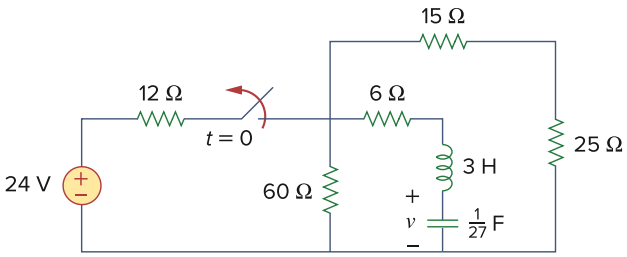



**Figure 8.75**
For Prob. 8.21.

* An asterisk indicates a challenging problem.

## Section 8.4 Source-Free Parallel *RLC* Circuit

**8.22** Assuming $R = 2\ \text{k}\Omega$, design a parallel *RLC* circuit that has the characteristic equation

$$s^2 + 100s + 10^6 = 0.$$

**8.23** For the network in Fig. 8.76, what value of $C$ is needed to make the response underdamped with unity neper frequency ($\alpha = 1$)?

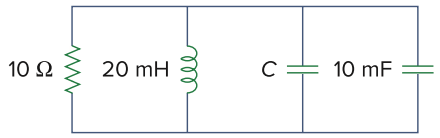



**Figure 8.76**
For Prob. 8.23.

**8.24** The switch in Fig. 8.77 moves from position A to position $B$ at $t = 0$ (please note that the switch must connect to point $B$ before it breaks the connection at $A$, a make-before-break switch). Determine $i(t)$ for $t > 0$.

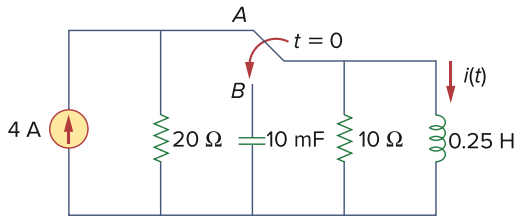



**Figure 8.77**
For Prob. 8.24.

**8.25** Using Fig. 8.78, design a problem to help other students better understand source-free *RLC* circuits.

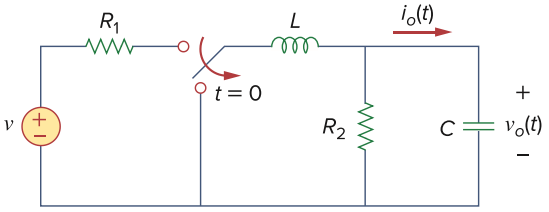



**Figure 8.78**
For Prob. 8.25.

## Section 8.5 Step Response of a Series *RLC* Circuit

**8.26** The step response of an *RLC* circuit is given by

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} + 5i = 10$$

Given that $i(0) = 2$ and $di(0)/dt = 4$, solve for $i(t)$.

**8.27** A branch voltage in an *RLC* circuit is described by

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 8v = 24$$

If the initial conditions are $v(0) = 0 = dv(0)/dt$, find $v(t)$.

**8.28** A series *RLC* circuit is described by

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 10$$

Find the response when $L = 0.5$ H, $R = 4\ \Omega$, and $C = 0.2$ F. Let $i(0) = 1$, $di(0)/dt = 0$.

**8.29** Solve the following differential equations subject to the specified initial conditions

(a) $d^2v/dt^2 + 4v = 12$, $v(0) = 0$, $dv(0)/dt = 2$
(b) $d^2i/dt^2 + 5\,di/dt + 4i = 8$, $i(0) = -1$, $di(0)/dt = 0$
(c) $d^2v/dt^2 + 2\,dv/dt + v = 3$, $v(0) = 5$, $dv(0)/dt = 1$
(d) $d^2i/dt^2 + 2\,di/dt + 5i = 10$, $i(0) = 4$, $di(0)/dt = -2$

**8.30** The step responses of a series *RLC* circuit are

$$v_C = 40 - 10e^{-2000t} - 10e^{-4000t}\ \text{V}, \qquad t > 0$$
$$i_L(t) = 3e^{-2000t} + 6e^{-4000t}\ \text{mA}, \qquad t > 0$$

(a) Find $C$. (b) Determine what type of damping is exhibited by the circuit.

**8.31** Consider the circuit in Fig. 8.79. Find $v_L(0^+)$ and $v_C(0^+)$.

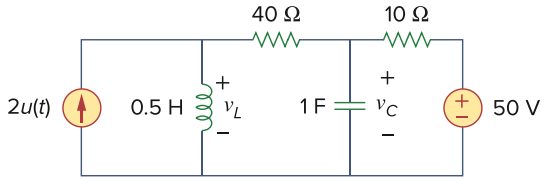



**Figure 8.79**
For Prob. 8.31.

**8.32** For the circuit in Fig. 8.80, find $v(t)$ for $t > 0$.

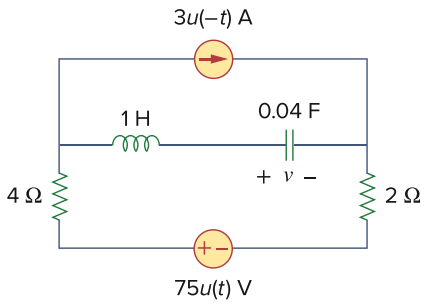



**Figure 8.80**
For Prob. 8.32.

**8.33** Find $v(t)$ for $t > 0$ in the circuit of Fig. 8.81.

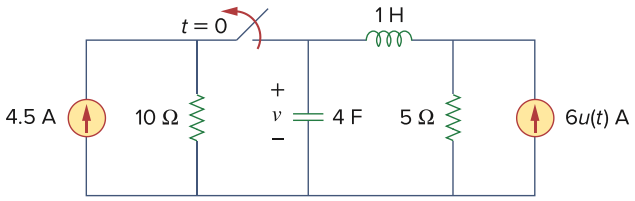



**Figure 8.81**
For Prob. 8.33.

**8.34** Calculate $i(t)$ for $t > 0$ in the circuit of Fig. 8.82.

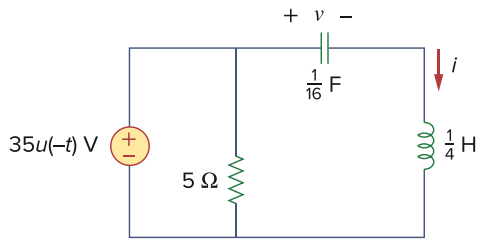



**Figure 8.82**
For Prob. 8.34.

**8.35** Using Fig. 8.83, design a problem to help other students better understand the step response of series *RLC* circuits.

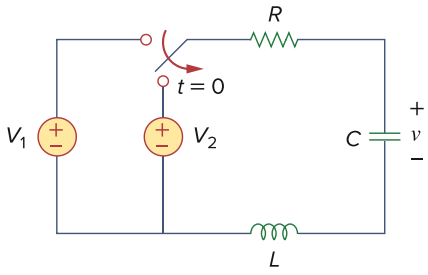



**Figure 8.83**
For Prob. 8.35.

**8.36** Obtain $v(t)$ and $i(t)$ for $t > 0$ in the circuit of Fig. 8.84.

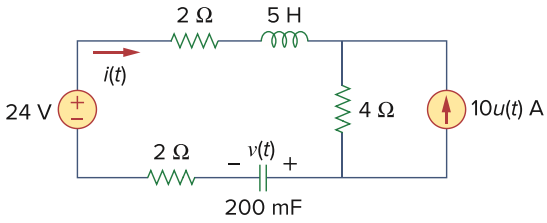



**Figure 8.84**
For Prob. 8.36.

***8.37** For the network in Fig. 8.85, solve for $i(t)$ for $t > 0$.

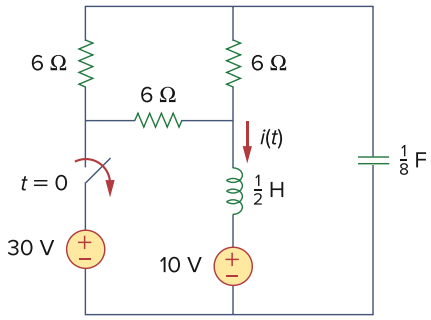



**Figure 8.85**
For Prob. 8.37.

**8.38** Refer to the circuit in Fig. 8.86. Calculate $i(t)$ for $t > 0$.

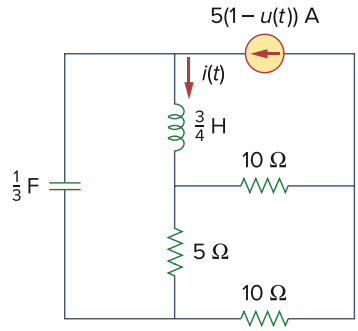



**Figure 8.86**
For Prob. 8.38.

**8.39** Determine $v(t)$ for $t > 0$ in the circuit of Fig. 8.87.

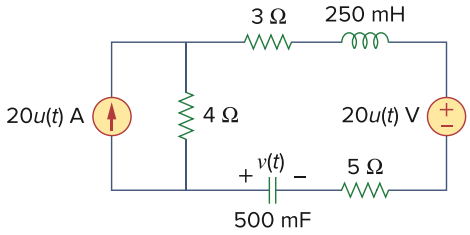



**Figure 8.87**
For Prob. 8.39.

**8.40** The switch in the circuit of Fig. 8.88 is moved from position $a$ to $b$ at $t = 0$. Assume that the voltage across the capacitor is equal to zero at $t = 0$ and that the switch is a make before break switch. Determine $i(t)$ for all $t > 0$.

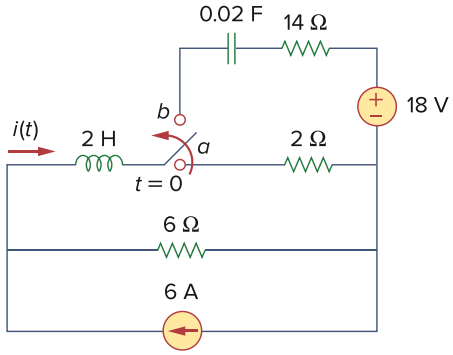



**Figure 8.88**
For Prob. 8.40.

*8.41 For the network in Fig. 8.89, find $i(t)$ for $t > 0$.

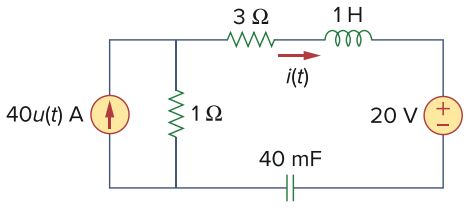



**Figure 8.89**
For Prob. 8.41.

*8.42 Given the network in Fig. 8.90, find $v(t)$ for $t > 0$.

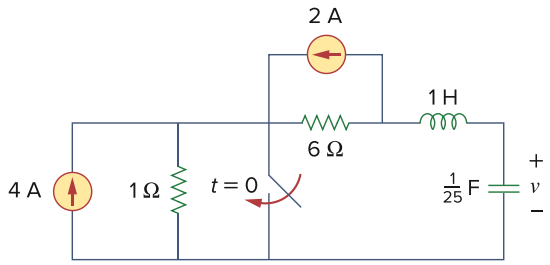



**Figure 8.90**
For Prob. 8.42.

8.43 The switch in Fig. 8.91 is opened at $t = 0$ after the circuit has reached steady state. Choose $R$ and $C$ such that $\alpha = 8$ Np/s and $\omega_d = 30$ rad/s.

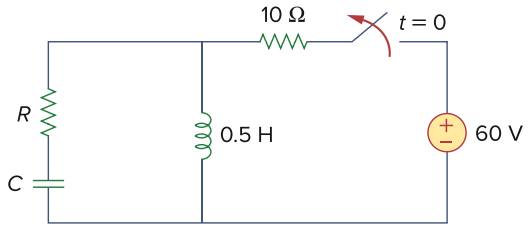



**Figure 8.91**
For Prob. 8.43.

8.44 A series *RLC* circuit has the following parameters: $R = 1$ kΩ, $L = 1$ H, and $C = 10$ nF. What type of damping does this circuit exhibit?

## Section 8.6 Step Response of a Parallel *RLC* Circuit

8.45 In the circuit of Fig. 8.92, find $v(t)$ and $i(t)$ for $t > 0$.

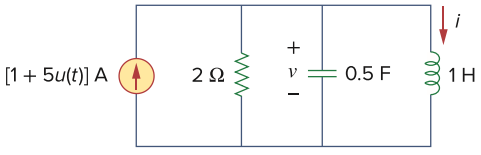



**Figure 8.92**
For Prob. 8.45.

8.46 Using Fig. 8.93, design a problem to help other students better understand the step response of a parallel *RLC* circuit.

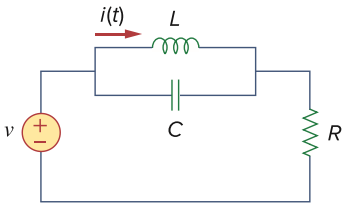



**Figure 8.93**
For Prob. 8.46.

8.47 Find the output voltage $v_o(t)$ in the circuit of Fig. 8.94.

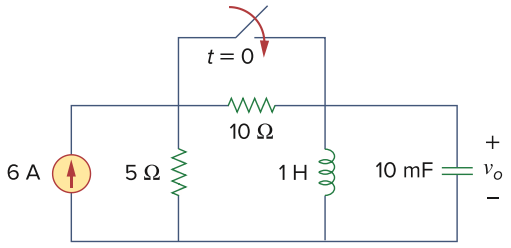



**Figure 8.94**
For Prob. 8.47.

8.48 Given the circuit in Fig. 8.95, find $i(t)$ and $v(t)$ for $t > 0$.

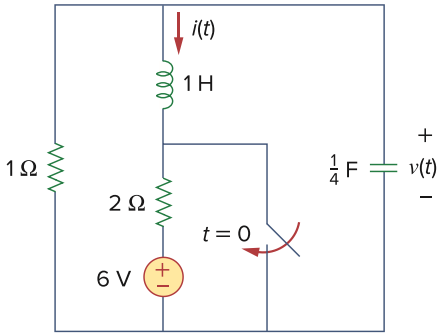



**Figure 8.95**
For Prob. 8.48.

8.49 Determine $i(t)$ for $t > 0$ in the circuit of Fig. 8.96.

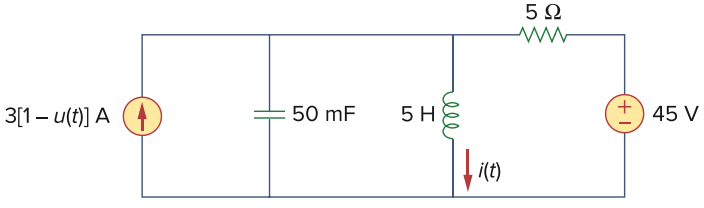



**Figure 8.96**
For Prob. 8.49.

**8.50** For the circuit in Fig. 8.97, find $i(t)$ for $t > 0$.

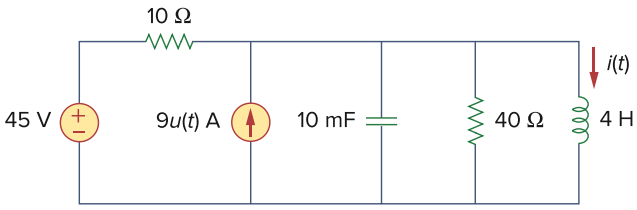



**Figure 8.97**
For Prob. 8.50.

**8.51** Find $v(t)$ for $t > 0$ in the circuit of Fig. 8.98.

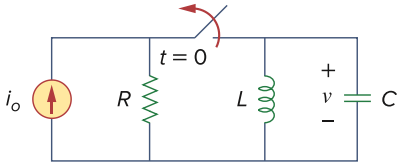



**Figure 8.98**
For Prob. 8.51.

**8.52** The step response of a parallel *RLC* circuit is

$$v = 10 + 20e^{-300t}(\cos 400t - 2\sin 400t)\ \text{V}, \qquad t \geq 0$$

when the inductor is 25 mH. Find $R$ and $C$.

## Section 8.7 General Second-Order Circuits

**8.53** After being open for a day, the switch in the circuit of Fig. 8.99 is closed at $t = 0$. Find the differential equation describing $i(t)$, $t > 0$.

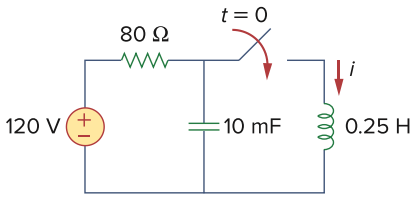



**Figure 8.99**
For Prob. 8.53.

**8.54** Using Fig. 8.100, design a problem to help other students better understand general second-order circuits.

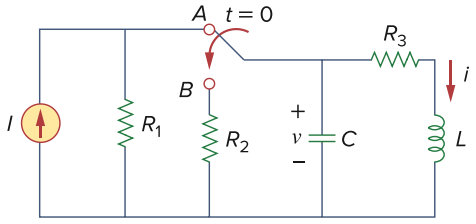



**Figure 8.100**
For Prob. 8.54.

**8.55** For the circuit in Fig. 8.101, find $v(t)$ for $t > 0$. Assume that $i(0^+) = 2$ A.

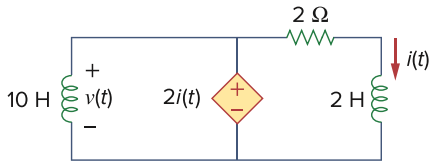



**Figure 8.101**
For Prob. 8.55.

**8.56** In the circuit of Fig. 8.102, find $i(t)$ for $t > 0$.

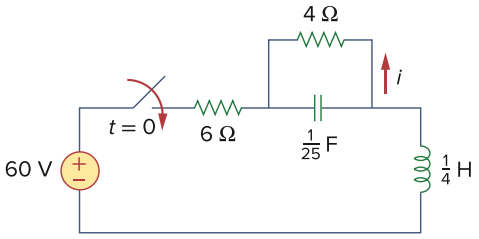



**Figure 8.102**
For Prob. 8.56.

**8.57** Given the circuit shown in Fig. 8.103, determine the characteristic equation of the circuit and the values for $i(t)$ and $v(t)$ for all $t > 0$.

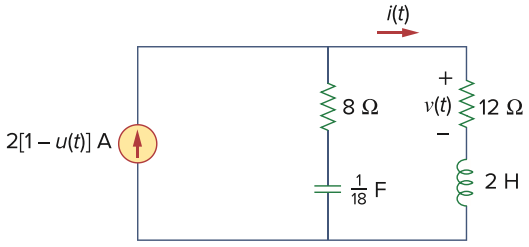



**Figure 8.103**
For Prob. 8.57.

**8.58** In the circuit of Fig. 8.104, the switch has been in position 1 for a long time but moved to position 2 at $t = 0$. Find:

(a) $v(0^+)$, $dv(0^+)/dt$,
(b) $v(t)$ for $t \geq 0$.

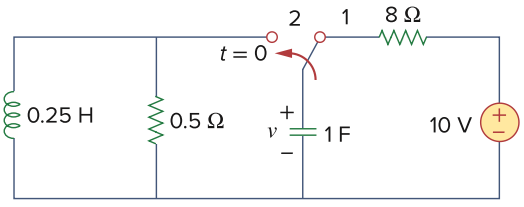



**Figure 8.104**
For Prob. 8.58.

**8.59** The switch in Fig. 8.105 has been in position 1 for $t < 0$. At $t = 0$, it is moved from position 1 to the top of the capacitor at $t = 0$. Please note that the switch is a make before break switch, it stays in contact with position 1 until it makes contact with the top of the capacitor and then breaks the contact at position 1. Given that the initial voltage across the capacitor is equal to zero, determine $v(t)$.

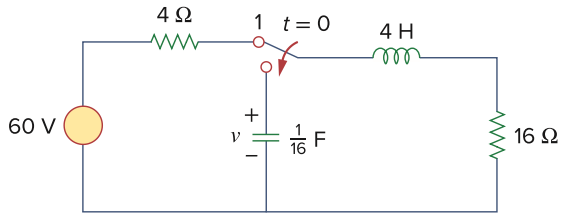



**Figure 8.105**
For Prob. 8.59.

**8.60** Obtain $i_1$ and $i_2$ for $t > 0$ in the circuit of Fig. 8.106.

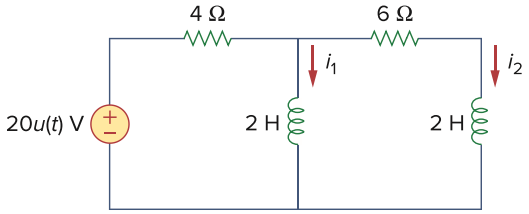



**Figure 8.106**
For Prob. 8.60.

**8.61** For the circuit in Prob. 8.5, find $i$ and $v$ for $t > 0$.

**8.62** Find the response $v_R(t)$ for $t > 0$ in the circuit of Fig. 8.107. Let $R = 8\ \Omega$, $L = 2$ H, and $C = 125$ mF.

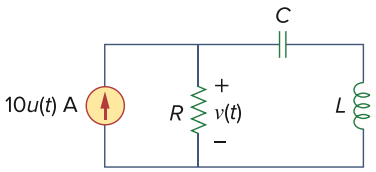



**Figure 8.107**
For Prob. 8.62.

## Section 8.8 Second-Order Op Amp Circuits

**8.63** For the op amp circuit in Fig. 8.108, find the differential equation for $i(t)$.

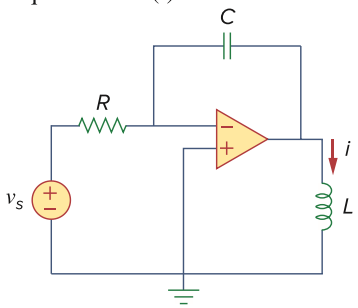



**Figure 8.108**
For Prob. 8.63.

**8.64** Using Fig. 8.109, design a problem to help other students better understand second-order op amp circuits.

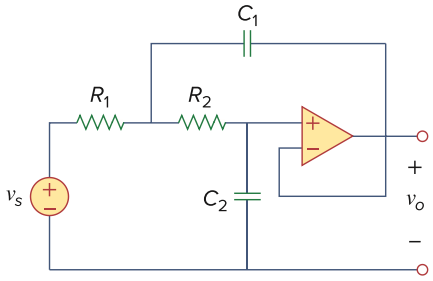



**Figure 8.109**
For Prob. 8.64.

**8.65** Determine the differential equation for the op amp circuit in Fig. 8.110. If $v_1(0^+) = 2$ V and $v_2(0^+) = 0$ V, find $v_o$ for $t > 0$. Let $R = 100$ kΩ and $C = 1\ \mu$F.

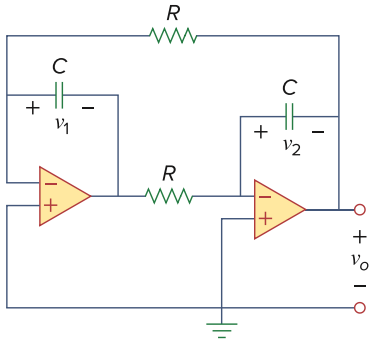



**Figure 8.110**
For Prob. 8.65.

**8.66** Obtain the differential equations for $v_o(t)$ in the op amp circuit of Fig. 8.111.

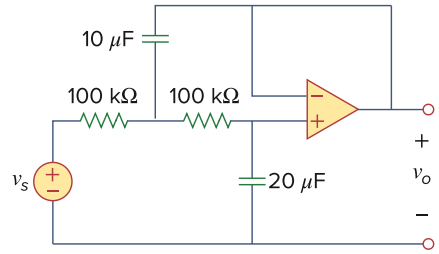



**Figure 8.111**
For Prob. 8.66.

***8.67** In the op amp circuit of Fig. 8.112, determine $v_o(t)$ for $t > 0$. Let $v_{in} = u(t)$ V, $R_1 = R_2 = 10$ kΩ, $C_1 = C_2 = 100$ μF.

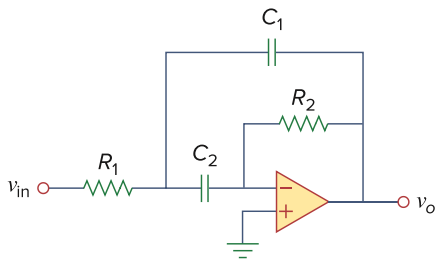



**Figure 8.112**
For Prob. 8.67.

## Section 8.9 *PSpice* Analysis of *RLC* Circuit

**8.68** For the step function $v_s = u(t)$, use *PSpice* or *MultiSim* to find the response $v(t)$ for $0 < t < 6$ s in the circuit of Fig. 8.113.

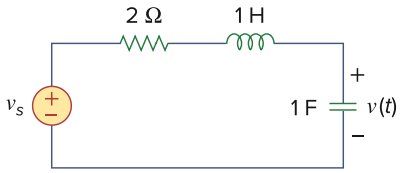



**Figure 8.113**
For Prob. 8.68.

**8.69** Given the source-free circuit in Fig. 8.114, use *PSpice* or *MultiSim* to get $i(t)$ for $0 < t < 20$ s. Take $v(0) = 30$ V and $i(0) = 2$ A.

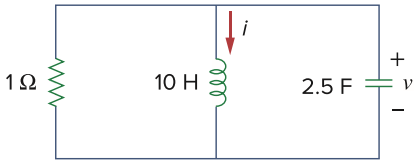



**Figure 8.114**
For Prob. 8.69.

**8.70** For the circuit in Fig. 8.115, use *PSpice* or *MultiSim* to obtain $v(t)$ for $0 < t < 4$ s. Assume that the capacitor voltage and inductor current at $t = 0$ are both zero.

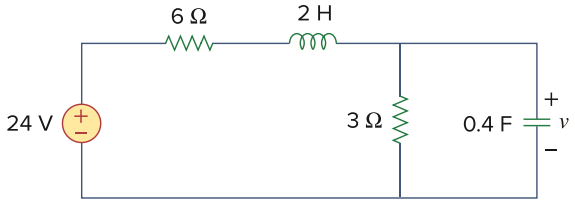



**Figure 8.115**
For Prob. 8.70.

**8.71** Obtain $v(t)$ for $0 < t < 4$ s in the circuit of Fig. 8.116 using *PSpice* or *MultiSim*.

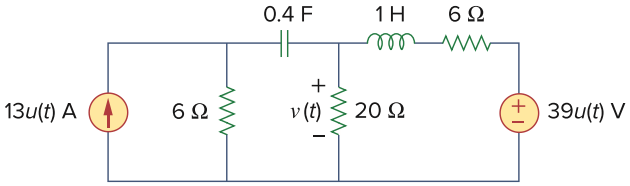



**Figure 8.116**
For Prob. 8.71.

**8.72** The switch in Fig. 8.117 has been in position 1 for a long time. At $t = 0$, it is switched to position 2. Use *PSpice* or *MultiSim* to find $i(t)$ for $0 < t < 0.2$ s.

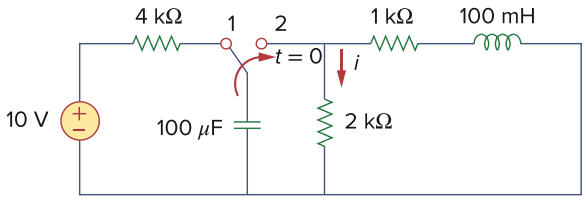



**Figure 8.117**
For Prob. 8.72.

**8.73** Design a problem, to be solved using *PSpice* or *MultiSim*, to help other students better understand source-free *RLC* circuits.

## Section 8.10 Duality

**8.74** Draw the dual of the circuit shown in Fig. 8.118.

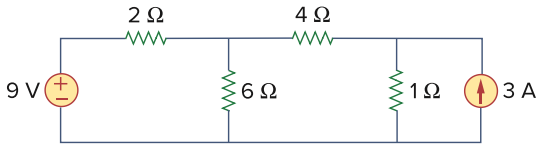



**Figure 8.118**
For Prob. 8.74.

**8.75** Obtain the dual of the circuit in Fig. 8.119.

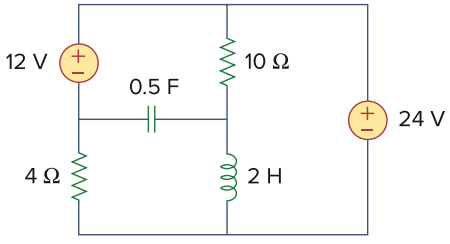



**Figure 8.119**
For Prob. 8.75.

**8.76** Find the dual of the circuit in Fig. 8.120.

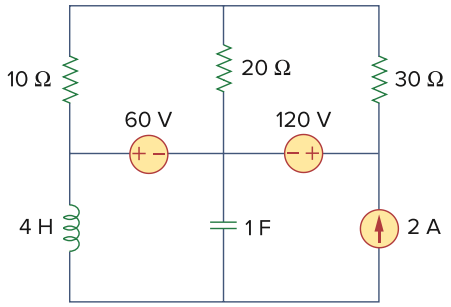



**Figure 8.120**
For Prob. 8.76.

**8.77** Draw the dual of the circuit in Fig. 8.121.

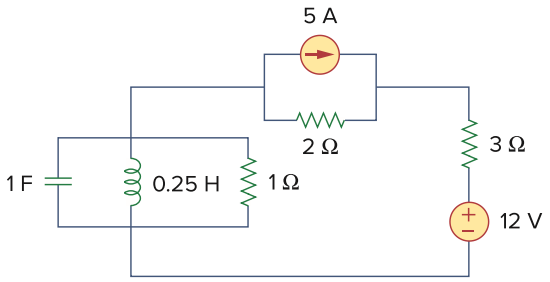



**Figure 8.121**
For Prob. 8.77.

## Section 8.11 Applications

**8.78** An automobile airbag igniter is modeled by the circuit in Fig. 8.122. Determine the time it takes the voltage across the igniter to reach its first peak after switching from $A$ to $B$. Let $R = 3\ \Omega$, $C = 1/30$ F, and $L = 60$ mH.

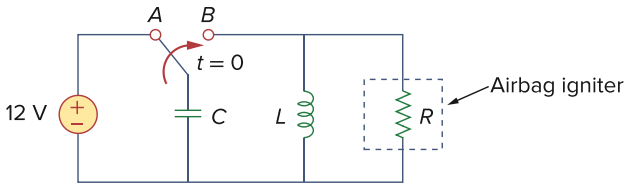



**Figure 8.122**
For Prob. 8.78.

**8.79** A load is modeled as a 100-mH inductor in parallel with a 12-Ω resistor. A capacitor is needed to be connected to the load so that the network is critically damped at 60 Hz. Calculate the size of the capacitor.

# Comprehensive Problems

**8.80** A mechanical system is modeled by a series $RLC$ circuit. It is desired to produce an overdamped response with time constants 0.1 and 0.5 ms. If a series 50-kΩ resistor is used, find the values of $L$ and $C$.

**8.81** An oscillogram can be adequately modeled by a second-order system in the form of a parallel $RLC$ circuit. It is desired to give an underdamped voltage across a 200-Ω resistor. If the damped frequency is 4 kHz and the time constant of the envelope is 0.25 s, find the necessary values of $L$ and $C$.

**8.82** The circuit in Fig. 8.123 is the electrical analog of body functions used in medical schools to study convulsions. The analog is as follows:

$C_1$ = Volume of fluid in a drug
$C_2$ = Volume of blood stream in a specified region
$R_1$ = Resistance in the passage of the drug from the input to the blood stream
$R_2$ = Resistance of the excretion mechanism, such as kidney, etc.
$v_0$ = Initial concentration of the drug dosage
$v(t)$ = Percentage of the drug in the blood stream

Find $v(t)$ for $t > 0$ given that $C_1 = 0.5\ \mu$F, $C_2 = 5\ \mu$F, $R_1 = 5$ MΩ, $R_2 = 2.5$ MΩ, and $v_0 = 60u(t)$ V.

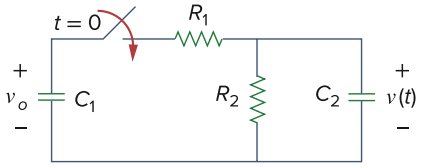



**Figure 8.123**
For Prob. 8.82.

8.83 Figure 8.124 shows a typical tunnel-diode oscillator circuit. The diode is modeled as a nonlinear resistor with $i_D = f(v_D)$, i.e., the diode current is a nonlinear function of the voltage across the diode. Derive the differential equation for the circuit in terms of $v$ and $i_D$.

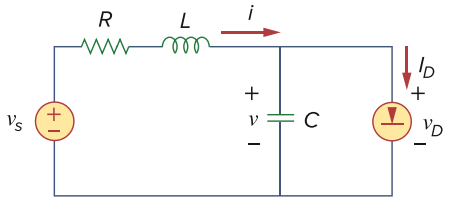



**Figure 8.124**
For Prob. 8.83.

PART TWO

# AC Circuits

OUTLINE


9 Sinusoids and Phasors

10 Sinusoidal Steady-State Analysis

11 AC Power Analysis

12 Three-Phase Circuits

13 Magnetically Coupled Circuits

14 Frequency Response

NASA

chapter

9

# Sinusoids and Phasors

*He who knows not, and knows not that he knows not, is a fool—shun him.*
*He who knows not, and knows that he knows not, is a c hild—teach him.*
*He who knows, and knows not that he knows, is asleep—wake him up.*
*He who knows, and knows that he knows, is wise—follow him.*

—Persian Proverb

## Enhancing Your Skills and Your Career

**ABET EC 2000 criteria (3.d), *"an ability to function on multi-disciplinary teams."***

The "ability to function on multidisciplinary teams" is inherently critical for the working engineer. Engineers rarely, if ever, work by themselves. Engineers will always be part of some team. One of the things I like to remind students is that you do not have to like everyone on a team; you just have to be a successful part of that team.

Most frequently, these teams include individuals from a variety of engineering disciplines, as well as individuals from nonengineering disciplines such as marketing and finance.

Students can easily develop and enhance this skill by working in study groups in every course they take. Clearly, working in study groups in nonengineering courses, as well as engineering courses outside your discipline, will also give you experience with multidisciplinary teams.

Photo by Charles Alexander

## Historical

George Westinghouse. Photo © Bettmann/Corbis

**Nikola Tesla** (1856–1943) and **George Westinghouse** (1846–1914) helped establish alternating current as the primary mode of electricity transmission and distribution.

Today it is obvious that ac generation is well established as the form of electric power that makes widespread distribution of electric power efficient and economical. However, at the end of the 19th century, which was the better—ac or dc—was hotly debated and had extremely outspoken supporters on both sides. The dc side was led by Thomas Edison, who had earned a lot of respect for his many contributions. Power generation using ac really began to build after the successful contributions of Tesla. The real commercial success in ac came from George Westinghouse and the outstanding team, including Tesla, he assembled. In addition, two other big names were C. F. Scott and B. G. Lamme.

The most significant contribution to the early success of ac was the patenting of the polyphase ac motor by Tesla in 1888. The induction motor and polyphase generation and distribution systems doomed the use of dc as the prime energy source.

### Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Better understand sinusoids.
2. Understand phasors.
3. Understand the phasor relationships for circuit elements.
4. Know and understand the concepts of impedance and admittance.
5. Understand Kirchhoff's laws in the frequency domain.
6. Comprehend the concept of phase-shift.
7. Understand the concept of AC bridges.

## 9.1 Introduction

Thus far our analysis has been limited for the most part to dc circuits: those circuits excited by constant or time-invariant sources. We have restricted the forcing function to dc sources for the sake of simplicity, for pedagogic reasons, and also for historic reasons. Historically, dc sources were the main means of providing electric power up until the late 1800s. At the end of that century, the battle of direct current versus alternating current began. Both had their advocates among the electrical engineers of the time. Because ac is more efficient and economical to transmit over long distances, ac systems ended up the winner. Thus, it is in keeping with the historical sequence of events that we considered dc sources first.

We now begin the analysis of circuits in which the source voltage or current is time-varying. In this chapter, we are particularly interested in sinusoidally time-varying excitation, or simply, excitation by a *sinusoid*.

A sinusoid is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as *alternating current* (*ac*). Such a current reverses at regular time intervals and has alternately positive and negative values. Circuits driven by sinusoidal current or voltage sources are called *ac circuits*.

We are interested in sinusoids for a number of reasons. First, nature itself is characteristically sinusoidal. We experience sinusoidal variation in the motion of a pendulum, the vibration of a string, the ripples on the ocean surface, and the natural response of underdamped second-order systems, to mention but a few. Second, a sinusoidal signal is easy to generate and transmit. It is the form of voltage generated throughout the world and supplied to homes, factories, laboratories, and so on. It is the dominant form of signal in the communications and electric power industries. Third, through Fourier analysis, any practical periodic signal can be represented by a sum of sinusoids. Sinusoids, therefore, play an important role in the analysis of periodic signals. Lastly, a sinusoid is easy to handle mathematically. The derivative and integral of a sinusoid are themselves sinusoids. For these and other reasons, the sinusoid is an extremely important function in circuit analysis.

A sinusoidal forcing function produces both a transient response and a steady-state response, much like the step function, which we studied in Chapters 7 and 8. The transient response dies out with time so that only the steady-state response remains. When the transient response has become negligibly small compared with the steady-state response, we say that the circuit is operating at sinusoidal steady state. It is this *sinusoidal steady-state response* that is of main interest to us in this chapter.

We begin with a basic discussion of sinusoids and phasors. We then introduce the concepts of impedance and admittance. The basic circuit laws, Kirchhoff's and Ohm's, introduced for dc circuits, will be applied to ac circuits. Finally, we consider applications of ac circuits in phase-shifters and bridges.

## 9.2 Sinusoids

Consider the sinusoidal voltage

$$v(t) = V_m \sin \omega t \tag{9.1}$$

where

$V_m$ = the *amplitude* of the sinusoid
$\omega$ = the *angular frequency* in radians/s
$\omega t$ = the *argument* of the sinusoid

The sinusoid is shown in Fig. 9.1(a) as a function of its argument and in Fig. 9.1(b) as a function of time. It is evident that the sinusoid repeats itself every $T$ seconds; thus, $T$ is called the *period* of the sinusoid. From the two plots in Fig. 9.1, we observe that $\omega T = 2\pi$,

$$T = \frac{2\pi}{\omega} \tag{9.2}$$

## Historical

© Hulton Archives/Getty Images

**Heinrich Rudorf Hertz** (1857–1894), a German experimental physicist, demonstrated that electromagnetic waves obey the same fundamental laws as light. His work confirmed James Clerk Maxwell's celebrated 1864 theory and prediction that such waves existed.

Hertz was born into a prosperous family in Hamburg, Germany. He attended the University of Berlin and did his doctorate under the prominent physicist Hermann von Helmholtz. He became a professor at Karlsruhe, where he began his quest for electromagnetic waves. Hertz successfully generated and detected electromagnetic waves; he was the first to show that light is electromagnetic energy. In 1887, Hertz noted for the first time the photoelectric effect of electrons in a molecular structure. Although Hertz only lived to the age of 37, his discovery of electromagnetic waves paved the way for the practical use of such waves in radio, television, and other communication systems. The unit of frequency, the hertz, bears his name.

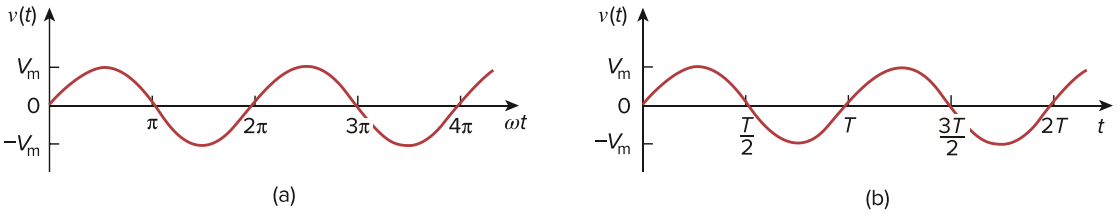



**Figure 9.1**
A sketch of $V_m \sin \omega t$: (a) as a function of $\omega t$, (b) as a function of $t$.

The fact that $v(t)$ repeats itself every $T$ seconds is shown by replacing $t$ by $t + T$ in Eq. (9.1). We get

$$v(t+T) = V_m \sin \omega(t+T) = V_m \sin \omega\left(t + \frac{2\pi}{\omega}\right)$$
$$= V_m \sin(\omega t + 2\pi) = V_m \sin \omega t = v(t) \tag{9.3}$$

Hence,

$$\boxed{v(t+T) = v(t)} \tag{9.4}$$

that is, $v$ has the same value at $t + T$ as it does at $t$ and $v(t)$ is said to be *periodic*. In general,

A **periodic function** is one that satisfies $f(t) = f(t + nT)$, for all $t$ and for all integers $n$.

As mentioned, the *period* $T$ of the periodic function is the time of one complete cycle or the number of seconds per cycle. The reciprocal of this quantity is the number of cycles per second, known as the *cyclic frequency* $f$ of the sinusoid. Thus,

$$\boxed{f = \frac{1}{T}} \tag{9.5}$$

From Eqs. (9.2) and (9.5), it is clear that

$$\omega = 2\pi f \tag{9.6}$$

While $\omega$ is in radians per second (rad/s), $f$ is in hertz (Hz).

The unit of $f$ is named after the German physicist Heinrich R. Hertz (1857–1894).

Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi) \tag{9.7}$$

where $(\omega t + \phi)$ is the argument and $\phi$ is the *phase*. Both argument and phase can be in radians or degrees.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t \qquad \text{and} \qquad v_2(t) = V_m \sin(\omega t + \phi) \tag{9.8}$$

shown in Fig. 9.2. The starting point of $v_2$ in Fig. 9.2 occurs first in time. Therefore, we say that $v_2$ *leads* $v_1$ by $\phi$ or that $v_1$ *lags* $v_2$ by $\phi$. If $\phi \neq 0$, we also say that $v_1$ and $v_2$ are *out of phase*. If $\phi = 0$, then $v_1$ and $v_2$ are said to be *in phase;* they reach their minima and maxima at exactly the same time. We can compare $v_1$ and $v_2$ in this manner because they operate at the same frequency; they do not need to have the same amplitude.

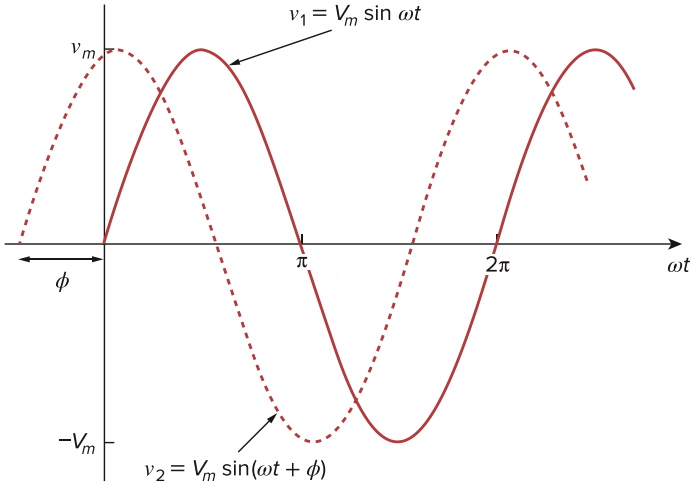



**Figure 9.2**
Two sinusoids with different phases.

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes. This is achieved by using the following trigonometric identities:

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned} \tag{9.9}$$

With these identities, it is easy to show that

$$\begin{aligned} \sin(\omega t \pm 180°) &= -\sin \omega t \\ \cos(\omega t \pm 180°) &= -\cos \omega t \\ \sin(\omega t \pm 90°) &= \pm \cos \omega t \\ \cos(\omega t \pm 90°) &= \mp \sin \omega t \end{aligned} \tag{9.10}$$

Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

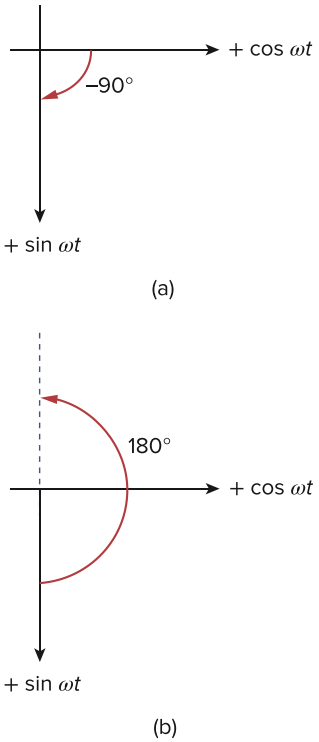



**Figure 9.3**
A graphical means of relating cosine and sine: (a) $\cos(\omega t - 90°) = \sin \omega t$, (b) $\sin(\omega t + 180°) = -\sin \omega t$.

A graphical approach may be used to relate or compare sinusoids as an alternative to using the trigonometric identities in Eqs. (9.9) and (9.10). Consider the set of axes shown in Fig. 9.3(a). The horizontal axis represents the magnitude of cosine, while the vertical axis (pointing down) denotes the magnitude of sine. Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates. This graphical technique can be used to relate two sinusoids. For example, we see in Fig. 9.3(a) that subtracting 90° from the argument of $\cos \omega t$ gives $\sin \omega t$, or $\cos(\omega t - 90°) = \sin \omega t$. Similarly, adding 180° to the argument of $\sin \omega t$ gives $-\sin \omega t$, or $\sin(\omega t + 180°) = -\sin \omega t$, as shown in Fig. 9.3(b).

The graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form. To add $A \cos \omega t$ and $B \sin \omega t$, we note that $A$ is the magnitude of $\cos \omega t$ while $B$ is the magnitude of $\sin \omega t$, as shown in Fig. 9.4(a). The magnitude and argument of the resultant sinusoid in cosine form is readily obtained from the triangle. Thus,

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta) \tag{9.11}$$

where

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1} \frac{B}{A} \tag{9.12}$$

For example, we may add $3 \cos \omega t$ and $-4 \sin \omega t$ as shown in Fig. 9.4(b) and obtain

$$3 \cos \omega t - 4 \sin \omega t = 5 \cos(\omega t + 53.1°) \tag{9.13}$$

Compared with the trigonometric identities in Eqs. (9.9) and (9.10), the graphical approach eliminates memorization. However, we must not confuse the sine and cosine axes with the axes for complex numbers to be discussed in the next section. Something else to note in Figs. 9.3 and 9.4 is that although the natural tendency is to have the vertical axis point up, the positive direction of the sine function is down in the present case.

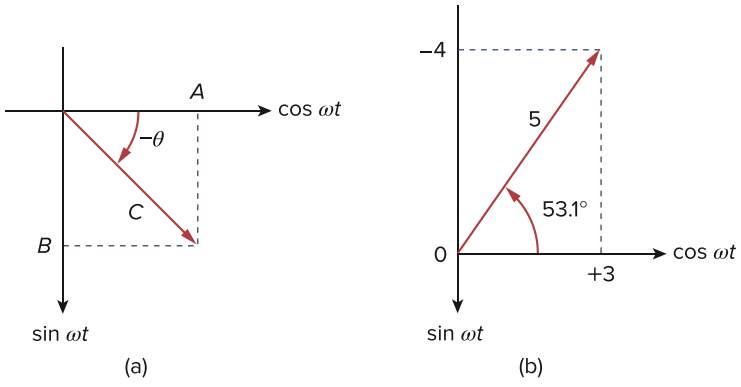



**Figure 9.4**
(a) Adding $A \cos \omega t$ and $B \sin \omega t$, (b) adding $3 \cos \omega t$ and $-4 \sin \omega t$.

## Example 9.1

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10°)\ \text{V}.$$

**Solution:**

The amplitude is $V_m = 12$ V.
The phase is $\phi = 10°$.
The angular frequency is $\omega = 50$ rad/s.
The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.
The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

## Practice Problem 9.1

Given the sinusoid $45\cos(5\ \pi t + 36°)$, calculate its amplitude, phase, angular frequency, period, and frequency.

**Answer:** 45, 36°, 15.708 rad/s, 400 ms, 2.5 Hz.

## Example 9.2

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50°)$ and $v_2 = 12\sin(\omega t - 10°)$. State which sinusoid is leading.

**Solution:**
Let us calculate the phase in three ways. The first two methods use trigonometric identities, while the third method uses the graphical approach.

**METHOD 1** In order to compare $v_1$ and $v_2$, we must e xpress them in the same form. If we e xpress them in cosine form with positive amplitudes,

$$v_1 = -10\cos(\omega t + 50°) = 10\cos(\omega t + 50° - 180°)$$
$$v_1 = 10\cos(\omega t - 130°) \quad \text{or} \quad v_1 = 10\cos(\omega t + 230°) \qquad \textbf{(9.2.1)}$$

and

$$v_2 = 12\sin(\omega t - 10°) = 12\cos(\omega t - 10° - 90°)$$
$$v_2 = 12\cos(\omega t - 100°) \qquad \textbf{(9.2.2)}$$

It can be deduced from Eqs. (9.2.1) and (9.2.2) that the phase difference between $v_1$ and $v_2$ is 30°. We can write $v_2$ as

$$v_2 = 12\cos(\omega t - 130° + 30°) \quad \text{or} \quad v_2 = 12\cos(\omega t + 260°) \qquad \textbf{(9.2.3)}$$

Comparing Eqs. (9.2.1) and (9.2.3) shows clearly that $v_2$ leads $v_1$ by 30°.

**METHOD 2** Alternatively, we may express $v_1$ in sine form:

$$v_1 = -10\cos(\omega t + 50°) = 10\sin(\omega t + 50° - 90°)$$
$$= 10\sin(\omega t - 40°) = 10\sin(\omega t - 10° - 30°)$$

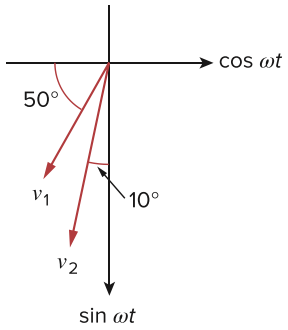



**Figure 9.5**
For Example 9.2.

But $v_2 = 12\sin(\omega t - 10°)$. Comparing the two shows that $v_1$ lags $v_2$ by 30°. This is the same as saying that $v_2$ leads $v_1$ by 30°.

**METHOD 3** We may regard $v_1$ as simply $-10\cos\omega t$ with a phase shift of +50°. Hence, $v_1$ is as shown in Fig. 9.5. Similarly, $v_2$ is $12\sin\omega t$ with a phase shift of $-10°$, as shown in Fig. 9.5. It is easy to see from Fig. 9.5 that $v_2$ leads $v_1$ by 30°, that is, $90° - 50° - 10°$.

## Practice Problem 9.2

Find the phase angle between

$$i_1 = -4\sin(377t + 55°) \quad \text{and} \quad i_2 = 5\cos(377t - 65°)$$

Does $i_1$ lead or lag $i_2$?

**Answer:** 210°, $i_1$ leads $i_2$.

## 9.3 Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise. The notion of solving ac circuits using phasors was first introduced by Charles Steinmetz in 1893. Before we completely define phasors and apply them to circuit analysis, we need to be thoroughly familiar with complex numbers.

Charles Proteus Steinmetz (1865–1923) was a German-Austrian mathematician and electrical engineer.

Appendix B presents a short tutorial on complex numbers.

A complex number $z$ can be written in rectangular form as

$$z = x + jy \tag{9.14a}$$

where $j = \sqrt{-1}$; $x$ is the real part of $z$; $y$ is the imaginary part of $z$. In this context, the variables $x$ and $y$ do not represent a location as in two-dimensional vector analysis but rather the real and imaginary parts of $z$ in the complex plane. Nevertheless, we note that there are some resemblances between manipulating complex numbers and manipulating two-dimensional vectors.

The complex number $z$ can also be written in polar or exponential form as

$$z = r\angle\phi = re^{j\phi} \tag{9.14b}$$

## Historical

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**Charles Proteus Steinmetz** (1865–1923), a German-Austrian mathematician and engineer, introduced the phasor method (covered in this chapter) in ac circuit analysis. He is also noted for his work on the theory of hysteresis.

Steinmetz was born in Breslau, Germany, and lost his mother at the age of one. As a youth, he was forced to leave Germany because of his political activities just as he was about to complete his doctoral dissertation in mathematics at the University of Breslau. He migrated to Switzerland and later to the United States, where he was employed by General Electric in 1893. That same year, he published a paper in which complex numbers were used to analyze ac circuits for the first time. This led to one of his many textbooks, *Theory and Calculation of ac Phenomena,* published by McGraw-Hill in 1897. In 1901, he became the president of the American Institute of Electrical Engineers, which later became the IEEE.

where $r$ is the magnitude of $z$, and $\phi$ is the phase of $z$. We notice that $z$ can be represented in three ways:

$$\begin{aligned} z &= x + jy && \text{Rectangular form} \\ z &= r\angle\phi && \text{Polar form} \\ z &= re^{j\phi} && \text{Exponential form} \end{aligned} \tag{9.15}$$

The relationship between the rectangular form and the polar form is shown in Fig. 9.6, where the $x$ axis represents the real part and the $y$ axis represents the imaginary part of a complex number. Given $x$ and $y$, we can get $r$ and $\phi$ as

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x} \tag{9.16a}$$

On the other hand, if we know $r$ and $\phi$, we can obtain $x$ and $y$ as

$$x = r\cos\phi, \qquad y = r\sin\phi \tag{9.16b}$$

Thus, $z$ may be written as

$$\boxed{z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)} \tag{9.17}$$

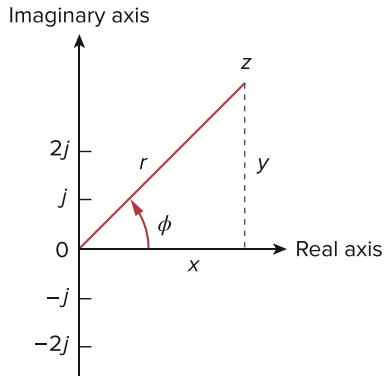



**Figure 9.6**
Representation of a complex number $z = x + jy = r\angle\phi$.

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$$z = x + jy = r\angle\phi, \qquad z_1 = x_1 + jy_1 = r_1\angle\phi_1$$
$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

the following operations are important.

**Addition:**

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{9.18a}$$

**Subtraction:**

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{9.18b}$$

**Multiplication:**

$$z_1 z_2 = r_1 r_2 \underline{/\phi_1 + \phi_2} \tag{9.18c}$$

**Division:**

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{/\phi_1 - \phi_2} \tag{9.18d}$$

**Reciprocal:**

$$\frac{1}{z} = \frac{1}{r} \underline{/-\phi} \tag{9.18e}$$

**Square Root:**

$$\sqrt{z} = \sqrt{r} \underline{/\phi/2} \tag{9.18f}$$

**Complex Conjugate:**

$$z^* = x - jy = r \underline{/-\phi} = re^{-j\phi} \tag{9.18g}$$

Note that from Eq. (9.18e),

$$\frac{1}{j} = -j \tag{9.18h}$$

These are the basic properties of complex numbers we need. Other properties of complex numbers can be found in Appendix B.

The idea of phasor representation is based on Euler's identity. In general,

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j \sin\phi} \tag{9.19}$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$; we may write

$$\cos\phi = \text{Re}(e^{j\phi}) \tag{9.20a}$$

$$\sin\phi = \text{Im}(e^{j\phi}) \tag{9.20b}$$

where Re and Im stand for the *real part of* and the *imaginary part of*. Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we use Eq. (9.20a) to express $v(t)$ as

$$v(t) = V_m \cos(\omega t + \phi) = \text{Re}(V_m e^{j(\omega t + \phi)}) \tag{9.21}$$

or

$$v(t) = \text{Re}(V_m e^{j\phi} e^{j\omega t}) \tag{9.22}$$

Thus,

$$\boxed{v(t) = \text{Re}(\mathbf{V} e^{j\omega t})} \tag{9.23}$$

where

$$\mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi} \tag{9.24}$$

$\mathbf{V}$ is thus the *phasor representation* of the sinusoid $v(t)$, as we said earlier. In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid. Either Eq. (9.20a) or Eq. (9.20b) can be used to develop the phasor, but the standard convention is to use Eq. (9.20a).

A phasor may be regarded as a mathematical equivalent of a sinusoid with the time dependence dropped.

One way of looking at Eqs. (9.23) and (9.24) is to consider the plot of the *sinor* $\mathbf{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius $V_m$ at an angular velocity $\omega$ in the counterclockwise direction, as shown in Fig. 9.7(a). We may regard $v(t)$ as the projection of the sinor $\mathbf{V}e^{j\omega t}$ on the real axis, as shown in Fig. 9.7(b). The value of the sinor at time $t = 0$ is the phasor $\mathbf{V}$ of the sinusoid $v(t)$. The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present. It is therefore important, when dealing with phasors, to keep in mind the frequency $\omega$ of the phasor; otherwise we can make serious mistakes.

If we use sine for the phasor instead of cosine, then $v(t) = V_m \sin(\omega t + \phi) = \text{Im}(V_m e^{j(\omega t+\phi)})$ and the corresponding phasor is the same as that in Eq. (9.24).

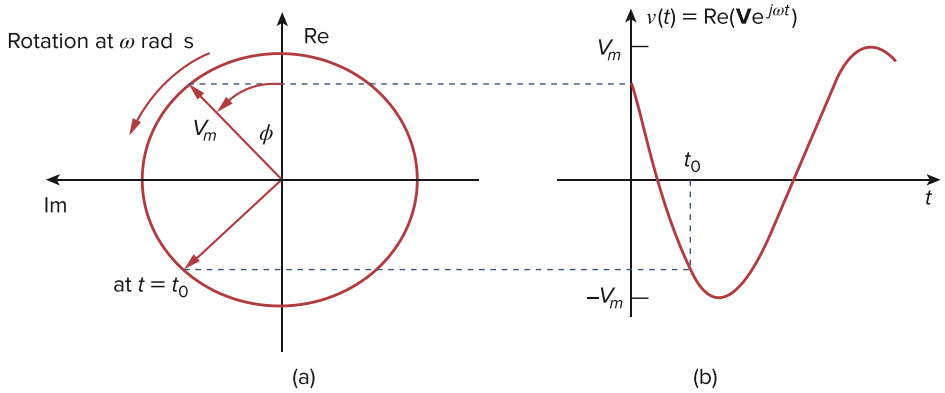



**Figure 9.7**
Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

Equation (9.23) states that to obtain the sinusoid corresponding to a given phasor $\mathbf{V}$, multiply the phasor by the time factor $e^{j\omega t}$ and take the real part. As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. Because a phasor has magnitude and phase ("direction"), it behaves as a vector and is printed in boldface. For example, phasors $\mathbf{V} = V_m\angle\phi$ and $\mathbf{I} = I_m\angle-\theta$ are graphically represented in Fig. 9.8. Such a graphical representation of phasors is known as a *phasor diagram*.

We use lightface italic letters such as $z$ to represent complex numbers but boldface letters such as $\mathbf{V}$ to represent phasors, because phasors are vector-like quantities.

Equations (9.21) through (9.23) reveal that to get the phasor corresponding to a sinusoid, we first express the sinusoid in the cosine form so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor $e^{j\omega t}$, and whatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m\angle\phi \tag{9.25}$$

(Time-domain representation) (Phasor-domain representation)

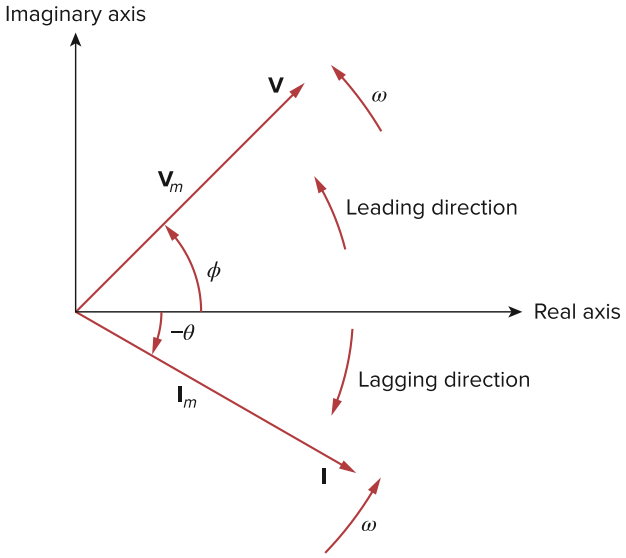



**Figure 9.8**
A phasor diagram showing $\mathbf{V} = V_m \angle \phi$ and $\mathbf{I} = I_m \angle -\theta$.

Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we obtain the corresponding phasor as $\mathbf{V} = V_m \angle \phi$. Equation (9.25) is also demonstrated in Table 9.1, where the sine function is considered in addition to the cosine function. From Eq. (9.25), we see that to get the phasor representation of a sinusoid, we express it in cosine form and take the magnitude and phase. Given a phasor, we obtain the time domain representation as the cosine function with the same magnitude as the phasor and the argument as $\omega t$ plus the phase of the phasor. The idea of expressing information in alternate domains is fundamental to all areas of engineering.

**TABLE 9.1**

Sinusoid-phasor transformation.

| Time domain representation | Phasor domain representation |
|---|---|
| $V_m \cos(\omega t + \phi)$ | $V_m \angle \phi$ |
| $V_m \sin(\omega t + \phi)$ | $V_m \angle \phi - 90^\circ$ |
| $I_m \cos(\omega t + \theta)$ | $I_m \angle \theta$ |
| $I_m \sin(\omega t + \theta)$ | $I_m \angle \theta - 90^\circ$ |

Note that in Eq. (9.25) the frequency (or time) factor $e^{j\omega t}$ is suppressed, and the frequency is not explicitly shown in the phasor domain representation because $\omega$ is constant. However, the response depends on $\omega$. For this reason, the phasor domain is also known as the *frequency domain.*

From Eqs. (9.23) and (9.24), $v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi)$, so that

$$\begin{aligned} \frac{dv}{dt} &= -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ) \\ &= \text{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \text{Re}(j\omega \mathbf{V} e^{j\omega t}) \end{aligned} \tag{9.26}$$

This shows that the derivative $v(t)$ is transformed to the phasor domain as $j\omega\mathbf{V}$

$$\underbrace{\frac{dv}{dt}}_{\text{(Time domain)}} \quad \Leftrightarrow \quad \underbrace{j\omega\mathbf{V}}_{\text{(Phasor domain)}} \tag{9.27}$$

Differentiating a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$.

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $\mathbf{V}/j\omega$

$$\underbrace{\int v\,dt}_{\text{(Time domain)}} \quad \Leftrightarrow \quad \underbrace{\frac{\mathbf{V}}{j\omega}}_{\text{(Phasor domain)}} \tag{9.28}$$

Integrating a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$.

Equation (9.27) allows the replacement of a derivative with respect to time with multiplication of $j\omega$ in the phasor domain, whereas Eq. (9.28) allows the replacement of an integral with respect to time with division by $j\omega$ in the phasor domain. Equations (9.27) and (9.28) are useful in finding the steady-state solution, which does not require knowing the initial values of the variable involved. This is one of the important applications of phasors.

Besides time differentiation and integration, another important use of phasors is found in summing sinusoids of the same frequency. This is best illustrated with an example, and Example 9.6 provides one.

Adding sinusoids of the same frequency is equivalent to adding their corresponding phasors.

The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

1. $v(t)$ is the *instantaneous or time domain* representation, while $\mathbf{V}$ is the *frequency or phasor domain* representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not. (This fact is often forgotten by students.)
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

Finally, we should bear in mind that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

## Example 9.3

Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

**Solution:**

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j \sin 50^\circ) = 25.71 + j30.64$$

$$20\angle -30^\circ = 20[\cos(-30^\circ) + j \sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$

$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$

$$= 0.565\angle -160.13^\circ$$

## Practice Problem 9.3

Evaluate the following complex numbers:

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$

**Answer:** (a) $-15.5 - j13.67$, (b) $8.293 + j7.2$.

## Example 9.4

Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

**Solution:**

(a) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle -40^\circ \text{ A}$$

(b) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4\sin(30t + 50^\circ) = 4\cos(30t + 50^\circ + 90^\circ)$$

$$= 4\cos(30t + 140^\circ) \text{ V}$$

The phasor form of $v$ is

$$\mathbf{V} = 4\angle 140^\circ \text{ V}$$

## Practice Problem 9.4

Express these sinusoids as phasors:

(a) $v = -14\sin(5t - 22^\circ)$ V
(b) $i = -8\cos(16t + 15^\circ)$ A

**Answer:** (a) $\mathbf{V} = 14\angle 68^\circ$ V, (b) $\mathbf{I} = 8\angle -165^\circ$ A.

## Example 9.5

Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A
(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

**Solution:**

(a) $\mathbf{I} = -3 + j4 = 5\angle 126.87^\circ$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ)\ \text{A}$$

(b) Because $j = 1\angle 90^\circ$,

$$\mathbf{V} = j8\angle -20^\circ = (1\angle 90^\circ)(8\angle -20^\circ)$$
$$= 8\angle 90^\circ - 20^\circ = 8\angle 70^\circ\ \text{V}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ)\text{V}$$

## Practice Problem 9.5

Find the sinusoids corresponding to these phasors:

(a) $\mathbf{V} = -25\angle 40^\circ$ V
(b) $\mathbf{I} = j(12 - j5)$ A

**Answer:** (a) $v(t) = 25\cos(\omega t - 140^\circ)$ V or $25\cos(\omega t + 220^\circ)$ V,
(b) $i(t) = 13\cos(\omega t + 67.38^\circ)$ A.

## Example 9.6

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ A and $i_2(t) = 5\sin(\omega t - 20^\circ)$ A, find their sum.

**Solution:**

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\angle 30^\circ$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is

$$\mathbf{I}_2 = 5\angle -110^\circ$$

If we let $i = i_1 + i_2$, then

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 4\angle 30^\circ + 5\angle -110^\circ$$
$$= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698$$
$$= 3.218\angle -56.97^\circ\ \text{A}$$

Transforming this to the time domain, we get

$$i(t) = 3.218\cos(\omega t - 56.97^\circ)\ \text{A}$$

Of course, we can find $i_1 + i_2$ using Eq. (9.9), but that is the hard way.

## Practice Problem 9.6

If $v_1 = -10\sin(\omega t - 30^\circ)$ V and $v_2 = 20\cos(\omega t + 45^\circ)$ V, find $v = v_1 + v_2$.

**Answer:** $v(t) = 29.77\cos(\omega t + 49.98^\circ)$ V.

## Example 9.7

Using the phasor approach, determine the current $i(t)$ in a circuit described by the integrodifferential equation

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

**Solution:**
We transform each term in the equation from time domain to phasor domain. Keeping Eqs. (9.27) and (9.28) in mind, we obtain the phasor form of the given equation as

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\angle 75^\circ$$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\angle 75^\circ$$

$$\mathbf{I} = \frac{50\angle 75^\circ}{4 - j10} = \frac{50\angle 75^\circ}{10.77\angle -68.2^\circ} = 4.642\angle 143.2^\circ\ \text{A}$$

Converting this to the time domain,

$$i(t) = 4.642\cos(2t + 143.2^\circ)\ \text{A}$$

Keep in mind that this is only the steady-state solution, and it does not require knowing the initial values.

## Practice Problem 9.7

Find the voltage $v(t)$ in a circuit described by the integrodifferential equation

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 50\cos(5t - 30^\circ)$$

using the phasor approach.

**Answer:** $v(t) = 5.3\cos(5t - 88^\circ)$ V.

## 9.4 Phasor Relationships for Circuit Elements

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements $R$, $L$, and $C$. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element. Again, we will assume the passive sign convention.

We begin with the resistor. If the current through a resistor $R$ is $i = I_m \cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \phi) \tag{9.29}$$

The phasor form of this voltage is

$$\mathbf{V} = RI_m \angle \phi \tag{9.30}$$

But the phasor representation of the current is $\mathbf{I} = I_m \angle \phi$. Hence,

$$\mathbf{V} = R\mathbf{I} \tag{9.31}$$

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 9.9 illustrates the voltage-current relations of a resistor. We should note from Eq. (9.31) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 9.10.

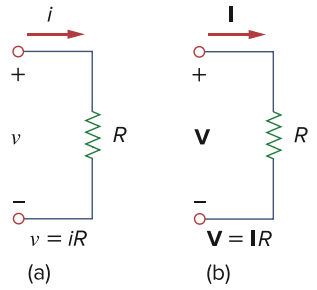



**Figure 9.9**
Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

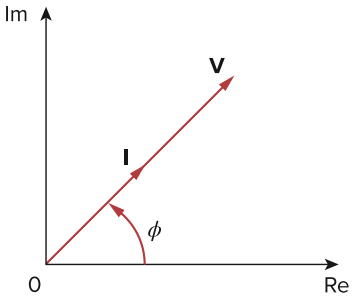



**Figure 9.10**
Phasor diagram for the resistor.

For the inductor $L$, assume the current through it is $i = I_m \cos(\omega t + \phi)$. The voltage across the inductor is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \tag{9.32}$$

Recall from Eq. (9.10) that $-\sin A = \cos(A + 90°)$. We can write the voltage as

$$v = \omega L I_m \cos(\omega t + \phi + 90°) \tag{9.33}$$

which transforms to the phasor

$$\mathbf{V} = \omega L I_m e^{j(\phi + 90°)} = \omega L I_m e^{j\phi} e^{j90°} = \omega L I_m \angle \phi + 90° \tag{9.34}$$

But $I_m \angle \phi = \mathbf{I}$, and from Eq. (9.19), $e^{j90°} = j$. Thus,

$$\mathbf{V} = j\omega L\mathbf{I} \tag{9.35}$$

showing that the voltage has a magnitude of $\omega L I_m$ and a phase of $\phi + 90°$. The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°. Figure 9.11 shows the voltage-current relations for the inductor. Figure 9.12 shows the phasor diagram.

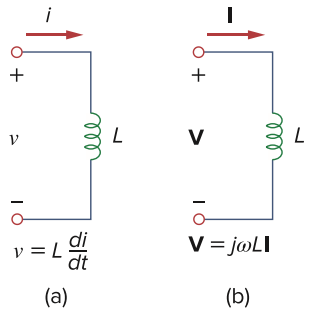



**Figure 9.11**
Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.

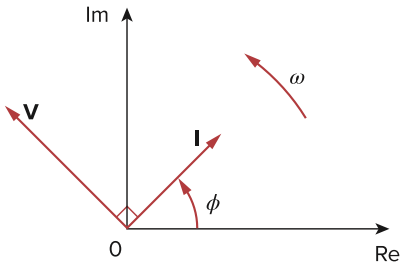



**Figure 9.12**
Phasor diagram for the inductor; **I** lags **V**.

For the capacitor $C$, assume the voltage across it is $v = V_m \cos(\omega t + \phi)$. The current through the capacitor is

$$i = C\frac{dv}{dt} \tag{9.36}$$

By following the same steps as we took for the inductor or by applying Eq. (9.27) on Eq. (9.36), we obtain

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.37}$$

Although it is equally correct to say that the inductor voltage leads the current by 90°, convention gives the current phase relative to the voltage.

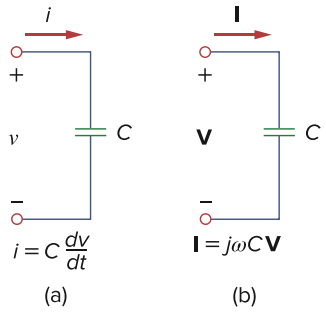



**Figure 9.13**
Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

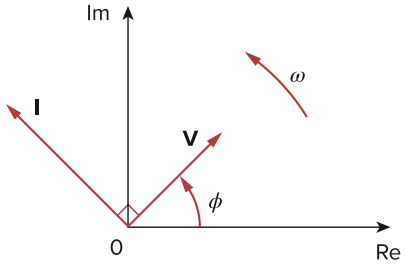



**Figure 9.14**
Phasor diagram for the capacitor; **I** leads **V**.

showing that the current and voltage are 90° out of phase. To be specific, the current leads the voltage by 90°. Figure 9.13 shows the voltage-current relations for the capacitor; Fig. 9.14 gives the phasor diagram. Table 9.2 summarizes the time domain and phasor domain representations of the circuit elements.

**TABLE 9.2**

Summary of voltage-current relationships.

| Element | Time domain | Frequency domain |
|---|---|---|
| $R$ | $v = Ri$ | $\mathbf{V} = R\mathbf{I}$ |
| $L$ | $v = L\frac{di}{dt}$ | $\mathbf{V} = j\omega L\mathbf{I}$ |
| $C$ | $i = C\frac{dv}{dt}$ | $\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$ |

## Example 9.8

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

**Solution:**
For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\angle 45°$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\angle 45°}{j60 \times 0.1} = \frac{12\angle 45°}{6\angle 90°} = 2\angle -45° \text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°) \text{ A}$$

## Practice Problem 9.8

If voltage $v = 25\sin(100t - 15°)$ V is applied to a 50$\mu$F capacitor, calculate the current through the capacitor.

**Answer:** $125\sin(100t + 75°)$ mA.

## 9.5 Impedance and Admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.38}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \tag{9.39}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{ZI}} \tag{9.40}$$

where $\mathbf{Z}$ is a frequency-dependent quantity known as *impedance*, measured in ohms.

The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage $\mathbf{V}$ to the phasor current $\mathbf{I}$, measured in ohms ($\Omega$).

The impedance represents the opposition that the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

The impedances of resistors, inductors, and capacitors can be readily obtained from Eq. (9.39). Table 9.3 summarizes their impedances. From the table we notice that $\mathbf{Z}_L = j\omega L$ and $\mathbf{Z}_C = -j/\omega C$. Consider two extreme cases of angular frequency. When $\omega = 0$ (i.e., for dc sources), $\mathbf{Z}_L = 0$ and $\mathbf{Z}_C \to \infty$, confirming what we already know—that the inductor acts like a short circuit, while the capacitor acts like an open circuit. When $\omega \to \infty$ (i.e., for high frequencies), $\mathbf{Z}_L \to \infty$ and $\mathbf{Z}_C = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit. Figure 9.15 illustrates this.

As a complex quantity, the impedence may be expressed in rectangular form as

$$\mathbf{Z} = R \pm jX \tag{9.41}$$

where $R = \text{Re}\,\mathbf{Z}$ is the *resistance* and $X = \text{Im}\,\mathbf{Z}$ is the *reactance*. The reactance, $X$, is just a magnitude, a positive value, but when used as a vector, a $j$ is associated with inductance and a $-j$ is associated with capacitance. Thus, impedance $\mathbf{Z} = R + jX$ is said to be *inductive* or lagging since current lags voltage, while impedance $\mathbf{Z} = R - jX$ is capacitive or leading because current leads voltage. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as

$$\mathbf{Z} = |\mathbf{Z}| \angle\theta \tag{9.42}$$

**TABLE 9.3**

Impedances and admittances of passive elements.

| Element | Impedance | Admittance |
|---|---|---|
| $R$ | $\mathbf{Z} = R$ | $\mathbf{Y} = \frac{1}{R}$ |
| $L$ | $\mathbf{Z} = j\omega L$ | $\mathbf{Y} = \frac{1}{j\omega L}$ |
| $C$ | $\mathbf{Z} = \frac{1}{j\omega C}$ | $\mathbf{Y} = j\omega C$ |

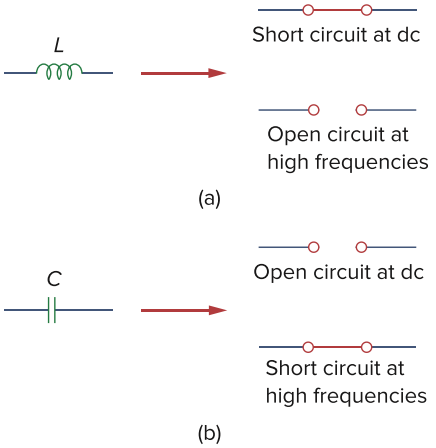



**Figure 9.15**
Equivalent circuits at dc and high frequencies: (a) inductor, (b) capacitor.

Comparing Eqs. (9.41) and (9.42), we infer that

$$\mathbf{Z} = R \pm jX = |\mathbf{Z}|\angle\theta \tag{9.43}$$

where

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{\pm X}{R} \tag{9.44}$$

and

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta \tag{9.45}$$

It is sometimes convenient to work with the reciprocal of impedance, known as *admittance*.

The **admittance** $\mathbf{Y}$ is the reciprocal of impedance, measured in siemens (S).

The admittance $\mathbf{Y}$ of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} \tag{9.46}$$

The admittances of resistors, inductors, and capacitors can be obtained from Eq. (9.39). They are also summarized in Table 9.3.

As a complex quantity, we may write $\mathbf{Y}$ as

$$\mathbf{Y} = G + jB \tag{9.47}$$

where $G = \text{Re}\,\mathbf{Y}$ is called the *conductance* and $B = \text{Im}\,\mathbf{Y}$ is called the *susceptance*. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). From Eqs. (9.41) and (9.47),

$$G + jB = \frac{1}{R + jX} \tag{9.48}$$

By rationalization,

$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} \tag{9.49}$$

Equating the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2} \tag{9.50}$$

showing that $G \neq 1/R$ as it is in resistive circuits. Of course, if $X = 0$, then $G = 1/R$.

## Example 9.9

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

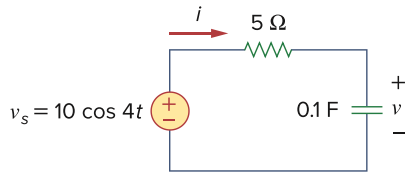



**Figure 9.16**
For Example 9.9.

**Solution:**
From the voltage source $10\cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \tag{9.9.1}$$
$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1} \tag{9.9.2}$$
$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V}$$

Converting $\mathbf{I}$ and $\mathbf{V}$ in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789\cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47\cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

## Practice Problem 9.9

Refer to Fig. 9.17. Determine $v(t)$ and $i(t)$.

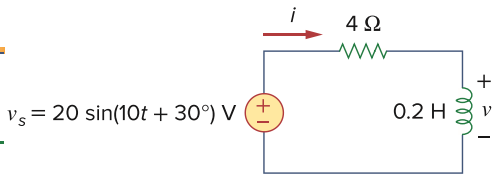



**Figure 9.17**
For Practice Prob. 9.9.

**Answer:** $8.944\sin(10t + 93.43^\circ)$ V, $4.472\sin(10t + 3.43^\circ)$ A.

## 9.6 †Kirchhoff's Laws in the Frequency Domain

We cannot do circuit analysis in the frequenc y domain without Kirch - hoff's current and v oltage laws. Therefore, we need to e xpress them in the frequency domain.

For KVL, let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop. Then

$$v_1 + v_2 + \cdots + v_n = 0 \tag{9.51}$$

In the sinusoidal steady state, each v oltage may be written in cosine form, so that Eq. (9.51) becomes

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0 \tag{9.52}$$

This can be written as

$$\text{Re}(V_{m1}e^{j\theta_1}e^{j\omega t}) + \text{Re}(V_{m2}e^{j\theta_2}e^{j\omega t}) + \cdots + \text{Re}(V_{mn}e^{j\theta_n}e^{j\omega t}) = 0$$

or

$$\text{Re}[(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \cdots + V_{mn}e^{j\theta_n})e^{j\omega t}] = 0 \tag{9.53}$$

If we let $\mathbf{V}_k = V_{mk}e^{j\theta_k}$, then

$$\text{Re}[(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n)e^{j\omega t}] = 0 \tag{9.54}$$

Because $e^{j\omega t} \neq 0$,

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0 \tag{9.55}$$

indicating that Kirchhoff's voltage law holds for phasors.

By following a similar procedure, we can show that Kirchhoff's current law holds for phasors. If we let $i_1, i_2, \ldots, i_n$ be the current leaving or entering a closed surface in a network at time $t$, then

$$i_1 + i_2 + \cdots + i_n = 0 \tag{9.56}$$

If $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ are the phasor forms of the sinusoids $i_1, i_2, \ldots, i_n$, then

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0 \tag{9.57}$$

which is Kirchhoff's current law in the frequency domain.

Once we have shown that both KVL and KCL hold in the frequency domain, it is easy to do many things, such as impedance combination, nodal and mesh analyses, superposition, and source transformation.

## 9.7 Impedance Combinations

Consider the $N$ series-connected impedances shown in Fig. 9.18. The same current $\mathbf{I}$ flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N) \tag{9.58}$$

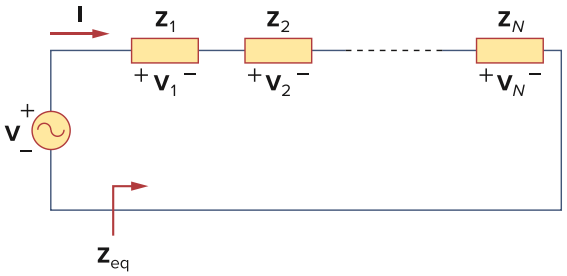



**Figure 9.18**
$N$ impedances in series.

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{\text{eq}} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

or

$$\boxed{\mathbf{Z}_{\text{eq}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N} \tag{9.59}$$

showing that the total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances.

If $N = 2$, as shown in Fig. 9.19, the current through the impedances is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2} \tag{9.60}$$

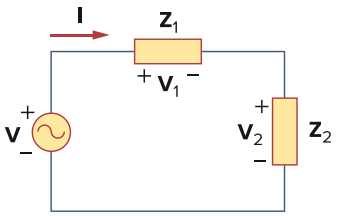



**Figure 9.19**
Voltage division.

Because $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$, then

$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}} \tag{9.61}$$

which is the *voltage-division* relationship.

In the same manner, we can obtain the equivalent impedance or admittance of the $N$ parallel-connected impedances shown in Fig. 9.20. The voltage across each impedance is the same. Applying KCL at the top node,

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right) \tag{9.62}$$

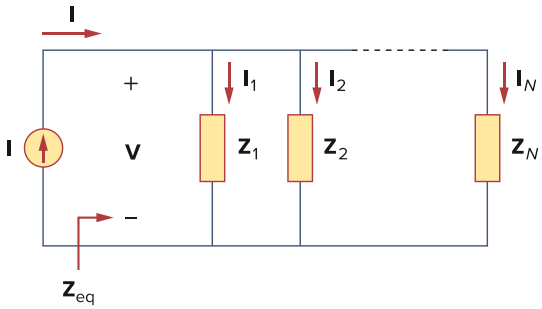



**Figure 9.20**
$N$ impedances in parallel.

The equivalent impedance is

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N} \tag{9.63}$$

and the equivalent admittance is

$$\boxed{\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N} \tag{9.64}$$

This indicates that the equivalent admittance of a parallel connection of admittances is the sum of the individual admittances.

When $N = 2$, as shown in Fig. 9.21, the equivalent impedance becomes

$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \tag{9.65}$$

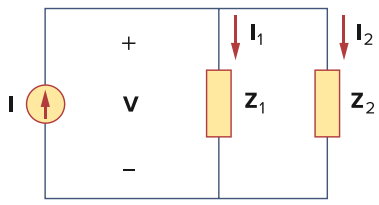



**Figure 9.21**
Current division.

Also, since

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

the currents in the impedances are

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \tag{9.66}$$

which is the *current-division* principle.

The delta-to-wye and wye-to-delta transformations that we applied to resisti ve circuits are also v alid for impedances. With reference to Fig. 9.22, the conversion formulas are as follows.

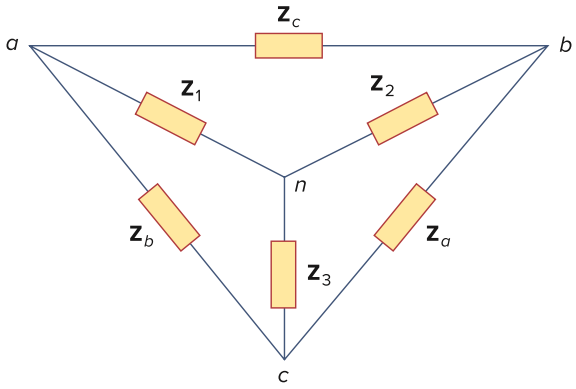



**Figure 9.22**
Superimposed *Y* and Δ networks.

*Y-Δ Conversion:*

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2} \tag{9.67}$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-*Y Conversion:*

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c} \tag{9.68}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

A delta or wye circuit is said to be **balanced** if it has equal impedances in all three branches.

When a $\Delta$-$Y$ circuit is balanced, Eqs. (9.67) and (9.68) become

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \quad \text{or} \quad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta \tag{9.69}$$

where $\mathbf{Z}_Y = \mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3$ and $\mathbf{Z}_\Delta = \mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c$.

As you see in this section, the principles of voltage division, current division, circuit reduction, impedance equivalence, and $Y$-$\Delta$ transformation all apply to ac circuits. Chapter 10 will show that other circuit techniques—such as superposition, nodal analysis, mesh analysis, source transformation, the Thevenin theorem, and the Norton theorem—are all applied to ac circuits in a manner similar to their application in dc circuits.

## Example 9.10

Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at $\omega = 50$ rad/s.

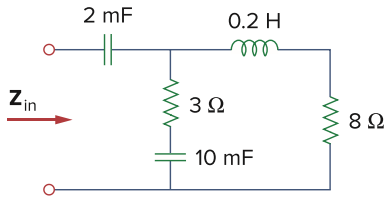



**Figure 9.23**
For Example 9.10.

**Solution:**
Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor
$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor
$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

## Practice Problem 9.10

Determine the input impedance of the circuit in Fig. 9.24 at $\omega = 20$ rad/s.

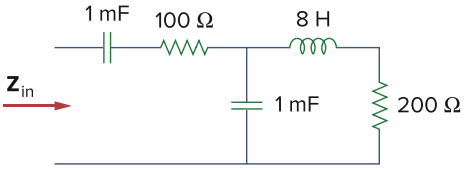



**Figure 9.24**
For Practice Prob. 9.10.

**Answer:** $(109.53 - j105.29)\ \Omega$.

## Example 9.11

Determine $v_o(t)$ in the circuit of Fig. 9.25.

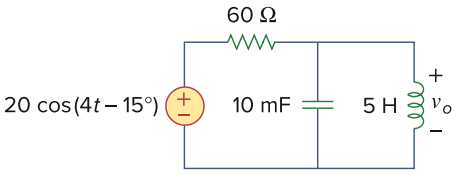



**Figure 9.25**
For Example 9.11.

**Solution:**
To do the analysis in the frequency domain, we must first transform the time domain circuit in Fig. 9.25 to the phasor domain equivalent in Fig. 9.26. The transformation produces

$$v_s = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad \mathbf{V}_s = 20\angle -15^\circ \text{ V}, \qquad \omega = 4$$

$$10 \text{ mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}}$$

$$= -j25\ \Omega$$

$$5 \text{ H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

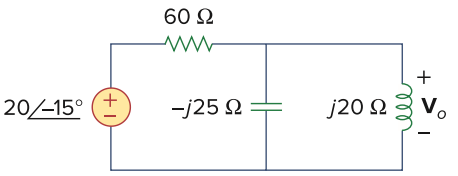



**Figure 9.26**
The frequency domain equivalent of the circuit in Fig. 9.25.

Let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\angle -15^\circ)$$

$$= (0.8575\angle 30.96^\circ)(20\angle -15^\circ) = 17.15\angle 15.96^\circ \text{ V}$$

We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ) \text{ V}$$

## Practice Problem 9.11

Calculate $v_o$ in the circuit of Fig. 9.27.

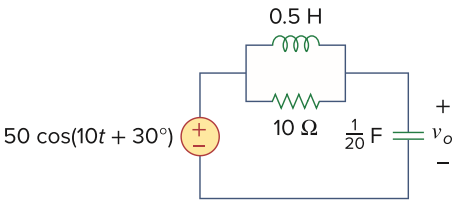



**Figure 9.27**
For Practice Prob. 9.11.

**Answer:** $v_o(t) = 35.36\cos(10t - 105^\circ)$ V.

## Example 9.12

Find current **I** in the circuit of Fig. 9.28.

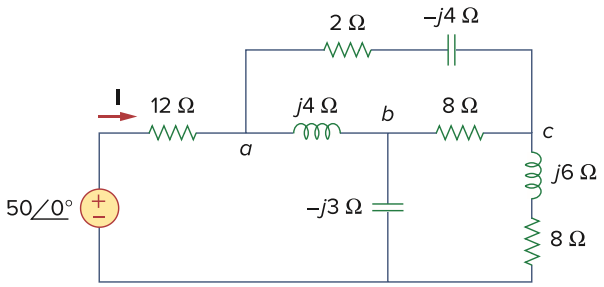



**Figure 9.28**
For Example 9.12.

**Solution:**
The delta netw ork connected to nodes $a$, $b$, and $c$ can be con verted to the $Y$ network of Fig. 9.29. We obtain the $Y$ impedances as follows using Eq. (9.68):

$$\mathbf{Z}_{an} = \frac{j4(2 - j4)}{j4 + 2 - j4 + 8} = \frac{4(4 + j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2 - j4)}{10} = (1.6 - j3.2)\ \Omega$$

The total impedance at the source terminals is

$$\begin{aligned}\mathbf{Z} &= 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \parallel (\mathbf{Z}_{cn} + j6 + 8) \\ &= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8) \\ &= 13.6 + j0.8 + \frac{j0.2(9.6 + j2.8)}{9.6 + j3} \\ &= 13.6 + j1 = 13.64\angle 4.204^\circ\ \Omega\end{aligned}$$

The desired current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{50\angle 0^\circ}{13.64\angle 4.204^\circ} = 3.666\angle -4.204^\circ\ \text{A}$$

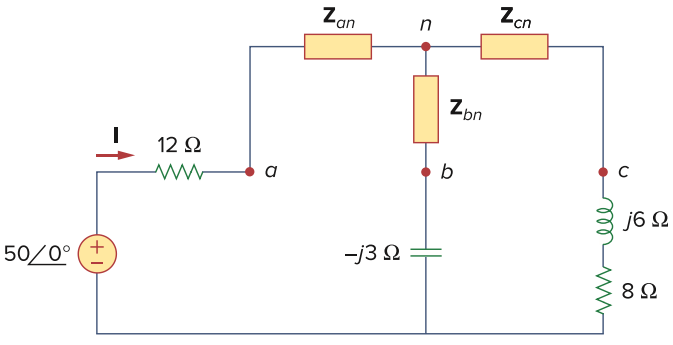



**Figure 9.29**
The circuit in Fig. 9.28 after delta-to-wye transformation.

## Practice Problem 9.12

Find **I** in the circuit of Fig. 9.30.

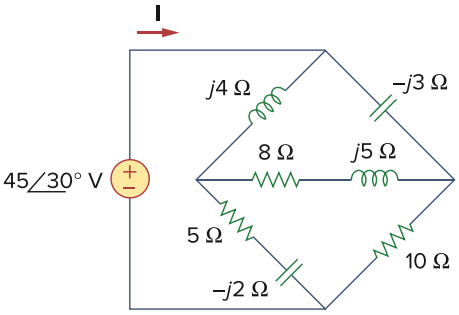



**Figure 9.30**
For Practice Prob. 9.12.

**Answer:** $12.728\angle 63.8°$ A.

## 9.8 †Applications

In Chapters 7 and 8, we sa w certain uses of *RC*, *RL*, and *RLC* circuits in dc applications. These circuits also have ac applications; among them are coupling circuits, phase-shifting circuits, filters, resonant circuits, ac bridge circuits, and transformers. This list of applications is ine xhaustive. We will consider some of them later. It will suffice here to observe two simple ones: *RC* phase-shifting circuits, and ac bridge circuits.

### 9.8.1 Phase-Shifters

A phase-shifting circuit is often emplo yed to correct an undesirable phase shift already present in a circuit or to produce special desired effects. An *RC* circuit is suitable for this purpose because its capacitor causes the circuit current to lead the applied v oltage. Two commonly used *RC* circuits are shown in Fig. 9.31. (*RL* circuits or any reactive circuits could also serve the same purpose.)

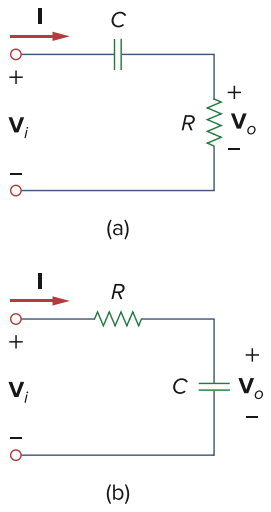



**Figure 9.31**
Series *RC* shift circuits: (a) leading output, (b) lagging output.

In Fig. 9.31(a), the circuit current **I** leads the applied v oltage $\mathbf{V}_i$ by some phase angle $\theta$, where $0 < \theta < 90°$, depending on the values of $R$ and $C$. If $X_C = -1/\omega C$, then the total impedance is $\mathbf{Z} = R + jX_C$, and the phase shift is given by

$$\theta = \tan^{-1}\frac{X_C}{R} \tag{9.70}$$

This shows that the amount of phase shift depends on the v alues of $R$, $C$, and the operating frequenc y. Since the output v oltage $\mathbf{V}_o$ across the resistor is in phase with the current, $\mathbf{V}_o$ leads (positive phase shift) $\mathbf{V}_i$ as shown in Fig. 9.32(a).

In Fig. 9.31(b), the output is taken across the capacitor. The current **I** leads the input v oltage $\mathbf{V}_i$ by $\theta$, but the output v oltage $v_o(t)$ across the capacitor lags (negative phase shift) the input v oltage $v_i(t)$ as illustrated in Fig. 9.32(b).

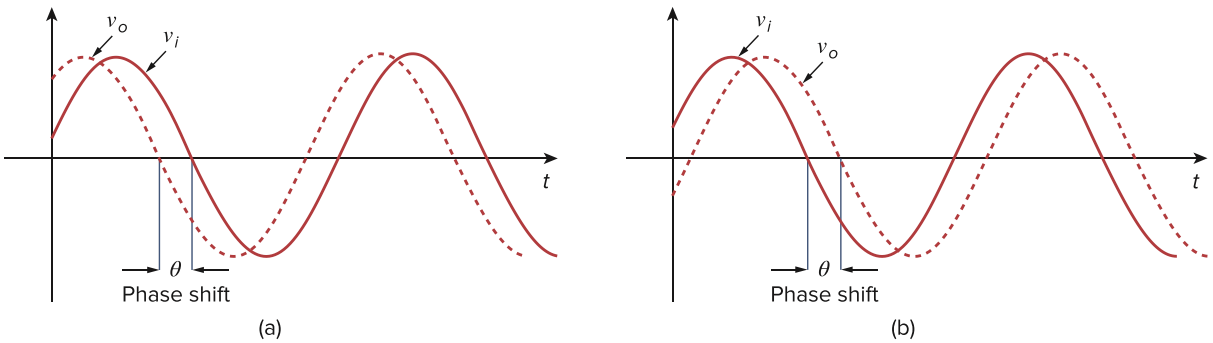



**Figure 9.32**
Phase shift in *RC* circuits: (a) leading output, (b) lagging output.

We should keep in mind that the simple *RC* circuits in Fig. 9.31 also act as voltage dividers. Therefore, as the phase shift $\theta$ approaches 90°, the output voltage $\mathbf{V}_o$ approaches zero. For this reason, these simple *RC* circuits are used only when small amounts of phase shift are required. If it is desired to have phase shifts greater than 60°, simple *RC* networks are cascaded, thereby providing a total phase shift equal to the sum of the individual phase shifts. In practice, the phase shifts due to the stages are not equal, because the succeeding stages load down the earlier stages unless op amps are used to separate the stages.

## Example 9.13

Design an *RC* circuit to provide a phase of 90° leading.

**Solution:**
If we select circuit components of equal ohmic value, say $R = |X_C| = 20\ \Omega$, at a particular frequency, according to Eq. (9.70), the phase shift is exactly 45°. By cascading two similar *RC* circuits in Fig. 9.31(a), we obtain the circuit in Fig. 9.33, providing a positive or leading phase shift of 90°, as we shall soon show. Using the series-parallel combination technique, $\mathbf{Z}$ in Fig. 9.33 is obtained as

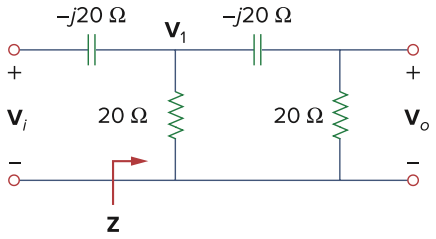



**Figure 9.33**
An *RC* phase shift circuit with 90° leading phase shift; for Example 9.13.

$$\mathbf{Z} = 20 \parallel (20 - j20) = \frac{20(20 - j20)}{40 - j20} = 12 - j4\ \Omega \qquad \textbf{(9.13.1)}$$

Using voltage division,

$$\mathbf{V}_1 = \frac{\mathbf{Z}}{\mathbf{Z} - j20}\mathbf{V}_i = \frac{12 - j4}{12 - j24}\mathbf{V}_i = \frac{\sqrt{2}}{3}\underline{/45^\circ}\ \mathbf{V}_i \qquad \textbf{(9.13.2)}$$

and

$$\mathbf{V}_o = \frac{20}{20 - j20}\mathbf{V}_1 = \frac{\sqrt{2}}{2}\underline{/45^\circ}\ \mathbf{V}_1 \qquad \textbf{(9.13.3)}$$

Substituting Eq. (9.13.2) into Eq. (9.13.3) yields

$$\mathbf{V}_o = \left(\frac{\sqrt{2}}{2}\underline{/45^\circ}\right)\left(\frac{\sqrt{2}}{3}\underline{/45^\circ}\ \mathbf{V}_i\right) = \frac{1}{3}\underline{/90^\circ}\ \mathbf{V}_i$$

Thus, the output leads the input by 90° but its magnitude is only about 33 percent of the input.

## Practice Problem 9.13

Design an *RC* circuit to provide a 90° lagging phase shift of the output voltage relative to the input voltage. If an ac voltage of 60 V rms is applied, what is the output voltage?

**Answer:** Figure 9.34 shows a typical design; 20 V rms.

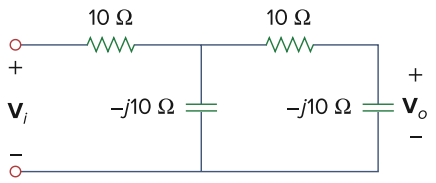



**Figure 9.34**
For Practice Prob. 9.13.

## Example 9.14

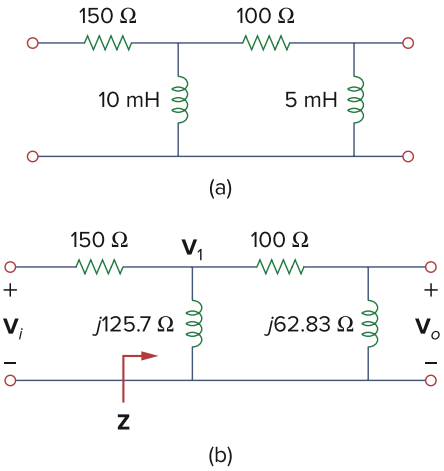



**Figure 9.35**
For Example 9.14.

For the *RL* circuit shown in Fig. 9.35(a), calculate the amount of phase shift produced at 2 kHz.

**Solution:**
At 2 kHz, we transform the 10- and 5-mH inductances to the corresponding impedances.

$$10 \text{ mH} \quad \Rightarrow \quad X_L = \omega L = 2\pi \times 2 \times 10^3 \times 10 \times 10^{-3} = 40\pi = 125.7\ \Omega$$

$$5 \text{ mH} \quad \Rightarrow \quad X_L = \omega L = 2\pi \times 2 \times 10^3 \times 5 \times 10^{-3} = 20\pi = 62.83\ \Omega$$

Consider the circuit in Fig. 9.35(b).The impedance $\mathbf{Z}$ is the parallel combination of $j125.7\ \Omega$ and $100 + j62.83\ \Omega$. Hence,

$$\mathbf{Z} = j125.7 \parallel (100 + j62.83) = \frac{j125.7(100 + j62.83)}{100 + j188.5} = 69.56 \angle 60.1^\circ\ \Omega \tag{9.14.1}$$

Using voltage division,

$$\mathbf{V}_1 = \frac{\mathbf{Z}}{\mathbf{Z} + 150}\mathbf{V}_i = \frac{69.56 \angle 60.1^\circ}{184.7 + j60.3}\mathbf{V}_i = 0.3582 \angle 42.02^\circ\ \mathbf{V}_i \tag{9.14.2}$$

and

$$\mathbf{V}_o = \frac{j62.832}{100 + j62.832}\mathbf{V}_1 = 0.532 \angle 57.86^\circ\ \mathbf{V}_1 \tag{9.14.3}$$

Combining Eqs. (9.14.2) and (9.14.3),

$$\mathbf{V}_o = (0.532 \angle 57.86^\circ)(0.3582 \angle 42.02^\circ)\mathbf{V}_i = 0.1906 \angle 100^\circ\ \mathbf{V}_i$$

showing that the output is about 19 percent of the input in magnitude but leading the input by 100°. If the circuit is terminated by a load, the load will affect the phase shift.

## Practice Problem 9.14

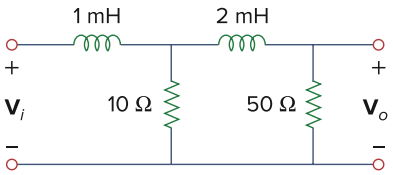



**Figure 9.36**
For Practice Prob. 9.14.

Refer to the *RL* circuit in Fig. 9.36. If 10 V is applied to the input, find the magnitude and the phase shift produced at 5 kHz. Specify whether the phase shift is leading or lagging.

**Answer:** 1.7161 V, 120.39°, lagging.

### 9.8.2 AC Bridges

An ac bridge circuit is used in measuring the inductance $L$ of an inductor or the capacitance $C$ of a capacitor. It is similar in form to the Wheatstone bridge for measuring an unknown resistance (discussed in Section 4.10) and follows the same principle. To measure $L$ and $C$, however, an ac

source is needed as well as an ac meter instead of the galvanometer. The ac meter may be a sensitive ac ammeter or voltmeter.

Consider the general ac bridge circuit displayed in Fig. 9.37. The bridge is *balanced* when no current flows through the meter. This means that $\mathbf{V}_1 = \mathbf{V}_2$. Applying the voltage division principle,

$$\mathbf{V}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \mathbf{V}_2 = \frac{\mathbf{Z}_x}{\mathbf{Z}_3 + \mathbf{Z}_x}\mathbf{V}_s \tag{9.71}$$

Thus,

$$\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{\mathbf{Z}_x}{\mathbf{Z}_3 + \mathbf{Z}_x} \quad \Rightarrow \quad \mathbf{Z}_2\mathbf{Z}_3 = \mathbf{Z}_1\mathbf{Z}_x \tag{9.72}$$

or

$$\boxed{\mathbf{Z}_x = \frac{\mathbf{Z}_3}{\mathbf{Z}_1}\mathbf{Z}_2} \tag{9.73}$$

This is the balanced equation for the ac bridge and is similar to Eq. (4.30) for the resistance bridge except that the $R$'s are replaced by $\mathbf{Z}$'s.

Specific ac bridges for measuring $L$ and $C$ are shown in Fig. 9.38, where $L_x$ and $C_x$ are the unknown inductance and capacitance to be measured while $L_s$ and $C_s$ are a standard inductance and capacitance (the values of which are known to great precision). In each case, two resistors, $R_1$ and $R_2$, are varied until the ac meter reads zero. Then the bridge is balanced. From Eq. (9.73), we obtain

$$L_x = \frac{R_2}{R_1}L_s \tag{9.74}$$

and

$$C_x = \frac{R_1}{R_2}C_s \tag{9.75}$$

Notice that the balancing of the ac bridges in Fig. 9.38 does not depend on the frequency $f$ of the ac source, since $f$ does not appear in the relationships in Eqs. (9.74) and (9.75).

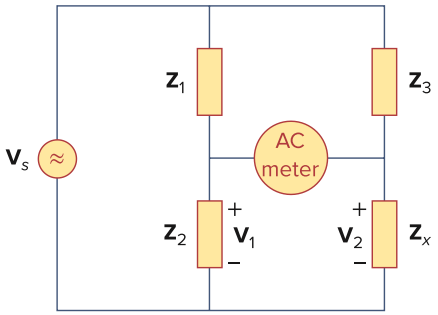
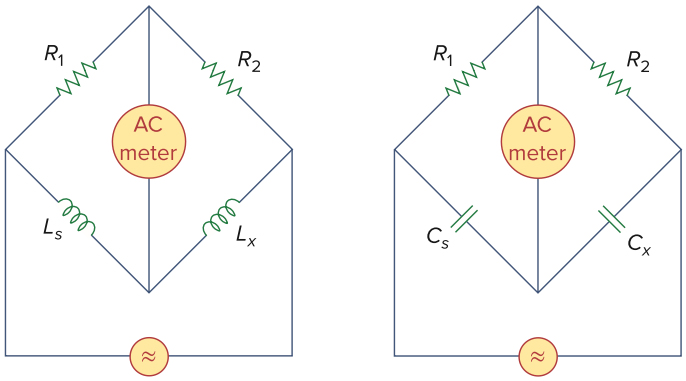



**Figure 9.37**
A general ac bridge.




(a) (b)

**Figure 9.38**
Specific ac bridges: (a) for measuring $L$, (b) for measuring $C$.

## Example 9.15

The ac bridge circuit of Fig. 9.37 balances when $\mathbf{Z}_1$ is a 1-kΩ resistor, $\mathbf{Z}_2$ is a 4.2-kΩ resistor, $\mathbf{Z}_3$ is a parallel combination of a 1.5-MΩ resistor and a 12-pF capacitor, and $f = 2$ kHz. Find: (a) the series components that make up $\mathbf{Z}_x$, and (b) the parallel components that make up $\mathbf{Z}_x$.

### Solution:

1. **Define.** The problem is clearly stated.
2. **Present.** We are to determine the unknown components subject to the fact that they balance the given quantities. Given that a parallel and series equivalent exists for this circuit, we need to find both.
3. **Alternative.** Although there are alternative techniques that can be used to find the unknown values, a straightforward equality works best. Once we have answers, we can check them by using hand techniques such as nodal analysis or just using *PSpice*.
4. **Attempt.** From Eq. (9.73),

$$\mathbf{Z}_x = \frac{\mathbf{Z}_3}{\mathbf{Z}_1}\mathbf{Z}_2 \tag{9.15.1}$$

where $\mathbf{Z}_x = R_x + jX_x$,

$$\mathbf{Z}_1 = 1000\ \Omega, \qquad \mathbf{Z}_2 = 4200\ \Omega \tag{9.15.2}$$

and

$$\mathbf{Z}_3 = R_3 \parallel \frac{1}{j\omega C_3} = \frac{\dfrac{R_3}{j\omega C_3}}{R_3 + 1/j\omega C_3} = \frac{R_3}{1 + j\omega R_3 C_3}$$

Since $R_3 = 1.5$ MΩ and $C_3 = 12$ pF,

$$\mathbf{Z}_3 = \frac{1.5 \times 10^6}{1 + j2\pi \times 2 \times 10^3 \times 1.5 \times 10^6 \times 12 \times 10^{-12}} = \frac{1.5 \times 10^6}{1 + j0.2262}$$

or

$$\mathbf{Z}_3 = 1.427 - j0.3228\ \text{MΩ} \tag{9.15.3}$$

(a) Assuming that $\mathbf{Z}_x$ is made up of series components, we substitute Eqs. (9.15.2) and (9.15.3) in Eq. (9.15.1) and obtain

$$\begin{aligned} R_x + jX_x &= \frac{4200}{1000}(1.427 - j0.3228) \times 10^6 \\ &= (5.993 - j1.356)\ \text{MΩ} \end{aligned} \tag{9.15.4}$$

Equating the real and imaginary parts yields $R_x = 5.993$ MΩ and a capacitive reactance

$$X_x = \frac{1}{\omega C} = 1.356 \times 10^6$$

or

$$C = \frac{1}{\omega X_x} = \frac{1}{2\pi \times 2 \times 10^3 \times 1.356 \times 10^6} = 58.69\ \text{pF}$$

(b) $Z_x$ remains the same as in Eq. (9.15.4) but $R_x$ and $X_x$ are in parallel. Assuming an *RC* parallel combination,

$$\mathbf{Z}_x = (5.993 - j1.356)\ \text{M}\Omega$$
$$= R_x \parallel \frac{1}{j\omega C_x} = \frac{R_x}{1 + j\omega R_x C_x}$$

By equating the real and imaginary parts, we obtain

$$R_x = \frac{\text{Real}(\mathbf{Z}_x)^2 + \text{Imag}(\mathbf{Z}_x)^2}{\text{Real}(\mathbf{Z}_x)} = \frac{5.993^2 + 1.356^2}{5.993} = \mathbf{6.3\ M\Omega}$$

$$C_x = -\frac{\text{Imag}(\mathbf{Z}_x)}{\omega[\text{Real}(\mathbf{Z}_x)^2 + \text{Imag}(\mathbf{Z}_x)^2]}$$

$$= -\frac{-1.356}{2\pi(2000)(5.917^2 + 1.356^2)} = \mathbf{2.852\ \mu F}$$

We have assumed a parallel *RC* combination which works in this case.

5. **Evaluate.** Let us now use *PSpice* to see if we indeed have the correct equalities. Running *PSpice* with the equivalent circuits, an open circuit between the "bridge" portion of the circuit, and a 10-volt input voltage yields the following voltages at the ends of the "bridge" relative to a reference at the bottom of the circuit:

```
FREQ          VM($N_0002)    VP($N_0002)
2.000E + 03   9.993E + 00   -8.634E - 03
2.000E + 03   9.993E + 00   -8.637E - 03
```

Because the voltages are essentially the same, then no measurable current can flow through the "bridge" portion of the circuit for any element that connects the two points together and we have a balanced bridge, which is to be expected. This indicates we have properly determined the unknowns.

There is a very important problem with what we have done! Do you know what that is? We have what can be called an ideal, "theoretical" answer, but one that really is not very good in the real world. The difference between the magnitudes of the upper impedances and the lower impedances is much too large and would never be accepted in a real bridge circuit. For greatest accuracy, the overall magnitude of the impedances must at least be within the same relative order. To increase the accuracy of the solution of this problem, I would recommend increasing the magnitude of the top impedances to be in the range of 500 kΩ to 1.5 MΩ. One additional real-world comment: The size of these impedances also creates serious problems in making actual measurements, so the appropriate instruments must be used in order to minimize their loading (which would change the actual voltage readings) on the circuit.

6. **Satisfactory?** Because we solved for the unknown terms and then tested to see if they worked, we validated the results. They can now be presented as a solution to the problem.

### Practice Problem 9.15

In the ac bridge circuit of Fig. 9.37, suppose that balance is achieved when $\mathbf{Z}_1$ is a 4.8-k$\Omega$ resistor, $\mathbf{Z}_2$ is a 10-$\Omega$ resistor in series with a 0.25-$\mu$H inductor, $\mathbf{Z}_3$ is a 12-k$\Omega$ resistor, and $f = 6$ MHz. Determine the series components that make up $\mathbf{Z}_x$.

**Answer:** A 25-$\Omega$ resistor in series with a 0.625-$\mu$H inductor.

## 9.9 Summary

1. A sinusoid is a signal in the form of the sine or cosine function. It has the general form

$$v(t) = V_m \cos(\omega t + \phi)$$

where $V_m$ is the amplitude, $\omega = 2\pi f$ is the angular frequency, $(\omega t + \phi)$ is the argument, and $\phi$ is the phase.

2. A phasor is a complex quantity that represents both the magnitude and the phase of a sinusoid. Given the sinusoid $v(t) = V_m \cos(\omega t + \phi)$, its phasor $\mathbf{V}$ is

$$\mathbf{V} = V_m \angle \phi$$

3. In ac circuits, voltage and current phasors always have a fixed relation to one another at any moment of time. If $v(t) = V_m \cos(\omega t + \phi_v)$ represents the voltage through an element and $i(t) = I_m \cos(\omega t + \phi_i)$ represents the current through the element, then $\phi_i = \phi_v$ if the element is a resistor, $\phi_i$ leads $\phi_v$ by 90° if the element is a capacitor, and $\phi_i$ lags $\phi_v$ by 90° if the element is an inductor.

4. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage across it to the phasor current through it:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R(\omega) + jX(\omega)$$

The admittance $\mathbf{Y}$ is the reciprocal of impedance:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = G(\omega) + jB(\omega)$$

Impedances are combined in series or in parallel the same way as resistances in series or parallel; that is, impedances in series add while admittances in parallel add.

5. For a resistor $\mathbf{Z} = R$, for an inductor $\mathbf{Z} = jX = j\omega L$, and for a capacitor $\mathbf{Z} = -jX = 1/j\omega C$.

6. Basic circuit laws (Ohm's and Kirchhoff's) apply to ac circuits in the same manner as they do for dc circuits; that is,

$$\mathbf{V} = \mathbf{ZI}$$

$$\Sigma \mathbf{I}_k = 0 \quad \text{(KCL)}$$

$$\Sigma \mathbf{V}_k = 0 \quad \text{(KVL)}$$

7. The techniques of voltage/current division, series/parallel combination of impedance/admittance, circuit reduction, and $Y$-$\Delta$ transformation all apply to ac circuit analysis.
8. AC circuits are applied in phase-shifters and bridges.

## Review Questions

**9.1** Which of the following is *not* a right way to express the sinusoid $A\cos\omega t$?

(a) $A\cos 2\pi ft$ (b) $A\cos(2\pi t/T)$
(c) $A\cos\omega(t-T)$ (d) $A\sin(\omega t-90°)$

**9.2** A function that repeats itself after fixed intervals is said to be:

(a) a phasor (b) harmonic
(c) periodic (d) reactive

**9.3** Which of these frequencies has the shorter period?

(a) 1 krad/s (b) 1 kHz

**9.4** If $v_1 = 30\sin(\omega t+10°)$ and $v_2 = 20\sin(\omega t+50°)$, which of these statements are true?

(a) $v_1$ leads $v_2$ (b) $v_2$ leads $v_1$
(c) $v_2$ lags $v_1$ (d) $v_1$ lags $v_2$
(e) $v_1$ and $v_2$ are in phase

**9.5** The voltage across an inductor leads the current through it by 90°.

(a) True (b) False

**9.6** The imaginary part of impedance is called:

(a) resistance (b) admittance
(c) susceptance (d) conductance
(e) reactance

**9.7** The impedance of a capacitor increases with increasing frequency.

(a) True (b) False

**9.8** At what frequency will the output voltage $v_o(t)$ in Fig. 9.39 be equal to the input voltage $v(t)$ ?

(a) 0 rad/s (b) 1 rad/s (c) 4 rad/s
(d) $\infty$ rad/s (e) none of the above

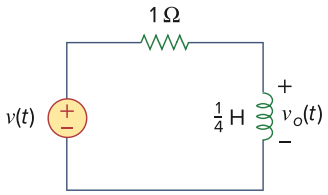



**Figure 9.39**
For Review Question 9.8.

**9.9** A series $RC$ circuit has $|V_R| = 12$ V and $|V_C| = 5$ V. The magnitude of the supply voltage is:

(a) −7 V (b) 7 V (c) 13 V (d) 17 V

**9.10** A series $RCL$ circuit has $R = 30\ \Omega$, $X_C = 50\ \Omega$, and $X_L = 90\ \Omega$. The impedance of the circuit is:

(a) $30 + j140\ \Omega$ (b) $30 + j40\ \Omega$
(c) $30 - j40\ \Omega$ (d) $-30 - j40\ \Omega$
(e) $-30 + j40\ \Omega$

*Answers: 9.1d, 9.2c, 9.3b, 9.4b,d, 9.5a, 9.6e, 9.7b, 9.8d, 9.9c, 9.10b.*

## Problems

### Section 9.2 Sinusoids

**9.1** Given the sinusoidal voltage $v(t) = 50\cos(30t+10°)$ V, find: (a) the amplitude $V_m$, (b) the period $T$, (c) the frequency $f$, and (d) $v(t)$ at $t = 10$ ms.

**9.2** A current source in a linear circuit has

$$i_s = 15\cos(25\pi t + 25°)\ \text{A}$$

(a) What is the amplitude of the current?
(b) What is the angular frequency?
(c) Find the frequency of the current.
(d) Calculate $i_s$ at $t = 2$ ms.

**9.3** Express the following functions in cosine form:

(a) $10\sin(\omega t+30°)$ (b) $-9\sin(8t)$
(c) $-20\sin(\omega t+45°)$

**9.4** Design a problem to help other students better understand sinusoids.

**9.5** Given $v_1 = 45\sin(\omega t + 30°)$ V and $v_2 = 50\cos(\omega t - 30°)$ V, determine the phase angle between the two sinusoids and which one lags the other.

**9.6** For the following pairs of sinusoids, determine which one leads and by how much.

(a) $v(t) = 10\cos(4t - 60°)$ and $i(t) = 4\sin(4t + 50°)$

(b) $v_1(t) = 4\cos(377t + 10°)$ and $v_2(t) = -20\cos 377t$

(c) $x(t) = 13\cos 2t + 5\sin 2t$ and $y(t) = 15\cos(2t - 11.8°)$

## Section 9.3 Phasors

**9.7** If $f(\phi) = \cos\phi + j\sin\phi$, show that $f(\phi) = e^{j\phi}$.

**9.8** Calculate these complex numbers and express your results in rectangular form:

(a) $\dfrac{60\angle 45°}{7.5 - j10} + j2$

(b) $\dfrac{32\angle -20°}{(6 - j8)(4 + j2)} + \dfrac{20}{-10 + j24}$

(c) $20 + (16\angle -50°)(5 + j12)$

**9.9** Evaluate the following complex numbers and leave your results in polar form:

(a) $5\angle 30° \left(6 - j8 + \dfrac{3\angle 60°}{2 + j}\right)$

(b) $\dfrac{(10\angle 60°)(35\angle -50°)}{(2 + j6) - (5 + j)}$

**9.10** Design a problem to help other students better understand phasors.

**9.11** Find the phasors corresponding to the following signals:

(a) $v(t) = 21\cos(4t - 15°)$V

(b) $i(t) = -8\sin(10t + 70°)$ mA

(c) $v(t) = 120\sin(10t - 50°)$ V

(d) $i(t) = -60\cos(30t + 10°)$ mA

**9.12** Let $\mathbf{X} = 4\angle 40°$ and $\mathbf{Y} = 20\angle -30°$. Evaluate the following quantities and express your results in polar form:

(a) $(\mathbf{X} + \mathbf{Y})\mathbf{X}^*$

(b) $(\mathbf{X} - \mathbf{Y})^*$

(c) $(\mathbf{X} + \mathbf{Y})/\mathbf{X}$

**9.13** Evaluate the following complex numbers:

(a) $\dfrac{2 + j3}{1 - j6} + \dfrac{7 - j8}{-5 + j11}$

(b) $\dfrac{(5\angle 10°)(10\angle -40°)}{(4\angle -80°)(-6\angle 50°)}$

(c) $\begin{vmatrix} 2 + j3 & -j2 \\ -j2 & 8 - j5 \end{vmatrix}$

**9.14** Simplify the following expressions:

(a) $\dfrac{(5 - j6) - (2 + j8)}{(-3 + j4)(5 - j) + (4 - j6)}$

(b) $\dfrac{(240\angle 75° + 160\angle -30°)(60 - j80)}{(67 + j84)(20\angle 32°)}$

(c) $\left(\dfrac{10 + j20}{3 + j4}\right)^2 \sqrt{(10 + j5)(16 - j20)}$

**9.15** Evaluate these determinants:

(a) $\begin{vmatrix} 10 + j6 & 2 - j3 \\ -5 & -1 + j \end{vmatrix}$

(b) $\begin{vmatrix} 20\angle -30° & -4\angle -10° \\ 16\angle 0° & 3\angle 40° \end{vmatrix}$

(c) $\begin{vmatrix} 1 - j & -j & 0 \\ j & 1 & -j \\ 1 & j & 1 + j \end{vmatrix}$

**9.16** Transform the following sinusoids to phasors:

(a) $-20\cos(4t + 135°)$ (b) $8\sin(20t + 30°)$

(c) $20\cos(2t) + 15\sin(2t)$

**9.17** Two voltages $v_1$ and $v_2$ appear in series so that their sum is $v = v_1 + v_2$. If $v_1 = 10\cos(50t - \pi/3)$ V and $v_2 = 12\cos(50t + 30°)$ V, find $v$.

**9.18** Obtain the sinusoids corresponding to each of the following phasors:

(a) $\mathbf{V}_1 = 60\angle 15°$ V, $\omega = 1$

(b) $\mathbf{V}_2 = 6 + j8$ V, $\omega = 40$

(c) $\mathbf{I}_1 = 2.8e^{-j\pi/3}$ A, $\omega = 377$

(d) $\mathbf{I}_2 = -0.5 - j1.2$ A, $\omega = 10^3$

**9.19** Using phasors, find:

(a) $3\cos(20t + 10°) - 5\cos(20t - 30°)$

(b) $40\sin 50t + 30\cos(50t - 45°)$

(c) $20\sin 400t + 10\cos(400t + 60°) - 5\sin(400t - 20°)$

**9.20** A linear network has a current input $7.5\cos(10t + 30°)$ A and a voltage output $120\cos(10t + 75°)$ V. Determine the associated impedance.

**9.21** Simplify the following:

(a) $f(t) = 5\cos(2t + 15°) - 4\sin(2t - 30°)$

(b) $g(t) = 8\sin t + 4\cos(t + 50°)$

(c) $h(t) = \int_0^t (10\cos 40t + 50\sin 40t)\, dt$

**9.22** An alternating voltage is given by $v(t) = 55\cos(5t + 45°)$ V. Use phasors to find

$$10v(t) + 4\frac{dv}{dt} - 2\int_{-\infty}^{t} v(t)\, dt$$

Assume that the value of the integral is zero at $t = -\infty$.

**9.23** Apply phasor analysis to evaluate the following:

(a) $v = [110\sin(20t + 30°) + 220\cos(20t - 90°)]$ V

(b) $i = [30\cos(5t + 60°) - 20\sin(5t + 60°)]$ A

**9.24** Find $v(t)$ in the following integrodifferential equations using the phasor approach:

(a) $v(t) + \int v\, dt = 10\cos t$

(b) $\frac{dv}{dt} + 5v(t) + 4\int v\, dt = 20\sin(4t + 10°)$

**9.25** Using phasors, determine $i(t)$ in the following equations:

(a) $2\frac{di}{dt} + 3i(t) = 4\cos(2t - 45°)$

(b) $10\int i\, dt + \frac{di}{dt} + 6i(t) = 5\cos(5t + 22°)$ A

**9.26** The loop equation for a series *RLC* circuit gives

$$\frac{di}{dt} + 2i + \int_{-\infty}^{t} i\, dt = \cos 2t \text{ A}$$

Assuming that the value of the integral at $t = -\infty$ is zero, find $i(t)$ using the phasor method.

**9.27** A parallel *RLC* circuit has the node equation

$$\frac{dv}{dt} + 50v + 100\int v\, dt = 110\cos(377t - 10°) \text{ V}$$

Determine $v(t)$ using the phasor method. You may assume that the value of the integral at $t = -\infty$ is zero.

## Section 9.4 Phasor Relationships for Circuit Elements

**9.28** Determine the current that flows through an 20-Ω resistor connected to a voltage source $v_s = 120\cos(377t + 37°)$ V.

**9.29** Given that $v_c(0) = 2\cos(155°)$ V, what is the instantaneous voltage across a 2-μF capacitor when the current through it is $i = 4\sin(10^6 t + 25°)$ A?

**9.30** A voltage $v(t) = 100\cos(60t + 20°)$ V is applied to a parallel combination of a 40-kΩ resistor and a 50-μF capacitor. Find the steady-state currents through the resistor and the capacitor.

**9.31** A series *RLC* circuit has $R = 80\ \Omega$, $L = 240$ mH, and $C = 5$ mF. If the input voltage is $v(t) = 115\cos 2t$, find the current flowing through the circuit.

**9.32** Using Fig. 9.40, design a problem to help other students better understand phasor relationships for circuit elements.

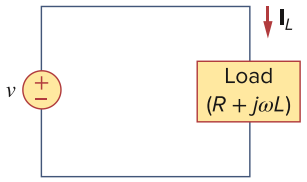



**Figure 9.40**
For Prob. 9.32.

**9.33** A series *RL* circuit is connected to a 220-V ac source. If the voltage across the resistor is 170 V, find the voltage across the inductor.

**9.34** What value of $\omega$ will cause the forced response, $v_o$, in Fig. 9.41 to be zero?

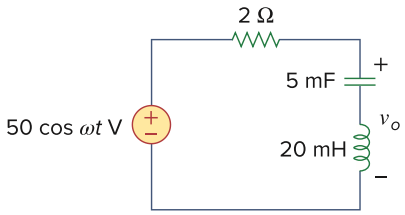



**Figure 9.41**
For Prob. 9.34.

## Section 9.5 Impedance and Admittance

**9.35** Find the steady-state current $i$ in the circuit of Fig. 9.42, when $v_s(t) = 115\cos 200t$ V.

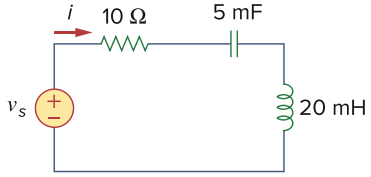



**Figure 9.42**
For Prob. 9.35.

**9.36** Using Fig. 9.43, design a problem to help other students better understand impedance.

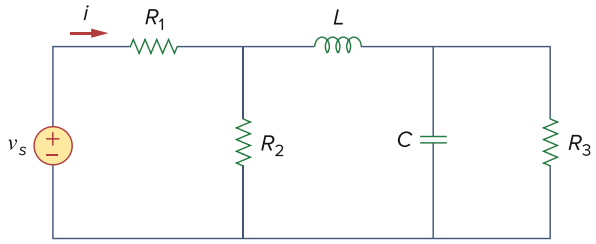



**Figure 9.43**
For Prob. 9.36.

**9.37** Determine the admittance **Y** for the circuit in Fig. 9.44.

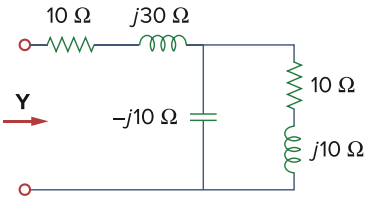



**Figure 9.44**
For Prob. 9.37.

**9.38** Using Fig. 9.45, design a problem to help other students better understand admittance.

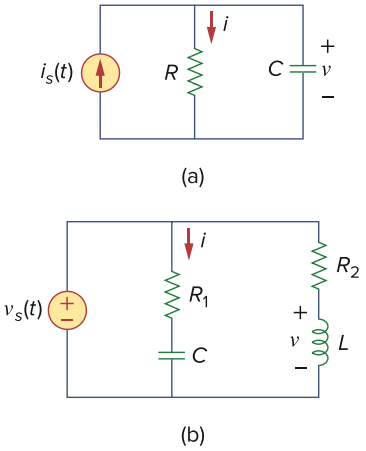



**Figure 9.45**
For Prob. 9.38.

**9.39** For the circuit shown in Fig. 9.46, find $Z_{eq}$ and use that to find current **I**. Let $\omega = 10$ rad/s.

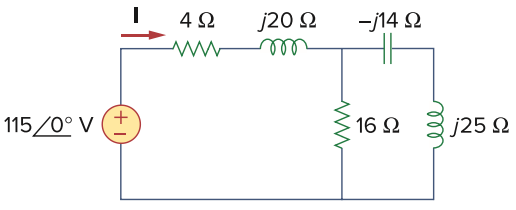



**Figure 9.46**
For Prob. 9.39.

**9.40** In the circuit of Fig. 9.47, find $i_o$ when:

(a) $\omega = 1$ rad/s (b) $\omega = 5$ rad/s
(c) $\omega = 10$ rad/s

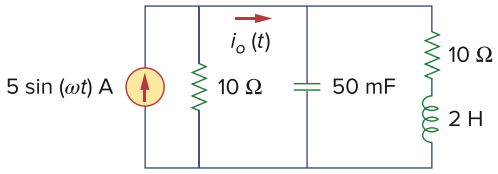



**Figure 9.47**
For Prob. 9.40.

**9.41** Find $v(t)$ in the *RLC* circuit of Fig. 9.48.

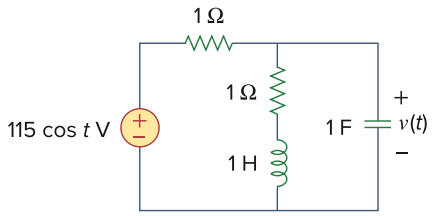



**Figure 9.48**
For Prob. 9.41.

**9.42** Calculate $v_o(t)$ in the circuit of Fig. 9.49.

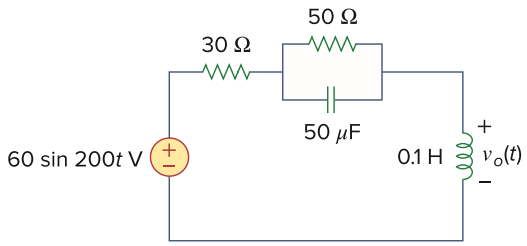



**Figure 9.49**
For Prob. 9.42.

**9.43** Find current $\mathbf{I}_o$ in the circuit shown in Fig. 9.50.

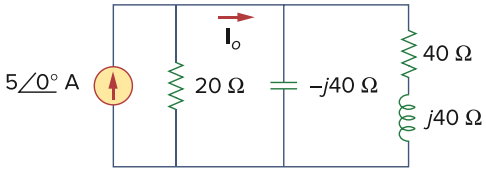



**Figure 9.50**
For Prob. 9.43.

**9.44** Calculate $i(t)$ in the circuit of Fig. 9.51.

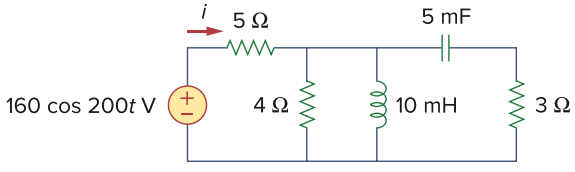



**Figure 9.51**
For prob. 9.44.

**9.45** Find current $\mathbf{I}_o$ in the network of Fig. 9.52.

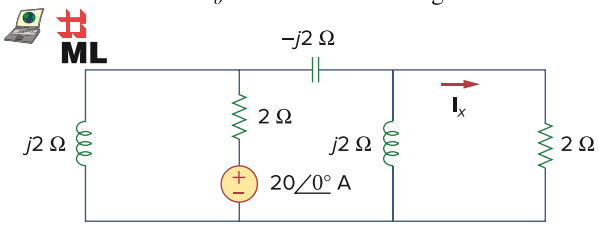



**Figure 9.52**
For Prob. 9.45.

**9.46** If $v_s = 100 \sin(10t + 18°)$ V in the circuit of Fig. 9.53, find $i_o$.

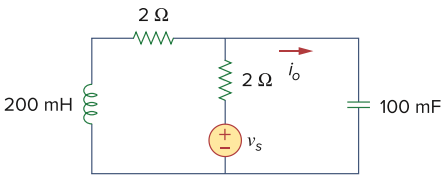



**Figure 9.53**
For Prob. 9.46.

**9.47** In the circuit of Fig. 9.54, determine the value of $i_s(t)$.

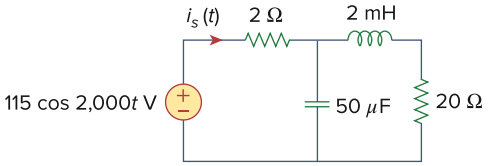



**Figure 9.54**
For Prob. 9.47.

**9.48** Given that $v_s(t) = 20 \sin(100t - 40°)$ in Fig. 9.55, determine $i_x(t)$.

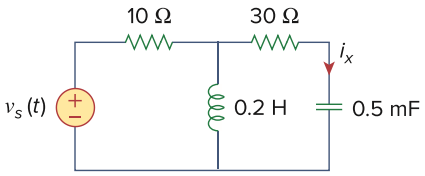



**Figure 9.55**
For Prob. 9.48.

**9.49** Find $v_s(t)$ in the circuit of Fig. 9.56 if the current $i_x$ through the 1-Ω resistor is $8 \sin 200t$ A.

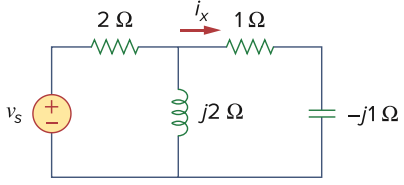



**Figure 9.56**
For Prob. 9.49.

**9.50** Determine $v_x$ in the circuit of Fig. 9.57. Let $i_s(t) = 5 \cos(100t + 40°)$ A.

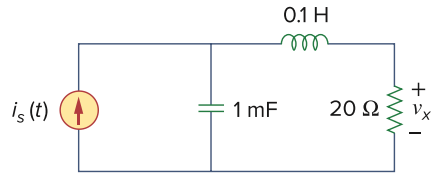



**Figure 9.57**
For Prob. 9.50.

**9.51** If the voltage $v_o$ across the 2-Ω resistor in the circuit of Fig. 9.58 is $90 \cos 2t$ V, obtain $i_s$.

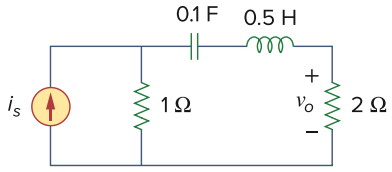



**Figure 9.58**
For Prob. 9.51.

**9.52** If $\mathbf{V}_o = 8\angle 30°$ V in the circuit of Fig. 9.59, find $\mathbf{I}_s$.

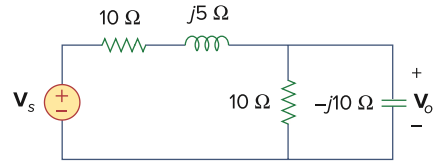



**Figure 9.59**
For Prob. 9.52.

**9.53** Find $\mathbf{I}_o$ in the circuit of Fig. 9.60.

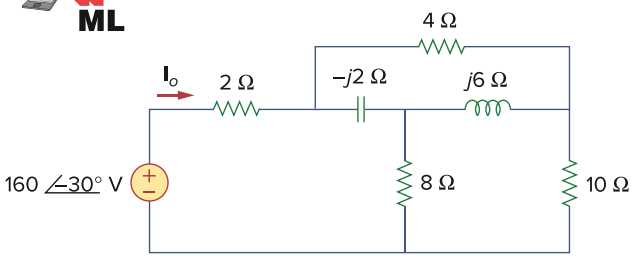



**Figure 9.60**
For Prob. 9.53.

**9.54** In the circuit of Fig. 9.61, find $\mathbf{V}_s$ if $\mathbf{I}_o = 30\angle 0°$ A.

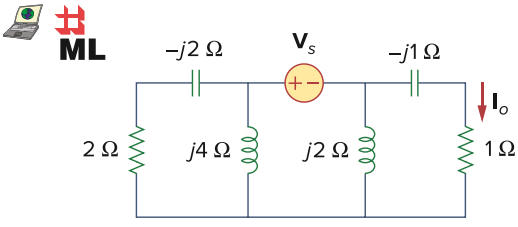



**Figure 9.61**
For Prob. 9.54.

***9.55** Find **Z** in the network of Fig. 9.62, given that $\mathbf{V}_o = 4\angle 0°$ V.

ML

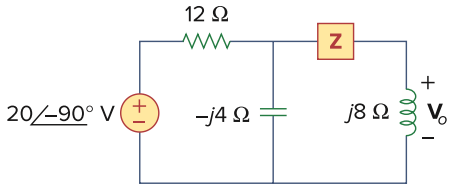



**Figure 9.62**
For Prob. 9.55.

## Section 9.7 Impedance Combinations

**9.56** At $\omega = 377$ rad/s, find the input impedance of the circuit shown in Fig. 9.63.

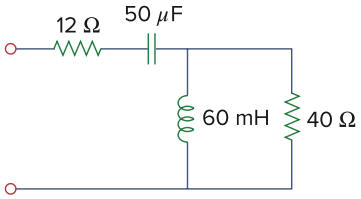



**Figure 9.63**
For Prob. 9.56.

**9.57** At $\omega = 1$ rad/s, obtain the input admittance in the circuit of Fig. 9.64.

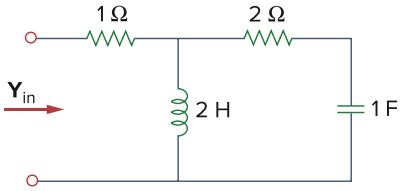



**Figure 9.64**
For Prob. 9.57.

**9.58** Using Fig. 9.65, design a problem to help other students better understand impedance combinations.

eXd

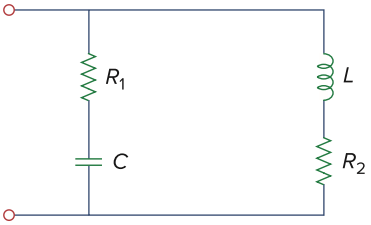



**Figure 9.65**
For Prob. 9.58.

* An asterisk indicates a challenging problem.

**9.59** For the network in Fig. 9.66, find $\mathbf{Z}_{in}$. Let $\omega = 100$ rad/s.

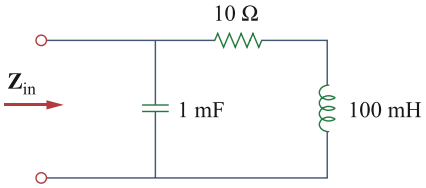



**Figure 9.66**
For Prob. 9.59.

**9.60** Obtain $\mathbf{Z}_{in}$ for the circuit in Fig. 9.67.

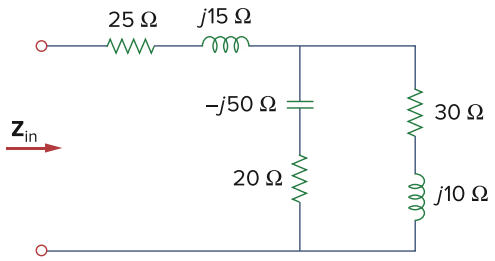



**Figure 9.67**
For Prob. 9.60.

**9.61** Find $\mathbf{Z}_{eq}$ in the circuit of Fig. 9.68.

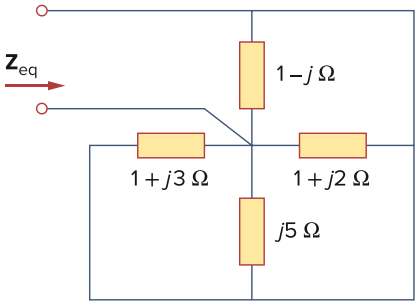



**Figure 9.68**
For Prob. 9.61.

**9.62** For the circuit in Fig. 9.69, find the input impedance $\mathbf{Z}_{in}$ at 10 krad/s.

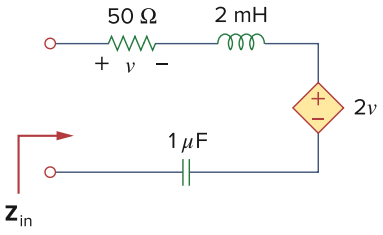



**Figure 9.69**
For Prob. 9.62.

**9.63** For the circuit in Fig. 9.70, find the value of $\mathbf{Z}_T$.

ML

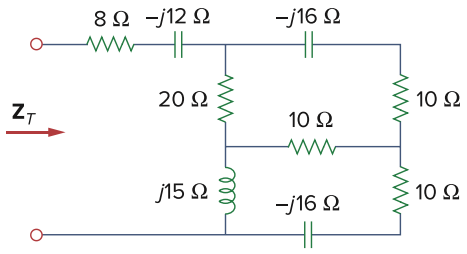



**Figure 9.70**
For Prob. 9.63.

**9.64** Find $\mathbf{Z}_T$ and $\mathrm{V}_o$ in the circuit in Fig. 9.71. Let the value of the inductance equal $j20\ \Omega$.

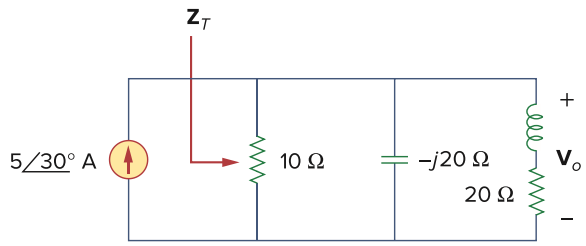



**Figure 9.71**
For Prob. 9.64.

**9.65** Determine $\mathbf{Z}_T$ and $\mathbf{I}$ for the circuit in Fig. 9.72.

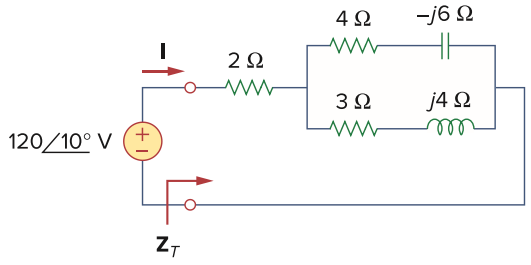



**Figure 9.72**
For Prob. 9.65.

**9.66** For the circuit in Fig. 9.73, calculate $\mathbf{Z}_T$ and $\mathbf{V}_{ab}$.

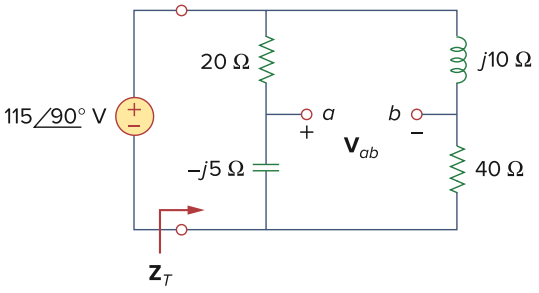



**Figure 9.73**
For Prob. 9.66.

**9.67** At $\omega = 10^3$ rad/s, find the input admittance of each of the circuits in Fig. 9.74.

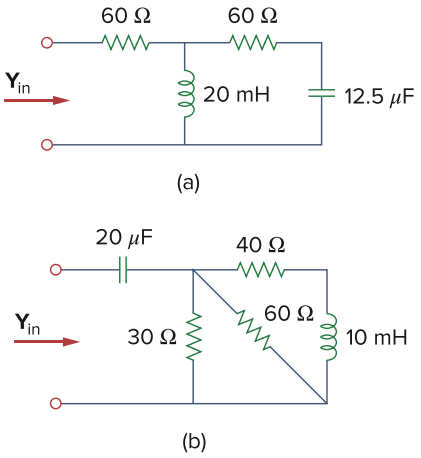



**Figure 9.74**
For Prob. 9.67.

**9.68** Determine $\mathbf{Y}_{eq}$ for the circuit in Fig. 9.75.

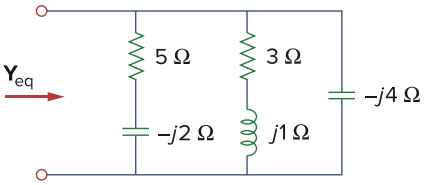



**Figure 9.75**
For Prob. 9.68.

**9.69** Find the equivalent admittance $\mathbf{Y}_{eq}$ of the circuit in Fig. 9.76.

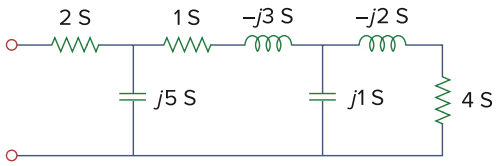



**Figure 9.76**
For Prob. 9.69.

**9.70** Find the equivalent impedance of the circuit in Fig. 9.77.

ML

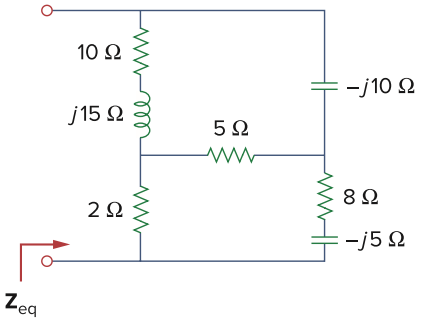



**Figure 9.77**
For Prob. 9.70.

**9.71** Obtain the equivalent impedance of the circuit in Fig. 9.78.

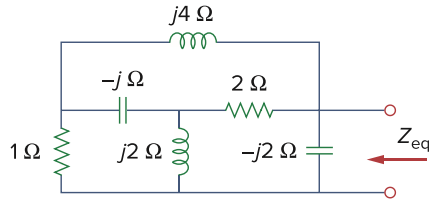



**Figure 9.78**
For Prob. 9.71.

**9.72** Calculate the value of $\mathbf{Z}_{ab}$ in the network of Fig. 9.79.

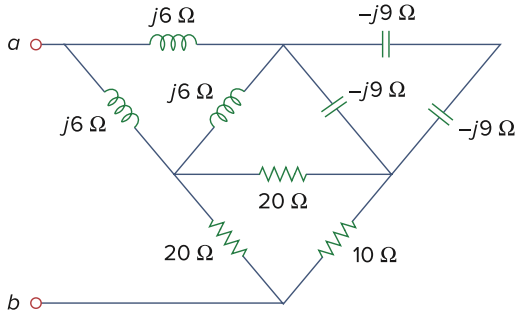



**Figure 9.79**
For Prob. 9.72.

**9.73** Determine the equivalent impedance of the circuit in Fig. 9.80.

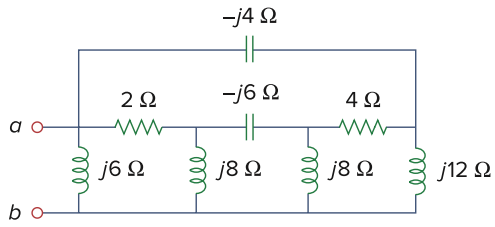



**Figure 9.80**
For Prob. 9.73.

## Section 9.8 Applications

**9.74** Design an *RL* circuit to provide a 90° leading phase shift.

**9.75** Design a circuit that will transform a sinusoidal voltage input to a cosinusoidal voltage output.

**9.76** For the following pairs of signals, determine if $v_1$ leads or lags $v_2$ and by how much.

(a) $v_1 = 10\cos(5t - 20°)$, $v_2 = 8\sin 5t$
(b) $v_1 = 19\cos(2t + 90°)$, $v_2 = 6\sin 2t$
(c) $v_1 = -4\cos 10t$, $v_2 = 15\sin 10t$

**9.77** Refer to the *RC* circuit in Fig. 9.81.

(a) Calculate the phase shift at 2 MHz.
(b) Find the frequency where the phase shift is 45°.

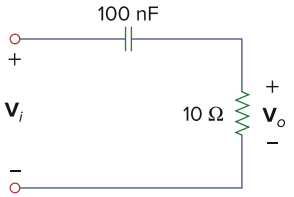



**Figure 9.81**
For Prob. 9.77.

**9.78** A coil with impedance $8 + j6\ \Omega$ is connected in series with a capacitive reactance $X$. The series combination is connected in parallel with a resistor $R$. Given that the equivalent impedance of the resulting circuit is $5\angle 0°\ \Omega$, find the value of $R$ and $X$.

**9.79** (a) Calculate the phase shift of the circuit in Fig. 9.82.
(b) State whether the phase shift is leading or lagging (output with respect to input).
(c) Determine the magnitude of the output when the input is 120 V.

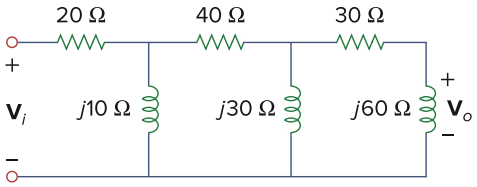



**Figure 9.82**
For Prob. 9.79.

**9.80** Consider the phase-shifting circuit in Fig. 9.83. Let $\mathbf{V}_i = 120$ V operating at 60 Hz. Find:

(a) $\mathbf{V}_o$ when $R$ is maximum
(b) $\mathbf{V}_o$ when $R$ is minimum
(c) the value of $R$ that will produce a phase shift of 45°

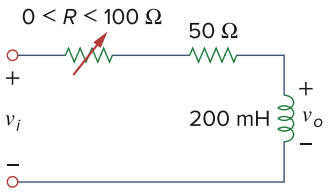



**Figure 9.83**
For Prob. 9.80.

**9.81** The ac bridge in Fig. 9.37 is balanced when $R_1 = 400\ \Omega$, $R_2 = 600\ \Omega$, $R_3 = 1.2$ kΩ, and $C_2 = 0.3\ \mu$F. Find $R_x$ and $C_x$. Assume $R_2$ and $C_2$ are in series.

**9.82** A capacitance bridge balances when $R_1 = 100\ \Omega$, $R_2 = 2$ kΩ, and $C_s = 40\ \mu$F. What is $C_x$, the capacitance of the capacitor under test?

**9.83** An inductive bridge balances when $R_1 = 1.2$ kΩ, $R_2 = 500\ \Omega$, and $L_s = 250$ mH. What is the value of $L_x$, the inductance of the inductor under test?

**9.84** The ac bridge shown in Fig. 9.84 is known as a *Maxwell bridge* and is used for accurate measurement of inductance and resistance of a coil in terms of a standard capacitance $C_s$. Show that when the bridge is balanced,

$$L_x = R_2R_3C_s \qquad \text{and} \qquad R_x = \frac{R_2}{R_1}R_3$$

Find $L_x$ and $R_x$ for $R_1 = 40\ \text{k}\Omega$, $R_2 = 1.6\ \text{k}\Omega$, $R_3 = 4\ \text{k}\Omega$, and $C_s = 0.45\ \mu\text{F}$.

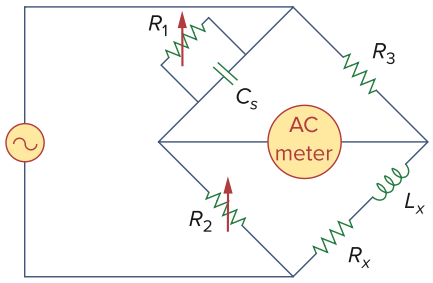



**Figure 9.84**
Maxwell bridge; For Prob. 9.84.

**9.85** The ac bridge circuit of Fig. 9.85 is called a *Wien bridge*. It is used for measuring the frequency of a source. Show that when the bridge is balanced,

$$f = \frac{1}{2\pi\sqrt{R_2R_4C_2C_4}}$$

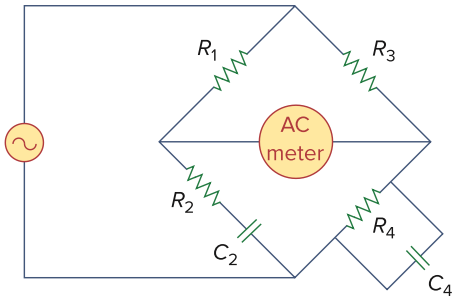



**Figure 9.85**
Wien bridge; For Prob. 9.85.

## Comprehensive Problems

**9.86** The circuit shown in Fig. 9.86 is used in a television receiver. What is the total impedance of this circuit?

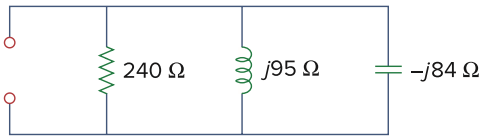



**Figure 9.86**
For Prob. 9.86.

**9.87** The network in Fig. 9.87 is part of the schematic describing an industrial electronic sensing device. What is the total impedance of the circuit at 4 kHz?

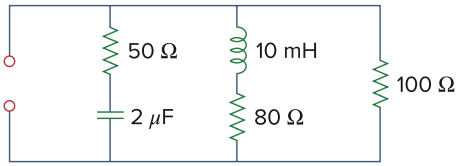



**Figure 9.87**
For Prob. 9.87.

**9.88** A series audio circuit is shown in Fig. 9.88.

(a) What is the impedance of the circuit?

(b) If the frequency were halved, what would be the impedance of the circuit?

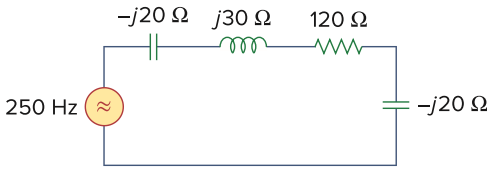



**Figure 9.88**
For Prob. 9.88.

**9.89** An industrial load is modeled as a series combination of an inductor and a resistance as shown in Fig. 9.89. Calculate the value of a capacitor $C$ across the series combination so that the net impedance is resistive at a frequency of 2 kHz.

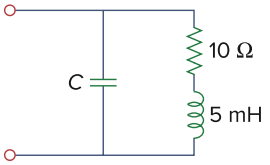



**Figure 9.89**
For Prob. 9.89.

**9.90** An industrial coil is modeled as a series combination of an inductance $L$ and resistance $R$, as shown in Fig. 9.90. Since an ac voltmeter measures only the magnitude of a sinusoid, the following

measurements are taken at 60 Hz when the circuit operates in the steady state:

$$|\mathbf{V}_s| = 145 \text{ V}, \qquad |\mathbf{V}_1| = 50 \text{ V}, \qquad |\mathbf{V}_o| = 110 \text{ V}$$

Use these measurements to determine the values of $L$ and $R$.

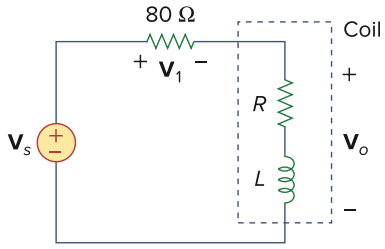



**Figure 9.90**
For Prob. 9.90.

**9.91** Figure 9.91 shows a series combination of an inductance and a resistance. If it is desired to connect a capacitor in parallel with the series combination such that the net impedance is resistive at 10 kHz, what is the required value of $C$?

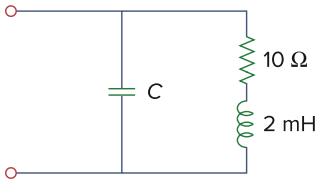



**Figure 9.91**
For Prob. 9.91.

**9.92** A transmission line has a series impedance of $\mathbf{Z} = 100\angle 75^\circ\ \Omega$ and a shut admittance of $\mathbf{Y} = 450\angle 48^\circ\ \mu\text{S}$. Find: (a) the characteristic impedance $\mathbf{Z}_o = \sqrt{\mathbf{Z}/\mathbf{Y}}$, (b) the propagation constant $\gamma = \sqrt{\mathbf{ZY}}$.

**9.93** A power transmission system is modeled as shown in Fig. 9.92. Given the source voltage and circuit elements

$\mathbf{V}_s = 115\angle 0^\circ$ V, source impedance
$\mathbf{Z}_s = (1 + j0.5)\ \Omega$, line impedance
$\mathbf{Z}_t = (0.4 + j0.3)\ \Omega$, and load impedance
$\mathbf{Z}_L = (23.2 + j18.9)\ \Omega$, find the load current $\mathbf{I}_L$.

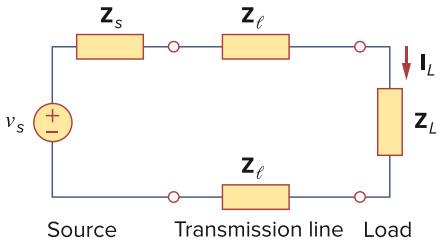



**Figure 9.92**
For Prob. 9.93.

chapter

# 10

# Sinusoidal Steady-State Analysis

*Three men are my friends—he that loves me, he that hates me, he that is indifferent to me. Who loves me, teaches me tenderness; who hates me, teaches me caution; who is indifferent to me, teaches me self-reliance.*

—J. E. Dinger

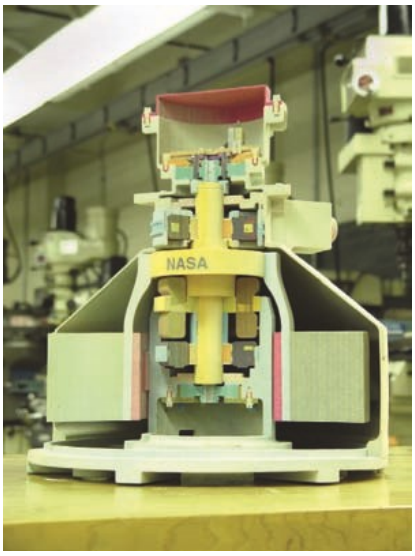
## Enhancing Your Career

### Career in Software Engineering

Software engineering is that aspect of engineering that deals with the practical application of scientific knowledge in the design, construction, and validation of computer programs and the associated documentation required to develop, operate, and maintain them. It is a branch of electrical engineering that is becoming increasingly important as more and more disciplines require one form of software package or another to perform routine tasks and as programmable microelectronic systems are used in more and more applications.

The role of a software engineer should not be confused with that of a computer scientist; the software engineer is a practitioner, not a theoretician. A software engineer should have good computer-programming skills and be familiar with programming languages, in particular C++, which is becoming increasingly popular. Because hardware and software are closely interlinked, it is essential that a software engineer have a thorough understanding of hardware design. Most important, the software engineer should have some specialized knowledge of the area in which the software development skill is to be applied.

All in all, the field of software engineering offers a great career to those who enjoy programming and developing software packages. The higher rewards will go to those having the best preparation, with the most interesting and challenging opportunities going to those with graduate education.




A three-dimensional printing of the output of an AutoCAD model of a NASA flywheel.
© Ansoft Corporation

## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Analyze electrical circuits in the frequency domain using nodal analysis.
2. Analyze electrical circuits in the frequency domain using mesh analysis.
3. Apply the superposition principle to frequency domain electrical circuits.
4. Apply source transformation in frequency domain circuits.
5. Understand how Thevenin and Norton equivalent circuits can be used in the frequency domain.
6. Analyze electrical circuits with op amps.

## 10.1 Introduction

In Chapter 9, we learned that the forced or steady-state response of circuits to sinusoidal inputs can be obtained by using phasors. We also know that Ohm's and Kirchhoff's laws are applicable to ac circuits. In this chapter, we want to see how nodal analysis, mesh analysis, Thevenin's theorem, Norton's theorem, superposition, and source transformations are applied in analyzing ac circuits. Since these techniques were already introduced for dc circuits, our major effort here will be to illustrate with examples.

Analyzing ac circuits usually requires three steps.

### Steps to Analyze AC Circuits:

1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain.

Step 1 is not necessary if the problem is specified in the frequency domain. In step 2, the analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved. Having read Chapter 9, we are adept at handling step 3.

Frequency domain analysis of an ac circuit via phasors is much easier than analysis of the circuit in the time domain.

Toward the end of the chapter, we learn how to apply *PSpice* in solving ac circuit problems. We finally apply ac circuit analysis to two practical ac circuits: oscillators and ac transistor circuits.

## 10.2 Nodal Analysis

The basis of nodal analysis is Kirchhoff's current law. Since KCL is valid for phasors, as demonstrated in Section 9.6, we can analyze ac circuits by nodal analysis. The following examples illustrate this.

## Example 10.1

Find $i_x$ in the circuit of Fig. 10.1 using nodal analysis.

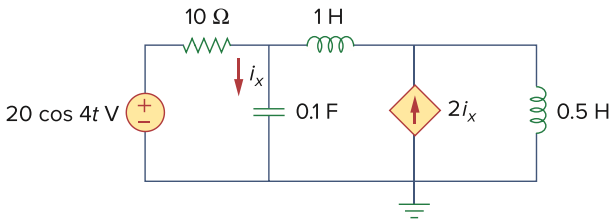



**Figure 10.1**
For Example 10.1.

**Solution:**
We first convert the circuit to the frequency domain:

$$\begin{aligned} 20\cos 4t &\Rightarrow 20\angle 0^\circ, \quad \Omega = 4 \text{ rad/s} \\ 1 \text{ H} &\Rightarrow j\Omega L = j4 \\ 0.5 \text{ H} &\Rightarrow j\Omega L = j2 \\ 0.1 \text{ F} &\Rightarrow \frac{1}{j\Omega C} = -j2.5 \end{aligned}$$

Thus, the frequency domain equivalent circuit is as shown in Fig. 10.2.

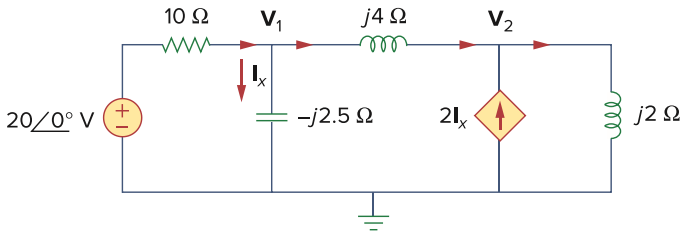



**Figure 10.2**
Frequency domain equivalent of the circuit in Fig. 10.1.

Applying KCL at node 1,

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

or

$$(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20 \qquad \textbf{(10.1.1)}$$

At node 2,

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

But $\mathbf{I}_x = \mathbf{V}_1/-j2.5$. Substituting this gives

$$\frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

By simplifying, we get

$$11\mathbf{V}_1 + 15\mathbf{V}_2 = 0 \qquad \textbf{(10.1.2)}$$

Equations (10.1.1) and (10.1.2) can be put in matrix form as

$$\begin{bmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

We obtain the determinants as

$$\Delta = \begin{vmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{vmatrix} = 15 - j5$$

$$\Delta_1 = \begin{vmatrix} 20 & j2.5 \\ 0 & 15 \end{vmatrix} = 300, \qquad \Delta_2 = \begin{vmatrix} 1 + j1.5 & 20 \\ 11 & 0 \end{vmatrix} = -220$$

$$\mathbf{V}_1 = \frac{\Delta_1}{\Delta} = \frac{300}{15 - j5} = 18.97 \angle 18.43^\circ \text{ V}$$

$$\mathbf{V}_2 = \frac{\Delta_2}{\Delta} = \frac{-220}{15 - j5} = 13.91 \angle 198.3^\circ \text{ V}$$

The current $\mathbf{I}_x$ is given by

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97 \angle 18.43^\circ}{2.5 \angle -90^\circ} = 7.59 \angle 108.4^\circ \text{ A}$$

Transforming this to the time domain,

$$i_x = 7.59 \cos(4t + 108.4^\circ) \text{ A}$$

## Practice Problem 10.1

Using nodal analysis, find $v_1$ and $v_2$ in the circuit of Fig. 10.3.

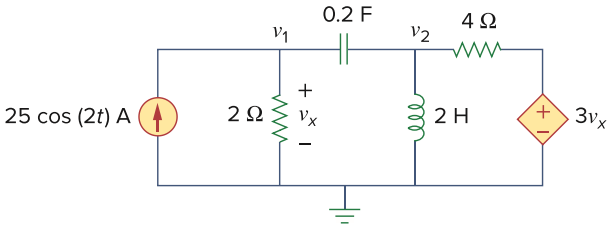



**Figure 10.3**
For Practice Prob. 10.1.

**Answer:** $v_1(t) = 28.31 \cos(2t + 60.01^\circ)$ V, $v_2(t) = 82.56 \cos(2t + 57.12^\circ)$ V.

## Example 10.2

Compute $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit of Fig. 10.4.

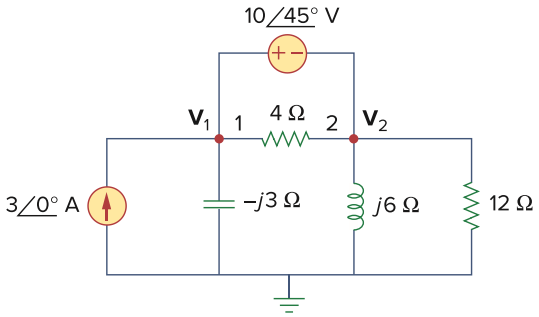



**Figure 10.4**
For Example 10.2.

**Solution:**

Nodes 1 and 2 form a supernode as shown in Fig. 10.5. Applying KCL at the supernode gives

$$3 = \frac{\mathbf{V}_1}{-j3} + \frac{\mathbf{V}_2}{j6} + \frac{\mathbf{V}_2}{12}$$

or

$$36 = j4\mathbf{V}_1 + (1 - j2)\mathbf{V}_2 \qquad \textbf{(10.2.1)}$$

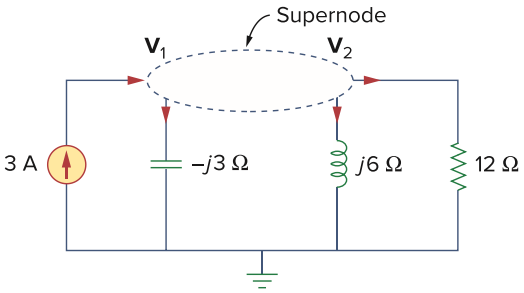



**Figure 10.5**
A supernode in the circuit of Fig. 10.4.

But a voltage source is connected between nodes 1 and 2, so that

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\angle 45^\circ \qquad \textbf{(10.2.2)}$$

Substituting Eq. (10.2.2) in Eq. (10.2.1) results in

$$36 - 40\angle 135^\circ = (1 + j2)\mathbf{V}_2 \quad \Rightarrow \quad \mathbf{V}_2 = 31.41\angle -87.18^\circ \text{ V}$$

From Eq. (10.2.2),

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\angle 45^\circ = 25.78\angle -70.48^\circ \text{ V}$$

## Practice Problem 10.2

Calculate $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit shown in Fig. 10.6.

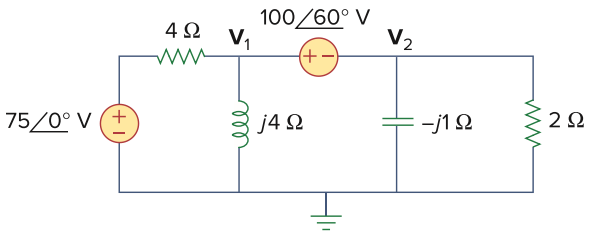



**Figure 10.6**
For Practice Prob. 10.2.

**Answer:** $\mathbf{V}_1 = 96.8\angle 69.66^\circ$ V, $\mathbf{V}_2 = 16.88\angle 165.72^\circ$ V.

## 10.3 Mesh Analysis

Kirchhoff's voltage law (KVL) forms the basis of mesh analysis. The validity of KVL for ac circuits was shown in Section 9.6 and is illustrated in the following examples. Keep in mind that the very nature of using mesh analysis is that it is to be applied to planar circuits.

## Example 10.3

Determine current $\mathbf{I}_o$ in the circuit of Fig. 10.7 using mesh analysis.

**Solution:**
Applying KVL to mesh 1, we obtain

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0 \qquad \textbf{(10.3.1)}$$

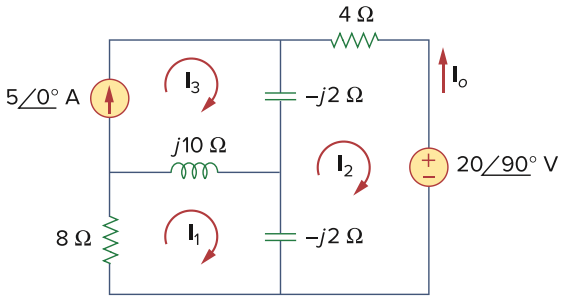



**Figure 10.7**
For Example 10.3.

For mesh 2,

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0 \qquad \textbf{(10.3.2)}$$

For mesh 3, $\mathbf{I}_3 = 5$. Substituting this in Eqs. (10.3.1) and (10.3.2), we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50 \qquad \textbf{(10.3.3)}$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10 \qquad \textbf{(10.3.4)}$$

Equations (10.3.3) and (10.3.4) can be put in matrix form as

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

from which we obtain the determinants

$$\Delta = \begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix} = 32(1 + j)(1 - j) + 4 = 68$$

$$\Delta_2 = \begin{vmatrix} 8 + j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240 = 416.17\angle -35.22^\circ$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\angle -35.22^\circ}{68} = 6.12\angle -35.22^\circ \text{ A}$$

The desired current is

$$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\angle 144.78^\circ \text{ A}$$

## Practice Problem 10.3

Find $\mathbf{I}_o$ in Fig. 10.8 using mesh analysis.

**Answer:** $5.969\angle 65.45^\circ$ A.

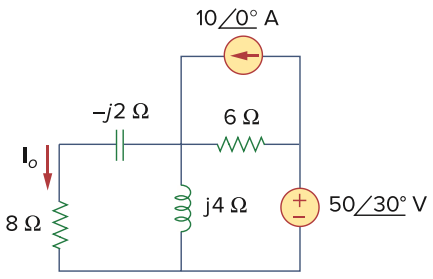



**Figure 10.8**
For Practice Prob. 10.3.

## Example 10.4

Solve for $\mathbf{V}_o$ in the circuit of Fig. 10.9 using mesh analysis.

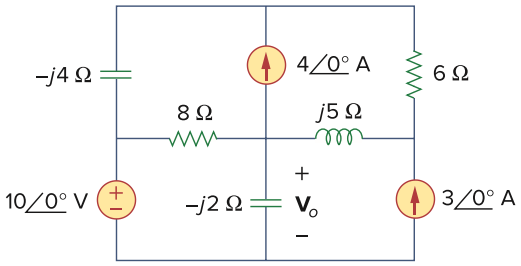



**Figure 10.9**
For Example 10.4.

**Solution:**
As shown in Fig. 10.10, meshes 3 and 4 form a supermesh due to the current source between the meshes. For mesh 1, KVL gives

$$-10 + (8 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - 8\mathbf{I}_3 = 0$$

or

$$(8 - j2)\mathbf{I}_1 + j2\mathbf{I}_2 - 8\mathbf{I}_3 = 10 \qquad \textbf{(10.4.1)}$$

For mesh 2,

$$\mathbf{I}_2 = -3 \qquad \textbf{(10.4.2)}$$

For the supermesh,

$$(8 - j4)\mathbf{I}_3 - 8\mathbf{I}_1 + (6 + j5)\mathbf{I}_4 - j5\mathbf{I}_2 = 0 \qquad \textbf{(10.4.3)}$$

Due to the current source between meshes 3 and 4, at node A,

$$\mathbf{I}_4 = \mathbf{I}_3 + 4 \qquad \textbf{(10.4.4)}$$

■ **METHOD 1** Instead of solving the above four equations, we reduce them to two by elimination.

Combining Eqs. (10.4.1) and (10.4.2),

$$(8 - j2)\mathbf{I}_1 - 8\mathbf{I}_3 = 10 + j6 \qquad \textbf{(10.4.5)}$$

Combining Eqs. (10.4.2) to (10.4.4),

$$-8\mathbf{I}_1 + (14 + j)\mathbf{I}_3 = -24 - j35 \qquad \textbf{(10.4.6)}$$

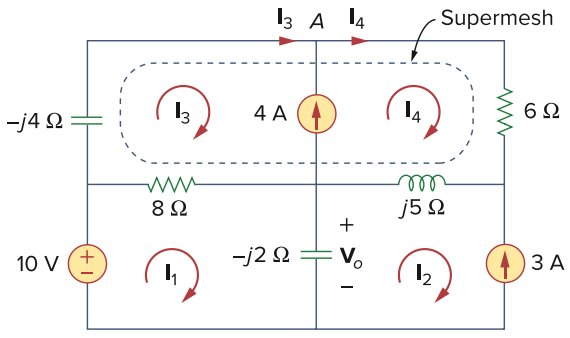



**Figure 10.10**
Analysis of the circuit in Fig. 10.9.

From Eqs. (10.4.5) and (10.4.6), we obtain the matrix equation

$$\begin{bmatrix} 8-j2 & -8 \\ -8 & 14+j \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10+j6 \\ -24-j35 \end{bmatrix}$$

We obtain the following determinants

$$\Delta = \begin{vmatrix} 8-j2 & -8 \\ -8 & 14+j \end{vmatrix} = 112 + j8 - j28 + 2 - 64 = 50 - j20$$

$$\Delta_1 = \begin{vmatrix} 10+j6 & -8 \\ -24-j35 & 14+j \end{vmatrix} = 140 + j10 + j84 - 6 - 192 - j280$$
$$= -58 - j186$$

Current $\mathbf{I}_1$ is obtained as

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{-58 - j186}{50 - j20} = 3.618 \angle 274.5^\circ \text{ A}$$

The required voltage $\mathbf{V}_o$ is

$$\mathbf{V}_o = -j2(\mathbf{I}_1 - \mathbf{I}_2) = -j2(3.618 \angle 274.5^\circ + 3)$$
$$= -7.2134 - j6.568 = 9.756 \angle 222.32^\circ \text{ V}$$

**METHOD 2** We can use *MATLAB* to solve Eqs. (10.4.1) to (10.4.4). We first cast the equations as

$$\begin{bmatrix} 8-j2 & j2 & -8 & 0 \\ 0 & 1 & 0 & 0 \\ -8 & -j5 & 8-j4 & 6+j5 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \\ \mathbf{I}_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 0 \\ 4 \end{bmatrix} \quad \textbf{(10.4.7a)}$$

or

$$\mathbf{AI} = \mathbf{B}$$

By inverting $\mathbf{A}$, we can obtain $\mathbf{I}$ as

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{B} \quad \textbf{(10.4.7b)}$$

We now apply *MATLAB* as follows:

```
>> A = [(8-j*2) j*2  -8        0;
        0       1    0         0;
        -8      -j*5 (8-j*4) (6+j*5);
        0       0     -1       1];
>> B = [10 -3 0 4]';
>> I = inv(A)*B

I =
  0.2828 - 3.6069i
  -3.0000
  -1.8690 - 4.4276i
   2.1310 - 4.4276i
>> Vo = -2*j*(I(1) - I(2))

Vo =
  -7.2138 - 6.5655i
```

as obtained previously.

## Practice Problem 10.4

Calculate current $\mathbf{I}_o$ in the circuit of Fig. 10.11.

**Answer:** $6.089\angle 5.94^\circ$ A.

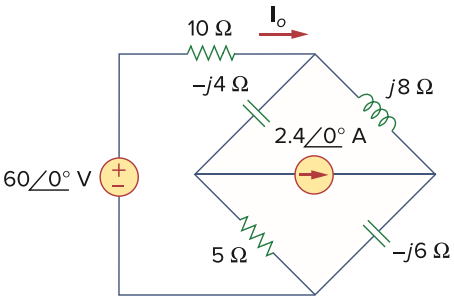



**Figure 10.11**
For Practice Prob. 10.4.

## 10.4 Superposition Theorem

Since ac circuits are linear, the superposition theorem applies to ac circuits the same way it applies to dc circuits. The theorem becomes important if the circuit has sources operating at *different* frequencies. In this case, since the impedances depend on frequency, we must have a different frequency domain circuit for each frequency. The total response must be obtained by adding the individual responses in the *time* domain. It is incorrect to try to add the responses in the phasor or frequency domain. Why? Because the exponential factor $e^{j\omega t}$ is implicit in sinusoidal analysis, and that factor would change for every angular frequency $\omega$. It would therefore not make sense to add responses at different frequencies in the phasor domain. Thus, when a circuit has sources operating at different frequencies, one must add the responses due to the individual frequencies in the time domain.

## Example 10.5

Use the superposition theorem to find $\mathbf{I}_o$ in the circuit in Fig. 10.7.

**Solution:**
Let

$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o \tag{10.5.1}$$

where $\mathbf{I}'_o$ and $\mathbf{I}''_o$ are due to the voltage and current sources, respectively. To find $\mathbf{I}'_o$, consider the circuit in Fig. 10.12(a). If we let $\mathbf{Z}$ be the parallel combination of $-j2$ and $8 + j10$, then

$$\mathbf{Z} = \frac{-j2(8 + j10)}{-2j + 8 + j10} = 0.25 - j2.25$$

and current $\mathbf{I}'_o$ is

$$\mathbf{I}'_o = \frac{j20}{4 - j2 + \mathbf{Z}} = \frac{j20}{4.25 - j4.25}$$

or

$$\mathbf{I}'_o = -2.353 + j2.353 \tag{10.5.2}$$

To get $\mathbf{I}''_o$, consider the circuit in Fig. 10.12(b). For mesh 1,

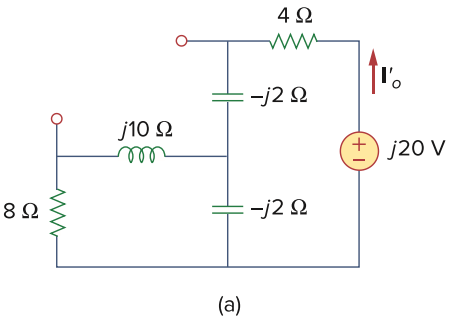
$$(8 + j8)\mathbf{I}_1 - j10\mathbf{I}_3 + j2\mathbf{I}_2 = 0 \tag{10.5.3}$$

For mesh 2,

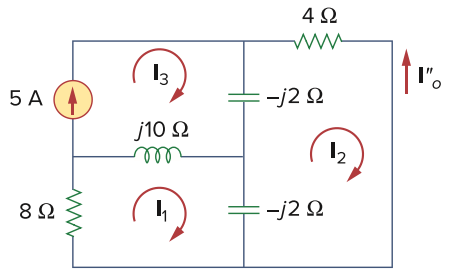
$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j2\mathbf{I}_3 = 0 \tag{10.5.4}$$

For mesh 3,

$$\mathbf{I}_3 = 5 \tag{10.5.5}$$




(a)




(b)

**Figure 10.12**
Solution of Example 10.5.

From Eqs. (10.5.4) and (10.5.5),

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j10 = 0$$

Expressing $\mathbf{I}_1$ in terms of $\mathbf{I}_2$ gives

$$\mathbf{I}_1 = (2 + j2)\mathbf{I}_2 - 5 \tag{10.5.6}$$

Substituting Eqs. (10.5.5) and (10.5.6) into Eq. (10.5.3), we get

$$(8 + j8)[(2 + j2)\mathbf{I}_2 - 5] - j50 + j2\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_2 = \frac{90 - j40}{34} = 2.647 - j1.176$$

Current $\mathbf{I}_o''$ is obtained as

$$\mathbf{I}_o'' = -\mathbf{I}_2 = -2.647 + j1.176 \tag{10.5.7}$$

From Eqs. (10.5.2) and (10.5.7), we write

$$\mathbf{I}_o = \mathbf{I}_o' + \mathbf{I}_o'' = -5 + j3.529 = 6.12\underline{/144.78^\circ}\ \text{A}$$

which agrees with what we got in Example 10.3. It should be noted that applying the superposition theorem is not the best way to solve this problem. It seems that we have made the problem twice as hard as the original one by using superposition. However, in Example 10.6, superposition is clearly the easiest approach.

## Practice Problem 10.5

Find current $\mathbf{I}_o$ in the circuit of Fig. 10.8 using the superposition theorem.

**Answer:** $5.97\underline{/65.45^\circ}$ A.

## Example 10.6

Find $v_o$ of the circuit of Fig. 10.13 using the superposition theorem.

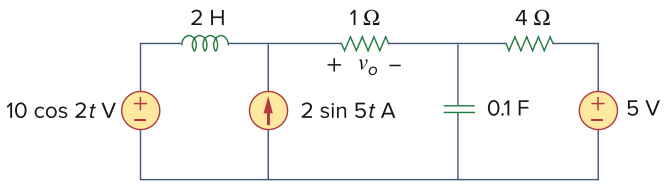



**Figure 10.13**
For Example 10.6.

**Solution:**

Because the circuit operates at three different frequencies ($\omega = 0$ for the dc voltage source), one way to obtain a solution is to use superposition, which breaks the problem into single-frequency problems. So we let

$$v_o = v_1 + v_2 + v_3 \tag{10.6.1}$$

where $v_1$ is due to the 5-V dc voltage source, $v_2$ is due to the 10 cos 2$t$ V voltage source, and $v_3$ is due to the 2 sin 5$t$ A current source.

To find $v_1$, we set to zero all sources except the 5-V dc source. We recall that at steady state, a capacitor is an open circuit to dc while an inductor is a short circuit to dc. There is an alternative way of looking at this. Because $\omega = 0$, $j\omega L = 0$, $1/j\omega C = \infty$. Either way, the equivalent circuit is as shown in Fig. 10.14(a). By voltage division,

$$-v_1 = \frac{1}{1+4}(5) = 1 \text{ V} \qquad \textbf{(10.6.2)}$$

To find $v_2$, we set to zero both the 5-V source and the 2 sin 5$t$ current source and transform the circuit to the frequency domain.

$$\begin{aligned} 10 \cos 2t \quad &\Rightarrow \quad 10\angle 0^\circ, \quad \omega = 2 \text{ rad/s} \\ 2 \text{ H} \quad &\Rightarrow \quad j\omega L = j4\ \Omega \\ 0.1 \text{ F} \quad &\Rightarrow \quad \frac{1}{j\omega C} = -j5\ \Omega \end{aligned}$$

The equivalent circuit is now as shown in Fig. 10.14(b). Let

$$\mathbf{Z} = -j5 \,\|\, 4 = \frac{-j5 \times 4}{4 - j5} = 2.439 - j1.951$$

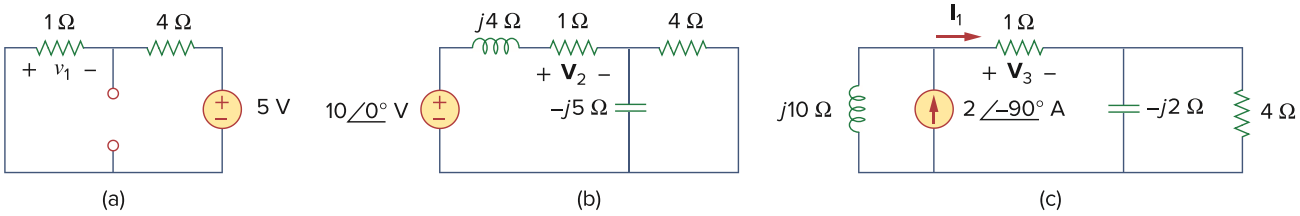



**Figure 10.14**
Solution of Example 10.6: (a) setting all sources to zero except the 5-V dc source, (b) setting all sources to zero except the ac voltage source, (c) setting all sources to zero except the ac current source.

By voltage division,

$$\mathbf{V}_2 = \frac{1}{1 + j4 + \mathbf{Z}}(10\angle 0^\circ) = \frac{10}{3.439 + j2.049} = 2.498\angle -30.79^\circ$$

In the time domain,

$$v_2 = 2.498 \cos(2t - 30.79^\circ) \qquad \textbf{(10.6.3)}$$

To obtain $v_3$, we set the voltage sources to zero and transform what is left to the frequency domain.

$$\begin{aligned} 2 \sin 5t \quad &\Rightarrow \quad 2\angle -90^\circ, \quad \omega = 5 \text{ rad/s} \\ 2 \text{ H} \quad &\Rightarrow \quad j\omega L = j10\ \Omega \\ 0.1 \text{ F} \quad &\Rightarrow \quad \frac{1}{j\omega C} = -j2\ \Omega \end{aligned}$$

The equivalent circuit is in Fig. 10.14(c). Let

$$\mathbf{Z}_1 = -j2 \,\|\, 4 = \frac{-j2 \times 4}{4 - j2} = 0.8 - j1.6\ \Omega$$

By current division,

$$\mathbf{I}_1 = \frac{j10}{j10 + 1 + \mathbf{Z}_1}(2\angle -90^\circ)\ \text{A}$$

$$\mathbf{V}_3 = \mathbf{I}_1 \times 1 = \frac{j10}{1.8 + j8.4}(-j2) = 2.328\angle -80^\circ\ \text{V}$$

In the time domain,

$$v_3 = 2.33\cos(5t - 80^\circ) = 2.33\sin(5t + 10^\circ)\ \text{V} \qquad \textbf{(10.6.4)}$$

Substituting Eqs. (10.6.2) to (10.6.4) into Eq. (10.6.1), we have

$$v_o(t) = -1 + 2.498\cos(2t - 30.79^\circ) + 2.33\sin(5t + 10^\circ)\ \text{V}$$

## Practice Problem 10.6

Calculate $v_o$ in the circuit of Fig. 10.15 using the superposition theorem.

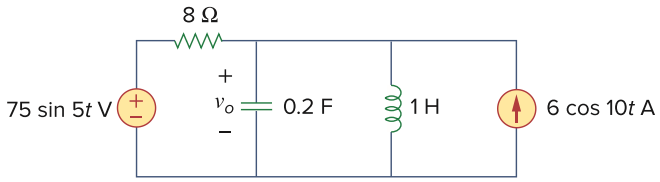



**Figure 10.15**
For Practice Prob. 10.6.

**Answer:** $11.577\sin(5t - 81.12^\circ) + 3.154\cos(10t - 86.24^\circ)$ V.

## 10.5 Source Transformation

As Fig. 10.16 shows, source transformation in the frequency domain involves transforming a voltage source in series with an impedance to a current source in parallel with an impedance, or vice versa. As we go from one source type to another, we must keep the following relationship in mind:

$$\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Leftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s} \qquad \textbf{(10.1)}$$

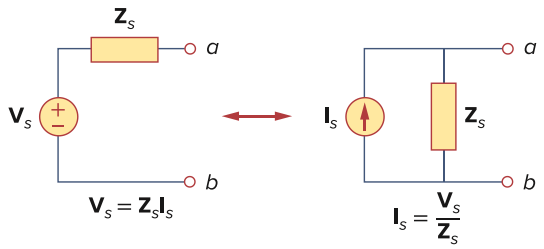



**Figure 10.16**
Source transformation.

## Example 10.7

Calculate $\mathbf{V}_x$ in the circuit of Fig. 10.17 using the method of source transformation.

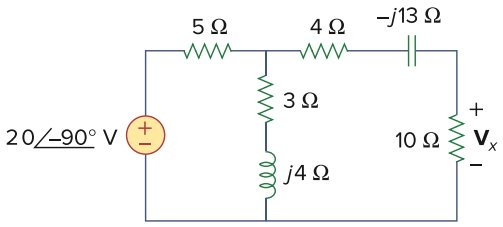



**Figure 10.17**
For Example 10.7.

**Solution:**
We transform the voltage source to a current source and obtain the cir - cuit in Fig. 10.18(a), where

$$\mathbf{I}_s = \frac{20\angle -90^\circ}{5} = 4\angle -90^\circ = -j4 \text{ A}$$

The parallel combination of 5-Ω resistance and $(3 + j4)$ impedance gives

$$\mathbf{Z}_1 = \frac{5(3 + j4)}{8 + j4} = 2.5 + j1.25 \ \Omega$$

Converting the current source to a voltage source yields the circuit in Fig. 10.18(b), where

$$\mathbf{V}_s = \mathbf{I}_s\mathbf{Z}_1 = -j4(2.5 + j1.25) = 5 - j10 \text{ V}$$

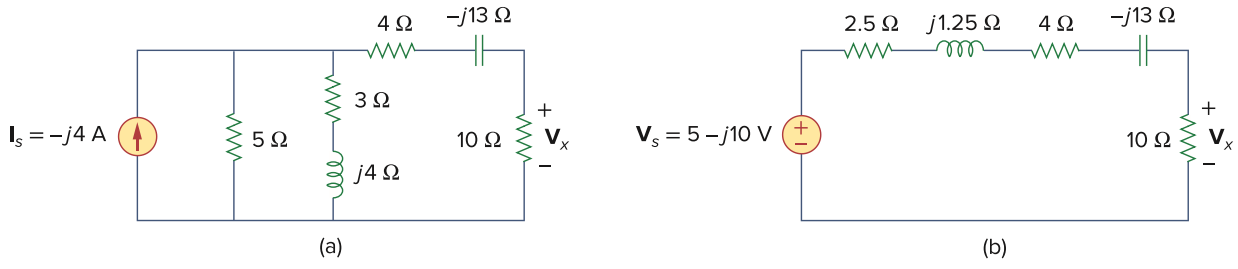



**Figure 10.18**
Solution of the circuit in Fig. 10.17.

By voltage division,

$$\mathbf{V}_x = \frac{10}{10 + 2.5 + j1.25 + 4 - j13}(5 - j10) = 5.519\angle -28^\circ \text{ V}$$

## Practice Problem 10.7

Find $\mathbf{I}_o$ in the circuit of Fig. 10.19 using the concept of source transformation.

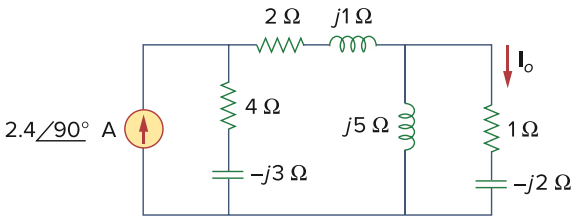



**Figure 10.19**
For Practice Prob. 10.7.

**Answer:** $1.9727/\underline{99.46^\circ}$ A.

## 10.6 Thevenin and Norton Equivalent Circuits

Thevenin's and Norton's theorems are applied to ac circuits in the same way as they are to dc circuits. The only additional effort arises from the need to manipulate complex numbers. The frequency domain version of a Thevenin equivalent circuit is depicted in Fig. 10.20, where a linear circuit is replaced by a voltage source in series with an impedance. The Norton equivalent circuit is illustrated in Fig. 10.21, where a linear cir - cuit is replaced by a current source in parallel with an impedance. Keep in mind that the two equivalent circuits are related as

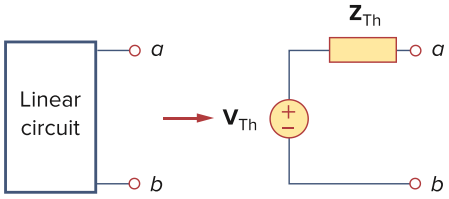



**Figure 10.20**
Thevenin equivalent.

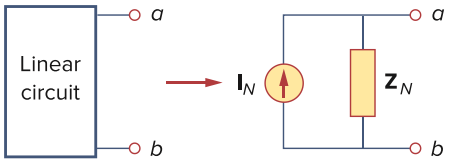



**Figure 10.21**
Norton equivalent.

$$\mathbf{V}_{\text{Th}} = \mathbf{Z}_N\mathbf{I}_N, \qquad \mathbf{Z}_{\text{Th}} = \mathbf{Z}_N \tag{10.2}$$

just as in source transformation. $\mathbf{V}_{\text{Th}}$ is the open-circuit voltage while $\mathbf{I}_N$ is the short-circuit current.

If the circuit has sources operating at dif ferent frequencies (see Example 10.6, for example), the Thevenin or Norton equivalent circuit must be determined at each frequenc y. This leads to entirely dif ferent equivalent circuits, one for each frequenc y, not one equi valent circuit with equivalent sources and equivalent impedances.

## Example 10.8

Obtain the Thevenin equivalent at terminals *a-b* of the circuit in Fig. 10.22.

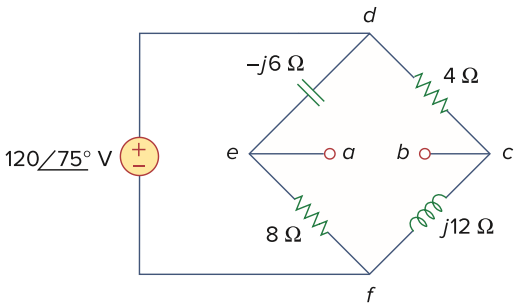



**Figure 10.22**
For Example 10.8.

**Solution:**

We find $\mathbf{Z}_{Th}$ by setting the voltage source to zero. As shown in Fig. 10.23(a), the 8 -Ω resistance is now in parallel with the $-j6$ reactance, so that their combination gives

$$\mathbf{Z}_1 = -j6 \,\|\, 8 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\ \Omega$$

Similarly, the 4 -Ω resistance is in parallel with the $j12$ reactance, and their combination gives

$$\mathbf{Z}_2 = 4 \,\|\, j12 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

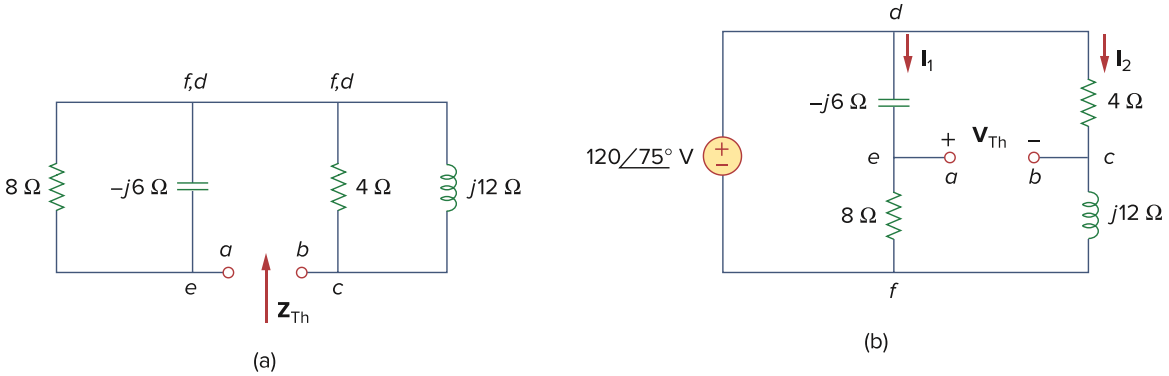



**Figure 10.23**
Solution of the circuit in Fig. 10.22: (a) finding $\mathbf{Z}_{Th}$, (b) finding $\mathbf{V}_{Th}$.

The Thevenin impedance is the series combination of $\mathbf{Z}_1$ and $\mathbf{Z}_2$; that is,

$$\mathbf{Z}_{Th} = \mathbf{Z}_1 + \mathbf{Z}_2 = 6.48 - j2.64\ \Omega$$

To find $\mathbf{V}_{Th}$, consider the circuit in Fig. 10.23(b). Currents $\mathbf{I}_1$ and $\mathbf{I}_2$ are obtained as

$$\mathbf{I}_1 = \frac{120\angle 75^\circ}{8 - j6}\ \text{A}, \qquad \mathbf{I}_2 = \frac{120\angle 75^\circ}{4 + j12}\ \text{A}$$

Applying KVL around loop *bcdeab* in Fig. 10.23(b) gives

$$\mathbf{V}_{Th} - 4\mathbf{I}_2 + (-j6)\mathbf{I}_1 = 0$$

or

$$\mathbf{V}_{Th} = 4\mathbf{I}_2 + j6\mathbf{I}_1 = \frac{480\angle 75^\circ}{4 + j12} + \frac{720\angle 75^\circ + 90^\circ}{8 - j6}$$

$$= 37.95\angle 3.43^\circ + 72\angle 201.87^\circ$$

$$= -28.936 - j24.55 = 37.95\angle 220.31^\circ\ \text{V}$$

## Practice Problem 10.8

Find the Thevenin equivalent at terminals *a-b* of the circuit in Fig. 10.24.

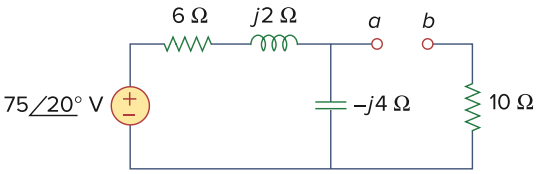



**Figure 10.24**
For Practice Prob. 10.8.

**Answer:** $\mathbf{Z}_{\text{Th}} = 12.4 - j3.2\ \Omega$, $\mathbf{V}_{\text{Th}} = 47.43\angle -51.57^\circ$ V.

## Example 10.9

Find the Thevenin equivalent of the circuit in Fig. 10.25 as seen from terminals *a-b*.

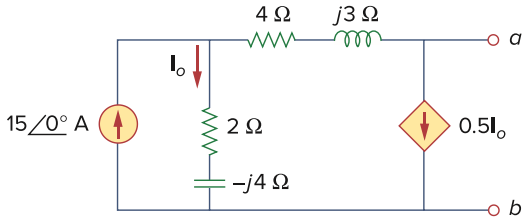



**Figure 10.25**
For Example 10.9.

**Solution:**
To find $\mathbf{V}_{\text{Th}}$, we apply KCL at node 1 in Fig. 10.26(a).

$$15 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \Rightarrow \quad \mathbf{I}_o = 10 \text{ A}$$

Applying KVL to the loop on the right -hand side in Fig. 10.26(a), we obtain

$$-\mathbf{I}_o(2 - j4) + 0.5\mathbf{I}_o(4 + j3) + \mathbf{V}_{\text{Th}} = 0$$

or

$$\mathbf{V}_{\text{Th}} = 10(2 - j4) - 5(4 + j3) = -j55$$

Thus, the Thevenin voltage is

$$\mathbf{V}_{\text{Th}} = 55\angle -90^\circ \text{ V}$$

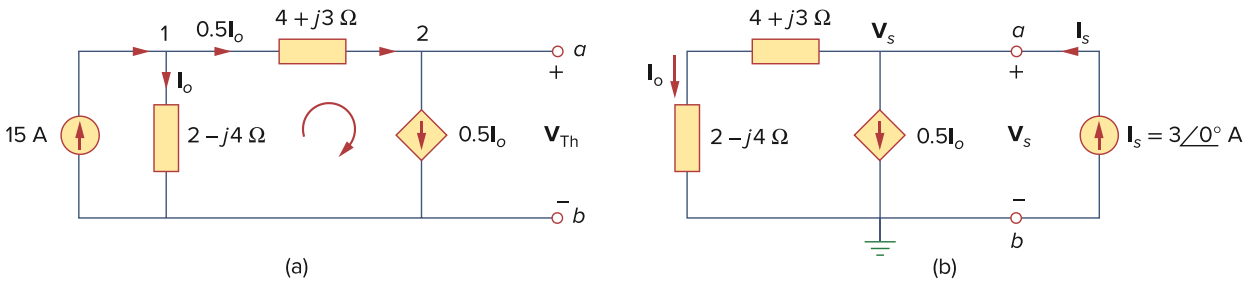



**Figure 10.26**
Solution of the problem in Fig. 10.25: (a) finding $\mathbf{V}_{\text{Th}}$, (b) finding $\mathbf{Z}_{\text{Th}}$.

To obtain $\mathbf{Z}_{\text{Th}}$, we remove the independent source. Due to the presence of the dependent current source, we connect a 3-A current source (3 is an arbitrary value chosen for convenience here, a number divisible by the sum of currents leaving the node) to terminals *a-b* as shown in Fig. 10.26(b). At the node, KCL gives

$$3 = \mathbf{I}_o + 0.5\mathbf{I}_o \qquad \Rightarrow \qquad \mathbf{I}_o = 2\text{A}$$

Applying KVL to the outer loop in Fig. 10.26(b) gives

$$\mathbf{V}_s = \mathbf{I}_o(4 + j3 + 2 - j4) = 2(6 - j)$$

The Thevenin impedance is

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{2(6 - j)}{3} = 4 - j0.6667\ \Omega$$

## Practice Problem 10.9

Determine the Thevenin equivalent of the circuit in Fig. 10.27 as seen from the terminals *a-b*.

**Answer:** $\mathbf{Z}_{\text{Th}} = 4.473\angle -7.64^\circ\ \Omega$, $\mathbf{V}_{\text{Th}} = 11.763\angle 72.9^\circ$ volts.

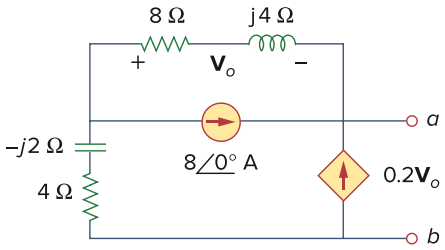



**Figure 10.27**
For Practice Prob. 10.9.

## Example 10.10

Obtain current $\mathbf{I}_o$ in Fig. 10.28 using Norton's theorem.

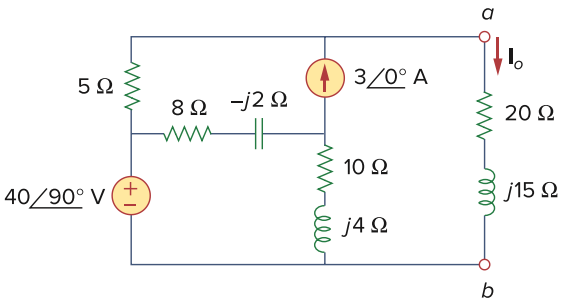



**Figure 10.28**
For Example 10.10.

**Solution:**

Our first objective is to find the Norton equivalent at terminals *a-b*. $\mathbf{Z}_N$ is found in the same way as $\mathbf{Z}_{\text{Th}}$. We set the sources to zero as shown in Fig. 10.29(a). As evident from the figure, the $(8 - j2)$ and $(10 + j4)$ impedances are short-circuited, so that

$$\mathbf{Z}_N = 5\ \Omega$$

To get $\mathbf{I}_N$, we short-circuit terminals *a-b* as in Fig. 10.29(b) and apply mesh analysis. Notice that meshes 2 and 3 form a supermesh because of the current source linking them. For mesh 1,

$$-j40 + (18 + j2)\mathbf{I}_1 - (8 - j2)\mathbf{I}_2 - (10 + j4)\mathbf{I}_3 = 0 \qquad \textbf{(10.10.1)}$$

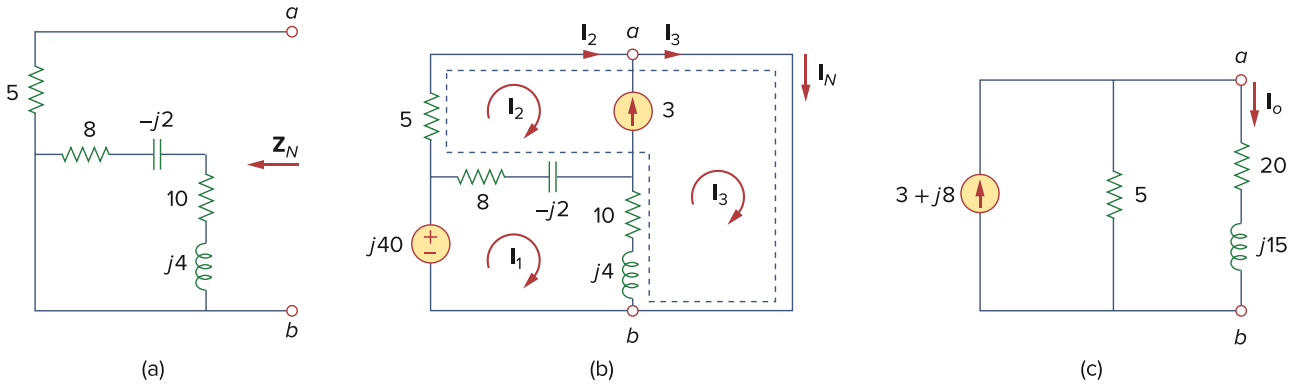



**Figure 10.29**
Solution of the circuit in Fig. 10.28: (a) finding $\mathbf{Z}_N$, (b) finding $\mathbf{V}_N$, (c) calculating $\mathbf{I}_o$.

For the supermesh,

$$(13 - j2)\mathbf{I}_2 + (10 + j4)\mathbf{I}_3 - (18 + j2)\mathbf{I}_1 = 0 \qquad \textbf{(10.10.2)}$$

At node $a$, due to the current source between meshes 2 and 3,

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 \qquad \textbf{(10.10.3)}$$

Adding Eqs. (10.10.1) and (10.10.2) gives

$$-j40 + 5\mathbf{I}_2 = 0 \quad \Rightarrow \quad \mathbf{I}_2 = j8$$

From Eq. (10.10.3),

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 = 3 + j8$$

The Norton current is

$$\mathbf{I}_N = \mathbf{I}_3 = (3 + j8)\ \text{A}$$

Figure 10.29(c) shows the Norton equivalent circuit along with the im - pedance at terminals $a$-$b$. By current division,

$$\mathbf{I}_o = \frac{5}{5 + 20 + j15}\mathbf{I}_N = \frac{3 + j8}{5 + j3} = 1.465\angle 38.48^\circ\ \text{A}$$

## Practice Problem 10.10

Determine the Norton equivalent of the circuit in Fig. 10.30 as seen from terminals $a$-$b$. Use the equivalent to find $\mathbf{I}_o$.

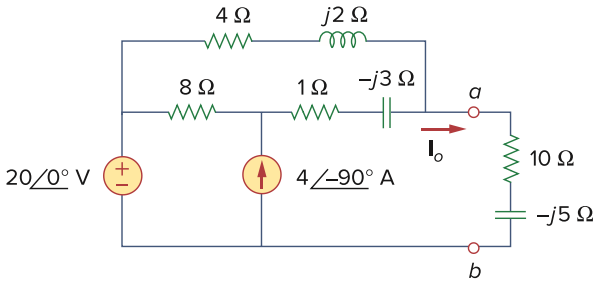



**Figure 10.30**
For Practice Prob. 10.10 and Prob. 10.35.

**Answer:** $\mathbf{Z}_N = 3.176 + j0.706\ \Omega$, $\mathbf{I}_N = 8.396\angle -32.68^\circ$ A, $\mathbf{I}_o = 1.9714\angle -2.10^\circ$ A.

## 10.7 Op Amp AC Circuits

The three steps stated in Section 10.1 also apply to op amp circuits, as long as the op amp is operating in the linear region. As usual, we will assume ideal op amps. (See Section 5.2.) As discussed in Chapter 5, the key to analyzing op amp circuits is to keep two important properties of an ideal op amp in mind:

1. No current enters either of its input terminals.
2. The voltage across its input terminals is zero.

The following examples will illustrate these ideas.

### Example 10.11

Determine $v_o(t)$ for the op amp circuit in Fig. 10.31(a) if $v_s = 3\cos 1000t$ V.

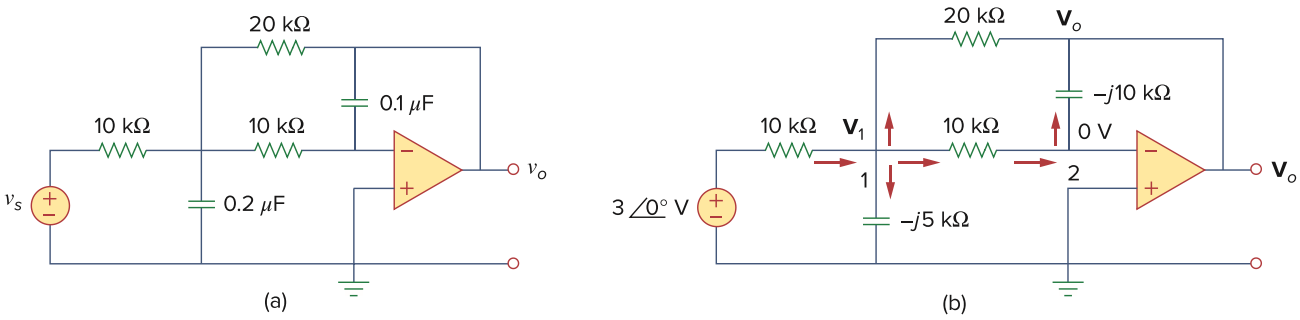



**Figure 10.31**
For Example 10.11: (a) the original circuit in the time domain, (b) its frequency domain equivalent.

**Solution:**
We first transform the circuit to the frequency domain, as shown in Fig. 10.31(b), where $\mathbf{V}_s = 3\angle 0°$, $\omega = 1000$ rad/s. Applying KCL at node 1, we obtain

$$\frac{3\angle 0° - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j5} + \frac{\mathbf{V}_1 - 0}{10} + \frac{\mathbf{V}_1 - \mathbf{V}_o}{20}$$

or

$$6 = (5 + j4)\mathbf{V}_1 - \mathbf{V}_o \qquad \textbf{(10.11.1)}$$

At node 2, KCL gives

$$\frac{\mathbf{V}_1 - 0}{10} = \frac{0 - \mathbf{V}_o}{-j10}$$

which leads to

$$\mathbf{V}_1 = -j\mathbf{V}_o \qquad \textbf{(10.11.2)}$$

Substituting Eq. (10.11.2) into Eq. (10.11.1) yields

$$6 = -j(5 + j4)\mathbf{V}_o - \mathbf{V}_o = (3 - j5)\mathbf{V}_o$$

$$\mathbf{V}_o = \frac{6}{3 - j5} = 1.029\angle 59.04°$$

Hence,

$$v_o(t) = 1.029\cos(1000t + 59.04°)\text{ V}$$

## Practice Problem 10.11

Find $v_o$ and $i_o$ in the op amp circuit of Fig. 10.32. Let $v_s = 12 \cos 5000t$ V.

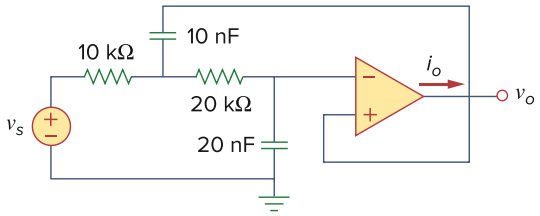



**Figure 10.32**
For Practice Prob. 10.11.

**Answer:** 4 sin 5,000$t$ V, 400 sin 5,000$t$ $\mu$A.

## Example 10.12

Compute the closed-loop gain and phase shift for the circuit in Fig. 10.33. Assume that $R_1 = R_2 = 10\ \text{k}\Omega$, $C_1 = 2\ \mu\text{F}$, $C_2 = 1\ \mu\text{F}$, and $\omega = 200$ rad/s.

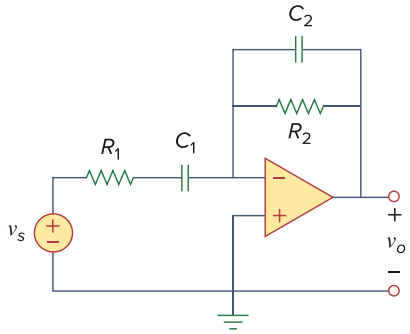



**Figure 10.33**
For Example 10.12.

**Solution:**
The feedback and input impedances are calculated as

$$\mathbf{Z}_f = R_2 \left\| \frac{1}{j\omega C_2} = \frac{R_2}{1 + j\omega R_2 C_2} \right.$$

$$\mathbf{Z}_i = R_1 + \frac{1}{j\omega C_1} = \frac{1 + j\omega R_1 C_1}{j\omega C_1}$$

Since the circuit in Fig. 10.33 is an inverting amplifier, the closed-loop gain is given by

$$\mathbf{G} = \frac{\mathbf{V}_o}{\mathbf{V}_s} = -\frac{\mathbf{Z}_f}{\mathbf{Z}_i} = \frac{-j\omega C_1 R_2}{(1 + j\omega R_1 C_1)(1 + j\omega R_2 C_2)}$$

Substituting the given values of $R_1$, $R_2$, $C_1$, $C_2$, and $\omega$, we obtain

$$\mathbf{G} = \frac{-j4}{(1 + j4)(1 + j2)} = 0.434 \angle 130.6°$$

Thus, the closed-loop gain is 0.434 and the phase shift is 130.6°.

## Practice Problem 10.12

Obtain the closed-loop gain and phase shift for the circuit in Fig. 10.34. Let $R = 10\ \text{k}\Omega$, $C = 1\ \mu\text{F}$, and $\omega = 1000$ rad/s.

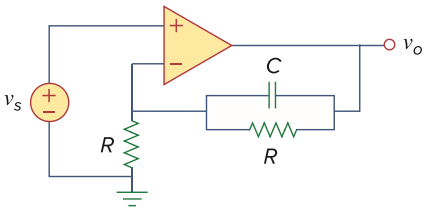



**Figure 10.34**
For Practice Prob. 10.12.

**Answer:** 1.0147, −5.6°.

## 10.8 AC Analysis Using *PSpice*

*PSpice* affords a big relief from the tedious task of manipulating complex numbers in ac circuit analysis. The procedure for using *PSpice* for ac analysis is quite similar to that required for dc analysis. The reader should read Section D.5 in Appendix D for a review of *PSpice* concepts for ac analysis. AC circuit analysis is done in the phasor or frequency domain, and all sources must have the same frequency. Although ac analysis with *PSpice* involves using AC Sweep, our analysis in this chapter requires a single frequency $f = \omega/2\pi$. The output file of *PSpice* contains voltage and current phasors. If necessary, the impedances can be calculated using the voltages and currents in the output file.

### Example 10.13

Obtain $v_o$ and $i_o$ in the circuit of Fig. 10.35 using *PSpice*.

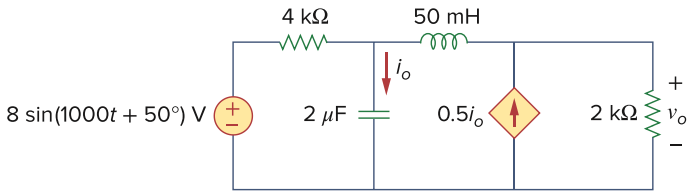



**Figure 10.35**
For Example 10.13.

**Solution:**
We first convert the sine function to cosine.

$$8\sin(1000t + 50°) = 8\cos(1000t + 50° - 90°)$$
$$= 8\cos(1000t - 40°)$$

The frequency $f$ is obtained from $\omega$ as

$$f = \frac{\omega}{2\pi} = \frac{1000}{2\pi} = 159.155 \text{ Hz}$$

The schematic for the circuit is shown in Fig. 10.36. Notice that the current-controlled current source F1 is connected such that its current flows from node 0 to node 3 in conformity with the original circuit in Fig. 10.35. Since we only want the magnitude and phase of $v_o$ and $i_o$, we set the attributes of IPRINT and VPRINT1 each to *AC = yes*, *MAG = yes*, *PHASE = yes*. As a single-frequency analysis, we select **Analysis/Setup/AC Sweep** and enter *Total Pts* = 1, *Start Freq* = 159.155, and *Final Freq* = 159.155. After saving the schematic, we simulate it by selecting **Analysis/Simulate.** The output file includes the source frequency in addition to the attributes checked for the pseudocomponents IPRINT and VPRINT1,

```
FREQ            IM(V_PRINT3)   IP(V_PRINT3)
1.592E+02       3.264E-03      -3.743E+01

FREQ            VM(3)          VP(3)
1.592E+02       1.550E+00      -9.518E+01
```

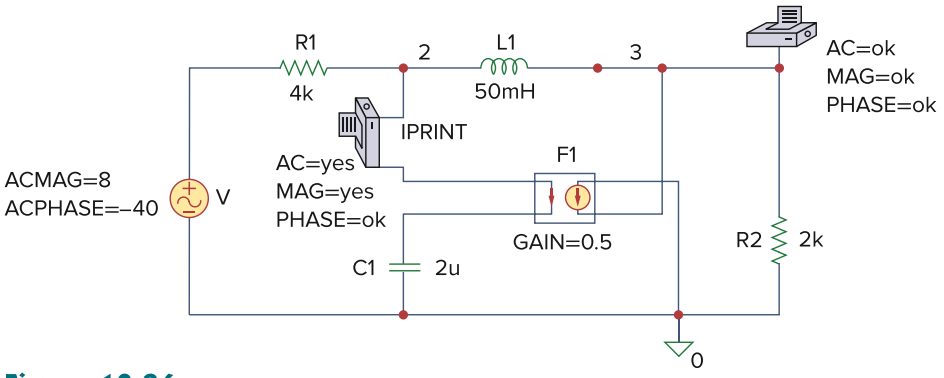



**Figure 10.36**
The schematic of the circuit in Fig. 10.35.

From this output file, we obtain

$$\mathbf{V}_o = 1.55\angle -95.18^\circ \text{ V}, \qquad \mathbf{I}_o = 3.264\angle -37.43^\circ \text{ mA}$$

which are the phasors for

$$v_o = 1.55\cos(1000t - 95.18^\circ) = 1.55\sin(1000t - 5.18^\circ) \text{ V}$$

and

$$i_o = 3.264\cos(1000t - 37.43^\circ) \text{ mA}$$

## Practice Problem 10.13

Use *PSpice* to obtain $v_o$ and $i_o$ in the circuit of Fig. 10.37.

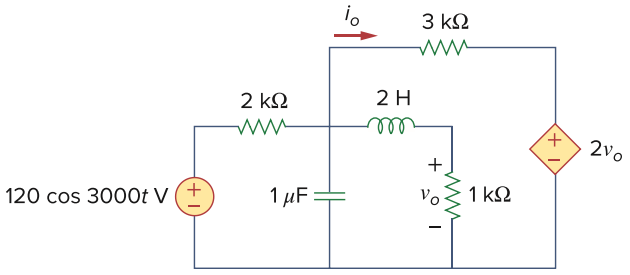



**Figure 10.37**
For Practice Prob. 10.13.

**Answer:** 3.219 cos(3,000*t* – 154.6°) V, 6.527 cos(3,000*t* – 55.12°) mA.

## Example 10.14

Find $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit of Fig. 10.38.

**Solution:**

1. **Define.** In its present form, the problem is clearly stated. Again, we must emphasize that time spent here will save lots of time and expense later on! One thing that might have created a problem for you is that, if the reference was missing for this problem, you would then need to ask the individual assigning the problem where

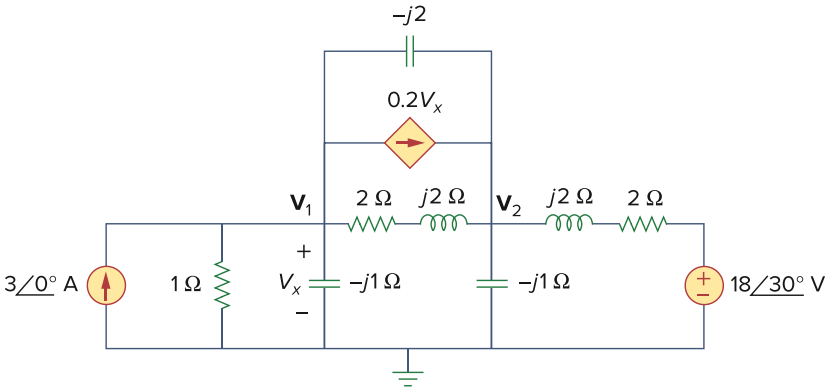



**Figure 10.38**
For Example 10.14.

it is to be located. If you could not do that, then you would need to assume where it should be and then clearly state what you did and why you did it.

2. **Present.** The given circuit is a frequency domain circuit and the unknown node voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ are also frequency domain values. Clearly, we need a process to solve for these unknowns in the frequency domain.
3. **Alternative.** We have two direct alternative solution techniques that we can easily use. We can do a straightforward nodal analysis approach or use *PSpice*. Since this example is in a section dedicated to using *PSpice* to solve problems, we will use *PSpice* to find $\mathbf{V}_1$ and $\mathbf{V}_2$. We can then use nodal analysis to check the answer.
4. **Attempt.** The circuit in Fig. 10.35 is in the time domain, whereas the one in Fig. 10.38 is in the frequency domain. Since we are not given a particular frequency and *PSpice* requires one, we select any frequency consistent with the given impedances. For example, if we select $\omega = 1$ rad/s, the corresponding frequency is $f = \omega/2\pi = 0.15916$ Hz. We obtain the values of the capacitance ($C = 1/\omega X_C$) and inductances ($L = X_L/\omega$). Making these changes results in the schematic in Fig. 10.39. To ease wiring, we have exchanged the positions of the voltage-controlled current source

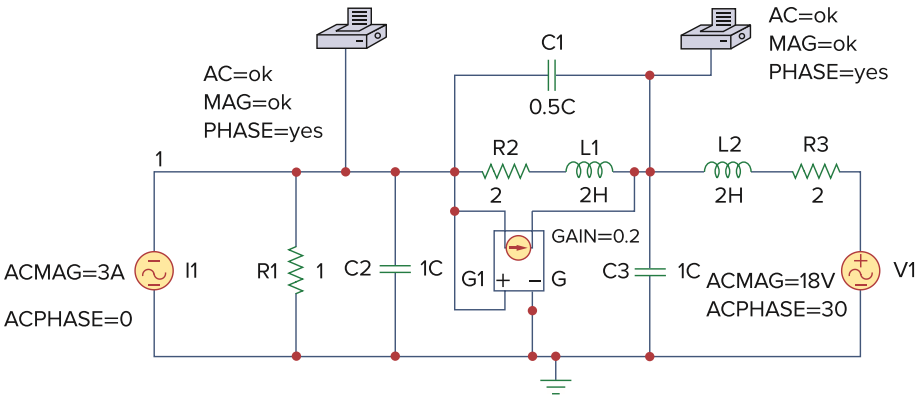



**Figure 10.39**
Schematic for the circuit in the Fig. 10.38.

G1 and the $2 + j2\ \Omega$ impedance. Notice that the current of G1 flows from node 1 to node 3, while the controlling voltage is across the capacitor C2, as required in Fig. 10.38. The attributes of pseudocomponents VPRINT1 are set as shown. As a single-frequency analysis, we select **Analysis/Setup/AC Sweep** and enter *Total Pts* = 1, *Start Freq* = 0.15916, and *Final Freq* = 0.15916. After saving the schematic, we select **Analysis/Simulate** to simulate the circuit. When this is done, the output file includes

```
FREQ          VM(1)          VP(1)
1.592E-01     2.708E+00      -5.673E+01

FREQ          VM(3)          VP(3)
1.592E-01     4.468E+00      -1.026E+02
```

from which we obtain,

$$\mathbf{V}_1 = \mathbf{2.708\angle -56.74^\circ\ V} \qquad \text{and} \qquad \mathbf{V}_2 = \mathbf{6.911\angle -80.72^\circ\ V}$$

5. **Evaluate.** One of the most important lessons to be learned is that when using programs such as *PSpice* you still need to validate the answer. There are many opportunities for making a mistake, including coming across an unknown "bug" in *PSpice* that yields incorrect results.

   So, how can we validate this solution? Obviously, we can rework the entire problem with nodal analysis, and perhaps using *MATLAB*, to see if we obtain the same results. There is another way we will use here: Write the nodal equations and substitute the answers obtained in the *PSpice* solution, and see if the nodal equations are satisfied.

   The nodal equations for this circuit are given below. Note we have substituted $\mathbf{V}_1 = \mathbf{V}_x$ into the dependent source.

$$-3 + \frac{\mathbf{V}_1 - 0}{1} + \frac{\mathbf{V}_1 - 0}{-j1} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{2 + j2} + 0.2\mathbf{V}_1 + \frac{\mathbf{V}_1 - \mathbf{V}_2}{-j2} = 0$$

$$(1 + j + 0.25 - j0.25 + 0.2 + j0.5)\mathbf{V}_1 - (0.25 - j0.25 + j0.5)\mathbf{V}_2 = 3$$

$$(1.45 + j1.25)\mathbf{V}_1 - (0.25 + j0.25)\mathbf{V}_2 = 3$$

$$1.9144\angle 40.76^\circ\ \mathbf{V}_1 - 0.3536\angle 45^\circ\ \mathbf{V}_2 = 3$$

   Now, to check the answer, we substitute the *PSpice* answers into this.

$$1.9144\angle 40.76^\circ \times 2.708\angle -56.74^\circ - 0.3536\angle 45^\circ \times 6.911\angle -80.72^\circ$$

$$= 5.184\angle -15.98^\circ - 2.444\angle -35.72^\circ$$

$$= 4.984 - j1.4272 - 1.9842 + j1.4269$$

$$= 3 - j0.0003 \qquad \text{[Answer checks]}$$

6. **Satisfactory?** Although we used only the equation from node 1 to check the answer, this is more than satisfactory to validate the answer from the *PSpice* solution. We can now present our work as a solution to the problem.

## Practice Problem 10.14

Obtain $\mathbf{V}_x$ and $\mathbf{I}_x$ in the circuit depicted in Fig. 10.40.

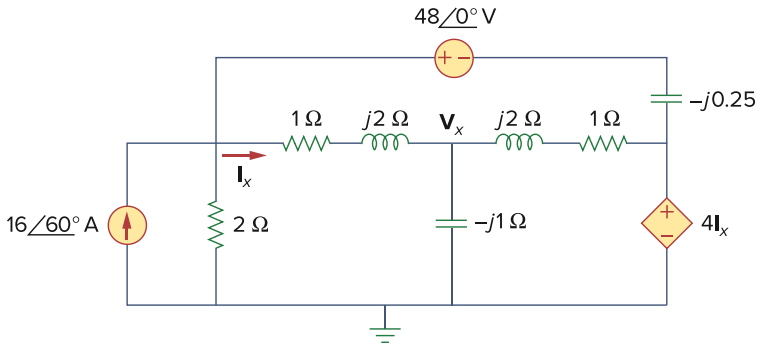



**Figure 10.40**
For Practice Prob. 10.14.

**Answer:** $39.37\angle 44.78°$ V, $10.336\angle 158°$ A.

## 10.9 Applications

The concepts learned in this chapter will be applied in later chapters to calculate electric power and determine frequency response. The concepts are also used in analyzing coupled circuits, three-phase circuits, ac transistor circuits, filters, oscillators, and other ac circuits. In this section, we apply the concepts to develop two practical ac circuits: the capacitance multiplier and the sine wave oscillators.

### 10.9.1 Capacitance Multiplier

The op amp circuit in Fig. 10.41 is known as a *capacitance multiplier*, for reasons that will become obvious. Such a circuit is used in integrated circuit technology to produce a multiple of a small physical capacitance $C$ when a large capacitance is needed. The circuit in Fig. 10.41 can be used to multiply capacitance values by a factor up to 1,000. For example, a 10-pF capacitor can be made to behave like a 100-nF capacitor.

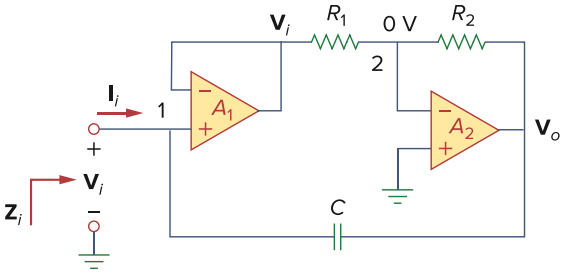



**Figure 10.41**
Capacitance multiplier.

In Fig. 10.41, the first op amp operates as a voltage follower, while the second one is an inverting amplifier. The voltage follower isolates the capacitance formed by the circuit from the loading imposed by the inverting amplifier. Since no current enters the input terminals of the op amp, the input current $\mathbf{I}_i$ flows through the feedback capacitor. Hence, at node 1,

$$\mathbf{I}_i = \frac{\mathbf{V}_i - \mathbf{V}_o}{1/j\omega C} = j\omega C(\mathbf{V}_i - \mathbf{V}_o) \tag{10.3}$$

Applying KCL at node 2 gives

$$\frac{\mathbf{V}_i - 0}{R_1} = \frac{0 - \mathbf{V}_o}{R_2}$$

or

$$\mathbf{V}_o = -\frac{R_2}{R_1}\mathbf{V}_i \tag{10.4}$$

Substituting Eq. (10.4) into (10.3) gives

$$\mathbf{I}_i = j\omega C\left(1 + \frac{R_2}{R_1}\right)\mathbf{V}_i$$

or

$$\frac{\mathbf{I}_i}{\mathbf{V}_i} = j\omega\left(1 + \frac{R_2}{R_1}\right)C \tag{10.5}$$

The input impedance is

$$\mathbf{Z}_i = \frac{\mathbf{V}_i}{\mathbf{I}_i} = \frac{1}{j\omega C_{\text{eq}}} \tag{10.6}$$

where

$$C_{\text{eq}} = \left(1 + \frac{R_2}{R_1}\right)C \tag{10.7}$$

Thus, by a proper selection of the values of $R_1$ and $R_2$, the op amp circuit in Fig. 10.41 can be made to produce an effective capacitance between the input terminal and ground, which is a multiple of the physical capacitance $C$. The size of the effective capacitance is practically limited by the inverted output voltage limitation. Thus, the larger the capacitance multiplication, the smaller is the allowable input voltage to prevent the op amps from reaching saturation.

A similar op amp circuit can be designed to simulate inductance. (See Prob. 10.89.) There is also an op amp circuit configuration to create a resistance multiplier.

## Example 10.15

Calculate $C_{\text{eq}}$ in Fig. 10.41 when $R_1 = 10\ \text{k}\Omega$, $R_2 = 1\ \text{M}\Omega$, and $C = 1\ \text{nF}$.

**Solution:**

From Eq. (10.7)

$$C_{\text{eq}} = \left(1 + \frac{R_2}{R_1}\right)C = \left(1 + \frac{1 \times 10^6}{10 \times 10^3}\right) 1\ \text{nF} = 101\ \text{nF}$$

**Practice Problem 10.15**

Determine the equivalent capacitance of the op amp circuit in Fig. 10.41 if $R_1 = 10\ \text{k}\Omega$, $R_2 = 10\ \text{M}\Omega$, and $C = 10\ \text{nF}$.

**Answer:** 10 $\mu$F.

### 10.9.2 Oscillators

We know that dc is produced by batteries. But how do we produce ac? One way is using *oscillators,* which are circuits that convert dc to ac.

An **oscillator** is a circuit that produces an ac waveform as output when powered by a dc input.

The only external source an oscillator needs is the dc power supply. Ironically, the dc power supply is usually obtained by converting the ac supplied by the electric utility company to dc. Having gone through the trouble of conversion, one may wonder why we need to use the oscillator to convert the dc to ac again. The problem is that the ac supplied by the utility company operates at a preset frequency of 60 Hz in the United States (50 Hz in some other nations), whereas many applications such as electronic circuits, communication systems, and microwave devices require internally generated frequencies that range from 0 to 10 GHz or higher. Oscillators are used for generating these frequencies.

This corresponds to $\omega = 2\pi f = 377$ rad/s.

In order for sine wave oscillators to sustain oscillations, they must meet the *Barkhausen criteria*:

1. The overall gain of the oscillator must be unity or greater. Therefore, losses must be compensated for by an amplifying device.
2. The overall phase shift (from input to output and back to the input) must be zero.

Three common types of sine wave oscillators are phase-shift, twin-$T$, and Wien-bridge oscillators. Here we consider only the Wien-bridge oscillator.

The *Wien-bridge oscillator* is widely used for generating sinusoids in the frequency range below 1 MHz. It is an $RC$ op amp circuit with only a few components, easily tunable and easy to design. As shown in Fig. 10.42, the oscillator essentially consists of a noninverting amplifier with two feedback paths: The positive feedback path to the noninverting input creates oscillations, while the negative feedback path to the inverting input controls the gain. If we define the impedances of the $RC$ series and parallel combinations as $\mathbf{Z}_s$ and $\mathbf{Z}_p$, then

$$\mathbf{Z}_s = R_1 + \frac{1}{j\omega C_1} = R_1 - \frac{j}{\omega C_1} \tag{10.8}$$

$$\mathbf{Z}_p = R_2 \| \frac{1}{j\omega C_2} = \frac{R_2}{1 + j\omega R_2 C_2} \tag{10.9}$$

The feedback ratio is

$$\frac{\mathbf{V}_2}{\mathbf{V}_o} = \frac{\mathbf{Z}_p}{\mathbf{Z}_s + \mathbf{Z}_p} \tag{10.10}$$

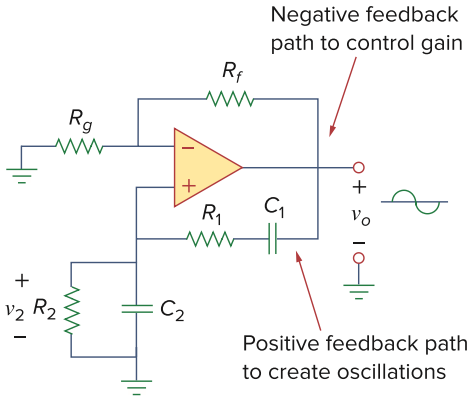



**Figure 10.42**
Wien-bridge oscillator.

Substituting Eqs. (10.8) and (10.9) into Eq. (10.10) gives

$$\frac{\mathbf{V}_2}{\mathbf{V}_o} = \frac{R_2}{R_2 + \left(R_1 - \dfrac{j}{\omega C_1}\right)(1 + j\omega R_2 C_2)} = \frac{\omega R_2 C_1}{\omega(R_2C_1 + R_1C_1 + R_2C_2) + j(\omega^2 R_1 C_1 R_2 C_2 - 1)} \tag{10.11}$$

To satisfy the second Barkhausen criterion, $\mathbf{V}_2$ must be in phase with $\mathbf{V}_o$, which implies that the ratio in Eq. (10.11) must be purely real. Hence, the imaginary part must be zero. Setting the imaginary part equal to zero gives the oscillation frequency $\omega_o$ as

$$\omega_o^2 R_1 C_1 R_2 C_2 - 1 = 0$$

or

$$\omega_o = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \tag{10.12}$$

In most practical applications, $R_1 = R_2 = R$ and $C_1 = C_2 = C$, so that

$$\omega_o = \frac{1}{RC} = 2\pi f_o \tag{10.13}$$

or

$$\boxed{f_o = \frac{1}{2\pi RC}} \tag{10.14}$$

Substituting Eq. (10.13) and $R_1 = R_2 = R$, $C_1 = C_2 = C$ into Eq. (10.11) yields

$$\frac{\mathbf{V}_2}{\mathbf{V}_o} = \frac{1}{3} \tag{10.15}$$

Thus, in order to satisfy the first Barkhausen criterion, the op amp must compensate by providing a gain of 3 or greater so that the overall gain is at least 1 or unity. We recall that for a noninverting amplifier,

$$\frac{\mathbf{V}_o}{\mathbf{V}_2} = 1 + \frac{R_f}{R_g} = 3 \tag{10.16}$$

or

$$R_f = 2R_g \tag{10.17}$$

Due to the inherent delay caused by the op amp, Wien-bridge oscillators are limited to operating in the frequency range of 1 MHz or less.

## Example 10.16

Design a Wien-bridge circuit to oscillate at 100 kHz.

**Solution:**

Using Eq. (10.14), we obtain the time constant of the circuit as

$$RC = \frac{1}{2\pi f_o} = \frac{1}{2\pi \times 100 \times 10^3} = 1.59 \times 10^{-6} \tag{10.16.1}$$

If we select $R = 10\ \text{k}\Omega$, then we can select $C = 159$ pF to satisfy Eq. (10.16.1). Since the gain must be 3, $R_f/R_g = 2$. We could select $R_f = 20\ \text{k}\Omega$ while $R_g = 10\ \text{k}\Omega$.

### Practice Problem 10.16

In the Wien-bridge oscillator circuit in Fig. 10.42, let $R_1 = R_2 = 2.5$ kΩ, $C_1 = C_2 = 1$ nF. Determine the frequency $f_o$ of the oscillator.

**Answer:** 63.66 kHz.

## 10.10 Summary

1. We apply nodal and mesh analysis to ac circuits by applying KCL and KVL to the phasor form of the circuits.
2. In solving for the steady-state response of a circuit that has independent sources with different frequencies, each independent source *must* be considered separately. The most natural approach to analyzing such circuits is to apply the superposition theorem. A separate phasor circuit for each frequency *must* be solved independently, and the corresponding response should be obtained in the time domain. The overall response is the sum of the time domain responses of all the individual phasor circuits.
3. The concept of source transformation is also applicable in the frequency domain.
4. The Thevenin equivalent of an ac circuit consists of a voltage source $\mathbf{V}_{Th}$ in series with the Thevenin impedance $\mathbf{Z}_{Th}$.
5. The Norton equivalent of an ac circuit consists of a current source $\mathbf{I}_N$ in parallel with the Norton impedance $\mathbf{Z}_N (= \mathbf{Z}_{Th})$.
6. *PSpice* is a simple and powerful tool for solving ac circuit problems. It relieves us of the tedious task of working with the complex numbers involved in steady-state analysis.
7. The capacitance multiplier and the ac oscillator provide two typical applications for the concepts presented in this chapter. A capacitance multiplier is an op amp circuit used in producing a multiple of a physical capacitance. An oscillator is a device that uses a dc input to generate an ac output.

## Review Questions

**10.1** The voltage $\mathbf{V}_o$ across the capacitor in Fig. 10.43 is:

(a) $5\angle 0°$ V (b) $7.071\angle 45°$ V

(c) $7.071\angle -45°$ V (d) $5\angle -45°$ V

**10.2** The value of the current $\mathbf{I}_o$ in the circuit of Fig. 10.44 is:

(a) $4\angle 0°$ A (b) $2.4\angle -90°$ A

(c) $0.6\angle 0°$ A (d) $-1$ A

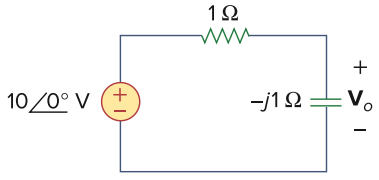



**Figure 10.43**
For Review Question 10.1.

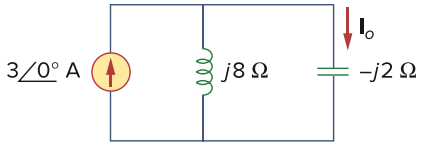



**Figure 10.44**
For Review Question 10.2.

**10.3** Using nodal analysis, the value of $\mathbf{V}_o$ in the circuit of Fig. 10.45 is:

(a) −24 V (b) −8 V

(c) 8 V (d) 24 V

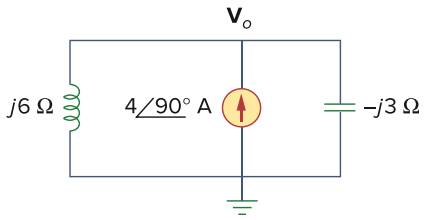



**Figure 10.45**
For Review Question 10.3.

**10.4** In the circuit of Fig. 10.46, current $i(t)$ is:

(a) 10 cos $t$ A (b) 10 sin $t$ A (c) 5 cos $t$ A

(d) 5 sin $t$ A (e) 4.472 cos($t$ − 63.43°) A

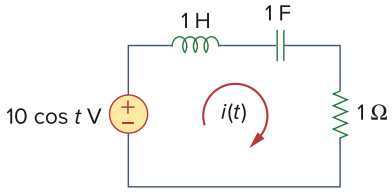



**Figure 10.46**
For Review Question 10.4.

**10.5** Refer to the circuit in Fig. 10.47 and observe that the two sources do not have the same frequency. The current $i_x(t)$ can be obtained by:

(a) source transformation

(b) the superposition theorem

(c) *PSpice*

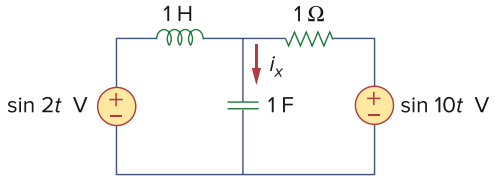



**Figure 10.47**
For Review Question 10.5.

**10.6** For the circuit in Fig. 10.48, the Thevenin impedance at terminals $a$-$b$ is:

(a) 1 Ω (b) 0.5 − $j$0.5 Ω

(c) 0.5 + $j$0.5 Ω (d) 1 + $j$2 Ω

(e) 1 − $j$2 Ω

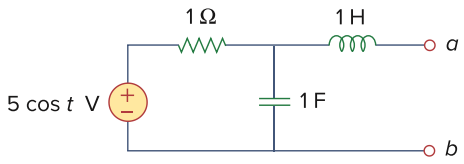



**Figure 10.48**
For Review Questions 10.6 and 10.7.

**10.7** In the circuit of Fig. 10.48, the Thevenin voltage at terminals $a$-$b$ is:

(a) 3.535∠−45° V (b) 3.535∠45° V

(c) 7.071∠−45° V (d) 7.071∠45° V

**10.8** Refer to the circuit in Fig. 10.49. The Norton equivalent impedance at terminals $a$-$b$ is:

(a) −$j$4 Ω (b) −$j$2 Ω

(c) $j$2 Ω (d) $j$4 Ω

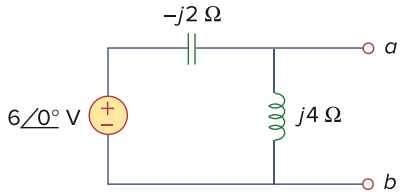



**Figure 10.49**
For Review Questions 10.8 and 10.9.

**10.9** The Norton current at terminals $a$-$b$ in the circuit of Fig. 10.49 is:

(a) 1∠0° A (b) 1.5∠−90° A

(c) 1.5∠90° A (d) 3∠90° A

**10.10** *PSpice* can handle a circuit with two independent sources of different frequencies.

(a) True (b) False

*Answers: 10.1c, 10.2a, 10.3d, 10.4a, 10.5b, 10.6c, 10.7a, 10.8a, 10.9d, 10.10b.*

# Problems

## Section 10.2 Nodal Analysis

**10.1** Determine $i$ in the circuit of Fig. 10.50.

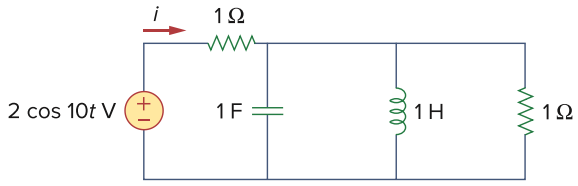



**Figure 10.50**
For Prob. 10.1.

**10.2** Using Fig. 10.51, design a problem to help other students better understand nodal analysis.

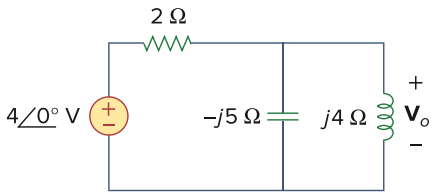



**Figure 10.51**
For Prob. 10.2.

**10.3** Determine $v_o$ in the circuit of Fig. 10.52.

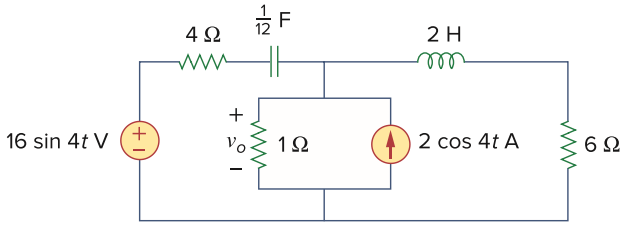



**Figure 10.52**
For Prob. 10.3.

**10.4** Compute $v_o(t)$ in the circuit of Fig. 10.53.

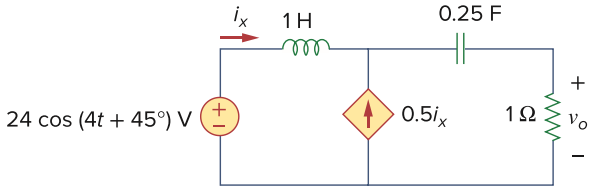



**Figure 10.53**
For Prob. 10.4.

**10.5** Find $i_o$ in the circuit of Fig. 10.54.

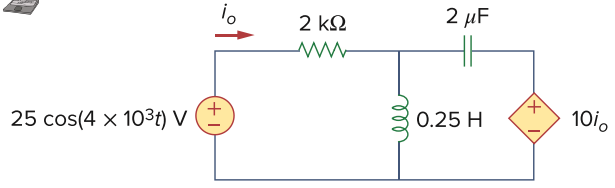



**Figure 10.54**
For Prob. 10.5.

**10.6** Determine $\mathbf{V}_x$ in Fig. 10.55.

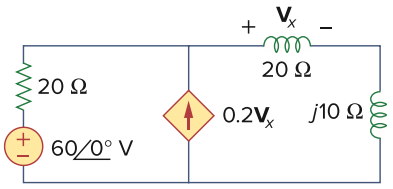



**Figure 10.55**
For Prob. 10.6.

**10.7** Use nodal analysis to find $\mathbf{V}$ in the circuit of Fig. 10.56.

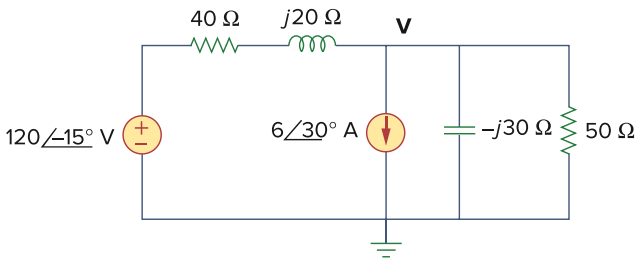



**Figure 10.56**
For Prob. 10.7.

**10.8** Use nodal analysis to find current $i_o$ in the circuit of Fig. 10.57. Let $i_s = 6\cos(200t + 15°)$ A.

ML

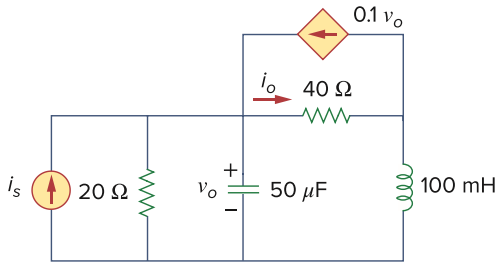



**Figure 10.57**
For Prob. 10.8.

**10.9** Use nodal analysis to find $v_o$ in the circuit of Fig. 10.58.

ML

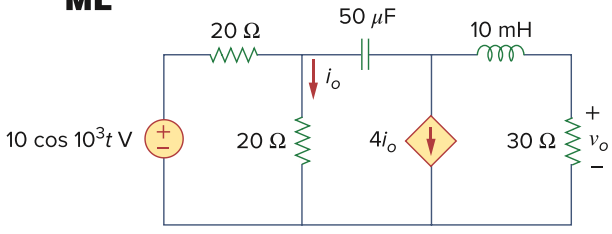



**Figure 10.58**
For Prob. 10.9.

**10.10** Use nodal analysis to find $v_o$ in the circuit of Fig. 10.59. Let $\omega = 2$ krad/s.

ML

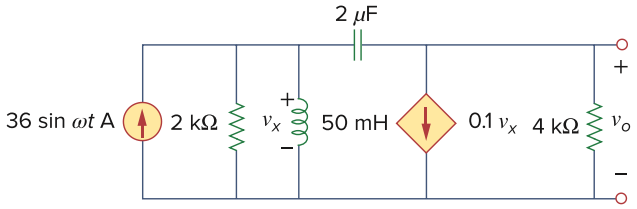



**Figure 10.59**
For Prob. 10.10.

**10.11** Using nodal analysis, find $i_o(t)$ in the circuit in Fig. 10.60.

ML

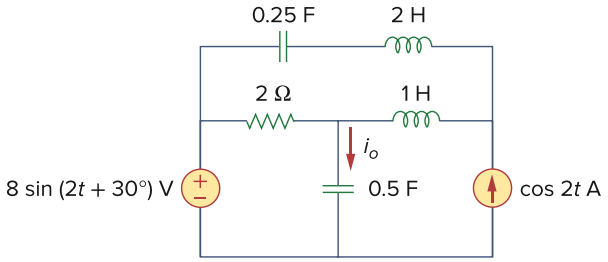



**Figure 10.60**
For Prob. 10.11.

**10.12** Using Fig. 10.61, design a problem to help other students better understand nodal analysis.

EDD

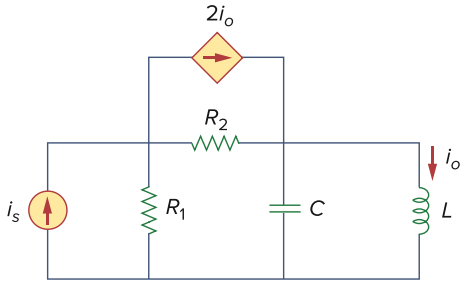



**Figure 10.61**
For Prob. 10.12.

**10.13** Determine $\mathbf{V}_x$ in the circuit of Fig. 10.62 using any method of your choice.

ML

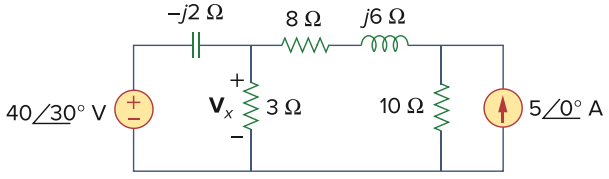



**Figure 10.62**
For Prob. 10.13.

**10.14** Calculate the voltage at nodes 1 and 2 in the circuit of Fig. 10.63 using nodal analysis.

ML

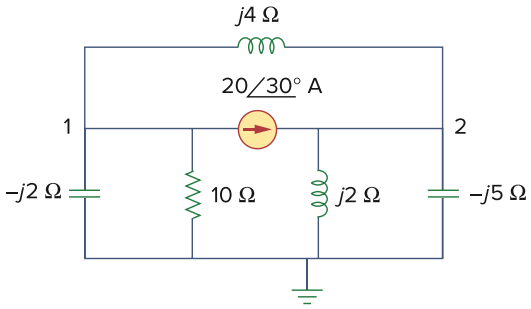



**Figure 10.63**
For Prob. 10.14.

**10.15** Solve for the current $\mathbf{I}$ in the circuit of Fig. 10.64 using nodal analysis.

ML

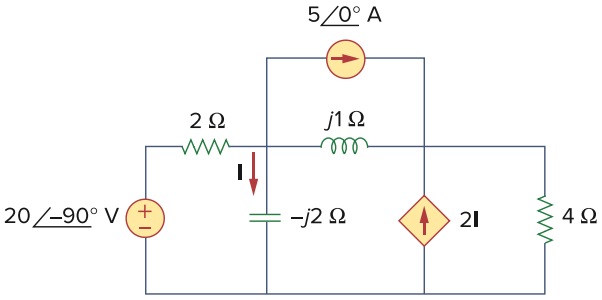



**Figure 10.64**
For Prob. 10.15.

**10.16** Use nodal analysis to find $\mathbf{V}_x$ in the circuit shown in Fig. 10.65.

ML

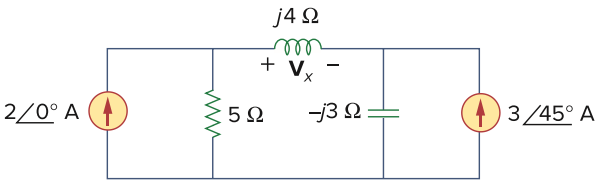



**Figure 10.65**
For Prob. 10.16.

**10.17** By nodal analysis, obtain current $\mathbf{I}_o$ in the circuit of Fig. 10.66.

ML

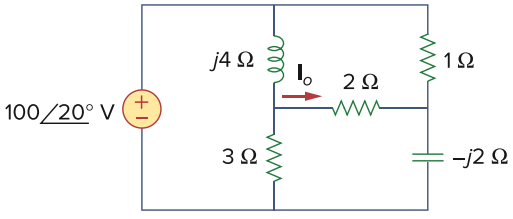



**Figure 10.66**
For Prob. 10.17.

**10.18** Use nodal analysis to obtain $\mathbf{V}_o$ in the circuit of Fig. 10.67 below.

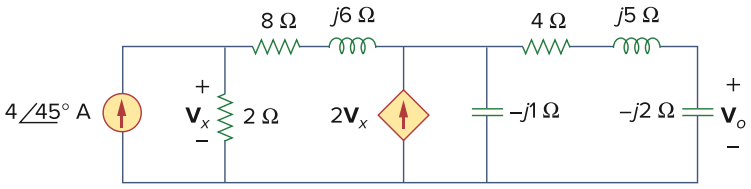



**Figure 10.67**
For Prob. 10.18.

**10.19** Obtain $\mathbf{V}_o$ in Fig. 10.68 using nodal analysis.

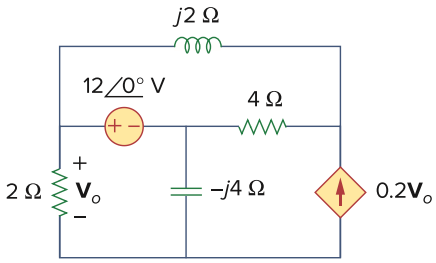



**Figure 10.68**
For Prob. 10.19.

**10.20** Refer to Fig. 10.69. If $v_s(t) = V_m \sin \omega t$ and $v_o(t) = A \sin(\omega t + \phi)$, derive the expressions for $A$ and $\phi$.

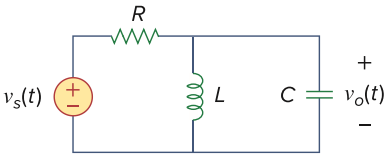



**Figure 10.69**
For Prob. 10.20.

**10.21** For each of the circuits in Fig. 10.70, find $\mathbf{V}_o/\mathbf{V}_i$ for $\omega = 0$, $\omega \to \infty$, and $\omega^2 = 1/LC$.

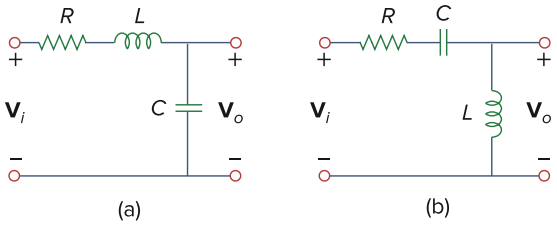



**Figure 10.70**
For Prob. 10.21.

**10.22** For the circuit in Fig. 10.71, determine $\mathbf{V}_o/\mathbf{V}_s$.

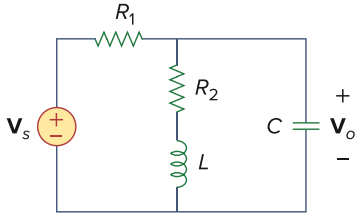



**Figure 10.71**
For Prob. 10.22.

**10.23** Using nodal analysis obtain $\mathbf{V}$ in the circuit of Fig. 10.72.

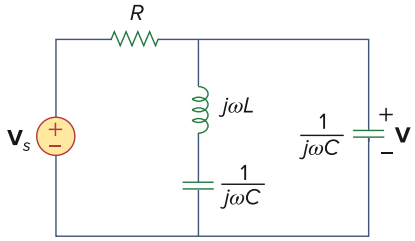



**Figure 10.72**
For Prob. 10.23.

## Section 10.3 Mesh Analysis

**10.24** Design a problem to help other students better understand mesh analysis.

**10.25** Solve for $i_o$ in Fig. 10.73 using mesh analysis.

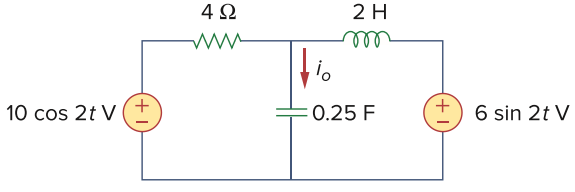



**Figure 10.73**
For Prob. 10.25.

**10.26** Use mesh analysis to find current $i_o$ in the circuit of Fig. 10.74.

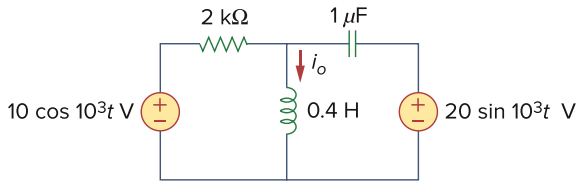



**Figure 10.74**
For Prob. 10.26.

**10.27** Using mesh analysis, find $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 10.75.

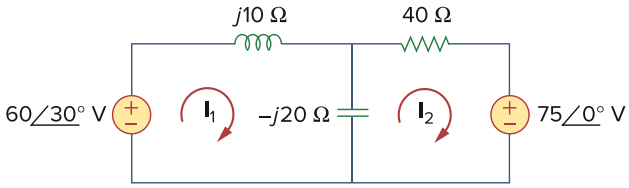



**Figure 10.75**
For Prob. 10.27.

**10.28** In the circuit of Fig. 10.76, determine the mesh currents $i_1$ and $i_2$. Let $v_1 = 10 \cos 4t$ V and $v_2 = 20 \cos(4t - 30°)$ V.

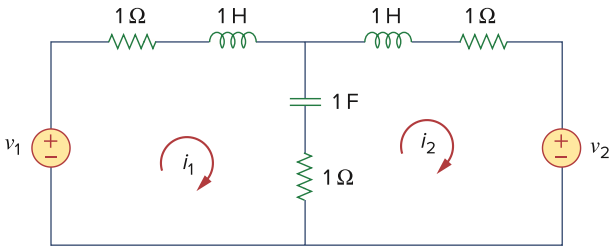



**Figure 10.76**
For Prob. 10.28.

**10.29** Using Fig. 10.77, design a problem to help other students better understand mesh analysis.

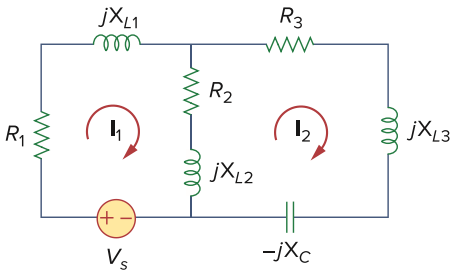



**Figure 10.77**
For Prob. 10.29.

**10.30** Use mesh analysis to find $v_o$ in the circuit of Fig. 10.78. Let $v_{s1} = 120 \cos(100t + 90°)$ V, $v_{s2} = 80 \cos 100t$ V.

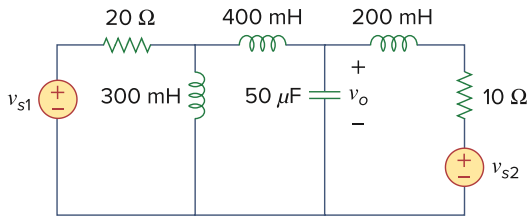



**Figure 10.78**
For Prob. 10.30.

**10.31** Use mesh analysis to determine current $\mathbf{I}_o$ in the circuit of Fig. 10.79 below.

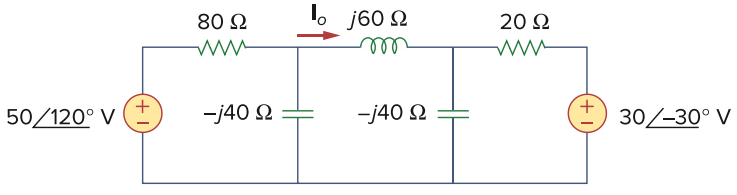



**Figure 10.79**
For Prob. 10.31.

**10.32** Determine $\mathbf{V}_o$ and $\mathbf{I}_o$ in the circuit of Fig. 10.80 using mesh analysis.
ML

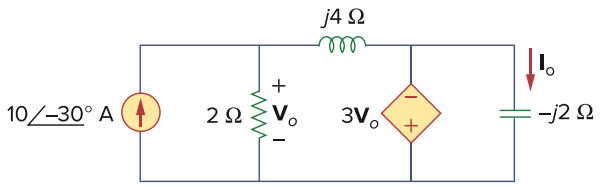



**Figure 10.80**
For Prob. 10.32.

**10.33** Compute $\mathbf{I}$ in Prob. 10.15 using mesh analysis.
ML

**10.34** Use mesh analysis to find $\mathbf{I}_o$ in Fig. 10.28 (for Example 10.10).
ML

**10.35** Calculate $\mathbf{I}_o$ in Fig. 10.30 (for Practice Prob. 10.10) using mesh analysis.
ML

**10.36** Compute $\mathbf{V}_o$ in the circuit of Fig. 10.81 using mesh analysis.
ML

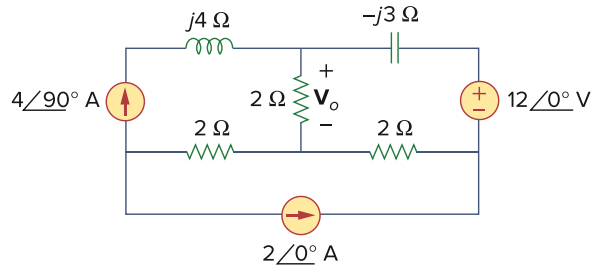



**Figure 10.81**
For Prob. 10.36.

**10.37** Use mesh analysis to find currents $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ in the circuit of Fig. 10.82.
ML

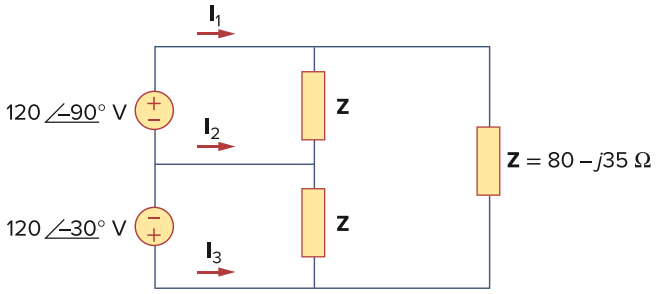



**Figure 10.82**
For Prob. 10.37.

**10.38** Using mesh analysis, obtain $\mathbf{I}_o$ in the circuit shown in Fig. 10.83.
ML

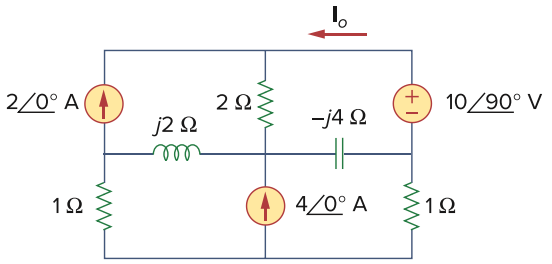



**Figure 10.83**
For Prob. 10.38.

**10.39** Find $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$, and $\mathbf{I}_x$ in the circuit of Fig. 10.84.
ML

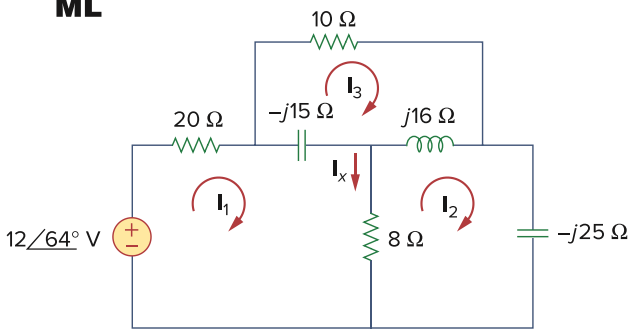



**Figure 10.84**
For Prob. 10.39.

## Section 10.4 Superposition Theorem

**10.40** Find $i_o$ in the circuit shown in Fig. 10.85 using superposition.

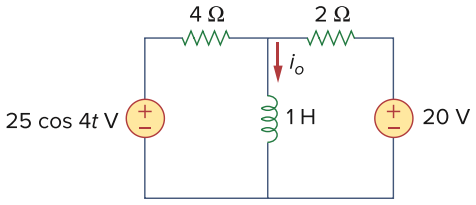



**Figure 10.85**
For Prob. 10.40.

**10.41** Find $v_o$ for the circuit in Fig. 10.86, assuming that $i_s(t) = 2 \sin(2t) + 3 \cos(4t)$ A.

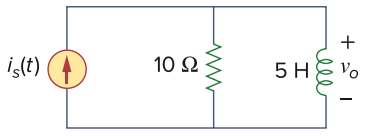



**Figure 10.86**
For Prob. 10.41.

**10.42** Using Fig. 10.87, design a problem to help other students better understand the superposition theorem.

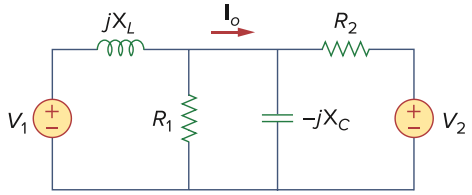



**Figure 10.87**
For Prob. 10.42.

**10.43** Using the superposition principle, find $i_x$ in the circuit of Fig. 10.88.

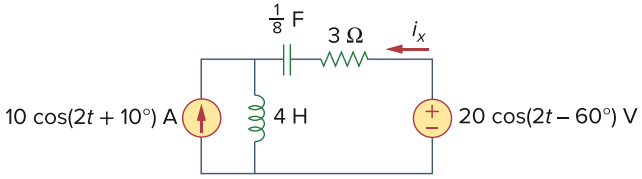



**Figure 10.88**
For Prob. 10.43.

**10.44** Use the superposition principle to obtain $v_x$ in the circuit of Fig. 10.89. Let $v_s = 50 \sin 2t$ V and $i_s = 12 \cos(6t + 10°)$ A.

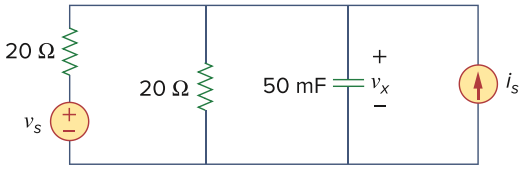



**Figure 10.89**
For Prob. 10.44.

**10.45** Use superposition to find $i(t)$ in the circuit of Fig. 10.90.

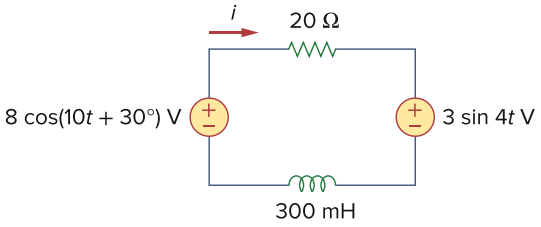



**Figure 10.90**
For Prob. 10.45.

**10.46** Solve for $v_o(t)$ in the circuit of Fig. 10.91 using the superposition principle.

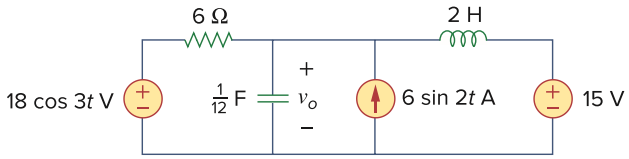



**Figure 10.91**
For Prob. 10.46.

**10.47** Determine $i_o$ in the circuit of Fig. 10.92, using the superposition principle.

ML

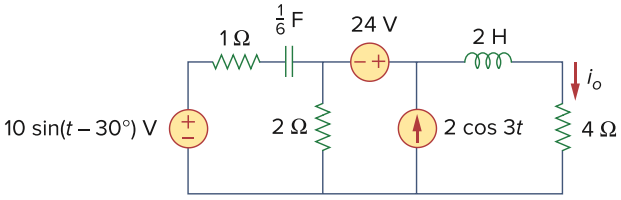



**Figure 10.92**
For Prob. 10.47.

**10.48** Find $i_o$ in the circuit of Fig. 10.93 using superposition.

ML

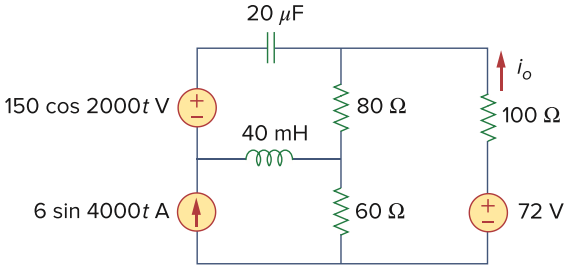



**Figure 10.93**
For Prob. 10.48.

## Section 10.5 Source Transformation

**10.49** Using source transformation, find $i$ in the circuit of Fig. 10.94.

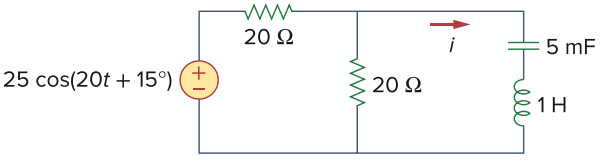



**Figure 10.94**
For Prob. 10.49.

**10.50** Using Fig. 10.95, design a problem to help other students understand source transformation.

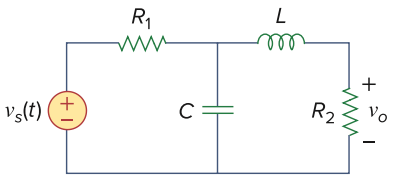



**Figure 10.95**
For Prob. 10.50.

**10.51** Use source transformation to find $\mathbf{I}_o$ in the circuit of Prob. 10.42.

**10.52** Use the method of source transformation to find $\mathbf{I}_x$ in the circuit of Fig. 10.96.

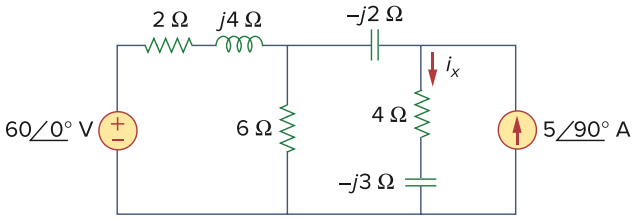



**Figure 10.96**
For Prob. 10.52.

**10.53** Use the concept of source transformation to find $\mathbf{V}_o$ in the circuit of Fig. 10.97.

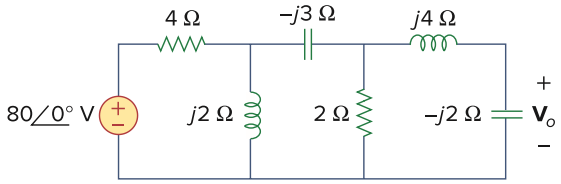
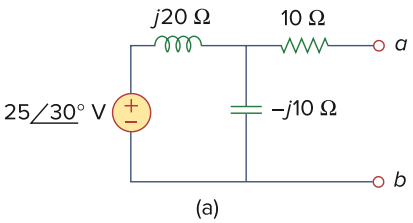



**Figure 10.97**
For Prob. 10.53.

**10.54** Rework Prob. 10.7 using source transformation.

## Section 10.6 Thevenin and Norton Equivalent Circuits

**10.55** Find the Thevenin and Norton equivalent circuits at terminals *a-b* for each of the circuits in Fig. 10.98.



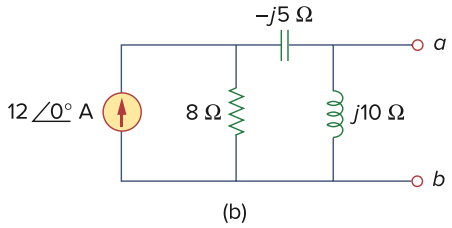
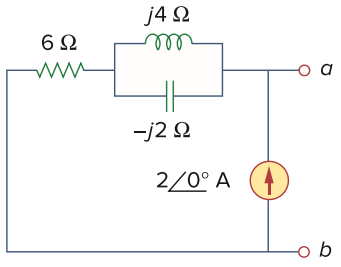




**Figure 10.98**
For Prob. 10.55.

**10.56** For each of the circuits in Fig. 10.99, obtain Thevenin and Norton equivalent circuits at terminals *a-b*.



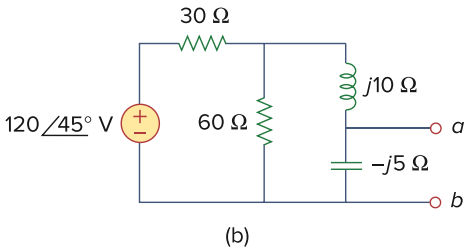




**Figure 10.99**
For Prob. 10.56.

**10.57** Using Fig. 10.100, design a problem to help other students better understand Thevenin and Norton equivalent circuits.

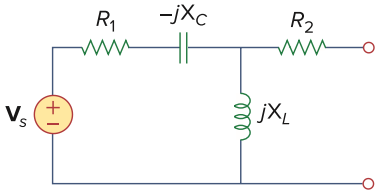



**Figure 10.100**
For Prob. 10.57.

**10.58** For the circuit depicted in Fig. 10.101, find the Thevenin equivalent circuit at terminals *a-b*.

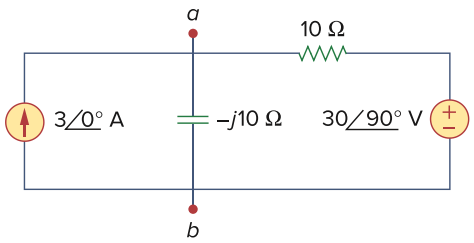



**Figure 10.101**
For Prob. 10.58.

**10.59** Calculate the output impedance of the circuit shown in Fig. 10.102.

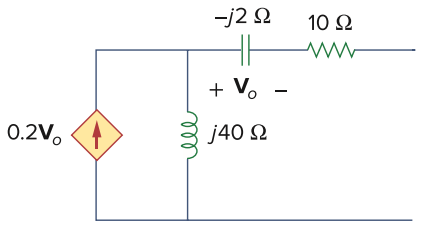



**Figure 10.102**
For Prob. 10.59.

**10.60** Find the Thevenin equivalent of the circuit in Fig. 10.103 as seen from:

(a) terminals *a-b* (b) terminals *c-d*

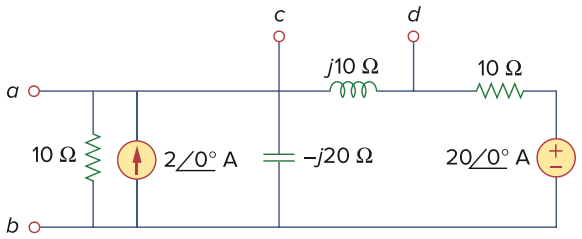



**Figure 10.103**
For Prob. 10.60.

**10.61** Find the Thevenin equivalent at terminals *a-b* of the circuit in Fig. 10.104.

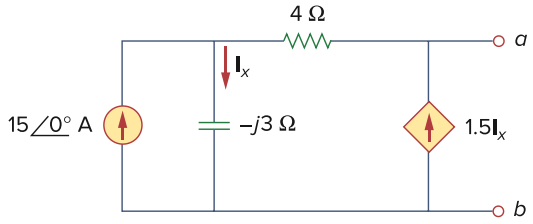



**Figure 10.104**
For Prob. 10.61.

**10.62** Using Thevenin's theorem, find $v_o$ in the circuit of Fig. 10.105.

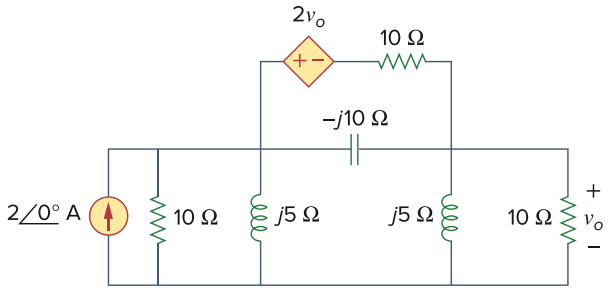



**Figure 10.105**
For Prob. 10.62.

**10.63** Obtain the Norton equivalent of the circuit depicted in Fig. 10.106 at terminals *a-b*.

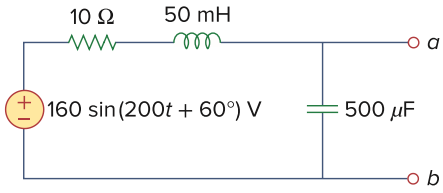



**Figure 10.106**
For Prob. 10.63.

**10.64** For the circuit shown in Fig. 10.107, find the Norton equivalent circuit at terminals *a-b*.

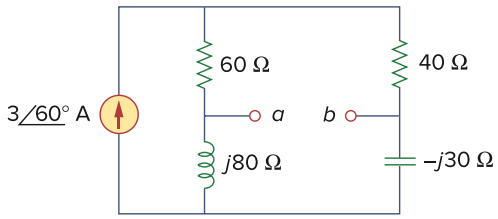



**Figure 10.107**
For Prob. 10.64.

**10.65** Using Fig. 10.108, design a problem to help other students better understand Norton's theorem.

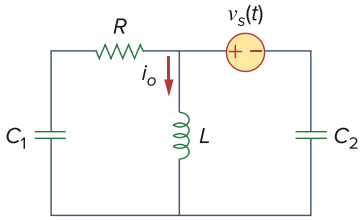



**Figure 10.108**
For Prob. 10.65.

**10.66** At terminals *a-b*, obtain Thevenin and Norton equivalent circuits for the network depicted in Fig. 10.109. Take $\omega = 10$ rad/s.

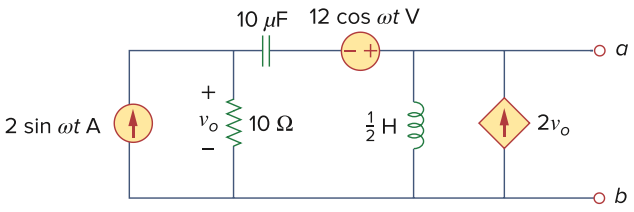



**Figure 10.109**
For Prob. 10.66.

**10.67** Find the Thevenin and Norton equivalent circuits at terminals *a-b* in the circuit of Fig. 10.110.

ML

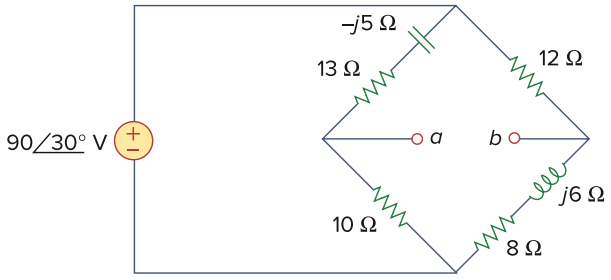



**Figure 10.110**
For Prob. 10.67.

**10.68** Find the Thevenin equivalent at terminals *a-b* in the circuit of Fig. 10.111.

ML

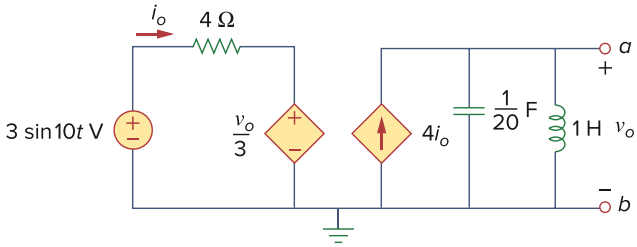



**Figure 10.111**
For Prob. 10.68.

## Section 10.7 Op Amp AC Circuits

**10.69** For the integrator shown in Fig. 10.112, obtain $\mathbf{V}_o/\mathbf{V}_s$. Find $v_o(t)$ when $v_s(t) = \mathbf{V}_m \sin \omega t$ and $\omega = 1/RC$.

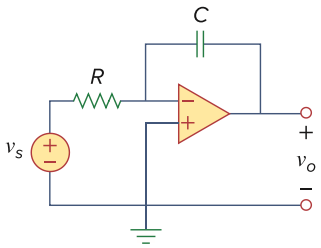



**Figure 10.112**
For Prob. 10.69.

**10.70** Using Fig. 10.113, design a problem to help other students better understand op amps in AC circuits.

eGd

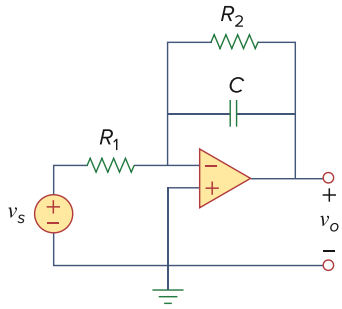



**Figure 10.113**
For Prob. 10.70.

**10.71** Find $v_o$ in the op amp circuit of Fig. 10.114.

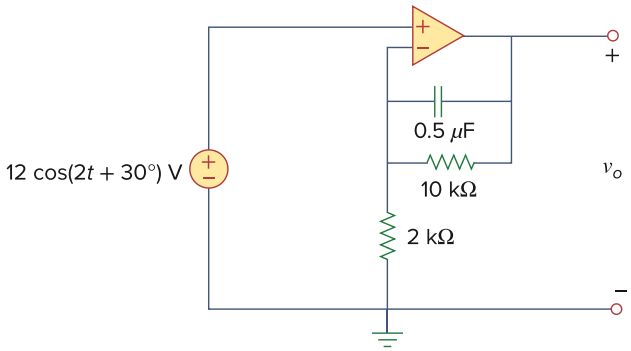



**Figure 10.114**
For Prob. 10.71.

**10.72** Compute $i_o(t)$ in the op amp circuit in Fig. 10.115 if $v_s = 4\cos(10^4 t)$ V.

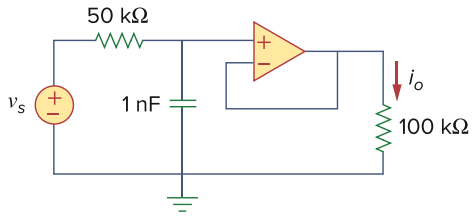



**Figure 10.115**
For Prob. 10.72.

**10.73** If the input impedance is defined as $\mathbf{Z}_{in} = \mathbf{V}_s/\mathbf{I}_s$, find the input impedance of the op amp circuit in Fig. 10.116 when $R_1 = 10$ kΩ, $R_2 = 20$ kΩ, $C_1 = 10$ nF, $C_2 = 20$ nF, and $\omega = 5000$ rad/s.

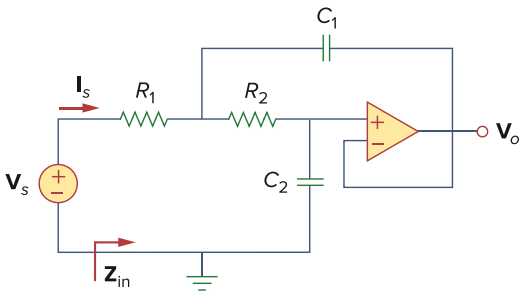



**Figure 10.116**
For Prob. 10.73.

**10.74** Evaluate the voltage gain $\mathbf{A}_v = \mathbf{V}_o/\mathbf{V}_s$ in the op amp circuit of Fig. 10.117. Find $\mathbf{A}_v$ at $\omega = 0$, $\omega \to \infty$, $\omega = 1/R_1C_1$, and $\omega = 1/R_2C_2$.

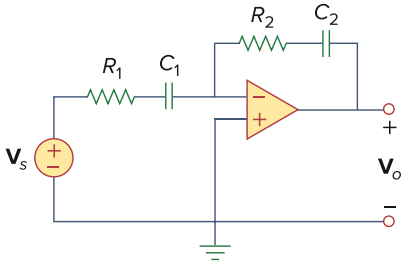



**Figure 10.117**
For Prob. 10.74.

**10.75** In the op amp circuit of Fig. 10.118, find the closed-loop gain and phase shift of the output voltage with respect to the input voltage if $C_1 = C_2 = 1$ nF, $R_1 = R_2 = 100$ kΩ, $R_3 = 20$ kΩ, $R_4 = 40$ kΩ, and $\omega = 2000$ rad/s.

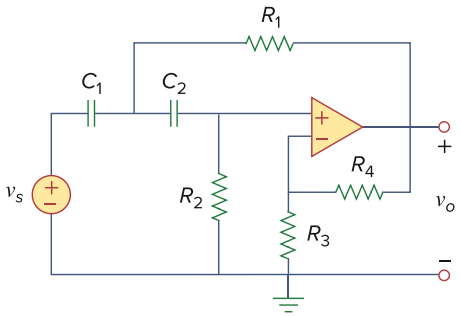



**Figure 10.118**
For Prob. 10.75.

**10.76** Determine $\mathbf{V}_o$ and $\mathbf{I}_o$ in the op amp circuit of Fig. 10.119.

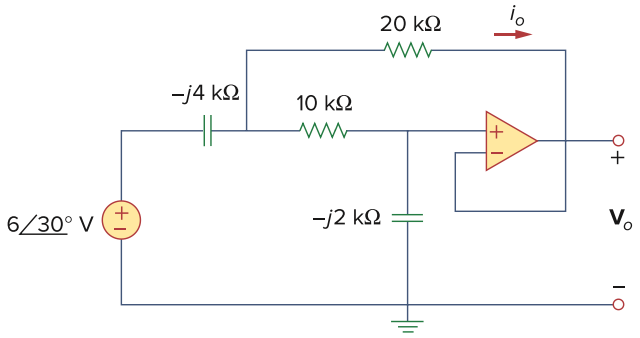



**Figure 10.119**
For Prob. 10.76.

**10.77** Compute the closed-loop gain $\mathbf{V}_o/\mathbf{V}_s$ for the op amp circuit of Fig. 10.120.

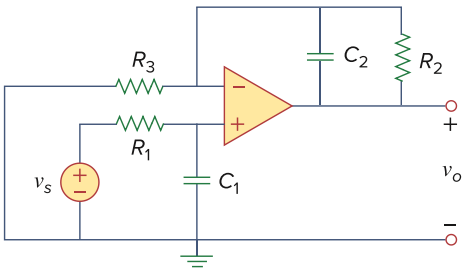



**Figure 10.120**
For Prob. 10.77.

**10.78** Determine $v_o(t)$ in the op amp circuit in Fig. 10.121 below.

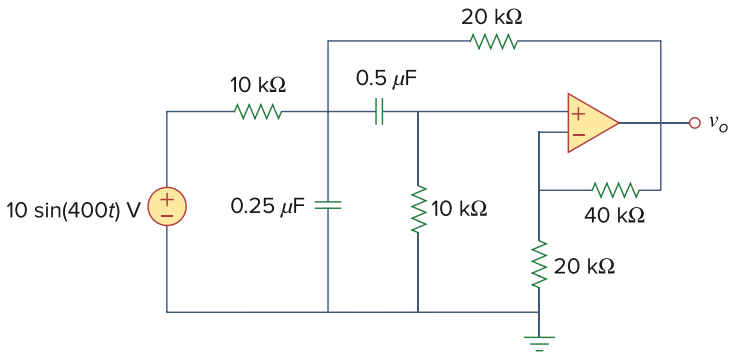



**Figure 10.121**
For Prob. 10.78.

**10.79** For the op amp circuit in Fig. 10.122, obtain $\mathbf{V}_o$.

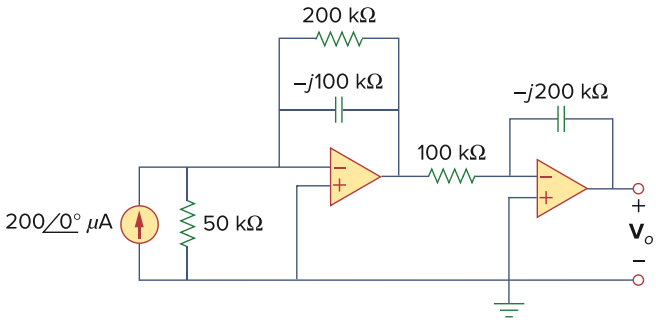



**Figure 10.122**
For Prob. 10.79.

**10.80** Obtain $v_o(t)$ for the op amp circuit in Fig. 10.123 if $v_s = 12\cos(1000t - 60°)$ V.

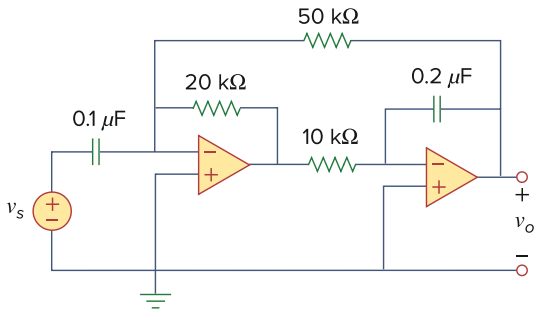



**Figure 10.123**
For Prob. 10.80.

## Section 10.8 AC Analysis Using *PSpice*

**10.81** Use *PSpice or MultiSim* to determine $\mathbf{V}_o$ in the circuit of Fig. 10.124. Assume $\omega = 1$ rad/s.

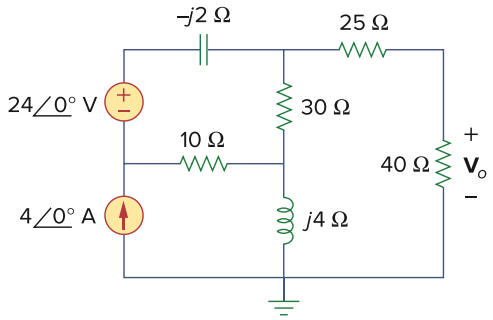



**Figure 10.124**
For Prob. 10.81.

**10.82** Solve Prob. 10.19 using *PSpice or MultiSim*.

**10.83** Use *PSpice or MultiSim* to find $v_o(t)$ in the circuit of Fig. 10.125. Let $i_s = 2\cos(10^3 t)$ A.

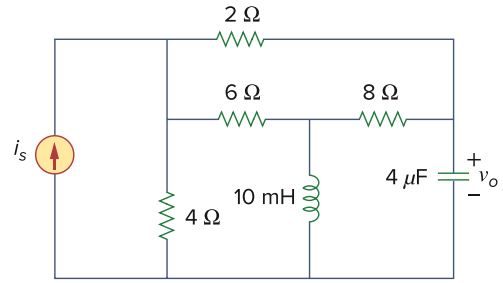



**Figure 10.125**
For Prob. 10.83.

**10.84** Obtain $\mathbf{V}_o$ in the circuit of Fig. 10.126 using *PSpice or MultiSim*.

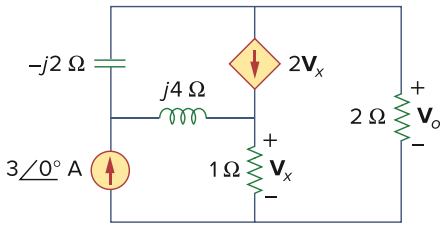



**Figure 10.126**
For Prob. 10.84.

**10.85** Using Fig. 10.127, design a problem to help other students better understand performing AC analysis with *PSpice or MultiSim*.

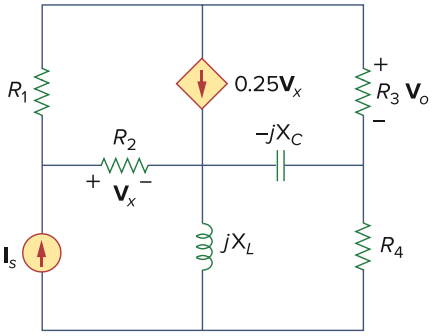



**Figure 10.127**
For Prob. 10.85.

**10.86** Use *PSpice or MultiSim* to find $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ in the network of Fig. 10.128.

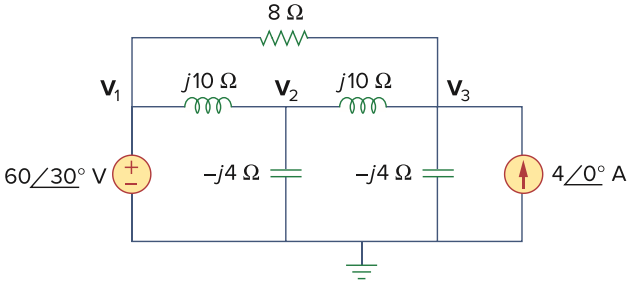



**Figure 10.128**
For Prob. 10.86.

**10.87** Determine $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ in the circuit of Fig. 10.129 using *PSpice or MultiSim*.

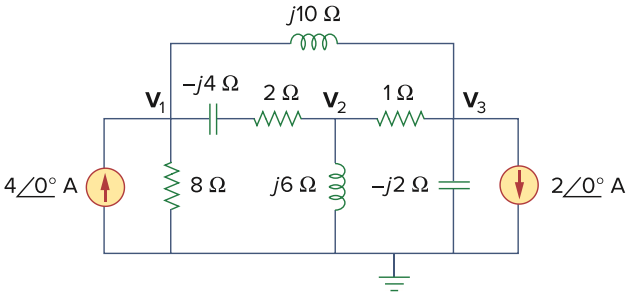



**Figure 10.129**
For Prob. 10.87.

**10.88** Use *PSpice or MultiSim* to find $v_o$ and $i_o$ in the circuit of Fig. 10.130 below.

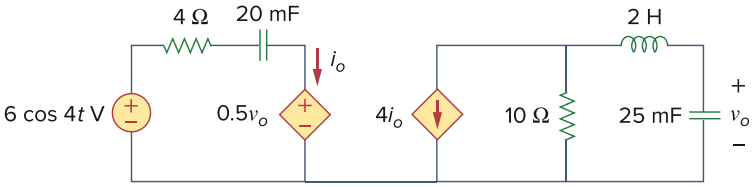



**Figure 10.130**
For Prob. 10.88.

## Section 10.9 Applications

**10.89** The op amp circuit in Fig. 10.131 is called an *inductance simulator*. Show that the input impedance is given by

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}_{\text{in}}}{\mathbf{I}_{\text{in}}} = j\omega L_{\text{eq}}$$

where

$$L_{\text{eq}} = \frac{R_1 R_3 R_4}{R_2 C}$$

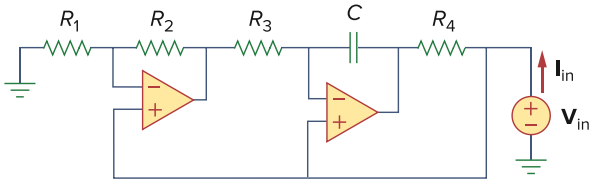



**Figure 10.131**
For Prob. 10.89.

**10.90** Figure 10.132 shows a Wien-bridge network. Show that the frequency at which the phase shift between the input and output signals is zero is $f = \frac{1}{2\pi} RC$, and that the necessary gain is $\mathbf{A}_v = \mathbf{V}_o/\mathbf{V}_i = 3$ at that frequency.

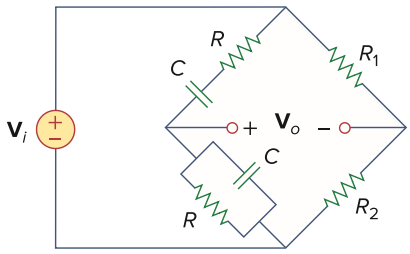



**Figure 10.132**
For Prob. 10.90.

**10.91** Consider the oscillator in Fig. 10.133.

(a) Determine the oscillation frequency.
(b) Obtain the minimum value of $R$ for which oscillation takes place.

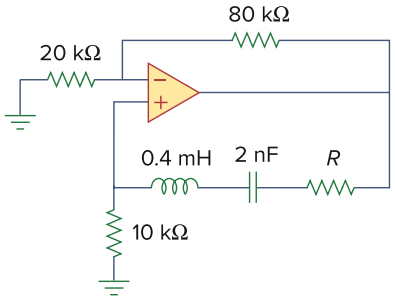



**Figure 10.133**
For Prob. 10.91.

**10.92** The oscillator circuit in Fig. 10.134 uses an ideal op amp.

(a) Calculate the minimum value of $R_o$ that will cause oscillation to occur.
(b) Find the frequency of oscillation.

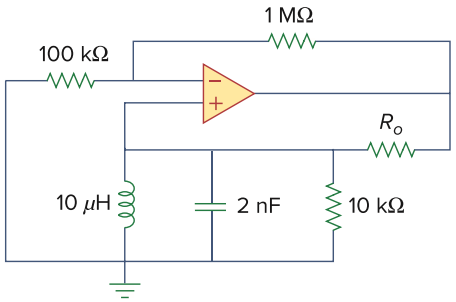



**Figure 10.134**
For Prob. 10.92.

**10.93** Figure 10.135 shows a *Colpitts oscillator*. Show that the oscillation frequency is

$$f_o = \frac{1}{2\pi\sqrt{LC_T}}$$

where $C_T = C_1C_2/(C_1 + C_2)$. Assume $R_i \gg X_{C_2}$.

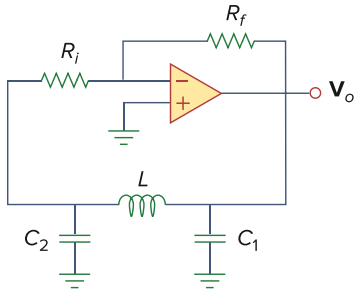



**Figure 10.135**
A Colpitts oscillator; for Prob. 10.93.

(*Hint:* Set the imaginary part of the impedance in the feedback circuit equal to zero.)

**10.94** Design a Colpitts oscillator that will operate at 50 kHz.

**10.95** Figure 10.136 shows a *Hartley oscillator*. Show that the frequency of oscillation is

$$f_o = \frac{1}{2\pi\sqrt{C(L_1 + L_2)}}$$

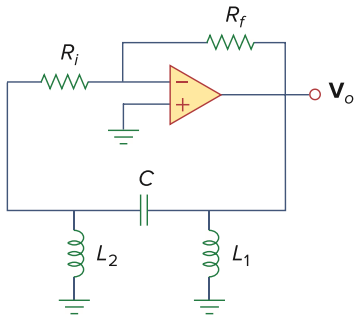



**Figure 10.136**
A Hartley oscillator; for Prob. 10.95.

**10.96** Refer to the oscillator in Fig. 10.137.

(a) Show that

$$\frac{\mathbf{V}_2}{\mathbf{V}_o} = \frac{1}{3 + j(\omega L/R - R/\omega L)}$$

(b) Determine the oscillation frequency $f_o$.

(c) Obtain the relationship between $R_1$ and $R_2$ in order for oscillation to occur.

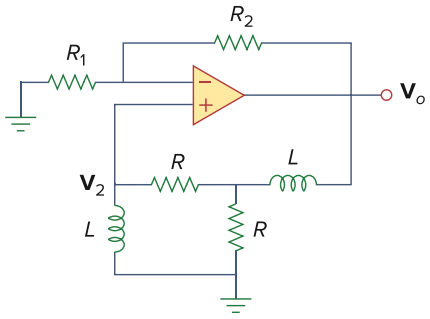



**Figure 10.137**
For Prob. 10.96.

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chapter

# 11

# AC Power Analysis

*Four things come not back: the spoken word; the sped arrow; time past; the neglected opportunity.*

—Al Halif Omar Ibn

## Enhancing Your Career

### Career in Power Systems

The discovery of the principle of an ac generator by Michael Faraday in 1831 was a major breakthrough in engineering; it provided a convenient way of generating the electric po wer that is needed in e very electronic, electrical, or electromechanical device we use now.

Electric power is obtained by con verting energy from sources such as fossil fuels (g as, oil, and coal), nuclear fuel (uranium), h ydro energy (water falling through a head), geothermal energy (hot water, steam), wind energy, tidal ener gy, and biomass ener gy (w astes). These v arious ways of generating electric po wer are studied in detail in the field of power engineering, which has become an indispensable subdiscipline of electrical engineering. An electrical engineer should be f amiliar with the analysis, generation, transmission, distrib ution, and cost of electric power.

The electric po wer industry is a v ery lar ge emplo yer of electrical engineers. The industry includes thousands of electric utility systems ranging from large, interconnected systems serving lar ge regional areas to small po wer companies serving indi vidual communities or f actories. Due to the comple xity of the po wer industry, there are numerous elec - trical engineering jobs in dif ferent areas of the industry: po wer plant (generation), transmission and distrib ution, maintenance, research, data acquisition and flow control, and management. Since electric po wer is used e verywhere, electric utility companies are e verywhere, of fering exciting training and steady emplo yment for men and w omen in thou - sands of communities throughout the world.

A pole-type transformer with a low-voltage, three-wire distribution system. © Dennis Wise/Getty Images RF

## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Fully understand instantaneous and average power.
2. Understand the basics of maximum average power.
3. Understand effective or rms values and how to calculate them and to understand their importance.
4. Understand apparent power (complex power), power, and reactive power and power factor.
5. Understand power factor correction and the importance of its use.

## 11.1 Introduction

Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. Our major concern in this chapter is power analysis.

Power analysis is of paramount importance. Power is the most important quantity in electric utilities, electronic, and communication systems, because such systems involve transmission of power from one point to another. Also, every industrial and household electrical device—every fan, motor, lamp, pressing iron, TV, personal computer—has a power rating that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance. The most common form of electric power is 50- or 60-Hz ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer.

We will begin by defining and deriving *instantaneous power* and *average power*. We will then introduce other power concepts. As practical applications of these concepts, we will discuss how power is measured and reconsider how electric utility companies charge their customers.

## 11.2 Instantaneous and Average Power

As mentioned in Chapter 2, the *instantaneous power* $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it. Assuming the passive sign convention,

$$p(t) = v(t)i(t) \tag{11.1}$$

The **instantaneous power** (in watts) is the power at any instant of time.

We can also think of the instantaneous power as the power absorbed by the element at a specific instant of time. Instantaneous quantities are denoted by lowercase letters.

It is the rate at which an element absorbs energy.

Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation, as

shown in Fig. 11.1. Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{11.2a}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{11.2b}$$

where $V_m$ and $I_m$ are the amplitudes (or peak values), and $\theta_v$ and $\theta_i$ are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i) \tag{11.3}$$

We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \tag{11.4}$$

and express Eq. (11.3) as

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i) \tag{11.5}$$

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is $2\omega$, which is twice the angular frequency of the voltage or current.

A sketch of $p(t)$ in Eq. (11.5) is shown in Fig. 11.2, where $T = 2\pi/\omega$ is the period of voltage or current. We observe that $p(t)$ is periodic, $p(t) = p(t + T_0)$, and has a period of $T_0 = T/2$, since its frequency is twice that of voltage or current. We also observe that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle. When $p(t)$ is positive, power is absorbed by the circuit. When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

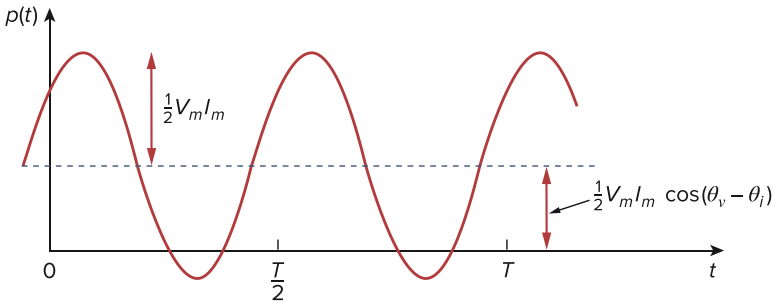



**Figure 11.2**
The instantaneous power $p(t)$ entering a circuit.

The instantaneous power changes with time and is therefore difficult to measure. The *average* power is more convenient to measure. In fact, the wattmeter, the instrument for measuring power, responds to average power.

The **average power**, in watts, is the average of the instantaneous power over one period.

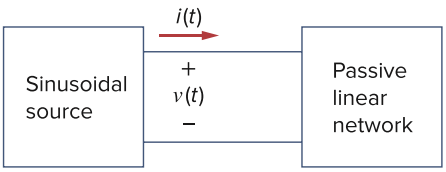



**Figure 11.1**
Sinusoidal source and passive linear circuit.

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt \tag{11.6}$$

Although Eq. (11.6) shows the averaging done over $T$, we would get the same result if we performed the integration over the actual period of $p(t)$ which is $T_0 = T/2$.

Substituting $p(t)$ in Eq. (11.5) into Eq. (11.6) gives

$$\begin{aligned} P &= \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt \\ &\quad + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt \\ &= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt \\ &\quad + \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt \end{aligned} \tag{11.7}$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is canceled by the area under it during the following negative half-cycle. Thus, the second term in Eq. (11.7) vanishes and the average power becomes

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) \tag{11.8}$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and current.

Note that $p(t)$ is time-varying while $P$ does not depend on time. To find the instantaneous power, we must necessarily have $v(t)$ and $i(t)$ in the time domain. But we can find the average power when voltage and current are expressed in the time domain, as in Eq. (11.8), or when they are expressed in the frequency domain. The phasor forms of $v(t)$ and $i(t)$ in Eq. (11.2) are $\mathbf{V} = V_m\angle\theta_v$ and $\mathbf{I} = I_m\angle\theta_i$, respectively. $P$ is calculated using Eq. (11.8) or using phasors $\mathbf{V}$ and $\mathbf{I}$. To use phasors, we notice that

$$\begin{aligned} \frac{1}{2}\mathbf{V}\mathbf{I}^* &= \frac{1}{2}V_m I_m \angle\theta_v - \theta_i \\ &= \frac{1}{2}V_m I_m[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)] \end{aligned} \tag{11.9}$$

We recognize the real part of this expression as the average power $P$ according to Eq. (11.8). Thus,

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\mathbf{V}\mathbf{I}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)} \tag{11.10}$$

Consider two special cases of Eq. (11.10). When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load $R$, and

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R \tag{11.11}$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. Equation (11.11) shows that a purely resistive circuit absorbs power at all times. When $\theta_v - \theta_i = \pm 90°$, we have a purely reactive circuit, and

$$P = \frac{1}{2} V_m I_m \cos 90° = 0 \tag{11.12}$$

showing that a purely reactive circuit absorbs no average power. In summary,

A resistive load ($R$) absorbs power at all times, while a reactive load ($L$ or $C$) absorbs zero average power.

## Example 11.1

Given that

$$v(t) = 120 \cos(377t + 45°) \text{ V} \quad \text{and} \quad i(t) = 10 \cos(377t - 10°) \text{ A}$$

find the instantaneous power and the average power absorbed by the passive linear network of Fig. 11.1.

**Solution:**

The instantaneous power is given by

$$p = vi = 1200 \cos(377t + 45°) \cos(377t - 10°)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35°) + \cos 55°]$$

or

$$p(t) = 344.2 + 600 \cos(754t + 35°) \text{ W}$$

The average power is

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2} 120(10) \cos[45° - (-10°)]$$
$$= 600 \cos 55° = 344.2 \text{ W}$$

which is the constant part of $p(t)$ above.

## Practice Problem 11.1

Calculate the instantaneous power and average power absorbed by the passive linear network of Fig. 11.1 if

$$v(t) = 330 \cos(10t + 20°) \text{ V} \quad \text{and} \quad i(t) = 33 \sin(10t + 60°) \text{ A}$$

**Answer:** $3.5 + 5.445 \cos(20t - 10°)$ kW, 3.5 kW.

## Example 11.2

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\angle 0°$ is applied across it.

**Solution:**

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\angle 0°}{30 - j70} = \frac{120\angle 0°}{76.16\angle -66.8°} = 1.576\angle 66.8° \text{ A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8°) = 37.24 \text{ W}$$

## Practice Problem 11.2

A current $\mathbf{I} = 33\angle 30°$ A flows through an impedance $\mathbf{Z} = 40\angle -22°\ \Omega$. Find the average power delivered to the impedance.

**Answer:** 20.19 kW.

## Example 11.3

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

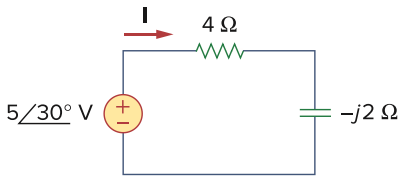



**Figure 11.3**
For Example 11.3.

**Solution:**
The current $\mathbf{I}$ is given by

$$\mathbf{I} = \frac{5\angle 30°}{4 - j2} = \frac{5\angle 30°}{4.472\angle -26.57°} = 1.118\angle 56.57° \text{ A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30° - 56.57°) = 2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I}_R = \mathbf{I} = 1.118\angle 56.57° \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57° \text{ V}$$

The average power absorbed by the resistor is

$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

## Practice Problem 11.3

In the circuit of Fig. 11.4, calculate the average power absorbed by the resistor and inductor. Find the average power supplied by the voltage source.

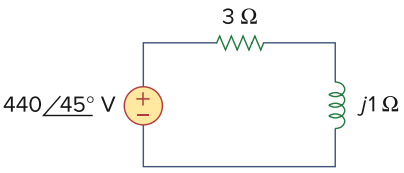



**Figure 11.4**
For Practice Prob. 11.3.

**Answer:** 29.04 kW, 0 W, 29.04 kW.

## Example 11.4

Determine the average power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).

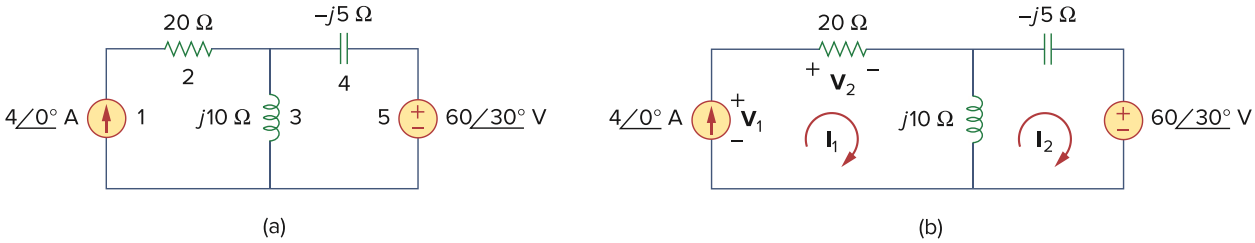



**Figure 11.5**
For Example 11.4.

**Solution:**
We apply mesh analysis as shown in Fig. 11.5(b). For mesh 1,

$$\mathbf{I}_1 = 4 \text{ A}$$

For mesh 2,

$$(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\angle 30^\circ = 0, \qquad \mathbf{I}_1 = 4 \text{ A}$$

or

$$j5\mathbf{I}_2 = -60\angle 30^\circ + j40 \quad \Rightarrow \quad \mathbf{I}_2 = -12\angle -60^\circ + 8$$
$$= 10.58\angle 79.1^\circ \text{ A}$$

For the voltage source, the current flowing from it is $\mathbf{I}_2 = 10.58\angle 79.1^\circ$ A and the voltage across it is $60\angle 30^\circ$ V, so that the average power is

$$P_5 = \frac{1}{2}(60)(10.58)\cos(30^\circ - 79.1^\circ) = 207.8 \text{ W}$$

Following the passive sign convention (see Fig. 1.8), this average power is absorbed by the source, in view of the direction of $\mathbf{I}_2$ and the polarity of the voltage source. That is, the circuit is delivering average power to the voltage source.

For the current source, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 183.9 + j20 = 184.984\angle 6.21^\circ \text{ V}$$

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21^\circ - 0) = -367.8 \text{ W}$$

It is negative according to the passive sign convention, meaning that the current source is supplying power to the circuit.

For the resistor, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is $20\mathbf{I}_1 = 80\angle 0^\circ$, so that the power absorbed by the resistor is

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

For the capacitor, the current through it is $\mathbf{I}_2 = 10.58\angle 79.1°$ and the voltage across it is $-j5\mathbf{I}_2 = (5\angle -90°)(10.58\angle 79.1°) = 52.9\angle 79.1° - 90°$. The average power absorbed by the capacitor is

$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90°) = 0$$

For the inductor, the current through it is $\mathbf{I}_1 - \mathbf{I}_2 = 2 - j10.39 = 10.58\angle -79.1°$. The voltage across it is $j10(\mathbf{I}_1 - \mathbf{I}_2) = 105.8\angle -79.1° + 90°$. Hence, the average power absorbed by the inductor is

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90° = 0$$

Notice that the inductor and the capacitor absorb zero average power and that the total power supplied by the current source equals the power absorbed by the resistor and the voltage source, or

$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

indicating that power is conserved.

## Practice Problem 11.4

Calculate the average power absorbed by each of the five elements in the circuit of Fig. 11.6.

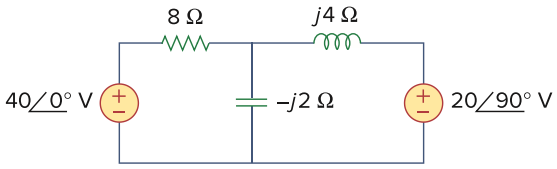



**Figure 11.6**
For Practice Prob. 11.4.

**Answer:** 40-V Voltage source: −60 W; $j20$-V Voltage source: −40 W; resistor: 100 W; others: 0 W.

## 11.3 Maximum Average Power Transfer

In Section 4.8 we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load $R_L$. Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $R_L = R_{\text{Th}}$. We now extend that result to ac circuits.

Consider the circuit in Fig. 11.7, where an ac circuit is connected to a load $\mathbf{Z}_L$ and is represented by its Thevenin equivalent. The load is usually represented by an impedance, which may model an electric motor, an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $\mathbf{Z}_{\text{Th}}$ and the load impedance $\mathbf{Z}_L$ are

$$\mathbf{Z}_{\text{Th}} = R_{\text{Th}} + jX_{\text{Th}} \tag{11.13a}$$

$$\mathbf{Z}_L = R_L + jX_L \tag{11.13b}$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + \mathbf{Z}_L} = \frac{\mathbf{V}_{\text{Th}}}{(R_{\text{Th}} + jX_{\text{Th}}) + (R_L + jX_L)} \tag{11.14}$$

From Eq. (11.11), the average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{\text{Th}}|^2 R_L/2}{(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2} \tag{11.15}$$

Our objective is to adjust the load parameters $R_L$ and $X_L$ so that $P$ is maximum. To do this we set $\partial P/\partial R_L$ and $\partial P/\partial X_L$ equal to zero. From Eq. (11.15), we obtain

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{\text{Th}}|^2 R_L (X_{\text{Th}} + X_L)}{[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2} \tag{11.16a}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{\text{Th}}|^2 [(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2 - 2R_L(R_{\text{Th}} + R_L)]}{2[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2} \tag{11.16b}$$

Setting $\partial P/\partial X_L$ to zero gives

$$X_L = -X_{\text{Th}} \tag{11.17}$$

and setting $\partial P/\partial R_L$ to zero results in

$$R_L = \sqrt{R_{\text{Th}}^2 + (X_{\text{Th}} + X_L)^2} \tag{11.18}$$

Combining Eqs. (11.17) and (11.18) leads to the conclusion that for maximum average power transfer, $\mathbf{Z}_L$ must be selected so that $X_L = -X_{\text{Th}}$ and $R_L = R_{\text{Th}}$, i.e.,

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}_{\text{Th}}^*} \tag{11.19}$$

For **maximum average power transfer,** the load impedance $\mathbf{Z}_L$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z}_{\text{Th}}$.

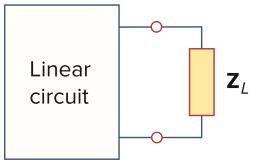
When $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^*$, we say that the load is matched to the source.

This result is known as the *maximum average power transfer theorem* for the sinusoidal steady state. Setting $R_L = R_{\text{Th}}$ and $X_L = -X_{\text{Th}}$ in Eq. (11.15) gives us the maximum average power as

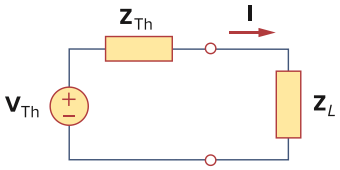
$$\boxed{P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}}} \tag{11.20}$$

In a situation in which the load is purely real, the condition for maximum power transfer is obtained from Eq. (11.18) by setting $X_L = 0$; that is,

$$R_L = \sqrt{R_{\text{Th}}^2 + X_{\text{Th}}^2} = |\mathbf{Z}_{\text{Th}}| \tag{11.21}$$




(a)




(b)

**Figure 11.7**
Finding the maximum average power transfer: (a) circuit with a load, (b) the Thevenin equivalent.

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance.

## Example 11.5

Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?

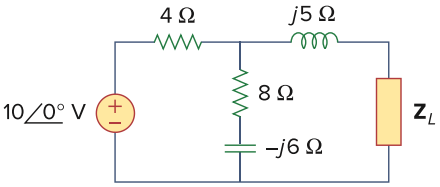



**Figure 11.8**
For Example 11.5.

**Solution:**
First we obtain the Thevenin equivalent at the load terminals. To get $\mathbf{Z}_{\text{Th}}$, consider the circuit shown in Fig. 11.9(a). We find

$$\mathbf{Z}_{\text{Th}} = j5 + 4 \,\|\, (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

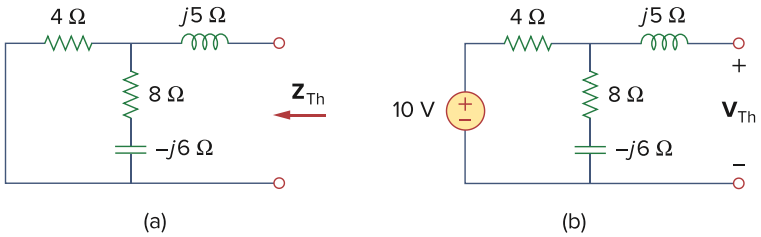



**Figure 11.9**
Finding the Thevenin equivalent of the circuit in Fig. 11.8.

To find $\mathbf{V}_{\text{Th}}$, consider the circuit in Fig. 11.8(b). By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\angle -10.3^\circ\ \text{V}$$

The load impedance draws the maximum power from the circuit when

$$\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = 2.933 - j4.467\ \Omega$$

According to Eq. (11.20), the maximum average power is

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}} = \frac{(7.454)^2}{8(2.933)} = 2.368\ \text{W}$$

## Practice Problem 11.5

For the circuit shown in Fig. 11.10, find the load impedance $\mathbf{Z}_L$ that absorbs the maximum average power. Calculate that maximum average power.

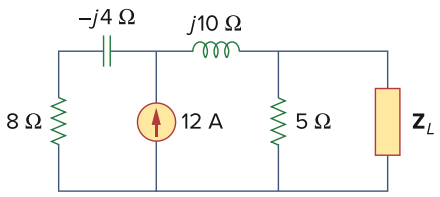



**Figure 11.10**
For Practice Prob. 11.5.

**Answer:** $3.415 - j0.7317\ \Omega$, 51.47 W.

## Example 11.6

In the circuit in Fig. 11.11, find the value of $R_L$ that will absorb the maximum average power. Calculate that power.

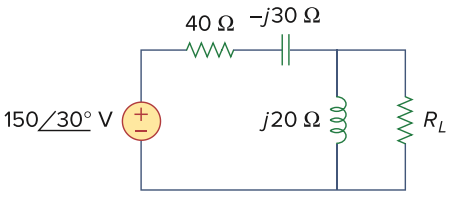



**Figure 11.11**
For Example 11.6.

**Solution:**
We first find the Thevenin equivalent at the terminals of $R_L$.

$$\mathbf{Z}_{\text{Th}} = (40 - j30) \| j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{j20}{j20 + 40 - j30}(150\angle 30°) = 72.76\angle 134°\ \text{V}$$

The value of $R_L$ that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_{\text{Th}}| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + R_L} = \frac{72.76\angle 134°}{33.66 + j22.35} = 1.8\angle 100.42°\ \text{A}$$

The maximum average power absorbed by $R_L$ is

$$P_{\max} = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}(1.8)^2(24.25) = 39.29\ \text{W}$$

## Practice Problem 11.6

In Fig. 11.12, the resistor $R_L$ is adjusted until it absorbs the maximum average power. Calculate $R_L$ and the maximum average power absorbed by it.

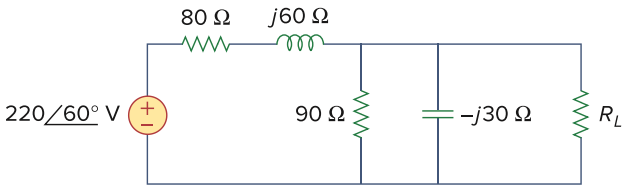



**Figure 11.12**
For Practice Prob. 11.6.

**Answer:** 30 Ω, 23.06 W.

## 11.4 Effective or RMS Value

The idea of *effective value* arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load.

The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

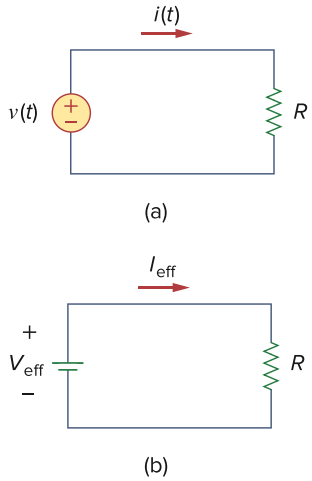



**Figure 11.13**
Finding the effective current: (a) ac circuit, (b) dc circuit.

In Fig. 11.13, the circuit in (a) is ac while that of (b) is dc. Our objecti ve is to find $I_{\text{eff}}$ that will transfer the same po wer to resistor $R$ as the sinusoid $i$. The average power absorbed by the resistor in the ac circuit is

$$P = \frac{1}{T}\int_0^T i^2 R\, dt = \frac{R}{T}\int_0^T i^2\, dt \tag{11.22}$$

while the power absorbed by the resistor in the dc circuit is

$$P = I_{\text{eff}}^2 R \tag{11.23}$$

Equating the expressions in Eqs. (11.22) and (11.23) and solving for $I_{\text{eff}}$, we obtain

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\, dt} \tag{11.24}$$

The effective value of the v oltage is found in the same w ay as current; that is,

$$V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2\, dt} \tag{11.25}$$

This indicates that the effective value is the (square) *root* of the *mean* (or average) of the *square* of the periodic signal. Thus, the effective value is often known as the *root-mean-square* value, or *rms* value for short; and we write

$$I_{\text{eff}} = \text{I}_{\text{rms}}, \qquad V_{\text{eff}} = V_{\text{rms}} \tag{11.26}$$

For any periodic function $x(t)$ in general, the rms value is given by

$$\boxed{X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\, dt}} \tag{11.27}$$

The **effective value** of a periodic signal is its root mean square (rms) value.

Equation 11.27 states that to find the rms value of $x(t)$, we first find its *square* $x^2$ and then find the *mean* of that, or

$$\frac{1}{T}\int_0^T x^2\, dt$$

and then the square *root* ($\sqrt{\quad}$) of that mean. The rms value of a constant is the constant itself. For the sinusoid $i(t) = I_m \cos \omega t$, the effective or rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t\, dt}$$

$$= \sqrt{\frac{I_m^2}{T}\int 0\, T\, \frac{1}{2}(1 + \cos 2\omega t)\, dt} = \frac{I_m}{\sqrt{2}} \tag{11.28}$$

Similarly, for $v(t) = V_m \cos \omega t$,

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \tag{11.29}$$

Keep in mind that Eqs. (11.28) and (11.29) are only v alid for sinusoidal signals.

The average power in Eq. (11.8) can be written in terms of the rms values.

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \tag{11.30}$$

Similarly, the average power absorbed by a resistor $R$ in Eq. (11.11) can be written as

$$P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} \tag{11.31}$$

When a sinusoidal voltage or current is specified, it is often in terms of its maximum (or peak) v alue or its rms v alue, since its a verage value is zero. The power industries specify phasor magnitudes in terms of their rms values rather than peak v alues. For instance, the 110 V available at every household is the rms value of the voltage from the power company. It is convenient in power analysis to e xpress voltage and current in their rms values. Also, analog v oltmeters and ammeters are designed to read directly the rms value of voltage and current, respectively.

## Example 11.7

Determine the rms v alue of the current w aveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.

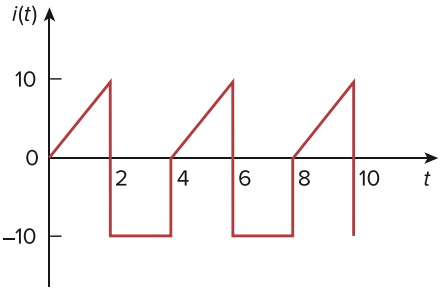



**Figure 11.14**
For Example 11.7.

**Solution:**
The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\Big|_0^2 + 100t\Big|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165 \text{ A}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3 \text{ W}$$

## Practice Problem 11.7

Find the rms value of the current waveform of Fig. 11.15. If the current flows through a 9-Ω resistor, calculate the average power absorbed by the resistor.

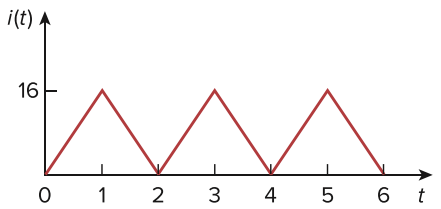



**Figure 11.15**
For Practice Prob. 11.7.

**Answer:** 9.238 A, 768 W.

## Example 11.8

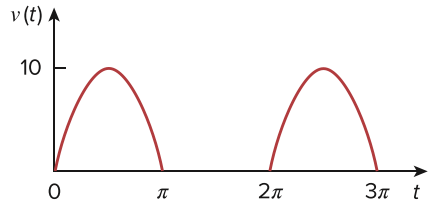



**Figure 11.16**
For Example 11.8.

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

**Solution:**
The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10 \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi}(10 \sin t)^2\,dt + \int_{\pi}^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence,

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^{\pi}\frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi}$$

$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\text{rms}} = 5 \text{ V}$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5 \text{ W}$$

## Practice Problem 11.8

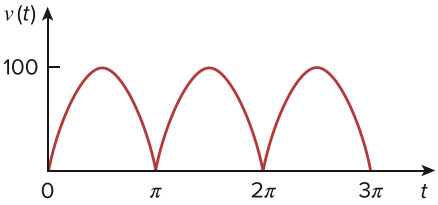



**Figure 11.17**
For Practice Prob. 11.8.

Find the rms value of the full-wave rectified sine wave in Fig. 11.17. Calculate the average power dissipated in a 6-Ω resistor.

**Answer:** 70.71 V, 833.3 W.

## 11.5 Apparent Power and Power Factor

In Section 11.2 we saw that if the voltage and current at the terminals of a circuit are

$$v(t) = V_m \cos(\omega t + \theta_v) \qquad \text{and} \qquad i(t) = I_m \cos(\omega t + \theta_i) \tag{11.32}$$

or, in phasor form, $\mathbf{V} = V_m\angle\theta_v$ and $\mathbf{I} = I_m\angle\theta_i$, the average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) \tag{11.33}$$

In Section 11.4, we saw that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i) \tag{11.34}$$

We have added a new term to the equation:

$$\boxed{S = V_{\text{rms}} I_{\text{rms}}} \tag{11.35}$$

The average power is a product of two terms. The product $V_{rms}I_{rms}$ is known as the *apparent power* $S$. The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

The **apparent power** (in VA) is the product of the rms values of voltage and current.

The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. It is measured in volt-amperes or VA to distinguish it from the average or real power, which is measured in watts. The power factor is dimensionless, since it is the ratio of the average power to the apparent power,

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i) \tag{11.36}$$

The angle $\theta_v - \theta_i$ is called the *power factor angle,* because it is the angle whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance if $\mathbf{V}$ is the voltage across the load and $\mathbf{I}$ is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle \theta_v - \theta_i \tag{11.37}$$

Alternatively, since

$$\mathbf{V}_{rms} = \frac{\mathbf{V}}{\sqrt{2}} = V_{rms} \angle \theta_v \tag{11.38a}$$

and

$$\mathbf{I}_{rms} = \frac{\mathbf{I}}{\sqrt{2}} = I_{rms} \angle \theta_i \tag{11.38b}$$

the impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{rms}}{\mathbf{I}_{rms}} = \frac{V_{rms}}{\mathbf{I}_{rms}} \angle \theta_v - \theta_i \tag{11.39}$$

The **power factor** is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

From Eq. (11.36), the power factor may be seen as that factor by which the apparent power must be multiplied to obtain the real or average power. The value of pf ranges between zero and unity. For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf = 1. This implies that the apparent power is equal to the average power. For a purely reactive load, $\theta_v - \theta_i = \pm 90°$ and pf = 0. In this case the average power is zero. In between these two extreme cases, pf is said to be *leading* or *lagging*. Leading power factor means that current leads voltage, which implies a capacitive load. Lagging power factor means that current lags voltage, implying an inductive load. Power factor affects the

From Eq. (11.36), the power factor may also be regarded as the ratio of the real power dissipated in the load to the apparent power of the load.

electric bills consumers pay the electric utility companies, as we will see in Section 11.9.2.

## Example 11.9

A series-connected load dra ws a current $i(t) = 4\cos(100\pi t + 10°)$ A when the applied v oltage is $v(t) = 120\cos(100\pi t - 20°)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

**Solution:**
The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20° - 10°) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\angle -20°}{4\angle 10°} = 30\angle -30° = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30°) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C}$$

or

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

## Practice Problem 11.9

Obtain the power factor and the apparent power of a load whose impedance is $\mathbf{Z} = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 155.56\cos(377t + 10°)$ V.

**Answer:** 0.8321 lagging, $167.69\angle 33.69°$ VA.

## Example 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

**Solution:**
The total impedance is

$$\mathbf{Z} = 6 + 4 \,\|\, (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\angle -13.24°\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\angle 0^\circ}{7\angle -13.24^\circ} = 4.286\angle 13.24^\circ \text{A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}} \text{pf} = (30)(4.286)0.9734 = 125 \text{ W}$$

or

$$P = I_{\text{rms}}^2 R = (4.286)^2(6.8) = 125 \text{ W}$$

where $R$ is the resistive part of $\mathbf{Z}$.

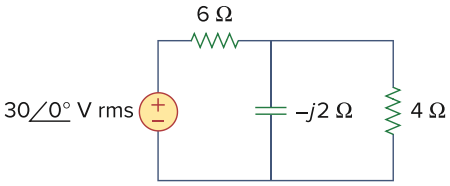



**Figure 11.18**
For Example 11.10.

## Practice Problem 11.10

Calculate the power factor of the entire circuit of Fig. 11.19 as seen by the source. What is the average power supplied by the source?

**Answer:** 0.936 lagging, 2.008 kW.

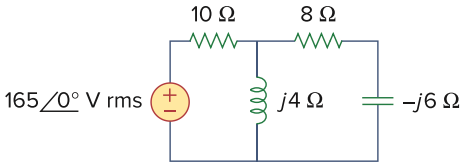



**Figure 11.19**
For Practice Prob. 11.10.

## 11.6 Complex Power

Considerable effort has been expended over the years to express power relations as simply as possible. Power engineers have coined the term *complex power,* which they use to find the total effect of parallel loads. Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load.

Consider the ac load in Fig. 11.20. Given the phasor form $\mathbf{V} = V_m\angle\theta_v$ and $\mathbf{I} = I_m\angle\theta_i$ of voltage $v(t)$ and current $i(t)$, the *complex power* $\mathbf{S}$ absorbed by the ac load is the product of the voltage and the complex conjugate of the current, or

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* \tag{11.40}$$

assuming the passive sign convention (see Fig. 11.20). In terms of the rms values,

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* \tag{11.41}$$

where

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}}\angle\theta_v \tag{11.42}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}}\angle\theta_i \tag{11.43}$$

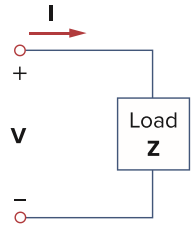



**Figure 11.20**
The voltage and current phasors associated with a load.

When working with the rms values of currents or voltages, we may drop the subscript rms if no confusion will be caused by doing so.

Thus, we may write Eq. (11.41) as

$$\begin{aligned}\mathbf{S} &= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + jV_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)\end{aligned} \tag{11.44}$$

This equation can also be obtained from Eq. (11.9). We notice from Eq. (11.44) that the magnitude of the complex power is the apparent power; hence, the complex power is measured in volt-amperes (VA). Also, we notice that the angle of the complex power is the power factor angle.

The complex power may be expressed in terms of the load impedance $\mathbf{Z}$. From Eq. (11.37), the load impedance $\mathbf{Z}$ may be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \underline{/\theta_v - \theta_i} \tag{11.45}$$

Thus, $\mathbf{V}_{\text{rms}} = \mathbf{Z}\mathbf{I}_{\text{rms}}$. Substituting this into Eq. (11.41) gives

$$\boxed{\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^*} \tag{11.46}$$

Since $\mathbf{Z} = R + jX$, Eq. (11.46) becomes

$$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ \tag{11.47}$$

where $P$ and $Q$ are the real and imaginary parts of the complex power; that is,

$$P = \text{Re}(\mathbf{S}) = \mathbf{I}_{\text{rms}}^2 R \tag{11.48}$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X \tag{11.49}$$

$P$ is the average or real power and it depends on the load's resistance $R$. $Q$ depends on the load's reactance $X$ and is called the *reactive* (or quadrature) power.

Comparing Eq. (11.44) with Eq. (11.47), we notice that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \tag{11.50}$$

The real power $P$ is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load. The reactive power $Q$ is a measure of the energy exchange between the source and the reactive part of the load. The unit of $Q$ is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt. We know from Chapter 6 that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source. Notice that:

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Thus,

**Complex power** (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power $P$ and its imaginary part is reactive power $Q$.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

$$\begin{aligned}\text{Complex Power} &= \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^* \\ &= |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}|\underline{/\theta_v - \theta_i} \\ \text{Apparent Power} &= S = |\mathbf{S}| = |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}| = \sqrt{P^2 + Q^2} \\ \text{Real Power} &= P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i) \\ \text{Reactive Power} &= Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i) \\ \text{Power Factor} &= \frac{P}{S} = \cos(\theta_v - \theta_i)\end{aligned} \tag{11.51}$$

This shows how the complex power contains *all* the relevant power information in a given load.

It is a standard practice to represent $\mathbf{S}$, $P$, and $Q$ in the form of a triangle, known as the *power triangle*, shown in Fig. 11.21(a). This is similar to the impedance triangle showing the relationship between $\mathbf{Z}$, $R$, and $X$, illustrated in Fig. 11.21(b). The power triangle has four items—the apparent/complex power, real power, reactive power, and the power factor angle. Given two of these items, the other two can easily be obtained from the triangle. As shown in Fig. 11.22, when $\mathbf{S}$ lies in the first quadrant, we have an inductive load and a lagging pf. When $\mathbf{S}$ lies in the fourth quadrant, the load is capacitive and the pf is leading. It is also possible for the complex power to lie in the second or third quadrant. This requires that the load impedance have a negative resistance, which is possible with active circuits.

**$\mathbf{S}$** contains *all* power information of a load. The real part of **$\mathbf{S}$** is the real power $P$; its imaginary part is the reactive power $Q$; its magnitude is the apparent power $S$; and the cosine of its phase angle is the power factor pf.

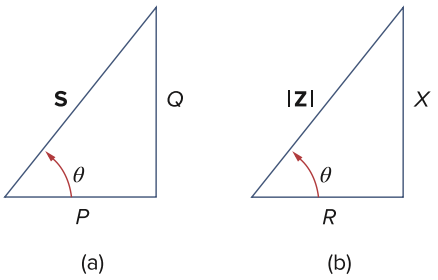



**Figure 11.21**
(a) Power triangle, (b) impedance triangle.

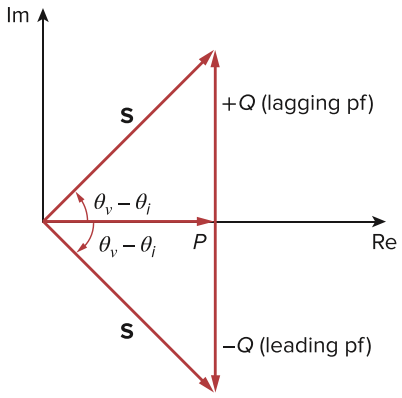



**Figure 11.22**
Power triangle.

## Example 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10°)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50°)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

**Solution:**

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\underline{/-10^\circ}, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\underline{/+50^\circ}$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* = \left(\frac{60}{\sqrt{2}}\underline{/-10^\circ}\right)\left(\frac{1.5}{\sqrt{2}}\underline{/-50^\circ}\right) = 45\underline{/-60^\circ}\ \text{VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45\ \text{VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\underline{/-60^\circ} = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5\ \text{W}$$

while the reactive power is

$$Q = -38.97\ \text{VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5\ \text{(leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\underline{/-10^\circ}}{1.5\underline{/+50^\circ}} = 40\underline{/-60^\circ}\ \Omega$$

which is a capacitive impedance.

## Practice Problem 11.11

For a load, $\mathbf{V}_{\text{rms}} = 110\underline{/85^\circ}$ V, $\mathbf{I}_{\text{rms}} = 3\underline{/15^\circ}$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

**Answer:** (a) $330\underline{/70^\circ}$ VA, 44 VA, (b) 112.87 W, 310.1 VAR, (c) 0.342 lagging, $(12.541 + j34.46)\ \Omega$.

## Example 11.12

A load $\mathbf{Z}$ draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

**Solution:**

(a) Given that $\text{pf} = \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13^\circ$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272\ \text{kW}$$

while the reactive power is

$$Q = S \sin\theta = 12{,}000 \times 0.517 = 6.204\text{kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + \text{j}6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^*$, we obtain

$$\mathbf{I}_{\text{rms}}^* = \frac{\mathbf{S}}{\mathbf{V}_{\text{rms}}} = \frac{10{,}272 + j6204}{120\angle 0^\circ} = 85.6 + j51.7 \text{ A} = 100\angle 31.13^\circ \text{ A}$$

Thus $\mathbf{I}_{\text{rms}} = 100\angle -31.13^\circ$ and the peak current is

$$I_m = \sqrt{2}I_{\text{rms}} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{120\angle 0^\circ}{100\angle -31.13^\circ} = 1.2\angle 31.13^\circ \ \Omega$$

which is an inductive impedance.

### Practice Problem 11.12

A sinusoidal source supplies 100 kVAR reactive power to load $\mathbf{Z} = 250\angle -75^\circ \ \Omega$. Determine: (a) the power factor, (b) the apparent power delivered to the load, and (c) the rms voltage.

**Answer:** (a) 0.2588 leading, (b) 103.53 kVA, (c) 5.087 kV.

## 11.7 †Conservation of AC Power

In fact, we already saw in Examples 11.3 and 11.4 that average power is conserved in ac circuits.

The principle of conservation of power applies to ac circuits as well as to dc circuits (see Section 1.5).

To see this, consider the circuit in Fig. 11.23(a), where two load impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are connected in parallel across an ac source $\mathbf{V}$. KCL gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 \tag{11.52}$$

The complex power supplied by the source is (from now on, unless otherwise specified, all values of voltages and currents will be assumed to be rms values)

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = \mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) = \mathbf{V}\mathbf{I}_1^* + \mathbf{V}\mathbf{I}_2^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.53}$$

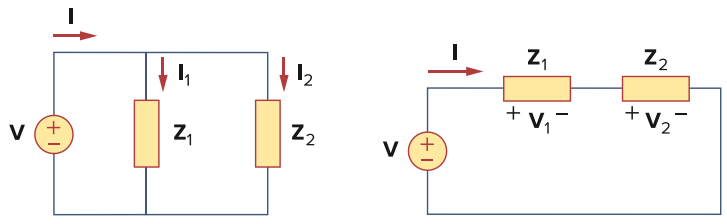



**Figure 11.23**
An ac voltage source supplied loads connected in: (a) parallel, (b) series.

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

If the loads are connected in series with the voltage source, as shown in Fig. 11.23(b), KVL yields

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \tag{11.54}$$

The complex power supplied by the source is

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = (\mathbf{V}_1 + \mathbf{V}_2)\mathbf{I}^* = \mathbf{V}_1\mathbf{I}^* + \mathbf{V}_2\mathbf{I}^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.55}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

We conclude from Eqs. (11.53) and (11.55) that whether the loads are connected in series or in parallel (or in general), the total power *supplied* by the source equals the total power *delivered* to the load. Thus, in general, for a source connected to $N$ loads,

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_N \tag{11.56}$$

In fact, all forms of ac power are conserved: instantaneous, real, reactive, and complex.

This means that the total complex power in a network is the sum of the complex powers of the individual components. (This is also true of real power and reactive power, but not true of apparent power.) This expresses the principle of conservation of ac power:

> The complex, real, and reactive powers of the sources equal the respective sums of the complex, real, and reactive powers of the individual loads.

From this we imply that the real (or reactive) power flow from sources in a network equals the real (or reactive) power flow into the other elements in the network.

## Example 11.13

Figure 11.24 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the $(4 + j2)\ \Omega$ impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

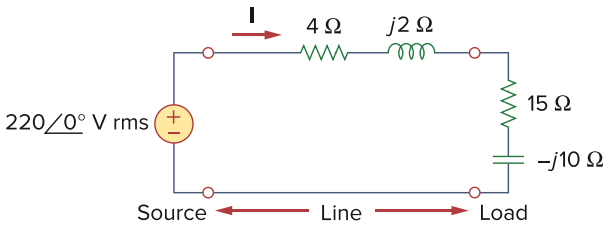



**Figure 11.24**
For Example 11.13.

**Solution:**

The total impedance is

$$\mathbf{Z} = (4 + j2) + (15 - j10) = 19 - j8 = 20.62\angle -22.83^\circ\ \Omega$$

The current through the circuit is

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{220\angle 0^\circ}{20.62\angle -22.83^\circ} = 10.67\angle 22.83^\circ \text{ A rms}$$

(a) For the source, the complex power is

$$\mathbf{S}_s = \mathbf{V}_s\mathbf{I}^* = (220\angle 0^\circ)(10.67\angle -22.83^\circ)$$
$$= 2347.4\angle -22.83^\circ = (2163.5 - j910.8) \text{ VA}$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).

(b) For the line, the voltage is

$$\mathbf{V}_{\text{line}} = (4 + j2)\mathbf{I} = (4.472\angle 26.57^\circ)(10.67\angle 22.83^\circ)$$
$$= 47.72\angle 49.4^\circ \text{ V rms}$$

The complex power absorbed by the line is

$$\mathbf{S}_{\text{line}} = \mathbf{V}_{\text{line}}\mathbf{I}^* = (47.72\angle 49.4^\circ)(10.67\angle -22.83^\circ)$$
$$= 509.2\angle 26.57^\circ = 455.4 + j227.7 \text{ VA}$$

or

$$\mathbf{S}_{\text{line}} = |\mathbf{I}|^2\mathbf{Z}_{\text{line}} = (10.67)^2(4 + j2) = 455.4 + j227.7 \text{ VA}$$

That is, the real power is 455.4 W and the reactive power is 227.76 VAR (lagging).

(c) For the load, the voltage is

$$\mathbf{V}_L = (15 - j10)\mathbf{I} = (18.03\angle -33.7^\circ)(10.67\angle 22.83^\circ)$$
$$= 192.38\angle -10.87^\circ \text{ V rms}$$

The complex power absorbed by the load is

$$\mathbf{S}_L = \mathbf{V}_L\mathbf{I}^* = (192.38\angle -10.87^\circ)(10.67\angle -22.83^\circ)$$
$$= 2053\angle -33.7^\circ = (1708 - j1139) \text{ VA}$$

The real power is 1708 W and the reactive power is 1139 VAR (leading). Note that $\mathbf{S}_s = \mathbf{S}_{\text{line}} + \mathbf{S}_L$, as expected. We have used the rms values of voltages and currents.

## Practice Problem 11.13

In the circuit in Fig. 11.25, the 60- Ω resistor absorbs an average power of 240 W. Find $\mathbf{V}$ and the complex power of each branch of the circuit. What is the overall complex power of the circuit? (Assume the current through the 60-Ω resistor has no phase shift.)

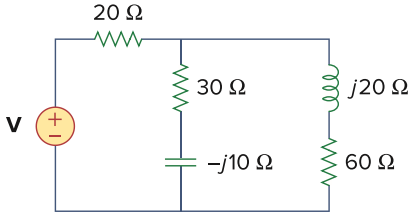



**Figure 11.25**
For Practice Prob. 11.13.

**Answer:** 240.7 $\angle 21.45^\circ$ V (rms); the 20- Ω resistor: 656 VA; the $(30 - j10)$ Ω impedance: $480 - j160$ VA; the $(60 + j20)$ Ω impedance: $240 + j80$ VA; overall: $1376 - j80$ VA.

## Example 11.14

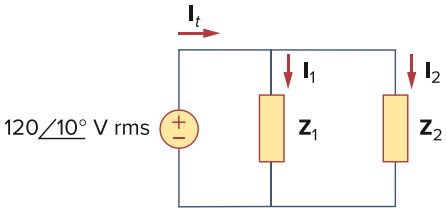



**Figure 11.26**
For Example 11.14.

In the circuit of Fig. 11.26, $\mathbf{Z}_1 = 60\angle -30°\ \Omega$ and $\mathbf{Z}_2 = 40\angle 45°\ \Omega$. Calculate the total: (a) apparent power, (b) real power, (c) reactive power, and (d) pf, supplied by the source and seen by the source.

**Solution:**
The current through $\mathbf{Z}_1$ is

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{120\angle 10°}{60\angle -30°} = 2\angle 40°\ \text{A rms}$$

while the current through $\mathbf{Z}_2$ is

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{120\angle 10°}{40\angle 45°} = 3\angle -35°\ \text{A rms}$$

The complex powers absorbed by the impedances are

$$\mathbf{S}_1 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_1^*} = \frac{(120)^2}{60\angle 30°} = 240\angle -30° = 207.85 - j120\ \text{VA}$$

$$\mathbf{S}_2 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_2^*} = \frac{(120)^2}{40\angle -45°} = 360\angle 45° = 254.6 + j254.6\ \text{VA}$$

The total complex power is

$$\mathbf{S}_t = \mathbf{S}_1 + \mathbf{S}_2 = 462.4 + j134.6\ \text{VA}$$

(a) The total apparent power is

$$|\mathbf{S}_t| = \sqrt{462.4^2 + 134.6^2} = 481.6\ \text{VA}.$$

(b) The total real power is

$$P_t = \text{Re}(\mathbf{S}_t) = 462.4\ \text{W or } P_t = P_1 + P_2.$$

(c) The total reactive power is

$$Q_t = \text{Im}(\mathbf{S}_t) = 134.6\ \text{VAR or } Q_t = Q_1 + Q_2.$$

(d) The pf $= P_t/|\mathbf{S}_t| = 462.4/481.6 = 0.96$ (lagging).
We may cross check the result by finding the complex power $\mathbf{S}_s$ supplied by the source.

$$\mathbf{I}_t = \mathbf{I}_1 + \mathbf{I}_2 = (1.532 + j1.286) + (2.457 - j1.721)$$
$$= 4 - j0.435 = 4.024\angle -6.21°\ \text{A rms}$$
$$\mathbf{S}_s = \mathbf{V}\mathbf{I}_t^* = (120\angle 10°)(4.024\angle 6.21°)$$
$$= 482.88\angle 16.21° = 463 + j135\ \text{VA}$$

which is the same as before.

## Practice Problem 11.14

Two loads connected in parallel are respectively 3 kW at a pf of 0.75 leading and 6 kW at a pf of 0.95 lagging. Calculate the pf of the combined two loads. Find the complex power supplied by the source.

**Answer:** 0.9972 (leading), $9 - j0.6742$ kVA.

## 11.8 Power Factor Correction

Most domestic loads (such as washing machines, air conditioners, and refrigerators) and industrial loads (such as induction motors) are inductive and operate at a low lagging power factor. Although the inductive nature of the load cannot be changed, we can increase its power factor.

The process of increasing the power factor without altering the voltage or current to the original load is known as **power factor correction.**

Alternatively, power factor correction may be viewed as the addition of a reactive element (usually a capacitor) in parallel with the load in order to make the power factor closer to unity.

An inductive load is modeled as a series combination of an inductor and a resistor.

Since most loads are inductive, as shown in Fig. 11.27(a), a load's power factor is improved or corrected by deliberately installing a capacitor in parallel with the load, as shown in Fig. 11.27(b). The effect of adding the capacitor can be illustrated using either the power triangle or the phasor diagram of the currents involved. Figure 11.28 shows the latter, where it is assumed that the circuit in Fig. 11.27(a) has a power factor of $\cos\theta_1$, while the one in Fig. 11.27(b) has a power factor of $\cos\theta_2$. It is evident from Fig. 11.28 that adding the capacitor has caused the phase angle between the supplied voltage and current to reduce from $\theta_1$ to $\theta_2$, thereby increasing the power factor. We also notice from the magnitudes of the vectors in Fig. 11.28 that with the same supplied voltage, the circuit in Fig. 11.27(a) draws larger current $I_L$ than the current $I$ drawn by the circuit in Fig. 11.27(b). Power companies charge more for larger currents, because they result in increased power losses (by a squared factor, since $P = I_L^2 R$). Therefore, it is beneficial to both the power company and the consumer that every effort is made to minimize current level or keep the power factor as close to unity as possible. By choosing a suitable size for the capacitor, the current can be made to be completely in phase with the voltage, implying unity power factor.

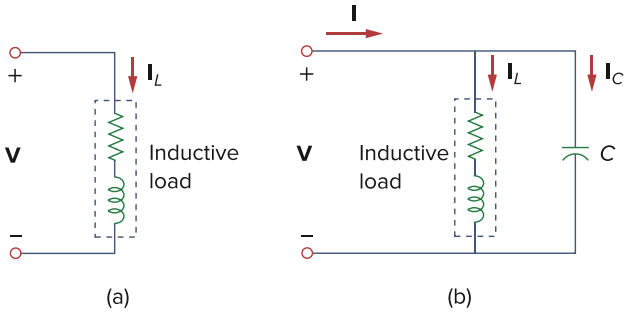



**Figure 11.27**
Power factor correction: (a) original inductive load, (b) inductive load with improved power factor.

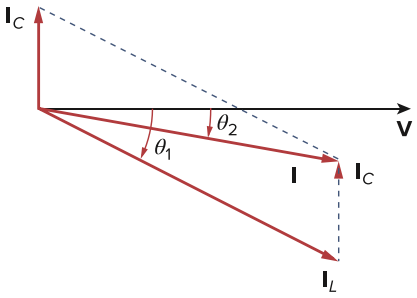



**Figure 11.28**
Phasor diagram showing the effect of adding a capacitor in parallel with the inductive load.

We can look at the power factor correction from another perspective. Consider the power triangle in Fig. 11.29. If the original inductive load has apparent power $S_1$, then

$$P = S_1 \cos\theta_1, \qquad Q_1 = S_1 \sin\theta_1 = P \tan\theta_1 \tag{11.57}$$

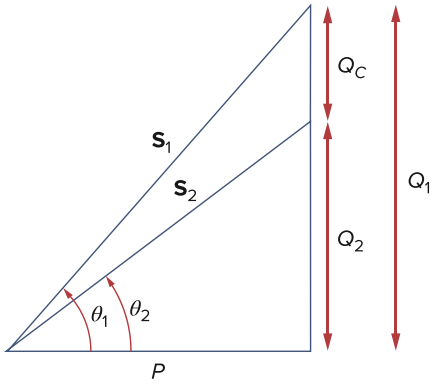



**Figure 11.29**
Power triangle illustrating power factor correction.

If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power (i.e., $P = S_2\cos\theta_2$), then the new reactive power is

$$Q_2 = P\tan\theta_2 \tag{11.58}$$

The reduction in the reactive power is caused by the shunt capacitor; that is,

$$Q_C = Q_1 - Q_2 = P(\tan\theta_1 - \tan\theta_2) \tag{11.59}$$

But from Eq. (11.46), $Q_C = V_{\text{rms}}^2/X_C = \omega C V_{\text{rms}}^2$. The value of the required shunt capacitance $C$ is determined as

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2} \tag{11.60}$$

Note that the real power $P$ dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

Although the most common situation in practice is that of an inductive load, it is also possible that the load is capacitive; that is, the load is operating at a leading power factor. In this case, an inductor should be connected across the load for power factor correction. The required shunt inductance $L$ can be calculated from

$$Q_L = \frac{V_{\text{rms}}^2}{X_L} = \frac{V_{\text{rms}}^2}{\omega L} \quad \Rightarrow \quad L = \frac{V_{\text{rms}}^2}{\omega Q_L} \tag{11.61}$$

where $Q_L = Q_1 - Q_2$, the difference between the new and old reactive powers.

## Example 11.15

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

**Solution:**
If the pf = 0.8, then

$$\cos\theta_1 = 0.8 \quad \Rightarrow \quad \theta_1 = 36.87^\circ$$

where $\theta_1$ is the phase difference between voltage and current. We obtain the apparent power from the real power and the pf as

$$S_1 = \frac{P}{\cos\theta_1} = \frac{4000}{0.8} = 5000\ \text{VA}$$

The reactive power is

$$Q_1 = S_1\sin\theta = 5000\sin 36.87 = 3000\ \text{VAR}$$

When the pf is raised to 0.95,

$$\cos\theta_2 = 0.95 \quad \Rightarrow \quad \theta_2 = 18.19^\circ$$

The real power $P$ has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos\theta_2} = \frac{4000}{0.95} = 4210.5 \text{ VA}$$

The new reactive power is

$$Q_2 = S_2 \sin\theta_2 = 1314.4 \text{ VAR}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{ VAR}$$

and

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu\text{F}$$

*Note:* Capacitors are normally purchased for voltages they expect to see. In this case, the maximum voltage this capacitor will see is about 170 V peak. We would suggest purchasing a capacitor with a voltage rating equal to, say, 200 V.

**Practice Problem 11.15**

Find the value of parallel capacitance needed to correct a load of 140 kVAR at 0.85 lagging pf to unity pf. Assume that the load is supplied by a 220-V (rms), 60-Hz line.

**Answer:** 7.673 mF.

## 11.9 †Applications

In this section, we consider two important application areas: how power is measured and how electric utility companies determine the cost of electricity consumption.

### 11.9.1 Power Measurement

Reactive power is measured by an instrument called the *varmeter*. The varmeter is often connected to the load in the same way as the wattmeter.

The average power absorbed by a load is measured by an instrument called the *wattmeter*.

The **wattmeter** is the instrument used for measuring the average power.

Some wattmeters do not have coils; the wattmeter considered here is the electromagnetic type.

Figure 11.30 shows a wattmeter that consists essentially of two coils: the current coil and the voltage coil. A current coil with very low impedance (ideally zero) is connected in series with the load (Fig. 11.31) and responds to the load current. The voltage coil with very high impedance (ideally infinite) is connected in parallel with the load as shown in Fig. 11.31 and responds to the load voltage. The current coil acts like a short circuit because of its low impedance; the voltage coil behaves like

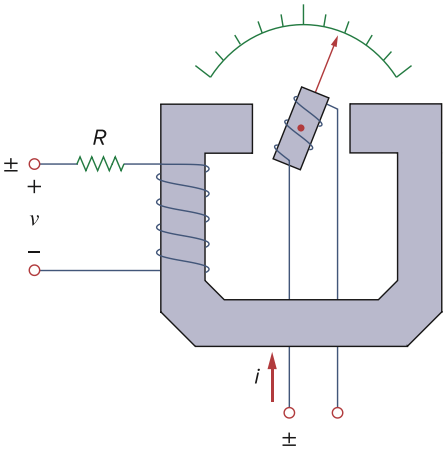



**Figure 11.30**
A wattmeter.

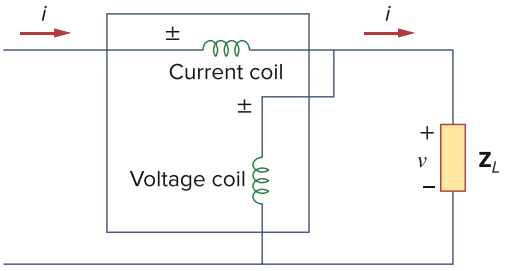



**Figure 11.31**
The wattmeter connected to the load.

an open circuit because of its high impedance. As a result, the presence of the w attmeter does not disturb the circuit or ha ve an ef fect on the power measurement.

When the two coils are energized, the mechanical inertia of the moving system produces a deflection angle that is proportional to the average value of the product $v(t)i(t)$. If the current and voltage of the load are $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$, their corresponding rms phasors are

$$\mathbf{V}_{\text{rms}} = \frac{V_m}{\sqrt{2}}\angle\theta_v \qquad \text{and} \qquad \mathbf{I}_{\text{rms}} = \frac{I_m}{\sqrt{2}}\angle\theta_i \tag{11.62}$$

and the wattmeter measures the average power given by

$$P = |\mathbf{V}_{\text{rms}}||\mathbf{I}_{\text{rms}}| \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \tag{11.63}$$

As shown in Fig. 11.31, each wattmeter coil has two terminals with one mark ed ±. To ensure upscale deflection, the ± terminal of the current coil is toward the source, while the ± terminal of the voltage coil is connected to the same line as the current coil. Re versing both coil connections still results in upscale deflection. However, reversing one coil and not the other results in downscale deflection and no wattmeter reading.

## Example 11.16

Find the wattmeter reading of the circuit in Fig. 11.32.

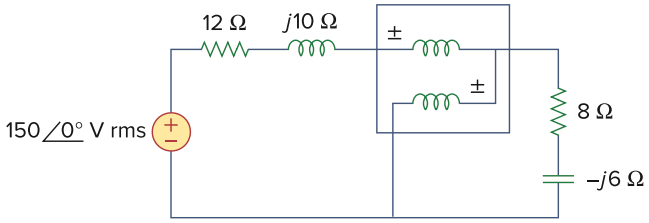



**Figure 11.32**
For Example 11.16.

**Solution:**

1. **Define.** The problem is clearly defined. Interestingly, this is a problem where the student could actually v alidate the results by doing the problem in the laboratory with a real wattmeter.

2. **Present.** This problem consists of finding the average power delivered to a load by an external source with a series impedance.
3. **Alternative.** This is a straightforward circuit problem where all we need to do is find the magnitude and phase of the current through the load and the magnitude and the phase of the voltage across the load. These quantities could also be found by using *PSpice*, which we will use as a check.
4. **Attempt.** In Fig. 11.32, the wattmeter reads the average power absorbed by the $(8 - j6)\ \Omega$ impedance because the current coil is in series with the impedance while the voltage coil is in parallel with it. The current through the circuit is

$$\mathbf{I}_{\text{rms}} = \frac{150\angle 0^\circ}{(12 + j10) + (8 - j6)} = \frac{150}{20 + j4}\ \text{A}$$

The voltage across the $(8 - j6)\ \Omega$ impedance is

$$\mathbf{V}_{\text{rms}} = \mathbf{I}_{\text{rms}}(8 - j6) = \frac{150(8 - j6)}{20 + j4}\ \text{V}$$

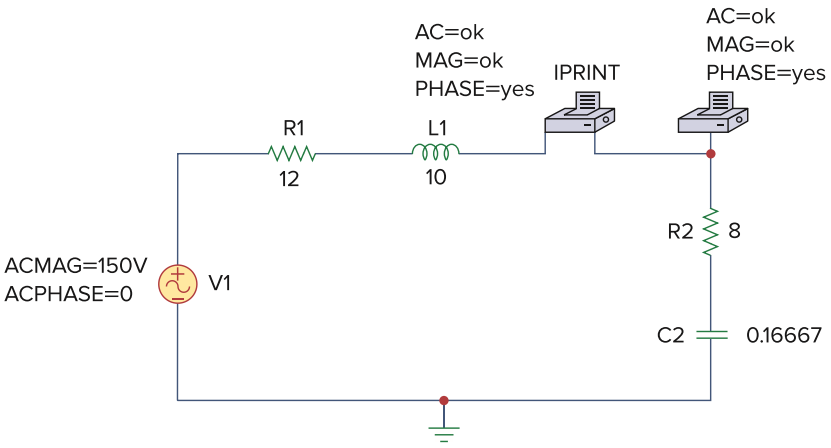
The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* = \frac{150(8 - j6)}{20 + j4} \cdot \frac{150}{20 - j4} = \frac{150^2(8 - j6)}{20^2 + 4^2}$$
$$= 423.7 - j324.6\ \text{VA}$$

The wattmeter reads

$$P = \text{Re}(\mathbf{S}) = \mathbf{432.7\ W}$$

5. **Evaluate.** We can check our results by using *PSpice*.




Simulation yields:

```
FREQ          IM(V_PRINT2)     IP(V_PRINT2)
1.592E-01     7.354E+00        -1.131E+01
```

and

```
FREQ          VM($N_0004)      VP($N_0004)
1.592E-01     7.354E+01        -4.818E+01
```

To check our answer , all we need is the magnitude of the current (7.354 A) flowing through the load resistor:

$$P = (I_L)^2R = (7.354)^2 8 = \mathbf{432.7\ W}$$

As expected, the answer does check!

6. **Satisfactory?** We ha ve satisf actorily solv ed the problem and the results can now be presented as a solution to the problem.

## Practice Problem 11.16

For the circuit in Fig. 11.33, find the wattmeter reading.

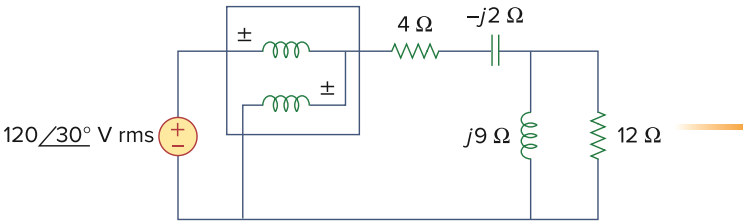



**Figure 11.33**
For Practice Prob. 11.16.

**Answer:** 1.437 kW.

### 11.9.2 Electricity Consumption Cost

In Section 1.7, we considered a simplified model of the way the cost of electricity consumption is determined. But the concept of po wer factor was not included in the calculations. Now we consider the importance of power factor in electricity consumption cost.

Loads with low power factors are costly to serve because they require large currents, as explained in Section 11.8. The ideal situation would be to draw minimum current from a supply so that $S = P$, $Q = 0$, and pf $= 1$. A load with nonzero $Q$ means that energy flows back and forth between the load and the source, gi ving rise to additional po wer losses. In vie w of this, power companies often encourage their customers to have power factors as close to unity as possible and penalize some customers who do not improve their load power factors.

Utility companies divide their customers into categories: as residential (domestic), commercial, and industrial, or as small po wer, medium power, and large power. They have different rate structures for each category. The amount of energy consumed in units of kilowatt-hours (kWh) is measured using a kilo watt-hour meter installed at the customer' s premises.

Although utility companies use different methods for charging customers, the tarif f or char ge to a consumer is often tw o-part. The first part is fixed and corresponds to the cost of generation, transmission, and distribution of electricity to meet the load requirements of the con - sumers. This part of the tarif f is generally e xpressed as a certain price

per kW of maximum demand. Or it may be based on kVA of maximum demand, to account for the power factor (pf) of the consumer. A pf penalty charge may be imposed on the consumer whereby a certain percentage of kW or kVA maximum demand is charged for every 0.01 fall in pf below a prescribed value, say 0.85 or 0.9. On the other hand, a pf credit may be given for every 0.01 that the pf exceeds the prescribed value.

The second part is proportional to the energy consumed in kWh; it may be in graded form, for example, the first 100 kWh at 16 cents/kWh, the next 200 kWh at 10 cents/kWh and so forth. Thus, the bill is determined based on the following equation:

$$\text{Total Cost} = \text{Fixed Cost} + \text{Cost of Energy} \tag{11.64}$$

## Example 11.17

A manufacturing industry consumes 200 MWh in one month. If the maximum demand is 1,600 kW, calculate the electricity bill based on the following two-part rate:

Demand charge: $5.00 per month per kW of billing demand.
Energy charge: 8 cents per kWh for the first 50,000 kWh, 5 cents per kWh for the remaining energy.

**Solution:**
The demand charge is

$$\$5.00 \times 1{,}600 = \$8{,}000 \tag{11.17.1}$$

The energy charge for the first 50,000 kWh is

$$\$0.08 \times 50{,}000 = \$4{,}000 \tag{11.17.2}$$

The remaining energy is 200,000 kWh − 50,000 kWh = 150,000 kWh, and the corresponding energy charge is

$$\$0.05 \times 150{,}000 = \$7{,}500 \tag{11.17.3}$$

Adding the results of Eqs. (11.17.1) to (11.17.3) gives

$$\text{Total bill for the month} = \$8{,}000 + \$4{,}000 + \$7{,}500 = \$19{,}500$$

It may appear that the cost of electricity is too high. But this is often a small fraction of the overall cost of production of the goods manufactured or the selling price of the finished product.

## Practice Problem 11.17

The monthly reading of a paper mill's meter is as follows:

Maximum demand: 48,000 kW
Energy consumed: 750 MWh

Using the two-part rate in Example 11.17, calculate the monthly bill for the paper mill.

**Answer:** $279,000.

## Example 11.18

A 300-kW load supplied at 13 kV (rms) operates 520 hours a month at 80 percent power factor. Calculate the average cost per month based on this simplified tariff:

Energy charge: 6 cents per kWh
Power-factor penalty: 0.1 percent of energy charge for every 0.01 that pf falls below 0.85.
Power-factor credit: 0.1 percent of energy charge for every 0.01 that pf exceeds 0.85.

**Solution:**
The energy consumed is

$$W = 300 \text{ kW} \times 520 \text{ h} = 156{,}000 \text{ kWh}$$

The operating power factor pf $= 80\% = 0.8$ is $5 \times 0.01$ below the prescribed power factor of 0.85. Since there is 0.1 percent energy charge for every 0.01, there is a power-factor penalty charge of 0.5 percent. This amounts to an energy charge of

$$\Delta W = 156{,}000 \times \frac{5 \times 0.1}{100} = 780 \text{ kWh}$$

The total energy is

$$W_t = W + \Delta W = 156{,}000 + 780 = 156{,}780 \text{ kWh}$$

The cost per month is given by

$$\text{Cost} = 6 \text{ cents} \times W_t = \$0.06 \times 156{,}780 = \$9{,}406.80$$

## Practice Problem 11.18

An 500-kW induction furnace at 0.88 power factor operates 20 hours per day for 26 days in a month. Determine the electricity bill per month based on the tariff in Example 11.18.

**Answer:** $15,553.20.

## 11.10 Summary

1. The instantaneous power absorbed by an element is the product of the element's terminal voltage and the current through the element:

$$p = vi.$$

2. Average or real power $P$ (in watts) is the average of instantaneous power $p$:

$$P = \frac{1}{T}\int_0^T p\, dt$$

If $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$, then $V_{\text{rms}} = V_m/\sqrt{2}$, $I_{\text{rms}} = I_m/\sqrt{2}$, and

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

Inductors and capacitors absorb no average power, while the average power absorbed by a resistor is $(1/2)I_m^2 R = I_{\text{rms}}^2 R$.

3. Maximum average power is transferred to a load when the load impedance is the complex conjugate of the Thevenin impedance as seen from the load terminals, $\mathbf{Z}_L = Z_{\text{Th}}^*$.
4. The effective value of a periodic signal $x(t)$ is its root-mean-square (rms) value.

$$X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2 dt}$$

For a sinusoid, the effective or rms value is its amplitude divided by $\sqrt{2}$.

5. The power factor is the cosine of the phase difference between voltage and current:

$$\text{pf} = \cos(\theta_v - \theta_i)$$

It is also the cosine of the angle of the load impedance or the ratio of real power to apparent power. The pf is lagging if the current lags voltage (inductive load) and is leading when the current leads voltage (capacitive load).

6. Apparent power $S$ (in VA) is the product of the rms values of voltage and current:

$$S = V_{\text{rms}} I_{\text{rms}}$$

It is also given by $S = |\mathbf{S}| = \sqrt{P^2 + Q^2}$, where $P$ is the real power and $Q$ is reactive power.

7. Reactive power (in VAR) is:

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

8. Complex power $\mathbf{S}$ (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. It is also the complex sum of real power $P$ and reactive power $Q$.

$$\mathbf{S} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* = V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} = P + jQ$$

Also,

$$\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*}$$

9. The total complex power in a network is the sum of the complex powers of the individual components. Total real power and reactive power are also, respectively, the sums of the individual real powers and the reactive powers, but the total apparent power is not calculated by the process.
10. Power factor correction is necessary for economic reasons; it is the process of improving the power factor of a load by reducing the overall reactive power.
11. The wattmeter is the instrument for measuring the average power. Energy consumed is measured with a kilowatt-hour meter.

## Review Questions

**11.1** The average power absorbed by an inductor is zero.

(a) True (b) False

**11.2** The Thevenin impedance of a network seen from the load terminals is $80 + j55\ \Omega$. For maximum power transfer, the load impedance must be:

(a) $-80 + j55\ \Omega$ (b) $-80 - j55\ \Omega$

(c) $80 - j55\ \Omega$ (d) $80 + j55\ \Omega$

**11.3** The amplitude of the voltage available in the 60-Hz, 120-V power outlet in your home is:

(a) 110 V (b) 120 V

(c) 170 V (d) 210 V

**11.4** If the load impedance is $20 - j20$, the power factor is

(a) $\angle -45°$ (b) 0 (c) 1

(d) 0.7071 (e) none of these

**11.5** A quantity that contains all the power information in a given load is the

(a) power factor (b) apparent power

(c) average power (d) reactive power

(e) complex power

**11.6** Reactive power is measured in:

(a) watts (b) VA

(c) VAR (d) none of these

**11.7** In the power triangle shown in Fig. 11.34(a), the reactive power is:

(a) 1000 VAR leading (b) 1000 VAR lagging

(c) 866 VAR leading (d) 866 VAR lagging

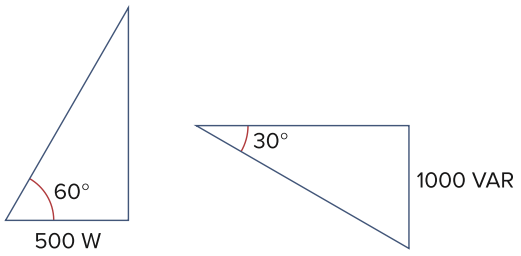



**Figure 11.34**
For Review Questions 11.7 and 11.8.

**11.8** For the power triangle in Fig. 11.34(b), the apparent power is:

(a) 2000 VA (b) 1000 VAR

(c) 866 VAR (d) 500 VAR

**11.9** A source is connected to three loads $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ in parallel. Which of these is not true?

(a) $P = P_1 + P_2 + P_3$ (b) $Q = Q_1 + Q_2 + Q_3$

(c) $S = S_1 + S_2 + S_3$ (d) $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$

**11.10** The instrument for measuring average power is the:

(a) voltmeter (b) ammeter

(c) wattmeter (d) varmeter

(e) kilowatt-hour meter

*Answers: 11.1a, 11.2c, 11.3c, 11.4d, 11.5e, 11.6c, 11.7d, 11.8a, 11.9c, 11.10c.*

## Problems[1]

### Section 11.2 Instantaneous and Average Power

**11.1** If $v(t) = 160 \cos 50t$ V and $i(t) = -33 \sin (50t - 30°)$A, calculate the instantaneous power and the average power.

**11.2** Given the circuit in Fig. 11.35, find the average power supplied or absorbed by each element.

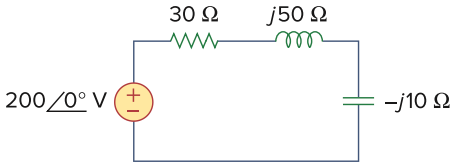



**Figure 11.35**
For Prob. 11.2.

**11.3** A load consists of a 60-Ω resistor in parallel with a 90-$\mu$F capacitor. If the load is connected to a voltage source $v_s(t) = 160 \cos 2000t$, find the average power delivered to the load.

**11.4** Using Fig. 11.36, design a problem to help other students better understand instantaneous and average power.

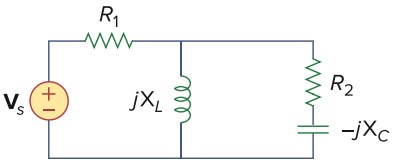



**Figure 11.36**
For Prob. 11.4.

[1] Starting with problem 11.22, unless otherwise specified, assume that all values of currents and voltages are rms.

**11.5** ssuming that $v_s = 8\cos(2t - 40°)$ V in the circuit of Fig. 11.37, find the average power delivered to each of the passive elements.

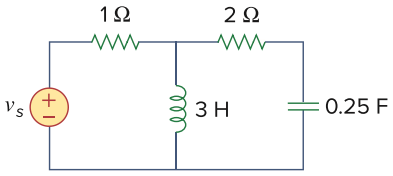



**Figure 11.37**
For Prob. 11.5.

**11.6** For the circuit in Fig. 11.38, $i_s = 6\cos 10^3 t$ A. Find the average power absorbed by the 50-Ω resistor.

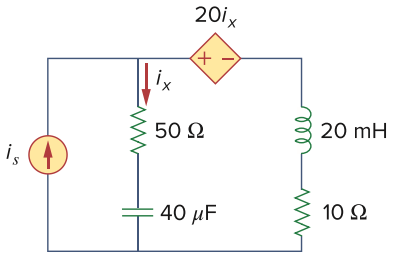



**Figure 11.38**
For Prob. 11.6.

**11.7** Given the circuit of Fig. 11.39, find the average power absorbed by the 10-Ω resistor.

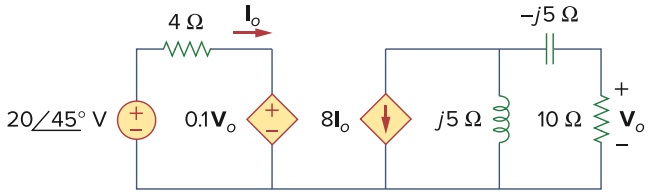



**Figure 11.39**
For Prob. 11.7.

**11.8** In the circuit of Fig. 11.40, determine the average power absorbed by the 40-Ω resistor.

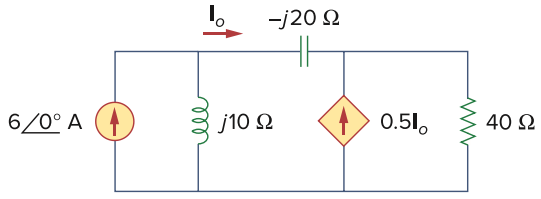



**Figure 11.40**
For Prob. 11.8.

**11.9** For the op amp circuit in Fig. 11.41, $\mathbf{V}_s = 2\angle 30°$ V. Find the average power absorbed by the 20-kΩ resistor.

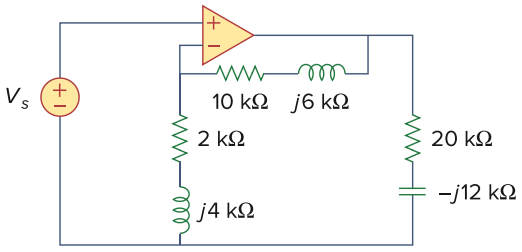



**Figure 11.41**
For Prob. 11.9.

**11.10** In the op amp circuit in Fig. 11.42, find the total average power absorbed by the resistors.

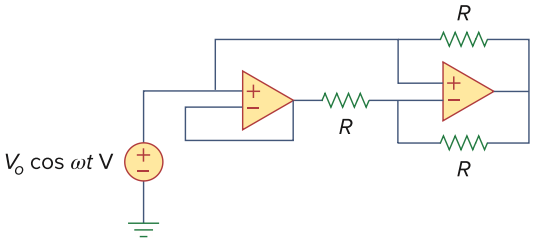



**Figure 11.42**
For Prob. 11.10.

**11.11** For the network in Fig. 11.43, assume that the port impedance is

$$\mathbf{Z}_{ab} = \frac{R}{\sqrt{1 + \omega^2 R^2 C^2}} \angle -\tan^{-1}\omega RC$$

Find the average power consumed by the network when $R = 10$ kΩ, $C = 200$ nF, and $i = 33\sin(377t + 22°)$ mA.

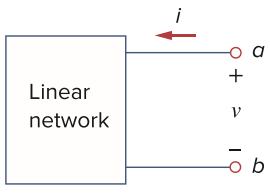



**Figure 11.43**
For Prob. 11.11.

## Section 11.3 Maximum Average Power Transfer

**11.12** For the circuit shown in Fig. 11.44, determine the load impedance $\mathbf{Z}_L$ for maximum power transfer (to $\mathbf{Z}_L$). Calculate the maximum power absorbed by the load.

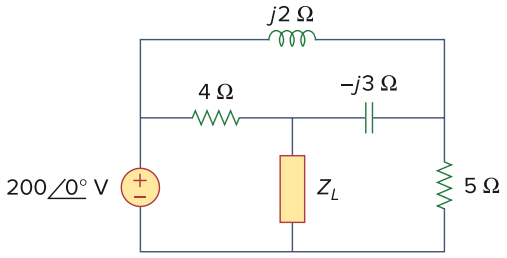



**Figure 11.44**
For Prob. 11.12.

**11.13** The Thevenin impedance of a source is $\mathbf{Z}_{\text{Th}} = 120 + j60\ \Omega$, while the peak Thevenin voltage is $\mathbf{V}_{\text{Th}} = 165 + j0$ V. Determine the maximum available average power from the source.

**11.14** Using Fig. 11.45, design a problem to help other students better understand maximum average power transfer to a load $\mathbf{Z}$.

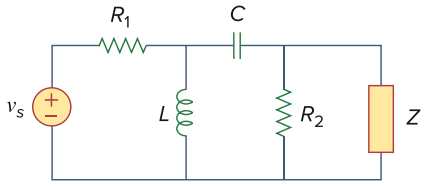



**Figure 11.45**
For Prob. 11.14.

**11.15** In the circuit of Fig. 11.46, find the value of $\mathbf{Z}_L$ that will absorb the maximum power and the value of the maximum power.

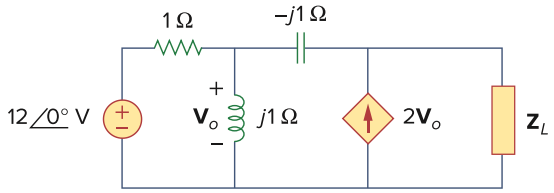



**Figure 11.46**
For Prob. 11.15.

**11.16** For the circuit in Fig. 11.47, find the value of $\mathbf{Z}_L$ that will receive the maximum power from the circuit. Then calculate the power delivered to the load $\mathbf{Z}_L$.

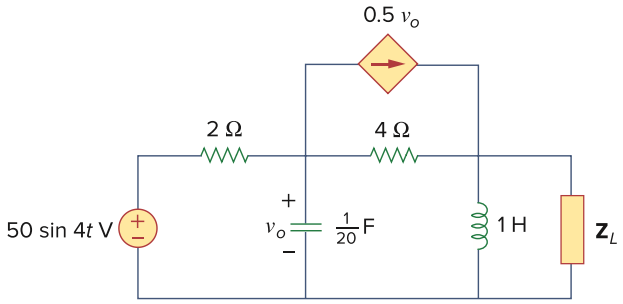



**Figure 11.47**
For Prob. 11.16.

**11.17** Calculate the value of $\mathbf{Z}_L$ in the circuit of Fig. 11.48 in order for $\mathbf{Z}_L$ to receive maximum average power. What is the maximum average power received by $\mathbf{Z}_L$?

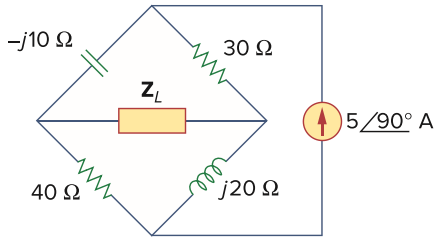



**Figure 11.48**
For Prob. 11.17.

**11.18** Find the value of $\mathbf{Z}_L$ in the circuit of Fig. 11.49 for maximum power transfer.

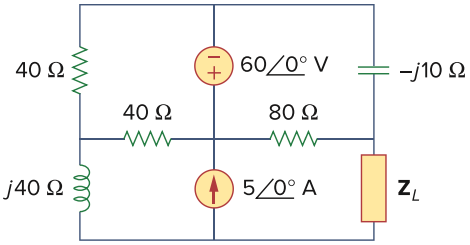



**Figure 11.49**
For Prob. 11.18.

**11.19** The variable resistor $R$ in the circuit of Fig. 11.50 is adjusted until it absorbs the maximum average power. Find $R$ and the maximum average power absorbed.

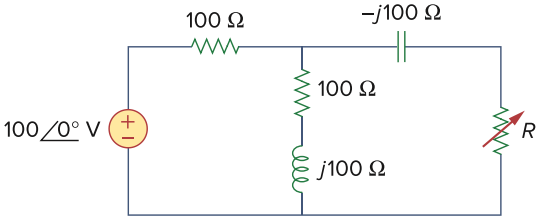



**Figure 11.50**
For Prob. 11.19.

**11.20** The load resistance $R_L$ in Fig. 11.51 is adjusted until it absorbs the maximum average power. Calculate the value of $R_L$ and the maximum average power.

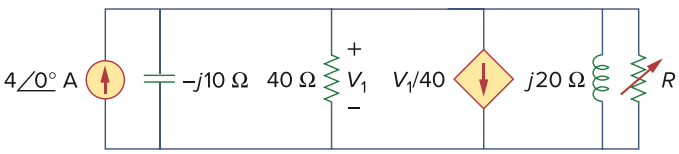



**Figure 11.51**
For Prob. 11.20.

**11.21** Assuming that the load impedance is to be purely resistive, what load should be connected to terminals *a*-*b* of the circuits in Fig. 11.52 so that the maximum power is transferred to the load?

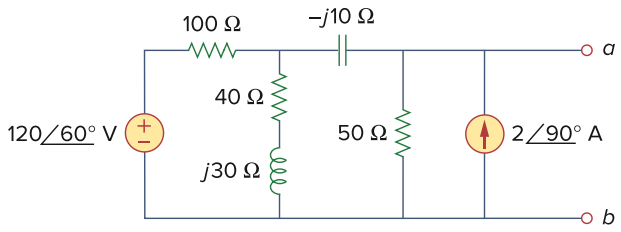



**Figure 11.52**
For Prob. 11.21.

## Section 11.4 Effective or RMS Value

**11.22** Find the rms value of the offset sine wave shown in Fig. 11.53.

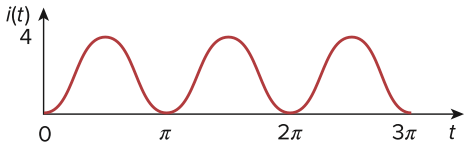



**Figure 11.53**
For Prob. 11.22.

**11.23** Using Fig. 11.54, design a problem to help other students better understand how to find the rms value of a waveshape.

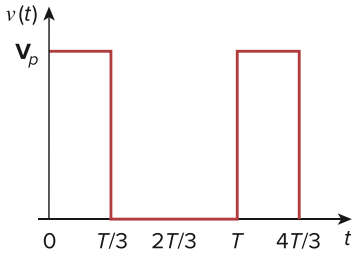



**Figure 11.54**
For Prob. 11.23.

**11.24** Determine the rms value of the waveform in Fig. 11.55.

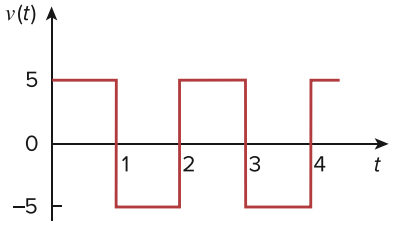



**Figure 11.55**
For Prob. 11.24.

**11.25** Find the rms value of the signal shown in Fig. 11.56.

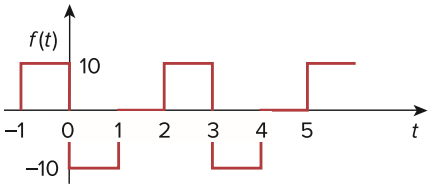



**Figure 11.56**
For Prob. 11.25.

**11.26** Find the effective value of the voltage waveform in Fig. 11.57.

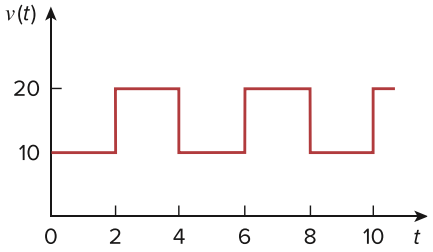



**Figure 11.57**
For Prob. 11.26.

**11.27** Calculate the rms value of the current waveform of Fig. 11.58.

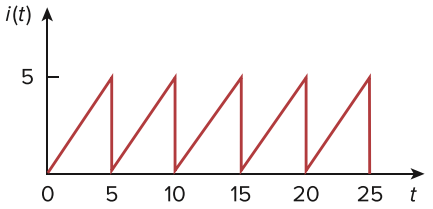



**Figure 11.58**
For Prob. 11.27.

**11.28** Find the rms value of the voltage waveform of Fig. 11.59 as well as the average power absorbed by a 2-Ω resistor when the voltage is applied across the resistor.

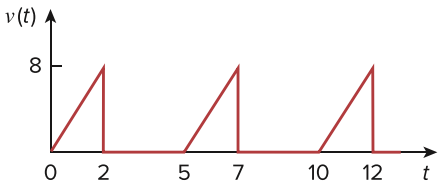



**Figure 11.59**
For Prob. 11.28.

**11.29** Calculate the effective value of the current waveform in Fig. 11.60 and the average power delivered to a 12-Ω resistor when the current runs through the resistor.

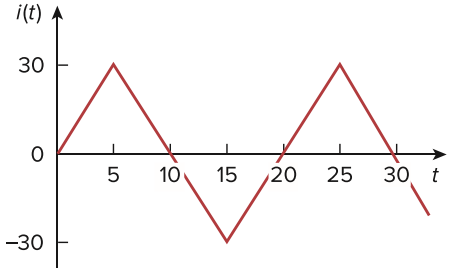



**Figure 11.60**
For Prob. 11.29.

**11.30** Compute the rms value of the waveform depicted in Fig. 11.61.

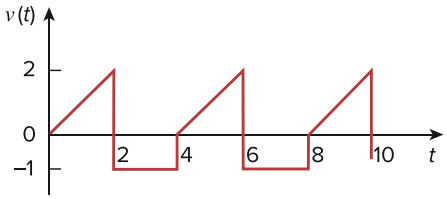



**Figure 11.61**
For Prob. 11.30.

**11.31** Find the rms value of the signal shown in Fig. 11.62.

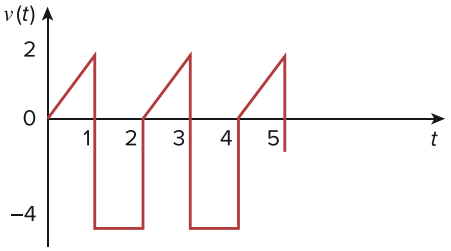



**Figure 11.62**
For Prob. 11.31.

**11.32** Obtain the rms value of the current waveform shown in Fig. 11.63.

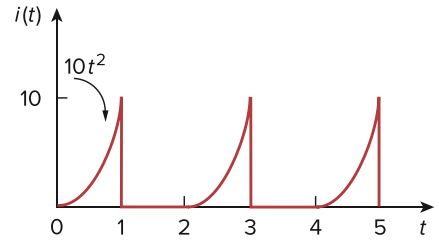



**Figure 11.63**
For Prob. 11.32.

**11.33** Determine the rms value for the waveform in Fig. 11.64.

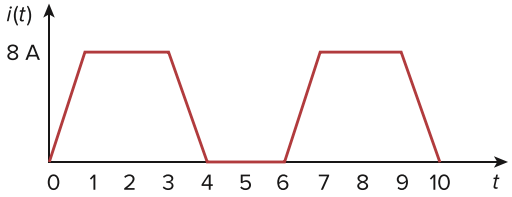



**Figure 11.64**
For Prob. 11.33.

**11.34** Find the effective value of $f(t)$ defined in Fig. 11.65.

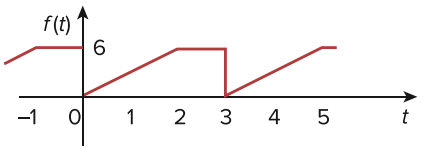



**Figure 11.65**
For Prob. 11.34.

**11.35** One cycle of a periodic voltage waveform is depicted in Fig. 11.66. Find the effective value of the voltage. Note that the cycle starts at $t = 0$ and ends at $t = 6$ s.

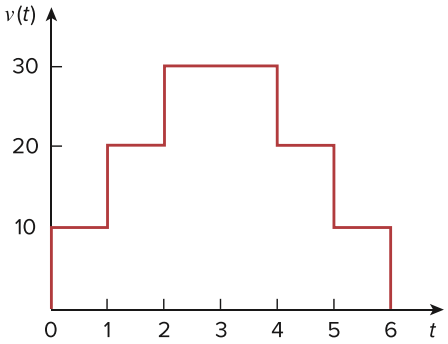



**Figure 11.66**
For Prob. 11.35.

**11.36** Calculate the rms value for each of the following functions:

(a) $i(t) = 10$ A (b) $v(t) = 4 + 3\cos 5t$ V
(c) $i(t) = 8 - 6\sin 2t$ A (d) $v(t) = 5\sin t + 4\cos t$ V

**11.37** Design a problem to help other students better understand how to determine the rms value of the sum of multiple currents.

## Section 11.5 Apparent Power and Power Factor

**11.38** For the power system in Fig. 11.67, find: (a) the average power, (b) the reactive power, (c) the power factor. Note that 440 V is an rms value.

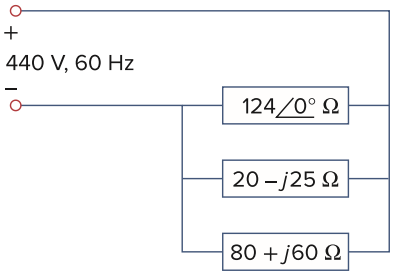



**Figure 11.67**
For Prob. 11.38.

**11.39** An ac motor with impedance $\mathbf{Z}_L = 2 + j1.2\ \Omega$ is supplied by a 220-V, 60-Hz source. (a) Find pf, $P$, and $Q$. (b) Determine the capacitor required to be connected in parallel with the motor so that the power factor is corrected to unity.

**11.40** Design a problem to help other students better understand apparent power and power factor.

**11.41** Obtain the power factor for each of the circuits in Fig. 11.68. Specify each power factor as leading or lagging.

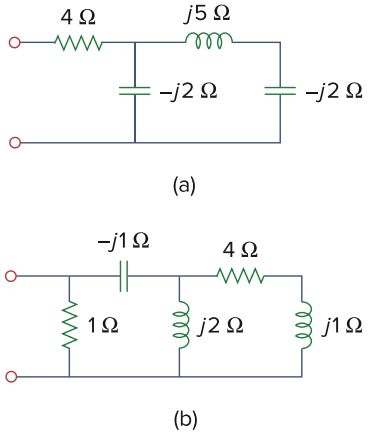



**Figure 11.68**
For Prob. 11.41.

## Section 11.6 Complex Power

**11.42** A 110-V rms, 60-Hz source is applied to a load impedance $\mathbf{Z}$. The apparent power entering the load is 120 VA at a power factor of 0.707 lagging.

(a) Calculate the complex power.
(b) Find the rms current supplied to the load.
(c) Determine $\mathbf{Z}$.
(d) Assuming that $\mathbf{Z} = R + j\omega L$, find the values of $R$ and $L$.

**11.43** Design a problem to help other students understand complex power.

**11.44** Find the complex power delivered by $v_s$ to the network in Fig. 11.69. Let $v_s = 100 \cos 2000t$ V.

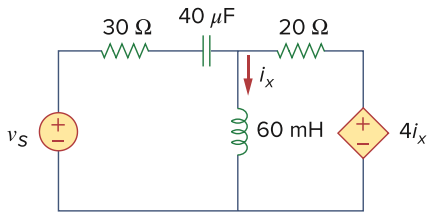



**Figure 11.69**
For Prob. 11.44.

**11.45** The voltage across a load and the current through it are given by

$$v(t) = 20 + 60 \cos 100t \text{ V}$$
$$i(t) = 1 - 0.5 \sin 100t \text{ A}$$

Find:

(a) the rms values of the voltage and of the current
(b) the average power dissipated in the load

**11.46** For the following voltage and current phasors, calculate the complex power, apparent power, real power, and reactive power. Specify whether the pf is leading or lagging.

(a) $\mathbf{V} = 220\angle 30°$ V rms, $I = 0.5\angle 60°$ A rms
(b) $\mathbf{V} = 250\angle -10°$ V rms,
$\mathbf{I} = 6.2\angle -25°$ A rms
(c) $\mathbf{V} = 120\angle 0°$ V rms, $\mathbf{I} = 2.4\angle -15°$ A rms
(d) $\mathbf{V} = 160\angle 45°$ V rms, $\mathbf{I} = 8.5\angle 90°$ A rms

**11.47** For each of the following cases, find the complex power, the average power, and the reactive power:

(a) $v(t) = 169.7 \sin (377t + 45°)$ V,
$i(t) = 5.657 \sin (377t)$ A
(b) $v(t) = 339.4 \sin (377t + 90°)$ V,
$i(t) = 5.657 \sin (377t + 45°)$ A
(c) $\mathbf{V} = 900\angle 90°$ V rms, $\mathbf{Z} = 75\angle 45°\ \Omega$
(d) $\mathbf{I} = 100\angle 60°$ A rms, $\mathbf{Z} = 50\angle 60°\ \Omega$

**11.48** Determine the complex power for the following cases:

(a) $P = 269$ W, $Q = 150$ VAR (capacitive)
(b) $Q = 2000$ VAR, pf = 0.9 (leading)
(c) $S = 600$ VA, $Q = 450$ VAR (inductive)
(d) $V_{\text{rms}} = 220$ V, $P = 1$ kW,
$|\mathbf{Z}| = 40\ \Omega$ (inductive)

**11.49** Find the complex power for the following cases:

(a) $P = 4$ kW, pf = 0.86 (lagging)
(b) $S = 2$ kVA, $P = 1.6$ kW (capacitive)
(c) $\mathbf{V}_{rms} = 208\angle 20°$ V, $\mathbf{I}_{rms} = 6.5\angle -50°$ A
(d) $\mathbf{V}_{rms} = 120\angle 30°$ V, $\mathbf{Z} = 40 + j60\ \Omega$

**11.50** Obtain the overall impedance for the following cases:

(a) $P = 1000$ W, pf = 0.8 (leading), $V_{rms} = 220$ V
(b) $P = 1500$ W, $Q = 2000$ VAR (inductive), $I_{rms} = 12$ A
(c) $\mathbf{S} = 4500\angle 60°$ VA, $\mathbf{V} = 120\angle 45°$ V

**11.51** For the entire circuit in Fig. 11.70, calculate:

(a) the power factor
(b) the average power delivered by the source
(c) the reactive power
(d) the apparent power
(e) the complex power

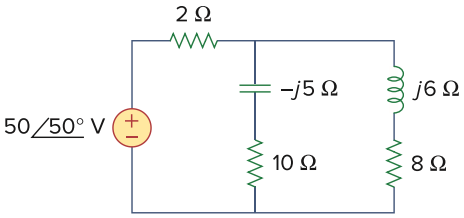



**Figure 11.70**
For Prob. 11.51.

**11.52** In the circuit of Fig. 11.71, device $A$ receives 2 kW at 0.8 pf lagging, device $B$ receives 3 kVA at 0.4 pf leading, while device $C$ is inductive and consumes 1 kW and receives 500 VAR.

(a) Determine the power factor of the entire system.
(b) Find $\mathbf{I}$ given that $\mathbf{V}_s = 120\angle 45°$ V rms.

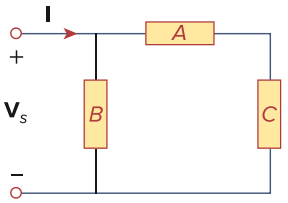



**Figure 11.71**
For Prob. 11.52.

**11.53** In the circuit of Fig. 11.72, load $A$ receives 4 kVA at 0.8 pf leading. Load $B$ receives 2.4 kVA at 0.6 pf lagging. Box $C$ is an inductive load that consumes 1 kW and receives 500 VAR.

(a) Determine $\mathbf{I}$.
(b) Calculate the power factor of the combination.

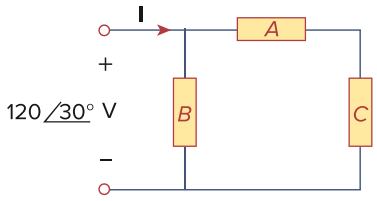



**Figure 11.72**
For Prob. 11.53.

## Section 11.7 Conservation of AC Power

**11.54** For the network in Fig. 11.73, find the complex power absorbed by each element.

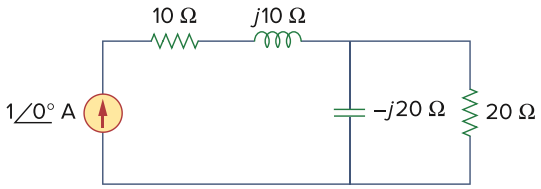



**Figure 11.73**
For Prob. 11.54.

**11.55** Using Fig. 11.74, design a problem to help other students better understand the conservation of AC power.

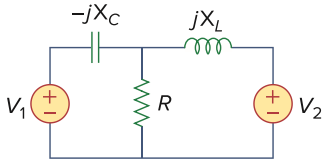



**Figure 11.74**
For Prob. 11.55.

**11.56** Obtain the complex power delivered by the source in the circuit of Fig. 11.75.

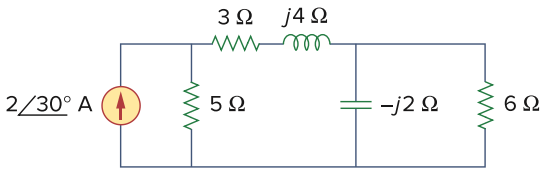



**Figure 11.75**
For Prob. 11.56.

**11.57** For the circuit in Fig. 11.76, find the average, reactive, and complex power delivered by the dependent current source.

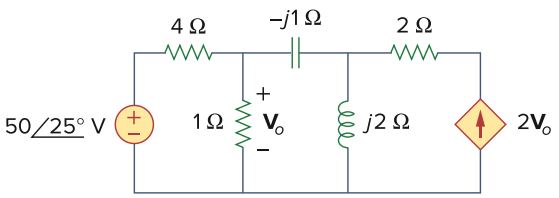



**Figure 11.76**
For Prob. 11.57.

**11.58** Obtain the complex power delivered to the 10-kΩ resistor in Fig. 11.77 below.

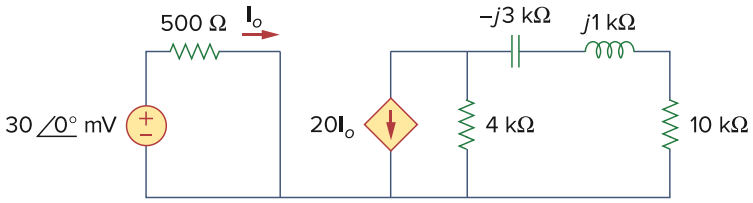



**Figure 11.77**
For Prob. 11.58.

**11.59** Calculate the reactive power in the inductor and capacitor in the circuit of Fig. 11.78.

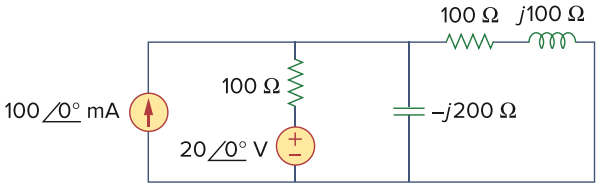



**Figure 11.78**
For Prob. 11.59.

**11.60** For the circuit in Fig. 11.79, find $\mathbf{V}_o$ and the input power factor.

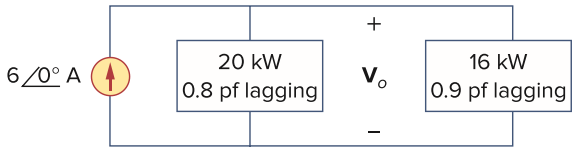



**Figure 11.79**
For Prob. 11.60.

**11.61** Given the circuit in Fig. 11.80, find $I_o$ and the overall complex power supplied.

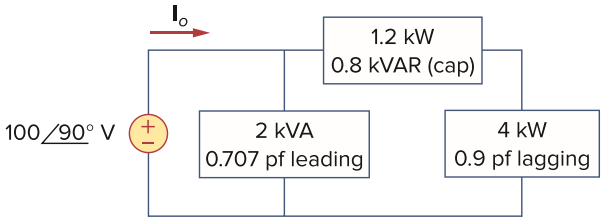



**Figure 11.80**
For Prob. 11.61.

**11.62** For the circuit in Fig. 11.81, find $\mathbf{V}_s$.

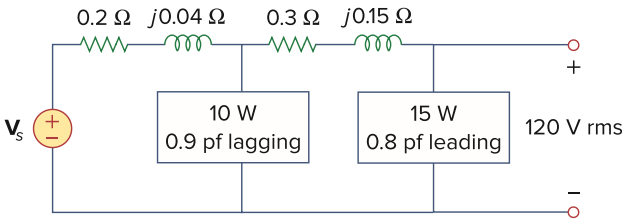



**Figure 11.81**
For Prob. 11.62.

**11.63** Find $\mathbf{I}_o$ in the circuit of Fig. 11.82.

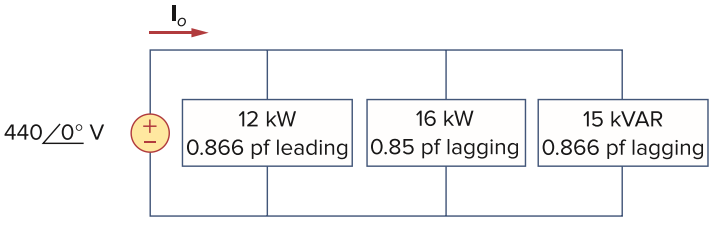



**Figure 11.82**
For Prob. 11.63.

**11.64** Determine $\mathbf{I}_s$ in the circuit of Fig. 11.83, if the voltage source supplies 6 kW and 1.2 kVAR (leading).

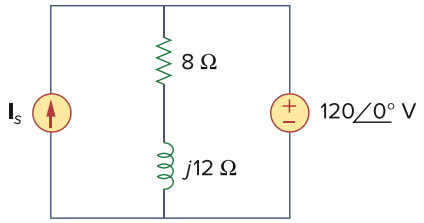



**Figure 11.83**
For Prob. 11.64.

**11.65** In the op amp circuit of Fig. 11.84, $v_s = 4\cos 10^4 t$ V. Find the average power delivered to the 50-kΩ resistor.

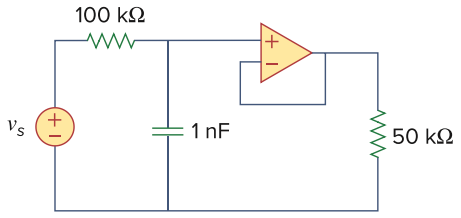



**Figure 11.84**
For Prob. 11.65.

**11.66** Obtain the average power absorbed by the 10-Ω resistor in the op amp circuit in Fig. 11.85.

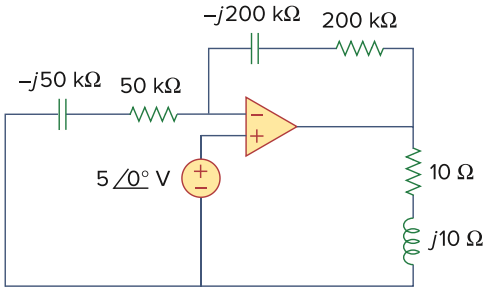



**Figure 11.85**
For Prob. 11.66.

**11.67** For the op amp circuit in Fig. 11.86, calculate:

(a) the complex power delivered by the voltage source

(b) the average power dissipated in the 10-Ω resistor

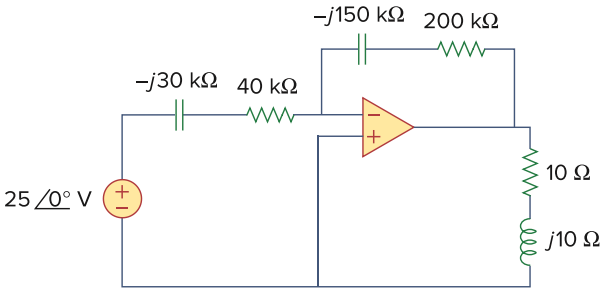



**Figure 11.86**
For Prob. 11.67.

**11.68** Compute the complex power supplied by the current source in the series *RLC* circuit in Fig. 11.87.

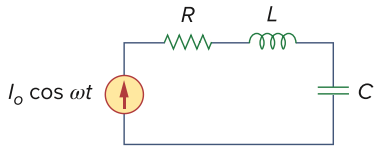



**Figure 11.87**
For Prob. 11.68.

## Section 11.8 Power Factor Correction

**11.69** Refer to the circuit shown in Fig. 11.88.

(a) What is the power factor?

(b) What is the average power dissipated?

(c) What is the value of the capacitance that will give a unity power factor when connected to the load?

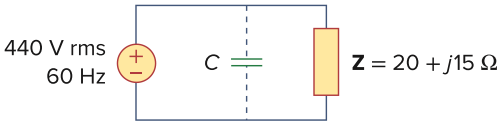



**Figure 11.88**
For Prob. 11.69.

**11.70** Design a problem to help other students better understand power factor correction.

**11.71** Three loads are connected in parallel to a $120\angle 0°$ V rms source. Load 1 absorbs 60 kVAR at pf = 0.85 lagging, load 2 absorbs 90 kW and 50 kVAR leading, and load 3 absorbs 100 kW at pf = 1. (a) Find the equivalent impedance. (b) Calculate the power factor of the parallel combination. (c) Determine the current supplied by the source.

**11.72** Two loads connected in parallel draw a total of 2.4 kW at 0.8 pf lagging from a 120-V rms, 60-Hz line. One load absorbs 1.5 kW at a 0.707 pf lagging. Determine: (a) the pf of the second load, (b) the parallel element required to correct the pf to 0.9 lagging for the two loads.

**11.73** A 240-V rms 60-Hz supply serves a load that is 10 kW (resistive), 15 kVAR (capacitive), and 22 kVAR (inductive). Find:

(a) the apparent power

(b) the current drawn from the supply

(c) the kVAR rating and capacitance required to improve the power factor to 0.96 lagging

(d) the current drawn from the supply under the new power-factor conditions

**11.74** A 120-V rms 60-Hz source supplies two loads connected in parallel, as shown in Fig. 11.89.

(a) Find the power factor of the parallel combination.

(b) Calculate the value of the capacitance connected in parallel that will raise the power factor to unity.

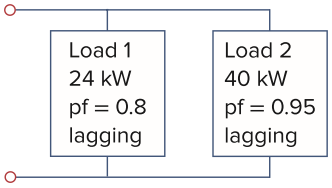



**Figure 11.89**
For Prob. 11.74.

**11.75** Consider the power system shown in Fig. 11.90. Calculate:

(a) the total complex power

(b) the power factor

(c) the parallel capacitance necessary to establish a unity power factor

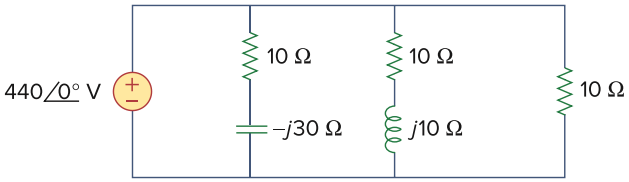



**Figure 11.90**
For Prob. 11.75.

## Section 11.9 Applications

**11.76** Obtain the wattmeter reading of the circuit in Fig. 11.91.

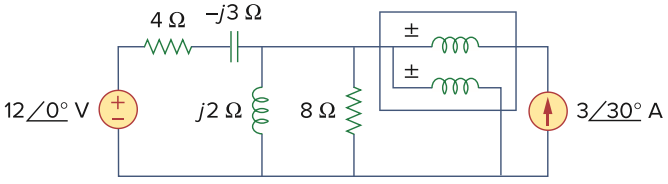



**Figure 11.91**
For Prob. 11.76.

**11.77** What is the reading of the wattmeter in the network of Fig. 11.92?

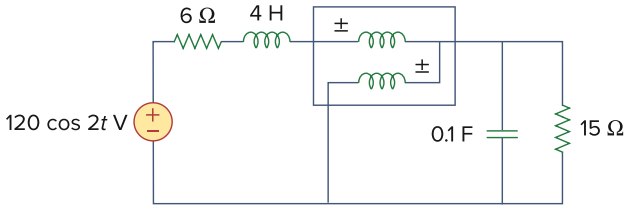



**Figure 11.92**
For Prob. 11.77.

**11.78** Find the wattmeter reading of the circuit shown in Fig. 11.93.

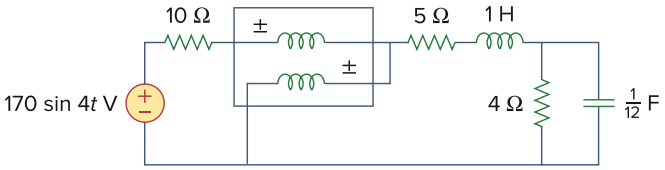



**Figure 11.93**
For Prob. 11.78.

**11.79** Determine the wattmeter reading of the circuit in Fig. 11.94.

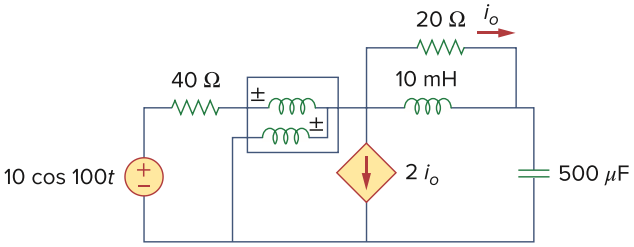



**Figure 11.94**
For Prob. 11.79.

**11.80** The circuit of Fig. 11.95 portrays a wattmeter connected into an ac network.

(a) Find the magnitude of the load current.

(b) Calculate the wattmeter reading.

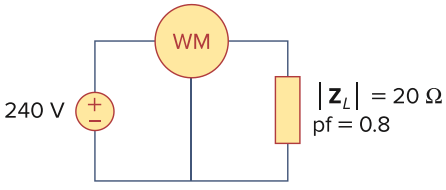



**Figure 11.95**
For Prob. 11.80.

**11.81** Design a problem to help other students better understand how to correct power factor to values other than unity.

**11.82** A 240-V rms 60-Hz source supplies a parallel combination of a 5-kW heater and a 30-kVA induction motor whose power factor is 0.82. Determine:

(a) the system apparent power

(b) the system reactive power

(c) the kVA rating of a capacitor required to adjust the system power factor to 0.9 lagging

(d) the value of the capacitor required

**11.83** Oscilloscope measurements indicate that the peak voltage across a load and the peak current through it are, respectively, $210\angle 60°$ V and $8\angle 25°$ A. Determine:

(a) the real power

(b) the apparent power

(c) the reactive power

(d) the power factor

**11.84** A consumer has an annual consumption of 1200 MWh with a maximum demand of 2.4 MVA. The maximum demand charge is $30 per kVA per annum, and the energy charge per kWh is 4 cents.

(a) Determine the annual cost of energy.

(b) Calculate the charge per kWh with a flat-rate tariff if the revenue to the utility company is to remain the same as for the two-part tariff.

**11.85** A regular household system of a single-phase three-wire circuit allows the operation of both 120-V and 240-V, 60-Hz appliances. The household circuit is modeled as shown in Fig. 11.96. Calculate:

(a) the currents $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_n$

(b) the total complex power supplied

(c) the overall power factor of the circuit

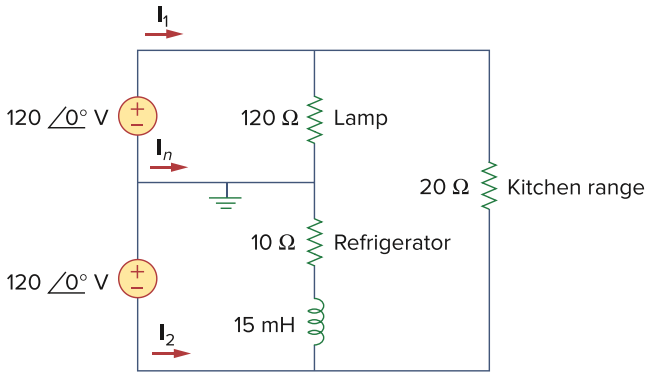



**Figure 11.96**
For Prob. 11.85.

## Comprehensive Problems

**11.86** A transmitter delivers maximum power to an antenna when the antenna is adjusted to represent a load of 75-Ω resistance in series with an inductance of 4 $\mu$H. If the transmitter operates at 4.12 MHz, find its internal impedance.

**11.87** In a TV transmitter, a series circuit has an impedance of 3 kΩ and a total current of 50 mA. If the voltage across the resistor is 80 V, what is the power factor of the circuit?

**11.88** A certain electronic circuit is connected to a 110-V ac line. The root-mean-square value of the current drawn is 2 A, with a phase angle of 55°.

(a) Find the true power drawn by the circuit.

(b) Calculate the apparent power.

**11.89** An industrial heater has a nameplate that reads:
210 V 60 Hz 12 kVA 0.78 pf lagging
Determine:

(a) the apparent and the complex power

(b) the impedance of the heater

* **11.90** A 2000-kW turbine-generator of 0.85 power factor operates at the rated load. An additional load of 300 kW at 0.8 power factor is added. What kVAR of capacitors is required to operate the turbine-generator but keep it from being overloaded?

**11.91** The nameplate of an electric motor has the following information:

Line voltage: 220 V rms
Line current: 15 A rms
Line frequency: 60 Hz
Power: 2700 W

Determine the power factor (lagging) of the motor. Find the value of the capacitance $C$ that must be connected across the motor to raise the pf to unity.

**11.92** As shown in Fig. 11.97, a 550-V feeder line supplies an industrial plant consisting of a motor drawing 90 kW at 0.8 pf (inductive), a capacitor with a rating of 20 kVAR, and lighting drawing 10 kW.

(a) Calculate the total reactive power and apparent power absorbed by the plant.

(b) Determine the overall pf.

(c) Find the magnitude of the current in the feeder line.

* An asterisk indicates a challenging problem.

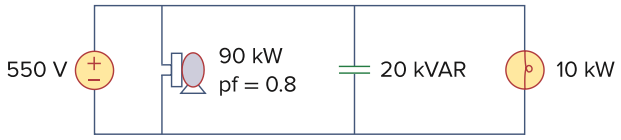



**Figure 11.97**
For Prob. 11.92.

**11.93** A factory has the following four major loads:

- A motor rated at 5 hp, 0.8 pf lagging (1hp = 0.7457 kW).
- A heater rated at 1.2 kW, 1.0 pf.
- Ten 120-W lightbulbs.
- A synchronous motor rated at 1.6 kVAR, 0.6 pf leading.

(a) Calculate the total real and reactive power.

(b) Find the overall power factor.

**11.94** A 1-MVA substation operates at full load at 0.7 power factor. It is desired to improve the power factor to 0.95 by installing capacitors. Assume that new substation and distribution facilities cost $120 per kVA installed, and capacitors cost $30 per kVA installed.

(a) Calculate the cost of capacitors needed.

(b) Find the savings in substation capacity released.

(c) Are capacitors economical for releasing the amount of substation capacity?

**1.95** A coupling capacitor is used to block dc current from an amplifier as shown in Fig. 11.98(a). The amplifier and the capacitor act as the source, while the speaker is the load as in Fig. 11.98(b).

(a) At what frequency is maximum power transferred to the speaker?

(b) If $V_s = 4.6$ V rms, how much power is delivered to the speaker at that frequency?

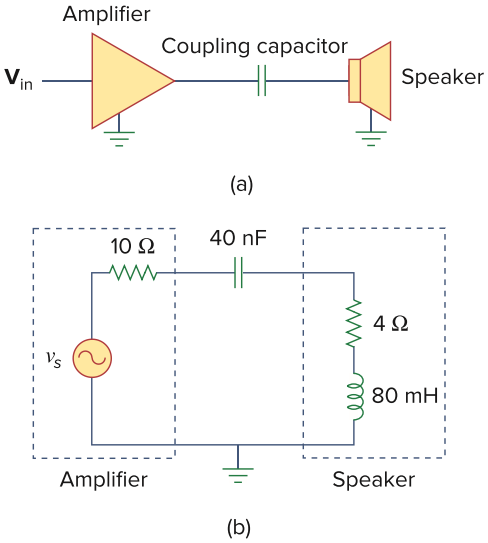



**Figure 11.98**
For Prob. 11.95.

**1.96** A power amplifier has an output impedance of $40 + j8\ \Omega$. It produces a no-load output voltage of 146 V at 300 Hz.

(a) Determine the impedance of the load that achieves maximum power transfer.

(b) Calculate the load power under this matching condition.

**1.97** A power transmission system is modeled as shown in Fig. 11.99. If $\mathbf{V}_s = 440\angle 0°$ rms, find the average power absorbed by the load.

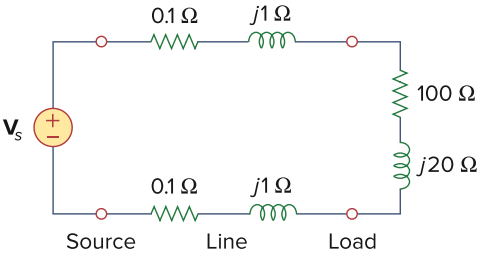



**Figure 11.99**
For Prob. 11.97.

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chapter

# 12

# Three-Phase Circuits

*He who cannot forgive others breaks the bridge over which he must pass himself.*

—G. Herbert

## Enhancing Your Skills and Your Career

### ABET EC 2000 criteria (3.e), *"an ability to identify, formulate, and solve engineering problems."*

Developing and enhancing your "ability to identify, formulate, and solve engineering problems" is a primary focus of this textbook. Following our six-step problem-solving process is the best way to practice this skill. Our recommendation is that you use this process whenever possible. You may be pleased to learn that this process works well for nonengineering courses.

### ABET EC 2000 criteria (f), *"an understanding of professional and ethical responsibility."*

"An understanding of professional and ethical responsibility" is required of every engineer. To some extent, this understanding is very personal for each of us. Let us identify some pointers to help you develop this understanding. One of my favorite examples is that an engineer has the responsibility to answer what I call the 'unasked question." For instance, assume that you own a car that has a problem with the transmission. In the process of selling that car, the prospective buyer asks you if there is a problem in the right-front wheel bearing. You answer no. However, as an engineer, you are required to inform the buyer that there is a problem with the transmission without being asked.

Your responsibility both professionally and ethically is to perform in a manner that does not harm those around you and to whom you are responsible. Clearly, developing this capability will take time and maturity on your part. I recommend practicing this by looking for professional and ethical components in your day-to-day activities.

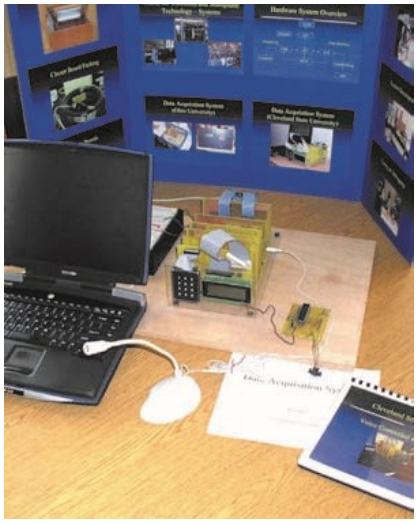


Photo by Charles Alexander

## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Understand balanced three-phase voltages.
2. Analyze balanced wye-wye circuits.
3. Understand and analyze balanced wye-delta circuits.
4. Analyze balanced delta-delta circuits.
5. Understand and analyze balanced delta-wye circuits.
6. Explain and analyze power in balanced three-phase circuits.
7. Analyze unbalanced three-phase circuits.

## 12.1 Introduction

So far in this text, we have dealt with single-phase circuits. A single-phase ac power system consists of a generator connected through a pair of wires (a transmission line) to a load. Figure 12.1(a) depicts a single-phase two-wire system, where $V_p$ is the rms magnitude of the source voltage and $\phi$ is the phase. What is more common in practice is a single-phase three-wire system, shown in Fig. 12.1(b). It contains two identical sources (equal magnitude and the same phase) that are connected to two loads by two outer wires and the neutral. For example, the normal household system is a single-phase three-wire system because the terminal voltages have the same magnitude and the same phase. Such a system allows the connection of both 120- and 240-V appliances.

*Historical note:* Thomas Edison invented a *three-wire system*, using three wires instead of four.

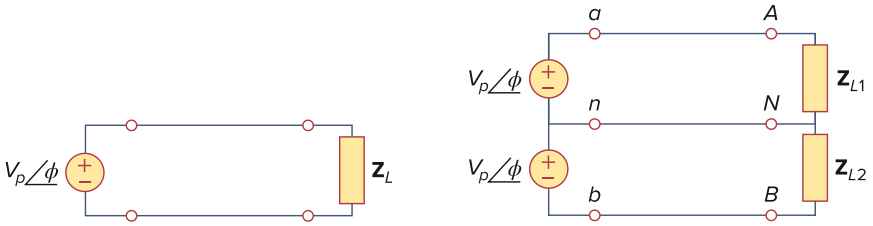



**Figure 12.1**
Single-phase systems: (a) two-wire type, (b) three-wire type.

Circuits or systems in which the ac sources operate at the same frequency but different phases are known as *polyphase*. Figure 12.2 shows a two-phase three-wire system, and Fig. 12.3 shows a three-phase four-wire system. As distinct from a single-phase system, a two-phase system is produced by a generator consisting of two coils placed perpendicular to each other so that the voltage generated by one lags the other by 90°. By the same token, a three-phase system is produced by a generator consisting of three sources having the same amplitude and frequency but out of phase with each other by 120°. Because the three-phase system is by far the most prevalent and most economical polyphase system, discussion in this chapter is mainly on three-phase systems.

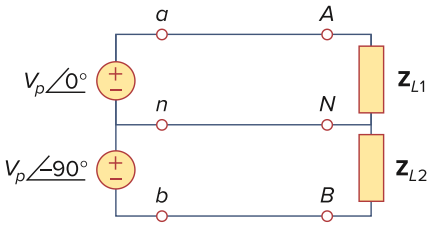


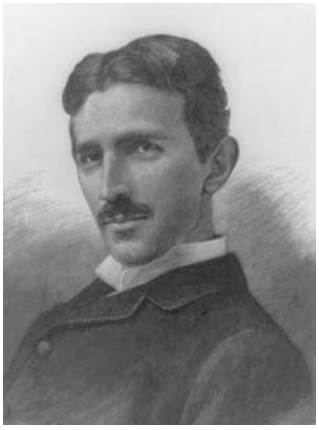

**Figure 12.2**
Two-phase three-wire system.

Three-phase systems are important for at least three reasons. First, nearly all electric power is generated and distributed in three-phase,

## Historical

**Nikola Tesla** (1856–1943) was a Croatian-American engineer whose inventions—among them the induction motor and the first polyphase ac power system—greatly influenced the settlement of the ac versus dc debate in favor of ac. He was also responsible for the adoption of 60 Hz as the standard for ac power systems in the United States.

Born in Austria-Hungary (now Croatia), to a clergyman, Tesla had an incredible memory and a keen affinity for mathematics. He moved to the United States in 1884 and first worked for Thomas Edison. At that time, the country was in the "battle of the currents" with George Westinghouse (1846–1914) promoting ac and Thomas Edison rigidly leading the dc forces. Tesla left Edison and joined Westinghouse because of his interest in ac. Through Westinghouse, Tesla gained the reputation and acceptance of his polyphase ac generation, transmission, and distribution system. He held 700 patents in his lifetime. His other inventions include high-voltage apparatus (the tesla coil) and a wireless transmission system. The unit of magnetic flux density, the tesla, was named in honor of him.



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at the operating frequency of 60 Hz (or $\omega = 377$ rad/s) in the United States or 50 Hz (or $\omega = 314$ rad/s) in some other parts of the world. When one-phase or two-phase inputs are required, they are taken from the three-phase system rather than generated independently. Even when more than three phases are needed—such as in the aluminum industry, where 48 phases are required for melting purposes—they can be provided by manipulating the three phases supplied. Second, the instantaneous power in a three-phase system can be constant (not pulsating), as we will see in Section 12.7. This results in uniform power transmission and less vibration of three-phase machines. Third, for the same amount of power, the three-phase system is more economical than the single-phase. The amount of wire required for a three-phase system is less than that required for an equivalent single-phase system.

We begin with a discussion of balanced three-phase voltages. Then we analyze each of the four possible configurations of balanced three-phase systems. We also discuss the analysis of unbalanced three-phase systems. We learn how to use *PSpice for Windows* to analyze a balanced or unbalanced three-phase system. Finally, we apply the concepts developed in this chapter to three-phase power measurement and residential electrical wiring.

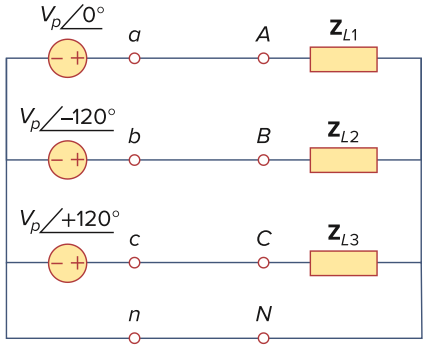



**Figure 12.3**
Three-phase four-wire system.

## 12.2 Balanced Three-Phase Voltages

Three-phase voltages are often produced with a three-phase ac generator (or alternator) whose cross-sectional view is shown in Fig. 12.4. The generator basically consists of a rotating magnet (called the *rotor*) surrounded by a stationary winding (called the *stator*). Three separate windings or coils with terminals $a$-$a'$, $b$-$b'$, and $c$-$c'$ are physically placed 120° apart around the stator. Terminals $a$ and $a'$, for example, stand for one of the ends of coils going into and the other end coming out of the

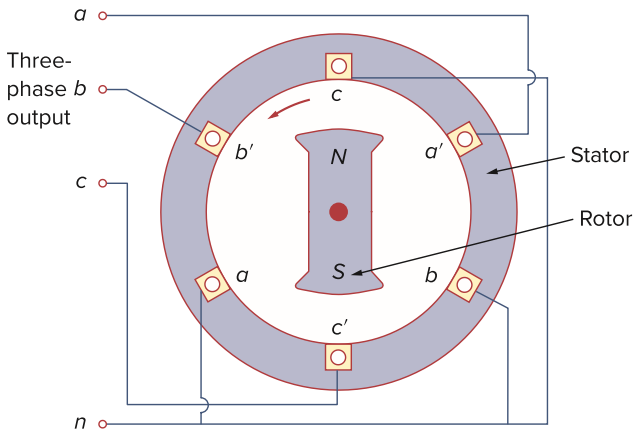



**Figure 12.4**
A three-phase generator.

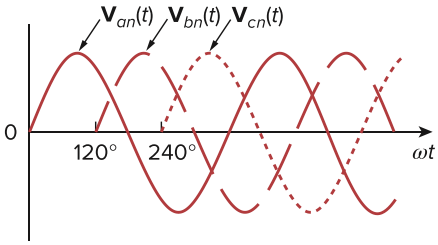



**Figure 12.5**
The generated voltages are 120° apart from each other.

page. As the rotor rotates, its magnetic field "cuts" the flux from the three coils and induces voltages in the coils. Because the coils are placed 120° apart, the induced voltages in the coils are equal in magnitude but out of phase by 120° (Fig. 12.5). Since each coil can be regarded as a single-phase generator by itself, the three-phase generator can supply power to both single-phase and three-phase loads.

A typical three-phase system consists of three voltage sources connected to loads by three or four wires (or transmission lines). (Three-phase current sources are very scarce.) A three-phase system is equivalent to three single-phase circuits. The voltage sources can be either wye-connected as shown in Fig. 12.6(a) or delta-connected as in Fig. 12.6(b).

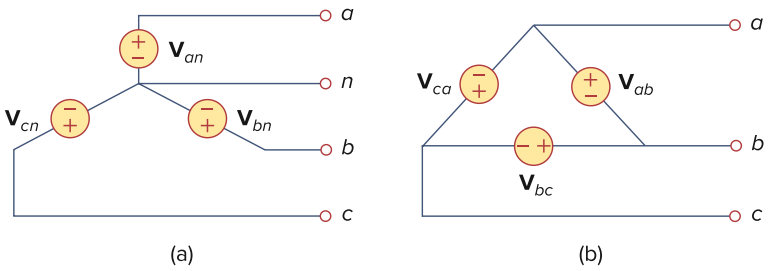



**Figure 12.6**
Three-phase voltage sources: (a) Y-connected source, (b) Δ-connected source.

Let us consider the wye-connected voltages in Fig. 12.6(a) for now. The voltages $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ are respectively between lines $a$, $b$, and $c$, and the neutral line $n$. These voltages are called *phase voltages*. If the voltage sources have the same amplitude and frequency $\omega$ and are out of phase with each other by 120°, the voltages are said to be *balanced*. This implies that

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0 \tag{12.1}$$

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}| \tag{12.2}$$

Thus,

As a common tradition in power systems, voltage and current in this chapter are in rms values unless otherwise stated.

**Balanced phase voltages** are equal in magnitude and are out of phase with each other by 120°.

Because the three-phase voltages are 120° out of phase with each other, there are two possible combinations. One possibility is shown in Fig. 12.7(a) and expressed mathematically as

$$\begin{aligned} \mathbf{V}_{an} &= V_p\angle 0° \\ \mathbf{V}_{bn} &= V_p\angle -120° \\ \mathbf{V}_{cn} &= V_p\angle -240° = V_p\angle +120° \end{aligned} \tag{12.3}$$

where $V_p$ is the effective or rms value of the phase voltages. This is known as the *abc sequence* or *positive sequence*. In this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{bn}$, which in turn leads $\mathbf{V}_{cn}$. This sequence is produced when the rotor in Fig. 12.4 rotates counterclockwise. The other possibility is shown in Fig. 12.7(b) and is given by

$$\begin{aligned} \mathbf{V}_{an} &= V_p\angle 0° \\ \mathbf{V}_{cn} &= V_p\angle -120° \\ \mathbf{V}_{bn} &= V_p\angle -240° = V_p\angle +120° \end{aligned} \tag{12.4}$$

This is called the *acb sequence* or *negative sequence*. For this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$, which in turn leads $\mathbf{V}_{bn}$. The *acb* sequence is produced when the rotor in Fig. 12.4 rotates in the clockwise direction. It is easy to show that the voltages in Eqs. (12.3) or (12.4) satisfy Eqs. (12.1) and (12.2). For example, from Eq. (12.3),

$$\begin{aligned} \mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} &= V_p\angle 0° + V_p\angle -120° + V_p\angle +120° \\ &= V_p(1.0 - 0.5 - j0.866 - 0.5 + j0.866) \\ &= 0 \end{aligned} \tag{12.5}$$

The phase sequence is the time order in which the voltages pass through their respective maximum values.

The phase sequence is determined by the order in which the phasors pass through a fixed point in the phase diagram.

In Fig. 12.7(a), as the phasors rotate in the counterclockwise direction with frequency $\omega$, they pass through the horizontal axis in a sequence *abcabca*.... Thus, the sequence is *abc* or *bca* or *cab*. Similarly, for the phasors in Fig. 12.7(b), as they rotate in the counterclockwise direction, they pass the horizontal axis in a sequence *acbacba*.... This describes the *acb* sequence. The phase sequence is important in three-phase power distribution. It determines the direction of the rotation of a motor connected to the power source, for example.

Like the generator connections, a three-phase load can be either wye-connected or delta-connected, depending on the end application. Figure 12.8(a) shows a wye-connected load, and Fig. 12.8(b) shows a delta-connected load. The neutral line in Fig. 12.8(a) may or may not be there, depending on whether the system is four- or three-wire. (And, of course, a neutral connection is topologically impossible for a delta connection.) A wye- or delta-connected load is said to be *unbalanced* if the phase impedances are not equal in magnitude or phase.

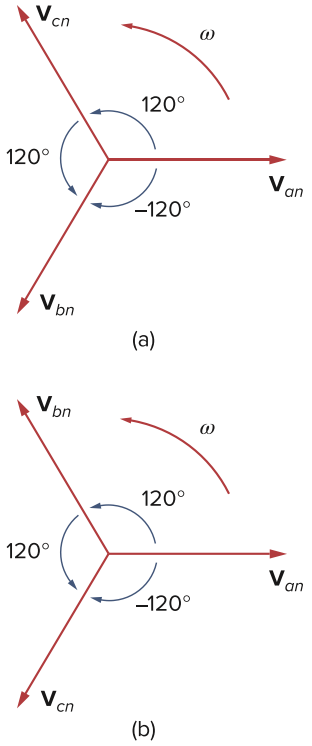



**Figure 12.7**
Phase sequences: (a) *abc* or positive sequence, (b) *acb* or negative sequence.

The phase sequence may also be regarded as the order in which the phase voltages reach their peak (or maximum) values with respect to time.

*Reminder:* As time increases, each phasor (or sinor) rotates at an angular velocity $\omega$.

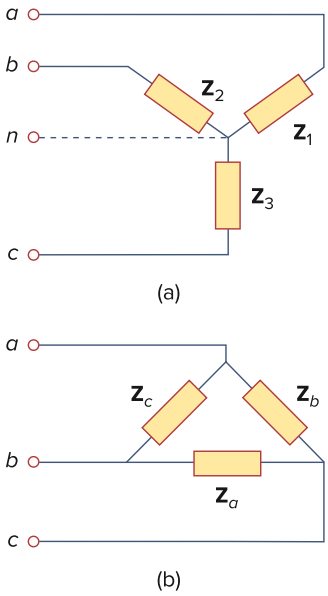



**Figure 12.8**
Two possible three-phase load configurations: (a) a Y-connected load, (b) a Δ-connected load.

*Reminder:* A Y-connected load consists of three impedances connected to a neutral node, while a Δ-connected load consists of three impedances connected around a loop. The load is balanced when the three impedances are equal in either case.

A **balanced load** is one in which the phase impedances are equal in magnitude and in phase.

For a *balanced* wye-connected load,

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_Y \tag{12.6}$$

where $\mathbf{Z}_Y$ is the load impedance per phase. For a *balanced* delta-connected load,

$$\mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c = \mathbf{Z}_\Delta \tag{12.7}$$

where $\mathbf{Z}_\Delta$ is the load impedance per phase in this case. We recall from Eq. (9.69) that

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \quad \text{or} \quad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta \tag{12.8}$$

so we know that a wye-connected load can be transformed into a delta-connected load, or vice versa, using Eq. (12.8).

Because both the three-phase source and the three-phase load can be either wye- or delta-connected, we have four possible connections:

- Y-Y connection (i.e.,Y-connected source with aY-connected load).
- Y-Δ connection.
- Δ-Δ connection.
- Δ-Y connection.

In subsequent sections, we will consider each of these possible configurations.

It is appropriate to mention here that a balanced delta-connected load is more common than a balanced wye-connected load. This is due to the ease with which loads may be added or removed from each phase of a delta-connected load. This is very difficult with a wye-connected load because the neutral may not be accessible. On the other hand, delta-connected sources are not common in practice because of the circulating current that will result in the delta-mesh if the three-phase voltages are slightly unbalanced.

## Example 12.1

Determine the phase sequence of the set of voltages

$$v_{an} = 200\cos(\omega t + 10°)$$
$$v_{bn} = 200\cos(\omega t - 230°), \qquad v_{cn} = 200\cos(\omega t - 110°)$$

**Solution:**
The voltages can be expressed in phasor form as

$$\mathbf{V}_{an} = 200\angle 10° \text{ V}, \qquad \mathbf{V}_{bn} = 200\angle -230° \text{ V}, \qquad \mathbf{V}_{cn} = 200\angle -110° \text{ V}$$

We notice that $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$ by 120° and $\mathbf{V}_{cn}$ in turn leads $\mathbf{V}_{bn}$ by 120°. Hence, we have an *acb* sequence.

## Practice Problem 12.1

Given that $\mathbf{V}_{bn} = 220\angle 30° \text{ V}$, find $\mathbf{V}_{an}$ and $\mathbf{V}_{cn}$, assuming a positive (*abc*) sequence.

**Answer:** $220\angle 150° \text{ V}$, $220\angle -90° \text{ V}$.

## 12.3 Balanced Wye-Wye Connection

We begin with the Y-Y system, because any balanced three-phase system can be reduced to an equivalent Y-Y system. Therefore, analysis of this system should be regarded as the key to solving all balanced three-phase systems.

A **balanced Y-Y system** is a three-phase system with a balanced Y-connected source and a balanced Y-connected load.

Consider the balanced four-wire Y-Y system of Fig. 12.9, where a Y-connected load is connected to a Y-connected source. We assume a balanced load so that load impedances are equal. Although the impedance $\mathbf{Z}_Y$ is the total load impedance per phase, it may also be regarded as the sum of the source impedance $\mathbf{Z}_s$, line impedance $\mathbf{Z}_\ell$, and load impedance $\mathbf{Z}_L$ for each phase, since these impedances are in series. As illustrated in Fig. 12.9, $\mathbf{Z}_s$ denotes the internal impedance of the phase winding of the generator; $\mathbf{Z}_\ell$ is the impedance of the line joining a phase of the source with a phase of the load; $\mathbf{Z}_L$ is the impedance of each phase of the load; and $\mathbf{Z}_n$ is the impedance of the neutral line. Thus, in general

$$\mathbf{Z}_Y = \mathbf{Z}_s + \mathbf{Z}_\ell + \mathbf{Z}_L \tag{12.9}$$

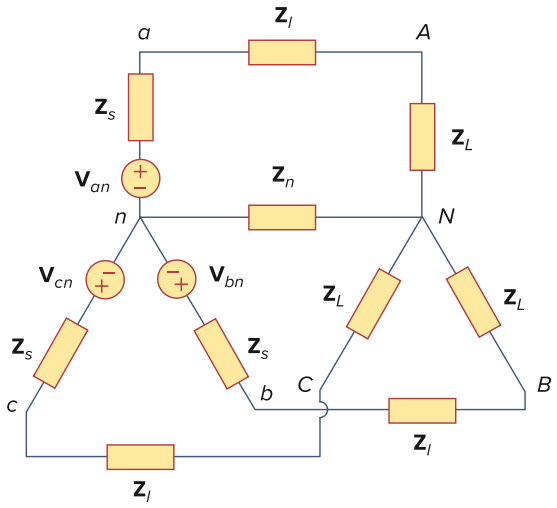



**Figure 12.9**
A balanced Y-Y system, showing the source, line, and load impedances.

$\mathbf{Z}_s$ and $\mathbf{Z}_\ell$ are often very small compared with $\mathbf{Z}_L$, so one can assume that $\mathbf{Z}_Y = \mathbf{Z}_L$ if no source or line impedance is given. In any event, by lumping the impedances together, the Y-Y system in Fig. 12.9 can be simplified to that shown in Fig. 12.10.

Assuming the positive sequence, the *phase* voltages (or line-to-neutral voltages) are

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -120^\circ, \qquad \mathbf{V}_{cn} = V_p\angle +120^\circ \tag{12.10}$$

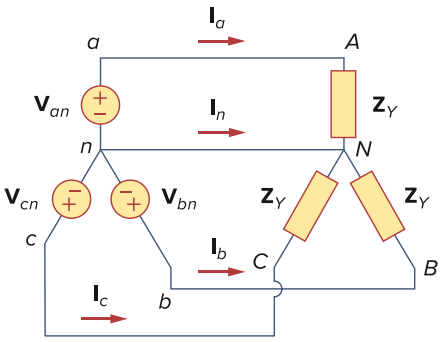



**Figure 12.10**
Balanced Y-Y connection.

The *line-to-line* voltages or simply *line* voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ are related to the phase voltages. For example,

$$\mathbf{V}_{ab} = \mathbf{V}_{an} + \mathbf{V}_{nb} = \mathbf{V}_{an} - \mathbf{V}_{bn} = V_p\angle 0° - V_p\angle -120° = V_p\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \sqrt{3}V_p\angle 30° \tag{12.11a}$$

Similarly, we can obtain

$$\mathbf{V}_{bc} = \mathbf{V}_{bn} - \mathbf{V}_{cn} = \sqrt{3}V_p\angle -90° \tag{12.11b}$$

$$\mathbf{V}_{ca} = \mathbf{V}_{cn} - \mathbf{V}_{an} = \sqrt{3}V_p\angle -210° \tag{12.11c}$$

Thus, the magnitude of the line voltages $V_L$ is $\sqrt{3}$ times the magnitude of the phase voltages $V_p$, or

$$\boxed{V_L = \sqrt{3}V_p} \tag{12.12}$$

where

$$V_p = |\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}| \tag{12.13}$$

and

$$V_L = |\mathbf{V}_{ab}| = |\mathbf{V}_{bc}| = |\mathbf{V}_{ca}| \tag{12.14}$$

Also the line voltages lead their corresponding phase voltages by 30°. Figure 12.11(a) illustrates this. Figure 12.11(a) also shows how to determine $\mathbf{V}_{ab}$ from the phase voltages, while Fig. 12.11(b) shows the same for the three line voltages. Notice that $\mathbf{V}_{ab}$ leads $\mathbf{V}_{bc}$ by 120°, and $\mathbf{V}_{bc}$ leads $\mathbf{V}_{ca}$ by 120°, so that the line voltages sum up to zero as do the phase voltages.

Applying KVL to each phase in Fig. 12.10, we obtain the line currents as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}, \qquad \mathbf{I}_b = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\angle -120°}{\mathbf{Z}_Y} = \mathbf{I}_a\angle -120° \tag{12.15}$$

$$\mathbf{I}_c = \frac{\mathbf{V}_{cn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\angle -240°}{\mathbf{Z}_Y} = \mathbf{I}_a\angle -240°$$

We can readily infer that the line currents add up to zero,

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0 \tag{12.16}$$

so that

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = 0 \tag{12.17a}$$

or

$$\mathbf{V}_{nN} = \mathbf{Z}_n\mathbf{I}_n = 0 \tag{12.17b}$$

that is, the voltage across the neutral wire is zero. The neutral line can thus be removed without affecting the system. In fact, in long distance power transmission, conductors in multiples of three are used with the earth itself acting as the neutral conductor. Power systems designed in this way are well grounded at all critical points to ensure safety.

While the *line* current is the current in each line, the *phase* current is the current in each phase of the source or load. In the Y-Y system, the line current is the same as the phase current. We will use single subscripts

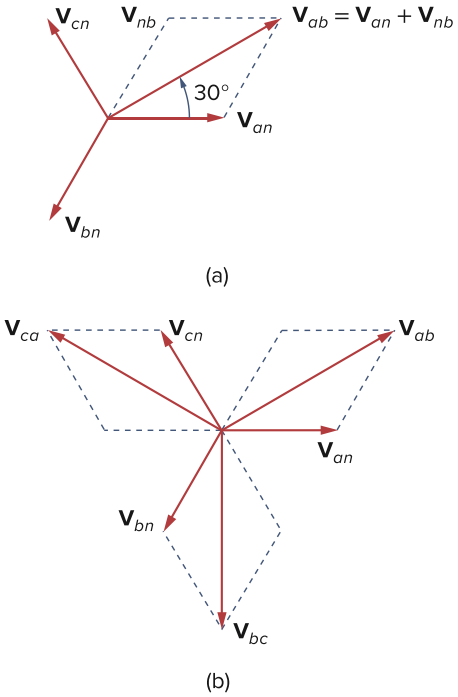



**Figure 12.11**
Phasor diagrams illustrating the relationship between line voltages and phase voltages.

for line currents because it is natural and con ventional to assume that line currents flow from the source to the load.

An alternative way of analyzing a balanced Y-Y system is to do so on a "per phase" basis. We look at one phase, say phase *a*, and analyze the single-phase equivalent circuit in Fig. 12.12. The single-phase analysis yields the line current $\mathbf{I}_a$ as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} \tag{12.18}$$

From $\mathbf{I}_a$, we use the phase sequence to obtain other line currents. Thus, as long as the system is balanced, we need only analyze one phase. We may do this even if the neutral line is absent, as in the three-wire system.

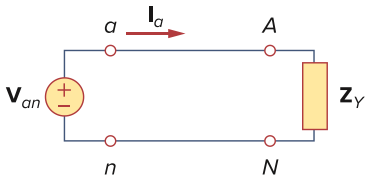



**Figure 12.12**
A single-phase equivalent circuit.

## Example 12.2

Calculate the line currents in the three-wire Y-Y system of Fig. 12.13.

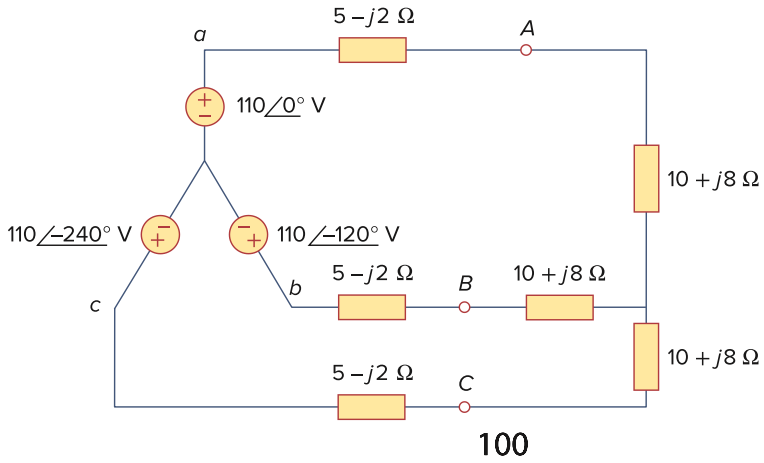



**Figure 12.13**
Three-wire Y-Y system; for Example 12.2.

**Solution:**
The three-phase circuit in Fig. 12.13 is balanced; we may replace it with its single-phase equivalent circuit such as in Fig. 12.12. We obtain $\mathbf{I}_a$ from the single-phase analysis as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}$$

where $\mathbf{Z}_Y = (5 - j2) + (10 + j8) = 15 + j6 = 16.155\angle 21.8°$. Hence,

$$\mathbf{I}_a = \frac{110\angle 0°}{16.155\angle 21.8°} = 6.81\angle -21.8° \text{ A}$$

In as much as the source voltages in Fig. 12.13 are in positive sequence, the line currents are also in positive sequence:

$$\mathbf{I}_b = \mathbf{I}_a\angle -120° = 6.81\angle -141.8° \text{ A}$$
$$\mathbf{I}_c = \mathbf{I}_a\angle -240° = 6.81\angle -261.8° \text{ A} = 6.81\angle 98.2° \text{ A}$$

### Practice Problem 12.2

A Y-connected balanced three-phase generator with an impedance of $0.4 + j0.3\ \Omega$ per phase is connected to a Y-connected balanced load with an impedance of $24 + j19\ \Omega$ per phase. The line joining the generator and the load has an impedance of $0.6 + j0.7\ \Omega$ per phase. Assuming a positive sequence for the source voltages and that $\mathbf{V}_{an} = 120\angle 30^\circ$ V, find: (a) the line voltages, (b) the line currents.

**Answer:** (a) $207.8\angle 60^\circ$ V, $207.8\angle -60^\circ$ V, $207.8\angle -180^\circ$ V, (b) $3.75\angle -8.66^\circ$ A, $3.75\angle -128.66^\circ$ A, $3.75\angle 111.34^\circ$ A.

## 12.4 Balanced Wye-Delta Connection

A **balanced Y-Δ system** consists of a balanced Y-connected source feeding a balanced Δ-connected load.

This is perhaps the most practical three-phase system, as the three-phase sources are usually Y-connected while the three-phase loads are usually Δ-connected.

The balanced Y-delta system is shown in Fig. 12.14, where the source is Y-connected and the load is Δ-connected. There is, of course, no neutral connection from source to load for this case. Assuming the positive sequence, the phase voltages are again

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -120^\circ, \qquad \mathbf{V}_{cn} = V_p\angle +120^\circ \tag{12.19}$$

As shown in Section 12.3, the line voltages are

$$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \sqrt{3}V_p\angle -90^\circ = \mathbf{V}_{BC}$$
$$\mathbf{V}_{ca} = \sqrt{3}V_p\angle 150^\circ = \mathbf{V}_{CA} \tag{12.20}$$

showing that the line voltages are equal to the voltages across the load impedances for this system configuration. From these voltages, we can obtain the phase currents as

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta} \tag{12.21}$$

These currents have the same magnitude but are out of phase with each other by 120°.

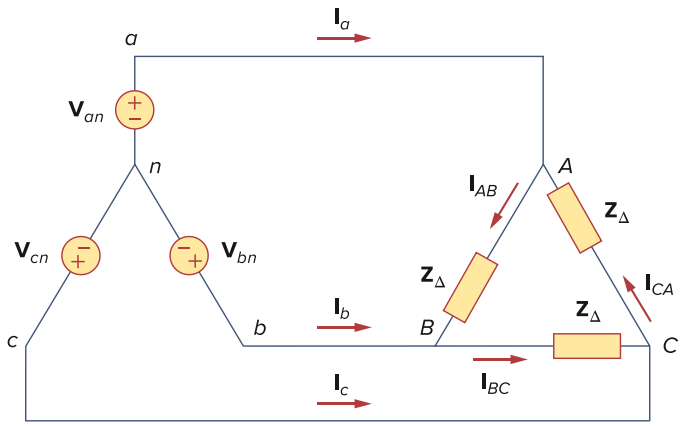



**Figure 12.14**
Balanced Y-Δ connection.

Another way to get these phase currents is to apply KVL. For example, applying KVL around loop *aABbna* gives

$$-\mathbf{V}_{an} + \mathbf{Z}_\Delta \mathbf{I}_{AB} + \mathbf{V}_{bn} = 0$$

or

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{an} - \mathbf{V}_{bn}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} \tag{12.22}$$

which is the same as Eq. (12.21). This is the more general way of finding the phase currents.

The line currents are obtained from the phase currents by applying KCL at nodes *A*, *B*, and *C*. Thus,

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC} \tag{12.23}$$

Since $\mathbf{I}_{CA} = \mathbf{I}_{AB}\angle -240°$,

$$\begin{aligned}\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA} &= \mathbf{I}_{AB}(1 - 1\angle -240°) \\ &= \mathbf{I}_{AB}(1 + 0.5 - j0.866) = \mathbf{I}_{AB}\sqrt{3}\angle -30°\end{aligned} \tag{12.24}$$

showing that the magnitude $I_L$ of the line current is $\sqrt{3}$ times the magnitude $I_p$ of the phase current, or

$$\boxed{I^L = \sqrt{3} I_p} \tag{12.25}$$

where

$$I_L = |\mathbf{I}_a| = |\mathbf{I}_b| = |\mathbf{I}_c| \tag{12.26}$$

and

$$I_p = |\mathbf{I}_{AB}| = |\mathbf{I}_{BC}| = |\mathbf{I}_{CA}| \tag{12.27}$$

Also, the line currents lag the corresponding phase currents by 30°, assuming the positive sequence. Figure 12.15 is a phasor diagram illustrating the relationship between the phase and line currents.

An alternative way of analyzing the Y-Δ circuit is to transform the Δ-connected load to an equi valent Y-connected load. Using the Δ-Y transformation formula in Eq. (12.8),

$$\boxed{\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta}{3}} \tag{12.28}$$

After this transformation, we now have a Y-Y system as in Fig. 12.10. The three-phase Y-Δ system in Fig. 12.14 can be replaced by the single-phase equivalent circuit in Fig. 12.16. This allows us to calculate only the line currents. The phase currents are obtained using Eq. (12.25) and utilizing the fact that each of the phase currents leads the corresponding line current by 30°.

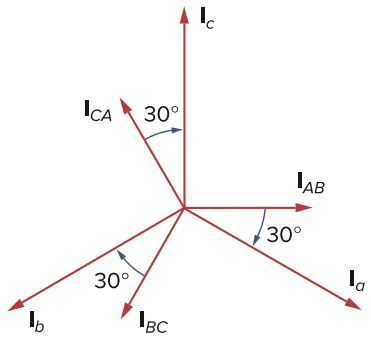



**Figure 12.15**
Phasor diagram illustrating the relationship between phase and line currents.

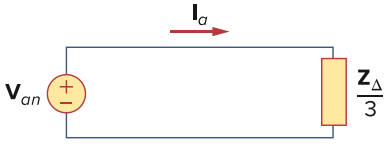



**Figure 12.16**
A single-phase equivalent circuit of a balanced Y-Δ circuit.

## Example 12.3

A balanced *abc*-sequence Y-connected source with $\mathbf{V}_{an} = 100\angle 10°$ V is connected to a Δ-connected balanced load $(8 + j4)$ Ω per phase. Calculate the phase and line currents.

**Solution:**
This can be solved in two ways.

■ **METHOD 1** The load impedance is

$$\mathbf{Z}_\Delta = 8 + j4 = 8.944\angle 26.57^\circ\ \Omega$$

If the phase voltage $\mathbf{V}_{an} = 100\angle 10^\circ$, then the line voltage is

$$\mathbf{V}_{ab} = \mathbf{V}_{an}\sqrt{3}\angle 30^\circ = 100\sqrt{3}\angle 10^\circ + 30^\circ = \mathbf{V}_{AB}$$

or

$$\mathbf{V}_{AB} = 173.2\angle 40^\circ\ \text{V}$$

The phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{173.2\angle 40^\circ}{8.944\angle 26.57^\circ} = 19.36\angle 13.43^\circ\ \text{A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\angle -120^\circ = 19.36\angle -106.57^\circ\ \text{A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\angle +120^\circ = 19.36\angle 133.43^\circ\ \text{A}$$

The line currents are

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ = \sqrt{3}(19.36)\angle 13.43^\circ - 30^\circ$$

$$= 33.53\angle -16.57^\circ\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 33.53\angle -136.57^\circ\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ = 33.53\angle 103.43^\circ\ \text{A}$$

■ **METHOD 2** Alternatively, using single-phase analysis,

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_\Delta/3} = \frac{100\angle 10^\circ}{2.981\angle 26.57^\circ} = 33.54\angle -16.57^\circ\ \text{A}$$

as above. Other line currents are obtained using the *abc* phase sequence.

### Practice Problem 12.3

One line voltage of a balanced Y-connected source is $\mathbf{V}_{AB} = 120\angle -20^\circ$ V. If the source is connected to a Δ-connected load of $20\angle 40^\circ\ \Omega$, find the phase and line currents. Assume the *abc* sequence.

**Answer:** $6\angle -60^\circ$ A, $6\angle -180^\circ$ A, $6\angle 60^\circ$ A, $10.392\angle -90^\circ$ A, $10.392\angle 150^\circ$ A, $10.392\angle 30^\circ$ A.

## 12.5 Balanced Delta-Delta Connection

A **balanced Δ-Δ system** is one in which both the balanced source and balanced load are Δ-connected.

The source as well as the load may be delta-connected as shown in Fig. 12.17. Our goal is to obtain the phase and line currents as usual.

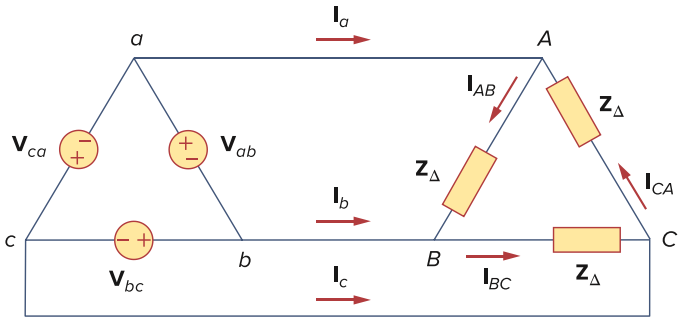



**Figure 12.17**
A balanced Δ-Δ connection.

Assuming a positive sequence, the phase voltages for a delta-connected source are

$$\mathbf{V}_{ab} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bc} = V_p\angle -120^\circ, \qquad \mathbf{V}_{ca} = V_p\angle +120^\circ \tag{12.29}$$

The line voltages are the same as the phase voltages. From Fig. 12.17, assuming there is no line impedances, the phase voltages of the delta-connected source are equal to the voltages across the impedances; that is,

$$\mathbf{V}_{ab} = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \mathbf{V}_{BC}, \qquad \mathbf{V}_{ca} = \mathbf{V}_{CA} \tag{12.30}$$

Hence, the phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{Z_\Delta} = \frac{\mathbf{V}_{ab}}{Z_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{Z_\Delta} = \frac{\mathbf{V}_{bc}}{Z_\Delta}$$
$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{Z_\Delta} = \frac{\mathbf{V}_{ca}}{Z_\Delta} \tag{12.31}$$

Because the load is delta-connected just as in the previous section, some of the formulas derived there apply here. The line currents are obtained from the phase currents by applying KCL at nodes *A*, *B*, and *C*, as we did in the previous section:

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC} \tag{12.32}$$

Also, as shown in the last section, each line current lags the corresponding phase current by 30°; the magnitude $I_L$ of the line current is $\sqrt{3}$ times the magnitude $I_p$ of the phase current,

$$I_L = \sqrt{3} I_p \tag{12.33}$$

An alternative way of analyzing the Δ-Δ circuit is to convert both the source and the load to their Y equivalents. We already know that $\mathbf{Z}_Y = \mathbf{Z}_\Delta/3$. To convert a Δ-connected source to a Y-connected source, see the next section.

## Example 12.4

A balanced Δ-connected load having an impedance $20 - j15\ \Omega$ is connected to a Δ-connected, positive-sequence generator having $\mathbf{V}_{ab} = 330\angle 0^\circ$ V. Calculate the phase currents of the load and the line currents.

**Solution:**
The load impedance per phase is

$$\mathbf{Z}_\Delta = 20 - j15 = 25\angle -36.57^\circ\ \Omega$$

Since $\mathbf{V}_{AB} = \mathbf{V}_{ab}$, the phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{330\angle 0^\circ}{25\angle -36.87^\circ} = 13.2\angle 36.87^\circ\ \text{A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\angle -120^\circ = 13.2\angle -83.13^\circ\ \text{A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\angle +120^\circ = 13.2\angle 156.87^\circ\text{A}$$

For a delta load, the line current always lags the corresponding phase current by 30° and has a magnitude $\sqrt{3}$ times that of the phase current. Hence, the line currents are

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ = (13.2\angle 36.87^\circ)(\sqrt{3}\angle -30^\circ)$$

$$= 22.86\angle 6.87^\circ\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 22.86\angle -113.13^\circ\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ = 22.86\angle 126.87^\circ\ \text{A}$$

### Practice Problem 12.4

A positive-sequence, balanced Δ-connected source supplies a balanced Δ-connected load. If the impedance per phase of the load is $18 + j12\ \Omega$ and $\mathbf{I}_a = 9.609\angle 35^\circ$ A, find $\mathbf{I}_{AB}$ and $\mathbf{V}_{AB}$.

**Answer:** $5.548\angle 65^\circ$ A, $120\angle 98.69^\circ$ V.

## 12.6 Balanced Delta-Wye Connection

A **balanced Δ-Y system** consists of a balanced Δ-connected source feeding a balanced Y-connected load.

Consider the Δ-Y circuit in Fig. 12.18. Again, assuming the *abc* sequence, the phase voltages of a delta-connected source are

$$\mathbf{V}_{ab} = V_p\angle 0^\circ, \qquad \mathbf{V}_{bc} = V_p\angle -120^\circ$$

$$\mathbf{V}_{ca} = V_p\angle +120^\circ \tag{12.34}$$

These are also the line voltages as well as the phase voltages.

We can obtain the line currents in man y ways. One way is to apply KVL to loop *aANBba* in Fig. 12.18, writing

$$-\mathbf{V}_{ab} + \mathbf{Z}_Y\mathbf{I}_a - \mathbf{Z}_Y\mathbf{I}_b = 0$$

or

$$\mathbf{Z}_Y(\mathbf{I}_a - \mathbf{I}_b) = \mathbf{V}_{ab} = V_p\angle 0^\circ$$

Thus,

$$\mathbf{I}_a - \mathbf{I}_b = \frac{V_p\angle 0^\circ}{\mathbf{Z}_Y} \tag{12.35}$$

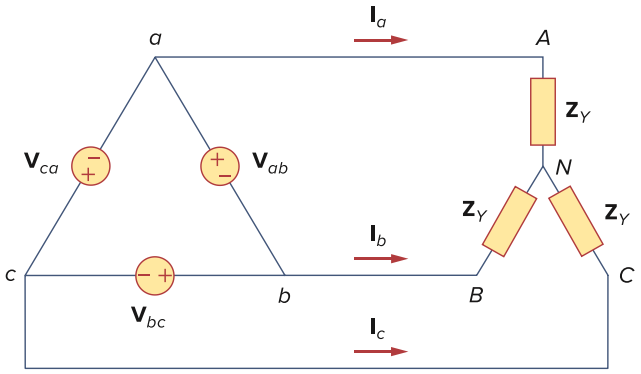



**Figure 12.18**
A balanced Δ-Y connection.

But $\mathbf{I}_b$ lags $\mathbf{I}_a$ by 120°, since we assumed the *abc* sequence; that is, $\mathbf{I}_b = \mathbf{I}_a\angle -120°$. Hence,

$$\begin{aligned}\mathbf{I}_a - \mathbf{I}_b &= \mathbf{I}_a(1 - 1\angle -120°) \\ &= \mathbf{I}_a\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \mathbf{I}_a\sqrt{3}\angle 30°\end{aligned} \tag{12.36}$$

Substituting Eq. (12.36) into Eq. (12.35) gives

$$\mathbf{I}_a = \frac{V_p/\sqrt{3}\angle -30°}{\mathbf{Z}_Y} \tag{12.37}$$

From this, we obtain the other line currents $\mathbf{I}_b$ and $\mathbf{I}_c$ using the positive phase sequence, i.e., $\mathbf{I}_b = \mathbf{I}_a\angle -120°$, $\mathbf{I}_c = \mathbf{I}_a\angle +120°$. The phase currents are equal to the line currents.

Another way to obtain the line currents is to replace the delta-connected source with its equivalent wye-connected source, as shown in Fig. 12.19. In Section 12.3, we found that the line-to-line voltages of a wye-connected source lead their corresponding phase voltages by 30°. Therefore, we obtain each phase voltage of the equivalent wye-connected source by dividing the corresponding line voltage of the delta-connected source by $\sqrt{3}$ and shifting its phase by $-30°$. Thus, the equivalent wye-connected source has the phase voltages

$$\mathbf{V}_{an} = \frac{V_p}{\sqrt{3}}\angle -30°$$

$$\mathbf{V}_{bn} = \frac{V_p}{\sqrt{3}}\angle -150°, \qquad \mathbf{V}_{cn} = \frac{V_p}{\sqrt{3}}\angle +90° \tag{12.38}$$

If the delta-connected source has source impedance $\mathbf{Z}_s$ per phase, the equivalent wye-connected source will have a source impedance of $\mathbf{Z}_s/3$ per phase, according to Eq. (9.69).

Once the source is transformed to wye, the circuit becomes a wye-wye system. Therefore, we can use the equivalent single-phase circuit shown in Fig. 12.20, from which the line current for phase *a* is

$$\mathbf{I}_a = \frac{V_p/\sqrt{3}\angle -30°}{\mathbf{Z}_Y} \tag{12.39}$$

which is the same as Eq. (12.37).

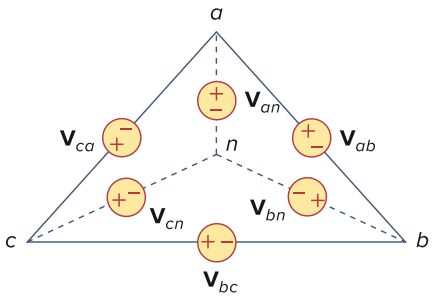



**Figure 12.19**
Transforming a Δ-connected source to an equivalent Y-connected source.

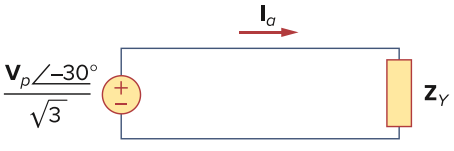



**Figure 12.20**
The single-phase equivalent circuit.

Alternatively, we may transform the wye-connected load to an equivalent delta-connected load. This results in a delta-delta system, which can be analyzed as in Section 12.5. Note that

$$\mathbf{V}_{AN} = \mathbf{I}_a \mathbf{Z}_Y = \frac{V_p}{\sqrt{3}} \angle -30^\circ$$
$$\mathbf{V}_{BN} = \mathbf{V}_{AN} \angle -120^\circ, \qquad \mathbf{V}_{CN} = \mathbf{V}_{AN} \angle +120^\circ \tag{12.40}$$

As stated earlier , the delta-connected load is more desirable than the wye-connected load. It is easier to alter the loads in any one phase of the delta-connected loads, as the individual loads are connected directly across the lines. However, the delta-connected source is hardly used in practice because any slight imbalance in the phase voltages will result in unwanted circulating currents.

Table 12.1 presents a summary of the formulas for phase currents and voltages and line currents and voltages for the four connections. Students are advised not to memorize the formulas but to understand how

**TABLE 12.1**

Summary of phase and line voltages/currents for balanced three-phase systems.[1]

| Connection | Phase voltages/currents | Line voltages/currents |
|---|---|---|
| Y-Y | $\mathbf{V}_{an} = V_p \angle 0^\circ$ | $\mathbf{V}_{ab} = \sqrt{3} V_p \angle 30^\circ$ |
| | $\mathbf{V}_{bn} = V_p \angle -120^\circ$ | $\mathbf{V}_{bc} = \mathbf{V}_{ab} \angle -120^\circ$ |
| | $\mathbf{V}_{cn} = V_p \angle +120^\circ$ | $\mathbf{V}_{ca} = \mathbf{V}_{ab} \angle +120^\circ$ |
| | Same as line currents | $\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_Y$ |
| | | $\mathbf{I}_b = \mathbf{I}_a \angle -120^\circ$ |
| | | $\mathbf{I}_c = \mathbf{I}_a \angle +120^\circ$ |
| Y-Δ | $\mathbf{V}_{an} = V_p \angle 0^\circ$ | $\mathbf{V}_{ab} = \mathbf{V}_{AB} = \sqrt{3} V_p \angle 30^\circ$ |
| | $\mathbf{V}_{bn} = V_p \angle -120^\circ$ | $\mathbf{V}_{bc} = \mathbf{V}_{BC} = \mathbf{V}_{ab} \angle -120^\circ$ |
| | $\mathbf{V}_{cn} = V_p \angle +120^\circ$ | $\mathbf{V}_{ca} = \mathbf{V}_{CA} = \mathbf{V}_{ab} \angle +120^\circ$ |
| | $\mathbf{I}_{AB} = \mathbf{V}_{AB}/\mathbf{Z}_\Delta$ | $\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3} \angle -30^\circ$ |
| | $\mathbf{I}_{BC} = \mathbf{V}_{BC}/\mathbf{Z}_\Delta$ | $\mathbf{I}_b = \mathbf{I}_a \angle -120^\circ$ |
| | $\mathbf{I}_{CA} = \mathbf{V}_{CA}/\mathbf{Z}_\Delta$ | $\mathbf{I}_c = \mathbf{I}_a \angle +120^\circ$ |
| Δ-Δ | $\mathbf{V}_{ab} = V_p \angle 0^\circ$ | Same as phase voltages |
| | $\mathbf{V}_{bc} = V_p \angle -120^\circ$ | |
| | $\mathbf{V}_{ca} = V_p \angle +120^\circ$ | |
| | $\mathbf{I}_{AB} = \mathbf{V}_{ab}/\mathbf{Z}_\Delta$ | $\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3} \angle -30^\circ$ |
| | $\mathbf{I}_{BC} = \mathbf{V}_{bc}/\mathbf{Z}_\Delta$ | $\mathbf{I}_b = \mathbf{I}_a \angle -120^\circ$ |
| | $\mathbf{I}_{CA} = \mathbf{V}_{ca}/\mathbf{Z}_\Delta$ | $\mathbf{I}_c = \mathbf{I}_a \angle +120^\circ$ |
| Δ-Y | $\mathbf{V}_{ab} = V_p \angle 0^\circ$ | Same as phase voltages |
| | $\mathbf{V}_{bc} = V_p \angle -120^\circ$ | |
| | $\mathbf{V}_{ca} = V_p \angle +120^\circ$ | |
| | Same as line currents | $\mathbf{I}_a = \dfrac{V_p \angle -30^\circ}{\sqrt{3}\mathbf{Z}_Y}$ |
| | | $\mathbf{I}_b = \mathbf{I}_a \angle -120^\circ$ |
| | | $\mathbf{I}_c = \mathbf{I}_a \angle +120^\circ$ |

[1]Positive or *abc* sequence is assumed.

they are derived. The formulas can always be obtained by directly applying KCL and KVL to the appropriate three-phase circuits.

### Example 12.5

A balanced Y-connected load with a phase impedance of $40 + j25\ \Omega$ is supplied by a balanced, positive sequence $\Delta$-connected source with a line voltage of 210 V. Calculate the phase currents. Use $\mathbf{V}_{ab}$ as a reference.

**Solution:**
The load impedance is

$$\mathbf{Z}_Y = 40 + j25 = 47.17\angle 32^\circ\ \Omega$$

and the source voltage is

$$\mathbf{V}_{ab} = 210\angle 0^\circ\ \text{V}$$

When the $\Delta$-connected source is transformed to a Y-connected source,

$$\mathbf{V}_{an} = \frac{\mathbf{V}_{ab}}{\sqrt{3}}\angle -30^\circ = 121.2\angle -30^\circ\ \text{V}$$

The line currents are

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} = \frac{121.2\angle -30^\circ}{47.12\angle 32^\circ} = 2.57\angle -62^\circ\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 2.57\angle -178^\circ\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle 120^\circ = 2.57\angle 58^\circ\ \text{A}$$

which are the same as the phase currents.

### Practice Problem 12.5

In a balanced $\Delta$-Y circuit, $\mathbf{V}_{ab} = 440\angle 15^\circ$ and $\mathbf{Z}_Y = (12 + j15)\ \Omega$. Calculate the line currents.

**Answer:** $13.224\angle -66.34^\circ$ A, $13.224\angle +173.66^\circ$ A, $13.224\angle 53.66^\circ$ A.

## 12.7 Power in a Balanced System

Let us now consider the power in a balanced three-phase system. We begin by examining the instantaneous power absorbed by the load. This requires that the analysis be done in the time domain. For a Y-connected load, the phase voltages are

$$v_{AN} = \sqrt{2}V_p \cos \omega t, \qquad v_{BN} = \sqrt{2}\ V_p \cos(\omega t - 120^\circ)$$
$$v_{CN} = \sqrt{2}\ V_p \cos(\omega t + 120^\circ) \tag{12.41}$$

where the factor $\sqrt{2}$ is necessary because $V_p$ has been defined as the rms value of the phase voltage. If $\mathbf{Z}_Y = Z\angle \theta$, the phase currents lag behind their corresponding phase voltages by $\theta$. Thus,

$$i_a = \sqrt{2}\, I_p \cos(\omega t - \theta), \qquad i_b = \sqrt{2}\, I_p \cos(\omega t - \theta - 120^\circ)$$
$$i_c = \sqrt{2}\, I_p \cos(\omega t - \theta + 120^\circ) \tag{12.42}$$

where $I_p$ is the rms value of the phase current. The total instantaneous power in the load is the sum of the instantaneous powers in the three phases; that is,

$$\begin{aligned} p &= p_a + p_b + p_c = v_{AN}i_a + v_{BN}i_b + v_{CN}i_c \\ &= 2V_pI_p[\cos \omega t \cos(\omega t - \theta) \\ &\quad + \cos(\omega t - 120°) \cos(\omega t - \theta - 120°) \\ &\quad + \cos(\omega t + 120°) \cos(\omega t - \theta + 120°)] \end{aligned} \tag{12.43}$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)] \tag{12.44}$$

gives

$$\begin{aligned} p &= V_pI_p[3 \cos \theta + \cos(2\omega t - \theta) + \cos(2\omega t - \theta - 240°) \\ &\quad + \cos(2\omega t - \theta + 240°)] \\ &= V_pI_p[3 \cos \theta + \cos \alpha + \cos \alpha \cos 240° + \sin \alpha \sin 240° \\ &\quad + \cos \alpha \cos 240° - \sin \alpha \sin 240°] \\ &\quad \text{where } \alpha = 2\omega t - \theta \\ &= V_pI_p\left[3 \cos \theta + \cos \alpha + 2\left(-\frac{1}{2}\right)\cos \alpha\right] = 3V_pI_p \cos \theta \end{aligned} \tag{12.45}$$

Thus the total instantaneous power in a balanced three-phase system is constant—it does not change with time as the instantaneous power of each phase does. This result is true whether the load is Y- or Δ-connected. This is one important reason for using a three-phase system to generate and distribute power. We will look into another reason a little later.

Since the total instantaneous power is independent of time, the average power per phase $P_p$ for either the Δ-connected load or the Y-connected load is $p/3$, or

$$P_p = V_pI_p \cos \theta \tag{12.46}$$

and the reactive power per phase is

$$Q_p = V_pI_p \sin \theta \tag{12.47}$$

The apparent power per phase is

$$S_p = V_pI_p \tag{12.48}$$

The complex power per phase is

$$\mathbf{S}_p = P_p + jQ_p = \mathbf{V}_p\mathbf{I}_p^* \tag{12.49}$$

where $\mathbf{V}_p$ and $\mathbf{I}_p$ are the phase voltage and phase current with magnitudes $V_p$ and $I_p$, respectively. The total average power is the sum of the average powers in the phases:

$$P = P_a + P_b + P_c = 3P_p = 3V_pI_p \cos \theta = \sqrt{3}V_LI_L \cos \theta \tag{12.50}$$

For a Y-connected load, $I_L = I_p$ but $V_L = \sqrt{3}V_p$, whereas for a Δ-connected load, $I_L = \sqrt{3}I_p$ but $V_L = V_p$. Thus, Eq. (12.50) applies for both Y-connected and Δ-connected loads. Similarly, the total reactive power is

$$Q = 3V_pI_p \sin \theta = 3Q_p = \sqrt{3}V_LI_L \sin \theta \tag{12.51}$$

and the total complex power is

$$\mathbf{S} = 3\mathbf{S}_p = 3\mathbf{V}_p\mathbf{I}_p^* = 3I_p^2\mathbf{Z}_p = \frac{3V_p^2}{\mathbf{Z}_p^*} \tag{12.52}$$

where $\mathbf{Z}_p = Z_p\angle\theta$ is the load impedance per phase. ($\mathbf{Z}_p$ could be $\mathbf{Z}_Y$ or $\mathbf{Z}_\Delta$.) Alternatively, we may write Eq. (12.52) as

$$\mathbf{S} = P + jQ = \sqrt{3}V_L I_L\angle\theta \tag{12.53}$$

Remember that $V_p$, $I_p$, $V_L$, and $I_L$ are all rms values and that $\theta$ is the angle of the load impedance or the angle between the phase voltage and the phase current.

A second major advantage of three-phase systems for power distribution is that the three-phase system uses a lesser amount of wire than the single-phase system for the same line voltage $V_L$ and the same absorbed power $P_L$. We will compare these cases and assume in both that the wires are of the same material (e.g., copper with resistivity $\rho$), of the same length $\ell$, and that the loads are resistive (i.e., unity power factor). For the two-wire single-phase system in Fig. 12.21(a), $I_L = P_L/V_L$, so the power loss in the two wires is

$$P_{\text{loss}} = 2I_L^2R = 2R\frac{P_L^2}{V_L^2} \tag{12.54}$$

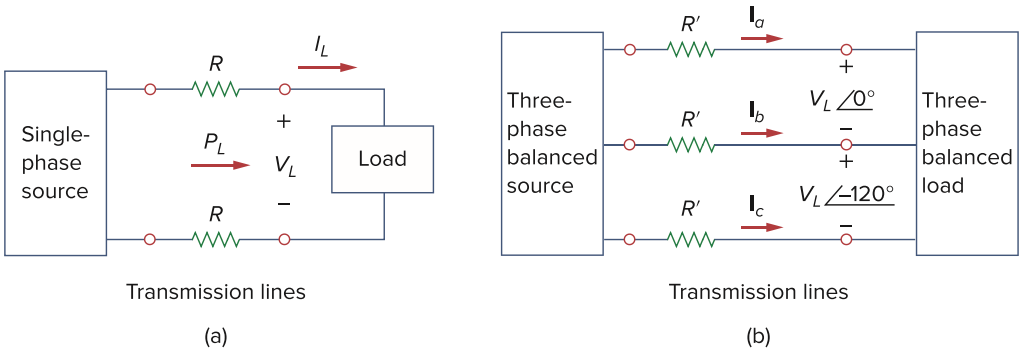



**Figure 12.21**
Comparing the power loss in (a) a single-phase system, and (b) a three-phase system.

For the three-wire three-phase system in Fig. 12.21(b), $I_L' = |\mathbf{I}_a| = |\mathbf{I}_b| = |\mathbf{I}_c| = P_L/\sqrt{3}V_L$ from Eq. (12.50). The power loss in the three wires is

$$P_{\text{loss}}' = 3(I_L')^2R' = 3R'\frac{P_L^2}{3V_L^2} = R'\frac{P_L^2}{V_L^2} \tag{12.55}$$

Equations (12.54) and (12.55) show that for the same total power delivered $P_L$ and same line voltage $V_L$,

$$\frac{P_{\text{loss}}}{P_{\text{loss}}'} = \frac{2R}{R'} \tag{12.56}$$

But from Chapter 2, $R = \rho\ell/\pi r^2$ and $R' = \rho\ell/\pi r'^2$, where $r$ and $r'$ are the radii of the wires. Thus,

$$\frac{P_{\text{loss}}}{P'_{\text{loss}}} = \frac{2r'^2}{r^2} \tag{12.57}$$

If the same power loss is tolerated in both systems, then $r^2 = 2r'^2$. The ratio of material required is determined by the number of wires and their volumes, so

$$\begin{aligned}\frac{\text{Material for single-phase}}{\text{Material for three-phase}} &= \frac{2(\pi r^2\ell)}{3(\pi r'^2\ell)} = \frac{2r^2}{3r'^2} \\ &= \frac{2}{3}(2) = 1.333\end{aligned} \tag{12.58}$$

since $r^2 = 2r'^2$. Equation (12.58) shows that the single-phase system uses 33 percent more material than the three-phase system or that the three-phase system uses only 75 percent of the material used in the equivalent single-phase system. In other words, considerably less material is needed to deliver the same power with a three-phase system than is required for a single-phase system.

## Example 12.6

Refer to the circuit in Fig. 12.13 (in Example 12.2). Determine the total average power, reactive power, and complex power at the source and at the load.

**Solution:**

It is sufficient to consider one phase, as the system is balanced. For phase $a$,

$$\mathbf{V}_p = 110\angle 0^\circ \text{ V} \qquad \text{and} \qquad \mathbf{I}_p = 6.81\angle -21.8^\circ \text{ A}$$

Thus, at the source, the complex power absorbed is

$$\begin{aligned}\mathbf{S}_s = -3\mathbf{V}_p\mathbf{I}_p^* &= -3(110\angle 0^\circ)(6.81\angle 21.8^\circ) \\ &= -2247\angle 21.8^\circ = -(2087 + j834.6) \text{ VA}\end{aligned}$$

The real or average power absorbed is $-2087$ W and the reactive power is $-834.6$ VAR.

At the load, the complex power absorbed is

$$\mathbf{S}_L = 3|\mathbf{I}_p|^2\mathbf{Z}_p$$

where $\mathbf{Z}_p = 10 + j8 = 12.81\angle 38.66^\circ$ and $\mathbf{I}_p = \mathbf{I}_a = 6.81\angle -21.8^\circ$. Hence,

$$\begin{aligned}\mathbf{S}_L &= 3(6.81)^2 12.81\angle 38.66^\circ = 1782\angle 38.66 \\ &= (1392 + j1113) \text{ VA}\end{aligned}$$

The real power absorbed is 1391.7 W and the reactive power absorbed is 1113.3 VAR. The difference between the two complex powers is absorbed by the line impedance $(5 - j2)\ \Omega$. To show that this is the case, we find the complex power absorbed by the line as

$$\mathbf{S}_\ell = 3|\mathbf{I}_p|^2\mathbf{Z}_\ell = 3(6.81)^2(5 - j2) = 695.6 - j278.3 \text{ VA}$$

which is the difference between $\mathbf{S}_s$ and $\mathbf{S}_L$; that is, $\mathbf{S}_s + \mathbf{S}_\ell + \mathbf{S}_L = 0$, as expected.

## Practice Problem 12.6

For the Y-Y circuit in Practice Prob. 12.2, calculate the complex power at the source and at the load.

**Answer:** $-(1054.2 + j843.3)$ VA, $(1012 + j801.6)$ VA.

## Example 12.7

A three-phase motor can be regarded as a balanced Y-load. A three-phase motor draws 5.6 kW when the line voltage is 220 V and the line current is 18.2 A. Determine the power factor of the motor.

**Solution:**
The apparent power is

$$S = \sqrt{3} V_L I_L = \sqrt{3}(220)(18.2) = 6935.13 \text{ VA}$$

Since the real power is

$$P = \text{S} \cos\theta = 5600 \text{ W}$$

the power factor is

$$\text{pf} = \cos\theta = \frac{P}{S} = \frac{5600}{6935.13} = 0.8075$$

## Practice Problem 12.7

Calculate the line current required for a 30-kW three-phase motor having a power factor of 0.85 lagging if it is connected to a balanced source with a line voltage of 550 V.

**Answer:** 37.05 A.

## Example 12.8

Two balanced loads are connected to a 240-kV rms 60-Hz line, as shown in Fig. 12.22(a). Load 1 draws 30 kW at a power factor of 0.6 lagging, while load 2 draws 45 kVAR at a power factor of 0.8 lagging. Assuming the *abc* sequence, determine: (a) the complex, real, and reactive powers absorbed by the combined load, (b) the line currents, and (c) the kVAR rating of the three capacitors Δ-connected in parallel with the load that will raise the power factor to 0.9 lagging and the capacitance of each capacitor.

**Solution:**

(a) For load 1, given that $P_1 = 30$ kW and $\cos\theta_1 = 0.6$, then $\sin\theta_1 = 0.8$. Hence,

$$S_1 = \frac{P_1}{\cos\theta_1} = \frac{30 \text{ kW}}{0.6} = 50 \text{ kVA}$$

and $Q_1 = S_1 \sin\theta_1 = 50(0.8) = 40$ kVAR. Thus, the complex power due to load 1 is

$$\mathbf{S}_1 = P_1 + jQ_1 = 30 + j40 \text{ kVA} \qquad \textbf{(12.8.1)}$$

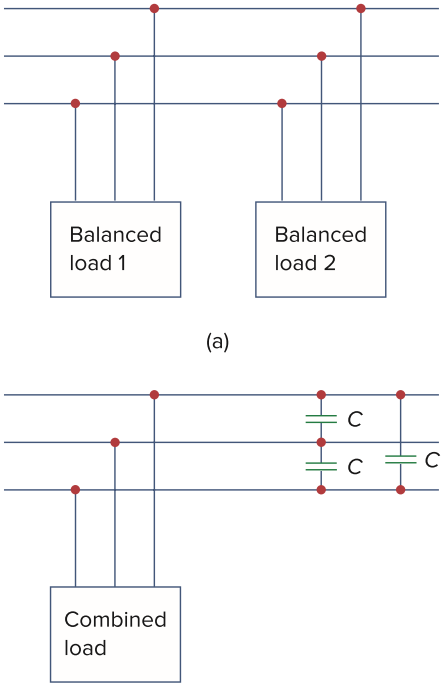



**Figure 12.22**
For Example 12.8: (a) The original balanced loads, (b) the combined load with improved power factor.

For load 2, if $Q_2 = 45$ kVAR and $\cos\theta_2 = 0.8$, then $\sin\theta_2 = 0.6$. We find

$$S_2 = \frac{Q_2}{\sin\theta_2} = \frac{45 \text{ kVA}}{0.6} = 75 \text{ kVA}$$

and $P_2 = S_2 \cos\theta_2\ 75(0.8) = 60$ kW. Therefore the complex power due to load 2 is

$$\mathbf{S}_2 = P_2 + jQ_2 = 60 + j45 \text{ kVA} \qquad \textbf{(12.8.2)}$$

From Eqs. (12.8.1) and (12.8.2), the total complex power absorbed by the load is

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 = 90 + j85 \text{ kVA} = 123.8\angle 43.36^\circ \text{ kVA} \qquad \textbf{(12.8.3)}$$

which has a power factor of $\cos 43.36^\circ = 0.727$ lagging. The real power is then 90 kW, while the reactive power is 85 kVAR.

It will help with the calculations to assume that the loads are wye connected and then to work with the phase voltages, i.e. the magnitude of $V_{AN} = \left(240/\sqrt{3}\right)$ kV.

(b) Since $S = 3\left((240 \text{ kV}/\sqrt{3})I_L\right) = \sqrt{3}(240 \text{ kV})I_L$

the magnitude of the line current is

$$I_L = \frac{S}{\sqrt{3}(240{,}000)} \qquad \textbf{(12.8.4)}$$

We apply this to each load keeping in mind that the magnitude of the phase voltages is equal to $\left(240/\sqrt{3}\right)$kV. For load 1,

$$I_{L1} = \frac{50{,}000}{\sqrt{3}\,240{,}000} = 120.28 \text{ mA}$$

Since the power factor is lagging, the line current lags the line voltage by $\theta_1 = \cos^{-1} 0.6 = 53.13^\circ$. Thus,

$$\mathbf{I}_{a1} = 120.28\angle -53.13^\circ$$

For load 2,

$$I_{L2} = \frac{75{,}000}{\sqrt{3}\,240{,}000} = 180.42 \text{ mA}$$

and the line current lags the line voltage by $\theta_2 = \cos^{-1} 0.8 = 36.87^\circ$. Hence,

$$\mathbf{I}_{a2} = 180.42\angle -36.87^\circ$$

The total line current is

$$\begin{aligned}\mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} &= 120.28\angle -53.13^\circ + 180.42\angle -36.87^\circ \\ &= (72.168 - j96.224) + (144.336 - j108.252) \\ &= 216.5 - j204.472 = 297.8\angle -43.36^\circ \text{ mA}\end{aligned}$$

Alternatively, we could obtain the current from the total complex power using Eq. (12.8.4) as

$$I_L = \frac{123{,}800}{\sqrt{3}\,240{,}000} = 297.82 \text{ mA}$$

and

$$\mathbf{I}_a = 297.82\angle -43.36^\circ \text{ mA}$$

which is the same as before. The other line currents, $\mathbf{I}_{b2}$ and $\mathbf{I}_{ca}$, can be obtained according to the *abc* sequence (i.e., $\mathbf{I}_b = 297.82\angle -163.36^\circ$ mA and $\mathbf{I}_c = 297.82\angle 76.64^\circ$ mA).

(c) We can find the reactive power needed to bring the power factor to 0.9 lagging using Eq. (11.59),

$$QC = P(\tan\theta_{\text{old}} - \tan\theta_{\text{new}})$$

where $P = 90$ kW, $\theta_{old} = 43.36°$, and $\theta_{new} = \cos^{-1} 0.9 = 25.84°$. Hence,

$$Q_C = 90{,}000(\tan 43.36° - \tan 25.84°) = 41.4 \text{ kVAR}$$

This reactive power is for the three capacitors. For each capacitor, the rating $Q'_C = 13.8$ kVAR. From Eq. (11.60), the required capacitance is

$$C = \frac{Q'_C}{\omega V_{rms}^2}$$

Since the capacitors are Δ-connected as shown in Fig. 12.22(b), $V_{rms}$ in the above formula is the line-to-line or line voltage, which is 240 kV. Thus,

$$C = \frac{13{,}800}{(2\pi 60)(240{,}000)^2} = 635.5 \text{ pF}$$

### Practice Problem 12.8

Assume that the two balanced loads in Fig. 12.22(a) are supplied by an 840-V rms 60-Hz line. Load 1 is Y-connected with $30 + j40\ \Omega$ per phase, while load 2 is a balanced three-phase motor drawing 48 kW at a power factor of 0.8 lagging. Assuming the *abc* sequence, calculate: (a) the complex power absorbed by the combined load, (b) the kVAR rating of each of the three capacitors Δ-connected in parallel with the load to raise the power factor to unity, and (c) the current drawn from the supply at unity power factor condition.

**Answer:** (a) $56.47 + j47.29$ kVA, (b) 15.76 kVAR, (c) 38.81 A.

## 12.8 †Unbalanced Three-Phase Systems

This chapter would be incomplete without mentioning unbalanced three-phase systems. An unbalanced system is caused by two possible situations: (1) The source voltages are not equal in magnitude and/or differ in phase by angles that are unequal, or (2) load impedances are unequal. Thus,

> An **unbalanced system** is due to unbalanced voltage sources or an unbalanced load.

To simplify analysis, we will assume balanced source voltages, but an unbalanced load.

Unbalanced three-phase systems are solved by direct application of mesh and nodal analysis. Figure 12.23 shows an example of an unbalanced three-phase system that consists of balanced source voltages (not shown in the figure) and an unbalanced Y-connected load (shown in the figure). Since the load is unbalanced, $\mathbf{Z}_A$, $\mathbf{Z}_B$, and $\mathbf{Z}_C$ are not equal. The line currents are determined by Ohm's law as

$$\mathbf{I}_a = \frac{\mathbf{V}_{AN}}{\mathbf{Z}_A}, \quad \mathbf{I}_b = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_B}, \quad \mathbf{I}_c = \frac{\mathbf{V}_{CN}}{\mathbf{Z}_C} \tag{12.59}$$

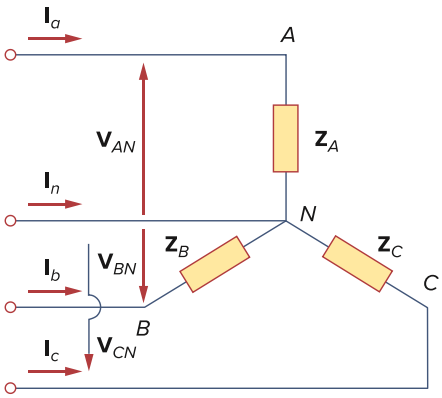



**Figure 12.23**
Unbalanced three-phase Y-connected load.

A special technique for handling unbalanced three-phase systems is the method of *symmetrical components*, which is beyond the scope of this text.

This set of unbalanced line currents produces current in the neutral line, which is not zero as in a balanced system. Applying KCL at node $N$ gives the neutral line current as

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) \tag{12.60}$$

In a three-wire system where the neutral line is absent, we can still find the line currents $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ using mesh analysis. At node $N$, KCL must be satisfied so that $\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0$ in this case. The same could be done for an unbalanced Δ-Y, Y-Δ, or Δ-Δ three-wire system. As mentioned earlier, in long distance power transmission, conductors in multiples of three (multiple three-wire systems) are used, with the earth itself acting as the neutral conductor.

To calculate power in an unbalanced three-phase system requires that we find the power in each phase using Eqs. (12.46) to (12.49). The total power is not simply three times the power in one phase but the sum of the powers in the three phases.

## Example 12.9

The unbalanced Y-load of Fig. 12.23 has balanced voltages of 100 V and the *acb* sequence. Calculate the line currents and the neutral current. Take $\mathbf{Z}_A = 15\ \Omega$, $\mathbf{Z}_B = 10 + j5\ \Omega$, $\mathbf{Z}_C = 6 - j8\ \Omega$.

**Solution:**

Using Eq. (12.59), the line currents are

$$\mathbf{I}_a = \frac{100\angle 0^\circ}{15} = 6.67\angle 0^\circ \text{ A}$$

$$\mathbf{I}_b = \frac{100\angle 120^\circ}{10 + j5} = \frac{100\angle 120^\circ}{11.18\angle 26.56^\circ} = 8.94\angle 93.44^\circ \text{ A}$$

$$\mathbf{I}_c = \frac{100\angle -120^\circ}{6 - j8} = \frac{100\angle -120^\circ}{10\angle -53.13^\circ} = 10\angle -66.87^\circ \text{ A}$$

Using Eq. (12.60), the current in the neutral line is

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = -(6.67 - 0.54 + j8.92 + 3.93 - j9.2)$$

$$= -10.06 + j0.28 = 10.06\angle 178.4^\circ \text{ A}$$

## Practice Problem 12.9

The unbalanced Δ-load of Fig. 12.24 is supplied by balanced line-to-line voltages of 440 V in the positive sequence. Find the line currents. Take $\mathbf{V}_{ab}$ as reference.

**Answer:** $39.71\angle -41.06^\circ$ A, $64.12\angle -139.8^\circ$ A, $70.13\angle 74.27^\circ$ A.

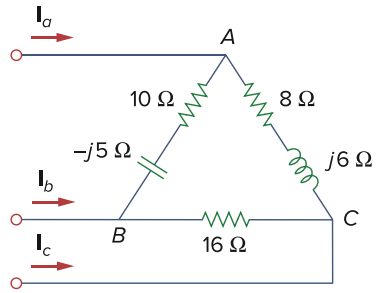



**Figure 12.24**
Unbalanced Δ-load; for Practice Prob. 12.9.

## Example 12.10

For the unbalanced circuit in Fig. 12.25, find: (a) the line currents, (b) the total complex power absorbed by the load, and (c) the total complex power absorbed by the source.

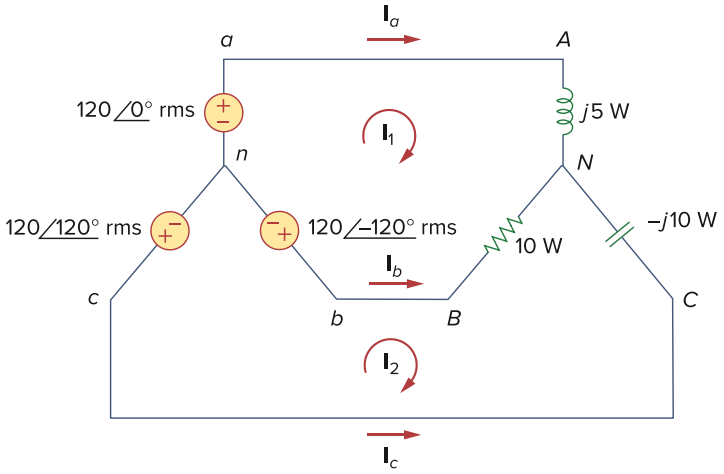



**Figure 12.25**
For Example 12.10.

**Solution:**

(a) We use mesh analysis to find the required currents. For mesh 1,

$$120\angle -120^\circ - 120\angle 0^\circ + (10 + j5)\mathbf{I}_1 - 10\mathbf{I}_2 = 0$$

or

$$(10 + j5)\mathbf{I}_1 - 10\mathbf{I}_2 = 120\sqrt{3}\angle 30^\circ \qquad \textbf{(12.10.1)}$$

For mesh 2,

$$120\angle 120^\circ - 120\angle -120^\circ + (10 - j10)\mathbf{I}_2 - 10\mathbf{I}_1 = 0$$

or

$$-10\mathbf{I}_1 + (10 - j10)\mathbf{I}_2 = 120\sqrt{3}\angle -90^\circ \qquad \textbf{(12.10.2)}$$

Equations (12.10.1) and (12.10.2) form a matrix equation:

$$\begin{bmatrix} 10 + j5 & -10 \\ -10 & 10 - j10 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120\sqrt{3}\angle 30^\circ \\ 120\sqrt{3}\angle -90^\circ \end{bmatrix}$$

The determinants are

$$\Delta = \begin{vmatrix} 10 + j5 & -10 \\ -10 & 10 - j10 \end{vmatrix} = 50 - j50 = 70.71\angle -45^\circ$$

$$\Delta_1 = \begin{vmatrix} 120\sqrt{3}\angle 30^\circ & -10 \\ 120\sqrt{3}\angle -90^\circ & 10 - j10 \end{vmatrix} = 207.85(13.66 - j13.66)$$
$$= 4015\angle -45^\circ$$

$$\Delta_2 = \begin{vmatrix} 10 + j5 & 120\sqrt{3}\angle 30^\circ \\ -10 & 120\sqrt{3}\angle -90^\circ \end{vmatrix} = 207.85(13.66 - j5)$$
$$= 3023.4\angle -20.1^\circ$$

The mesh currents are

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{4015.23\angle -45^\circ}{70.71\angle -45^\circ} = 56.78 \text{ A}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{3023.4\angle -20.1^\circ}{70.71\angle -45^\circ} = 42.75\angle 24.9^\circ \text{ A}$$

The line currents are

$$\mathbf{I}_a = \mathbf{I}_1 = 56.78 \text{ A}, \qquad \mathbf{I}_c = -\mathbf{I}_2 = 42.75\angle -155.1^\circ \text{ A}$$
$$\mathbf{I}_b = \mathbf{I}_2 - \mathbf{I}_1 = 38.78 + j18 - 56.78 = 25.46\angle 135^\circ \text{ A}$$

(b) We can now calculate the complex power absorbed by the load. For phase A,

$$\mathbf{S}_A = |\mathbf{I}_a|^2 \mathbf{Z}_A = (56.78)^2(j5) = j16{,}120 \text{ VA}$$

For phase B,

$$\mathbf{S}_B = |\mathbf{I}_b|^2 \mathbf{Z}_B = (25.46)^2(10) = 6480 \text{ VA}$$

For phase C,

$$\mathbf{S}_C = |\mathbf{I}_c|^2 \mathbf{Z}_C = (42.75)^2(-j10) = -j18{,}276 \text{ VA}$$

The total complex power absorbed by the load is

$$\mathbf{S}_L = \mathbf{S}_A + \mathbf{S}_B + \mathbf{S}_C = 6480 - j2156 \text{ VA}$$

(c) We check the result above by finding the power absorbed by the source. For the voltage source in phase *a*,

$$\mathbf{S}_a = -\mathbf{V}_{an}\mathbf{I}_a^* = -(120\angle 0^\circ)(56.78) = -6813.6 \text{ VA}$$

For the source in phase *b*,

$$\mathbf{S}_b = -\mathbf{V}_{bn}\mathbf{I}_b^* = -(120\angle -120^\circ)(25.46\angle -135^\circ)$$
$$= -3055.2\angle 105^\circ = 790 - j2951.1 \text{ VA}$$

For the source in phase *c*,

$$\mathbf{S}_c = -\mathbf{V}_{bn}\mathbf{I}_c^* = -(120\angle 120^\circ)(42.75\angle 155.1^\circ)$$
$$= -5130\angle 275.1^\circ = -456.03 + j5109.7 \text{ VA}$$

The total complex power absorbed by the three-phase source is

$$\mathbf{S}_s = \mathbf{S}_a + \mathbf{S}_b + \mathbf{S}_c = -6480 + j2156 \text{ VA}$$

showing that $\mathbf{S}_s + \mathbf{S}_L = 0$ and confirming the conservation principle of ac power.

## Practice Problem 12.10

Find the line currents in the unbalanced three-phase circuit of Fig. 12.26 and the real power absorbed by the load.

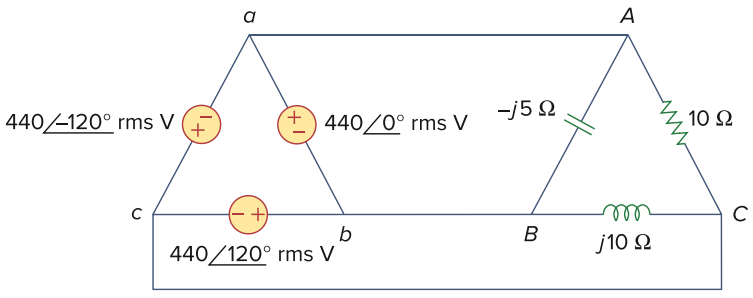



**Figure 12.26**
For Practice Prob. 12.10.

**Answer:** $128.01\angle 80.1^\circ$ A, $76.21\angle -60^\circ$ A, $85\angle -135^\circ$ A, 19.36 kW.

## 12.9 PSpice for Three-Phase Circuits

*PSpice* can be used to analyze three-phase balanced or unbalanced circuits in the same way it is used to analyze single-phase ac circuits. However, a delta-connected source presents two major problems to *PSpice*. First, a delta-connected source is a loop of voltage sources—which *PSpice* does not like. To avoid this problem, we insert a resistor of negligible resistance (say, 1 $\mu\Omega$ per phase) into each phase of the delta-connected source. Second, the delta-connected source does not provide a convenient node for the ground node, which is necessary to run *PSpice*. This problem can be eliminated by inserting balanced wye-connected large resistors (say, 1 M$\Omega$ per phase) in the delta-connected source so that the neutral node of the wye-connected resistors serves as the ground node 0. Example 12.12 will illustrate this.

### Example 12.11

For the balanced Y-$\Delta$ circuit in Fig. 12.27, use *PSpice* to find the line current $\mathbf{I}_{aA}$, the phase voltage $\mathbf{V}_{AB}$, and the phase current $\mathbf{I}_{AC}$. Assume that the source frequency is 60 Hz.

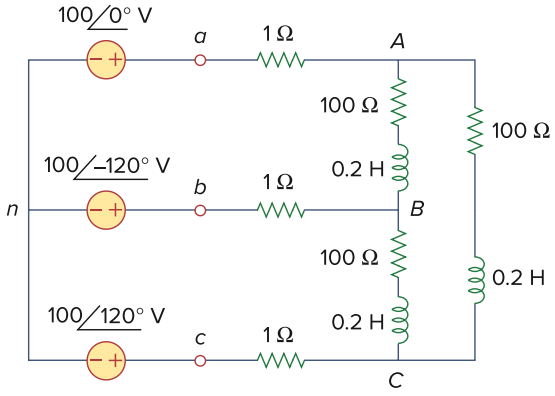



**Figure 12.27**
For Example 12.11.

**Solution:**

The schematic is shown in Fig. 12.28. The pseudocomponents IPRINT are inserted in the appropriate lines to obtain $\mathbf{I}_{aA}$ and $\mathbf{I}_{AC}$, while VPRINT2 is inserted between nodes A and B to print differential voltage $\mathbf{V}_{AB}$. We set the attributes of IPRINT and VPRINT2 each to *AC* = *yes*, *MAG* = *yes*, *PHASE* = *yes*, to print only the magnitude and phase of the currents and voltages. As a single-frequency analysis, we select **Analysis/Setup/AC Sweep** and enter *Total Pts* = 1, *Start Freq* = 60, and *Final Freq* = 60. Once the circuit is saved, it is simulated by selecting **Analysis/Simulate**. The output file includes the following:

```
FREQ          V(A,B)          VP(A,B)
6.000E+01     1.699E+02       3.081E+01

FREQ          IM(V_PRINT2)    IP(V_PRINT2)
6.000E+01     2.350E+00       -3.620E+01

FREQ          IM(V_PRINT3)    IP(V_PRINT3)
6.000E+01     1.357E+00       -6.620E+01
```

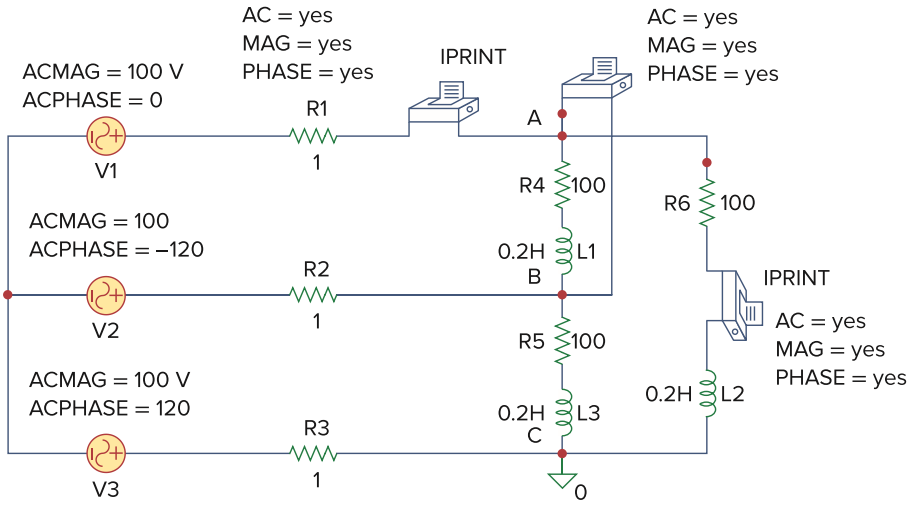



**Figure 12.28**
Schematic for the circuit in Fig. 12.27.

From this, we obtain

$$\mathbf{I}_{aA} = 2.35\angle -36.2^\circ \text{ A}$$

$$\mathbf{V}_{AB} = 169.9\angle 30.81^\circ \text{ V}, \quad \mathbf{I}_{AC} = 1.357\angle -66.2^\circ \text{ A}$$

## Practice Problem 12.11

Refer to the balanced Y-Y circuit of Fig. 12.29. Use *PSpice* to find the line current $\mathbf{I}_{bB}$ and the phase voltage $\mathbf{V}_{AN}$. Take $f = 100$ Hz.

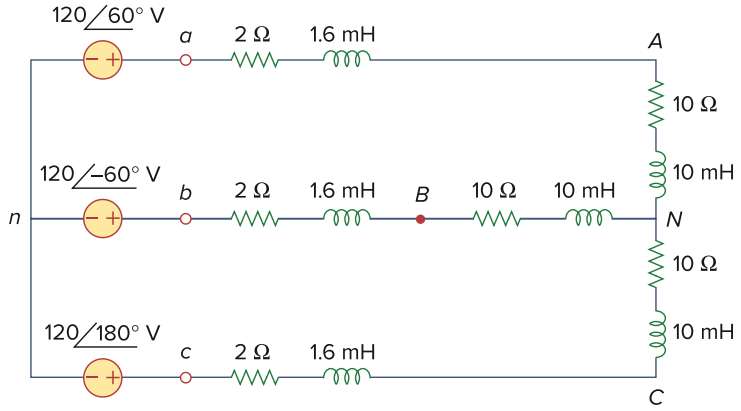



**Figure 12.29**
For Practice Prob. 12.11.

**Answer:** $100.9\angle 60.87^\circ$ V, $8.547\angle -91.27^\circ$ A.

## Example 12.12

Consider the unbalanced Δ-Δ circuit in Fig. 12.30. Use *PSpice* to find the generator current $\mathbf{I}_{ab}$, the line current $\mathbf{I}_{bB}$, and the phase current $\mathbf{I}_{BC}$.

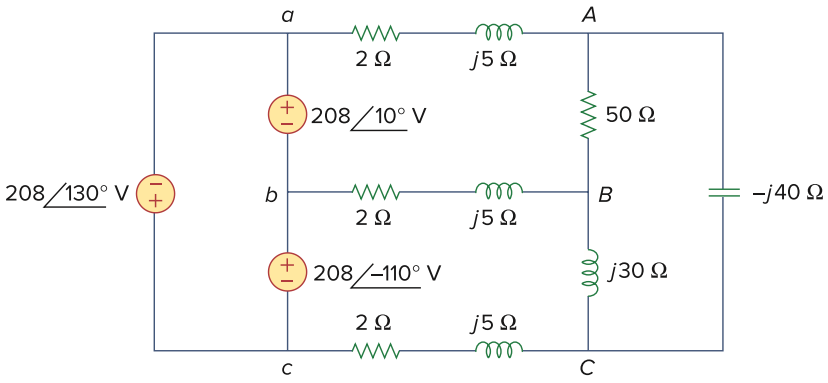



**Figure 12.30**
For Example 12.12.

**Solution:**

1. **Define.** The problem and solution process are clearly defined.
2. **Present.** We are to find the generator current flowing from *a* to *b*, the line current flowing from *b* to *B*, and the phase current flowing from *B* to *C*.
3. **Alternative.** Although there are different approaches to solving this problem, the use of *PSpice* is mandated. Therefore, we will not use another approach.
4. **Attempt.** As mentioned above, we avoid the loop of voltage sources by inserting a 1-$\mu\Omega$ series resistor in the delta-connected source. To provide a ground node 0, we insert balanced wye-connected resistors (1 MΩ per phase) in the delta-connected source, as shown in the schematic in Fig. 12.31. Three IPRINT pseudocomponents with their attributes are inserted to be able

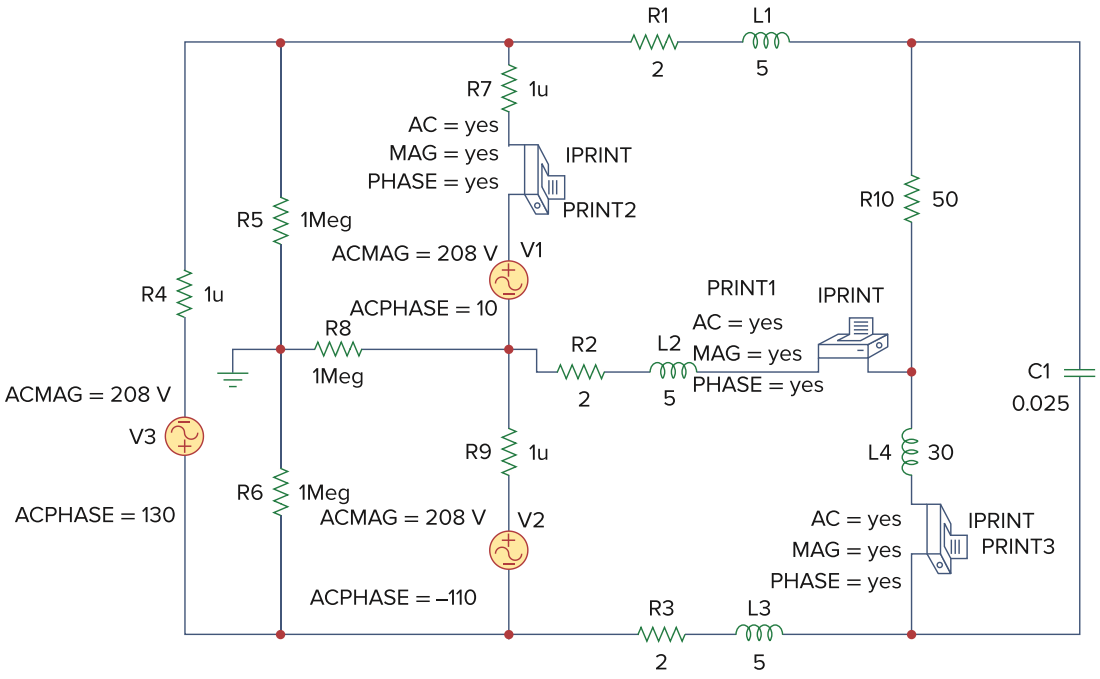



**Figure 12.31**
Schematic for the circuit in Fig. 12.30.

to get the required currents $\mathbf{I}_{ab}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{BC}$. Since the operating frequency is not given and the inductances and capacitances should be specified instead of impedances, we assume $\omega = 1$ rad/s so that $f = 1/2\pi = 0.159155$ Hz. Thus,

$$L = \frac{X_L}{\omega} \qquad \text{and} \qquad C = \frac{1}{\omega X_C}$$

We select **Analysis/Setup/AC Sweep** and enter *Total Pts* = 1, *Start Freq* = 0.159155, and *Final Freq* = 0.159155. Once the schematic is saved, we select **Analysis/Simulate** to simulate the circuit. The output file includes:

```
FREQ          IM(V_PRINT1)    IP(V_PRINT1)
1.592E-01     9.106E+00       1.685E+02

FREQ          IM(V_PRINT2)    IP(V_PRINT2)
1.592E-01     5.959E+00       -1.772E+02

FREQ          IM(V_PRINT3)    IP(V_PRINT3)
1.592E-01     5.500E+00       1.725E+02
```

which yields

$$I_{ab} = \mathbf{5.595}\underline{/\mathbf{-177.2}^\circ}\ \mathbf{A},\ I_{bB} = \mathbf{9.106}\underline{/\mathbf{168.5}^\circ}\ \mathbf{A}, \text{ and}$$
$$I_{BC} = \mathbf{5.5}\underline{/\mathbf{172.5}^\circ}\ \mathbf{A}$$

5. **Evaluate.** We can check our results by using mesh analysis. Let the loop *aABb* be loop 1, the loop *bBCc* be loop 2, and the loop *ACB* be loop 3, with the three loop currents all flowing in the clockwise direction. We then end up with the following loop equations:

Loop 1

$$(54 + j10)I_1 - (2 + j5)I_2 - (50)I_3 = 208\underline{/10^\circ} = 204.8 + j36.12$$

Loop 2

$$-(2 + j5)I_1 + (4 + j40)I_2 - (j30)I_3 = 208\underline{/-110^\circ}$$
$$= -71.14 - j195.46$$

Loop 3

$$-(50)I_1 - (j30)I_2 + (50 - j10)I_3 = 0$$

Using MATLAB to solve this we get,

```
>>Z = [(54+10i),(-2-5i),-50;(-2-5i),(4+40i),
-30i;-50,-30i,(50-10i)]

Z =

54.0000+10.0000i-2.0000-5.0000i-50.0000
-2.0000-5.0000i 4.0000 + 40.0000i 0-30.0000i
-50.0000 0-30.0000i 50.0000-10.0000i
```

```
>>V= [(204.8+36.12i);(-71.14-195.46i);0]

V=
1.0e+002*
2.0480+0.3612i
-0.7114-1.9546i
     0
>>I= inv(Z)*V

I=
8.9317+2.6983i
0.0096+4.5175i
5.4619+3.7964i
```

$$I_{bB} = -I_1 + I_2 = -(8.932 + j2.698) + (0.0096 + j4.518)$$
$$= -8.922 + j1.82 = \mathbf{9.106\angle 168.47^\circ\ A} \qquad \text{Answer checks}$$
$$I_{BC} = I_2 - I_3 = (0.0096 + j4.518) - (5.462 + j3.796)$$
$$= -5.452 + j0.722 = \mathbf{5.5\angle 172.46^\circ\ A} \qquad \text{Answer checks}$$

Now to solve for $I_{ab}$. If we assume a small internal impedance for each source, we can obtain a reasonably good estimate for $I_{ab}$. Adding in internal resistors of 0.01 Ω, and adding a fourth loop around the source circuit, we now get

Loop 1

$$(54.01 + j10)I_1 - (2 + j5)I_2 - (50)I_3 - 0.01I_4 = 208\angle 10^\circ$$
$$= 204.8 + j36.12$$

Loop 2

$$-(2 + j5)I_1 + (4.01 + j40)I_2 - (j30)I_3 - 0.01I_4$$
$$= 208\angle -110^\circ = -71.14 - j195.46$$

Loop 3

$$-(50)I_1 - (j30)I_2 + (50 - j10)I_3 = 0$$

Loop 4

$$-(0.01)I_1 - (0.01)I_2 + (0.03)I_4 = 0$$

```
>>Z= [(54.01+10i),(-2-5i),-50,-0.01;(-2-5i),
(4.01+40i),-30i,-0.01;-50,-30i,(50-10i),
0;-0.01,-0.01,0,0.03]

Z=
54.0100+10.0000i -2.0000-5.0000i, -50.0000 -0.0100
-2.0000-5.0000i 4.0100-40.0000i 0-30.0000i 0.0100
-50.0000 0-30.0000i 50.0000-10.0000i 0
-0.0100 -0.0100 0 0.0300

>>V= [(204.8+36.12i);(-71.14-195.46i);0;0]
```

```
V =

1.0e+002*

2.0480+0.3612i
-0.7114-1.9546i
     0
     0
>>I = inv(Z)*V
I =

8.9309+2.6973i
0.0093+4.5159i
5.4623+3.7954i
2.9801+2.4044i
```

$$I_{ab} = -I_1 + I_4 = -(8.931 + j2.697) + (2.98 + j2.404)$$
$$= -5.951 - j0.293 = \mathbf{5.958\angle -177.18^\circ\ A}. \quad \text{Answer checks.}$$

6. **Satisfactory?** We have a satisfactory solution and an adequate check for the solution. We can now present the results as a solution to the problem.

## Practice Problem 12.12

For the unbalanced circuit in Fig. 12.32, use *PSpice* to find the generator current $\mathbf{I}_{ca}$, the line current $\mathbf{I}_{cC}$, and the phase current $\mathbf{I}_{AB}$.

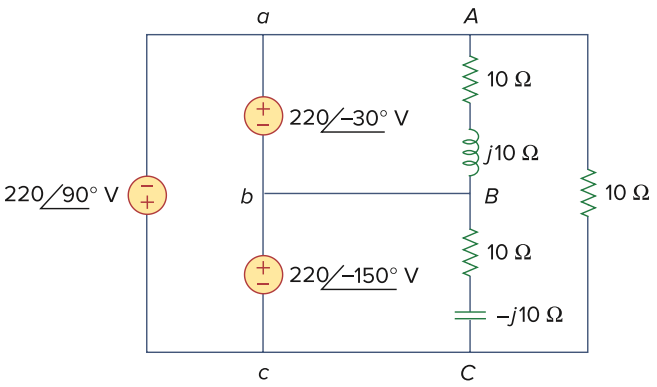



**Figure 12.32**
For Practice Prob. 12.12.

**Answer:** $24.68\angle -90^\circ$ A, $37.25\angle 83.79^\circ$ A, $15.55\angle -75.01^\circ$ A.

## 12.10 †Applications

Both wye and delta source connections have important practical applications. The wye source connection is used for long distance transmission of electric power, where resistive losses ( $I^2R$) should be minimal. This

is due to the fact that the wye connection gives a line voltage that is $\sqrt{3}$ greater than the delta connection; hence, for the same power, the line current is $\sqrt{3}$ smaller. In addition, delta connected are also undesirable due to the potential of having disastrous circulating currents. Sometimes, using transformers, we create the equivalent of delta connect source. This conversion from three-phase to single-phase is required in residential wiring, because household lighting and appliances use single-phase power. Three-phase power is used in industrial wiring where a large power is required. In some applications, it is immaterial whether the load is wye- or delta-connected. For example, both connections are satisfactory with induction motors. In fact, some manufacturers connect a motor in delta for 220 V and in wye for 440 V so that one line of motors can be readily adapted to two different voltages.

Here we consider two practical applications of those concepts covered in this chapter: power measurement in three-phase circuits and residential wiring.

### 12.10.1 Three-Phase Power Measurement

Section 11.9 presented the wattmeter as the instrument for measuring the average (or real) power in single-phase circuits. A single wattmeter can also measure the average power in a three-phase system that is balanced, so that $P_1 = P_2 = P_3$; the total power is three times the reading of that one wattmeter. However, two or three single-phase wattmeters are necessary to measure power if the system is unbalanced. The *three-wattmeter method* of power measurement, shown in Fig. 12.33, will work regardless of whether the load is balanced or unbalanced, wye- or delta-connected. The three-wattmeter method is well suited for power measurement in a three-phase system where the power factor is constantly changing. The total average power is the algebraic sum of the three wattmeter readings,

$$P_T = P_1 + P_2 + P_3 \tag{12.61}$$

where $P_1$, $P_2$, and $P_3$ correspond to the readings of wattmeters $W_1$, $W_2$, and $W_3$, respectively. Notice that the common or reference point $o$ in Fig. 12.33 is selected arbitrarily. If the load is wye-connected, point $o$ can be connected to the neutral point $n$. For a delta-connected load, point $o$ can be connected to any point. If point $o$ is connected to point $b$, for example, the voltage coil in wattmeter $W_2$ reads zero and $P_2 = 0$, indicating that wattmeter $W_2$ is not necessary. Thus, two wattmeters are sufficient to measure the total power.

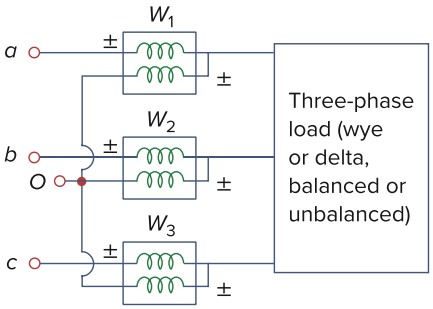



**Figure 12.33**
Three-wattmeter method for measuring three-phase power.

The *two-wattmeter method* is the most commonly used method for three-phase power measurement. The two wattmeters must be properly connected to any two phases, as shown typically in Fig. 12.34. Notice that the current coil of each wattmeter measures the line current, while the respective voltage coil is connected between the line and the third line and measures the line voltage. Also notice that the ± terminal of the voltage coil is connected to the line to which the corresponding current coil is connected. Although the individual wattmeters no longer read the power taken by any particular phase, the algebraic sum of the two wattmeter readings equals the total average power absorbed by the load, regardless of whether it is wye- or delta-connected, balanced or

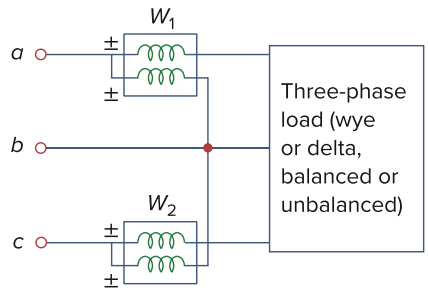



**Figure 12.34**
Two-wattmeter method for measuring three-phase power.

unbalanced. The total real power is equal to the algebraic sum of the two wattmeter readings,

$$P_T = P_1 + P_2 \tag{12.62}$$

We will show here that the method works for a balanced three-phase system.

Consider the balanced, wye-connected load in Fig. 12.35. Our objective is to apply the two-wattmeter method to find the average power absorbed by the load. Assume the source is in the *abc* sequence and the load impedance $\mathbf{Z}_Y = Z_Y\angle\theta$. Due to the load impedance, each voltage coil leads its current coil by $\theta$, so that the power factor is $\cos\theta$. We recall that each line voltage leads the corresponding phase voltage by 30°. Thus, the total phase difference between the phase current $\mathbf{I}_a$ and line voltage $\mathbf{V}_{ab}$ is $\theta + 30°$, and the average power read by wattmeter $W_1$ is

$$P_1 = \text{Re}[\mathbf{V}_{ab}\mathbf{I}_a^*] = V_{ab}I_a\cos(\theta + 30°) = V_L I_L\cos(\theta + 30°) \tag{12.63}$$

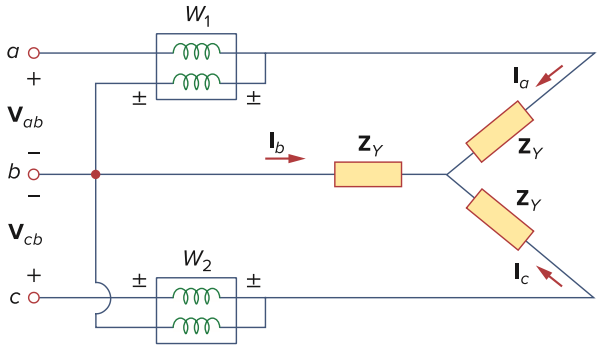



**Figure 12.35**
Two-wattmeter method applied to a balanced wye load.

Similarly, we can show that the average power read by wattmeter 2 is

$$P_2 = \text{Re}[\mathbf{V}_{cb}\mathbf{I}_c^*] = V_{cb}I_c\cos(\theta - 30°) = V_L I_L\cos(\theta - 30°) \tag{12.64}$$

We now use the trigonometric identities

$$\begin{aligned}\cos(A + B) &= \cos A\cos B - \sin A\sin B\\ \cos(A - B) &= \cos A\cos B + \sin A\sin B\end{aligned} \tag{12.65}$$

to find the sum and the difference of the two wattmeter readings in Eqs. (12.63) and (12.64):

$$\begin{aligned}P_1 + P_2 &= V_L I_L[\cos(\theta + 30°) + \cos(\theta - 30°)]\\ &= V_L I_L(\cos\theta\cos 30° - \sin\theta\sin 30°\\ &\qquad + \cos\theta\cos 30° + \sin\theta\sin 30°)\\ &= V_L I_L 2\cos 30°\cos\theta = \sqrt{3}V_L I_L\cos\theta\end{aligned} \tag{12.66}$$

since $2\cos 30° = \sqrt{3}$. Comparing Eq. (12.66) with Eq. (12.50) shows that the sum of the wattmeter readings gives the total average power,

$$\boxed{P_T = P_1 + P_2} \tag{12.67}$$

Similarly,

$$\begin{aligned} P_1 - P_2 &= V_L I_L[\cos(\theta + 30^\circ) - \cos(\theta - 30^\circ)] \\ &= V_l I_L(\cos\theta \cos 30^\circ - \sin\theta \sin 30^\circ \\ &\qquad -\cos\theta \cos 30^\circ - \sin\theta \sin 30^\circ) \\ &= -V_L I_L 2 \sin 30^\circ \sin\theta \\ P_2 - P_1 &= V_L I_L \sin\theta \end{aligned} \tag{12.68}$$

since $2 \sin 30^\circ = 1$. Comparing Eq. (12.68) with Eq. (12.51) shows that the difference of the wattmeter readings is proportional to the total reactive power, or

$$\boxed{Q_T = \sqrt{3}(P_2 - P_1)} \tag{12.69}$$

From Eqs. (12.67) and (12.69), the total apparent power can be obtained as

$$S_T = \sqrt{P_T^2 + Q_T^2} \tag{12.70}$$

Dividing Eq. (12.69) by Eq. (12.67) gives the tangent of the power factor angle as

$$\tan\theta = \frac{Q_T}{P_T} = \sqrt{3}\,\frac{P_2 - P_1}{P_2 + P_1} \tag{12.71}$$

from which we can obtain the power factor as pf $= \cos\theta$. Thus, the two-wattmeter method not only provides the total real and reactive powers, it can also be used to compute the power factor. From Eqs. (12.67), (12.69), and (12.71), we conclude that:

1. If $P_2 = P_1$, the load is resistive.
2. If $P_2 > P_1$, the load is inductive.
3. If $P_2 < P_1$, the load is capacitive.

Although these results are derived from a balanced wye-connected load, they are equally valid for a balanced delta-connected load. However, the two-wattmeter method cannot be used for power measurement in a three-phase four-wire system unless the current through the neutral line is zero. We use the three-wattmeter method to measure the real power in a three-phase four-wire system.

## Example 12.13

Three wattmeters $W_1$, $W_2$, and $W_3$ are connected, respectively, to phases *a, b,* and *c* to measure the total power absorbed by the unbalanced wye-connected load in Example 12.9 (see Fig. 12.23). (a) Predict the wattmeter readings. (b) Find the total power absorbed.

**Solution:**

Part of the problem is already solved in Example 12.9. Assume that the wattmeters are properly connected as in Fig. 12.36.

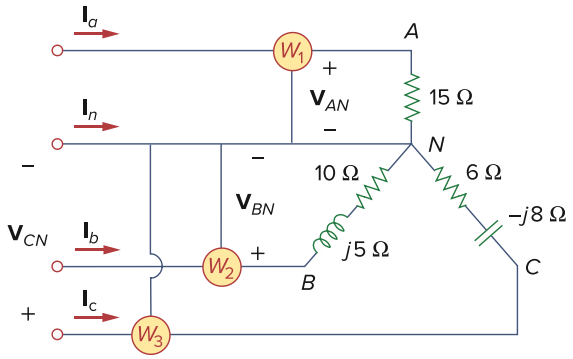



**Figure 12.36**
For Example 12.13.

(a) From Example 12.9,

$$\mathbf{V}_{AN} = 100\angle 0°, \qquad \mathbf{V}_{BN} = 100\angle 120°, \qquad \mathbf{V}_{CN} = 100\angle -120° \text{ V}$$

while

$$\mathbf{I}_a = 6.67\angle 0°, \qquad \mathbf{I}_b = 8.94\angle 93.44°, \qquad \mathbf{I}_c = 10\angle -66.87° \text{ A}$$

We calculate the wattmeter readings as follows:

$$\begin{aligned} P_1 &= \text{Re}(\mathbf{V}_{AN}\mathbf{I}_a^*) = V_{AN}I_a\cos(\theta_{\mathbf{V}_{AN}} - \theta_{\mathbf{I}_a}) \\ &= 100 \times 6.67 \times \cos(0° - 0°) = 667 \text{ W} \\ P_2 &= \text{Re}(\mathbf{V}_{BN}\mathbf{I}_b^*) = V_{BN}I_b\cos(\theta_{\mathbf{V}_{BN}} - \theta_{\mathbf{I}_b}) \\ &= 100 \times 8.94 \times \cos(120° - 93.44°) = 800 \text{ W} \\ P_3 &= \text{Re}(\mathbf{V}_{CN}\mathbf{I}_c^*) = V_{CN}I_c\cos(\theta_{\mathbf{V}_{CN}} - \theta_{\mathbf{I}_c}) \\ &= 100 \times 10 \times \cos(-120° + 66.87°) = 600 \text{ W} \end{aligned}$$

(b) The total power absorbed is

$$P_T = P_1 + P_2 + P_3 = 667 + 800 + 600 = 2067 \text{ W}$$

We can find the power absorbed by the resistors in Fig. 12.36 and use that to check or confirm this result

$$\begin{aligned} P_T &= |I_a|^2(15) + |I_b|^2(10) + |I_c|^2(6) \\ &= 6.67^2(15) + 8.94^2(10) + 10^2(6) \\ &= 667 + 800 + 600 = 2067 \text{ W} \end{aligned}$$

which is exactly the same thing.

## Practice Problem 12.13

Repeat Example 12.13 for the network in Fig. 12.24 (see Practice Prob. 12.9). *Hint:* Connect the reference point $o$ in Fig. 12.33 to point $B$.

**Answer:** (a) 13.175 kW, 0 W, 29.91 kW, (b) 43.08 kW.

## Example 12.14

The two-wattmeter method produces w attmeter readings $P_1 = 1560$ W and $P_2 = 2100$ W when connected to a delta-connected load. If the line voltage is 220 V, calculate: (a) the per-phase average power, (b) the per-phase reactive power, (c) the power factor, and (d) the phase impedance.

**Solution:**
We can apply the given results to the delta-connected load.
(a) The total real or average power is

$$P_T = P_1 + P_2 = 1560 + 2100 = 3660 \text{ W}$$

The per-phase average power is then

$$P_p = \frac{1}{3}P_T = 1220 \text{ W}$$

(b) The total reactive power is

$$Q_T = \sqrt{3}(P_2 - P_1) = \sqrt{3}(2100 - 1560) = 935.3 \text{ VAR}$$

so that the per-phase reactive power is

$$Q_p = \frac{1}{3}Q_T = 311.77 \text{ VAR}$$

(c) The power angle is

$$\theta = \tan^{-1}\frac{Q_T}{P_T} = \tan^{-1}\frac{935.3}{3660} = 14.33^\circ$$

Hence, the power factor is

$$\cos\theta = 0.9689(\text{lagging})$$

It is a lagging pf because $Q_T$ is positive or $P_2 > P_1$.
(d) The phase impedance is $\mathbf{Z}_p = Z_p\underline{/\theta}$. We know that $\theta$ is the same as the pf angle; that is, $\theta = 14.33^\circ$.

$$Z_p = \frac{V_p}{I_p}$$

We recall that for a delta-connected load, $V_p = V_L = 220$ V. From Eq. (12.46),

$$P_p = V_p I_p \cos\theta \quad \Rightarrow \quad I_p = \frac{1220}{220 \times 0.9689} = 5.723 \text{ A}$$

Hence,

$$Z_p = \frac{V_p}{I_p} = \frac{220}{5.723} = 38.44\ \Omega$$

and

$$\mathbf{Z}_p = 38.44\underline{/14.33^\circ}\ \Omega$$

**Practice Problem 12.14**

Let the line voltage $V_L = 208$ V and the wattmeter readings of the balanced system in Fig. 12.35 be $P_1 = -560$ W and $P_2 = 800$ W. Determine:

(a) the total average power
(b) the total reactive power
(c) the power factor
(d) the phase impedance

Is the impedance inductive or capacitive?

**Answer:** (a) 240 W, (b) 2.356 kV AR, (c) 0.1014, (d) $18.25\underline{/84.18^\circ}\ \Omega$, inductive.

## Example 12.15

The three-phase balanced load in Fig. 12.35 has impedance per phase of $\mathbf{Z}_Y = 8 + j6\ \Omega$. If the load is connected to 208-V lines, predict the readings of the wattmeters $W_1$ and $W_2$. Find $P_T$ and $Q_T$.

**Solution:**
The impedance per phase is

$$\mathbf{Z}_Y = 8 + j6 = 10\angle 36.87^\circ\ \Omega$$

so that the pf angle is 36.87°. Since the line voltage $V_L = 208$ V, the line current is

$$I_L = \frac{V_p}{|\mathbf{Z}_Y|} = \frac{208/\sqrt{3}}{10} = 12\text{ A}$$

Then

$$P_1 = V_L I_L \cos(\theta + 30^\circ) = 208 \times 12 \times \cos(36.87^\circ + 30^\circ)$$
$$= 980.48\text{ W}$$

$$P_2 = V_L I_L \cos(\theta - 30^\circ) = 208 \times 12 \times \cos(36.87^\circ - 30^\circ)$$
$$= 2478.1\text{ W}$$

Thus, wattmeter 1 reads 980.48 W, while wattmeter 2 reads 2478.1 W. Since $P_2 > P_1$, the load is inductive. This is evident from the load $\mathbf{Z}_Y$ itself. Next,

$$P_T = P_1 + P_2 = 3.459\text{ kW}$$

and

$$Q_T = \sqrt{3}(P_2 - P_1) = \sqrt{3}(1497.6)\text{ VAR} = 2.594\text{ kVAR}$$

## Practice Problem 12.15

If the load in Fig. 12.35 is delta-connected with impedance per phase of $\mathbf{Z}_p = 30 + j40\ \Omega$ and $V_L = 220$ V, predict the readings of the wattmeters $W_1$ and $W_2$. Calculate $P_T$ and $Q_T$.

**Answer:** 200.6 W, 1.5418 kW, 1.7424 kW, 2.323 kVAR.

### 12.10.2 Residential Wiring

In the United States, most household lighting and appliances operate on 120-V, 60-Hz, single-phase alternating current. (The electricity may also be supplied at 110, 115, or 117 V, depending on the location.) The local power company supplies the house with a three-wire ac system. Typically, as in Fig. 12.37, the line voltage of, say, 12,000 V is stepped down to 120/240 V with a transformer (more details on transformers in the next chapter). The three wires coming from the transformer are typically colored red (hot), black (hot), and white (neutral). As shown in Fig. 12.38, the two 120-V voltages are opposite in phase and hence add up to zero. That is, $\mathbf{V}_W = 0\angle 0^\circ, \mathbf{V}_B = 120\angle 0^\circ, \mathbf{V}_R = 120\angle 180^\circ = -\mathbf{V}_B$.

$$\mathbf{V}_{BR} = \mathbf{V}_B - \mathbf{V}_R = \mathbf{V}_B - (-\mathbf{V}_B) = 2\mathbf{V}_B = 240\angle 0^\circ \qquad \textbf{(12.72)}$$

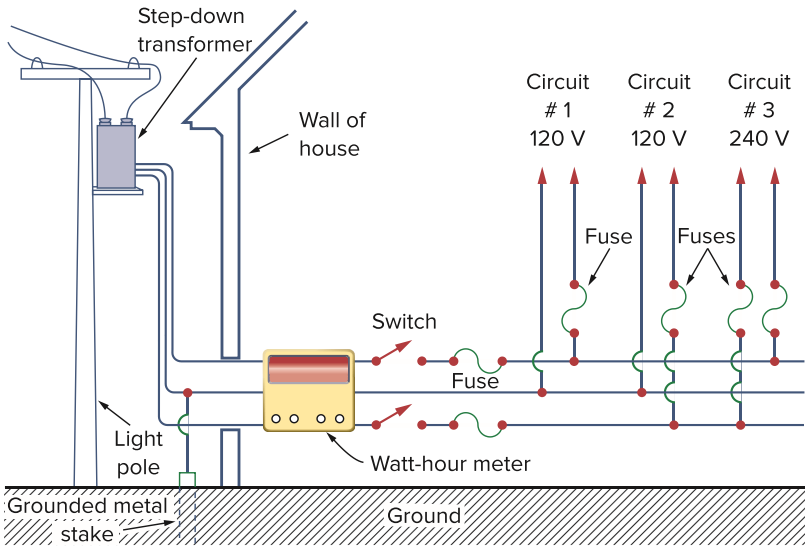



**Figure 12.37**
A 120/240 household power system.
Source: A. Marcus and C. M. Thomson, *Electricity for Technicians,* 2nd edition, © 1975, p. 324. Pearson Education, Inc., Upper Saddle River, NJ.

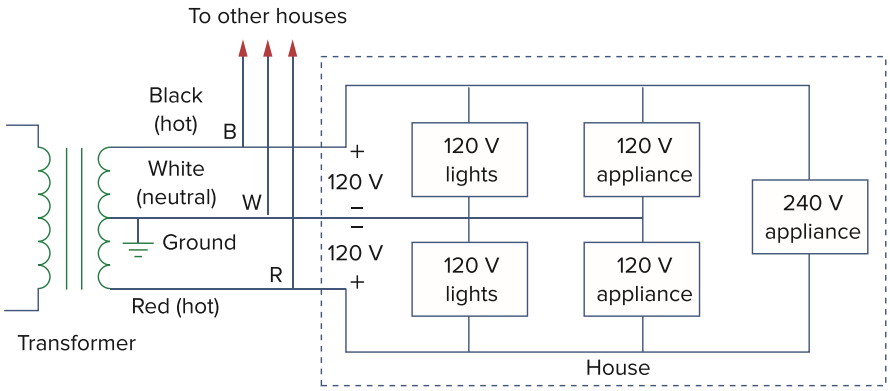



**Figure 12.38**
Single-phase three-wire residential wiring.

Since most appliances are designed to operate with 120 V, the lighting and appliances are connected to the 120-V lines, as illustrated in Fig. 12.39 for a room. Notice in Fig. 12.37 that all appliances are connected in parallel. Heavy appliances that consume large currents, such as air conditioners, dishwashers, ovens, and laundry machines, are connected to the 240-V power line.

Because of the dangers of electricity, house wiring is carefully regulated by a code drawn by local ordinances and by the National Electrical Code (NEC). To avoid trouble, insulation, grounding, fuses, and circuit breakers are used. Modern wiring codes require a third wire for a separate ground. The ground wire does not carry power like the neutral wire but enables appliances to have a separate ground connection. Figure 12.40 shows the connection of the receptacle to a 120-V rms line and to the ground. As shown in the figure, the neutral line is connected to the ground (the earth) at many critical locations. Although the ground line seems redundant, grounding is important for many reasons. First, it is required by NEC. Second, grounding provides

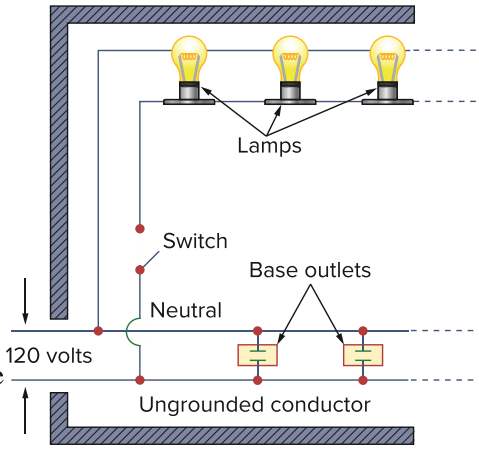



**Figure 12.39**
A typical wiring diagram of a room.
Source: A. Marcus and C. M. Thomson, *Electricity for Technicians,* 2nd edition, © 1975, p. 325. Pearson Education, Inc., Upper Saddle River, NJ.

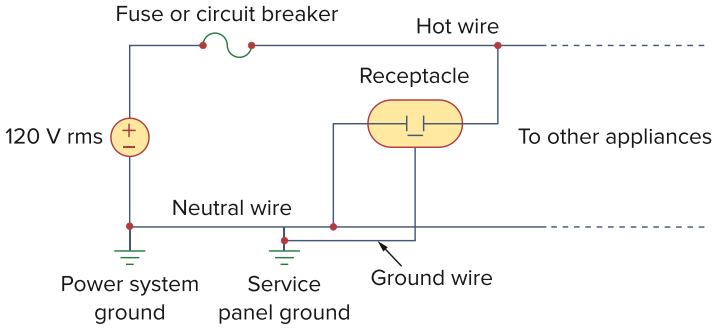



**Figure 12.40**
Connection of a receptacle to the hot line and to the ground.

a convenient path to ground for lightning that strikes the power line. Third, grounds minimize the risk of electric shock. What causes shock is the passage of current from one part of the body to another. The human body is like a big resistor $R$. If $V$ is the potential difference between the body and the ground, the current through the body is determined by Ohm's law as

$$I = \frac{V}{R} \tag{12.73}$$

The value of $R$ varies from person to person and depends on whether the body is wet or dry. How great or how deadly the shock is depends on the amount of current, the pathway of the current through the body, and the length of time the body is exposed to the current. Currents less than 1 mA may not be harmful to the body, but currents greater than 10 mA can cause severe shock. A modern safety device is the *ground-fault circuit interrupter* (GFCI), used in outdoor circuits and in bathrooms, where the risk of electric shock is greatest. It is essentially a circuit breaker that opens when the sum of the currents $i_R$, $i_W$, and $i_B$ through the red, white, and the black lines is not equal to zero, or $i_R + i_W + i_B \neq 0$.

The best way to avoid electric shock is to follow safety guidelines concerning electrical systems and appliances. Here are some of them:

- Never assume that an electrical circuit is dead. Always check to be sure.
- Use safety devices when necessary, and wear suitable clothing (insulated shoes, gloves, etc.).
- Never use two hands when testing high-voltage circuits, since the current through one hand to the other hand has a direct path through your chest and heart.
- Do not touch an electrical appliance when you are wet. Remember that water conducts electricity.
- Be extremely careful when working with electronic appliances such as radio and TV because these appliances have large capacitors in them. The capacitors take time to discharge after the power is disconnected.
- Always have another person present when working on a wiring system, just in case of an accident.

## 12.11 Summary

1. The phase sequence is the order in which the phase voltages of a three-phase generator occur with respect to time. In an *abc* sequence of balanced source voltages, $\mathbf{V}_{an}$ leads $\mathbf{V}_{bn}$ by 120°, which in turn leads $\mathbf{V}_{cn}$ by 120°. In an *acb* sequence of balanced voltages, $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$ by 120°, which in turn leads $\mathbf{V}_{bn}$ by 120°.
2. A balanced wye- or delta-connected load is one in which the three-phase impedances are equal.
3. The easiest way to analyze a balanced three-phase circuit is to transform both the source and the load to a Y-Y system and then analyze the single-phase equivalent circuit. Table 12.1 presents a summary of the formulas for phase currents and voltages and line currents and voltages for the four possible configurations.
4. The line current $I_L$ is the current flowing from the generator to the load in each transmission line in a three-phase system. The line voltage $V_L$ is the voltage between each pair of lines, excluding the neutral line if it exists. The phase current $I_p$ is the current flowing through each phase in a three-phase load. The phase voltage $V_p$ is the voltage of each phase. For a wye-connected load,
$$V_L = \sqrt{3}V_p \qquad \text{and} \qquad I_L = I_p$$
For a delta-connected load,
$$V_L = V_p \qquad \text{and} \qquad I_L = \sqrt{3}I_p$$
5. The total instantaneous power in a balanced three-phase system is constant and equal to the average power.
6. The total complex power absorbed by a balanced three-phase Y-connected or Δ-connected load is
$$\mathbf{S} = P + jQ = \sqrt{3}V_L I_L \underline{/\theta}$$
where $\theta$ is the angle of the load impedances.
7. An unbalanced three-phase system can be analyzed using nodal or mesh analysis.
8. *PSpice* is used to analyze three-phase circuits in the same way as it is used for analyzing single-phase circuits.
9. The total real power is measured in three-phase systems using either the three-wattmeter method or the two-wattmeter method.
10. Residential wiring uses a 120/240-V, single-phase, three-wire system.

## Review Questions

**12.1** What is the phase sequence of a three-phase motor for which $\mathbf{V}_{AN} = 220\underline{/-100°}$ V and $\mathbf{V}_{BN} = 220\underline{/140°}$ V?

(a) *abc* (b) *acb*

**12.2** If in an *acb* phase sequence, $V_{an} = 100\underline{/-20°}$, then $\mathbf{V}_{cn}$ is:

(a) $100\underline{/-140°}$ (b) $100\underline{/100°}$

(c) $100\underline{/-50°}$ (d) $100\underline{/10°}$

**12.3** Which of these is not a required condition for a balanced system:

(a) $|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$

(b) $\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0$

(c) $V_{an} + V_{bn} + V_{cn} = 0$

(d) Source voltages are 120° out of phase with each other.

(e) Load impedances for the three phases are equal.

**12.4** In a Y-connected load, the line current and phase current are equal.

(a) True (b) False

**12.5** In a Δ-connected load, the line current and phase current are equal.

(a) True (b) False

**12.6** In a Y-Y system, a line voltage of 220 V produces a phase voltage of:

(a) 381 V (b) 311 V (c) 220 V
(d) 156 V (e) 127 V

**12.7** In a Δ-Δ system, a phase voltage of 100 V produces a line voltage of:

(a) 58 V (b) 71 V (c) 100 V
(d) 173 V (e) 141 V

**12.8** When a Y-connected load is supplied by voltages in *abc* phase sequence, the line voltages lag the corresponding phase voltages by 30°.

(a) True (b) False

**12.9** In a balanced three-phase circuit, the total instantaneous power is equal to the average power.

(a) True (b) False

**12.10** The total power supplied to a balanced Δ-load is found in the same way as for a balanced Y-load.

(a) True (b) False

*Answers: 12.1a, 12.2a, 12.3c, 12.4a, 12.5b,12.6e, 12.7c, 12.8b, 12.9a, 12.10a.*

# Problems[1]

## Section 12.2 Balanced Three-Phase Voltages

**12.1** If $\mathbf{V}_{ab} = 400$ V in a balanced Y-connected three-phase generator, find the phase voltages, assuming the phase sequence is:

(*a*) *abc* (*b*) *acb*

**12.2** What is the phase sequence of a balanced three-phase circuit for which $\mathbf{V}_{an} = 120\angle 30°$ V and $\mathbf{V}_{cn} = 120\angle -90°$ V? Find $\mathbf{V}_{bn}$.

**12.3** Given a balanced Y-connected three-phase generator with a line-to-line voltage of $\mathbf{V}_{ab} = 100\angle 45°$ V and $\mathbf{V}_{bc} = 100\angle 165°$ V, determine the phase sequence and the value of $\mathbf{V}_{ca}$.

**12.4** A three-phase system with *abc* sequence and $V_L = 440$ V feeds a Y-connected load with $Z_L = 40\angle 30°$ Ω. Find the line currents.

**12.5** For a Y-connected load, the time-domain expressions for three line-to-neutral voltages at the terminals are:

$$v_{AN} = 120\cos(\omega t + 32°)\text{ V}$$
$$v_{BN} = 120\cos(\omega t - 88°)\text{ V}$$
$$v_{CN} = 120\cos(\omega t + 152°)\text{ V}$$

Write the time-domain expressions for the line-to-line voltages $v_{AB}$, $v_{BC}$, and $v_{CA}$.

## Section 12.3 Balanced Wye-Wye Connection

**12.6** Using Fig. 12.41, design a problem to help other students better understand balanced wye-wye connected circuits.

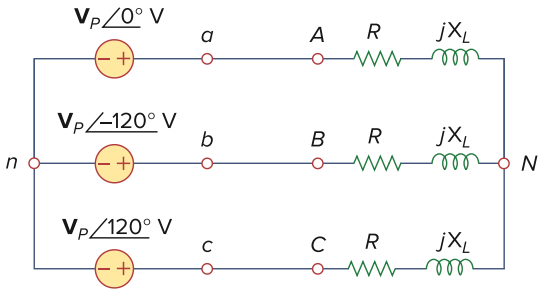



**Figure 12.41**
For Prob. 12.6.

**12.7** Obtain the line currents in the three-phase circuit of Fig. 12.42 on the next page.

**12.8** In a balanced three-phase Y-Y system, the source is an *acb* sequence of voltages and $\mathbf{V}_{cn} = 120\angle 35°$ V rms. The line impedance per phase is $(1 + j2)$ Ω, while the per-phase impedance of the load is $(11 + j14)$ Ω. Calculate the line currents and the load voltages.

**12.9** A balanced Y-Y four-wire system has phase voltages

$$\mathbf{V}_{an} = 120\angle 0°, \qquad \mathbf{V}_{bn} = 120\angle -120°$$
$$\mathbf{V}_{cn} = 120\angle 120°\text{ V}$$

The load impedance per phase is $19 + j13$ Ω, and the line impedance per phase is $1 + j2$ Ω. Solve for the line currents and neutral current.

[1] Remember that unless stated otherwise, all given voltages and currents are rms values.

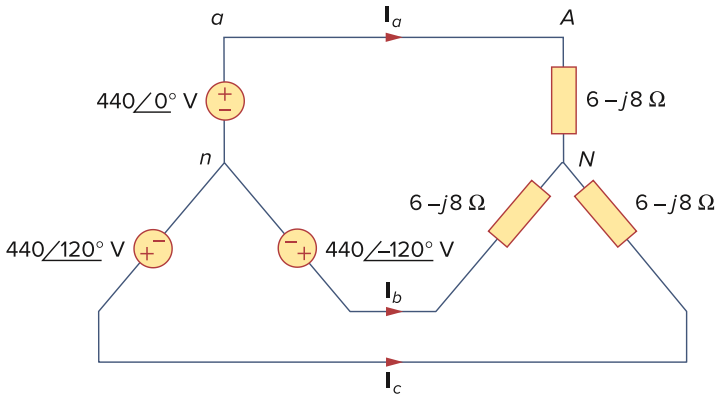



**Figure 12.42**
For Prob. 12.7.

**12.10** For the circuit in Fig. 12.43, determine the current in the neutral line.

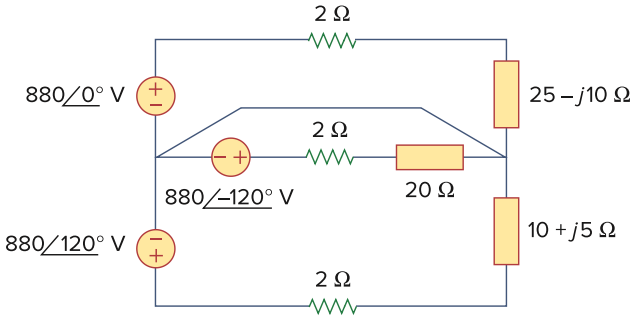



**Figure 12.43**
For Prob. 12.10.

## Section 12.4 Balanced Wye-Delta Connection

**12.11** In the Y-Δ system shown in Fig. 12.44, the source is a positive sequence with $\mathbf{V}_{an} = 440\angle 0°$ V and phase impedance $\mathbf{Z}_p = 2 - j3\ \Omega$. Calculate the line voltage $\mathbf{V}_L$ and the line current $\mathbf{I}_L$.

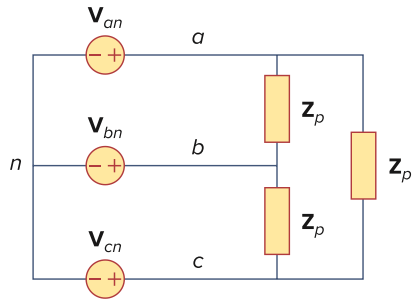



**Figure 12.44**
For Prob. 12.11.

**12.12** Using Fig. 12.45, design a problem to help other students better understand wye-delta connected circuits.

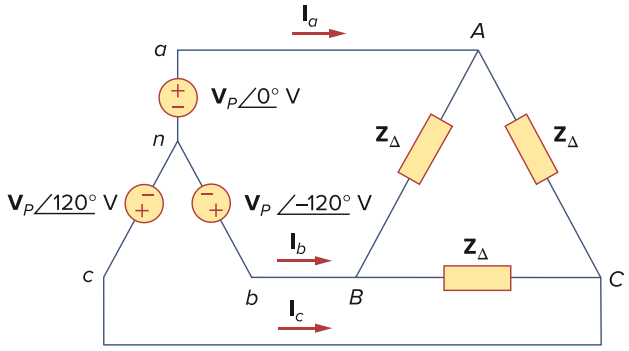



**Figure 12.45**
For Prob. 12.12.

**12.13** In the balanced three-phase Y-Δ system in Fig. 12.46, find the line current $I_L$ and the average power delivered to the load.

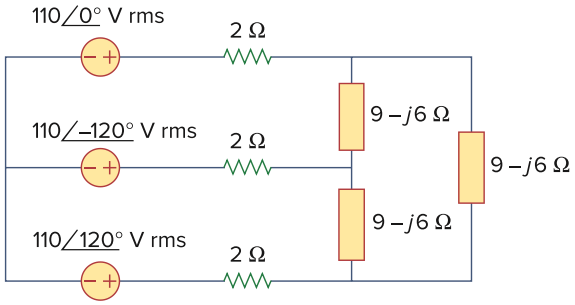



**Figure 12.46**
For Prob. 12.13.

**12.14** Obtain the line currents in the three-phase circuit of Fig. 12.47 on the next page.

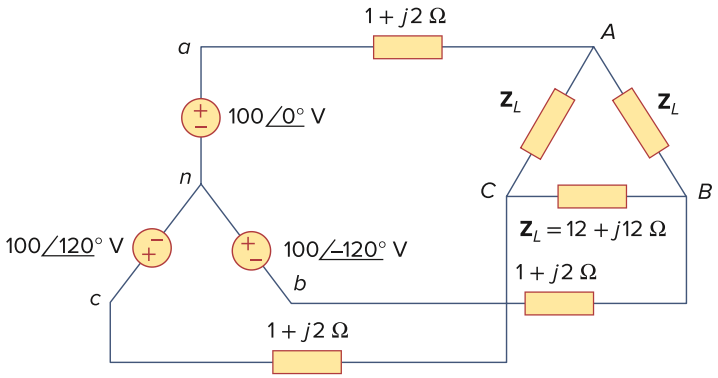



**Figure 12.47**
For Prob. 12.14.

**12.15** The circuit in Fig. 12.48 is excited by a balanced three-phase source with a line voltage of 210 V. If $\mathbf{Z}_l = 1 + j1\ \Omega$, $\mathbf{Z}_\Delta = 24 - j30\ \Omega$, and $\mathbf{Z}_Y = 12 + j5\ \Omega$, determine the magnitude of the line current of the combined loads.

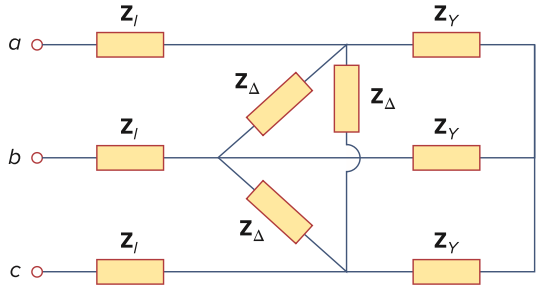



**Figure 12.48**
For Prob. 12.15.

**12.16** A balanced delta-connected load has a phase current $\mathbf{I}_{AC} = 5\angle -30°$ A.

(a) Determine the three line currents assuming that the circuit operates in the positive phase sequence.

(b) Calculate the load impedance if the line voltage is $\mathbf{V}_{AB} = 440\angle 0°$ V.

**12.17** A positive sequence wye-connected source where $\mathbf{V}_{an} = 120\angle 90°$ V, is connected to a delta-connected load where $\mathbf{Z}_L = (60 + j45)\ \Omega$. Determine the line currents.

**12.18** If $\mathbf{V}_{an} = 220\angle 60°$ V in the network of Fig. 12.49, find the load phase currents $\mathbf{I}_{AB}$, $\mathbf{I}_{BC}$, and $\mathbf{I}_{CA}$.

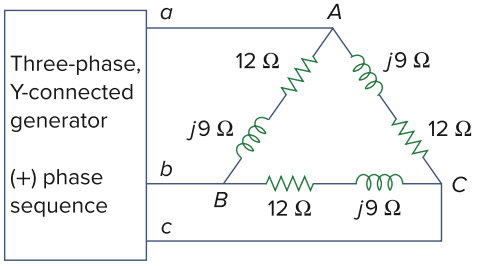



**Figure 12.49**
For Prob. 12.18.

## Section 12.5 Balanced Delta-Delta Connection

**12.19** For the Δ-Δ circuit of Fig. 12.50, calculate the phase and line currents.

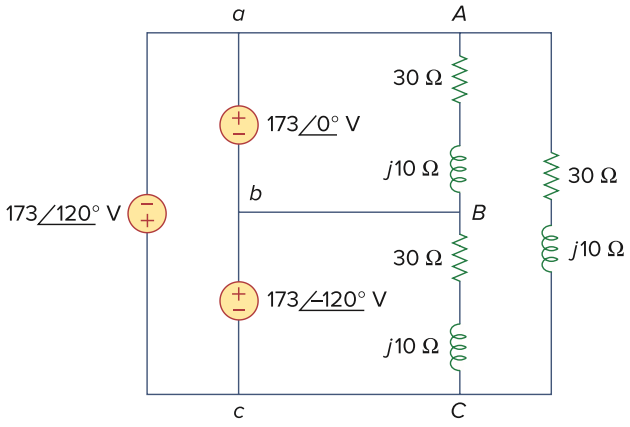



**Figure 12.50**
For Prob. 12.19.

**12.20** Using Fig. 12.51, design a problem to help other students better understand balanced delta-delta connected circuits.

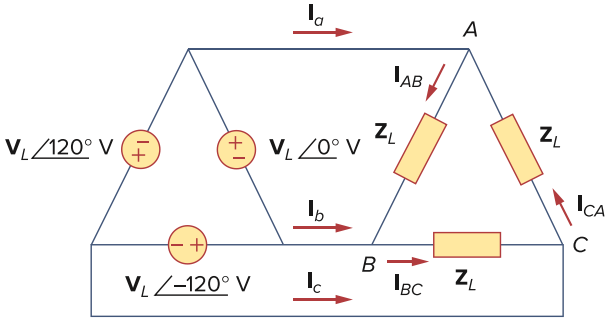



**Figure 12.51**
For Prob. 12.20.

**12.21** Three 440-V generators form a delta-connected source that is connected to a balanced delta-connected load of $Z_L = (8.66 + j5)\ \Omega$ per phase as shown in Fig. 12.52. Determine the value of $I_{BC}$ and $I_{aA}$. What is the pf of the load?

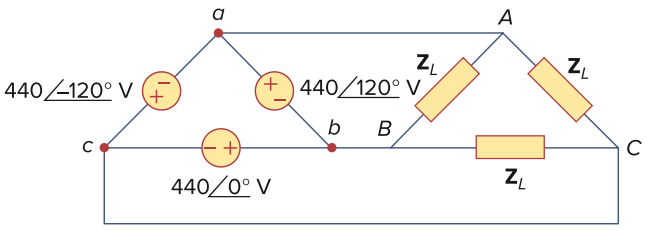



**Figure 12.52**
For Prob. 12.21.

**12.22** Find the line currents $I_{aA}$, $I_{bB}$, and $I_{cC}$ in the three-phase network of Fig. 12.53. Take $\mathbf{Z}_L = (114 + j87)\ \Omega$ and $\mathbf{Z}_l = (2 + j)\ \Omega$.

**12.23** A balanced delta connected source is connected to a balanced delta connected load where $Z_L = (80 + j60)\ \Omega$ and $Z_l = (2 + j)\ \Omega$. Given that the load voltages are $\mathbf{V}_{AB} = 100\angle 0°$ V, $\mathbf{V}_{BC} = 100\angle 120°$ V, and $\mathbf{V}_{CA} = 100\angle -120°$ V. Calculate the source voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$.

**12.24** A balanced delta-connected source has phase voltage $\mathbf{V}_{ab} = 880\angle 30°$ V and a positive phase sequence. If this is connected to a balanced delta-connected load, find the line and phase currents. Take the load impedance per phase as $60\angle 30°\ \Omega$ and line impedance per phase as $1 + j1\ \Omega$.

## Section 12.6 Balanced Delta-Wye Connection

**12.25** In the circuit of Fig. 12.54, if $\mathbf{V}_{ab} = 440\angle 10°$, $\mathbf{V}_{bc} = 440\angle -110°$, $\mathbf{V}_{ca} = 440\angle 130°$ V, find the line currents.

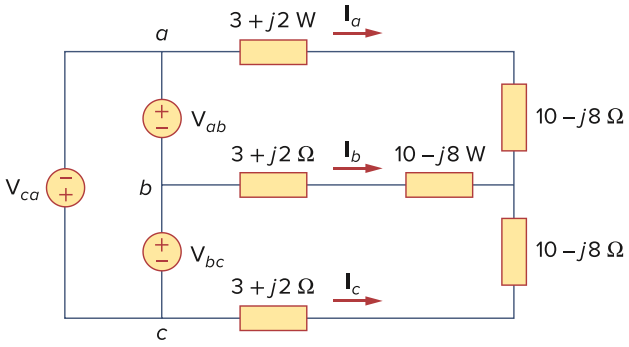



**Figure 12.54**
For Prob. 12.25.

**12.26** Using Fig. 12.55, design a problem to help other students better understand balanced delta connected sources delivering power to balanced wye connected loads.

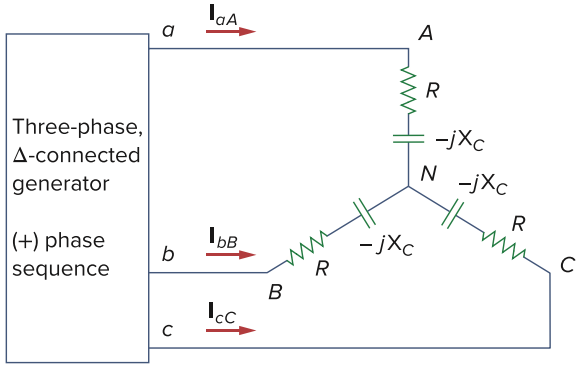



**Figure 12.55**
For Prob. 12.26.

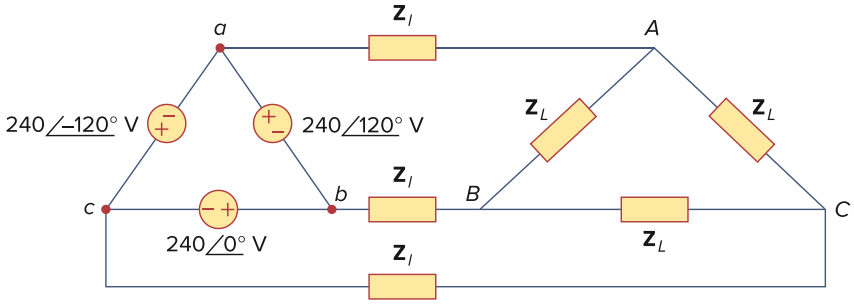



**Figure 12.53**
For Prob. 12.22.

**12.27** A Δ-connected source supplies power to a Y-connected load in a three-phase balanced system. Given that the line impedance is $2 + j1\ \Omega$ per phase while the load impedance is $6 + j4\ \Omega$ per phase, find the magnitude of the line voltage at the load. Assume the source phase voltage $\mathbf{V}_{ab} = 208\angle 0°$ V rms.

**12.28** The line-to-line voltages in a Y-load have a magnitude of 880 V and are in the positive sequence at 60 Hz. If the loads are balanced with $Z_1 = Z_2 = Z_3 = 25\angle 30°$, find all line currents and phase voltages.

## Section 12.7 Power in a Balanced System

**12.29** A balanced three-phase Y-Δ system has $\mathbf{V}_{an} = 240\angle 0°$ V rms and $\mathbf{Z}_\Delta = 51 + j45\ \Omega$. If the line impedance per phase is $0.4 + j1.2\ \Omega$, find the total complex power delivered to the load.

**12.30** In Fig. 12.56, the rms value of the line voltage is 208 V. Find the average power delivered to the load.

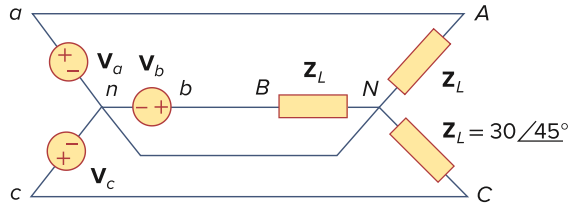



**Figure 12.56**
For Prob. 12.30.

**12.31** A balanced delta-connected load is supplied by a 60-Hz three-phase source with a line voltage of 480 V. Each load phase draws 24 kW at a lagging power factor of 0.8. Find:

(a) the load impedance per phase
(b) the line current
(c) the value of capacitance needed to be connected in parallel with each load phase to minimize the current from the source

**12.32** Design a problem to help other students better understand power in a balanced three-phase system.

**12.33** A three-phase source delivers 4.8 kVA to a wye-connected load with a phase voltage of 208 V and a power factor of 0.9 lagging. Calculate the source line current and the source line voltage.

**12.34** A balanced wye-connected load with a phase impedance of $10 - j16\ \Omega$ is connected to a balanced three-phase generator with a line voltage of 220 V. Determine the line current and the complex power absorbed by the load.

**12.35** Three equal impedances, $60 + j30\ \Omega$ each, are delta-connected to a 230-V rms, three-phase circuit. Another three equal impedances, $40 + j10\ \Omega$ each, are wye-connected across the same circuit at the same points. Determine:

(a) the line current
(b) the total complex power supplied to the two loads
(c) the power factor of the two loads combined

**12.36** A 4200-V, three-phase transmission line has an impedance of $4 + j\ \Omega$ per phase. If it supplies a load of 1 MVA at 0.75 power factor (lagging), find:

(a) the complex power
(b) the power loss in the line
(c) the voltage at the sending end

**12.37** The total power measured in a three-phase system feeding a balanced wye-connected load is 12 kW at a power factor of 0.6 leading. If the line voltage is 440 V, calculate the line current $I_L$ and the load impedance $\mathbf{Z}_Y$.

**12.38** Given the circuit in Fig. 12.57 below, find the total complex power absorbed by the load.

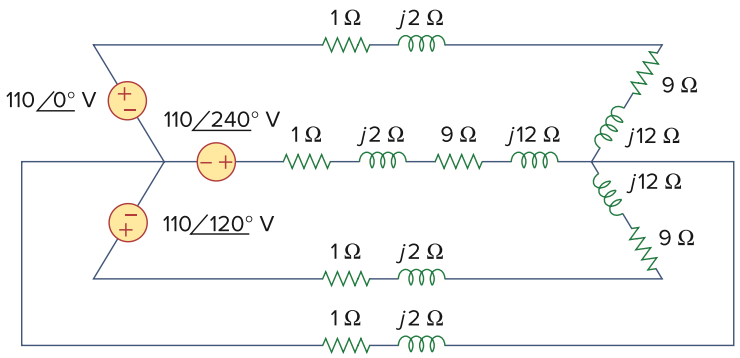



**Figure 12.57**
For Prob. 12.38.

**12.39** Find the real power absorbed by the load in Fig. 12.58.

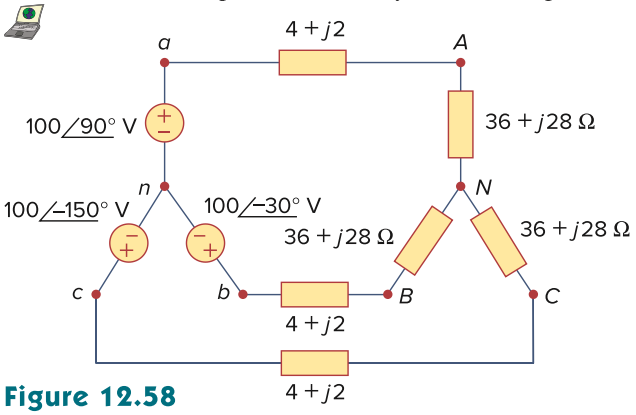



**Figure 12.58**
For Prob. 12.39.

**12.40** For the three-phase circuit in Fig. 12.59, find the average power absorbed by the delta-connected load with $\mathbf{Z}_\Delta = 21 + j24\ \Omega$.

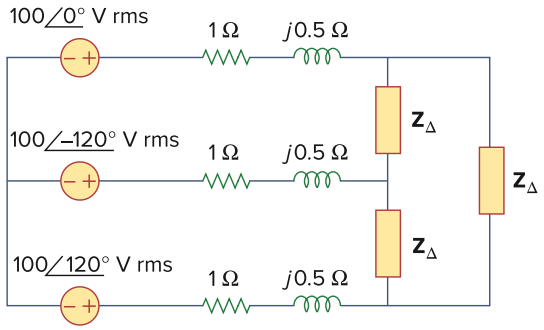



**Figure 12.59**
For Prob. 12.40.

**12.41** A balanced delta-connected load draws 5 kW at a power factor of 0.8 lagging. If the three-phase system has an effective line voltage of 400 V, find the line current.

**12.42** A balanced three-phase generator delivers 7.2 kW to a wye-connected load with impedance $30 - j40\ \Omega$ per phase. Find the line current $I_L$ and the line voltage $V_L$.

**12.43** Refer to Fig. 12.48. Obtain the complex power absorbed by the combined loads.

**12.44** A three-phase line has an impedance of $1 + j3\ \Omega$ per phase. The line feeds a balanced delta-connected load, which absorbs a total complex power of $12 + j5$ kVA. If the line voltage at the load end has a magnitude of 240 V, calculate the magnitude of the line voltage at the source end and the source power factor.

**12.45** A balanced wye-connected load is connected to the generator by a balanced transmission line with an impedance of $0.5 + j2\ \Omega$ per phase. If the load is rated at 450 kW, 0.708 power factor lagging, 440-V line voltage, find the line voltage at the generator.

**12.46** A three-phase load consists of three 100-Ω resistors that can be wye- or delta-connected. Determine which connection will absorb the most average power from a three-phase source with a line voltage of 110 V. Assume zero line impedance.

**12.47** The following three parallel-connected three-phase loads are fed by a balanced three-phase source:

Load 1: 250 kVA, 0.8 pf lagging
Load 2: 300 kVA, 0.95 pf leading
Load 3: 450 kVA, unity pf

If the line voltage is 13.8 kV, calculate the line current and the power factor of the source. Assume that the line impedance is zero.

**12.48** A balanced, positive-sequence wye-connected source has $\mathbf{V}_{an} = 240\angle 0°$ V rms and supplies an unbalanced delta-connected load via a transmission line with impedance $2 + j3\ \Omega$ per phase.

(a) Calculate the line currents if $\mathbf{Z}_{AB} = 40 + j15\ \Omega$, $\mathbf{Z}_{BC} = 60\ \Omega$, $\mathbf{Z}_{CA} = 18 - j12\ \Omega$.
(b) Find the complex power supplied by the source.

**12.49** Each phase load consists of a 20-Ω resistor and a 10-Ω inductive reactance. With a line voltage of 480 V rms, calculate the average power taken by the load if:

(a) the three-phase loads are delta-connected
(b) the loads are wye-connected

**12.50** A balanced three-phase source with $\mathbf{V}_L = 240$ V rms is supplying 8 kVA at 0.6 power factor lagging to two wye-connected parallel loads. If one load draws 3 kW at unity power factor, calculate the impedance per phase of the second load.

## Section 12.8 Unbalanced Three-Phase Systems

**12.51** Consider the wye-delta system shown in Fig. 12.60. Let $\mathbf{Z}_1 = 100\ \Omega$, $\mathbf{Z}_2 = j100\ \Omega$, and $\mathbf{Z}_3 = -j100\ \Omega$. Determine the phase currents, $\mathbf{I}_{AB}$, $\mathbf{I}_{BC}$, and $\mathbf{I}_{CA}$, and the line currents, $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$.

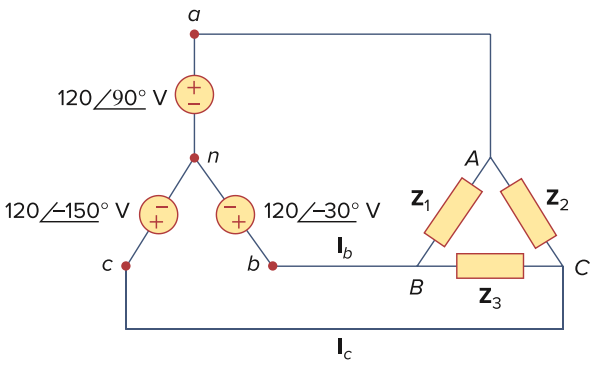



**Figure 12.60**
For Prob. 12.51.

**12.52** A four-wire wye-wye circuit has

$$\mathbf{V}_{an} = 220\angle 120°, \qquad \mathbf{V}_{bn} = 220\angle 0°$$
$$\mathbf{V}_{cn} = 220\angle -120° \text{ V}$$

If the impedances are

$$\mathbf{Z}_{AN} = 20\angle 60°, \qquad \mathbf{Z}_{BN} = 30\angle 0°$$
$$\mathbf{Z}_{cn} = 40\angle 30° \ \Omega$$

find the current in the neutral line.

**12.53** Using Fig. 12.61, design a problem that will help other students better understand unbalanced three-phase systems.

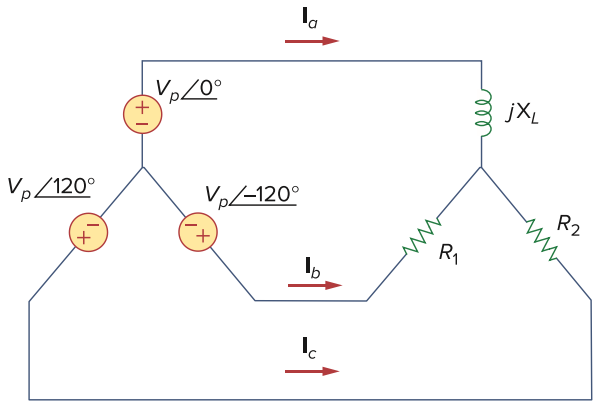



**Figure 12.61**
For Prob. 12.53.

**12.54** A balanced three-phase Y-source with $V_P = 880$ V rms drives a Y-connected three-phase load with phase impedance $\mathbf{Z}_A = 80\ \Omega$, $\mathbf{Z}_B = 60 + j90\ \Omega$, and $\mathbf{Z}_C = j80\ \Omega$. Calculate the line currents and total complex power delivered to the load. Assume that the neutrals are connected.

**12.55** A three-phase supply, with the line-to-line voltage of 240 V rms, has the unbalanced load as shown in Fig. 12.62. Find the line currents and the total complex power delivered to the load.

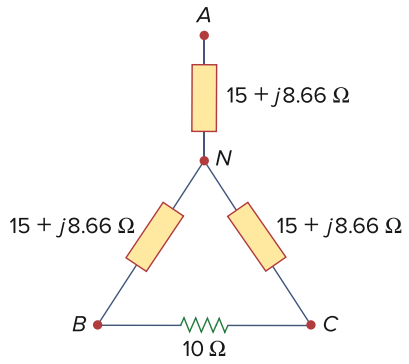



**Figure 12.62**
For Prob. 12.55.

**12.56** Using Fig. 12.63, design a problem to help other students to better understand unbalanced three-phase systems.

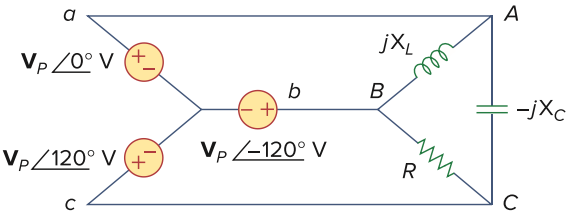



**Figure 12.63**
For Prob. 12.56.

**12.57** Determine the line currents for the three-phase circuit of Fig. 12.64. Let $\mathbf{V}_a = 220\angle 0°$, $\mathbf{V}_b = 220\angle -120°$, $\mathbf{V}_c = 220\angle 120°$ V.

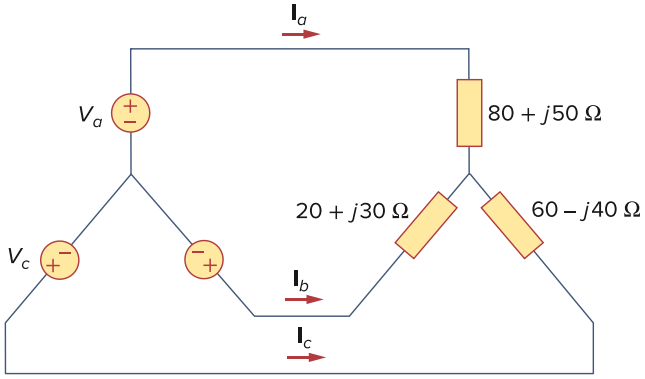



**Figure 12.64**
For Prob. 12.57.

## Section 12.9 *PSpice* for Three-Phase Circuits

**12.58** Solve Prob. 12.10 using *PSpice or MultiSim.*

**12.59** The source in Fig. 12.65 is balanced and exhibits a positive phase sequence. If $f = 60$ Hz, use *PSpice or MultiSim* to find $\mathbf{V}_{AN}$, $\mathbf{V}_{BN}$, and $\mathbf{V}_{CN}$.

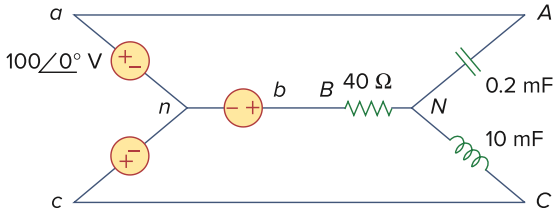



**Figure 12.65**
For Prob. 12.59.

**12.60** Use *PSpice or MultiSim* to determine $\mathbf{I}_o$ in the single-phase, three-wire circuit of Fig. 12.66. Let $\mathbf{Z}_1 = 15 - j10\ \Omega$, $\mathbf{Z}_2 = 30 + j20\ \Omega$, and $\mathbf{Z}_3 = 12 + j5\ \Omega$.

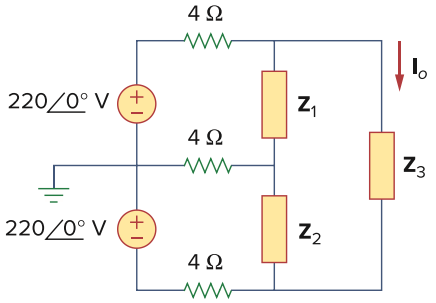



**Figure 12.66**
For Prob. 12.60.

**12.61** Given the circuit in Fig. 12.67, use *PSpice or MultiSim* to determine currents $\mathbf{I}_{aA}$ and voltage $\mathbf{V}_{BN}$.

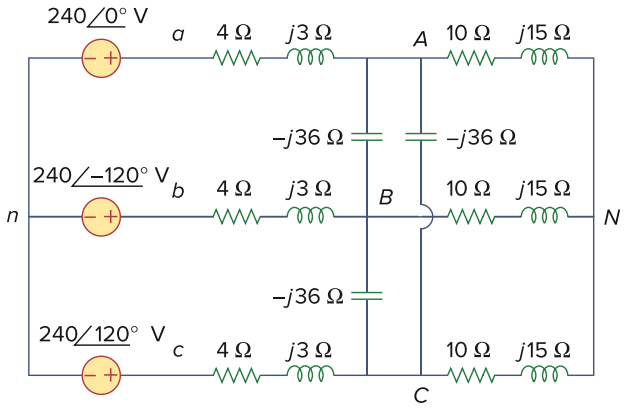



**Figure 12.67**
For Prob. 12.61.

**12.62** Using Fig. 12.68, design a problem to help other students better understand how to use *PSpice or MultiSim* to analyze three-phase circuits.

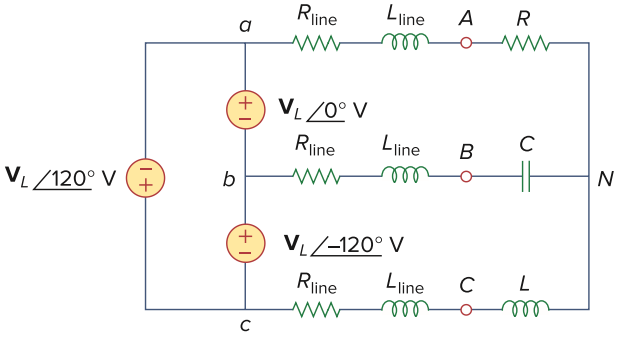



**Figure 12.68**
For Prob. 12.62.

**12.63** Use *PSpice or MultiSim* to find currents $\mathbf{I}_{aA}$ and $\mathbf{I}_{AC}$ in the unbalanced three-phase system shown in Fig. 12.69. Let

$$\mathbf{Z}_l = 2 + j, \qquad \mathbf{Z}_1 = 40 + j20\ \Omega,$$
$$\mathbf{Z}_2 = 50 - j30\ \Omega, \qquad \mathbf{Z}_3 = 25\ \Omega$$

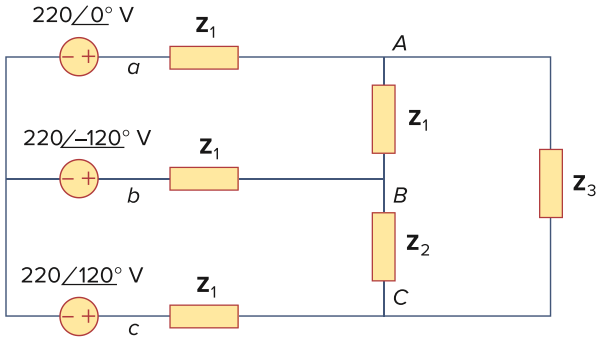



**Figure 12.69**
For Prob. 12.63.

**12.64** For the circuit in Fig. 12.58, use *PSpice or MultiSim* to find the line currents and the phase currents.

**12.65** A balanced three-phase circuit is shown in Fig. 12.70 on the next page. Use *PSpice or MultiSim* to find the line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$.

## Section 12.10 Applications

**12.66** A three-phase, four-wire system operating with a 480-V line voltage is shown in Fig. 12.71. The source voltages are balanced. The power absorbed by the resistive wye-connected load is measured by the three-wattmeter method. Calculate:

(a) the voltage to neutral

(b) the currents $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$, and $\mathbf{I}_n$

(c) the readings of the wattmeters

(d) the total power absorbed by the load

***12.67** As shown in Fig. 12.72, a three-phase four-wire line with a phase voltage of 120 V rms and positive phase sequence supplies a balanced motor load at 260 kVA at 0.85 pf lagging. The motor load is connected to the three main lines marked *a, b,* and *c*. In addition, incandescent lamps (unity pf) are connected as follows: 24 kW from line *c* to the neutral, 15 kW from line b to the neutral, and 9 kW from line *c* to the neutral.

(a) If three wattmeters are arranged to measure the power in each line, calculate the reading of each meter.

(b) Find the magnitude of the current in the neutral line.

* An asterisk indicates a challenging problem.

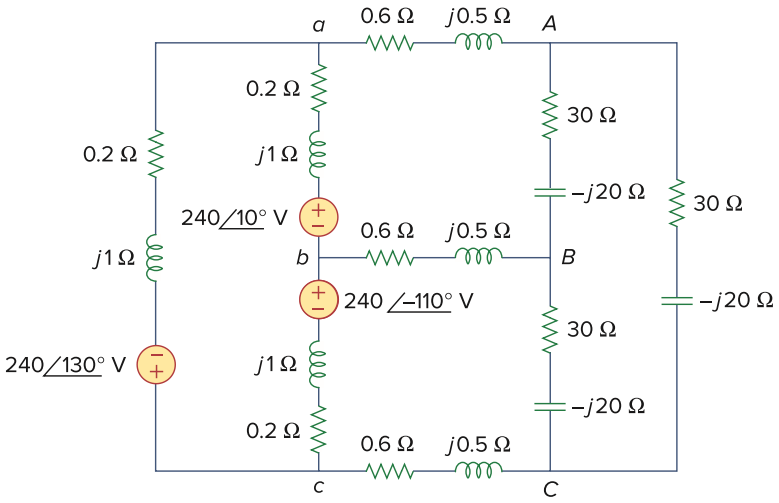



**Figure 12.70**
For Prob. 12.65.

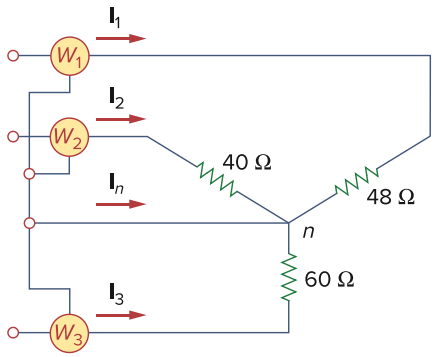



**Figure 12.71**
For Prob. 12.66.

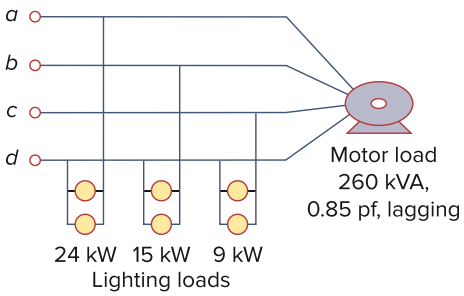



**Figure 12.72**
For Prob. 12.67.

**12.68** Meter readings for a three-phase wye-connected alternator supplying power to a motor indicate that the line voltages are 330 V, the line currents are 8.4 A, and the total line power is 4.5 kW. Find:

(a) the load in VA

(b) the load pf

(c) the phase current

(d) the phase voltage

**12.69** A certain store contains three balanced three-phase loads. The three loads are:

Load 1: 16 kVA at 0.85 pf lagging

Load 2: 12 kVA at 0.6 pf lagging

Load 3: 8 kW at unity pf

The line voltage at the load is 208 V rms at 60 Hz, and the line impedance is $0.4 + j0.8\ \Omega$. Determine the line current and the complex power delivered to the loads.

**12.70** The two-wattmeter method gives $P_1 = 1200$ W and $P_2 = -400$ W for a three-phase motor running on a 240-V line. Assume that the motor load is wye-connected and that it draws a line current of 6 A. Calculate the pf of the motor and its phase impedance.

**12.71** In Fig. 12.73, two wattmeters are properly connected to the unbalanced load supplied by a balanced source such that $\mathbf{V}_{ab} = 208\angle 0°$ V with positive phase sequence.

(a) Determine the reading of each wattmeter.

(b) Calculate the total apparent power absorbed by the load.

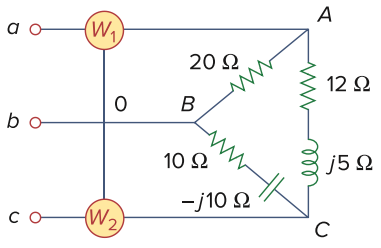



**Figure 12.73**
For Prob. 12.71.

**12.72** If wattmeters $W_1$ and $W_2$ are properly connected respectively between lines $a$ and $b$ and lines $b$ and $c$ to measure the power absorbed by the delta-connected load in Fig. 12.44, predict their readings.

**12.73** For the circuit displayed in Fig. 12.74, find the wattmeter readings.

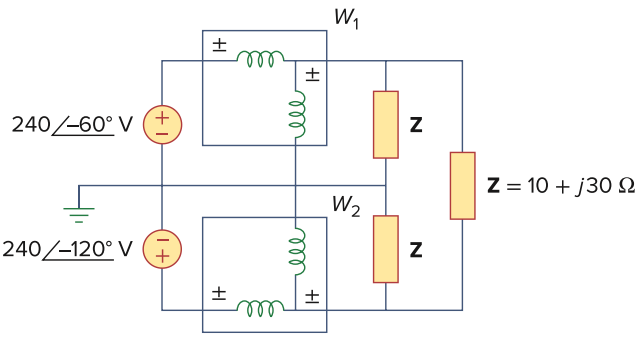



**Figure 12.74**
For Prob. 12.73.

**12.74** Predict the wattmeter readings for the circuit in Fig. 12.75.

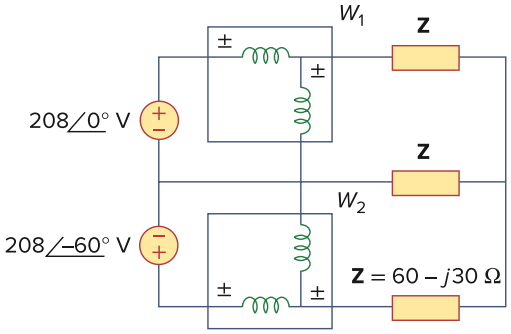



**Figure 12.75**
For Prob. 12.74.

**12.75** A man has a body resistance of 600 Ω. How much current flows through his ungrounded body:

(a) when he touches the terminals of a 12-V autobattery?

(b) when he sticks his finger into a 120-V light socket?

**12.76** Show that the $I^2R$ losses will be higher for a 120-V appliance than for a 240-V appliance if both have the same power rating.

## Comprehensive Problems

**12.77** A three-phase generator supplied 10 kVA at a power factor of 0.85 lagging. If 7,500 W are delivered to the load and line losses are 160 W per phase, what are the losses in the generator?

**12.78** A three-phase 440-V, 51-kW, 60-kVA inductive load operates at 60 Hz and is wye-connected. It is desired to correct the power factor to 0.95 lagging. What value of capacitor should be placed in parallel with each load impedance?

**12.79** A balanced three-phase generator has an *abc* phase sequence with phase voltage $\mathbf{V}_{an} = 554.3\angle 0°$ V. The generator feeds an induction motor which may be represented by a balanced Y-connected load with an impedance of $12 + j5\ \Omega$ per phase. Find the line currents and the load voltages. Assume a line impedance of 2 Ω per phase.

**12.80** A balanced three-phase source furnishes power to the following three loads:

Load 1: 6 kVA at 0.83 pf lagging

Load 2: unknown

Load 3: 8 kW at 0.7071 pf leading

If the line current is 84.6 A rms, the line voltage at the load is 208 V rms, and the combined load has a 0.8 pf lagging, determine the unknown load.

**12.81** A professional center is supplied by a balanced three-phase source. The center has four balanced three-phase loads as follows:

Load 1: 150 kVA at 0.8 pf leading

Load 2: 100 kW at unity pf

Load 3: 200 kVA at 0.6 pf lagging

Load 4: 80 kW and 95 kVAR (inductive)

If the line impedance is $0.02 + j0.05\ \Omega$ per phase and the line voltage at the loads is 480 V, find the magnitude of the line voltage at the source.

**12.82** A balanced three-phase system has a distribution wire with impedance $2 + j6\ \Omega$ per phase. The system supplies two three-phase loads that are connected in parallel. The first is a balanced wye-connected load that absorbs 400 kVA at a power factor of 0.8 lagging. The second load is a balanced delta-connected load with impedance of $10 + j8\ \Omega$ per phase. If the magnitude of the line voltage at the loads is 2400 V rms, calculate the magnitude of the line voltage at the source and the total complex power supplied to the two loads.

**12.83** A commercially available three-phase inductive motor operates at a full load of 120 hp (1 hp = 746 W) at 95 percent efficiency at a lagging power

factor of 0.707. The motor is connected in parallel to a 80-kW balanced three-phase heater at unity power factor. If the magnitude of the line voltage is 480 V rms, calculate the line current.

*12.84 Figure 12.76 displays a three-phase delta-connected motor load which is connected to a line voltage of 440 V and draws 4 kVA at a power factor of 72 percent lagging. In addition, a single 1.8 kVAR capacitor is connected between lines *a* and *b*, while a 800-W lighting load is connected between line *c* and neutral. Assuming the *abc* sequence and taking $\mathbf{V}_{an} = V_p\angle 0°$, find the magnitude and phase angle of currents $\mathbf{I}_a$, $\mathbf{I}_b$, $\mathbf{I}_c$, and $\mathbf{I}_n$.

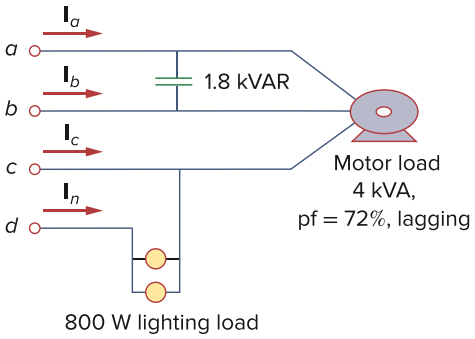



**Figure 12.76**
For Prob. 12.84.

12.85 Design a three-phase heater with suitable symmetric loads using wye-connected pure resistance. Assume that the heater is supplied by a 240-V line voltage and is to give 27 kW of heat.

12.86 For the single-phase three-wire system in Fig. 12.77, find currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{nN}$.

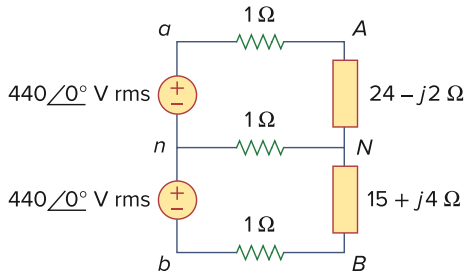



**Figure 12.77**
For Prob. 12.86.

12.87 Consider the single-phase three-wire system shown in Fig. 12.78. Find the current in the neutral wire and the complex power supplied by each source. Take $\mathbf{V}_s$ as a $220\angle 0°$-V, 60-Hz source.

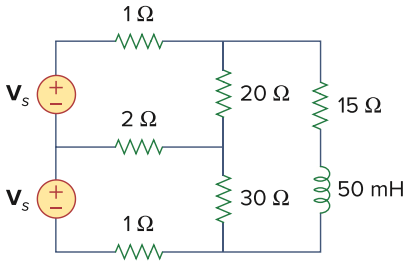



**Figure 12.78**
For Prob. 12.87.

chapter

# 14

# Frequency Response

*Dost thou love Life? Then do not squander Time; for that is the stuff Life is made.*

—Benjamin Franklin

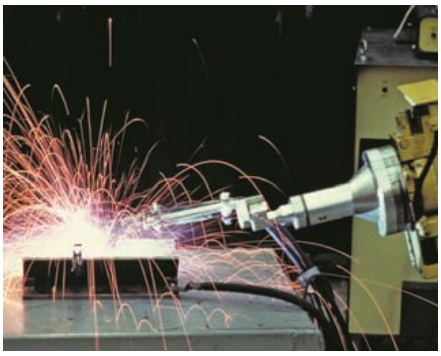
## Enhancing Your Career

### Career in Control Systems

Control systems are another area of electrical engineering where circuit analysis is used. A control system is designed to regulate the behavior of one or more variables in some desired manner. Control systems play major roles in our everyday life. Household appliances such as heating and air-conditioning systems, switch-controlled thermostats, washers and dryers, cruise controllers in automobiles, elevators, traffic lights, manufacturing plants, navigation systems—all utilize control systems. In the aerospace field, precision guidance of space probes, the wide range of operational modes of the space shuttle, and the ability to maneuver space vehicles remotely from earth all require knowledge of control systems. In the manufacturing sector, repetitive production line operations are increasingly performed by robots, which are programmable control systems designed to operate for many hours without fatigue.

Control engineering integrates circuit theory and communication theory. It is not limited to any specific engineering discipline but may involve environmental, chemical, aeronautical, mechanical, civil, and electrical engineering. For example, a typical task for a control system engineer might be to design a speed regulator for a disk drive head.

A thorough understanding of control systems techniques is essential to the electrical engineer and is of great value for designing control systems to perform the desired task.



A welding robot. © Vol. 1 PhotoDisc/Getty Images RF

## Learning Objectives

*By using the information and exercises in this chapter you will be able to:*

1. Understand what transfer functions are and how to determine them.
2. Understand the decibel scale, why we use it, and how to use it.
3. Understand Bode plots, and know why we use them and how to determine them.
4. Understand series and parallel resonance, why they are important, and how to find them.
5. Understand passive filters.
6. Understand active filters.
7. Discuss magnitude and frequency scaling and why they are important.

## 14.1 Introduction

In our sinusoidal circuit analysis, we have learned how to find voltages and currents in a circuit with a constant frequency source. If we let the amplitude of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's *frequency response*. The frequency response may be regarded as a complete description of the sinusoidal steady-state behavior of a circuit as a function of frequency.

The frequency response of a circuit is the variation in its behavior with change in signal frequency.

The frequency response of a circuit may also be considered as the variation of the gain and phase with frequency.

The sinusoidal steady-state frequency responses of circuits are of significance in many applications, especially in communications and control systems. A specific application is in electric filters that block out or eliminate signals with unwanted frequencies and pass signals of the desired frequencies. Filters are used in radio, TV, and telephone systems to separate one broadcast frequency from another.

We begin this chapter by considering the frequency response of simple circuits using their transfer functions. We then consider Bode plots, which are the industry-standard way of presenting frequency response. We also consider series and parallel resonant circuits and encounter important concepts such as resonance, quality factor, cutoff frequency, and bandwidth. We discuss different kinds of filters and network scaling. In the last section, we consider one practical application of resonant circuits and two applications of filters.

## 14.2 Transfer Function

The transfer function $\mathbf{H}(\omega)$ (also called the *network function*) is a useful analytical tool for finding the frequency response of a circuit. In fact, the frequency response of a circuit is the plot of the circuit's transfer function $\mathbf{H}(\omega)$ versus $\omega$, with $\omega$ varying from $\omega = 0$ to $\omega = \infty$.

A transfer function is the frequency-dependent ratio of a forced function to a forcing function (or of an output to an input). The idea of a transfer function was implicit when we used the concepts of impedance

and admittance to relate voltage and current. In general, a linear network can be represented by the block diagram shown in Fig. 14.1.

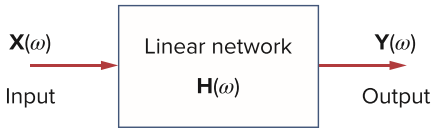



**Figure 14.1**
A block diagram representation of a linear network.

The transfer function $\mathbf{H}(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\mathbf{Y}(\omega)$ (an element voltage or current) to a phasor input $\mathbf{X}(\omega)$ (source voltage or current).

Thus,

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)} \tag{14.1}$$

assuming zero initial conditions. Since the input and output can be either voltage or current at any place in the circuit, there are four possible transfer functions:

In this context, $\mathbf{X}(\omega)$ and $\mathbf{Y}(\omega)$ denote the input and output phasors of a network; they should not be confused with the same symbolism used for reactance and admittance. The multiple usage of symbols is conventionally permissible due to lack of enough letters in the English language to express all circuit variables distinctly.

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} \tag{14.2a}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)} \tag{14.2b}$$

Some authors use $\mathbf{H}(j\omega)$ for transfer instead of $\mathbf{H}(\omega)$, since $\omega$ and $j$ are an inseparable pair.

$$\mathbf{H}(\omega) = \text{Transfer impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)} \tag{14.2c}$$

$$\mathbf{H}(\omega) = \text{Transfer admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)} \tag{14.2d}$$

where subscripts $i$ and $o$ denote input and output values. Being a complex quantity, $\mathbf{H}(\omega)$ has a magnitude $H(\omega)$ and a phase $\phi$; that is, $\mathbf{H}(\omega) = H(\omega)\underline{/\phi}$.

To obtain the transfer function using Eq. (14.2), we first obtain the frequency-domain equivalent of the circuit by replacing resistors, inductors, and capacitors with their impedances $R$, $j\omega L$, and $1/j\omega C$. We then use any circuit technique(s) to obtain the appropriate quantity in Eq. (14.2). We can obtain the frequency response of the circuit by plotting the magnitude and phase of the transfer function as the frequency varies. A computer is a real time-saver for plotting the transfer function.

The transfer function $\mathbf{H}(\omega)$ can be expressed in terms of its numerator polynomial $\mathbf{N}(\omega)$ and denominator polynomial $\mathbf{D}(\omega)$ as

$$\mathbf{H}(\omega) = \frac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)} \tag{14.3}$$

where $\mathbf{N}(\omega)$ and $\mathbf{D}(\omega)$ are not necessarily the same expressions for the input and output functions, respectively. The representation of $\mathbf{H}(\omega)$ in Eq. (14.3) assumes that common numerator and denominator factors in $\mathbf{H}(\omega)$ have canceled, reducing the ratio to lowest terms. The roots of $\mathbf{N}(\omega) = 0$ are called the *zeros* of $\mathbf{H}(\omega)$ and are usually represented as $j\omega = z_1, z_2, \ldots$. Similarly, the roots of $\mathbf{D}(\omega) = 0$ are the *poles* of $\mathbf{H}(\omega)$ and are represented as $j\omega = p_1, p_2, \ldots$.

A zero, as a *root* of the numerator polynomial, is a value that results in a zero value of the function. A pole, as a *root* of the denominator polynomial, is a value for which the function is infinite.

A zero may also be regarded as the value of $s = j\omega$ that makes $\mathbf{H}(s)$ zero, and a pole as the value of $s = j\omega$ that makes $\mathbf{H}(s)$ infinite.

To avoid complex algebra, it is expedient to replace $j\omega$ temporarily with $s$ when working with $\mathbf{H}(\omega)$ and replace $s$ with $j\omega$ at the end.

## Example 14.1

For the *RC* circuit in Fig. 14.2(a), obtain the transfer function $\mathbf{V}_o/\mathbf{V}_s$ and its frequency response. Let $v_s = V_m \cos \omega t$.

**Solution:**

The frequency-domain equivalent of the circuit is in Fig. 14.2(b). By voltage division, the transfer function is given by

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

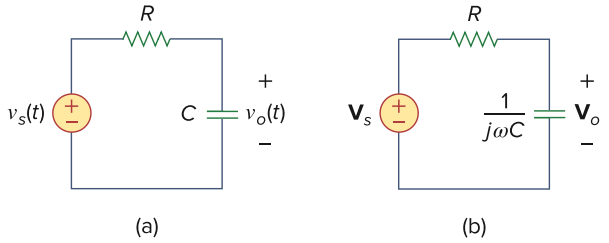



**Figure 14.2**
For Example 14.1: (a) time-domain *RC* circuit, (b) frequency-domain *RC* circuit.

Comparing this with Eq. (9.18e), we obtain the magnitude and phase of $\mathbf{H}(\omega)$ as

$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

where $\omega_0 = 1/RC$. To plot $H$ and $\phi$ for $0 < \omega < \infty$, we obtain their values at some critical points and then sketch.

At $\omega = 0$, $H = 1$ and $\phi = 0$. At $\omega = \infty$, $H = 0$ and $\phi = -90°$. Also, at $\omega = \omega_0$, $H = 1/\sqrt{2}$ and $\phi = -45°$. With these and a few more points as shown in Table 14.1, we find that the frequency response is as shown in Fig. 14.3. Additional features of the frequency response in Fig. 14.3 will be explained in Section 14.6.1 on low-pass filters.

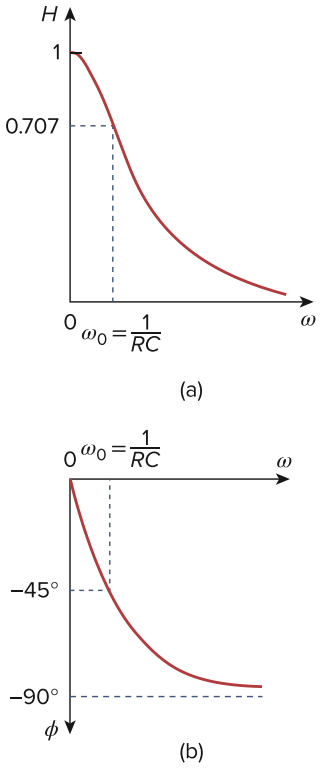



**Figure 14.3**
Frequency response of the *RC* circuit: (a) amplitude response, (b) phase response.

**TABLE 14.1**

For Example 14.1.

| $\omega/\omega_0$ | $H$ | $\phi$ | $\omega/\omega_0$ | $H$ | $\phi$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 10 | 0.1 | –84° |
| 1 | 0.71 | –45° | 20 | 0.05 | –87° |
| 2 | 0.45 | –63° | 100 | 0.01 | –89° |
| 3 | 0.32 | –72° | ∞ | 0 | –90° |

## Practice Problem 14.1

Obtain the transfer function $\mathbf{V}_o/\mathbf{V}_s$ of the *RL* circuit in Fig. 14.4, assuming $v_s = V_m \cos \omega t$. Sketch its frequency response.

**Answer:** $j\omega L/(R + j\omega L)$; see Fig. 14.5 for the response.

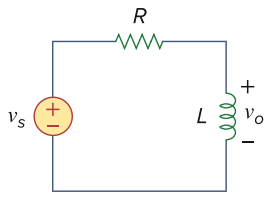



**Figure 14.4**
*RL* circuit for Practice Prob. 14.1.

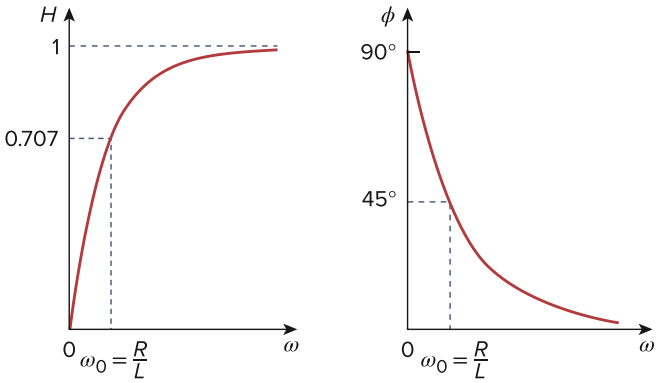



**Figure 14.5**
Frequency response of the *RL* circuit in Fig. 14.4.

## Example 14.2

For the circuit in Fig. 14.6, calculate the g ain $\mathbf{I}_o(\omega)/\mathbf{I}_i(\omega)$ and its poles and zeros.

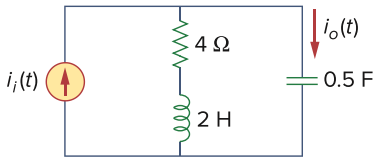



**Figure 14.6**
For Example 14.2.

**Solution:**
By current division,

$$\mathbf{I}_o(\omega) = \frac{4 + j2\omega}{4 + j2\omega + 1/j0.5\omega}\,\mathbf{I}_i(\omega)$$

or

$$\frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{1 + j2\omega + (j\omega)^2} = \frac{s(s+2)}{s^2 + 2s + 1}, \qquad s = j\omega$$

The zeros are at

$$s(s+2) = 0 \quad \Rightarrow \quad z_1 = 0,\ z_2 = -2$$

The poles are at

$$s^2 + 2s + 1 = (s+1)^2 = 0$$

Thus, there is a repeated pole (or double pole) at $p = -1$.

## Practice Problem 14.2

Find the transfer function $\mathbf{V}_o(\omega)/\mathbf{I}_i(\omega)$ for the circuit in Fig. 14.7. Obtain its zeros and poles.

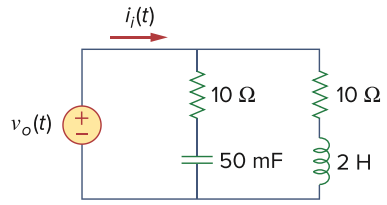


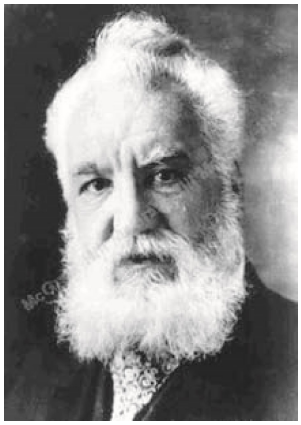

**Figure 14.7**
For Practice Prob. 14.2.

**Answer:** $\dfrac{10(s+2)(s+5)}{s^2 + 10s + 10}$, $s = j\omega$; zeros: −2, −5; poles: −1.127, −8.873.

## 14.3 †The Decibel Scale

It is not always easy to get a quick plot of the magnitude and phase of the transfer function as we did above. A more systematic way of obtaining the frequency response is to use Bode plots. Before we begin to construct Bode plots, we should take care of two important issues: the use of logarithms and decibels in expressing gain.

## Historical



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**Alexander Graham Bell** (1847–1922) inventor of the telephone, was a Scottish-American scientist.

Bell was born in Edinburgh, Scotland, a son of Alexander Melville Bell, a well-known speech teacher. Alexander the younger also became a speech teacher after graduating from the University of Edinburgh and the University of London. In 1866 he became interested in transmitting speech electrically. After his older brother died of tuberculosis, his father decided to move to Canada. Alexander was asked to come to Boston to work at the School for the Deaf. There he met Thomas A. Watson, who became his assistant in his electromagnetic transmitter experiment. On March 10, 1876, Alexander sent the famous first telephone message: "Watson, come here I want you." The bel, the logarithmic unit introduced in this chapter, is named in his honor.

Since Bode plots are based on logarithms, it is important that we keep the following properties of logarithms in mind:

1. $\log P_1P_2 = \log P_1 + \log P_2$
2. $\log P_1/P_2 = \log P_1 - \log P_2$
3. $\log P^n = n \log P$
4. $\log 1 = 0$

*Historical note:* The *bel* is named after Alexander Graham Bell, the inventor of the telephone.

In communications systems, gain is measured in *bels*. Historically, the bel is used to measure the ratio of two levels of power or power gain $G$; that is,

$$G = \text{Number of bels} = \log_{10} \frac{P_2}{P_1} \tag{14.4}$$

The *decibel* (dB) provides us with a unit of less magnitude. It is 1/10th of a bel and is given by

$$\boxed{G_{\text{dB}} = 10 \log_{10} \frac{P_2}{P_1}} \tag{14.5}$$

When $P_1 = P_2$, there is no change in power and the gain is 0 dB. If $P_2 = 2P_1$, the gain is

$$G_{\text{dB}} = 10 \log_{10} 2 \simeq 3 \text{ dB} \tag{14.6}$$

and when $P_2 = 0.5P_1$, the gain is

$$G_{\text{dB}} = 10 \log_{10} 0.5 \simeq -3 \text{ dB} \tag{14.7}$$

Equations (14.6) and (14.7) show another reason why logarithms are greatly used: The logarithm of the reciprocal of a quantity is simply negative the logarithm of that quantity.

Alternatively, the gain $G$ can be expressed in terms of voltage or current ratio. To do so, consider the network shown in Fig. 14.8. If $P_1$ is the input power, $P_2$ is the output (load) power, $R_1$ is the input resistance,

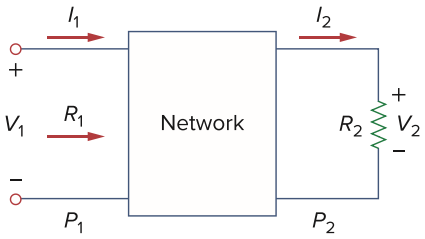



**Figure 14.8**
Voltage-current relationships for a four-terminal network.

and $R_2$ is the load resistance, then $P_1 = 0.5V_1^2/R_1$ and $P_2 = 0.5V_2^2/R_2$, and Eq. (14.5) becomes

$$G_{\text{dB}} = 10 \log_{10} \frac{P_2}{P_1} = 10 \log_{10} \frac{V_2^2/R_2}{V_1^2/R_1}$$
$$= 10 \log_{10} \left(\frac{V_2}{V_1}\right)^2 + 10 \log_{10} \frac{R_1}{R_2} \tag{14.8}$$

$$G_{\text{dB}} = 20 \log_{10} \frac{V_2}{V_1} - 10 \log_{10} \frac{R_2}{R_1} \tag{14.9}$$

For the case when $R_2 = R_1$, a condition that is often assumed when comparing voltage levels, Eq. (14.9) becomes

$$\boxed{G_{\text{dB}} = 20 \log_{10} \frac{V_2}{V_1}} \tag{14.10}$$

Instead, if $P_1 = I_1^2 R_1$ and $P_2 = I_2^2 R_2$, for $R_1 = R_2$, we obtain

$$G_{\text{dB}} = 20 \log_{10} \frac{I_2}{I_1} \tag{14.11}$$

Three things are important to note from Eqs. (14.5), (14.10), and (14.11):

1. That $10 \log_{10}$ is used for power, while $20 \log_{10}$ is used for voltage or current, because of the square relationship between them ($P = V^2/R = I^2R$).
2. That the dB value is a logarithmic measurement of the *ratio* of one variable to another *of the same type*. Therefore, it applies in expressing the transfer function $H$ in Eqs. (14.2a) and (14.2b), which are dimensionless quantities, but not in expressing $H$ in Eqs. (14.2c) and (14.2d).
3. It is important to note that we only use voltage and current magnitudes in Eqs. (14.10) and (14.11). Negative signs and angles will be handled independently as we will see in Section 14.4.

With this in mind, we now apply the concepts of logarithms and decibels to construct Bode plots.

## 14.4 Bode Plots

Obtaining the frequency response from the transfer function as we did in Section 14.2 is an uphill task. The frequency range required in frequency response is often so wide that it is inconvenient to use a linear scale for the frequency axis. Also, there is a more systematic way of locating the important features of the magnitude and phase plots of the transfer function. For these reasons, it has become standard practice to plot the transfer function on a pair of semilogarithmic plots: The magnitude in decibels is plotted against the logarithm of the frequency; on a separate plot, the phase in degrees is plotted against the logarithm of the frequency. Such semilogarithmic plots of the transfer function—known as *Bode plots*—have become the industry standard.

*Historical note:* Named after Hendrik W. Bode (1905–1982), an engineer with the Bell Telephone Laboratories, for his pioneering work in the 1930s and 1940s.

**Bode plots** are semilog plots of the magnitude (in decibels) and phase (in degrees) of a transfer function versus frequency.

Bode plots contain the same information as the nonlogarithmic plots discussed in the previous section, but they are much easier to construct, as we shall see shortly.

The transfer function can be written as

$$\mathbf{H} = H\angle\phi = He^{j\phi} \tag{14.12}$$

Taking the natural logarithm of both sides,

$$\ln \mathbf{H} = \ln H + \ln e^{j\phi} = \ln H + j\phi \tag{14.13}$$

Thus, the real part of $\ln\mathbf{H}$ is a function of the magnitude while the imaginary part is the phase. In a Bode magnitude plot, the gain

$$H_{\text{dB}} = 20 \log_{10} H \tag{14.14}$$

is plotted in decibels (dB) versus frequency. Table 14.2 provides a few values of $H$ with the corresponding values in decibels. In a Bode phase plot, $\phi$ is plotted in degrees versus frequency. Both magnitude and phase plots are made on semilog graph paper.

**TABLE 14.2**

Specific gain and their decibel values.*

| Magnitude $H$ | $20 \log_{10} H$ (dB) |
|---|---|
| 0.001 | −60 |
| 0.01 | −40 |
| 0.1 | −20 |
| 0.5 | −6 |
| $1/\sqrt{2}$ | −3 |
| 1 | 0 |
| $\sqrt{2}$ | 3 |
| 2 | 6 |
| 10 | 20 |
| 20 | 26 |
| 100 | 40 |
| 1000 | 60 |

* Some of these values are approximate.

A transfer function in the form of Eq. (14.3) may be written in terms of factors that have real and imaginary parts. One such representation might be

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots} \tag{14.15}$$

which is obtained by dividing out the poles and zeros in $\mathbf{H}(\omega)$. The representation of $\mathbf{H}(\omega)$ as in Eq. (14.15) is called the *standard form*. $\mathbf{H}(\omega)$ may include up to seven types of different factors that can appear in various combinations in a transfer function. These are:

1. A gain $K$
2. A pole $(j\omega)^{-1}$ or zero $(j\omega)$ at the origin
3. A simple pole $1/(1 + j\omega/p_1)$ or zero $(1 + j\omega/z_1)$
4. A quadratic pole $1/[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$ or zero $[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]$

In constructing a Bode plot, we plot each factor separately and then add them graphically. The factors can be considered one at a time and then combined additively because of the logarithms involved. It is this mathematical convenience of the logarithm that makes Bode plots a powerful engineering tool.

We will now make straight-line plots of the factors listed above. We shall find that these straight-line plots known as Bode plots approximate the actual plots to a reasonable degree of accuracy.

**Constant term:** For the gain $K$, the magnitude is $20 \log_{10} K$ and the phase is 0°; both are constant with frequency. Thus, the magnitude and phase plots of the gain are shown in Fig. 14.9. If $K$ is negative, the magnitude remains $20 \log_{10} |K|$ but the phase is ±180°.

**Pole/zero at the origin:** For the zero $(j\omega)$ at the origin, the magnitude is $20 \log_{10} \omega$ and the phase is 90°. These are plotted in Fig. 14.10, where we notice that the slope of the magnitude plot is 20 dB/decade, while the phase is constant with frequency.

The Bode plots for the pole $(j\omega)^{-1}$ are similar except that the slope of the magnitude plot is −20 dB/decade while the phase is −90°. In general,

The origin is where $\omega = 1$ or $\log \omega = 0$ and the gain is zero.

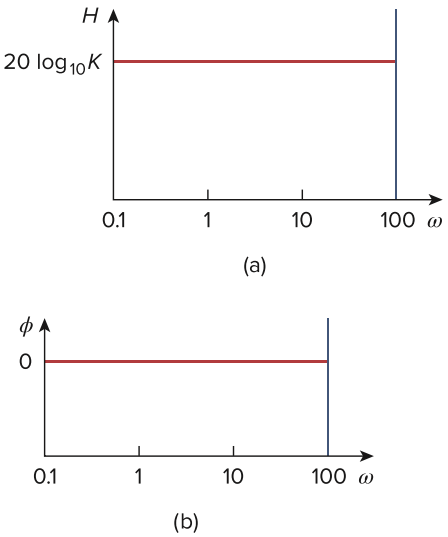



**Figure 14.9**
Bode plots for gain $K$: (a) magnitude plot, (b) phase plot.

for $(j\omega)^N$, where $N$ is an integer, the magnitude plot will have a slope of $20N$ dB/decade, while the phase is $90N$ degrees.

**Simple pole/zero:** For the simple zero $(1 + j\omega/z_1)$, the magnitude is $20 \log_{10}|1 + j\omega/z_1|$ and the phase is $\tan^{-1} \omega/z_1$. We notice that

$$H_{\text{dB}} = 20 \log_{10}\left|1 + \frac{j\omega}{z_1}\right| \quad \Rightarrow \quad 20 \log_{10} 1 = 0 \quad \text{as} \quad \omega \to 0 \tag{14.16}$$

$$H_{\text{dB}} = 20 \log_{10}\left|1 + \frac{j\omega}{z_1}\right| \quad \Rightarrow \quad 20 \log_{10} \frac{\omega}{z_1} \quad \text{as} \quad \omega \to \infty \tag{14.17}$$

showing that we can approximate the magnitude as zero (a straight line with zero slope) for small v alues of $\omega$ and by a straight line with slope 20 dB/decade for large values of $\omega$. The frequency $\omega = z_1$ where the two asymptotic lines meet is called the *corner frequency* or *break frequency*. Thus, the approximate magnitude plot is sho wn in Fig. 14.11(a), where the actual plot is also shown. Notice that the approximate plot is close to the actual plot except at the break frequency, where $\omega = z_1$ and the deviation is $20 \log_{10}|(1 + j1)| = 20 \log_{10} \sqrt{2} \simeq 3$ dB.

The phase $\tan^{-1}(\omega/z_1)$ can be expressed as

$$\phi = \tan^{-1}\left(\frac{\omega}{z_1}\right) = \begin{cases} 0, & \omega = 0 \\ 45^\circ, & \omega = z_1 \\ 90^\circ, & \omega \to \infty \end{cases} \tag{14.18}$$

As a straight-line approximation, we let $\phi \simeq 0$ for $\omega \leq z_1/10$, $\phi \simeq 45^\circ$ for $\omega = z_1$, and $\phi \simeq 90^\circ$ for $\omega \geq 10z_1$. As shown in Fig. 14.11(b) along with the actual plot, the straight-line plot has a slope of 45° per decade.

The Bode plots for the pole $1/(1 + j\omega/p_1)$ are similar to those in Fig. 14.11 except that the corner frequency is at $\omega = p_1$, the magnitude has a slope of $-20$ dB/decade, and the phase has a slope of $-45^\circ$ per decade.

**Quadratic pole/zer o:** The magnitude of the quadratic pole $1/[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$ is $-20 \log_{10}|1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2|$ and the phase is $-\tan^{-1}(2\zeta_2\omega/\omega_n)/(1 - \omega^2/\omega_n^2)$. But

$$H_{\text{dB}} = -20 \log_{10}\left|1 + \frac{j2\zeta_2\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right| \quad \Rightarrow \quad 0 \quad \text{as} \quad \omega \to 0 \tag{14.19}$$

A decade is an interval between two frequencies with a ratio of 10; e.g., between $\omega_0$ and $10\omega_0$, or between 10 and 100 Hz. Thus, 20 dB/decade means that the magnitude changes 20 dB whenever the frequency changes tenfold or one decade.

The special case of dc ($\omega = 0$) does not appear on Bode plots because $\log 0 = -\infty$, implying that zero frequency is infinitely far to the left of the origin of Bode plots.

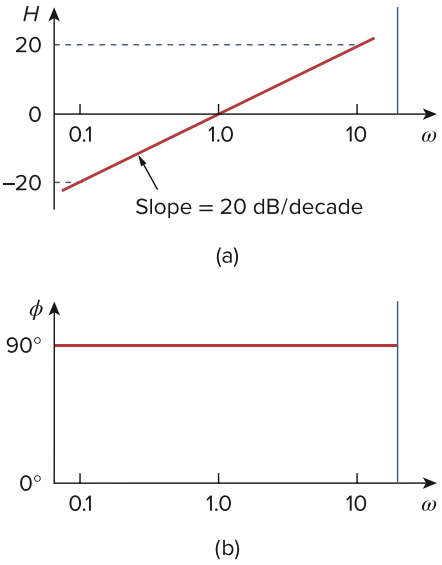



**Figure 14.10**
Bode plot for a zero $(j\omega)$ at the origin: (a) magnitude plot, (b) phase plot.

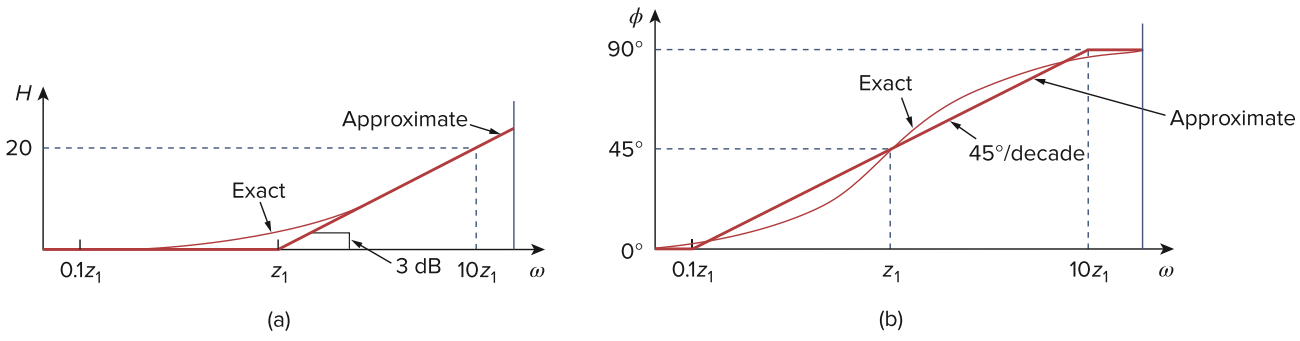



**Figure 14.11**
Bode plots of zero $(1 + j\omega/z_1)$: (a) magnitude plot, (b) phase plot.

and

$$H_{\text{dB}} = -20 \log_{10}\left|1 + \frac{j2\zeta_2\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right| \quad \Rightarrow \quad -40 \log_{10}\frac{\omega}{\omega_n} \quad \text{as} \quad \omega \to \infty \tag{14.20}$$

Thus, the amplitude plot consists of two straight asymptotic lines: one with zero slope for $\omega < \omega_n$ and the other with slope $-40$ dB/decade for $\omega > \omega_n$, with $\omega_n$ as the corner frequency. Figure 14.12(a) shows the approximate and actual amplitude plots. Note that the actual plot depends on the damping factor $\zeta_2$ as well as the corner frequency $\omega_n$. The significant peaking in the neighborhood of the corner frequency should be added to the straight-line approximation if a high level of accuracy is desired. However, we will use the straight-line approximation for the sake of simplicity.

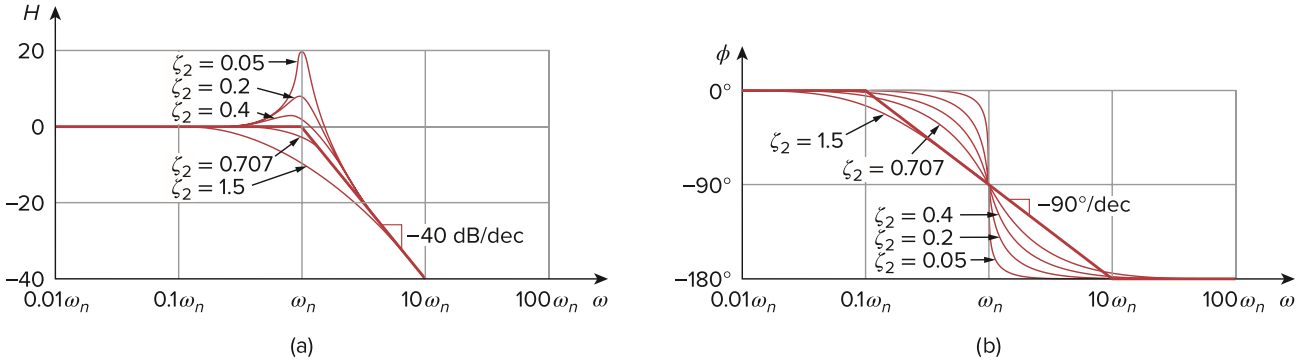



**Figure 14.12**
Bode plots of quadratic pole $[1 + j2\zeta\omega/\omega_n - \omega^2/\omega_n^2]^{-1}$: (a) magnitude plot, (b) phase plot.

There is another procedure for obtaining Bode plots that is faster and perhaps more efficient than the one we have just discussed. It consists in realizing that zeros cause an increase in slope, while poles cause a decrease. By starting with the low-frequency asymptote of the Bode plot, moving along the frequency axis, and increasing or decreasing the slope at each corner frequency, one can sketch the Bode plot immediately from the transfer function without the effort of making individual plots and adding them. This procedure can be used once you become proficient in the one discussed here.

Digital computers have rendered the procedure discussed here almost obsolete. Several software packages such as *PSpice, MATLAB,* Mathcad, and Micro-Cap can be used to generate frequency response plots. We will discuss *PSpice* later in the chapter.

The phase can be expressed as

$$\phi = -\tan^{-1}\frac{2\zeta_2\omega/\omega_n}{1 - \omega^2/\omega_n^2} = \begin{cases} 0, & \omega = 0 \\ -90°, & \omega = \omega_n \\ -180°, & \omega \to \infty \end{cases} \tag{14.21}$$

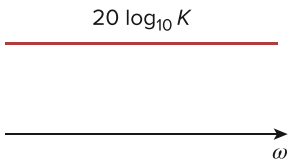
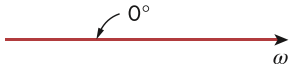
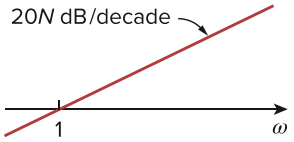
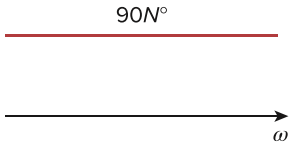
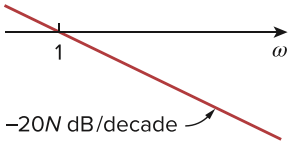
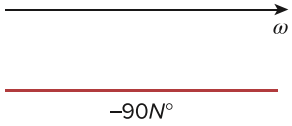
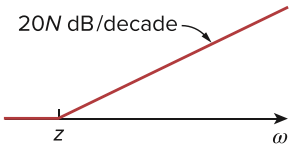
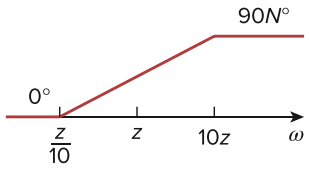
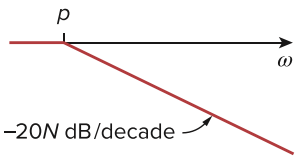
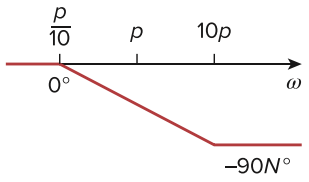
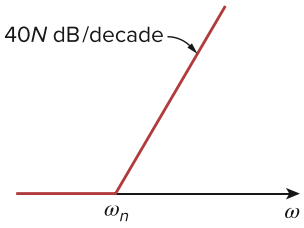
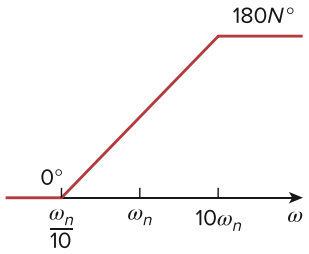
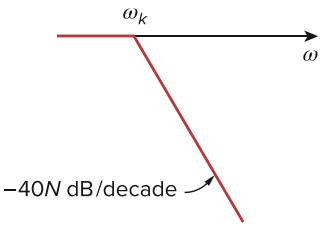
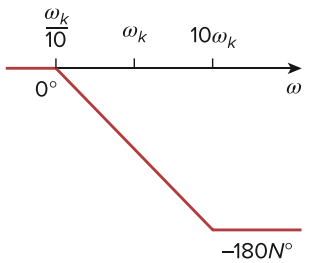
The phase plot is a straight line with a slope of $-90°$ per decade starting at $\omega_n/10$ and ending at $10\omega_n$, as shown in Fig. 14.12(b). We see again that the difference between the actual plot and the straight-line plot is due to the damping factor. Notice that the straight-line approximations for both magnitude and phase plots for the quadratic pole are the same as those for a double pole, that is, $(1 + j\omega/\omega_n)^{-2}$. We should expect this because the double pole $(1+j\omega/\omega_n)^{-2}$ equals the quadratic pole $1/[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$ when $\zeta_2 = 1$. Thus, the quadratic pole can be treated as a double pole as far as straight-line approximation is concerned.

For the quadratic zero $[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]$, the plots in Fig. 14.12 are inverted because the magnitude plot has a slope of 40 dB/decade while the phase plot has a slope of 90° per decade.

Table 14.3 presents a summary of Bode plots for the seven factors. Of course, not every transfer function has all seven factors. To sketch the Bode plots for a function $\mathbf{H}(\omega)$ in the form of Eq. (14.15), for example, we first record the corner frequencies on the semilog graph paper, sketch the factors one at a time as discussed above, and then combine additively

**TABLE 14.3**

Summary of Bode straight-line magnitude and phase plots.

| Factor | Magnitude | Phase |
|---|---|---|
| $K$ |  |  |
| $(j\omega)^N$ |  |  |
| $\frac{1}{(j\omega)^N}$ |  |  |
| $\left(\frac{1+j\omega}{z}\right)^N$ |  |  |
| $\frac{1}{(1+j\omega/p)^N}$ |  |  |
| $\left[1+\frac{2j\omega\zeta}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]^N$ |  |  |
| $\frac{1}{[1+2j\omega\zeta/\omega_k+(j\omega/\omega_k)^2]^N}$ |  |  |

the graphs of the f actors. The combined graph is often dra wn from left to right, changing slopes appropriately each time a corner frequenc y is encountered. The following examples illustrate this procedure.

## Example 14.3

Construct the Bode plots for the transfer function

$$\mathbf{H}(\omega) = \frac{200j\omega}{(j\omega + 2)(j\omega + 10)}$$

**Solution:**

We first put $\mathbf{H}(\omega)$ in the standard form by dividing out the poles and zeros. Thus,

$$\mathbf{H}(\omega) = \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

$$= \frac{10\,|j\omega|}{|1 + j\omega/2|\,|1 + j\omega/10|} \underline{/90^\circ - \tan^{-1}\omega/2 - \tan^{-1}\omega/10}$$

Hence, the magnitude and phase are

$$H_{\text{dB}} = 20\log_{10} 10 + 20\log_{10}|j\omega| - 20\log_{10}\left|1 + \frac{j\omega}{2}\right|$$

$$- 20\log_{10}\left|1 + \frac{j\omega}{10}\right|$$

$$\phi = 90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$

We notice that there are two corner frequencies at $\omega = 2, 10$. For both the magnitude and phase plots, we sketch each term as shown by the dotted lines in Fig. 14.13. We add them up graphically to obtain the overall plots shown by the solid curves.

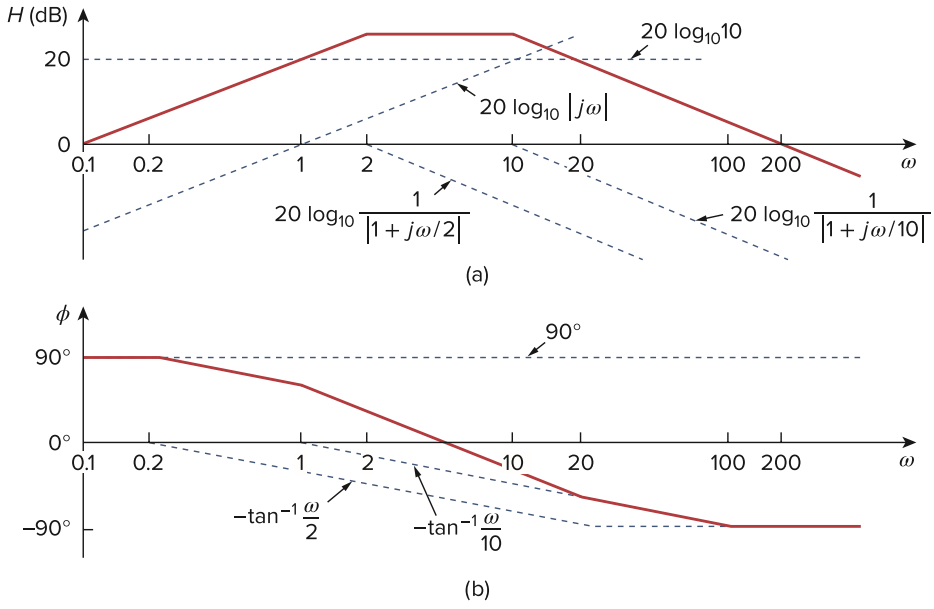



**Figure 14.13**
For Example 14.3: (a) magnitude plot, (b) phase plot.

## Practice Problem 14.3

Draw the Bode plots for the transfer function

$$\mathbf{H}(\omega) = \frac{5(j\omega + 2)}{j\omega(j\omega + 10)}$$

**Answer:** See Fig. 14.14.

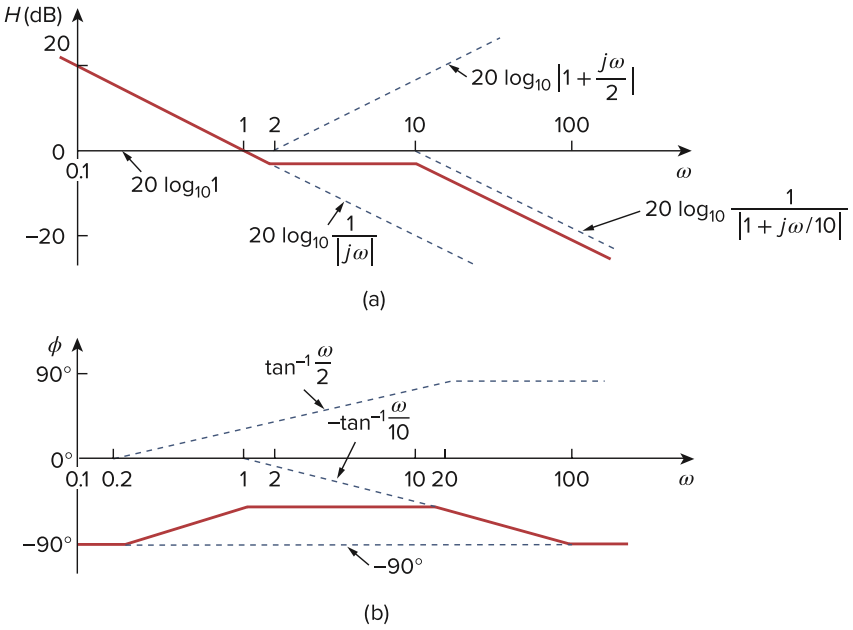



**Figure 14.14**
For Practice Prob. 14.3: (a) magnitude plot, (b) phase plot.

## Example 14.4

Obtain the Bode plots for

$$\mathbf{H}(\omega) = \frac{j\omega + 10}{j\omega(j\omega + 5)^2}$$

**Solution:**
Putting $\mathbf{H}(\omega)$ in the standard form, we get

$$\mathbf{H}(\omega) = \frac{0.4(1 + j\omega/10)}{j\omega(1 + j\omega/5)^2}$$

From this, we obtain the magnitude and phase as

$$H_{\text{dB}} = 20\log_{10} 0.4 + 20\log_{10}\left|1 + \frac{j\omega}{10}\right| - 20\log_{10}|j\omega| - 40\log_{10}\left|1 + \frac{j\omega}{5}\right|$$

$$\phi = 0° + \tan^{-1}\frac{\omega}{10} - 90° - 2\tan^{-1}\frac{\omega}{5}$$

There are two corner frequencies at $\omega = 5, 10$ rad/s. For the pole with corner frequency at $\omega = 5$, the slope of the magnitude plot is $-40$ dB/decade and that of the phase plot is $-90°$ per decade due to the power of 2. The

magnitude and the phase plots for the individual terms (in dotted lines) and the entire $\mathbf{H}(j\omega)$ (in solid lines) are in Fig. 14.15.

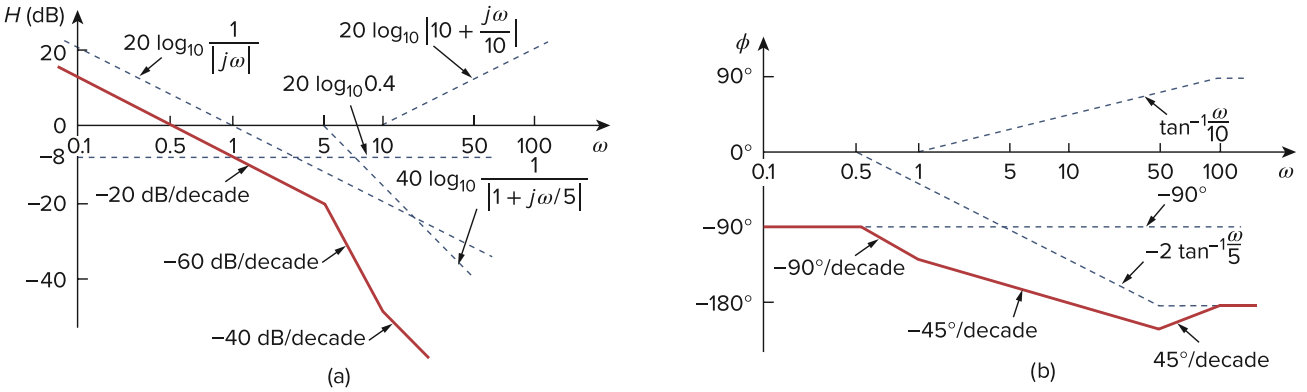



**Figure 14.15**
Bode plots for Example 14.4: (a) magnitude plot, (b) phase plot.

## Practice Problem 14.4

Sketch the Bode plots for

$$\mathbf{H}(\omega) = \frac{50\,j\omega}{(j\omega + 4)(j\omega + 10)^2}$$

**Answer:** See Fig. 14.16.

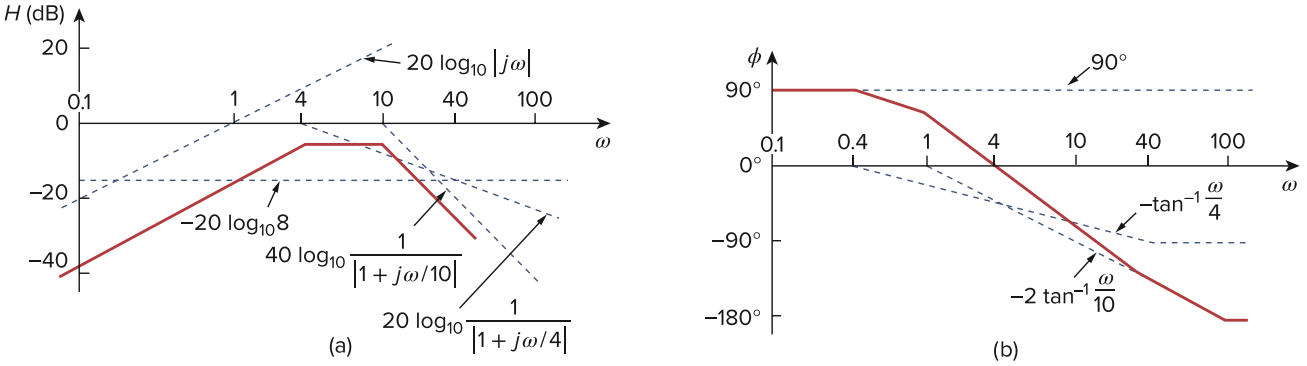



**Figure 14.16**
For Practice Prob. 14.4: (a) magnitude plot, (b) phase plot.

## Example 14.5

Draw the Bode plots for

$$\mathbf{H}(s) = \frac{s + 1}{s^2 + 12s + 100}$$

**Solution:**

1. **Define.** The problem is clearly stated and we follo w the technique outlined in the chapter.
2. **Present.** We are to develop the approximate Bode plot for the given function, $\mathbf{H}(s)$.
3. **Alternative.** The two most effective choices would be the approximation technique outlined in the chapter , which we will use here, and *MATLAB*, which can actually give us the exact Bode plots.

4. **Attempt.** We express $\mathbf{H}(s)$ as

$$\mathbf{H}(\omega) = \frac{1/100(1 + j\omega)}{1 + j\omega 1.2/10 + (j\omega/10)^2}$$

For the quadratic pole, $\omega_n = 10$ rad/s, which serves as the corner frequency. The magnitude and phase are

$$H_{\text{dB}} = -20 \log_{10} 100 + 20 \log_{10} |1 + j\omega| - 20 \log_{10} \left| 1 + \frac{j\omega 1.2}{10} - \frac{\omega^2}{100} \right|$$

$$\phi = 0° + \tan^{-1} \omega - \tan^{-1} \left[ \frac{\omega 1.2/10}{1 - \omega^2/100} \right]$$

Figure 14.17 shows the Bode plots. Notice that the quadratic pole is treated as a repeated pole at $\omega_k$, that is, $(1 + j\omega/\omega_k)^2$, which is an approximation.

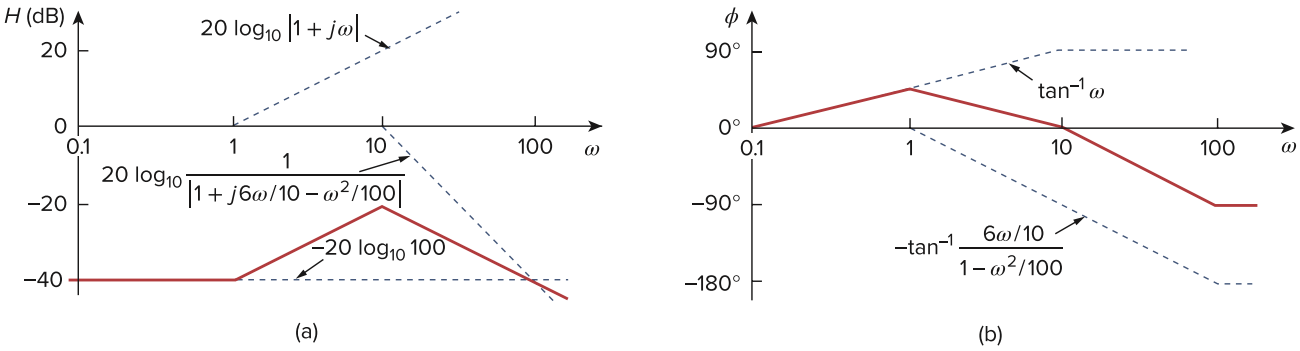



**Figure 14.17**
Bode plots for Example 14.5: (a) magnitude plot, (b) phase plot.

5. **Evaluate.** Although we could use *MATLAB* to validate the solution, we will use a more straightforward approach. First, we must realize that the denominator assumes that $\zeta = 0$ for the approximation, so we will use the following equation to check our answer:

$$\mathbf{H}(s) \simeq \frac{s + 1}{s^2 + 10^2}$$

We also note that we need to actually solve for $H_{\text{dB}}$ and the corresponding phase angle $\phi$. First, let $\omega = 0$.

$$H_{\text{dB}} = 20 \log_{10}(1/100) = -40 \qquad \text{and} \qquad \phi = 0°$$

Now try $\omega = 1$.

$$H_{\text{dB}} = 20 \log_{10}(1.4142/99) = -36.9 \text{ dB}$$

which is the expected 3 dB up from the corner frequency.

$$\phi = 45° \qquad \text{from} \qquad \mathbf{H}(j) = \frac{j + 1}{-1 + 100}$$

Now try $\omega = 100$.

$$H_{\text{dB}} = 20\log_{10}(100) - 20\log_{10}(9900) = 39.91 \text{ dB}$$

$\phi$ is 90° from the numerator minus 180°, which gives $-90°$. We now have checked three different points and got close agreement, and, because this is an approximation, we can feel confident that we have worked the problem successfully.

You can reasonably ask why did we not check at $\omega = 10$? If we just use the approximate value we used above, we end up with an infinite value, which is to be expected from $\zeta = 0$ (see Fig. 14.12a). If we used the actual value of $\mathbf{H}(j10)$ we will still end up being far from the approximate values, since $\zeta = 0.6$ and Fig. 14.12a shows a significant deviation from the approximation. We could have reworked the problem with $\zeta = 0.707$, which would have gotten us closer to the approximation. However, we really have enough points without doing this.

6. **Satisfactory?** We are satisfied the problem has been worked successfully and we can present the results as a solution to the problem.

## Practice Problem 14.5

Construct the Bode plots for

$$H(s) = \frac{10}{s(s^2 + 80s + 400)}$$

**Answer:** See Fig. 14.18.

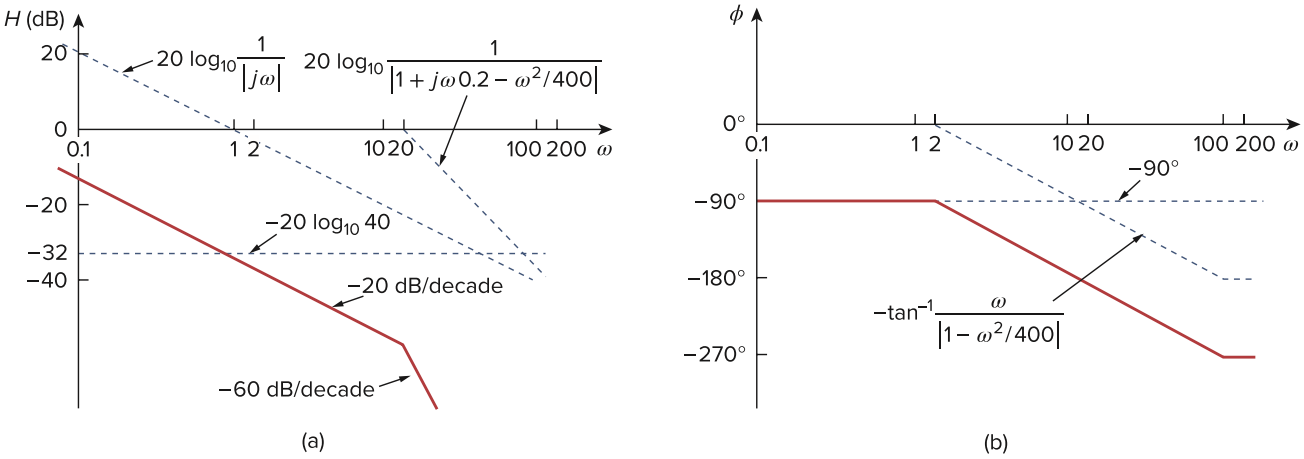



**Figure 14.18**
For Practice Prob. 14.5: (a) magnitude plot, (b) phase plot.

## Example 14.6

Given the Bode plot in Fig. 14.19, obtain the transfer function $\mathbf{H}(\omega)$.

**Solution:**

To obtain $\mathbf{H}(\omega)$ from the Bode plot, we keep in mind that a zero always causes an upward turn at a corner frequency, while a pole causes

a downward turn. We notice from Fig. 14.19 that there is a zero $j\omega$ at the origin, which should have intersected the frequency axis at $\omega = 1$. This is indicated by the straight line with slope +20 dB/decade. The fact that this straight line is shifted by 40 dB indicates that there is a 40-dB gain; that is,

$$40 = 20 \log_{10} K \quad \Rightarrow \quad \log_{10} K = 2$$

or

$$K = 10^2 = 100$$

In addition to the zero $j\omega$ at the origin, we notice that there are three factors with corner frequencies at $\omega = 1$, 5, and 20 rad/s. Thus, we have:

1. A pole at $p = 1$ with slope −20 dB/decade to cause a downward turn and counteract the zero at the origin.The pole at $p = 1$ is determined as $1/(1 + j\omega/1)$.
2. Another pole at $p = 5$ with slope −20 dB/decade causing a do wnward turn. The pole is $1/(1 + j\omega/5)$.
3. A third pole at $p = 20$ with slope −20 dB/decade causing a further downward turn. The pole is $1/(1 + j\omega/20)$.

Putting all these together gives the corresponding transfer function as

$$\mathbf{H}(\omega) = \frac{100\, j\omega}{(1 + j\omega/1)(1 + j\omega/5)(1 + j\omega/20)}$$

$$= \frac{j\omega 10^4}{(j\omega + 1)(j\omega + 5)(j\omega + 20)}$$

or

$$\mathbf{H}(s) = \frac{10^4 s}{(s + 1)(s + 5)(s + 20)}, \qquad s = j\omega$$

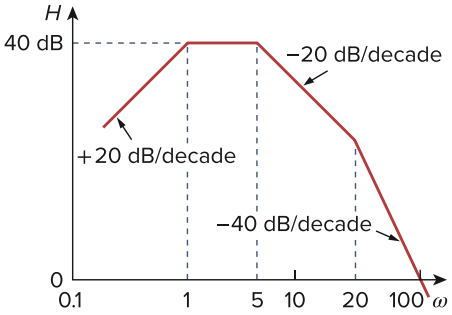



**Figure 14.19**
For Example 14.6.

## Practice Problem 14.6

Obtain the transfer function H( $\omega$) corresponding to the Bode plot in Fig. 14.20.

**Answer:** $\mathbf{H}(\omega) = \dfrac{2{,}000{,}000(s + 5)}{(s + 10)(s + 100)^2}$.

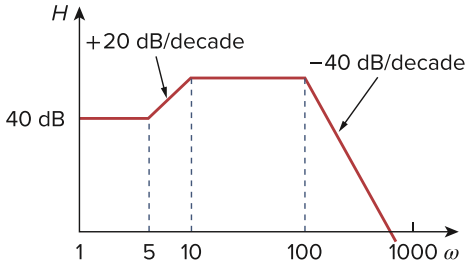



**Figure 14.20**
For Practice Prob. 14.6.

To see how to use *MATLAB* to produce Bode plots, refer to Section 14.11.

## 14.5 Series Resonance

The most prominent feature of the frequenc y response of a circuit may be the sharp peak (or *resonant peak*) e xhibited in its amplitude char - acteristic. The concept of resonance applies in se veral areas of science and engineering. Resonance occurs in an y system that has a comple x conjugate pair of poles; it is the cause of oscillations of stored ener gy from one form to another . It is the phenomenon that allo ws frequency

discrimination in communications networks. Resonance occurs in any circuit that has at least one inductor and one capacitor.

**Resonance** is a condition in an *RLC* circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.

Resonant circuits (series or parallel) are useful for constructing filters, as their transfer functions can be highly frequency selective. They are used in many applications such as selecting the desired stations in radio and TV receivers.

Consider the series *RLC* circuit shown in Fig. 14.21 in the frequency domain. The input impedance is

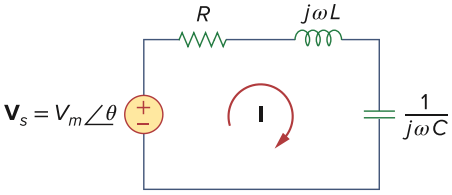



**Figure 14.21**
The series resonant circuit.

$$\mathbf{Z} = \mathbf{H}(\omega) = \frac{\mathbf{V}_s}{\mathbf{I}} = R + j\omega L + \frac{1}{j\omega C} \tag{14.22}$$

or

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{14.23}$$

Resonance results when the imaginary part of the transfer function is zero, or

$$\text{Im}(\mathbf{Z}) = \omega L - \frac{1}{\omega C} = 0 \tag{14.24}$$

The value of $\omega$ that satisfies this condition is called the *resonant frequency* $\omega_0$. Thus, the resonance condition is

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{14.25}$$

or

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}} \tag{14.26}$$

Since $\omega_0 = 2\pi f_0$,

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ Hz} \tag{14.27}$$

Note that at resonance:

1. The impedance is purely resistive, thus, $\mathbf{Z} = R$. In other words, the *LC* series combination acts like a short circuit, and the entire voltage is across $R$.
2. The voltage $\mathbf{V}_s$ and the current $\mathbf{I}$ are in phase, so that the power factor is unity.
3. The magnitude of the transfer function $\mathbf{H}(\omega) = \mathbf{Z}(\omega)$ is minimum.
4. The inductor voltage and capacitor voltage can be much more than the source voltage.

The frequency response of the circuit's current magnitude

$$I = |\mathbf{I}| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \tag{14.28}$$

Note No. 4 becomes evident from the fact that

$$|\mathbf{V}_L| = \frac{V_m}{R}\omega_0 L = QV_m$$

$$|\mathbf{V}_C| = \frac{V_m}{R}\frac{1}{\omega_0 C} = QV_m$$

where $Q$ is the quality factor, defined in Eq. (14.38).

is shown in Fig. 14.22; the plot only shows the symmetry illustrated in this graph when the frequency axis is a logarithm. The average power dissipated by the *RLC* circuit is

$$P(\omega) = \frac{1}{2} I^2 R \tag{14.29}$$

The highest power dissipated occurs at resonance, when $I = V_m/R$, so that

$$P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R} \tag{14.30}$$

At certain frequencies $\omega = \omega_1, \omega_2$, the dissipated power is half the maximum value; that is,

$$P(\omega_1) = P(\omega_2) = \frac{(V_m/\sqrt{2})^2}{2R} = \frac{V_m^2}{4R} \tag{14.31}$$

Hence, $\omega_1$ and $\omega_2$ are called the *half-power frequencies.*

The half-power frequencies are obtained by setting $Z$ equal to $\sqrt{2}R$, and writing

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}\,R \tag{14.32}$$

Solving for $\omega$, we obtain

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$
$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{14.33}$$

We can relate the half-power frequencies with the resonant frequency. From Eqs. (14.26) and (14.33),

$$\omega_0 = \sqrt{\omega_1 \omega_2} \tag{14.34}$$

showing that the resonant frequency is the geometric mean of the half-power frequencies. Notice that $\omega_1$ and $\omega_2$ are in general not symmetrical around the resonant frequency $\omega_0$, because the frequency response is not generally symmetrical. However, as will be explained shortly, symmetry of the half-power frequencies around the resonant frequency is often a reasonable approximation.

Although the height of the curve in Fig. 14.22 is determined by $R$, the width of the curve depends on other factors. The width of the response curve depends on the *bandwidth B*, which is defined as the difference between the two half-power frequencies,

$$B = \omega_2 - \omega_1 \tag{14.35}$$

This definition of bandwidth is just one of several that are commonly used. Strictly speaking, $B$ in Eq. (14.35) is a half-power bandwidth, because it is the width of the frequency band between the half-power frequencies.

The "sharpness" of the resonance in a resonant circuit is measured quantitatively by the *quality factor Q*. At resonance, the reactive energy

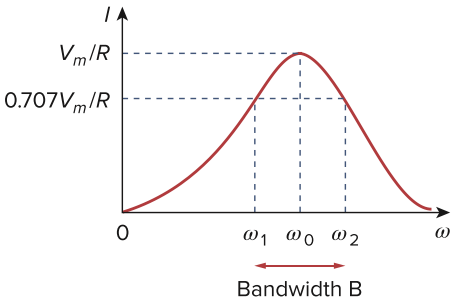



**Figure 14.22**
The current amplitude versus frequency for the series resonant circuit of Fig. 14.21.

Although the same symbol $Q$ is used for the reactive power, the two are not equal and should not be confused. $Q$ here is dimensionless, whereas reactive power $Q$ is in VAR. This may help distinguish between the two.

in the circuit oscillates between the inductor and the capacitor. The quality factor relates the maximum or peak energy stored to the energy dissipated in the circuit per cycle of oscillation:

$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}} \tag{14.36}$$

It is also regarded as a measure of the energy storage property of a circuit in relation to its energy dissipation property. In the series $RLC$ circuit, the peak energy stored is $\frac{1}{2}LI^2$, while the energy dissipated in one period is $\frac{1}{2}(I^2R)(1/f_0)$. Hence,

$$Q = 2\pi \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2R(1/f_0)} = \frac{2\pi f_0 L}{R} \tag{14.37}$$

or

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}} \tag{14.38}$$

Notice that the quality factor is dimensionless. The relationship between the bandwidth $B$ and the quality factor $Q$ is obtained by substituting Eq. (14.33) into Eq. (14.35) and utilizing Eq. (14.38).

$$\boxed{B = \frac{R}{L} = \frac{\omega_0}{Q}} \tag{14.39}$$

or $B = \omega_0^2 CR$. Thus,

The quality factor of a resonant circuit is the ratio of its resonant frequency to its bandwidth.

Keep in mind that Eqs. (14.33), (14.38), and (14.39) only apply to a series $RLC$ circuit.

As illustrated in Fig. 14.23, the higher the value of $Q$, the more selective the circuit is but the smaller the bandwidth. The *selectivity* of an $RLC$ circuit is the ability of the circuit to respond to a certain frequency and discriminate against all other frequencies. If the band of frequencies to be selected or rejected is narrow, the quality factor of the resonant circuit must be high. If the band of frequencies is wide, the quality factor must be low.

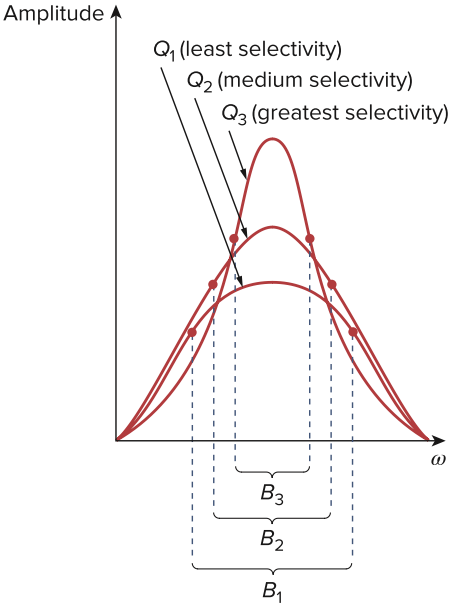



**Figure 14.23**
The higher the circuit $Q$, the smaller the bandwidth.

The quality factor is a measure of the selectivity (or "sharpness" of resonance) of the circuit.

A resonant circuit is designed to operate at or near its resonant frequency. It is said to be a *high-Q circuit* when its quality factor is equal to or greater than 10. For high-$Q$ circuits ($Q \geq 10$), the half-power frequencies are, for all practical purposes, symmetrical around the resonant frequency and can be approximated as

$$\boxed{\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}} \tag{14.40}$$

High-$Q$ circuits are used often in communications networks.

We see that a resonant circuit is characterized by five related parameters: the two half-power frequencies $\omega_1$ and $\omega_2$, the resonant frequency $\omega_0$, the bandwidth $B$, and the quality factor $Q$.

## Example 14.7

In the circuit of Fig. 14.24, $R = 2\ \Omega$, $L = 1$ mH, and $C = 0.4\ \mu$F. (a) Find the resonant frequency and the half-power frequencies. (b) Calculate the quality factor and bandwidth. (c) Determine the amplitude of the current at $\omega_0$, $\omega_1$, and $\omega_2$.

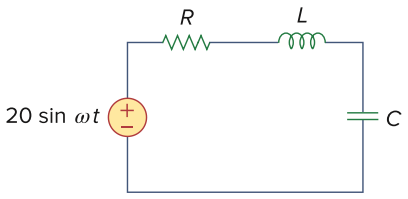



**Figure 14.24**
For Example 14.7.

**Solution:**

(a) The resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

**METHOD 1** The lower half-power frequency is

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= -\frac{2}{2 \times 10^{-3}} + \sqrt{(10^3)^2 + (50 \times 10^3)^2}$$

$$= -1 + \sqrt{1 + 2500} \text{ krad/s} = 49 \text{ krad/s}$$

Similarly, the upper half-power frequency is

$$\omega_2 = 1 + \sqrt{1 + 2500} \text{ krad/s} = 51 \text{ krad/s}$$

(b) The bandwidth is

$$B = \omega_2 - \omega_1 = 2 \text{ krad/s}$$

or

$$B = \frac{R}{L} = \frac{2}{10^{-3}} = 2 \text{ krad/s}$$

The quality factor is

$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25$$

**METHOD 2** Alternatively, we could find

$$Q = \frac{\omega_0 L}{R} = \frac{50 \times 10^3 \times 10^{-3}}{2} = 25$$

From $Q$, we find

$$B = \frac{\omega_0}{Q} = \frac{50 \times 10^3}{25} = 2 \text{ krad/s}$$

Since $Q > 10$, this is a high-$Q$ circuit and we can obtain the half-power frequencies as

$$\omega_1 = \omega_0 - \frac{B}{2} = 50 - 1 = 49 \text{ krad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 50 + 1 = 51 \text{ krad/s}$$

as obtained earlier.

(c) At $\omega = \omega_0$,

$$I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$$

At $\omega = \omega_1, \omega_2$,

$$I = \frac{V_m}{\sqrt{2}\,R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$$

## Practice Problem 14.7

A series-connected circuit has $R = 4\ \Omega$ and $L = 25$ mH. (a) Calculate the value of $C$ that will produce a quality factor of 50. (b) Find $\omega_1$, $\omega_2$, and $B$. (c) Determine the average power dissipated at $\omega = \omega_0, \omega_1, \omega_2$. Take $V_m = 100$ V.

**Answer:** (a) 0.625 $\mu$F, (b) 7920 rad/s, 8080 rad/s, 160 rad/s, (c) 1.25 kW, 0.625 kW, 0.625 kW.

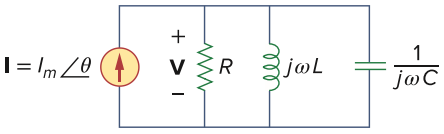



**Figure 14.25**
The parallel resonant circuit.

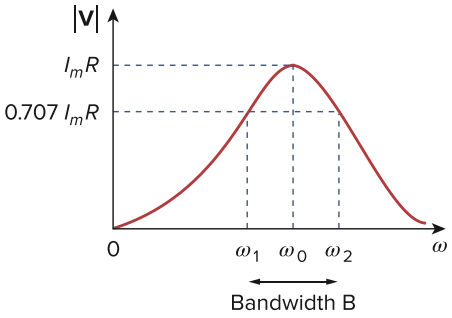



**Figure 14.26**
The current amplitude versus frequency for the series resonant circuit of Fig. 14.25.

## 14.6 Parallel Resonance

The parallel *RLC* circuit in Fig. 14.25 is the dual of the series *RLC* circuit. So we will avoid needless repetition. The admittance is

$$\mathbf{Y} = H(\omega) = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L} \tag{14.41}$$

or

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \tag{14.42}$$

Resonance occurs when the imaginary part of $\mathbf{Y}$ is zero,

$$\omega C - \frac{1}{\omega L} = 0 \tag{14.43}$$

or

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}} \tag{14.44}$$

which is the same as Eq. (14.26) for the series resonant circuit. The voltage $|\mathbf{V}|$ is sketched in Fig. 14.26 as a function of frequenc y. Notice that at resonance, the parallel *LC* combination acts like an open circuit, so that the entire current flows through $R$. Also, the inductor and capacitor current can be much more than the source current at resonance.

We exploit the duality between Figs. 14.21 and 14.25 by comparing Eq. (14.42) with Eq. (14.23). By replacing $R$, $L$, and $C$ in the expressions

We can see this from the fact that

$$|\mathbf{I}_L| = \frac{I_m R}{\omega_0 L} = QI_m$$

$$|\mathbf{I}_C| = \omega_0 C I_m R = QI_m$$

where $Q$ is the quality factor, defined in Eq. (14.47).

for the series circuit with $1/R$, $C$, and $L$ respectively, we obtain for the parallel circuit

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$
$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{14.45}$$

$$B = \omega_2 - \omega_1 = \frac{1}{RC} \tag{14.46}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L} \tag{14.47}$$

It should be noted that Eqs. (14.45) to (14.47) apply only to a parallel *RLC* circuit. Using Eqs. (14.45) and (14.47), we can e xpress the half-power frequencies in terms of the quality factor. The result is

$$\omega_1 = \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} - \frac{\omega_0}{2Q}, \qquad \omega_2 = \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} + \frac{\omega_0}{2Q} \tag{14.48}$$

Again, for high-$Q$ circuits ($Q \geq 10$)

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2} \tag{14.49}$$

Table 14.4 presents a summary of the characteristics of the series and parallel resonant circuits. Besides the series and parallel*RLC* considered here, other resonant circuits exist. Example 14.9 treats a typical example.

**TABLE 14.4**

Summary of the characteristics of resonant *RLC* circuits.

| Characteristic | Series circuit | Parallel circuit |
|---|---|---|
| Resonant frequency, $\omega_0$ | $\frac{1}{\sqrt{LC}}$ | $\frac{1}{\sqrt{LC}}$ |
| Quality factor, $Q$ | $\frac{\omega_0 L}{R}$ or $\frac{1}{\omega_0 RC}$ | $\frac{R}{\omega_0 L}$ or $\omega_0 RC$ |
| Bandwidth, $B$ | $\frac{\omega_0}{Q}$ | $\frac{\omega_0}{Q}$ |
| Half-power frequencies, $\omega_1$, $\omega_2$ | $\omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$ | $\omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$ |
| For $Q \geq 10$, $\omega_1$, $\omega_2$ | $\omega_0 \pm \frac{B}{2}$ | $\omega_0 \pm \frac{B}{2}$ |

## Example 14.8

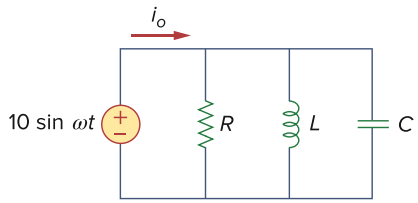



**Figure 14.27**
For Example 14.8.

In the parallel *RLC* circuit of Fig. 14.27, let $R = 8\ \text{k}\Omega$, $L = 0.2$ mH, and $C = 8\ \mu\text{F}$. (a) Calculate $\omega_0$, $Q$, and $B$. (b) Find $\omega_1$ and $\omega_2$. (c) Determine the power dissipated at $\omega_0$, $\omega_1$, and $\omega_2$.

**Solution:**

(a)

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \times 10^{-3} \times 8 \times 10^{-6}}} = \frac{10^5}{4} = 25\ \text{krad/s}$$

$$Q = \frac{R}{\omega_0 L} = \frac{8 \times 10^3}{25 \times 10^3 \times 0.2 \times 10^{-3}} = 1{,}600$$

$$B = \frac{\omega_0}{Q} = 15.625\ \text{rad/s}$$

(b) Due to the high value of $Q$, we can regard this as a high- $Q$ circuit, Hence,

$$\omega_1 = \omega_0 - \frac{B}{2} = 25{,}000 - 7.812 = 24{,}992\ \text{rad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 25{,}000 + 7.812 = 25{,}008\ \text{rad/s}$$

(c) At $\omega = \omega_0$, $\mathbf{Y} = 1/R$ or $\mathbf{Z} = R = 8\ \text{k}\Omega$. Then

$$\mathbf{I}_o = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{10\angle -90^\circ}{8{,}000} = 1.25\angle -90^\circ\ \text{mA}$$

Since the entire current flows through $R$ at resonance, the average power dissipated at $\omega = \omega_0$ is

$$P = \frac{1}{2}|\mathbf{I}_o|^2 R = \frac{1}{2}(1.25 \times 10^{-3})^2 (8 \times 10^3) = 6.25\ \text{mW}$$

or

$$P = \frac{V_m^2}{2R} = \frac{100}{2 \times 8 \times 10^3} = 6.25\ \text{mW}$$

At $\omega = \omega_1, \omega_2$,

$$P = \frac{V_m^2}{4R} = 3.125\ \text{mW}$$

## Practice Problem 14.8

A parallel resonant circuit has $R = 100\ \text{k}\Omega$, $L = 50$ mH, and $C = 2$ nF. Calculate $\omega_0$, $\omega_1$, $\omega_2$, $Q$, and $B$.

**Answer:** 100 krad/s, 97.5 krad/s,102.5 krad/s, 20, 5 krad/s.

## Example 14.9

Determine the resonant frequency of the circuit in Fig. 14.28.

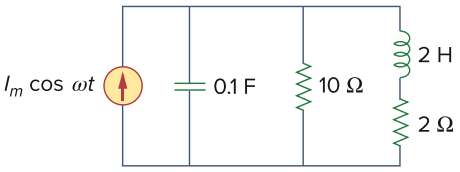



**Figure 14.28**
For Example 14.9.

**Solution:**
The input admittance is

$$\mathbf{Y} = j\omega 0.1 + \frac{1}{10} + \frac{1}{2 + j\omega 2} = 0.1 + j\omega 0.1 + \frac{2 - j\omega 2}{4 + 4\omega^2}$$

At resonance, $\text{Im}(\mathbf{Y}) = 0$ and

$$\omega_0 0.1 - \frac{2\omega_0}{4 + 4\omega_0^2} = 0 \qquad \Rightarrow \qquad \omega_0 = 2 \text{ rad/s}$$

## Practice Problem 14.9

Calculate the resonant frequency of the circuit in Fig. 14.29.

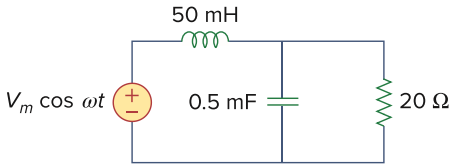



**Figure 14.29**
For Practice Prob. 14.9

**Answer:** 173.21 rad/s.

## 14.7 Passive Filters

The concept of filters has been an integral part of the evolution of electrical engineering from the beginning. Several technological achievements would not have been possible without electrical filters. Because of this prominent role of filters, much effort has been expended on the theory, design, and construction of filters and many articles and books have been written on them. Our discussion in this chapter should be considered introductory.

A filter is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.

As a frequency-selective device, a filter can be used to limit the frequency spectrum of a signal to some specified band of frequencies. Filters are the circuits used in radio and TV receivers to allow us to select one desired signal out of a multitude of broadcast signals in the environment.

A filter is a *passive filter* if it consists of only passive elements $R$, $L$, and $C$. It is said to be an *active filter* if it consists of active elements (such as transistors and op amps) in addition to passive elements $R$, $L$, and $C$. We consider passive filters in this section and active filters in the next section. $LC$ filters have been used in practical applications for more than eight decades. $LC$ filter technology feeds related areas such as equalizers, impedance-matching networks, transformers, shaping networks, power dividers, attenuators, and directional couplers, and is continuously providing practicing engineers with opportunities to innovate and experiment. Besides the $LC$ filters we study in these sections, there are other kinds of filters—such as digital filters, electromechanical filters, and microwave filters—which are beyond the level of this text.

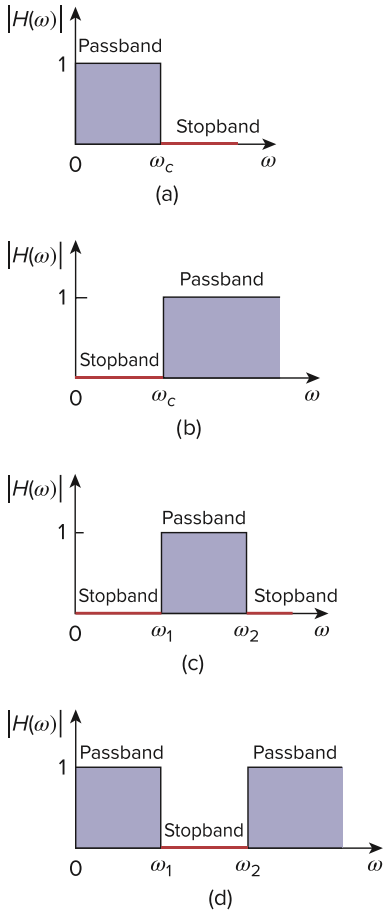



**Figure 14.30**
Ideal frequency response of four types of filters: (a) low-pass filter, (b) high-pass filter, (c) band-pass filter, (d) band-stop filter.

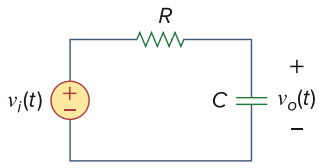



**Figure 14.31**
A low-pass filter.

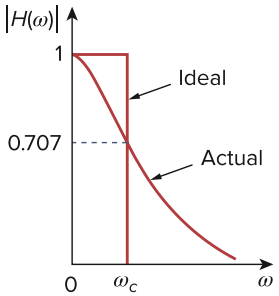



**Figure 14.32**
Ideal and actual frequency response of a low-pass filter.

As shown in Fig. 14.30, there are four types of filters whether passive or active:

1. A *low-pass filter* passes low frequencies and stops high frequencies, as shown ideally in Fig. 14.30(a).
2. A *high-pass filter* passes high frequencies and rejects low frequencies, as shown ideally in Fig. 14.30(b).
3. A *band-pass filter* passes frequencies within a frequenc y band and blocks or attenuates frequencies outside the band, as sho wn ideally in Fig. 14.30(c).
4. A *band-stop filter* passes frequencies outside a frequenc y band and blocks or attenuates frequencies within the band, as sho wn ideally in Fig. 14.30(d).

Table 14.5 presents a summary of the characteristics of these filters. Be aware that the characteristics in Table 14.5 are only valid for first- or second-order filters—but one should not have the impression that only these kinds of filter exist. We now consider typical circuits for realizing the filters shown in Table 14.5.

**TABLE 14.5**

Summary of the characteristics of ideal filters.

| Type of Filter | $H(0)$ | $H(\infty)$ | $H(\omega_c)$ or $H(\omega_0)$ |
|---|---|---|---|
| Low-pass | 1 | 0 | $1/\sqrt{2}$ |
| High-pass | 0 | 1 | $1/\sqrt{2}$ |
| Band-pass | 0 | 0 | 1 |
| Band-stop | 1 | 1 | 0 |

$\omega_c$ is the cutoff frequency for low-pass and high-pass filters; $\omega_0$ is the center frequency for band-pass and band-stop filters.

## 14.7.1 Low-Pass Filter

A typical low-pass filter is formed when the output of an *RC* circuit is taken off the capacitor as shown in Fig. 14.31. The transfer function (see also Example 14.1) is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$\mathbf{H}(\omega) = \frac{1}{1 + j\omega RC} \tag{14.50}$$

Note that $\mathbf{H}(0) = 1$, $\mathbf{H}(\infty) = 0$. Figure 14.32 shows the plot of $|H(\omega)|$, along with the ideal characteristic. The half-power frequency, which is equivalent to the corner frequency on the Bode plots but in the context of filters is usually known as the *cutoff frequency* $\omega_c$, is obtained by setting the magnitude of $\mathbf{H}(\omega)$ equal to $1/\sqrt{2}$, thus,

$$H(\omega_c) = \frac{1}{\sqrt{1 + \omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}}$$

or

$$\omega_c = \frac{1}{RC} \tag{14.51}$$

The cutoff frequency is also called the *rolloff frequency*.

A low-pass filter is designed to pass only frequencies from dc up to the cutoff frequency $\omega_c$.

A low-pass filter can also be formed when the output of an *RL* circuit is taken off the resistor. Of course, there are many other circuits for low-pass filters.

The cutoff frequency is the frequency at which the transfer function **H** drops in magnitude to 70.71% of its maximum value. It is also regarded as the frequency at which the power dissipated in a circuit is half of its maximum value.

### 14.7.2 High-Pass Filter

A high-pass filter is formed when the output of an *RC* circuit is taken off the resistor as shown in Fig. 14.33. The transfer function is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R}{R + 1/j\omega C}$$

$$\mathbf{H}(\omega) = \frac{j\omega RC}{1 + j\omega RC} \tag{14.52}$$

Note that $\mathbf{H}(0) = 0$, $\mathbf{H}(\infty) = 1$. Figure 14.34 shows the plot of $|H(\omega)|$. Again, the corner or cutoff frequency is

$$\omega_c = \frac{1}{RC} \tag{14.53}$$

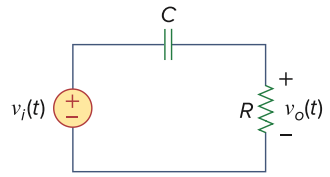



**Figure 14.33**
A high-pass filter.

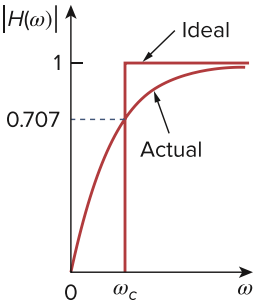



**Figure 14.34**
Ideal and actual frequency response of a high-pass filter.

A high-pass filter is designed to pass all frequencies above its cutoff frequency $\omega_c$.

A high-pass filter can also be formed when the output of an *RL* circuit is taken off the inductor.

### 14.7.3 Band-Pass Filter

The *RLC* series resonant circuit provides a band-pass filter when the output is taken off the resistor as shown in Fig. 14.35. The transfer function is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R}{R + j(\omega L - 1/\omega C)} \tag{14.54}$$

We observe that $\mathbf{H}(0) = 0$, $\mathbf{H}(\infty) = 0$. Figure 14.36 shows the plot of $|H(\omega)|$. The band-pass filter passes a band of frequencies $(\omega_1 < \omega < \omega_2)$ centered on $\omega_0$, the center frequency, which is given by

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{14.55}$$

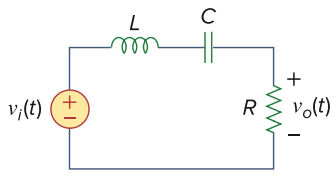



**Figure 14.35**
A band-pass filter.

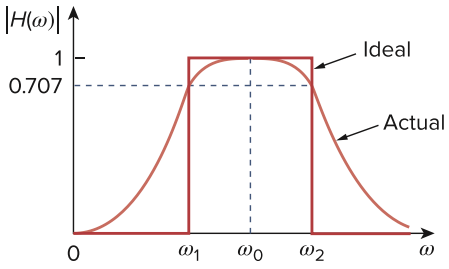



**Figure 14.36**
Ideal and actual frequency response of a band-pass filter.

A band-pass filter is designed to pass all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$.

Because the band-pass filter in Fig. 14.35 is a series resonant circuit, the half-power frequencies, the bandwidth, and the quality factor are determined as in Section 14.5. A band-pass filter can also be formed by cascading the low-pass filter (where $\omega_2 = \omega_c$) in Fig. 14.31 with the

high-pass filter (where $\omega_1 = \omega_c$) in Fig. 14.33. However, the result would not be the same as just adding the output of the low-pass filter to the input of the high-pass filter, because one circuit loads the other and alters the desired transfer function.

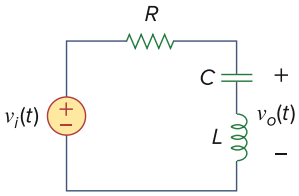



**Figure 14.37**
A band-stop filter.

### 14.7.4 Band-Stop Filter

A filter that prevents a band of frequencies between two designated values ($\omega_1$ and $\omega_2$) from passing is variably known as a *band-stop, band-reject*, or *notch* filter. A band-stop filter is formed when the output *RLC* series resonant circuit is taken off the*LC* series combination as shown in Fig. 14.37. The transfer function is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{j(\omega L - 1/\omega C)}{R + j(\omega L - 1/\omega C)} \tag{14.56}$$

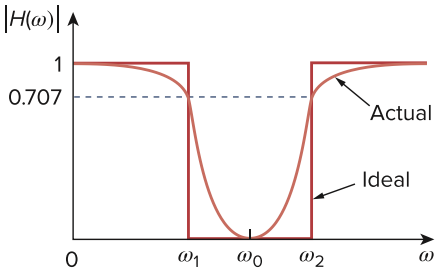



**Figure 14.38**
Ideal and actual frequency response of a band-stop filter.

Notice that $\mathbf{H}(0) = 1$, $\mathbf{H}(\infty) = 1$. Figure 14.38 shows the plot of $|H(\omega)|$. Again, the center frequency is given by

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{14.57}$$

while the half-power frequencies, the bandwidth, and the qualityfactor are calculated using the formulas in Section 14.5 for a series reso - nant circuit. Here, $\omega_0$ is called the *frequency of rejection*, while the corresponding bandwidth ($B = \omega_2 - \omega_1$) is known as the *bandwidth of rejection*. Thus,

A band-stop filter is designed to stop or eliminate all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$.

Notice that adding the transfer functions of the band-pass and the band-stop gives unity at any frequency for the same values of *R, L,* and *C*. Of course, this is not true in general b ut true for the circuits treated here. This is due to the fact that the characteristic of one is the inverse of the other.

In concluding this section, we should note that:

1. From Eqs. (14.50), (14.52), (14.54), and (14.56), the maximum gain of a passive filter is unity. To generate a gain greater than unity, one should use an active filter as the next section shows.
2. There are other ways to get the types of filters treated in this section.
3. The filters treated here are the simple types. Many other filters have sharper and complex frequency responses.

## Example 14.10

Determine what type of filter is shown in Fig. 14.39. Calculate the corner or cutoff frequency. Take $R = 2\ \text{k}\Omega$, $L = 2\ \text{H}$, and $C = 2\ \mu\text{F}$.

**Solution:**
The transfer function is

$$\mathbf{H}(s) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R \| 1/sC}{sL + R \| 1/sC}, \qquad s = j\omega \tag{14.10.1}$$

But

$$R \parallel \frac{1}{sC} = \frac{R/sC}{R + 1/sC} = \frac{R}{1 + sRC}$$

Substituting this into Eq. (14.10.1) gives

$$\mathbf{H}(s) = \frac{R/(1 + sRC)}{sL + R/(1 + sRC)} = \frac{R}{s^2RLC + sL + R}, \qquad s = j\omega$$

or

$$\mathbf{H}(\omega) = \frac{R}{-\omega^2RLC + j\omega L + R} \tag{14.10.2}$$

Because $\mathbf{H}(0) = 1$ and $\mathbf{H}(\infty) = 0$, we conclude from Table 14.5 that the circuit in Fig. 14.39 is a second-order low-pass filter. The magnitude of $\mathbf{H}$ is

$$H = \frac{R}{\sqrt{(R - \omega^2RLC)^2 + \omega^2L^2}} \tag{14.10.3}$$

The corner frequency is the same as the half-power frequency, that is, where $\mathbf{H}$ is reduced by a factor of $1/\sqrt{2}$. Because the dc value of $H(\omega)$ is 1, at the corner frequency, Eq. (14.10.3) becomes after squaring

$$H^2 = \frac{1}{2} = \frac{R^2}{(R - \omega_c^2RLC)^2 + \omega_c^2L^2}$$

or

$$2 = (1 - \omega_c^2LC)^2 + \left(\frac{\omega_cL}{R}\right)^2$$

Substituting the values of $R$, $L$, and $C$, we obtain

$$2 = (1 - \omega_c^2 4 \times 10^{-6})^2 + (\omega_c 10^{-3})^2$$

Assuming that $\omega_c$ is in krad/s,

$$2 = (1 - 4\omega_c^2)^2 + \omega_c^2 \qquad \text{or} \qquad 16\omega_c^4 - 7\omega_c^2 - 1 = 0$$

Solving the quadratic equation in $\omega_c^2$, we get $\omega_c^2 = 0.5509$ and $-0.1134$. Because $\omega_c$ is real,

$$\omega_c = 0.742 \text{ krad/s} = 742 \text{ rad/s}$$

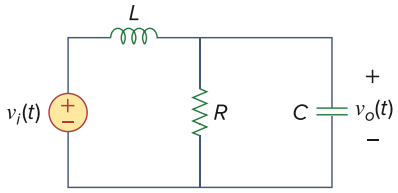



**Figure 14.39**
For Example 14.10.

## Practice Problem 14.10

For the circuit in Fig. 14.40, obtain the transfer function $\mathbf{V}_o(\omega)/\mathbf{V}_i(\omega)$. Identify the type of filter the circuit represents and determine the corner frequency. Take $R_1 = 100\ \Omega = R_2$, $L = 2$ mH.

**Answer:** $\dfrac{R_2}{R_1 + R_2}\left(\dfrac{j\omega}{j\omega + \omega_c}\right)$, high-pass filter

$\omega_c = \dfrac{R_1R_2}{(R_1 + R_2)L} = 25$ krad/s.

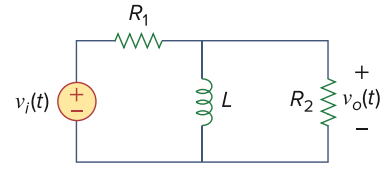



**Figure 14.40**
For Practice Prob. 14.10.

## Example 14.11

If the band-stop filter in Fig. 14.37 is to reject a 200-Hz sinusoid while passing other frequencies, calculate the values of $L$ and $C$. Take $R = 150\ \Omega$ and the bandwidth as 100 Hz.

**Solution:**
We use the formulas for a series resonant circuit in Section 14.5.

$$B = 2\pi(100) = 200\pi \text{ rad/s}$$

But

$$B = \frac{R}{L} \quad \Rightarrow \quad L = \frac{R}{B} = \frac{150}{200\pi} = 0.2387 \text{ H}$$

Rejection of the 200-Hz sinusoid means that $f_0$ is 200 Hz, so that $\omega_0$ in Fig. 14.38 is

$$\omega_0 = 2\pi f_0 = 2\pi(200) = 400\pi$$

Given that $\omega_0 = 1/\sqrt{LC}$,

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{(400\pi)^2(0.2387)} = 2.653\ \pi\text{F}$$

## Practice Problem 14.11

Design a band-pass filter of the form in Fig. 14.35 with a lower cutoff frequency of 20.1 kHz and an upper cutoff frequency of 20.3 kHz. Take $R = 30\ \text{k}\Omega$. Calculate $L$, $C$, and $Q$.

**Answer:** 23.87 H, 2.6 pF, 101.

## 14.8 Active Filters

There are three major limitations to the passive filters considered in the previous section. First, they cannot generate gain greater than 1; passive elements cannot add energy to the network. Second, they may require bulky and expensive inductors. Third, they perform poorly at frequencies below the audio frequency range (300 Hz $< f <$ 3,000Hz). Nevertheless, passive filters are useful at high frequencies.

Active filters consist of combinations of resistors, capacitors, and op amps. They offer some advantages over passive $RLC$ filters. First, they are often smaller and less expensive, because they do not require inductors. This makes feasible the integrated circuit realizations of filters. Second, they can provide amplifier gain in addition to providing the same frequency response as $RLC$ filters. Third, active filters can be combined with buffer amplifiers (voltage followers) to isolate each stage of the filter from source and load impedance effects. This isolation allows designing the stages independently and then cascading them to realize the desired transfer function. (Bode plots, being logarithmic, may be added when transfer functions are cascaded.) However, active filters are less reliable and less stable. The practical limit

of most active filters is about 100 kHz—most active filters operate well below that frequency.

Filters are often classified according to their order (or number of poles) or their specific design type.

### 14.8.1 First-Order Low-Pass Filter

One type of first-order filter is shown in Fig. 14.41. The components selected for $Z_i$ and $Z_f$ determine whether the filter is low-pass or high-pass, but one of the components must be reactive.

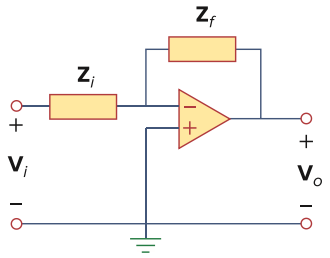



**Figure 14.41**
A general first-order active filter.

Figure 14.42 shows a typical active low-pass filter. For this filter, the transfer function is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = -\frac{\mathbf{Z}_f}{\mathbf{Z}_i} \tag{14.58}$$

where $\mathbf{Z}_i = R_i$ and

$$\mathbf{Z}_f = R_f \parallel \frac{1}{j\omega C_f} = \frac{R_f / j\omega C_f}{R_f + 1/j\omega C_f} = \frac{R_f}{1 + j\omega C_f R_f} \tag{14.59}$$

Therefore,

$$\mathbf{H}(\omega) = -\frac{R_f}{R_i}\frac{1}{1 + j\omega C_f R_f} \tag{14.60}$$

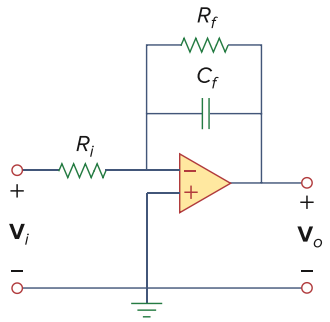



**Figure 14.42**
Active first-order low-pass filter.

We notice that Eq. (14.60) is similar to Eq. (14.50), except that there is a low frequency ($\omega \to 0$) gain or dc gain of $-R_f/R_i$. Also, the corner frequency is

$$\omega_c = \frac{1}{R_f C_f} \tag{14.61}$$

which does not depend on $R_i$. This means that several inputs with different $R_i$ could be summed if required, and the corner frequency would remain the same for each input.

### 14.8.2 First-Order High-Pass Filter

Figure 14.43 shows a typical high-pass filter. As before,

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = -\frac{\mathbf{Z}_f}{\mathbf{Z}_i} \tag{14.62}$$

where $\mathbf{Z}_i = R_i + 1/j\omega C_i$ and $\mathbf{Z}_f = R_f$ so that

$$\mathbf{H}(\omega) = -\frac{R_f}{R_i + 1/j\omega C_i} = -\frac{j\omega C_i R_f}{1 + j\omega C_i R_i} \tag{14.63}$$

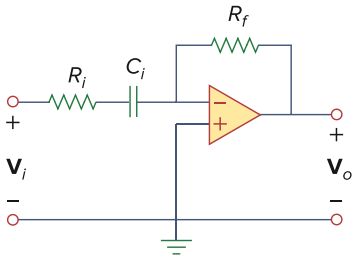



**Figure 14.43**
Active first-order high-pass filter.

This is similar to Eq. (14.52), except that at very high frequencies ($\omega \to \infty$), the gain tends to $-R_f/R_i$. The corner frequency is

$$\omega_c = \frac{1}{R_i C_i} \tag{14.64}$$

### 14.8.3 Band-Pass Filter

The circuit in Fig. 14.42 may be combined with that in Fig. 14.43 to form a band-pass filter that will have a gain $K$ over the required range of frequencies. By cascading a unity-gain low-pass filter, a unity-gain

This way of creating a band-pass filter, not necessarily the best, is perhaps the easiest to understand.

high-pass filter, and an inverter with gain $-R_f/R_i$, as shown in the block diagram of Fig. 14.44(a), we can construct a band-pass filter whose frequency response is that in Fig. 14.44(b). The actual construction of the band-pass filter is shown in Fig. 14.45.

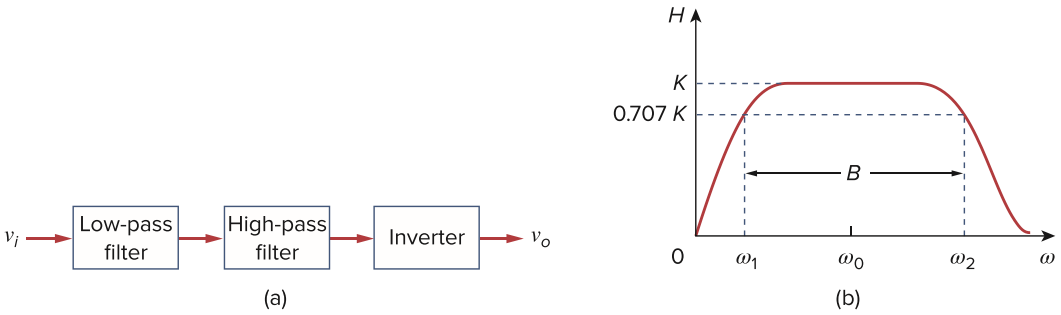



**Figure 14.44**
Active band-pass filter: (a) block diagram, (b) frequency response.

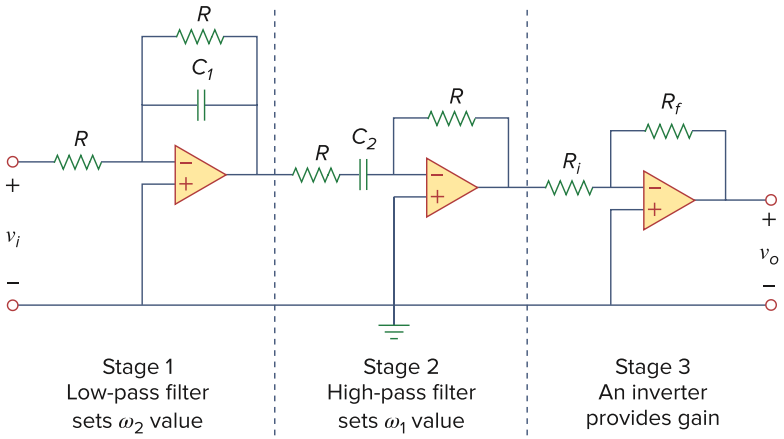



**Figure 14.45**
Active band-pass filter.

The analysis of the band-pass filter is relatively simple. Its transfer function is obtained by multiplying Eqs. (14.60) and (14.63) with the gain of the inverter; that is,

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \left(-\frac{1}{1+j\omega C_1 R}\right)\left(-\frac{j\omega C_2 R}{1+j\omega C_2 R}\right)\left(-\frac{R_f}{R_i}\right)$$
$$= -\frac{R_f}{R_i}\frac{1}{1+j\omega C_1 R}\frac{j\omega C_2 R}{1+j\omega C_2 R} \tag{14.65}$$

The low-pass section sets the upper corner frequency as

$$\omega_2 = \frac{1}{RC_1} \tag{14.66}$$

while the high-pass section sets the lower corner frequency as

$$\omega_1 = \frac{1}{RC_2} \tag{14.67}$$

With these values of $\omega_1$ and $\omega_2$, the center frequency, bandwidth, and quality factor are found as follows:

$$\omega_0 = \sqrt{\omega_1 \omega_2} \tag{14.68}$$

$$B = \omega_2 - \omega_1 \tag{14.69}$$

$$Q = \frac{\omega_0}{B} \tag{14.70}$$

To find the passband gain $K$, we write Eq. (14.65) in the standard form of Eq. (14.15),

$$\mathbf{H}(\omega) = -\frac{R_f}{R_i} \frac{j\omega/\omega_1}{(1 + j\omega/\omega_1)(1 + j\omega/\omega_2)} = -\frac{Rf}{R_i} \frac{j\omega\omega_2}{(\omega_1 + j\omega)(\omega_2 + j\omega)} \tag{14.71}$$

At the center frequency $\omega_0 = \sqrt{\omega_1 \omega_2}$, the magnitude of the transfer function is

$$|\mathbf{H}(\omega_0)| = \left| \frac{R_f}{R_i} \frac{j\omega_0\omega_2}{(\omega_1 + j\omega_0)(\omega_2 + j\omega_0)} \right| = \frac{R_f}{R_i} \frac{\omega_2}{\omega_1 + \omega_2} \tag{14.72}$$

Thus, the passband gain is

$$K = \frac{R_f}{R_i} \frac{\omega_2}{\omega_1 + \omega_2} \tag{14.73}$$

### 14.8.4 Band-Reject (or Notch) Filter

A band-reject filter may be constructed by parallel combination of a low-pass filter and a high-pass filter and a summing amplifier, as shown in the block diagram of Fig. 14.46(a). The circuit is designed such that the lower cutoff frequency $\omega_1$ is set by the low-pass filter while the upper cutoff frequency $\omega_2$ is set by the high-pass filter. The gap between $\omega_1$ and $\omega_2$ is the bandwidth of the filter. As shown in Fig. 14.46(b), the filter passes frequencies below $\omega_1$ and above $\omega_2$. The block diagram in Fig. 14.46(a) is actually constructed as shown in Fig. 14.47. The transfer function is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = -\frac{R_f}{R_i}\left( -\frac{1}{1 + j\omega C_1 R} - \frac{j\omega C_2 R}{1 + j\omega C_2 R} \right) \tag{14.74}$$

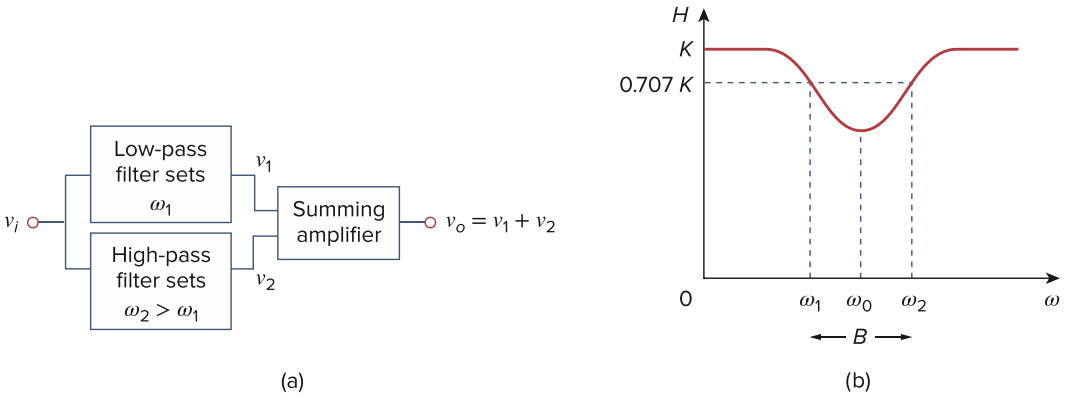



**Figure 14.46**
Active band-reject filter: (a) block diagram, (b) frequency response.

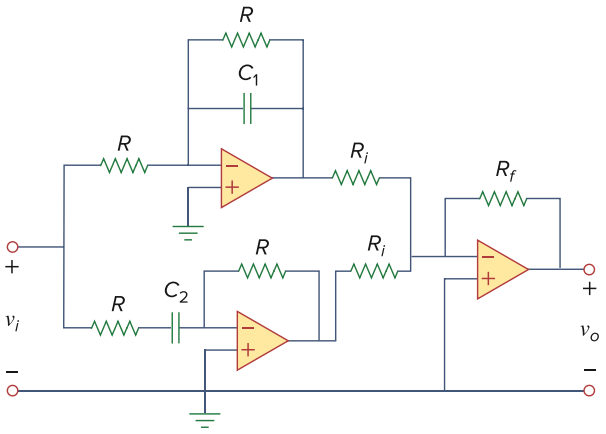



**Figure 14.47**
Active band-reject filter.

The formulas for calculating the values of $\omega_1$, $\omega_2$, the center frequency, bandwidth, and quality factor are the same as in Eqs. (14.66) to (14.70).

To determine the passband gain $K$ of the filter, we can write Eq. (14.74) in terms of the upper and lower corner frequencies as

$$\begin{aligned}\mathbf{H}(\omega) &= \frac{R_f}{R_i}\left(\frac{1}{1+j\omega/\omega_2}+\frac{j\omega/\omega_1}{1+j\omega/\omega_1}\right)\\ &= \frac{R_f}{R_i}\frac{(1+j2\omega/\omega_1+(j\omega)^2/\omega_1\omega_1)}{(1+j\omega/\omega_2)(1+j\omega/\omega_1)}\end{aligned} \tag{14.75}$$

Comparing this with the standard form in Eq. (14.15) indicates that in the two passbands ($\omega \to 0$ and $\omega \to \infty$) the gain is

$$K = \frac{R_f}{R_i} \tag{14.76}$$

We can also find the gain at the center frequency by finding the magnitude of the transfer function at $\omega_0 = \sqrt{\omega_1\omega_2}$, writing

$$\begin{aligned}H(\omega_0) &= \left|\frac{R_f}{R_i}\frac{(1+j2\omega_0/\omega_1+(j\omega_0)^2/\omega_1\omega_1)}{(1+j\omega_0/\omega_2)(1+j\omega_0/\omega_1)}\right|\\ &= \frac{R_f}{R_i}\frac{2\omega_1}{\omega_1+\omega_2}\end{aligned} \tag{14.77}$$

Again, the filters treated in this section are only typical. There are many other active filters that are more complex.

## Example 14.12

Design a low-pass active filter with a dc gain of 4 and a corner frequency of 500 Hz.

**Solution:**
From Eq. (14.61), we find

$$\omega_c = 2\pi f_c = 2\pi(500) = \frac{1}{R_f C_f} \tag{14.12.1}$$

The dc gain is

$$H(0) = -\frac{R_f}{R_i} = -4 \qquad \textbf{(14.12.2)}$$

We have two equations and three unknowns. If we select $C_f = 0.2\ \mu$F, then

$$R_f = \frac{1}{2\pi(500)0.2 \times 10^{-6}} = 1.59\ \text{k}\Omega$$

and

$$R_i = \frac{R_f}{4} = 397.5\ \Omega$$

We use a 1.6-k$\Omega$ resistor for $R_f$ and a 400-$\Omega$ resistor for $R_i$. Figure 14.42 shows the filter.

## Practice Problem 14.12

Design a high-pass filter with a high-frequency gain of 5 and a corner frequency of 2 kHz. Use a 50-nF capacitor in your design.

**Answer:** $R_i = 1{,}600\ \Omega$ and $R_f = 8\ \text{k}\Omega$.

## Example 14.13

Design a band-pass filter in the form of Fig. 14.45 to pass frequencies between 250 and 3,000 Hz and with $K = 10$. Select $R = 20\ \text{k}\Omega$.

**Solution:**

1. **Define.** The problem is clearly stated and the circuit to be used in the design is specified.
2. **Present.** We are ask ed to use the op amp circuit specified in Fig. 14.45 to design a band-pass filter. We are given the value of $R$ to use (20 k$\Omega$). In addition, the frequency range of the signals to be passed is 250 Hz to 3 kHz.
3. **Alternative.** We will use the equations developed in Section 14.8.3 to obtain a solution. We will then use the resulting transfer function to validate the answer.
4. **Attempt.** Because $\omega_1 = 1/RC_2$, we obtain

$$C_2 = \frac{1}{R\omega_1} = \frac{1}{2\pi f_1 R} = \frac{1}{2\pi \times 250 \times 20 \times 10^3} = \textbf{31.83 nF}$$

Similarly, since $\omega_2 = 1/RC_1$,

$$C_1 = \frac{1}{R\omega_2} = \frac{1}{2\pi f_2 R} = \frac{1}{2\pi \times 3{,}000 \times 20 \times 10^3} = \textbf{2.65 nF}$$

From Eq. (14.73),

$$\frac{R_f}{R_i} = K\frac{\omega_1 + \omega_2}{\omega_2} = K\frac{f_1 + f_2}{f_2} = \frac{10(3{,}250)}{3{,}000} = 10.83$$

If we select $R_i = \mathbf{10\ k\Omega}$, then $R_f = 10.83R_i \simeq \mathbf{108.3\ k\Omega}$.

5. **Evaluate.** The output of the first op amp is given by

$$\frac{V_i - 0}{20\ \text{k}\Omega} + \frac{V_1 - 0}{20\ \text{k}\Omega} + \frac{s2.65 \times 10^{-9}(V_1 - 0)}{1} = 0 \rightarrow V_1 = -\frac{V_i}{1 + 5.3 \times 10^{-5}s}$$

The output of the second op amp is given by

$$\frac{V_1 - 0}{20\ \text{k}\Omega + \dfrac{1}{s31.83\ \text{nF}}} + \frac{V_2 - 0}{20\ \text{k}\Omega} = 0 \rightarrow$$

$$V_2 = -\frac{6.366 \times 10^{-4} sV_1}{1 + 6.366 \times 10^{-4}s} = \frac{6.366 \times 10^{-4}\, sV_i}{(1 + 6.366 \times 10^{-4}s)(1 + 5.3 \times 10^{-5}s)}$$

The output of the third op amp is given by

$$\frac{V_2 - 0}{10\ \text{k}\Omega} + \frac{V_o - 0}{108.3\ \text{k}\Omega} = 0 \rightarrow V_o = 10.83V_2 \rightarrow j2\pi \times 25°$$

$$V_o = -\frac{6.894 \times 10^{-3} sV_i}{(1 + 6.366 \times 10^{-4}s)(1 + 5.3 \times 10^{-5}s)}$$

Let $j2\pi \times 25°$ and solve for the magnitude of $V_o/V_i$.

$$\frac{V_o}{V_i} = \frac{-j10.829}{(1 + j1)(1)}$$

$|V_o/V_i| = \mathbf{(0.7071)10.829}$, which is the lower corner frequency point.

Let $s = j2\pi \times 3000 = j18.849\ \text{k}\Omega$. We then get

$$\frac{V_o}{V_i} = \frac{-j129.94}{(1 + j12)(1 + j1)} = \frac{129.94\angle -90°}{(12.042\angle 85.24°)(1.4142\angle 45°)} = \mathbf{(0.7071)10.791\angle -18.61°}$$

Clearly this is the upper corner frequency and the answer checks.

6. **Satisfactory?** We have satisfactorily designed the circuit and can present the results as a solution to the problem.

## Practice Problem 14.13

Design a notch filter based on Fig. 14.47 for $\omega_0 = 20$ krad/s, $K = 5$, and $Q = 10$. Use $R = R_i = 10\ \text{k}\Omega$.

**Answer:** $C_1 = 4.762$ nF, $C_2 = 5.263$ nF, and $R_f = 50\ \text{k}\Omega$.

## 14.9 Scaling

In designing and analyzing filters and resonant circuits or in circuit analysis in general, it is sometimes convenient to work with element values of 1 Ω, 1 H, or 1 F, and then transform the values to realistic values by

*scaling*. We have taken advantage of this idea by not using realistic element values in most of our examples and problems; mastering circuit analysis is made easy by using convenient component values. We have thus eased calculations, knowing that we could use scaling to then make the values realistic.

There are two ways of scaling a circuit: *magnitude* or *impedance scaling*, and *frequency scaling*. Both are useful in scaling responses and circuit elements to values within the practical ranges. While magnitude scaling leaves the frequency response of a circuit unaltered, frequency scaling shifts the frequency response up or down the frequency spectrum.

### 14.9.1 Magnitude Scaling

Magnitude scaling is the process of increasing all impedances in a network by a factor, the frequency response remaining unchanged.

Recall that impedances of individual elements $R$, $L$, and $C$ are given by

$$\mathbf{Z}_R = R, \qquad \mathbf{Z}_L = j\omega L, \qquad \mathbf{Z}_C = \frac{1}{j\omega C} \tag{14.78}$$

In magnitude scaling, we multiply the impedance of each circuit element by a factor $K_m$ and let the frequency remain constant. This gives the new impedances as

$$\mathbf{Z}'_R = K_m\mathbf{Z}_R = K_m R, \qquad \mathbf{Z}'_L = K_m\mathbf{Z}_L = j\omega K_m L$$
$$\mathbf{Z}'_C = K_m\mathbf{Z}_C = \frac{1}{j\omega C/K_m} \tag{14.79}$$

Comparing Eq. (14.79) with Eq. (14.78), we notice the following changes in the element values: $R \to K_m R$, $L \to K_m L$, and $C \to C/K_m$. Thus, in magnitude scaling, the new values of the elements and frequency are

$$\boxed{\begin{aligned} R' &= K_m R, & L' &= K_m L \\ C' &= \frac{C}{K_m}, & \omega' &= \omega \end{aligned}} \tag{14.80}$$

The primed variables are the new values and the unprimed variables are the old values. Consider the series or parallel *RLC* circuit. We now have

$$\omega'_0 = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{K_m LC/K_m}} = \frac{1}{\sqrt{LC}} = \omega_0 \tag{14.81}$$

showing that the resonant frequency, as expected, has not changed. Similarly, the quality factor and the bandwidth are not affected by magnitude scaling. Also, magnitude scaling does not affect transfer functions in the forms of Eqs. (14.2a) and (14.2b), which are dimensionless quantities.

### 14.9.2 Frequency Scaling

Frequency scaling is equivalent to relabeling the frequency axis of a frequency response plot. It is needed when translating frequencies such as a resonant frequency, a corner frequency, a bandwidth, etc., to a realistic level. It can be used to bring capacitance and inductance values into a range that is convenient to work with.

**Frequency scaling** is the process of shifting the frequency response of a network up or down the frequency axis while leaving the impedance the same.

We achieve frequency scaling by multiplying the frequency by a factor $K_f$ while keeping the impedance the same.

From Eq. (14.78), we see that the impedances of $L$ and $C$ are frequency-dependent. If we apply frequenc y scaling to $\mathbf{Z}_L(\omega)$ and $\mathbf{Z}_C(\omega)$ in Eq. (14.78), we obtain

$$\mathbf{Z}_L = j(\omega K_f)L' = j\omega L \quad \Rightarrow \quad L' = \frac{L}{K_f} \tag{14.82a}$$

$$Z_C = \frac{1}{j(\omega K_f)C'} = \frac{1}{j\omega C} \quad \Rightarrow \quad C' = \frac{C}{K_f} \tag{14.82b}$$

since the impedance of the inductor and capacitor must remain the same after frequency scaling. We notice the following changes in the element values: $L \to L/K_f$ and $C \to C/K_f$. The value of $R$ is not affected, since its impedance does not depend on frequency. Thus, in frequency scaling, the new values of the elements and frequency are

$$\begin{aligned} R' &= R, & L' &= \frac{L}{K_f} \\ C' &= \frac{C}{K_f}, & \omega' &= K_f\omega \end{aligned} \tag{14.83}$$

Again, if we consider the series or parallel $RLC$ circuit, for the resonant frequency

$$\omega_0' = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{(L/K_f)(C/K_f)}} = \frac{K_f}{\sqrt{LC}} = K_f\omega_0 \tag{14.84}$$

and for the bandwidth

$$B' = K_f B \tag{14.85}$$

but the quality factor remains the same ($Q' = Q$).

### 14.9.3 Magnitude and Frequency Scaling

If a circuit is scaled in magnitude and frequency at the same time, then

$$\begin{aligned} R' &= K_m R, & L' &= \frac{K_m}{K_f}L \\ C' &= \frac{1}{K_m K_f}C, & \omega' &= K_f\omega \end{aligned} \tag{14.86}$$

These are more general formulas than those in Eqs. (14.80) and (14.83). We set $K_m = 1$ in Eq. (14.86) when there is no magnitude scaling or $K_f = 1$ when there is no frequency scaling.

## Example 14.14

A fourth-order Butterworth low-pass filter is shown in Fig. 14.48(a). The filter is designed such that the cutoff frequency $\omega_c = 1$ rad/s. Scale the circuit for a cutoff frequency of 50 kHz using 10-kΩ resistors.

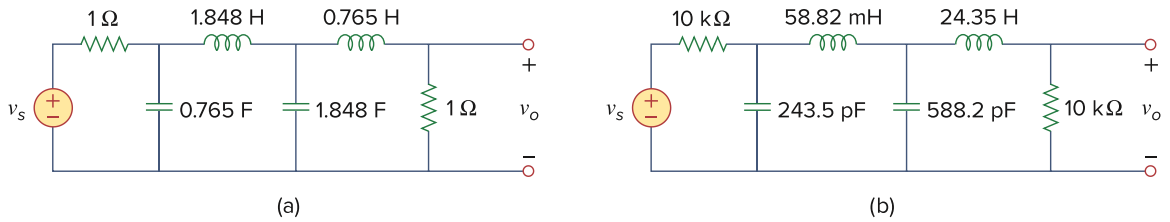



**Figure 14.48**
For Example 14.14: (a) Normalized Butterworth low-pass filter, (b) scaled version of the same low-pass filter.

**Solution:**
If the cutoff frequency is to shift from $\omega_c = 1$ rad/s to $\omega_c' = 2\pi(50)$ krad/s, then the frequency scale factor is

$$K_f = \frac{\omega_c'}{\omega_c} = \frac{100\pi \times 10^3}{1} = \pi \times 10^5$$

Also, if each 1-Ω resistor is to be replaced by a 10-k Ω resistor, then the magnitude scale factor must be

$$K_m = \frac{R'}{R} = \frac{10 \times 10^3}{1} = 10^4$$

Using Eq. (14.86),

$$L_1' = \frac{K_m}{K_f} L_1 = \frac{10^4}{\pi \times 10^5}(1.848) = 58.82 \text{ mH}$$

$$L_2' = \frac{K_m}{K_f} L_2 = \frac{10^4}{\pi \times 10^5}(0.765) = 24.35 \text{ mH}$$

$$C_1' = \frac{C_1}{K_m K_f} = \frac{0.765}{\pi \times 10^9} = 243.5 \text{ pF}$$

$$C_2' = \frac{C_2}{K_m K_f} = \frac{1.848}{\pi \times 10^9} = 588.2 \text{ pF}$$

The scaled circuit is shown in Fig. 14.48(b). This circuit uses practical values and will provide the same transfer function as the prototype in Fig. 14.48(a), but shifted in frequency.

## Practice Problem 14.14

A third-order Butterworth filter normalized to $\omega_c = 1$ rad/s is shown in Fig. 14.49. Scale the circuit to a cutoff frequency of 10 kHz. Use 15-nF capacitors.

**Answer:** $R_1' = R_2' = 1.061$ kΩ, $C_1' = C_2' = 15$ nF, $L' = 33.77$ mH.

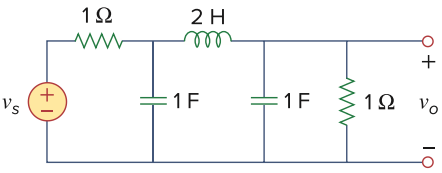



**Figure 14.49**
For Practice Prob. 14.14.

## 14.10 Frequency Response Using *PSpice*

*PSpice* is a useful tool in the hands of the modern circuit designer for obtaining the frequency response of circuits. The frequency response is obtained using the AC Sweep as discussed in Section D.5 (Appendix D). This requires that we specify in the AC Sweep dialog box *Total Pts, Start Freq, End Freq*, and the sweep type. *Total Pts* is the number of points in the frequency sweep, and *Start Freq* and *End Freq* are, respectively, the starting and final frequencies, in hertz. In order to know what frequencies to select for *Start Freq* and *End Freq*, one must have an idea of the frequency range of interest by making a rough sketch of the frequency response. In a complex circuit where this may not be possible, one may use a trial-and-error approach.

There are three types of sweeps:

*Linear:* The frequency is varied linearly from *Start Freq* to *End Freq* with *Total Pts* equally spaced points (or responses).

*Octave:* The frequency is swept logarithmically by octaves from *Start Freq* to *End Freq* with *Total Pts* per octave. An octave is a factor of 2 (e.g., 2 to 4, 4 to 8, 8 to 16).

*Decade:* The frequency is varied logarithmically by decades from *Start Freq* to *End Freq* with *Total Pts* per decade. A decade is a factor of 10 (e.g., from 2 to 20 Hz, 20 to 200 Hz, 200 Hz to 2 kHz).

It is best to use a linear sweep when displaying a narrow frequency range of interest, as a linear sweep displays the frequency range well in a narrow range. Conversely, it is best to use a logarithmic (octave or decade) sweep for displaying a wide frequency range of interest—if a linear sweep is used for a wide range, all the data will be crowded at the high- or low-frequency end and insufficient data at the other end.

With the above specifications, *PSpice* performs a steady-state sinusoidal analysis of the circuit as the frequency of all the independent sources is varied (or swept) from *Start Freq* to *End Freq*.

The *PSpice* A/D program produces a graphical output. The output data type may be specified in the *Trace Command Box* by adding one of the following suffixes to V or I:

M Amplitude of the sinusoid.

P Phase of the sinusoid.

dB Amplitude of the sinusoid in decibels, that is, 20 log (amplitude).

10

### Example 14.15

Determine the frequency response of the circuit shown in Fig. 14.50.

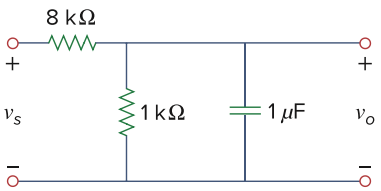



**Figure 14.50**
For Example 14.15.

**Solution:**

We let the input voltage *vs* be a sinusoid of amplitude 1 V and phase 0°. Figure 14.51 is the schematic for the circuit. The capacitor is rotated 270° counterclockwise to ensure that pin 1 (the positive terminal) is on top. The voltage marker is inserted to the output voltage across the capacitor. To perform a linear sweep for $1 < f < 1{,}000$ Hz with 50 points, we select **Analysis/Setup/AC Sweep, DCLICK** *Linear*, type 50 in the *Total Pts* box, type 1 in the *Start Freq* box, and type 1000 in the *End Freq* box. After saving the file, we select **Analysis/Simulate** to simulate the circuit. If there are no errors, the *PSpice* A/D window will

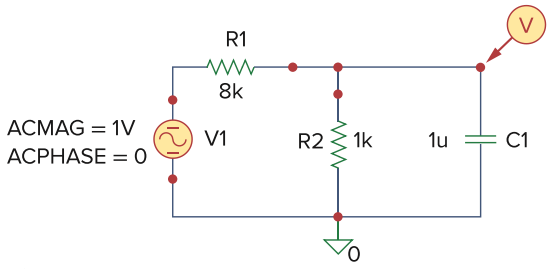



**Figure 14.51**
The schematic for the circuit in Fig. 14.50.

display the plot of V(C1:1), which is the same as $V_o$ or $H(\omega) = V_o/1$, as shown in Fig. 14.52(a). This is the magnitude plot, since V(C1:1) is the same as VM(C1:1). To obtain the phase plot, select **Trace/Add** in the *PSpice* A/D menu and type VP(C1:1) in the **Trace Command** box. Figure 14.52(b) shows the result. By hand, the transfer function is

$$H(\omega) = \frac{V_o}{V_s} = \frac{1{,}000}{9{,}000 + j\omega 8}$$

or

$$H(\omega) = \frac{1}{9 + j16\pi \times 10^{-3}}$$

showing that the circuit is a low-pass filter as demonstrated in Fig. 14.52. Notice that the plots in Fig. 14.52 are similar to those in Fig. 14.3 (note that the horizontal axis in Fig. 14.52 is logrithic while the horizontal axis in Fig. 14.3 is linear.)

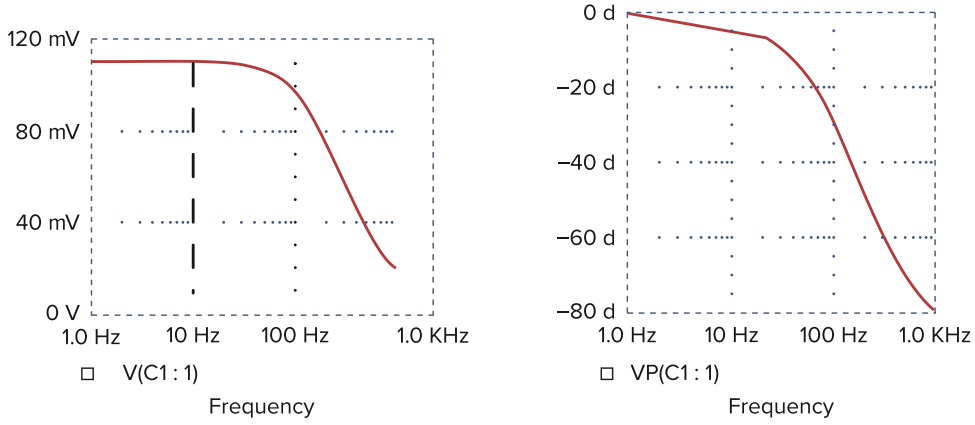



**Figure 14.52**
For Example 14.15: (a) magnitude plot, (b) phase plot of the frequency response.

## Practice Problem 14.15

Obtain the frequency response of the circuit in Fig. 14.53 using *PSpice*. Use a linear frequency sweep and consider 1 $< f <$ 1,000 Hz with 100 points.

**Answer:** See Fig. 14.54.

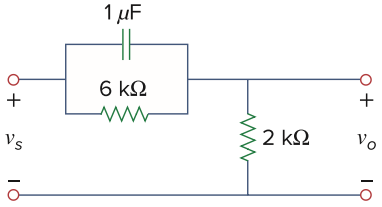



**Figure 14.53**
For Practice Prob. 14.15.

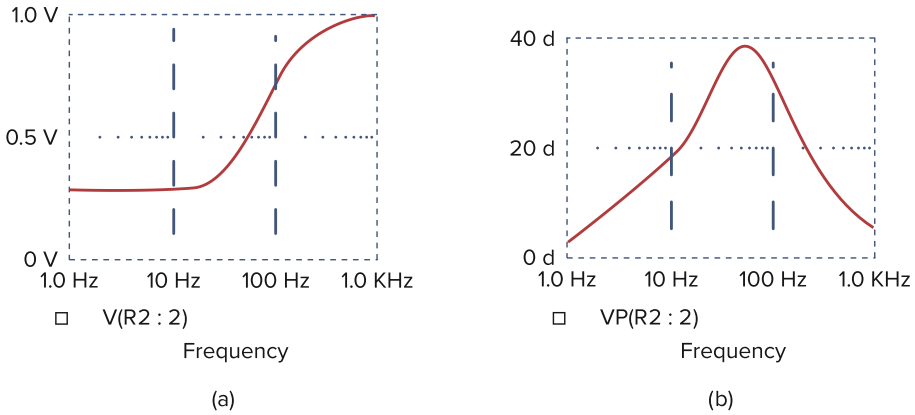



**Figure 14.54**
For Practice Problem 14.15: (a) magnitude plot, (b) phase plot of the frequency response.

## Example 14.16

Use *PSpice* to generate the gain and phase Bode plots of $V$ in the circuit of Fig. 14.55.

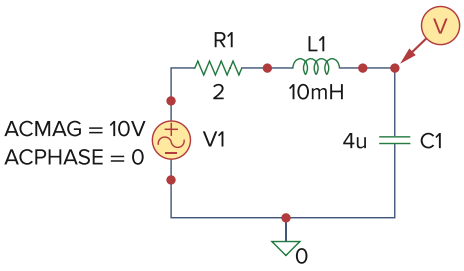



**Figure 14.55**
For Example 14.16.

**Solution:**

The circuit treated in Example 14.15 is first-order while the one in this example is second-order. Since we are interested in Bode plots, we use decade frequency sweep for $300 < f < 3{,}000$ Hz with 50 points per de - cade. We select this range because we know that the resonant frequency of the circuit is within the range. Recall that

$$\omega_0 = \frac{1}{\sqrt{LC}} = 5 \text{ krad/s} \qquad \text{or} \qquad f_0 = \frac{\omega}{2\pi} = 795.8 \text{ Hz}$$

After drawing the circuit as in Fig. 14.55, we select **Analysis/Setup/AC Sweep, DCLICK** *Decade*, enter 50 in the *Total Pts* box, 300 as the *Start Freq*, and 3,000 in the *End Freq* box. Upon saving the file, we simulate it by selecting **Analysis/Simulate**. This will automatically bring up the *PSpice* A/D window and display V(C1:1) if there are no errors. Since we are interested in the Bode plot, we select **Trace/Add** in the *PSpice* A/D menu and type dB(V(C1:1)) in the **Trace Command** box. The result is the Bode magnitude plot in Fig. 14.56(a). For the phase plot, we select

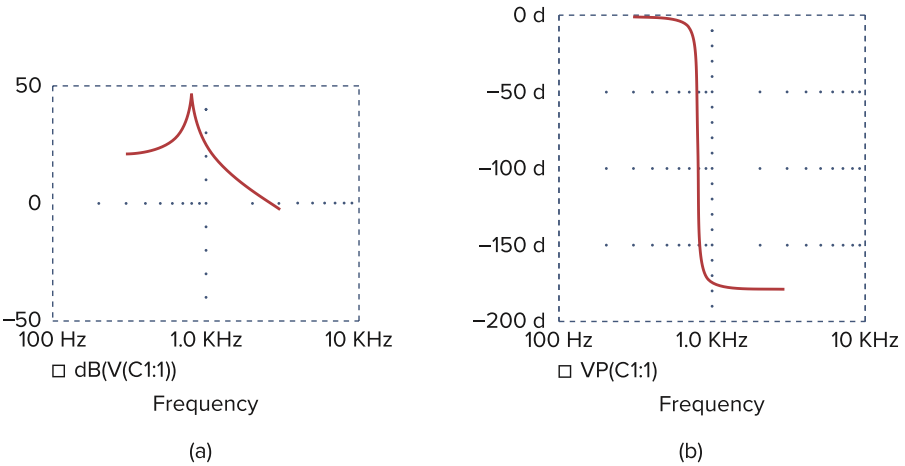



**Figure 14.56**
For Example 14.16: (a) Bode plot, (b) phase plot of the response.

**Trace/Add** in the *PSpice* A/D menu and type VP(C1:1) in the **Trace Command** box. The result is the Bode phase plot of Fig. 14.56(b). Notice that the plots confirm the resonant frequency of 795.8 Hz.

## Practice Problem 14.16

Consider the network in Fig. 14.57. Use *PSpice* to obtain the Bode plots for $V_o$ over a frequency from 1 to 100 kHz using 20 points per decade.

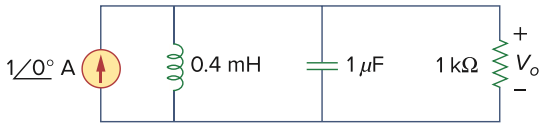



**Figure 14.57**
For Practice Prob. 14.16.

**Answer:** See Fig. 14.58.

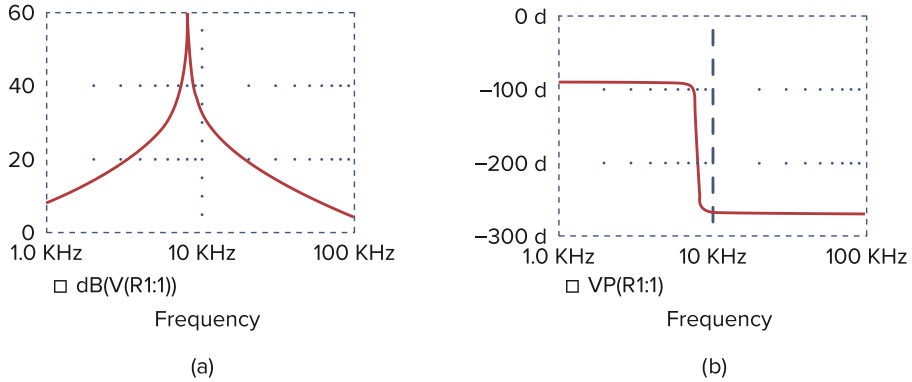



**Figure 14.58**
For Practice Prob. 14.16: Bode (a) magnitude plot, (b) phase plot.

## 14.11 Computation Using *MATLAB*

*MATLAB* is a software package that is widely used for engineering computation and simulation. A review of *MATLAB* is provided in Appendix E for the beginner. This section shows how to use the software to numerically perform most of the operations presented in this chapter and Chapter 15. The key to describing a system in *MATLAB* is to specify the numerator (num) and denominator (den) of the transfer function of the system. Once this is done, we can use several *MATLAB* commands to obtain the system's Bode plots (frequency response) and the system's response to a given input.

The command **bode** produces the Bode plots (both magnitude and phase) of a given transfer function $H(s)$. The format of the command is **bode** (num, den), where num is the numerator of $H(s)$ and den is its denominator. The frequency range and number of points are automatically selected. For example, consider the transfer function in Example 14.3. It is better to first write the numerator and denominator in polynomial forms.

Thus,

$$H(s) = \frac{200\,j\omega}{(j\omega + 2)(j\omega + 10)} = \frac{200s}{s^2 + 12s + 20}, \qquad s = j\omega$$

Using the following commands, the Bode plots are generated as shown in Fig. 14.59. If necessary, the command **logspace** can be included to generate a logarithmically spaced frequency and the command **semilogx** can be used to produce a semilog scale.

```
>> num = [200 0]; % specify the numerator of H(s)
>> den = [1 12 20]; % specify the denominator of H(s)
>> bode(num, den); % determine and draw Bode plots
```

The step response $y(t)$ of a system is the output when the input $x(t)$ is the unit step function. The command **step** plots the step response of a system given the numerator and denominator of the transfer function of that system. The time range and number of points are automatically selected. For example, consider a second-order system with the transfer function

$$H(s) = \frac{12}{s^2 + 3s + 12}$$

We obtain the step response of the system shown in Fig. 14.60 by using the following commands.

```
>> n = 12;
>> d = [1 3 12];
>> step(n,d);
```

We can verify the plot in Fig. 14.60 by obtaining $y(t) = x(t) * u(t)$ or $Y(s) = X(s)H(s)$.

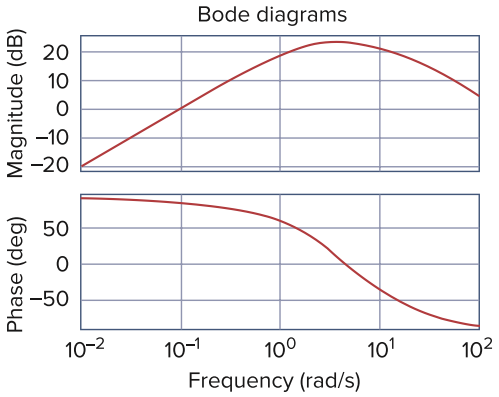



**Figure 14.59**
Magnitude and phase plots.

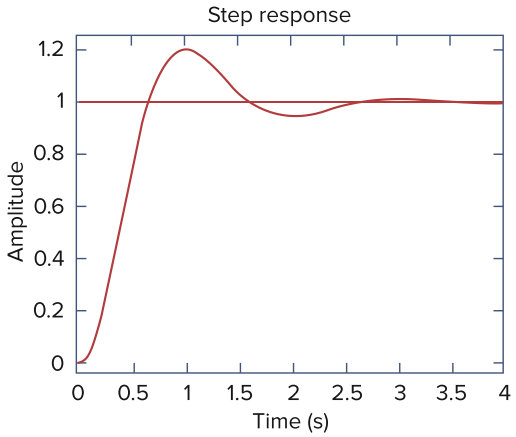



**Figure 14.60**
The Step response of $H(s) = 12/(s^2 + 3s + 12)$.

The command **lsim** is a more general command than **step**. It calculates the time response of a system to any arbitrary input signal. The format of the command is $y =$ **lsim** (num, den, $x$, $t$), where $x(t)$ is the input signal, $t$ is the time vector, and $y(t)$ is the output generated. For example, assume a system is described by the transfer function

$$H(s) = \frac{s + 4}{s^3 + 2s^2 + 5s + 10}$$

To find the response $y(t)$ of the system to input $x(t) = 10e^{-t}u(t)$, we use the following *MATLAB* commands. Both the response $y(t)$ and the input $x(t)$ are plotted in Fig. 14.61.

```
>> t = 0:0.02:5; % time vector 0 < t < 5 with increment
       0.02
>> x = 10*exp(-t);
>> num = [1 4];
>> den = [1 2 5 10];
>> y = lsim(num,den,x,t);
>> plot(t,x,t,y)
```

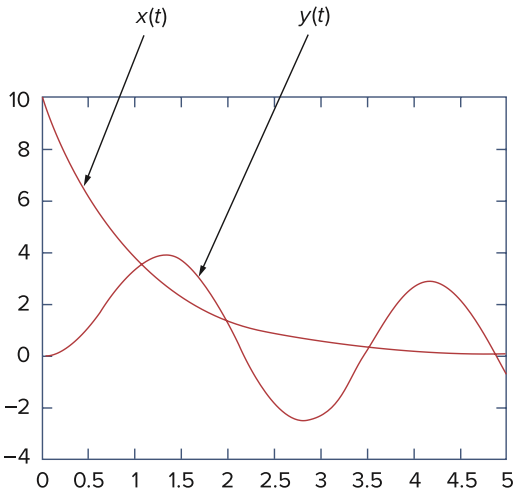



**Figure 14.61**
The response of the system described by $H(s) = (s + 4)/(s^2 + 2s^2 + 5s + 10)$ to an exponential input.

## 14.12 †Applications

Resonant circuits and filters are widely used, particularly in electronics, power systems, and communications systems. For example, a Notch filter with a cutoff frequency at 60 Hz may be used to eliminate the 60-Hz power line noise in various communications electronics. Filtering of signals in communications systems is necessary in order to select the desired signal from a host of others in the same range (as in the case of radio receivers discussed next) and also to minimize the effects of noise and interference on the desired signal. In this section, we consider one practical application of resonant circuits and two applications of filters. The focus of each application is not to understand the details of how each device works but to see how the circuits considered in this chapter are applied in the practical devices.

### 14.12.1 Radio Receiver

Series and parallel resonant circuits are commonly used in radio and TV receivers to tune in stations and to separate the audio signal from the radio-frequency carrier wave. As an example, consider the block diagram

of an AM radio receiver shown in Fig. 14.62. Incoming amplitude-modulated radio waves (thousands of them at different frequencies from different broadcasting stations) are received by the antenna. A resonant circuit (or a band-pass filter) is needed to select just one of the incoming waves. The selected signal is very weak and is amplified in stages in order to generate an audible audio-frequency wave. Thus, we have the radiofrequency (RF) amplifier to amplify the selected broadcast signal, the intermediate frequency (IF) amplifier to amplify an internally generated signal based on the RF signal, and the audio amplifier to amplify the audio signal just before it reaches the loudspeaker. It is much easier to amplify the signal at three stages than to build an amplifier to provide the same amplification for the entire band.

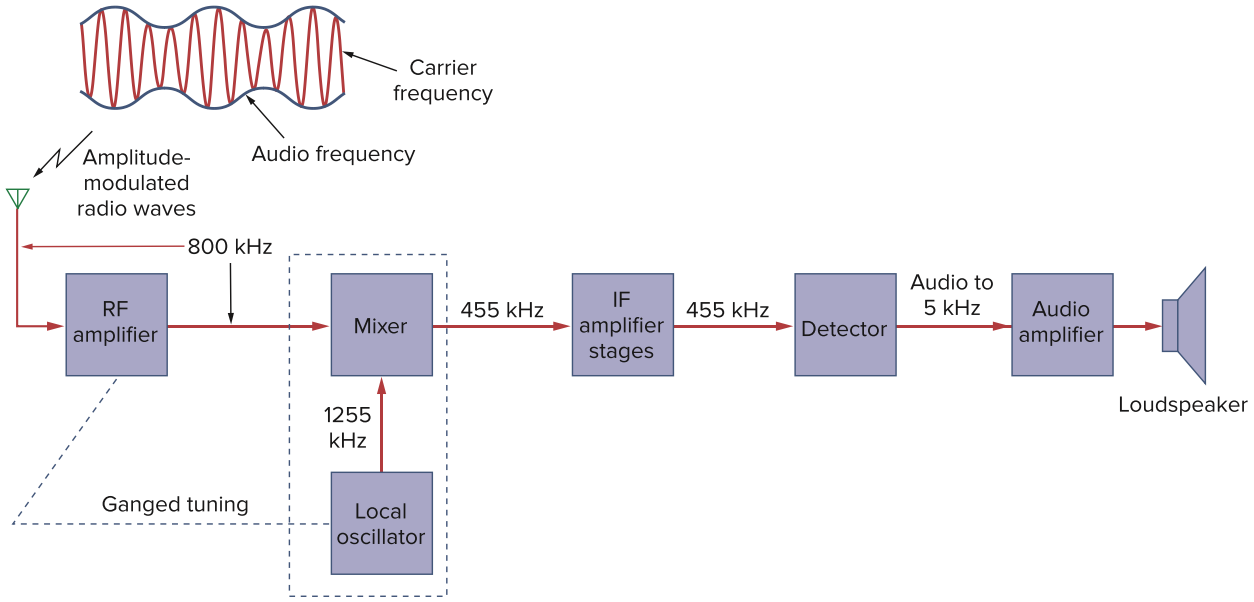



**Figure 14.62**
A simplified block diagram of a superheterodyne AM radio receiver.

The type of AM receiver shown in Fig. 14.62 is known as the *superheterodyne receiver*. In the early development of radio, each amplification stage had to be tuned to the frequency of the incoming signal. This way, each stage must have several tuned circuits to cover the entire AM band (540 to 1600 kHz). To avoid the problem of having several resonant circuits, modern receivers use a *frequency mixer* or *heterodyne* circuit, which always produces the same IF signal (445 kHz) but retains the audio frequencies carried on the incoming signal. To produce the constant IF frequency, the rotors of two separate variable capacitors are mechanically coupled with one another so that they can be rotated simultaneously with a single control; this is called *ganged tuning*. A *local oscillator* ganged with the RF amplifier produces an RF signal that is combined with the incoming wave by the frequency mixer to produce an output signal that contains the sum and the difference frequencies of the two signals. For example, if the resonant circuit is tuned to receive an 800-kHz incoming signal, the local oscillator must produce a 1,255-kHz signal, so that the sum (1,255 + 800 = 2,055 kHz) and the difference (1,255 – 800 = 455 kHz) of frequencies are available at the output of the mixer. However, only

the difference, 455 kHz, is used in practice. This is the only frequency to which all the IF amplifier stages are tuned, regardless of the station dialed. The original audio signal (containing the "intelligence") is extracted in the detector stage. The detector basically removes the IF signal, leaving the audio signal. The audio signal is amplified to drive the loudspeaker, which acts as a transducer converting the electrical signal to sound.

Our major concern here is the tuning circuit for the AM radio receiver. The operation of the FM radio receiver is different from that of the AM receiver discussed here, and in a much different range of frequencies, but the tuning is similar.

## Example 14.17

The resonant or tuner circuit of an AM radio is portrayed in Fig. 14.63. Given that $L = 1\ \mu$H, what must be the range of $C$ to have the resonant frequency adjustable from one end of the AM band to another?

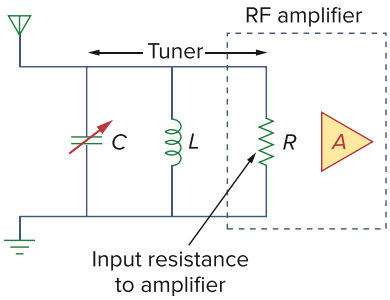



**Figure 14.63**
The tuner circuit for Example 14.17.

**Solution:**
The frequency range for AM broadcasting is 540 to 1,600 kHz. We consider the low and high ends of the band. Since the resonant circuit in Fig. 14.63 is a parallel type, we apply the ideas in Section 14.6. From Eq. (14.44),

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}}$$

or

$$C = \frac{1}{4\pi^2 f_0^2 L}$$

For the high end of the AM band, $f_0 = 1{,}600$ kHz, and the corresponding $C$ is

$$C_1 = \frac{1}{4\pi^2 \times 1{,}600^2 \times 10^6 \times 10^{-6}} = 9.9 \text{ nF}$$

For the low end of the AM band, $f_0 = 540$ kHz, and the corresponding $C$ is

$$C_2 = \frac{1}{4\pi^2 \times 540^2 \times 10^6 \times 10^{-6}} = 86.9 \text{ nF}$$

Thus, $C$ must be an adjustable (gang) capacitor varying from 9.9 to 86.9 nF.

## Practice Problem 14.17

For an FM radio receiver, the incoming wave is in the frequency range from 88 to 108 MHz. The tuner circuit is a parallel $RLC$ circuit with a 4-$\mu$H coil. Calculate the range of the variable capacitor necessary to cover the entire band.

**Answer:** From 0.543 to 0.818 pF.

### 14.12.2 Touch-Tone Telephone

A typical application of filtering is the touch-tone telephone set shown in Fig. 14.64. The keypad has 12 buttons arranged in four rows and three columns. The arrangement provides 12 distinct signals by using seven tones divided into two groups: the low-frequency group (697 to 941 Hz) and the high-frequency group (1,209 to 1,477 Hz). Pressing a button generates a sum of two sinusoids corresponding to its unique pair of frequencies. For example, pressing the number 6 button generates sinusoidal tones with frequencies 770 and 1,477 Hz.

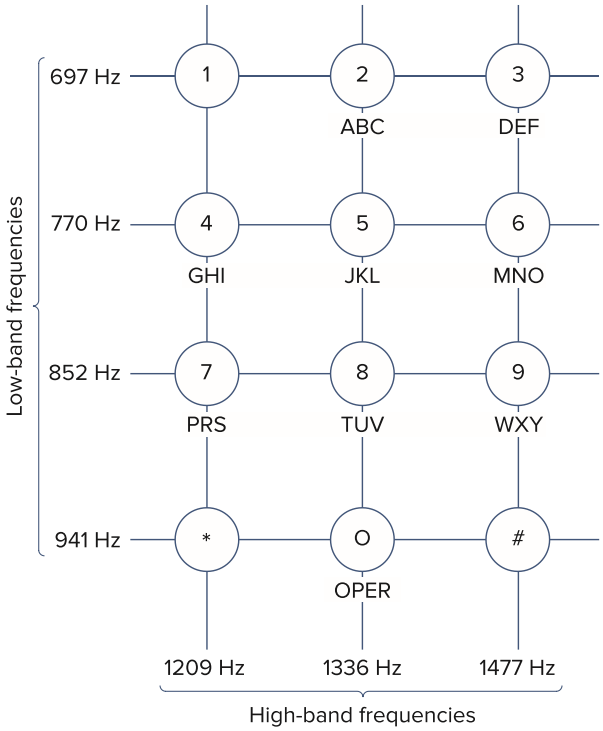



**Figure 14.64**
Frequency assignments for touch-tone dialing.

When a caller dials a telephone number, a set of signals is transmitted to the telephone office, where the touch-tone signals are decoded by detecting the frequencies they contain. Figure 14.65 shows the block diagram for the detection scheme. The signals are first amplified and separated into their respective groups by the low-pass (LP) and high-pass (HP) filters. The limiters (L) are used to convert the separated tones into square waves. The individual tones are identified using seven band-pass (BP) filters, each filter passing one tone and rejecting other tones. Each filter is followed by a detector (D), which is energized when its input voltage exceeds a certain level. The outputs of the detectors provide the required dc signals needed by the switching system to connect the caller to the party being called.

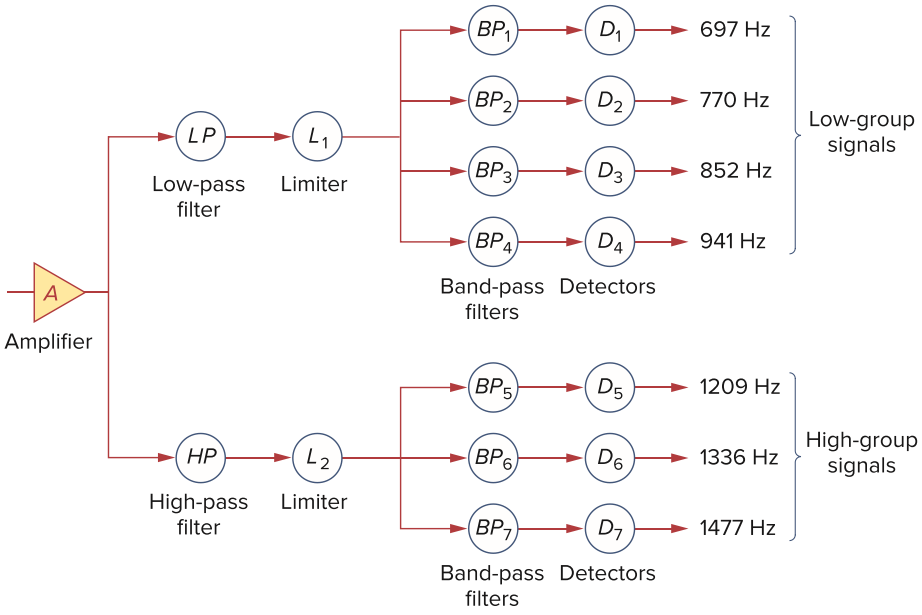



**Figure 14.65**
Block diagram of detection scheme.

## Example 14.18

Using the standard 600-Ω resistor used in telephone circuits and a series *RLC* circuit, design the band-pass filter $BP_2$ in Fig. 14.65.

**Solution:**
The band-pass filter is the series *RLC* circuit in Fig. 14.35. Inasmuch as $BP_2$ passes frequencies 697 to 852 Hz and is centered at $f_0 = 770$ Hz, its bandwidth is

$$B = 2\pi(f_2 - f_1) = 2\pi(852 - 697) = 973.89 \text{ rad/s}$$

From Eq. (14.39),

$$L = \frac{R}{B} = \frac{600}{973.89} = 0.616 \text{ H}$$

From Eq. (14.27) or (14.55),

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{4\pi^2 f_0^2 L} = \frac{1}{4\pi^2 \times 770^2 \times 0.616} = 69.36 \text{ nF}$$

## Practice Problem 14.18

Repeat Example 14.18 for band-pass filter $BP_6$.

**Answer:** 356 mH, 39.83 nF.

### 14.12.3 Crossover Network

Another typical application of filters is the *crossover network* that couples an audio amplifier to woofer and tweeter speakers, as shown in Fig. 14.66(a). The network basically consists of one high-pass *RC* filter

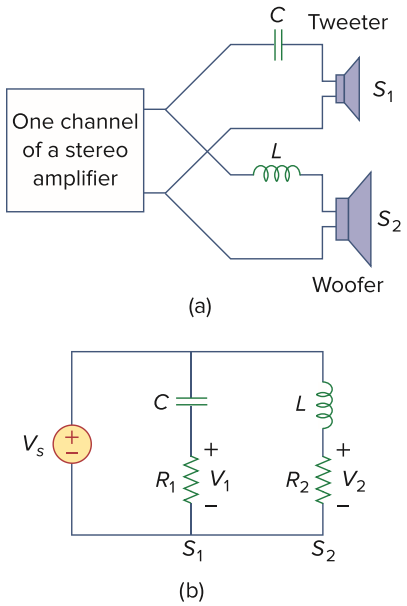



**Figure 14.66**
(a) A crossover network for two loudspeakers, (b) equivalent model.

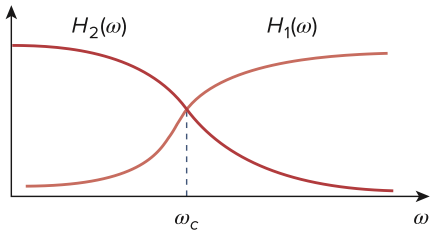



**Figure 14.67**
Frequency responses of the crossover network in Fig. 14.66.

and one low-pass *RL* filter. It routes frequencies higher than a prescribed crossover frequency $f_c$ to the tweeter (high-frequency loudspeaker) and frequencies below $f_c$ into the woofer (low-frequency loudspeaker). These loudspeakers have been designed to accommodate certain frequency responses. A woofer is a low-frequency loudspeaker designed to reproduce the lower part of the frequency range, up to about 3 kHz. A tweeter can reproduce audio frequencies from about 3 kHz to about 20 kHz. The two speaker types can be combined to reproduce the entire audio range of interest and provide the optimum in frequency response.

By replacing the amplifier with a voltage source, the approximate equivalent circuit of the crossover network is shown in Fig. 14.66(b), where the loudspeakers are modeled by resistors. As a high-pass filter, the transfer function $V_1/V_s$ is given by

$$H_1(\omega) = \frac{V_1}{V_s} = \frac{j\omega R_1 C}{1 + j\omega R_1 C} \tag{14.87}$$

Similarly, the transfer function of the low-pass filter is given by

$$H_2(\omega) = \frac{V_2}{V_s} = \frac{R_2}{R_2 + j\omega L} \tag{14.88}$$

The values of $R_1$, $R_2$, $L$, and $C$ may be selected such that the two filters have the same cutoff frequency, known as the *crossover frequency*, as shown in Fig. 14.67.

The principle behind the crossover network is also used in the resonant circuit for a TV receiver, where it is necessary to separate the video and audio bands of RF carrier frequencies. The lower-frequency band (picture information in the range from about 30 Hz to about 4 MHz) is channeled into the receiver's video amplifier, while the high-frequency band (sound information around 4.5 MHz) is channeled to the receiver's sound amplifier.

## Example 14.19

In the crossover network of Fig. 14.66, suppose each speaker acts as a 6-Ω resistance. Find $C$ and $L$ if the crossover frequency is 2.5 kHz.

**Solution:**
For the high-pass filter,

$$\omega_c = 2\pi f_c = \frac{1}{R_1 C}$$

or

$$C = \frac{1}{2\pi f_c R_1} = \frac{1}{2\pi \times 2.5 \times 10^3 \times 6} = 10.61\ \mu\text{F}$$

For the low-pass filter,

$$\omega_c = 2\pi f_c = \frac{R_2}{L}$$

or

$$L = \frac{R_2}{2\pi f_c} = \frac{6}{2\pi \times 2.5 \times 10^3} = 382\ \mu\text{H}$$

### Practice Problem 14.19

If each speaker in Fig. 14.66 has an 8Ω resistance and $C = 10\ \mu$F, find $L$ and the crossover frequency.

**Answer:** 0.64 mH, 1.989 kHz.

## 14.13 Summary

1. The transfer function $\mathbf{H}(\omega)$ is the ratio of the output response $\mathbf{Y}(\omega)$ to the input excitation $\mathbf{X}(\omega)$; that is, $\mathbf{H}(\omega) = \mathbf{Y}(\omega)/\mathbf{X}(\omega)$.
2. The frequency response is the variation of the transfer function with frequency.
3. Zeros of a transfer function $\mathbf{H}(s)$ are the values of $s = j\omega$ that make $H(s) = 0$, while poles are the values of $s$ that make $H(s) \to \infty$.
4. The decibel is the unit of logarithmic gain. For a voltage or current gain $G$, its decibel equivalent is $G_{\text{dB}} = 20 \log_{10} G$.
5. Bode plots are semilog plots of the magnitude and phase of the transfer function as it varies with frequency. The straight-line approximations of $H$ (in dB) and $\phi$ (in degrees) are constructed using the corner frequencies defined by the poles and zeros of $\mathbf{H}(\omega)$.
6. The resonant frequency is that frequency at which the imaginary part of a transfer function vanishes. For series and parallel $RLC$ circuits.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

7. The half-power frequencies ($\omega_1$, $\omega_2$) are those frequencies at which the power dissipated is one-half of that dissipated at the resonant frequency. The geometric mean between the half-power frequencies is the resonant frequency, or

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

8. The bandwidth is the frequency band between half-power frequencies:

$$B = \omega_2 - \omega_1$$

9. The quality factor is a measure of the sharpness of the resonance peak. It is the ratio of the resonant (angular) frequency to the bandwidth,

$$Q = \frac{\omega_0}{B}$$

10. A filter is a circuit designed to pass a band of frequencies and reject others. Passive filters are constructed with resistors, capacitors, and inductors. Active filters are constructed with resistors, capacitors, and an active device, usually an op amp.
11. Four common types of filters are low-pass, high-pass, band-pass, and band-stop. A low-pass filter passes only signals whose frequencies are below the cutoff frequency $\omega_c$. A high-pass filter passes only signals whose frequencies are above the cutoff frequency $\omega_c$. A band-pass filter passes only signals whose frequencies are within a prescribed

range ($\omega_1 < \omega < \omega_2$). A band-stop filter passes only signals whose frequencies are outside a prescribed range ($\omega_1 > \omega > \omega_2$).

12. Scaling is the process whereby unrealistic element values are magnitude-scaled by a factor $K_m$ and/or frequency-scaled by a factor $K_f$ to produce realistic values.

$$R' = K_m R, \qquad L' = \frac{K_m}{K_f} L, \qquad C' = \frac{1}{K_m K_f} C$$

13. *PSpice* can be used to obtain the frequency response of a circuit if a frequency range for the response and the desired number of points within the range are specified in the AC Sweep.
14. The radio receiver—one practical application of resonant circuits—employs a band-pass resonant circuit to tune in one frequency among all the broadcast signals picked up by the antenna.
15. The touch-tone telephone and the crossover network are two typical applications of filters. The touch-tone telephone system employs filters to separate tones of different frequencies to activate electronic switches. The crossover network separates signals in different frequency ranges so that they can be delivered to different devices such as tweeters and woofers in a loudspeaker system.

## Review Questions

**14.1** A zero of the transfer function

$$H(s) = \frac{10(s+1)}{(s+2)(s+3)}$$

is at

(a) 10 (b) −1 (c) −2 (d) −3

**14.2** On the Bode magnitude plot, the slope of $1/(5 + j\omega)^2$ for large values of $\omega$ is

(a) 20 dB/decade (b) 40 dB/decade
(c) −40 dB/decade (d) −20 dB/decade

**14.3** On the Bode phase plot for $0.5 < \omega < 50$, the slope of $[1 + j10\omega - \omega^2/25]^2$ is

(a) 45°/decade (b) 90°/decade
(c) 135°/decade (d) 180°/decade

**14.4** How much inductance is needed to resonate at 5 kHz with a capacitance of 12 nF?

(a) 2,652 H (b) 11.844 H
(c) 3.333 H (d) 84.43 mH

**14.5** The difference between the half-power frequencies is called the:

(a) quality factor (b) resonant frequency
(c) bandwidth (d) cutoff frequency

**14.6** In a series *RLC* circuit, which of these quality factors has the steepest magnitude response curve near resonance?

(a) $Q = 20$ (b) $Q = 12$
(c) $Q = 8$ (d) $Q = 4$

**14.7** In a parallel *RLC* circuit, the bandwidth $B$ is directly proportional to $R$.

(a) True (b) False

**14.8** When the elements of an *RLC* circuit are both magnitude-scaled and frequency-scaled, which quality is unaffected?

(a) resistor (b) resonant frequency
(c) bandwidth (d) quality factor

**14.9** What kind of filter can be used to select a signal of one particular radio station?

(a) low-pass (b) high-pass
(c) band-pass (d) band-stop

**14.10** A voltage source supplies a signal of constant amplitude, from 0 to 40 kHz, to an RC low-pass filter. A load resistor, connected in parallel across the capacitor, experiences the maximum voltage at:

(a) dc (b) 10 kHz
(c) 20 kHz (d) 40 kHz

*Answers: 14.1b, 14.2c, 14.3d, 14.4d, 14.5c, 14.6a, 14.7b, 14.8d, 14.9c, 14.10a.*

# Problems

## Section 14.2 Transfer Function

**14.1** Find the transfer function $I_o/I_i$ of the *RL* circuit in Fig. 14.68. Express it using $\omega_0 = R/L$.

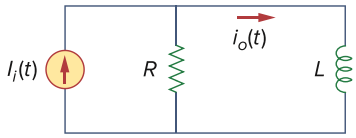



**Figure 14.68**
For Prob. 14.1.

**14.2** Using Fig. 14.69, design a problem to help other students better understand how to determine transfer functions.

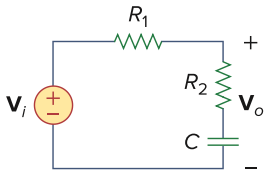



**Figure 14.69**
For Prob. 14.2.

**14.3** For the circuit shown in Fig. 14.70, find $\mathbf{H}(s) = \mathbf{V}_o(s)/\mathbf{I}_i(s)$.

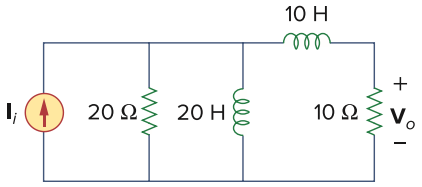



**Figure 14.70**
For Prob. 14.3.

**14.4** Find the transfer function $\mathbf{H}(s) = \mathbf{V}_o/\mathbf{V}_i$ of the circuit shown in Fig. 14.71.

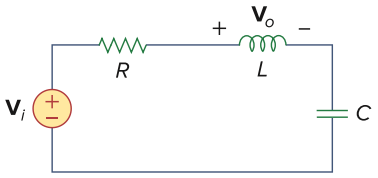



**Figure 14.71**
For Prob. 14.4.

**14.5** For the circuit shown in Fig. 14.72, find $\mathbf{H}(s) = \mathbf{V}_o/\mathbf{I}_s$.

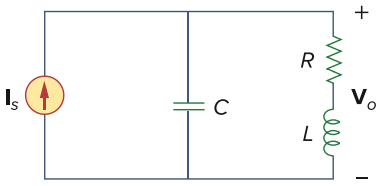



**Figure 14.72**
For Prob. 14.5.

**14.6** For the circuit shown in Fig. 14.73, find $\mathbf{H}(s) = \mathbf{V}_o(s)/\mathbf{V}_s(s)$.

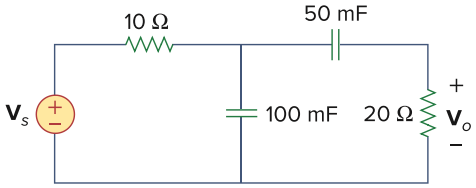



**Figure 14.73**
For Prob. 14.6.

## Section 14.3 The Decibel Scale

**14.7** Calculate $|\mathbf{H}(\omega)|$ if $H_{\text{dB}}$ equals

(a) 0.1 dB (b) −5 dB (c) 215 dB

**14.8** Design a problem to help other students calculate the magnitude in dB and phase in degrees of a variety of transfer functions at a single value of $\omega$.

## Section 14.4 Bode Plots

**14.9** A ladder network has a voltage gain of

$$\mathbf{H}(\omega) = \frac{10}{(1 + j\omega)(10 + j\omega)}$$

Sketch the Bode plots for the gain.

**14.10** Design a problem to help other students better understand how to determine the Bode magnitude and phase plots of a given transfer function in terms of $j\omega$.

**14.11** Sketch the Bode plots for

$$\mathbf{H}(\omega) = \frac{0.2(10 + j\omega)}{j\omega(2 + j\omega)}$$

**14.12** A transfer function is given by

$$T(s) = \frac{100(s + 10)}{s(s + 10)}$$

Sketch the magnitude and phase Bode plots.

**14.13** Construct the Bode plots for

$$G(s) = \frac{0.1(s + 1)}{s^2(s + 10)}, \qquad s = j\omega$$

**14.14** Draw the Bode plots for

$$\mathbf{H}(\omega) = \frac{250(j\omega + 1)}{j\omega(-\omega^2 + 10\,j\omega + 25)}$$

**14.15** Construct the Bode magnitude and phase plots for

$$H(s) = \frac{2(s + 1)}{(s + 2)(s + 10)}, \qquad s = j\omega$$

**14.16** Sketch Bode magnitude and phase plots for

$$H(s) = \frac{1.6}{s(s^2 + s + 16)}, \qquad s = j\omega$$

**14.17** Sketch the Bode plots for

$$G(s) = \frac{s}{(s + 2)^2(s + 1)}, \qquad s = j\omega$$

**14.18** A linear network has this transfer function

$$H(s) = \frac{7s^2 + s + 4}{s^3 + 8s^2 + 14s + 5}, \qquad s = j\omega$$

Use *MATLAB* or equivalent to plot the magnitude and phase (in degrees) of the transfer function. Take $0.1 < \omega < 10$ rad/s.

**14.19** Sketch the asymptotic Bode plots of the magnitude and phase for

$$H(s) = \frac{80s}{(s + 10)(s + 20)(s + 40)}, \qquad s = j\omega$$

**14.20** Design a more complex problem than given in Prob. 14.10, to help other students better understand how to determine the Bode magnitude and phase plots of a given transfer function in terms of $j\omega$. Include at least a second order repeated root.

**14.21** Sketch the magnitude Bode plot for

$$H(s) = \frac{10s(s + 20)}{(s + 1)(s^2 + 60s + 400)}, \qquad s = j\omega$$

**14.22** Find the transfer function $\mathbf{H}(\omega)$ with the Bode magnitude plot shown in Fig. 14.74.

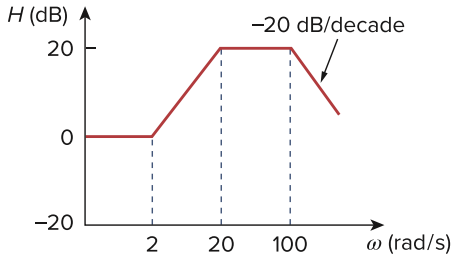



**Figure 14.74**
For Prob. 14.22.

**14.23** The Bode magnitude plot of $\mathbf{H}(\omega)$ is shown in Fig. 14.75. Find $\mathbf{H}(\omega)$.

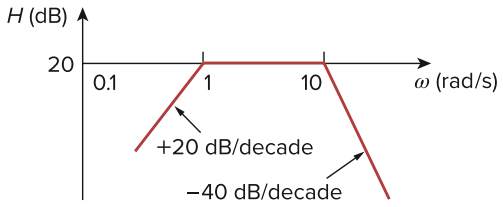



**Figure 14.75**
For Prob. 14.23.

**14.24** The magnitude plot in Fig. 14.76 represents the transfer function of a preamplifier. Find $H(s)$.

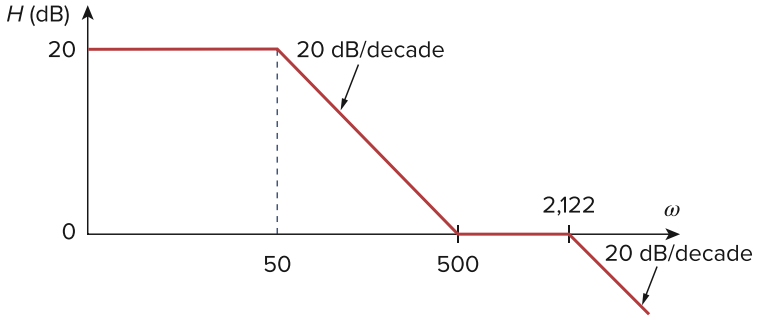



**Figure 14.76**
For Prob. 14.24.

## Section 14.5 Series Resonance

**14.25** A series *RLC* network has $R = 2$ kΩ, $L = 40$ mH, and $C = 1$ μF. Calculate the impedance at resonance and at one-fourth, one-half, twice, and four times the resonant frequency.

**14.26** Design a problem to help other students better understand $\omega_0$, $Q$, and $B$ at resonance in series *RLC* circuits.

**14.27** Design a series *RLC* resonant circuit with $\omega_0 =$ 40 rad/s and $B = 10$ rad/s.

**14.28** Design a series *RLC* circuit with $B = 20$ rad/s and $\omega_0 = 1{,}000$ rad/s. Find the circuit's $Q$. Let $R = 10\ \Omega$.

**14.29** Let $v_s = 20\cos(at)$ V in the circuit of Fig. 14.77. Find $\omega_0$, $Q$, and $B$, as seen by the capacitor.

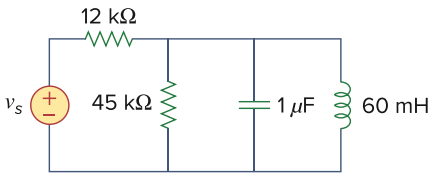



**Figure 14.77**
For Prob. 14.29.

**14.30** A circuit consisting of a coil with inductance 10 mH and resistance 20 Ω is connected in series with a capacitor and a generator with an rms voltage of 120 V. Find:

(a) the value of the capacitance that will cause the circuit to be in resonance at 15 kHz

(b) the current through the coil at resonance

(c) the $Q$ of the circuit

## Section 14.6 Parallel Resonance

**14.31** Design a parallel resonant *RLC* circuit with $\omega_0 = 100$ krad/s and a bandwidth of 10 krad/s. Additionally what is the value of $Q$?

**14.32** Design a problem to help other students better understand the quality factor, the resonant frequency, and bandwidth of a parallel *RLC* circuit.

**14.33** A parallel resonant circuit with a bandwidth of 40 krad/s and the half-power frequencies are $\omega_1 =$ 4.98 Mrad/s and $\omega_2 = 5.02$ Mrad/s, calculate the quality factor and resonant frequency.

**14.34** A parallel *RLC* circuit has $R = 100$ kΩ, $L =$ 100 mH, and a $C = 10$ μF. Determine the value of $Q$, the resonant frequency, and the bandwidth. If $R = 200$ kΩ, how does that affect the values of $Q$, resonant frequency, and the bandwidth?

**14.35** A parallel *RLC* circuit has $R = 10$ kΩ, $L = 100$ mH, and a resonant frequency of 200 krad/s. Calculate the value of $C$, the value of the quality factor, and the bandwidth.

**14.36** It is expected that a parallel *RLC* resonant circuit has a midband admittance of $25 \times 10^{-3}$ S, quality factor of 120, and a resonant frequency of 200 krad/s. Calculate the values of $R$, $L$, and $C$. Find the bandwidth and the half-power frequencies.

**14.37** Rework Prob. 14.25 if the elements are connected in parallel.

**14.38** Find the resonant frequency of the circuit in Fig. 14.78.

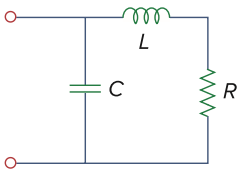



**Figure 14.78**
For Prob. 14.38.

**14.39** For the "tank" circuit in Fig. 14.79, find the resonant frequency.

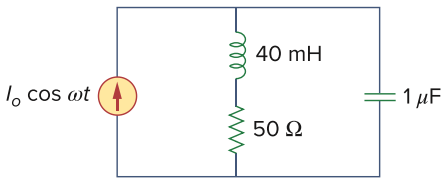



**Figure 14.79**
For Probs. 14.39, 14.71, and 14.91.

**14.40** A parallel resonance circuit has a resistance of 2 kΩ and half-power frequencies of 86 kHz and 90 kHz. Determine:

(a) the capacitance

(b) the inductance

(c) the resonant frequency

(d) the bandwidth

(e) the quality factor

**14.41** Using Fig. 14.80, design a problem to help other students better understand the quality factor, the resonant frequency, and bandwidth of *RLC* circuits.

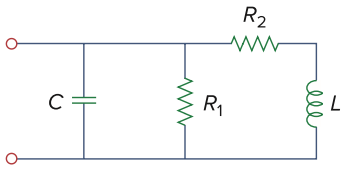



**Figure 14.80**
For Prob. 14.41.

**14.42** For the circuits in Fig. 14.81, find the resonant frequency $\omega_0$, the quality factor $Q$, and the bandwidth $B$.

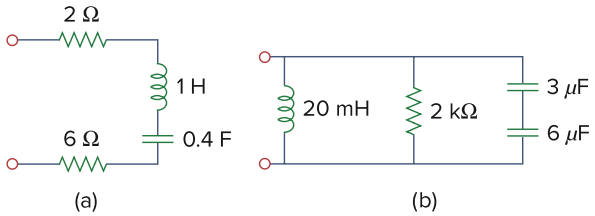



**Figure 14.81**
For Prob. 14.42.

**14.43** Calculate the resonant frequency of each of the circuits in Fig. 14.82.

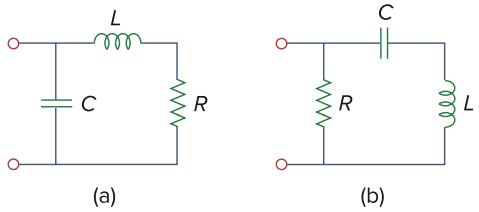



**Figure 14.82**
For Prob. 14.43.

***14.44** For the circuit in Fig. 14.83, find:

(a) the resonant frequency $\omega_0$

(b) $\mathbf{Z}_{\text{in}}(\omega_0)$

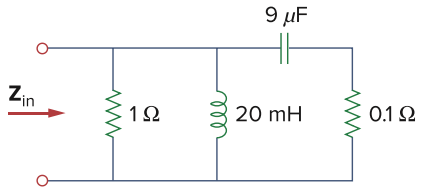



**Figure 14.83**
For Prob. 14.44.

**14.45** For the circuit shown in Fig. 14.84, find $\omega_0$, $B$, and $Q$, as seen by the voltage across the inductor.

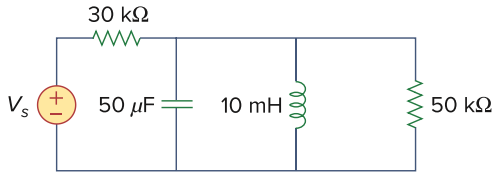



**Figure 14.84**
For Prob. 14.45.

**14.46** For the network illustrated in Fig. 14.85, find

(a) the transfer function $\mathbf{H}(\omega) = \mathbf{V}_o(\omega)/\mathbf{I}(\omega)$,

(b) the magnitude of $\mathbf{H}$ at $\omega_0 = 1$ rad/s.

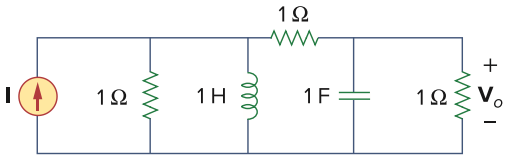



**Figure 14.85**
For Probs. 14.46, 14.78, and 14.92.

## Section 14.7 Passive Filters

**14.47** Show that a series $LR$ circuit is a low-pass filter if the output is taken across the resistor. Calculate the corner frequency $f_c$ if $L = 2$ mH and $R = 10$ kΩ.

**14.48** Find the transfer function $\mathbf{V}_o/\mathbf{V}_s$ of the circuit in Fig. 14.86. Show that the circuit is a low-pass filter.

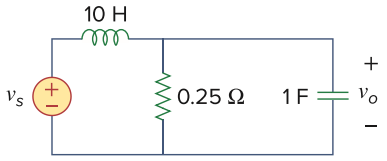



**Figure 14.86**
For Prob. 14.48.

**14.49** Design a problem to help other students better understand low-pass filters described by transfer functions.

**14.50** Determine what type of filter is in Fig. 14.87. Calculate the corner frequency $f_c$.

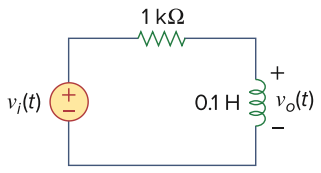



**Figure 14.87**
For Prob. 14.50.

* An asterisk indicates a challenging problem.

**14.51** Design an *RL* low-pass filter that uses a 40-mH coil and has a cutoff frequency of 5 kHz.

**14.52** Design a problem to help other students better understand passive high-pass filters.

**14.53** Design a series *RLC* type band-pass filter with cutoff frequencies of 10 kHz and 11 kHz. Assuming $C = 80$ pF, find $R$, $L$, and $Q$.

**14.54** Design a passive band-stop filter with $\omega_0 = 10$ rad/s and $Q = 20$.

**14.55** Determine the range of frequencies that will be passed by a series *RLC* band-pass filter with $R = 10\ \Omega$, $L = 25$ mH, and $C = 0.4\ \mu$F. Find the quality factor.

**14.56** (a) Show that for a band-pass filter,

$$\mathbf{H}(s) = \frac{sB}{s^2 + sB + \omega_0^2}, \qquad s = j\omega$$

where $B$ = bandwidth of the filter and $\omega_0$ is the center frequency.

(b) Similarly, show that for a band-stop filter,

$$\mathbf{H}(s) = \frac{s^2 + \omega_0^2}{s^2 + sB + \omega_0^2}, \qquad s = j\omega$$

**14.57** Determine the center frequency and bandwidth of the band-pass filters in Fig. 14.88.

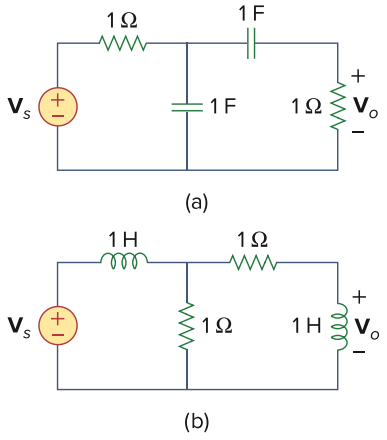



**Figure 14.88**
For Prob. 14.57.

**14.58** The circuit parameters for a series *RLC* band-stop filter are $R = 250\ \Omega$, $L = 1$ mH, $C = 40$ pF. Calculate:

(a) the center frequency

(b) the half-power frequencies

(c) the quality factor

**14.59** Find the bandwidth and center frequency of the band-stop filter of Fig. 14.89.

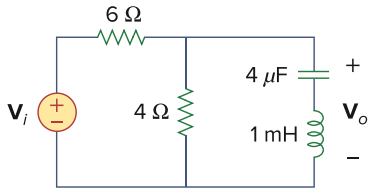



**Figure 14.89**
For Prob. 14.59.

## Section 14.8 Active Filters

**14.60** Obtain the transfer function of a high-pass filter with a passband gain of 100 and a cutoff frequency of 40 rad/s.

**14.61** Find the transfer function for each of the active filters in Fig. 14.90.

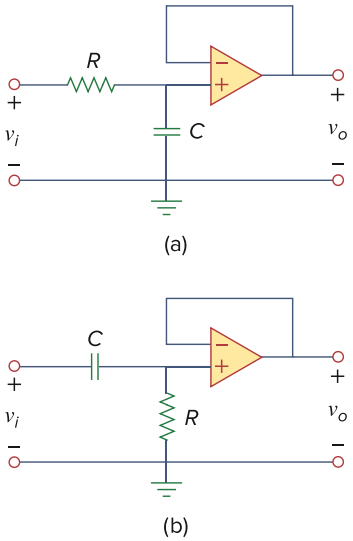



**Figure 14.90**
For Probs. 14.61 and 14.62.

**14.62** The filter in Fig. 14.90(b) has a 3-dB cutoff frequency at 1 kHz. If its input is connected to a 120-mV variable frequency signal, find the output voltage at:

(a) 200 Hz (b) 2 kHz (c) 10 kHz

**14.63** Design an active first-order high-pass filter with

$$\mathbf{H}(s) = -\frac{100s}{s + 10}, \qquad s = j\omega$$

Use a 1-$\mu$F capacitor.

**14.64** Obtain the transfer function of the active filter in Fig. 14.91 on the next page. What kind of filter is it?

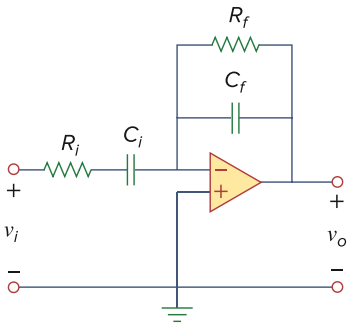



**Figure 14.91**
For Prob. 14.64.

**14.65** A high-pass filter is shown in Fig. 14.92. Show that the transfer function is

$$\mathbf{H}(\omega) = \left(1 + \frac{R_f}{R_i}\right)\frac{j\omega RC}{1 + j\omega RC}$$

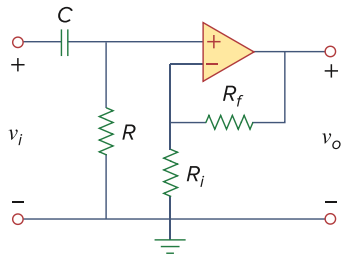



**Figure 14.92**
For Prob. 14.65.

**14.66** A "general" first-order filter is shown in Fig. 14.93.

(a) Show that the transfer function is

$$\mathbf{H}(s) = \frac{R_4}{R_3 + R_4} \times \frac{s + (1/R_1C)[R_1/R_2 - R_3/R_4]}{s + 1/R_2C},$$

$$s = j\omega$$

(b) What condition must be satisfied for the circuit to operate as a high-pass filter?

(c) What condition must be satisfied for the circuit to operate as a low-pass filter?

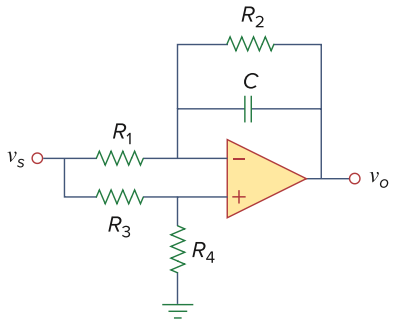



**Figure 14.93**
For Prob. 14.66.

**14.67** Design an active low-pass filter with dc gain of 0.25 and a corner frequency of 500 Hz.

**14.68** Design a problem to help other students better understand the design of active high-pass filters when specifying a high-frequency gain and a corner frequency.

**14.69** Design the filter in Fig. 14.94 to meet the following requirements:

(a) It must attenuate a signal at 2 kHz by 3 dB compared with its value at 10 MHz.

(b) It must provide a steady-state output of $v_o(t) = 10\sin(2\pi \times 10^8 t + 180°)$ V for an input $v_s(t) = 4\sin(2\pi \times 10^8 t)$ V.

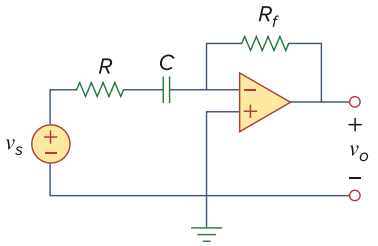



**Figure 14.94**
For Prob. 14.69.

***14.70** A second-order active filter known as a Butterworth filter is shown in Fig. 14.95.

(a) Find the transfer function $\mathbf{V}_o/\mathbf{V}_i$.

(b) Show that it is a low-pass filter.

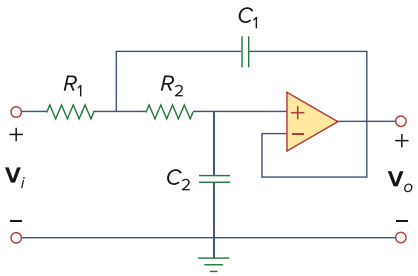



**Figure 14.95**
For Prob. 14.70.

## Section 14.9 Scaling

**14.71** Use magnitude and frequency scaling on the circuit of Fig. 14.79 to obtain an equivalent circuit in which the inductor and capacitor have magnitude 1 H and 1 F respectively.

**14.72** Design a problem to help other students better understand magnitude and frequency scaling.

**14.73** Calculate the values of $R$, $L$, and $C$ that will result in $R = 12$ kΩ, $L = 40$ μH, and $C = 300$ nF respectively when magnitude-scaled by 800 and frequency-scaled by 1000.

**14.74** A circuit has $R_1 = 3\ \Omega$, $R_2 = 10\ \Omega$, $L = 2$H, and $C = 1/10$ F. After the circuit is magnitude-scaled by 100 and frequency-scaled by $10^6$, find the new values of the circuit elements.

**14.75** In an *RLC* circuit, $R = 20\ \Omega$, $L = 4$ H, and $C = 1$ F. The circuit is magnitude-scaled by 10 and frequency-scaled by $10^5$. Calculate the new values of the elements.

**14.76** Given a parallel *RLC* circuit with $R = 5$ kΩ, $L = 10$ mH, and $C = 20\ \mu$F, if the circuit is magnitude-scaled by $K_m = 500$ and frequency-scaled by $K_f = 10^5$, find the resulting values of $R$, $L$, and $C$.

**14.77** A series *RLC* circuit has $R = 10\ \Omega$, $\omega_0 = 40$ rad/s, and $B = 5$ rad/s. Find $L$ and $C$ when the circuit is scaled:

(a) in magnitude by a factor of 600,

(b) in frequency by a factor of 1,000,

(c) in magnitude by a factor of 400 and in frequency by a factor of $10^5$.

**14.78** Redesign the circuit in Fig. 14.85 so that all resistive elements are scaled by a factor of 1,000 and all frequency-sensitive elements are frequency-scaled by a factor of $10^4$.

***14.79** Refer to the network in Fig. 14.96.

(a) Find $\mathbf{Z}_{in}(s)$.

(b) Scale the elements by $K_m = 10$ and $K_f = 100$. Find $\mathbf{Z}_{in}(s)$ and $\omega_0$.

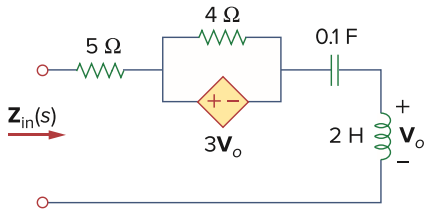



**Figure 14.96**
For Prob. 14.79.

**14.80** (a) For the circuit in Fig. 14.97, draw the new circuit after it has been scaled by $K_m = 200$ and $K_f = 10^4$.

(b) Obtain the Thevenin equivalent impedance at terminals $a$-$b$ of the scaled circuit at $\omega = 10^4$ rad/s.

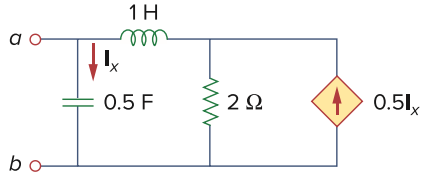



**Figure 14.97**
For Prob. 14.80.

**14.81** The circuit shown in Fig. 14.98 has the impedance

$$Z(s) = \frac{1{,}000(s+1)}{(s+1+j50)(s+1-j50)}, \qquad s = j\omega$$

Find:

(a) the values of $R$, $L$, $C$, and $G$

(b) the element values that will raise the resonant frequency by a factor of $10^3$ by frequency scaling

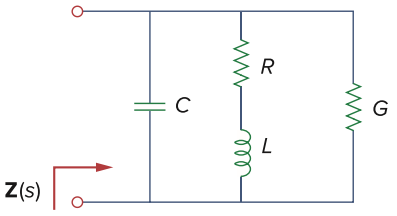



**Figure 14.98**
For Prob. 14.81.

**14.82** Scale the low-pass active filter in Fig. 14.99 so that its corner frequency increases from 1 rad/s to 200 rad/s. Use a 1-$\mu$F capacitor.

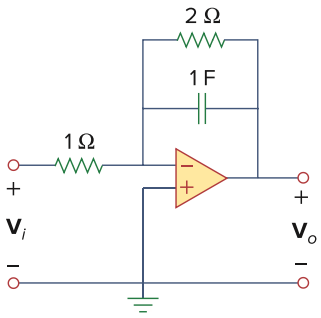



**Figure 14.99**
For Prob. 14.82.

**14.83** The op amp circuit in Fig. 14.100 is to be magnitude-scaled by 100 and frequency-scaled by $10^5$. Find the resulting element values.

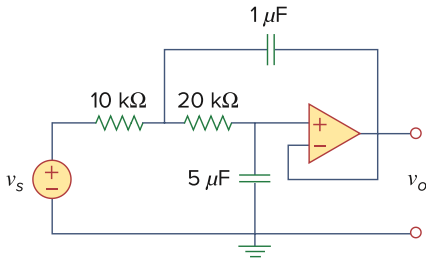



**Figure 14.100**
For Prob. 14.83.

## Section 14.10 Frequency Response Using *PSpice*

**14.84** Using *PSpice or MultiSim,* obtain the frequency response of the circuit in Fig. 14.101 on the next page.

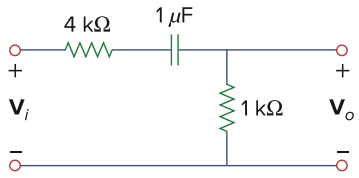



**Figure 14.101**
For Prob. 14.84.

**14.85** Use *PSpice or MultiSim* to obtain the magnitude and phase plots of $\mathbf{V}_o/\mathbf{I}_s$ of the circuit in Fig. 14.102.

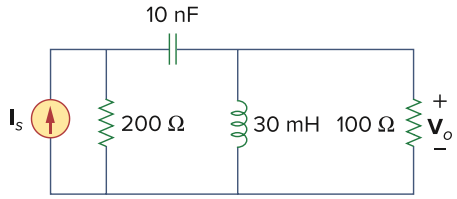



**Figure 14.102**
For Prob. 14.85.

**14.86** Using Fig. 14.103, design a problem to help other students better understand how to use *PSpice* to obtain the frequency response (magnitude and phase of I) in electrical circuits.

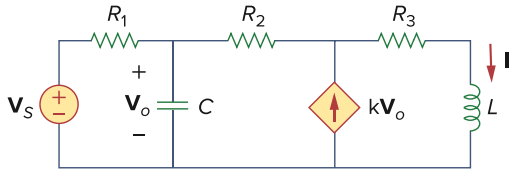



**Figure 14.103**
For Prob. 14.86.

**14.87** In the interval $0.1 < f < 100$ Hz, plot the response of the network in Fig. 14.104. Classify this filter and obtain $\omega_0$.

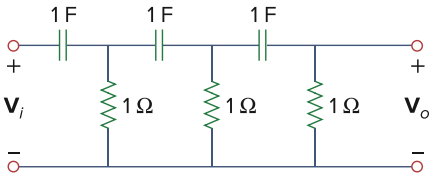



**Figure 14.104**
For Prob. 14.87.

**14.88** Use *PSpice or MultiSim* to generate the magnitude and phase Bode plots of $\mathbf{V}_o$ in the circuit of Fig. 14.105.

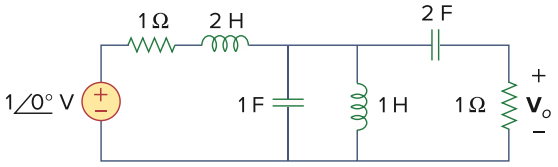



**Figure 14.105**
For Prob. 14.88.

**14.89** Obtain the magnitude plot of the response $\mathbf{V}_o$ in the network of Fig. 14.106 for the frequency interval $100 < f < 1{,}000$ Hz.

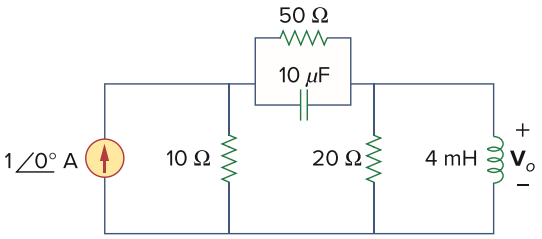



**Figure 14.106**
For Prob. 14.89.

**14.90** Obtain the frequency response of the circuit in Fig. 14.40 (see Practice Problem 14.10). Take $R_1 = R_2 = 100\ \Omega$, $L = 2$ mH. Use $1 < f < 100{,}000$ Hz.

**14.91** For the "tank" circuit of Fig. 14.79, obtain the frequency response (voltage across the capacitor) using *PSpice or MultiSim*. Determine the resonant frequency of the circuit.

**14.92** Using *PSpice or MultiSim*, plot the magnitude of the frequency response of the circuit in Fig. 14.85.

## Section 14.12 Applications

**14.93** For the phase shifter circuit shown in Fig. 14.107, find $\mathbf{H} = V_o/V_s$.

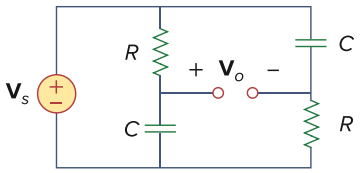



**Figure 14.107**
For Prob. 14.93.

**14.94** For an emergency situation, an engineer needs to make an *RC* high-pass filter. He has one 10-pF capacitor, one 30-pF capacitor, one 1.8-kΩ resistor, and one 3.3-kΩ resistor available. Find the greatest cutoff frequency possible using these elements.

**14.95** A series-tuned antenna circuit consists of a variable capacitor (40 pF to 360 pF) and a 240-$\mu$H antenna coil that has a dc resistance of 12 Ω.

(a) Find the frequency range of radio signals to which the radio is tunable.

(b) Determine the value of $Q$ at each end of the frequency range.

**14.96** The crossover circuit in Fig. 14.108 is a low-pass filter that is connected to a woofer. Find the transfer function $\mathbf{H}(\omega) = \mathbf{V}_o(\omega)/\mathbf{V}_i(\omega)$.

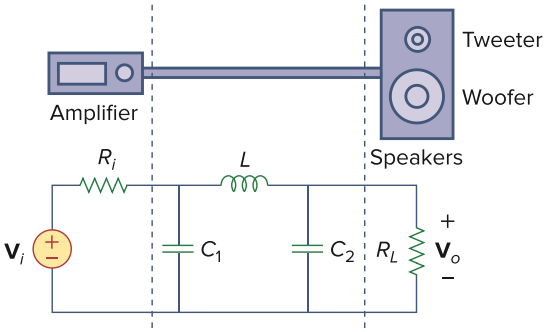



**Figure 14.108**
For Prob. 14.96.

**14.97** The crossover circuit in Fig. 14.109 is a high-pass filter that is connected to a tweeter. Determine the transfer function $\mathbf{H}(\omega) = \mathbf{V}_o(\omega)/\mathbf{V}_i(\omega)$.

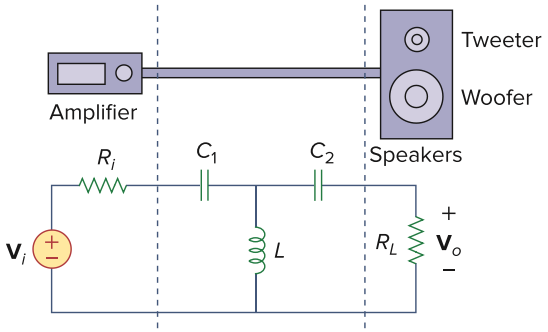



**Figure 14.109**
For Prob. 14.97.

## Comprehensive Problems

**14.98** A certain electronic test circuit produced a resonant curve with half-power points at 432 Hz and 454 Hz. If $Q = 20$, what is the resonant frequency of the circuit?

**14.99** In an electronic device, a series circuit is employed that has a resistance of 100 Ω, a capacitive reactance of 5 kΩ, and an inductive reactance of 300 Ω when used at 2 MHz. Find the resonant frequency and bandwidth of the circuit.

**14.100** In a certain application, a simple *RC* low-pass filter is designed to reduce high frequency noise. If the desired corner frequency is 20 kHz and $C = 0.5\ \mu$F, find the value of $R$.

**14.101** In an amplifier circuit, a simple *RC* high-pass filter is needed to block the dc component while passing the time-varying component. If the desired rolloff frequency is 15 Hz and $C = 10\ \mu$F, find the value of $R$.

**14.102** Practical *RC* filter design should allow for source and load resistances as shown in Fig. 14.110. Let $R = 4$ kΩ and $C = 40$-nF. Obtain the cutoff frequency when:

(a) $R_s = 0$, $R_L = \infty$,

(b) $R_s = 1$ kΩ, $R_L = 5$ kΩ.

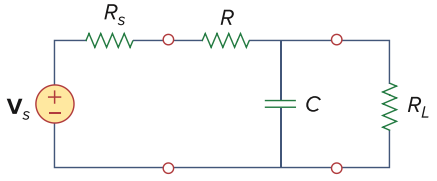



**Figure 14.110**
For Prob. 14.102.

**14.103** The *RC* circuit in Fig. 14.111 is used for a lead compensator in a system design. Obtain the transfer function of the circuit.

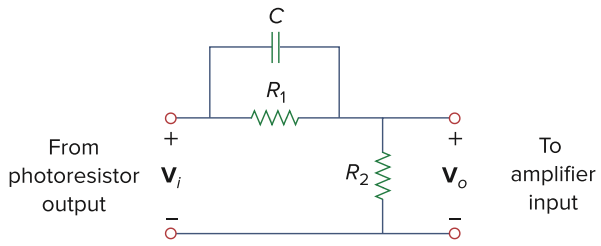



**Figure 14.111**
For Prob. 14.103.

**14.104** A low-quality-factor, double-tuned band-pass filter is shown in Fig. 14.112. Use *PSpice or MultiSim* to generate the magnitude plot of $\mathbf{V}_o(\omega)$.

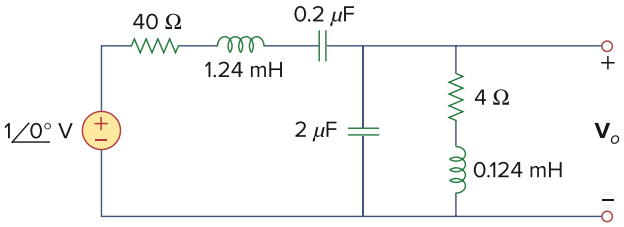



**Figure 14.112**
For Prob. 14.104.