lecture #13

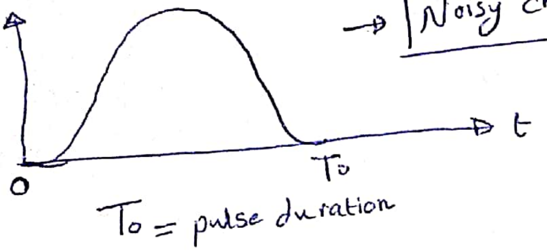
- Performance analysis of Digital Communications system :-

- optimum linear Detector for Binary polar signaling.
  (Reciever)

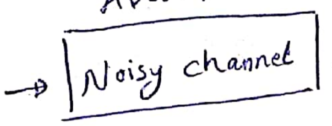
"1" → $+p(t)$

"0" → $-p(t)$

"ADWGN"

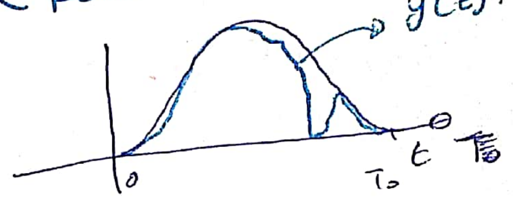


To = pulse duration

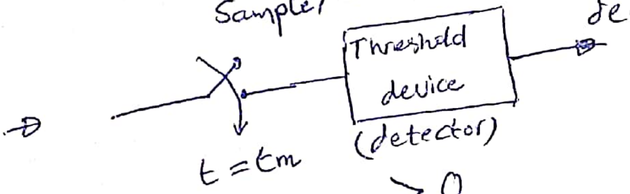


Rx

$y(t) = \pm p(t) + n(t)$ ← Recieved Signal



Sample, $y(t) = \pm p(t) + N(t)$



$\gtrless 0$ ← for polar

$p_o(t) = \pm p(t) * H(t)$

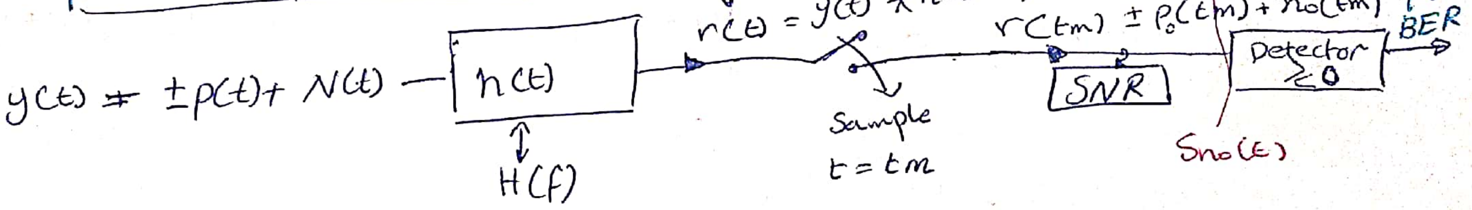
$N(t) * H(t).$

The optimal structure

disicion variable

$r(t) = y(t) * h(t) = \pm p_o(t) + n_o(t)$

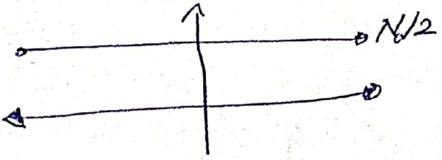
$r(t_m) = \pm p_o(t_m) + n_o(t_m)$



$n(t)$ additive white gaussian Noise

$E[n(t)] = 0$

Double Side Noise PSD :-



$$SNR = \frac{|\pm P_o(t)|^2}{\sigma_n^2} = \frac{\text{Power of desired part}}{\text{Noise power}}$$

$$\sigma_n^2 = E\left[n_o^2(t)\right] = \int_{-\infty}^{\infty} S_{n_o}(f)\, df$$

$$= \int_{-\infty}^{\infty} S_n(f)\, |H(f)|^2\, df$$

$$S_n(f) = \frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df.$$

$$*** \; SNR = \frac{p^2(t_m)}{\sigma_n^2} = \frac{p^2(t_m)}{\frac{N}{2}\int_{-\infty}^{\infty}[H(f)]^2 \, df}.$$

target: to find $H(f)$ and t_m such that SNR is Maximum. and the BER is minimum.

P_e = the propability of error

$$P_e = P(\text{"error"} \cap \text{"1"}_{Txd}) \cup P(\text{"error} \cap \text{"0"}_{Txd})$$

$$= P_r(\text{"error"} \cap \text{"1"} \; Txd) + P(\text{"error} \cap \text{"0"} \; Txd).$$

$$= P(\text{"error"}/\text{"1" was Txed}) \, P(\text{"1" was transmitted}) + P(\text{"error"} \,|\, \text{"0" was Txed}) \, P(\text{"0" was txed})$$

$$= 0.5 * P(\text{"error"}/\text{"1" was txed"} + 0.5 \, P(\text{"error"}/\text{"0" was txed})$$

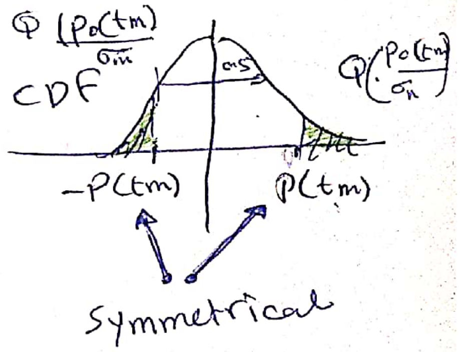
$$= 0.5 \, P_r\Big(+P_o(t_m) + \underbrace{n_o(t_m)}_{R.V} < 0\Big) + 0.5 \, P\big(-P_o(t_m) + n_o(t_m) > 0\big)$$

$$= 0.5 P\big(n_o(t_m) < -p(t_m)\big) + 0.5\big(n_o(t_m) > p(t_m)\big)$$

$$N(t) \sim (0, \sigma_n^2)$$

$$\Rightarrow \; \underline{\underline{1 - F(x) = \Phi(x)}}$$

same area Under the curve.

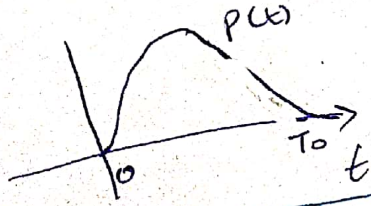


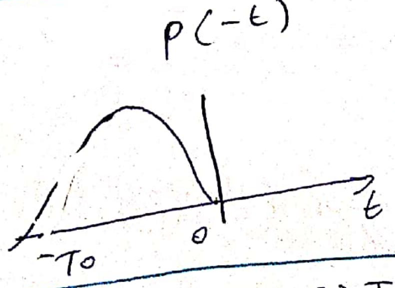
$$0.5 \, Q\left(\frac{P_o(t_m)}{\sigma_n}\right) + 0.5 \, Q\left(\frac{P_o(t_m)}{\sigma_n}\right)$$

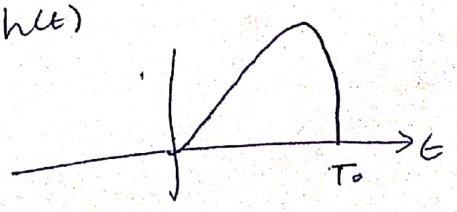
$$= Q\left(\frac{P_o(t_m)}{\sigma_n}\right) = \boxed{Q(\sqrt{SNR})}$$

** The filter that Maximizes SNR and minimized P_e is called matched filter.

$$h(t) = P(T_o - t).$$





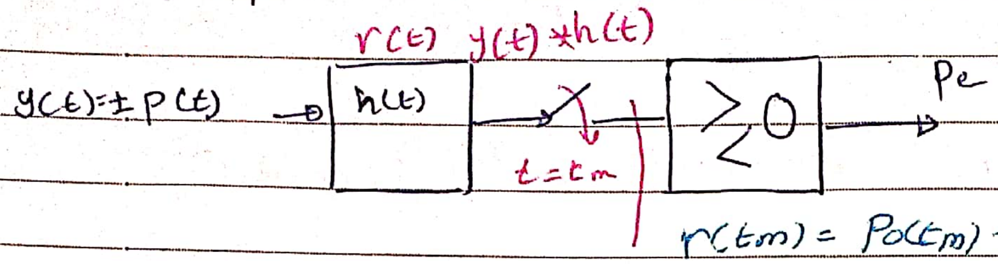


$$\boxed{H(f) \longrightarrow P(-f) \, e^{-j(2\pi f) T_o}}$$

Optimum Linear Rx for polar Signaling.

"1" → $p(t)$

"0" → $-p(t)$



$r(t_m) = p_o(t_m) + n(t_m)$

$r(t_m)$ → disicion variable

$Q = (\sqrt{SNR}) = Q\left(\frac{p_o(t_m)}{\sigma_n}\right)$

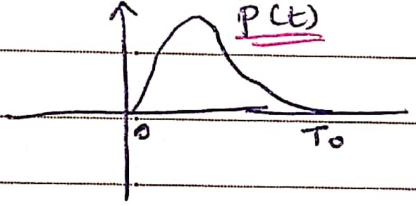
$\sigma_n^2 = \frac{N}{2}\int_{-\infty}^{\infty} |H(t)|^2 dt$

$h(t) \overset{?}{\longleftrightarrow} H(f)$

t_m? such that Pe is minimum.

$H(f) = P(-f)\, e^{-j\pi f T_o}$

$h(t) = p(T_o - t)$

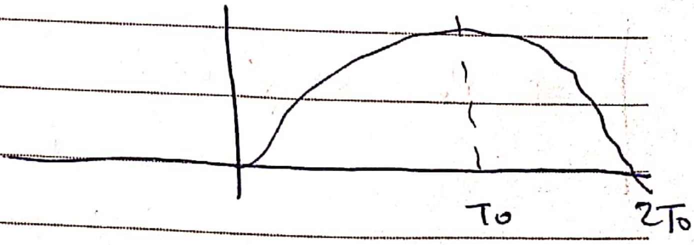


$p_o(t) = h(t) * p(t)$

$p(T_o - t) * p(t)$

width T_o width T_o

↓ width $2T_o$



$t_m = T_o$; to achieve the maximum over lapping ((متطبقين على بعض))

$$p_e = Q\left(\sqrt{\frac{p_o(t_o)}{\sigma_n}}\right)$$

$$\sigma_N^2 = \frac{N}{2}\int_{-\infty}^{\infty}|H(f)|^2 df = \frac{N}{2}\int_{-\infty}^{\infty}|P(f)|^2 = \frac{NE_p}{2}$$

$$p_o(T_o) \Rightarrow p_o(t) = p(t) * h(t) = FT^{-1}\{P(f)H(f)\}$$

$$\int_{-\infty}^{\infty} p(f)\,H(f)\,e^{+j2\pi ft}\,df$$

$$p_o(T_o) = \int_{-\infty}^{\infty} P(f)\,H(f)\,e^{j2\pi fT_o}\,df$$

$$p_o(T_o) = \int_{-\infty}^{\infty} P(f)\,P(-f)\,\cancel{e^{j2\pi fT_o}}\,\cancel{e^{-j2\pi fT_o}}$$

$$p_o(T_o) = \int_{-\infty}^{\infty} |P(f)|^2 = E_p$$

$$\frac{p_o(t_o)}{\sigma_N^2} = \frac{E_p}{\sqrt{\frac{N}{2}E_p}} = \sqrt{2E_pN}$$

$$\boxed{\Rightarrow p_e = Q\left(\sqrt{2E_pN}\right)}$$

→ pulse energy.

* polar signaling.

* channel is ADWG

* Matched filter

* corrolation Rx :- an alternative to the matched filter.

$r(t_m) = y(t_m) * h(t_m)$

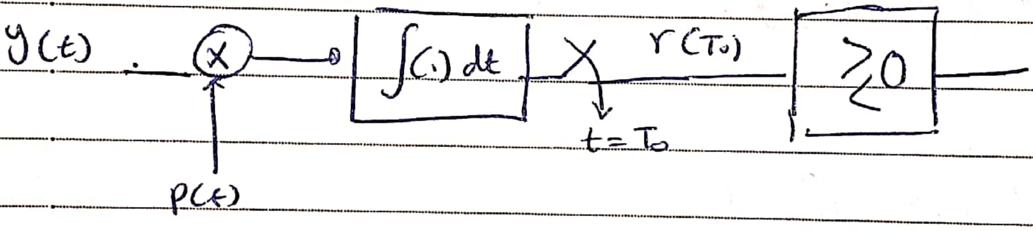
$r(t) = y(t) * h(t)$

$$= \int_{-\infty}^{\infty} y(x) h(t-x)\, dx$$

$$= \int_{-\infty}^{\infty} y(x)\ p(T_0 + x - t)$$

$$r(T_0) = \int_{-\infty}^{\infty} y(x)\ p(\cancel{T_0} + x - \cancel{T_0}) = \int_{-\infty}^{\infty} y(x) \underbrace{p(x)}_{(0,\,T_0)} dx$$

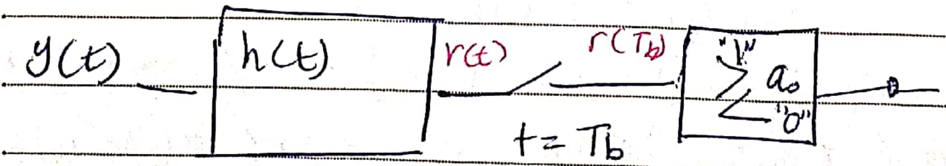
$$\boxed{r(T_0) = \int_{0}^{T_0} y(x)\, p(x)\, dx}$$



Optimum Linear Rx for General Binary Signaling.

"1" → $p(t)$ $\quad 0 \le t \le T_b$

"0" → $q(t)$ $\quad 0 \le t \le T_b$



$p_o(T_b) > q_o(T_b)$

$$y(t) = \begin{cases} p(t) + n(t) & 0 \le t \le T_b \ ; \text{"1"} \\ q(t) + n(t) & 0 \le t \le T_b \ ; \text{"0"} \end{cases}$$

$\rightarrow p(t) * h(t)$ $\qquad$ $n(t) * h(t)$

$$r(t) = \begin{cases} p_o(t) + n_o(t) & \text{"1"} \\ q_o(t) + n_o(t) & \text{"0"} \end{cases}$$

$\sim N(0, \sigma_n^2)$ gaussian.

$$r(T_b) = \begin{cases} p_o(T_b) + n_o(T_b) & \text{"1"} \\ q_o(T_b) + n_o(T_b) & \text{"0"} \end{cases}$$

$\sim N(0, \sigma_n^2)$

the disicion variable

$r(T_b) \sim N(p_o(T_b), \sigma_N^2)$; "1"

$r(T_b) \sim N(q_o(T_b), \sigma_N^2)$; "0"

$$\sigma_N^2 = \frac{N}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

$$p_o(T_b) = \int P(f) \, H(f) \, e^{j T_b f 2\pi} \, df$$

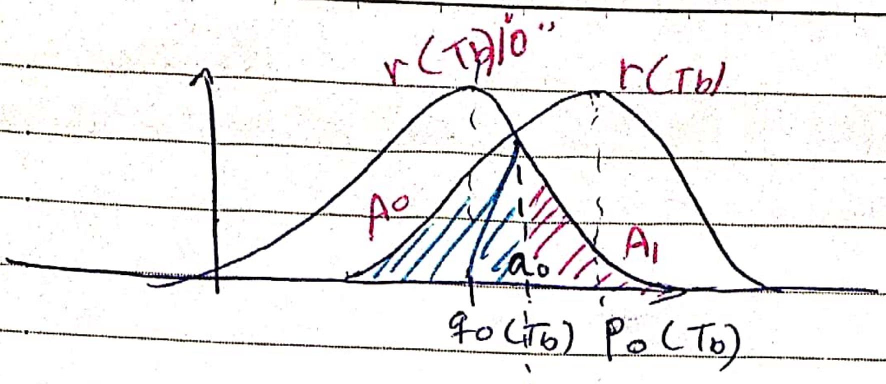
$$q_o(T_b) = \int_{-\infty}^{\infty} Q(f) \, H(f) \, e^{j 2b f 2\pi} \, df$$

$$p_o(T_b) > q_o(T_b)$$

$P_e = P\{\text{error} / 1 \text{txed}\} * 0.5 + P\{\text{error} / 0 \text{txed}\} * 0.5$

$0.5 \, Pr\{N(p_o(T_b), \sigma_n^2) < a_0\} + 0.5 \, Pr\{N(q_o(T_b), \sigma_n^2) > a_0\}$

$\swarrow r(T_b) / \text{"1"}$ $\qquad$ $(r^b / \text{"0"})$



$$A_1 = Q\left(\frac{a_0 - q_0(T_b)}{\sigma_N}\right)$$

$$Q\left(\frac{P_0(T_b) - a_0}{\sigma_N}\right) \quad \& \quad A_0 = 1 - Q\left(\frac{a_0 - P_0(T_b)}{\sigma_N}\right)$$

$\downarrow$

Rule $\boxed{1 - Q(x - a) = Q(a - x)}$

$$P_e = \frac{1}{2} A_0 + \frac{1}{2} A_1 = \frac{1}{2} Q\left(\frac{P_0(t) - a_0}{\sigma_N}\right) + \frac{1}{2} Q\left(\frac{q_0(T_b) - a_0}{\sigma_N}\right)$$

Find $\underline{\underline{a_0}}$

$$\frac{dP_e}{da_0} = \text{----} = 0$$

$$\frac{P_0(T_b) + q_0(T_b)}{2} \rightarrow \text{حرج}$$

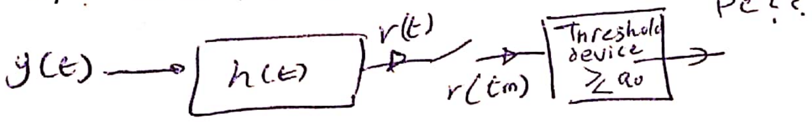
①

General Binary Signaling

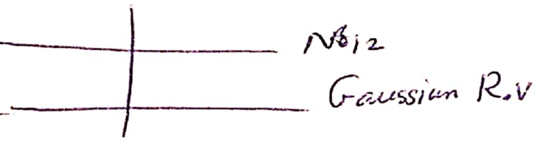
"1" $\rightarrow p(t)$ $\quad 0 \leq t \leq T_b \quad R_b = \frac{1}{T_b}$

"0" $\rightarrow q(t)$

Optimume Rx (detector)



n(t) noise



$$\sigma_n^2 = \frac{N}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$$

● optimum threshold $\quad a_0 = \frac{p_0(T_b) + q_0(T_b)}{2}$

$$P_e = \frac{1}{2} Q\left(\frac{a_0 - q_0}{\sigma_n}\right) + \frac{1}{2} Q\left(\frac{p_0(T_b) - a_0}{\sigma_n}\right)$$

$$\Rightarrow \frac{1}{2} Q\left(\frac{\frac{p_0(T_b)}{2} + \frac{q_0(T_b}{2} - q_0(T_b)}{\sigma_n}\right)$$

$$+ \frac{1}{2} Q\left(\frac{p_0(T_b) - \frac{p(T_b}{2} - \frac{q_0(T_b)}{2}}{\sigma_n}\right)$$

$$= Q\left(\frac{p_0(T_b) - q_0(T_b)}{2\sigma_n}\right)$$

- optimume matched filter $\quad \Rightarrow h(t) = p(T_b - t) - q(T_b - t).$

$$H(f) = (P(-f) - Q(-f))\, e^{-j2\pi f T_b}$$

$$p_0(T_b) = p_0(t)\Big|_{t=T_b}$$

$$p_0(t) = p(t) * h(t)\Big|_{t=T_b} \rightarrow F^{-1}\{P(f)\, H(f)\}\Big|_{t=T_b} = \int_{-\infty}^{\infty} P(f)\underbrace{H(f)}_{\text{matched filter}} e^{+j2\pi f T_b} df$$

$$p_0(T_b) = \int_{-\infty}^{\infty} P(f)\left[P(-f) - Q(-f)\right] e^{(1)} df$$

$$= \underline{E_p} - \underline{E_{pq}} \rightarrow \overset{\text{correlation.}}{\int_{-\infty}^{\infty} P(f).Q(-f)} = \int_0^{T_b} q(t)\, p(t)\, dt$$

$$\int_{-\infty}^{\infty} |P(f)|^2 df = \int_0^{T_b} (p(t))^2 dt$$

$$q_0(T_b) = E_{pq} - E_q$$

②

$$\sigma_n^2 = \frac{N}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N}{2}\int_{-\infty}^{\infty} (P(-f) - Q(-f))^2 df = \frac{N}{2}\left[E_p + E_q - 2E_{qp}\right]$$

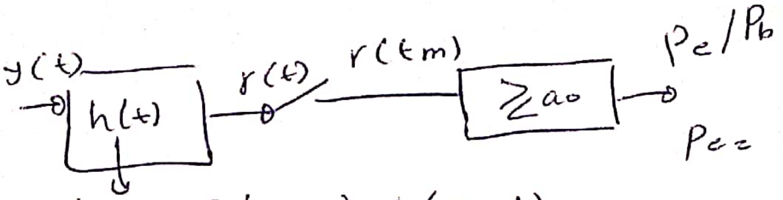
$$\Rightarrow P_e = Q\left(\frac{E_p - E_{pq} - E_{pq} - E_q}{2\sqrt{\frac{N}{2}(E_p + E_q - 2E_{pq})}}\right) = \boxed{Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N}}\right)}$$

$$a_0 = \frac{q_0(T_b) + p_0(T_b)}{2} = \frac{E_{pq} - E_q + E_p - E_{pq}}{2} \quad \boxed{a_0 = \frac{E_p - E_q}{2}}$$

optimum Reciever

"1" → $p(t)$

"0" → $q(t)$



$$P_e = Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N}}\right)$$

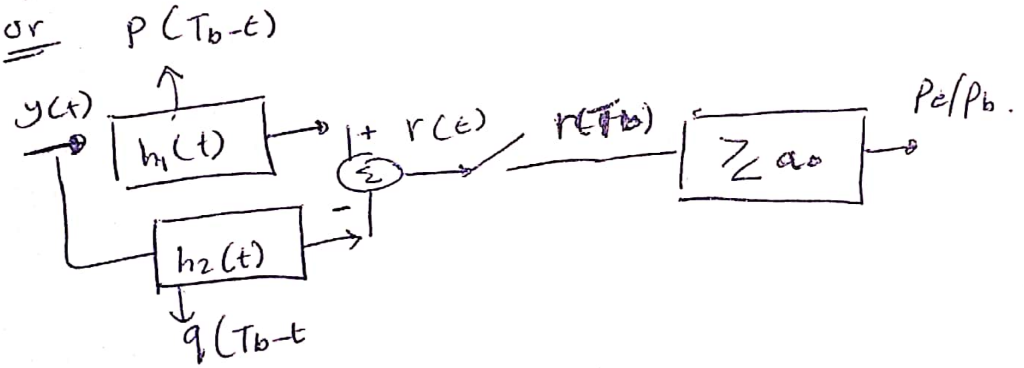
$$h(t) = p(T_b - t) - q(T_b - t)$$

---

* energy per bit ⇒ $E_b = E_p P_r(\text{"1"}) + E_q P_r(\text{"0"})$

For equal prob. $P_r(\text{"1"}) = P_r(\text{"0"})$ ⇒ $\boxed{E_b = \frac{E_p + E_q}{2}}$

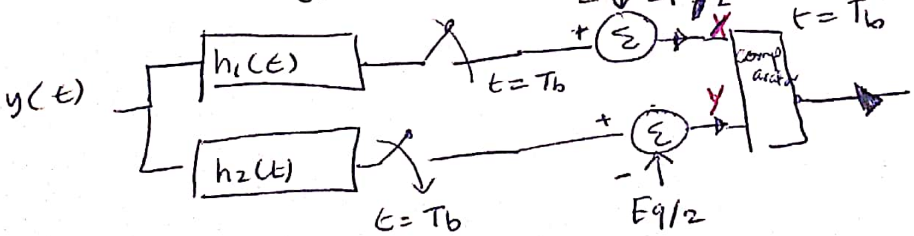
or



$$r(t) = y(t) * h_1(t) - y(t) * h_2(t)\Big|_{t=T_b} \gtrless \frac{E_p}{2} - \frac{E_q}{2}$$

$$r(T_b) \gtrless \frac{E_p}{2} - \frac{E_q}{2}$$

$$\overbrace{(y(t) * h_1(t))\Big|_{t=T_b} - \frac{E_p}{2}}^{X} \gtrless \overbrace{y(t) * h_2(t)\Big|_{t=T_b} - \frac{E_q}{2}}^{Y}$$



X > Y ⇒ "1"

X < Y ⇒ "0"

Special cases scw

$\boxed{1}$ * For polar signaling

"1" → $p(t)$

"0" → $-p(t) \Rightarrow q(t)$

$$E_q = E_p = \int_{0}^{T_b} p(t)^2$$

$$\boxed{a_0 = \frac{E_p - E_q}{2} = 0}$$

$$E_b = \frac{E_p + E_q}{2} = E_p$$

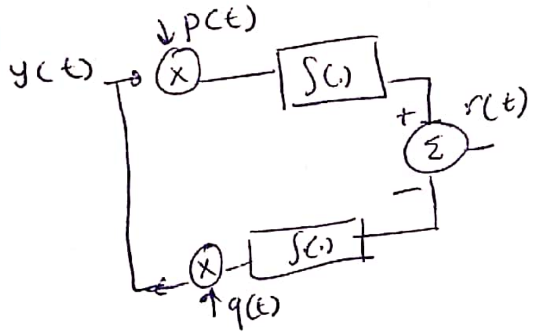
$$P_e = Q\left(\sqrt{\frac{E_p + E_p + 2E_p}{2N}}\right)$$

$$E_{pq} = \int -(p(t)^2) = -E_p$$

$$P_e = \left(Q\left(\sqrt{\frac{2E_p}{N}}\right)\right) = \boxed{Q\left(\sqrt{\frac{2E_b}{N}}\right)}$$

$$h(t) = p(T_b - t) - (-p(T_b - t) = \cancel{2}p(T_b - t)$$ ⇒ matched filter

or



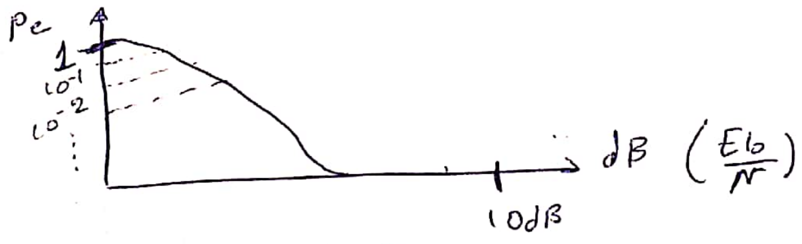
average signal power per bit

$$S = \frac{E_b}{T_b}$$

$\frac{E_b}{N}$ → Express the performance of the system.

Polar $P_e = Q\sqrt{\frac{(2E_b)}{N}}$

semi-log scale



Hw ↗ draw this figure

-5 → 20dB   step = 2dB

$E_bN = [-6 \,.\, -4 \,.\, -2 \qquad 20]$

$P_e = Q\sqrt{2\,10^{E_b/N10}}$

Plot semilog (

Optimum Rx for general Binary signaling :-

الخميس
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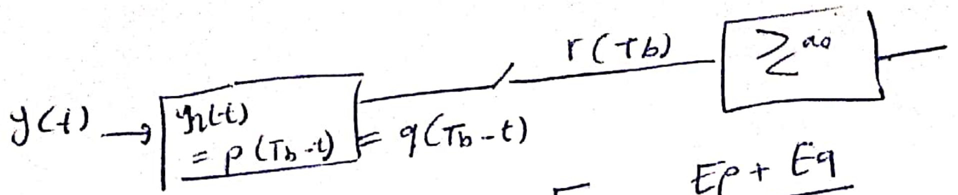
"1" → $p(t)$ $\qquad R_b = \frac{1}{T_b}$

"0" → $-p(t)$ $\qquad 0 \le t \le T_b$ $\qquad a_0 = \frac{E_p - E_q}{2}$



$E_p = \int_0^{T_b} p^2(t)\,dt$ $\qquad E_b = \frac{E_p + E_q}{2}$

↗ Energy per bit.

$E_{pq} = \int_0^{T_b} p(t)\,q(t)\,dt.$

$E_q = \int_0^{T_b} q^2(t)\,dt$

$BER = P_e = P_b = Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N}}\right)$

① polar

$P_e = Q\left(\sqrt{\frac{2E_b}{N}}\right)$

$a_0 = 0$

$h(t) = p(T_b - t)$

② on-off

"1" → $p(t)$

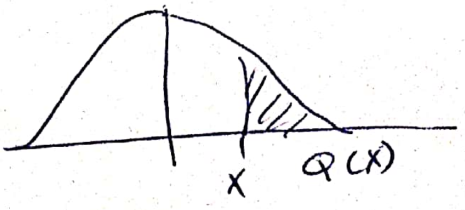
"0" → No pulse ⇒ $q(t) = 0$

$E_p = \int_0^{T_b} p^2(t)\,dt$ $\qquad E_q = 0$

$a_0 = \frac{E_p}{2}$ $\qquad E_{pq} = \int p(t)\,(0)\,dt = 0$

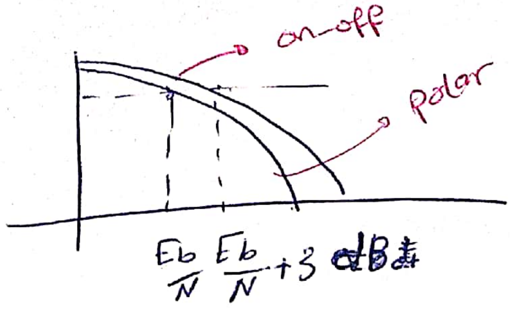
$E_b = \frac{E_p}{2}$

$P_e = Q\left(\sqrt{\frac{E_p + 0 - 2(0)}{2N}}\right)$ $\qquad Q\left(\sqrt{\frac{E_p}{2N}}\right) = Q\left(\sqrt{\frac{E_p}{N}}\right)$



performance of ON-off is worst than that of the polar

by 3DB   3dB on-off



$\boxed{3}$ orthogonal Binary signaling

"1" → $p(t)$ → E_p

"0" → $q(t)$ → E_q

$$E_{pq} = \int_0^{T_b} p(t)\, q(t)\, dt = 0$$

$$a_0 = \frac{E_p}{2} - \frac{E_q}{2}$$

$$a_0 = \frac{E_p - E_q}{2}$$

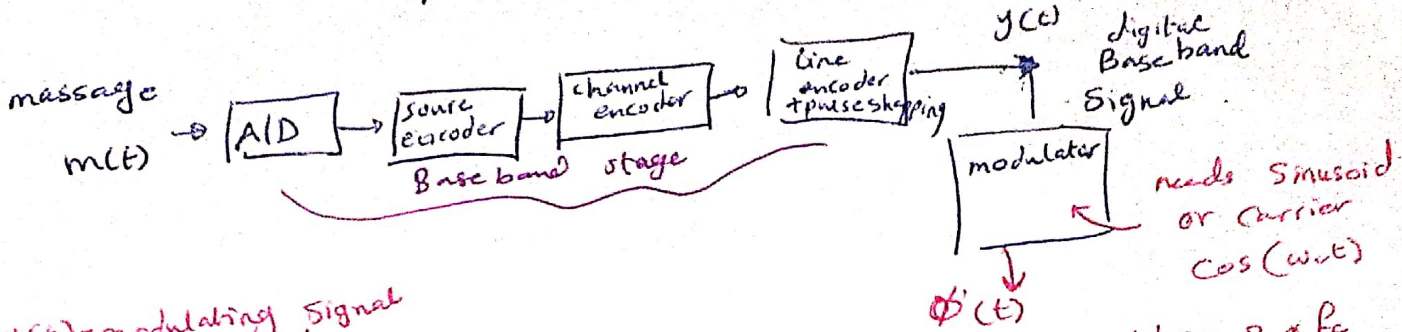
$$h(t) = p(T_b - t) - q(T_b - t)$$

$$E_b = \frac{E_p + E_q}{2}$$

$$P_e = Q\left(\sqrt{\frac{E_p + E_q - 2(0)}{2N}}\right) = Q\left(\sqrt{\frac{E_b}{N}}\right)$$

lecture #14

• Digital carrier:- (based pass) system



y(t) = modulating signal

φ(t) = modulated signal.

W c = 2 * fc

fc → 500kHz → 50GHz

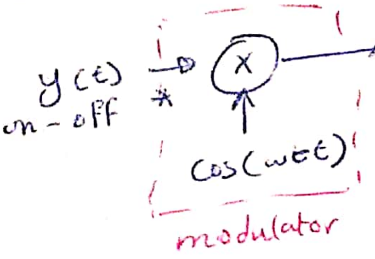
[I] Binary modulation

Ex:- assume ON/off signal y(t)

$$y(t)_{on\,off} = \sum_k a_k\, p(t-kT_b)$$

$a_k = +1$ or 0

$\phi(t) = y(t)\cos(w_c t)$ — ASK



$$\phi(t)_{ASK} = \sum_k a_k\,(p(t-kT_b))\cos(W_c t).$$

$0 < t < T_b$

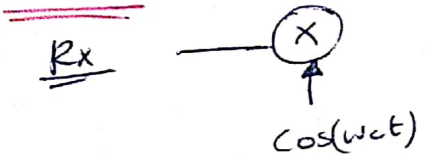
"1" → $p(t)\cos(w_c t)$.

"0" → 0

amplitude shift keying ASK

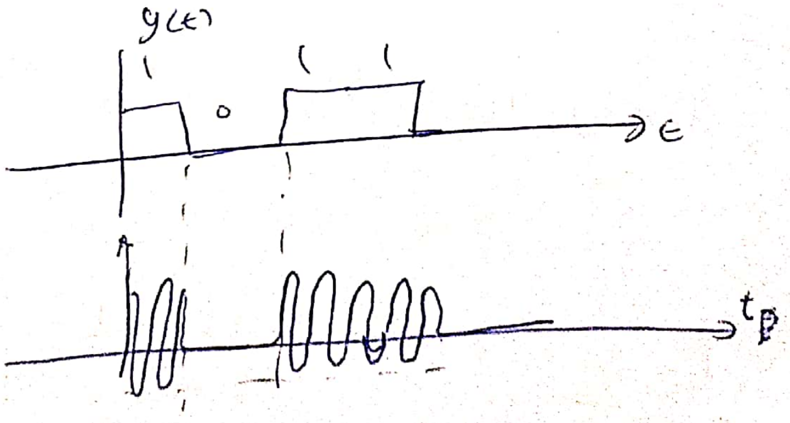
Binary

• Coherent :-



* non-coherent.
Related to the envelope detection.

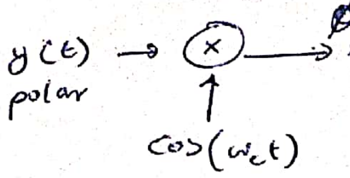
M-ask
↳ coherent
↳ non-coherent.



Example : polar Signaling

$$y(t)_{\text{~~on-off~~}} = \sum_k a_k \, p(t - kT_b)$$

$$a_k = +1, -1$$

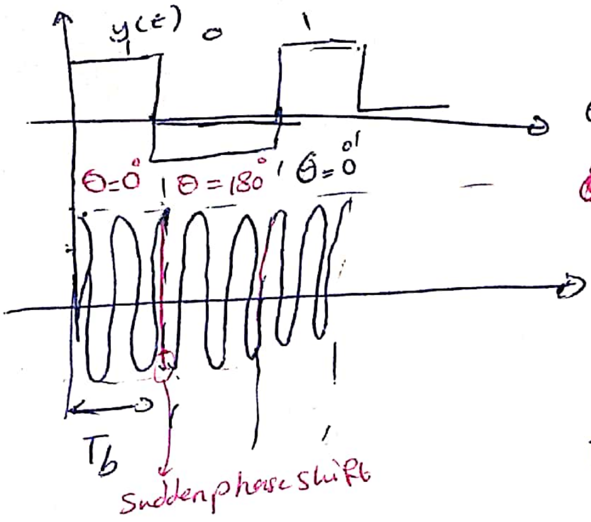


$$\phi(t)_{psk} = y(t) . \cos(w_c t) = \sum_k a_k \, p(t - kT_b) \cos(w_c t)$$

$$\boxed{\begin{aligned} \text{"1"} &\rightarrow p(t) \cos(w_c t) \\ \text{"0"} &\rightarrow -p(t) \cos(w_c t) \end{aligned}}.$$

$$\rightarrow p(t) \cos(w_c t + 180^\circ).$$

BPSK



non-coherent detection can't be used. (information in the phase shift).

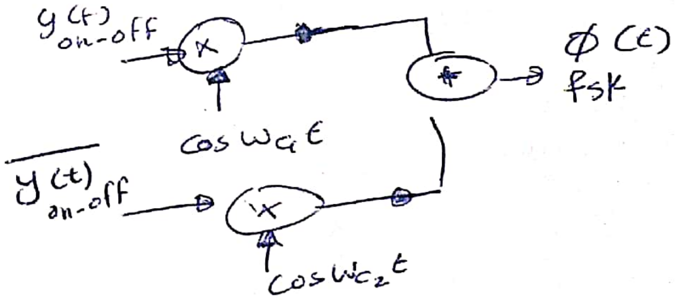
psk can be detected only coherentlly.

"1" → phase $\underline{0^\circ}$

"0" → phase $\underline{180^\circ}$

Ex : on off signal

$$y(t) = \sum_K a_k \, p(t - kT_b)$$



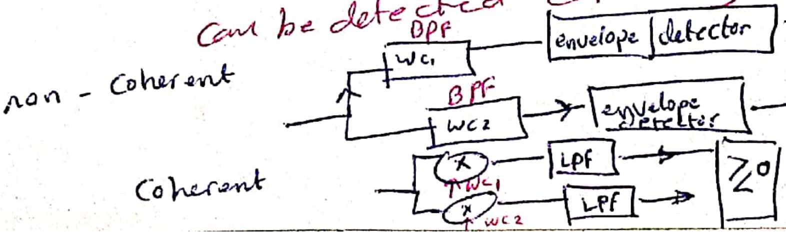
~~φ cos~~

$$\phi(t) = y(t)_{on-off} \cos(w_{c_1} t) + \overline{y(t)}_{on-off} \cos(w_{c_2} t)$$

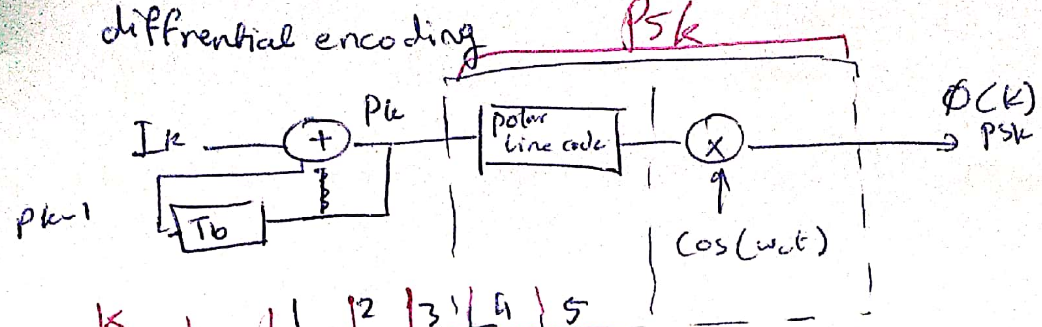
$$\sum a_k \, p(t - kT_b) \cos(w_{c_1} t) + \sum (1 - a_k) \, p(t - kT_b) \cos(w_{c_2} t)$$

$$\left.\begin{aligned} \text{"1"} &\longrightarrow \phi(t) = p(t) \cos(w_{c_1} t) \\ \text{"0"} &\longrightarrow \phi(t) = p(t) \cos(w_{c_2} t) \end{aligned}\right\} \boxed{FSK}$$

can be detected coherentlly or non-coherentlly.



diffrential encoding



k		1	2	3	4	5
I_k		"1"	"0"	"1"	"1"	"0"
P_k	0 initial	1	1	0	1	1
θk	π	0	0	π	0	0
$\theta k - \theta k_{-1}$		π	0	π	π	0
$\hat{I}_k$		"1"	"0"	"1"	"1"	"0"

"1" → 0°

"0" → 180°

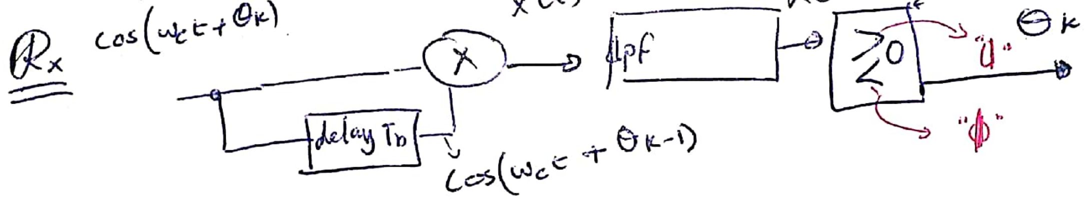
DPSK

$\theta_k - \theta_{k-1} = \pi$ → "1"

$\theta_k - \theta_{k-1} = 0$ → "0".

$x(t) = \frac{1}{2}\cos(\theta k - \theta k\text{-}1) + \frac{1}{2}\cos(2W_c t + \theta + \theta k\text{-}1)$

$\tilde{x}(t) = \frac{1}{2}\cos(\theta k - \theta k\text{-}1)$



$$\theta_k - \theta_{k-1} = \begin{cases} \pi \\ 0 \end{cases} \qquad \hat{x}(k) \begin{cases} -\frac{1}{2} \\ \frac{1}{2} \end{cases}$$

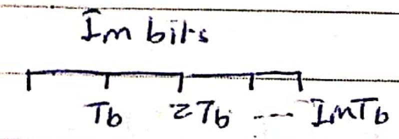
19/11/2019 . الثلاثاء

Lecture #15 M-ary Digital carrier Modulation.

Last lecture B-Ask , B-FSK , B-PSK

- for higher data Bit rate
  transmit $I_m = \log_2 M$ over each signaling period T_b

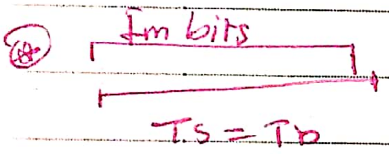
$$\text{data bit rate} = \frac{I_m\,[\text{bits}]}{T_b\,[\text{sec}]} = I_m R_b \ \text{bits/s}$$



$$\text{data Rate} = \frac{I_m}{I_m T_b} = R_b \ \text{bps}$$

data Rate أكبر بس ال BW أقل.

$$BW \propto R_s = \frac{1}{T_s} = \frac{R_b}{I_m}$$ spectral efficiency I_m



data Rate $= I_m R_b$ bps

$BW = R_b$

$+ BW \propto R_b$

$$S.e = \frac{I_m R_b}{R_b} = I_m\ (\text{bps})/Hz$$

① M-ary Ask

Recall B-ask

"1" → $p(t)\cos(\omega_c t)$

"0" → 0

M-ary ASK

transmit $I_m = \log M$ bits over 1 T_b Signaling Period
Using One of the² Following → Band pass signals

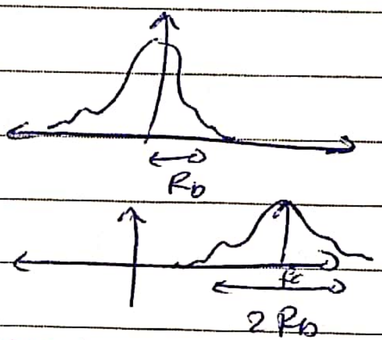
6 $0, p(t)\cos(w_c t), 2p(t)\cos(w_c t), \dots (M-1)p(t)\cos(w$

4-ary ASK

$I_m = 2$ bits $M = 4$ symbols

$00 \rightarrow 0$ $0 \leq t \leq T_b$
$01 \rightarrow p(t)\cos(w_c t)$
$10 \rightarrow 2p(t)\cos(w_c t)$
$11 \rightarrow 3p(t)\cos(w_c t)$.

↓ high avg power



[2] M-ary FSK (orthogonal Signaling)

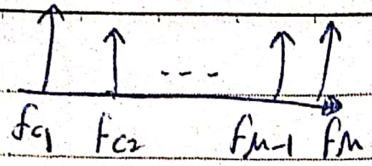
Recall "1" $\rightarrow p(t)\cos(2\pi f_{c1} t)$ $0 \leq t \leq T_b$
"0" $\rightarrow p(t)\cos(2\pi f_{c2} t)$

M-ary

transmit I_m bits Using Signals $p(t)\cos(w_{c_i} t)$ $i = 1, 2, \dots, M$

4-ary fsk $M = 4$ $I_m = 2$ bits/symbol

$00 \rightarrow p(t)\cos(w_{c1} t)$ $0 \leq t \leq T_b$
$01 \rightarrow p(t)\cos(w_{c2} t)$
$10 \rightarrow p(t)\cos(w_{c3} t)$
$11 \rightarrow p(t)\cos(w_{c4} t)$



dis- needs Big BW

adv- power less than Ask.

$BW \sim f_M - f_1$

for equally seperated freq.

$f_i = f_1 + (i-1) \cdot \underset{\hookrightarrow \text{Seperation}}{\delta f}$ $i = 1, 2, 3, \ldots, M$

$f_1 = f_1$

$f_2 = f_1 + \delta f$

$f_3 = f_1 + 2\delta f$

δf ↙ Small

Save BW

But Reciever performance ↓

δf such that (symbols are orthogonal)

for orthogonal symbols chose f_M & f_N such that

$$\int_0^{T_b} p(t)\cos(2\pi f_M t) \cdot p(t)\cos(2\pi f_N t) = 0$$

chose $p(t) = A$ $0 \le t < T_b$ (rect pulse)

$$A^2 \int_0^{T_b} \left[\frac{1}{2}\cos\left(2\pi(f_m + f_N)t\right) + \frac{1}{2}\cos\left(f_M - f_N\right)\right] dt$$

$$\frac{A^2}{2}\int_0^{T_b} \cos\left(2\pi(f_m + f_N)\,t\right) + \int_0^{T_b} \cos\pi\left(f_m - f_N\right)t\,]\,dt$$

$$\underbrace{\frac{A^2}{2}\left[\frac{T_b \sin\left(2\pi\left(f_m + f_N\right)(T_b)\right)}{2\pi\left(f_m + f_N\right)T_b}\right.}_{\simeq \text{Zero}} + \frac{A^2}{2}\,\frac{T_b \sin\left(\pi(f_M - f_N)\,T_b\right.}{2\pi\left(f_M - f_N\right)T_b}$$

$$\frac{A^2}{2} \sin \frac{(2\pi(f_M - f_N) T_b)}{2\pi(f_M - f_N)^{m}} = 0$$

; I need every two frequincies orthogonal

$$= \sin(2\pi(f_M - f_N) T_b) = 0$$

$$f_M = f_1 + (M-1)\,\delta f$$

$$f_N = f_1 + (N-1)\,\delta f$$

$$f_M - f_N = (M-N)\,\delta f$$

$$\sin(2\pi(M-N)\delta f\, T_b = 0$$

$$\cancel{2\pi}(M-N)T_b = +k\cancel{\pi}$$

$$\delta f = \frac{\boxed{k}}{2\boxed{(M-N)}T_b}$$

k=1 minimum

→ 1 (minimum) value.

$$\boxed{\delta f = \frac{1}{2T_b}} \leftrightarrow \boxed{T_s = T_b}$$

$$BW = (f_M - f_1) + 2B$$

↳ Bw for Baseband signal

3 M-ary psk

Recall B-psk $p(t)\cos(w_c t + \theta_i)$ $\theta_i = 0, \pi$

$\theta_i = 0 \to$ "1", $\pi \to$ "0"

M-ary psk transmit I m bit using one of $p(t)\cos(w_c t + \theta_i)$

$$\theta_i = 0^\circ + \frac{2\pi}{M}(i-1)$$

4-ary psk Q-PSK

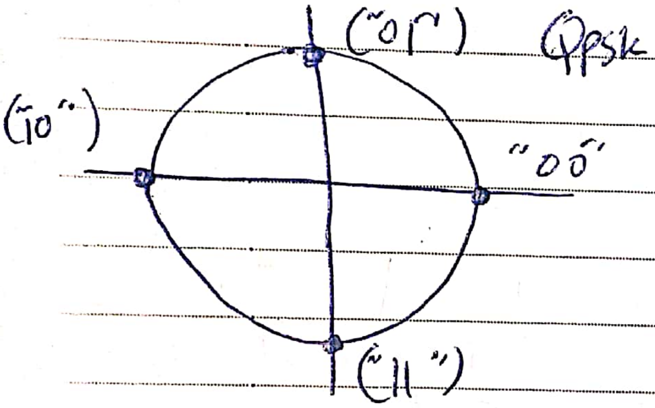
$\theta_0 = 0^\circ$

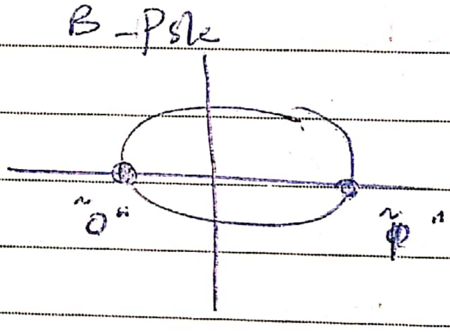
"00" → $p(t) \cos(w_c t + 0)$

'01' → $p(t) \cos(w_c t + \frac{\pi}{2})$

"10" → $p(t) \cos(w_c t + \pi)$

"11" → $p(t) \cos(w_c + \frac{3\pi}{2})$





$p(t) \cos(w_c t + \theta_i)$

lecture #15

الخميس ١١/٢١

1. M-ary ASK
2. M-ary FSK

▨ "Im bits" $\rightarrow p(t)\cos(2\pi f_c t)$

$$f_i = f_1 + (i-1)\delta f$$

$$\delta f = \frac{1}{2T_b}$$

3. M-arry PSK $\rightarrow p(t)\cos(w_c t + \theta_i)$

$$\theta_i = \theta_0 + (i-1)\left(\frac{2\pi}{M}\right)$$

$$p(t)\cdot\cos(w_c t + \theta_i) = [p(t)\cos\theta_i]\cdot\cos(w_c t) + [-p(t)\sin(\theta_i)]^{\sin}\cdot\sin(w_c t)]$$

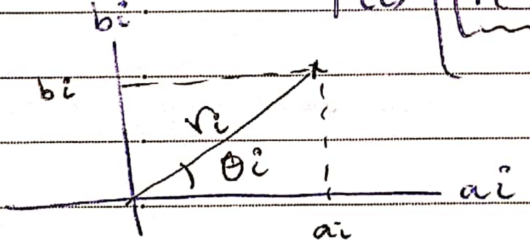
4. Mary QAM (Qudrature Amplitude modulation).

Im bits $\rightarrow \underline{\underline{r_i}}\ p(t)\cos(w_c t - \underline{\underline{\theta_i}})$

$$r_i = \sqrt{a_i^2 + b_i^2} \qquad \theta_i = \tan^{-1}\left(\frac{b_i}{a_i}\right)$$

$$p(t)\left[\left[\overset{\cos\theta_i}{\underbrace{r_i\cos(w_c t)}}\right]^{\cos(w} + r_i\sin\theta_i\sin w_c t\right]$$

$$a_i\cos(w_c t) + b_i\sin(w_c t).$$



$a_i = \pm1, \pm2, \ldots$

$b_i = \pm1, \pm2, \ldots$

Ex- 4-ary QAM $M = 4$ $Im = 2$

$a_i = \pm1$ $b_i = \pm1$

(a_i, b_i) signal

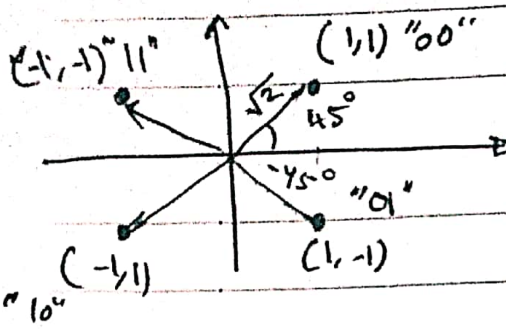
"00" $\rightarrow (+1, +1) \rightarrow [\cos w_c t + \sin(w_c t)]\,p(t)$

"01" $\rightarrow (+1, -1) \rightarrow [\cos(w_c t) - \sin(w_c t)]\,p(t)$

"10" $\rightarrow (-1, +1) \rightarrow [-\cos(w_c t) + \sin(w_c t)]\,p(t)$

"11" $\rightarrow (-1, -1) \rightarrow [-\cos(w_c t) - \sin(w_c t)]\,p(t)$

4_psk ⇒ QAM same performance.



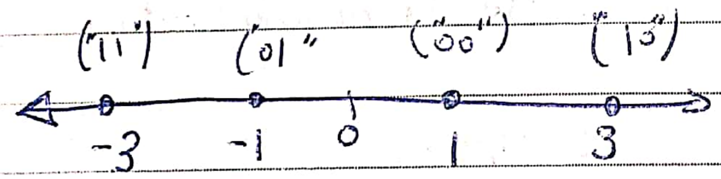
$\boxed{5}$ M-ary PAM (pulse amplitude Modulation).

"Im bits" → $a_i\, p(t) \cos(\omega_c t)$

$a_i = \pm 1, \pm 3, \pm 5, \dots \pm N-1$

Ex 4-ary Pam M = 4 Im = 2

	a_i	
00 →	+1	$p(t) \cos(\omega_c t)$ ~~[illegible]~~
01 →	-1	$-p(t) \cos(\omega_c t)$
10 →	+3	$3\, p(t) \cos(\omega_c t)$
11 →	-3	$-3 p(t) \cos(\omega_c t)$



geometrical Representation

Ex 16-QAM M = 16 Im = 4

$a_i = \pm 1, \pm 2$

$b_i = \pm 1, \pm 2$

$a_i + j b_i$

0000 → $1 + j1$

0001 $1 - j1$

⋮ $-1 + j$

  $-1 - j$

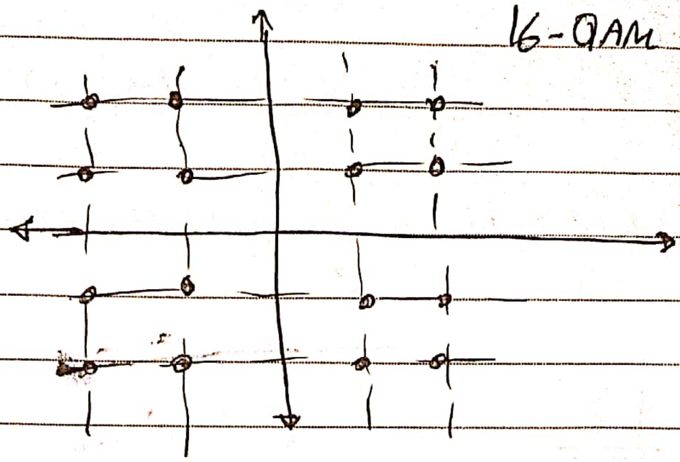
1111 $1 + j2$

  $1 - j2$

  $-1 + j2$

  $-1 - j2$

  ↓



Lecture #16

Coherent Recievers for Binary Digital Carrier modulations.

B-PSK :- $\longrightarrow$ baseband.

"1" → $p(t)\cos(\omega_c t) * \sqrt{2} = d(t)$

"0" → $-p(t)\cos(\omega_c t) \Rightarrow * \sqrt{2} = -d(t)$

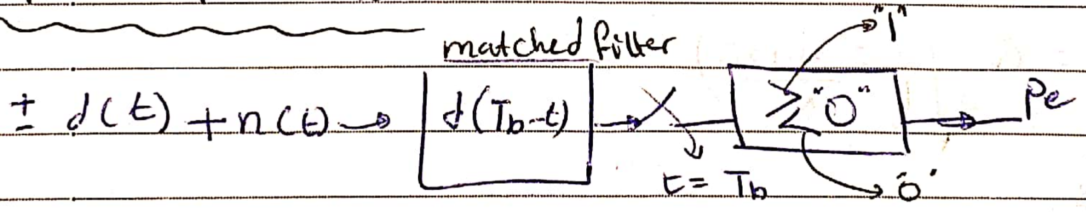
For polar Optimum Reciever matched filter

$\longrightarrow$ RF pulse/ Band pass pulse.

"1" → $d(t)$ $0 < t \leq T_b$

"0" → $-d(t)$

Optimum Rx structure



$$P_e = Q\left(\sqrt{\frac{2E_d}{N}}\right)$$

$$E_d = \int_0^{T_b} d^2(t)\,dt = 2\int_0^{T_b} \cos^2\omega_c t \,.\, p^2(t)\,dt$$

$$= \int_0^{T_b} 2\,p(t)\left[\frac{1}{2}+\frac{1}{2}\cos(2w_ct)\right]dt$$

$$\int_0^{T_b} p^2(t)\,dt + \overset{0}{\cancel{\int_0^{T_b} p(t)\cos(2w_ct)\,dt}} \rightarrow f_c \text{ in MHz or GHz.}$$

$$= \underline{E_p}$$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{2E_p}{N}}\right)$$

P energy perbit assume equally propability.

$$\boxed{E_b = E_d = E_p} \qquad p_e = Q\left(\sqrt{\frac{2E_b}{N}}\right)$$

∴ for polarand B-psk have the same performance.

$\underline{h}$ → fading coeficient $\qquad h = |h|\,e^{jB_n}$

$$p_e = Q\sqrt{\left(\frac{2E_b|h|^2}{N}\right)} \quad \underline{R.V}$$

↑ $\underline{R.V}$

$$\overline{P_e} = E\left[p_e\right]$$

$$= \int p_e\, f_{P_e}(p_e)\,dp_e) = E\left[\qquad\right]$$

$$= \int Q\left(\sqrt{\frac{2E_b|h|^2}{N}}\right) f_{\underset{|h|^2}{X}}(X)$$

26/11

lecture #16 Comp.　　　　　　　　　　　　　　　　الشرقاد

Coherent $\underline{Rx}$ for BFSK ← orthogonal

"1" $\longrightarrow \sqrt{2}\,p(t)\cos(2\pi f_1 t) = d(t)$ $0 \le t < T_b$

"0" $\longrightarrow \sqrt{2}\,p(t)\cos(2\pi f_2 (t) = q(t)$

$$f_2 = f_1 + \delta_f = f_1 + \frac{1}{2T_b}$$

$$\Delta f = f_2 - f_1 = \frac{1}{2T_b} \longrightarrow$$

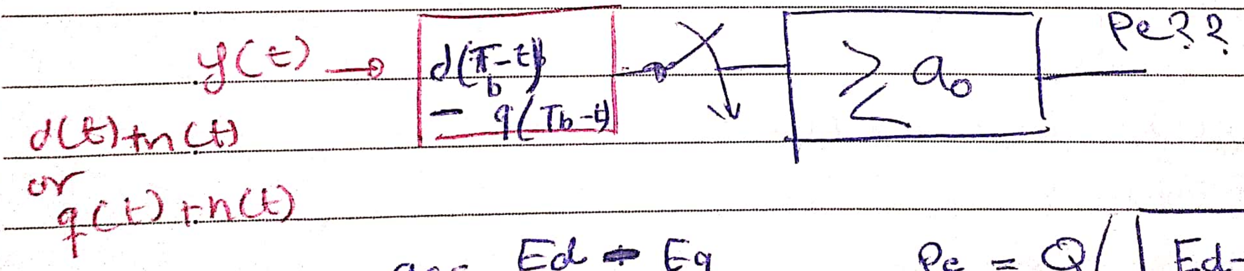
Minimum freq. Separation for orthogonal BFSK

↓ minimum Shift BFSK $\triangleq$ MSK

optimum Rx

"1" $\longrightarrow d(t)$

"0" $\longrightarrow q(t)$



$$a_0 = \frac{E_d - E_q}{2} \qquad Pe = Q\left(\sqrt{\frac{E_d+E_q-2E_{dq}}{2N}}\right)$$

$$E_d = \int_0^{T_b} d^2(t)\,dt = \int_0^{T_b} 2p^2(t)\cos(2\pi f t)\,dt = \underline{E_p}$$

$E_q = E_p$

$E_{dq} =$ zero (orthogonal) $\longrightarrow \Delta f = \frac{1}{2T_b}$

Eb ⇒ Energy Per bit = $\frac{Ea + Eq}{2} = Ep$

$$P_e = Q\left(\sqrt{\frac{E_b}{N}}\right) = Q\sqrt{\left(\frac{E_p}{N}\right)}$$

lecture #17 optimum Coherent Rx for M-ary digital communication systems.

- transmit $I_m = \log_2 M$ bits Using 1 of the Signals is

$S_1(t)$

$S_2(t)$

$\vdots$

$S_M(t)$

↳ ⇒ $y(t) = S_i(t) + n(t)$ $0 \le t \le T_0$

Recieved signal

- Space representation

Signals (in time domain) $S_1(t), S_2(t), \ldots, S_M(t)$ ↓

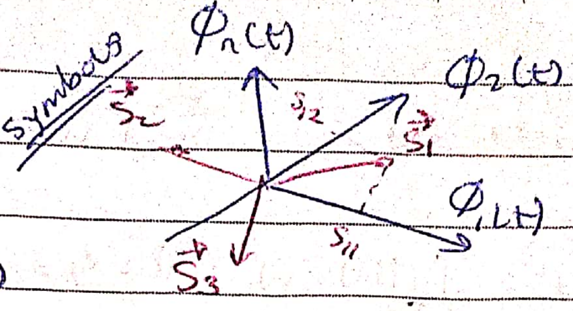
Space domain

n-dimensional space ←

$\vec{S_1}, \vec{S_2}, \ldots \vec{S_M}$

$\vec{S_i} = (S_{i1}, S_{i2}, \ldots, S_{i\underline{\underline{n}}}$

ex = $\underline{\underline{n=3}}$ dimensions



$\phi_1(t), \phi_2(t), \phi_3(t)$ --- $\phi_n(t)$

↳ called the $\underline{\underline{\text{orthonormal}}}$ basis function (basis set).

Energy = 1

$$\langle \phi_i(t) . \phi_k(t) \rangle = \int_{t \in T_b} \phi_i(t) \, \phi_k(t) \, dt .$$

orthogonal

$$= \begin{cases} \underline{\underline{0}} , & i \neq k \\ \underline{\underline{1}} , & i = k \end{cases}$$

normal

$$S_i(t) = S_{i1}\phi_1(t) + S_{i2}\,\phi_2(t) + \cdots + S_{iN}\phi_n(t)$$

$$\Rightarrow S_i(t) = \sum_{k=1}^{N} S_{ik}\,\phi_k(t)$$

$$S_i(t) = \vec{S}_i = (S_{i1}, S_{i2}, --- , S_{in})$$

$$S_{ik} = \int_{T_b} S_i(t)\,\phi_k(t)\,dt .$$

$i = 1, 2, --- , M$

$k = 1, 2, --- , N$

$$S_2(t) \rightarrow \vec{S}_2 = (S_{21}, S_{22}, \ldots , S_{2n})$$

$$S_{21} = \int S_2(t)\,\phi_1(t)\,dt$$

$$S_{22} = \int S_2(t)\,\phi_2(t)\,dt .$$

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time domain

① $E_{si} = \int_{t \in T_0} s_i^2(t)\, dt)$

Cross Product
↓

② $E_{s_i, s_j} \langle s_i(t), s_j(t) \rangle = \int_{T_0} s_i(t) \cdot s_j(t)$

③ $d_{s_i s_j} = \sqrt{\int (s_i(t) - s_j(t))^2}$

الطول

space domain

① $E_{s_i} = \|\vec{s}\|^2 = s_{i1}^2 + s_{i2}^2 + \cdots, s_{in}^2 = \sum_{k=1}^{N} s_{ik}^2$

② $E_{s_i s_j} = \langle \vec{s_i} \cdot \vec{s_j} \rangle$

$s_{i1} s_{j1} + s_{i2} s_{j2} + \cdots + s_{ni} \cdot s_{jn}$

$= \sum_{k=1}^{n} s_{ik} s_{jk}$

distance between

③ $d_{s_i s_j} = \|\vec{s_i} - \vec{s_j}\|$

$= \sqrt{\sum_{k=1}^{n} (s_{ik} - s_{jk})^2}$

↑

Euclidean distance.

M-ary digital Com. system

transmit $I_M = \log_2 M$ bits using one of the following $s_1(t), --- s_M(t)$, over $(\underline{0 \le t \le T_b})$

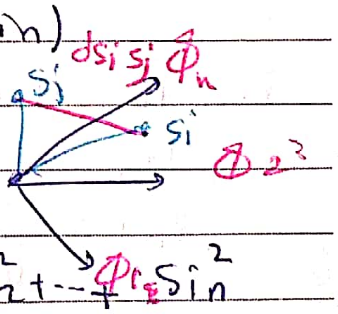
time domain $\Longrightarrow$ space domain
with dimension $n \le m$
we need orthonormal basic function
$\{\phi_i(t)\} = i = 1, 2, 3, --- M$
$\langle \phi_i(t), \phi_j(t) \rangle = \begin{cases} 1, & i=j \\ 0, & i \ne j \end{cases}$

decomposition:

$$s_i(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + ---$$
$$= \sum_{k=1}^{n} s_{ik}\phi_k(t)$$

$$s_i(t) \longrightarrow \vec{s_i} = (s_{i1}, s_{i2}, ---, s_{in})$$

$$s_{ik} = \int_0^{T_b} s_i(t)\,\phi_k(t)\,dt$$



$$E_{s_i} = \int_0^{T_b} s_i^2(t) = \|s_i\|^2 = s_{i1}^2 + s_{i2}^2 + --- + s_{in}^2$$

$$d_{s_i s_j} = \| \vec{s_i} - \vec{s_j} \|$$

we use the gram-schmidt orthogonalization to find the basis set $\{\phi_i(t), i = 1, 2, 5, --- n\}$ from $s_1(t), s_2(t), --- s_M(t)$

Steps: ① $\phi_1(t) = \dfrac{s_1(t)}{\sqrt{E_{s_1}}}$, $E_{s_1} = \int_0^{T_0} s_1^2(t)\,dt$

s_{11}

$$s_1(t) = \sqrt{E_{s_1}}\ \phi(t)$$

② define $g_2(t) = S_2(t) - S_{21}\phi_1(t)$

where $S_{21} = \int S_2(t)\,\phi_1(t)\,dt$.

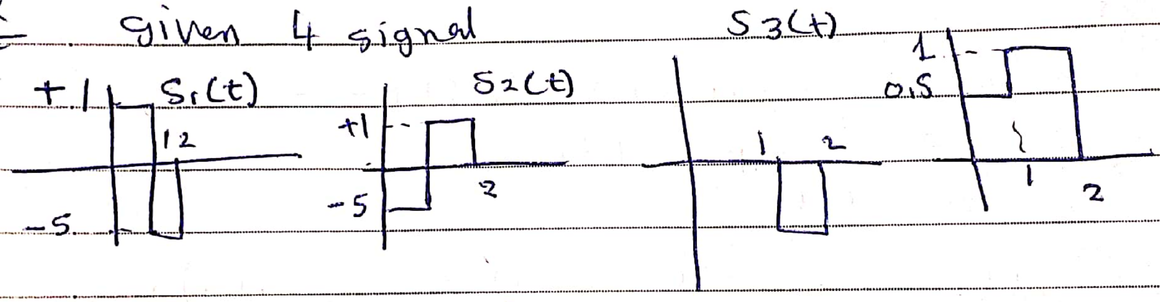
$\phi_2(t) = \dfrac{g_2(t)}{\sqrt{E_{g2}}}$ $\quad E_{g2} = \int_0^{T_0} g^2(t)\,dt$.

③ define $g_3(t) = S_3(t) - S_{31}\phi_1(t) - S_{32}\phi_1(t)$

$\phi_3(t) = \dfrac{g_3(t)}{\sqrt{E_{g3}}}$

Stop when $g_i(t) = 0$

Ex given 4 signal



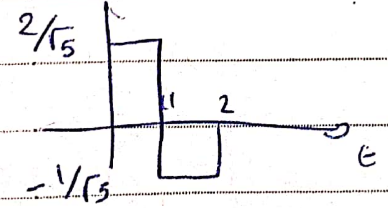
Find the set of basis functions $\phi_1(t), \cdots \phi_4(t)$

Steps

$\phi_1(t) = \dfrac{S_1(t)}{\sqrt{E_{S1}}}$ $\frac{\sqrt{5}}{4}$, $E_{S1} = \int_0^1 (1)^2\,dt) + \int_1^2 (0.5)^2\,dt$

$= 1 + .25 = 1.25 = \boxed{\sqrt{5}/2 = \sqrt{E_{gs1}}}$

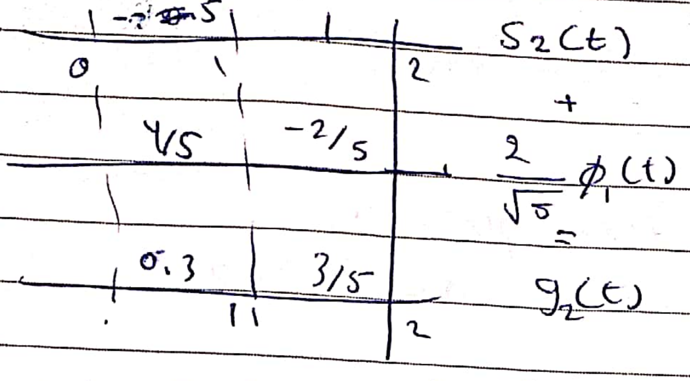
$\phi_1(t) = \dfrac{2}{\sqrt{5}} S_1(t)$



Step 2 :- define $g_2(t) = S_2(t) - S_{21}\phi_1(t)$

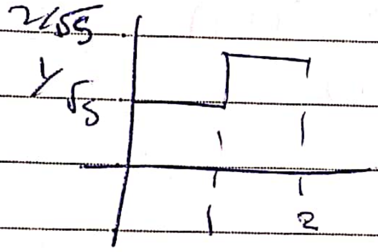
$S_{21} = \int_0^2 S_2(t)\phi_1(t) * \frac{2}{\sqrt{5}}$

$g_2(t) = S_2(t) + \frac{2}{\sqrt{5}}\phi_1(t)$



$\phi_2(t) = \frac{g_2(t)}{\sqrt{}}$ — $\int_0^2 g_2^2(t)\,dt = \int_0^1 (.3)^2 + \int_1^2 \frac{9}{25}\,dt$

$\phi_2(t) = \frac{\sqrt{3}}{3} g_2(t)$ $= \frac{9}{100} + \frac{9}{25} = \frac{9}{20}$



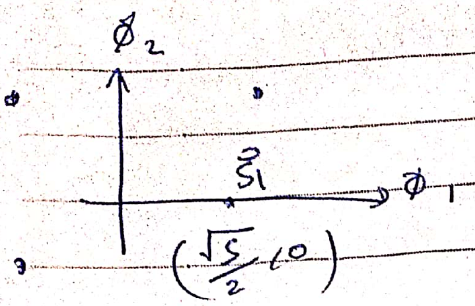
Step 3 :-

$g_3(t) = S_3(t) - S_{31}\phi_1(t) - S_{23}\phi_2(t)$ ------ = zero

we have just $\phi_1(t)$, $\phi_2(t)$:- $\vec{S}_3 = (S_{31}, S_{32}) = (\cdot\cdot, \cdot$

$\vec{S}_1 = (S_{11}, S_{12}) = (\sqrt{5}/2, 0)$ $\vec{S}_4 = (S_{41}, S_{42}) = (\cdot\cdot,$

$\vec{S}_2 = (S_{21}, S_{22}) = (-\frac{2}{\sqrt{5}}, \text{--})$



$s_1(t) \Rightarrow \phi_1(t) = \frac{s_1(t)}{\sqrt{E_{s1}}}$

$s_1(t) = \underline{\underline{\phi_1(t)}} \sqrt{E_{s1}}$

نفس الاتجاه

Tx

Rx :-

$s_i(t) \longrightarrow y(t) = s_i(t) + n(t)$

$0 \leq t < T_0$ ∴ deterministic ↗ ↘ R.P

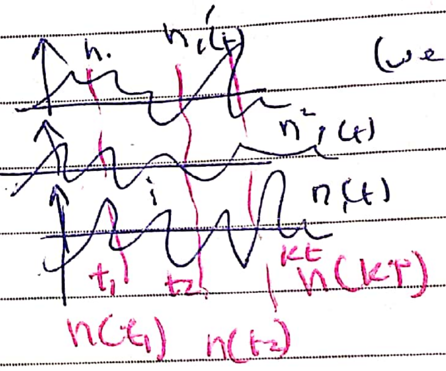
① $n(t)$

① zero mean

$E(n(t)) = 0$

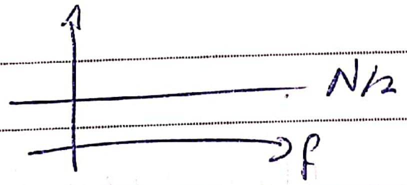
$n(t)$ R.P

Family of sample functions



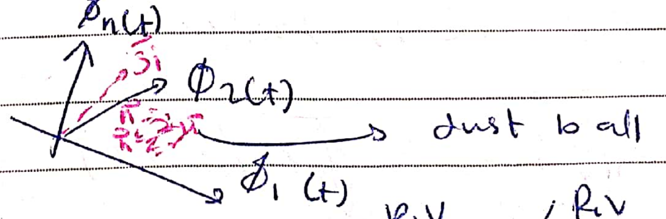
(we define samples)

② white



$R_n(t_j - t_i) = R(\tau) = \frac{N}{2}\delta(\tau)$

③ Gaussian



R.P
$n(t) \longrightarrow \vec{n} = (n_1, n_2, \cdots n_n)$
R.V R.V R.V

$$n(t) = \sum_{k=1}^{n} n_k \phi_k(t)$$

$$n_k = \int_0^{Tb} \underset{\uparrow \, R.P}{n(t)} \, \phi_k(t) \, dt$$

$$E[nk] = 0 = \int_0^{T_b} E[\overset{0}{\cancel{nt)}}] \, \phi_k t \; dt.$$

n(t) is gaussian Rp so, nk is gaussion

$$n_k \sim N(0, \sigma_k^2)$$

$$\vec{n} = (n_1, n_2, ---, n_n)$$

↳ jointy gaussion R.vector

$$f_{\vec{n}}(\vec{n}) = f_{n_1, n_2, ---n_n}(n_1, n_2, ---) = \frac{|[C_n]^{-1}|^{1/2}}{2\pi \, (N/2)}$$

$$= e^{-\frac{(\vec{n} - \vec{\bar{n}})^T [C_n]^{-1} (n - \vec{\bar{n}})}{2}}$$

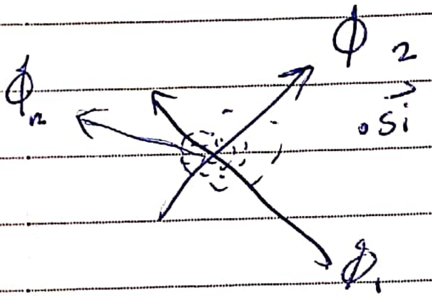
R.P

$n(t) \longrightarrow \vec{n} = (n_1, n_2, \dots, n_n)$

R vector

Gaussian $\quad n_k = \int_{T_0} n(t)\, \phi_k(t)\, dt$

$s_i(t) \longrightarrow \vec{s_i} = (s_{i1}, s_{i2}, \dots, s_{in})$



$E[n_k] = 0$

$f_{\vec{n}}(\vec{n}) = f(n_1, n_2, \dots n_n)$

$$= \frac{1}{(2\pi)^{n/2}\sqrt{|[C_n]|}}\, e^{-\frac{1}{2}[\vec{n}-\bar{\vec{n}})^T [C_n]^{-1} [\vec{n}-\bar{n}]}$$

$$[C_n] = \begin{bmatrix} \overset{\sigma_1^2}{C_{11}} & C_{12} & --- & C_{1n} \\ \vdots & \overset{\sigma_2^2}{C_{22}} & & \\ \vdots & & \overset{\sigma_3^2}{C_{33}} \;\; \dots & \overset{\sigma_n^2}{C_{nn}} \\ C_{n1} & & & \end{bmatrix}$$

$c_{ij} = E[(n_i - \bar{n}_i)(n_j - \bar{n}_j)]$

$E[n_i \cdot n_j] - \cancel{\bar{n}_i \cdot \bar{n}_j}$

$$C_{ij} = E[n_i n_j] = E\left[\int_{T_0} n(t)\,\phi_i(t)\,dt - \int_{T_0} n(\lambda)\,\phi_j(\lambda)\,d\lambda\right]$$

constants

$$= \iint \underset{\text{corrolation}}{E}\left[n(t)\,n(\lambda)\,\phi_i(t)\,\phi_j(\lambda)\,dt\,d\lambda\right.$$

$$= \iint R_n(t,\lambda)\,\phi_i(t)\,\phi_j(\lambda)\,dt\,d\lambda$$

$$= \iint R_n(t-\lambda)\,\phi_i(t)\,\phi_j(\lambda)\,dt\,d\lambda$$

$$= \iint \frac{N}{2}\,\delta(t-\lambda)\,\phi_i(t)\,\phi_j(\lambda)\,d\lambda\,dt$$

$$= \frac{N}{2}\int \phi_i(t)\,\phi_j(t)\,dt$$

$$= \begin{cases} 0, & i \neq j \\ \frac{N}{2}, & i = j \end{cases}$$

$C_{ij} = 0 \quad , i \neq j$

$C_{ii} = \sigma_1^2 = N/2$

$$= \quad C[n] = \begin{bmatrix} N/2 & 0 & 0 & \\ 0 & N/2 & & \\ 0 & 0 & N/2 & \cdots N/2 \end{bmatrix}$$

$n_k \sim N(0, N/2)$

$q_{ij} = 0 \quad i \neq j$

Uncorrelated

[illegible] independent

$$f_{n_1, n_2, \dots, n_n}(n_1, n_2, \dots, n_n) = f_{n_1} \cdot f_{n_2}(n_1) \cdot \dots$$

$$\underline{\underline{ex}} \leftarrow \prod_{i=1}^{n} f_{n_i}(n_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \frac{N}{2}}} e^{-n_i^2/\cancel{2}N/\cancel{2}}$$

$$= \frac{1}{(\pi N)^{n/2}} e^{-\sum_{i=1}^{n} n_i^2/N} = \boxed{\frac{1}{(\pi N)^{-n/2}} e^{-\|\vec{n}\|^2/N}}$$

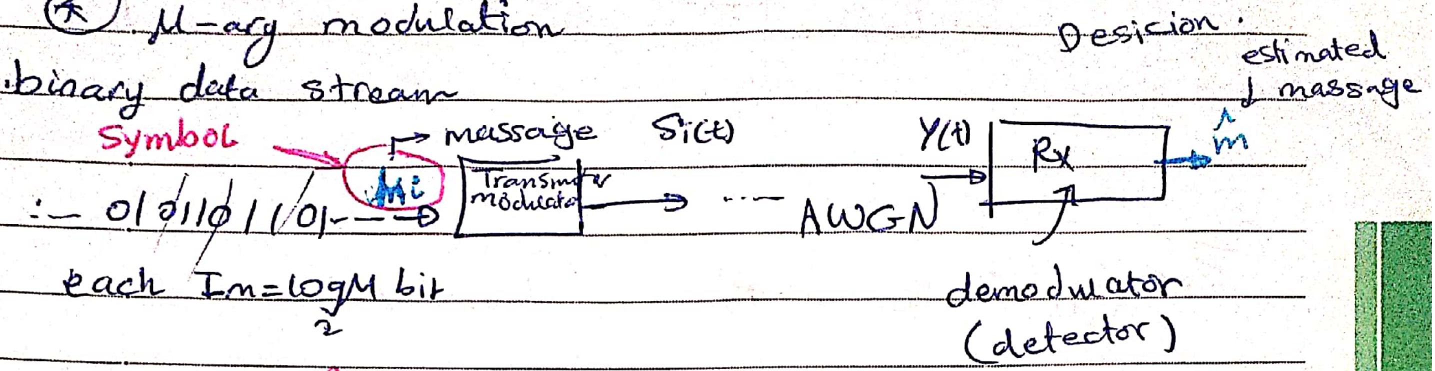
$n(t) \rightarrow \vec{n}$

Lecture #18

optimum Rx for AWGN channels.

⊛ M-ary modulation

binary data stream



m_1

m_2

$\vdots$

m_M

R.P R.P

$$y(t) = s_i(t) + n(t) \qquad 0 < t \leq T_0$$

Recieved signal

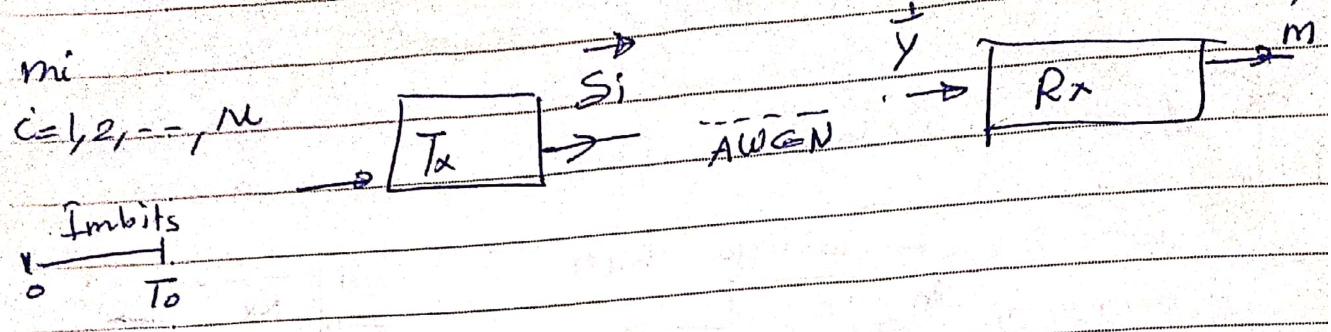
symbol (signal) wave form (determenstic).

* from knowladge of y(t), the Reciever must decide which massage has been transmitted.

$p(m_i)$ → Priori probability.

(1) ** Design the optimum Rx subject to minimizing the probability of error,

space domain



$\vec{S}_i = (s_{i1}, S_{i2}, S_{i3}, \dots, S_{in})$ $\quad i=1,2,\dots,M$

$\vec{n} = (n_1, n_2, \dots, n_n)$

$n_i \sim N(0, \frac{N}{2})$

$\tilde{n}$. Dimensional Gaussion R.V

$f_{\vec{n}}(n_1, n_2, \dots n_n) = \frac{1}{(\pi N)^{-n/2}} e^{-\|\vec{n}\|^2/N}$

$\vec{y} = \vec{S}_i + \vec{n}$

$y_l = S_{il} + n_l$

$\hookrightarrow N(0, N/2)$

$\rightarrow N(S_{il}, N/2)$

y_l $\forall l$ are independent

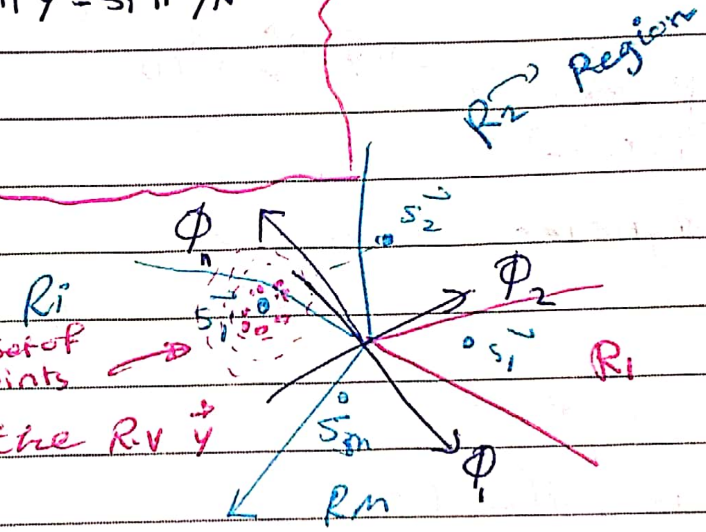
for all l

$$f_{\vec{y}} = f_{x_1 x_2 x_3,}(y_1, y_2, \dots y_n) = \prod_{i=1}^{n} f_{y_e}(y_e)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{\cancel{2}\pi N/\cancel{2}}} e^{-(y_l - s_i)^2/\cancel{2}N/\cancel{2}}$$

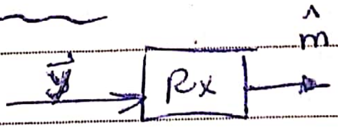
$$= \;\; = \frac{1}{(\pi N)^2} e^{-\sum_{l=1}^{n} (y_e - s_{ie})^2/N}$$

$$\boxed{= \frac{1}{(\pi N)^{n/2}} e^{-\|\vec{y} - \vec{s}_i\|^2/N}}$$

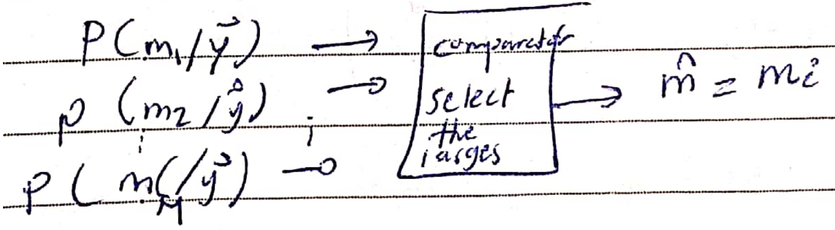


the set of all of these points represents the R.V $\vec{y}$

* Decision procedure..



Reciever decides that
$\hat{m} = m_i$ such that
$P(M_k/\vec{y})$; $k = 1, 2, \dots, M$
is max for $k = i$
given $\vec{y}$ is Recieved



The decision function $p(m_k/\vec{y})$

$$p(m_k/\vec{y}) = \frac{f_{\vec{y}}(\vec{y}/m_k)\,p(m_k)}{\underbrace{f_{\vec{y}}(\vec{y})}} \rightarrow \text{ignored}$$

(ثابتة للجميع) .

Decision function:

$$= f_{\vec{y}}(\vec{y}/m_k)\,p(m_k)$$

$$= \frac{p(m_k)}{\cancel{(\pi N)^{n/2}}}\, e^{-\|\vec{y}-\vec{s}_k\|^2/N}$$

ignored (ثابت لكل)

Energy

$$E_{s_i} = \int s_i^2(t)\,dt$$

or $\|s_i\|^2$

decision function

take the (Ln)

$$\ln(p\,m_k) - \frac{\|\vec{y}-\vec{s}_k\|^2}{N} \rightarrow \text{distance}$$

$$= \ln(P\,m_k) - \frac{1}{N}\langle \vec{y}-\vec{s}_k, \vec{y}-\vec{s}_k\rangle$$

$$\left\|\left(\ln(P\,m_k) - \frac{1}{N}\left[\cancel{\|y\|^2} + \|s_k\|^2 - 2\langle \vec{y}, \vec{s}_k\rangle\right]\right)\right\| \times N/2$$

ignored

$= E_{s_k}$

$$= \frac{N}{2}\ln(P\,m_k) - \frac{1}{2}\left[E_{s_k} - 2\langle \vec{y}, \vec{s}_k\rangle\right]$$

$$= \underbrace{\frac{N}{2}\ln(P\,m_k) - \frac{1}{2}\left[E_{s_k}\right]}_{a_k} + \langle \vec{y}, \vec{s}_k\rangle$$

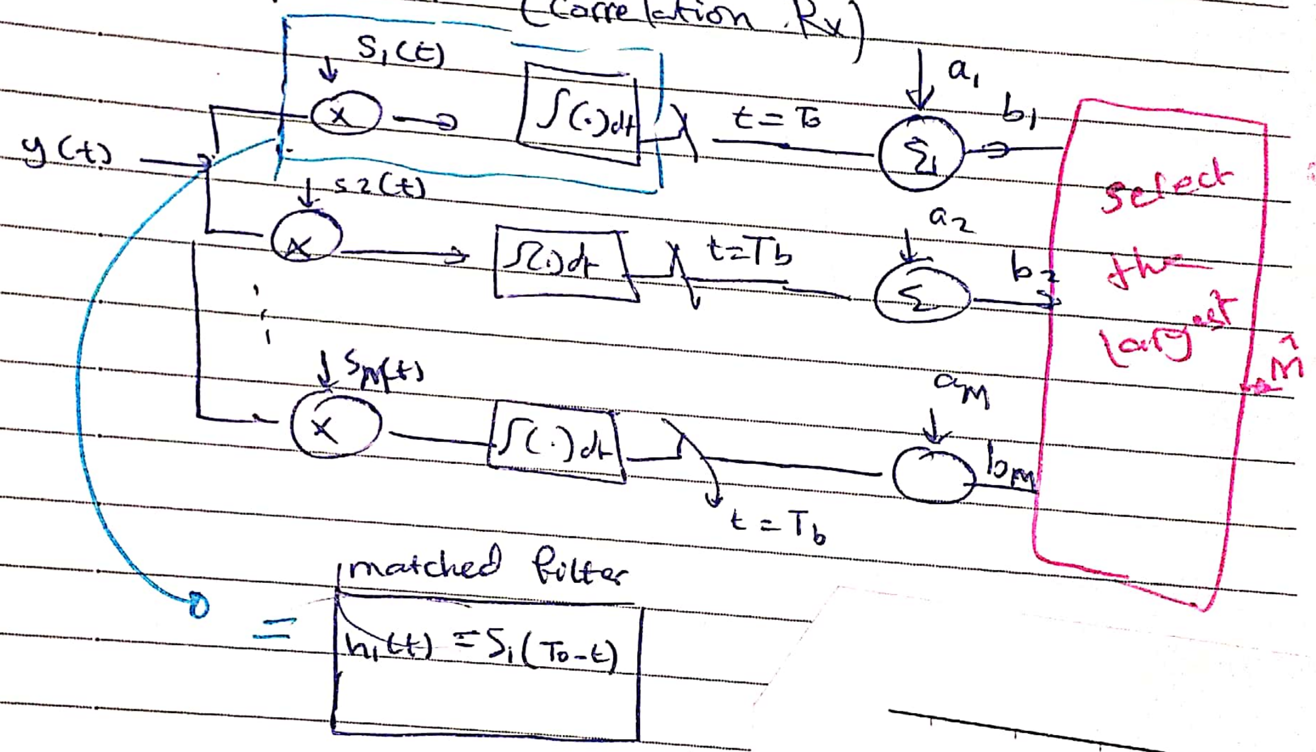
decision function

$b_k = a_k + \langle \vec{y} + \vec{s_k})$

where $a_k = \frac{n}{2} \ln(P_{mk}) - \frac{1}{2} E_{sk}$

$$\langle \vec{y}, \vec{s_k} \rangle = \int_0^{T_0} y(t)\, s_k(t)\, dt$$

optimum Reciever (correlation Rx)

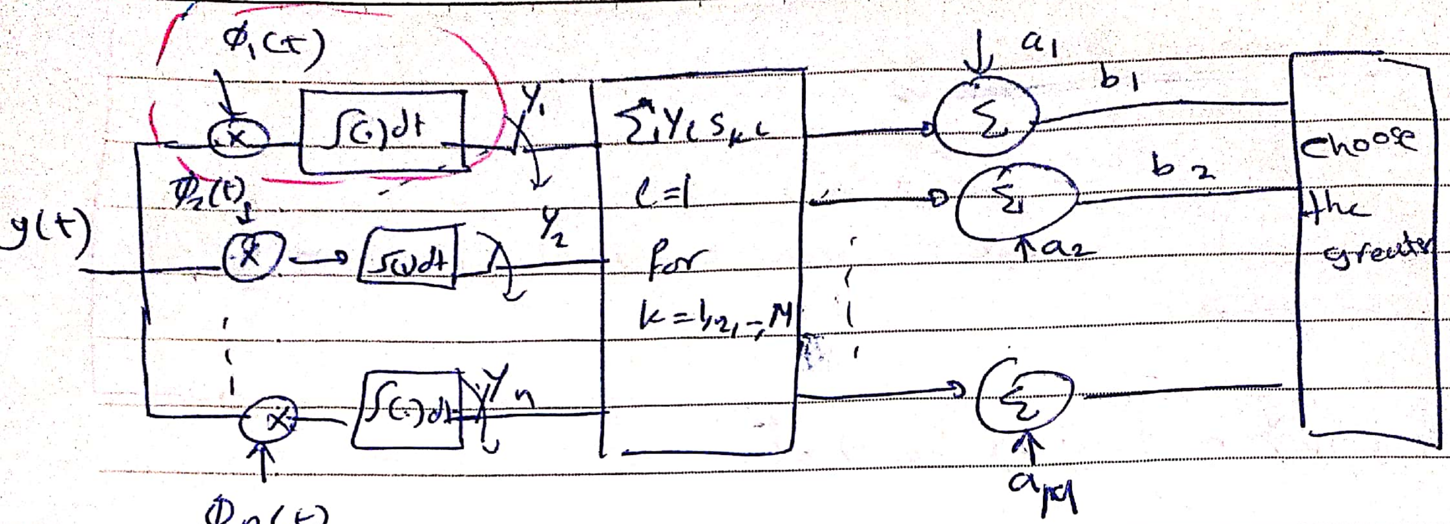


$b_k = a_k + \langle \vec{y}, \vec{s_k})$

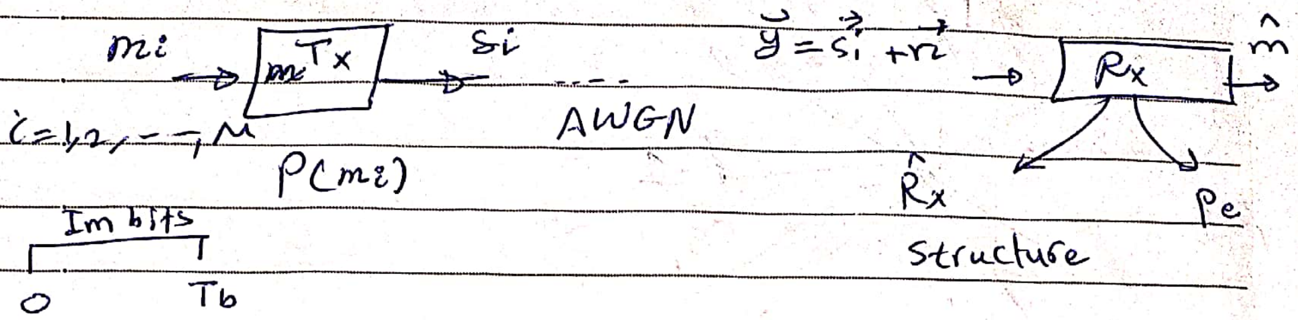
$$\langle \vec{y}, \vec{s_k} \rangle = \sum_{l=1}^{N} y_l\, s_{kl}$$

$$y_l = \int y(t)\, \phi_l(t)\, dt$$

matched filter $\boxed{\phi_1(T_b - t)}$



M-ary modulation AWGN



Decision procedure

$P(\vec{y}/m_k)$ for $k=1,2,\text{---},M$ is max for $k=i$

simplify the decision function

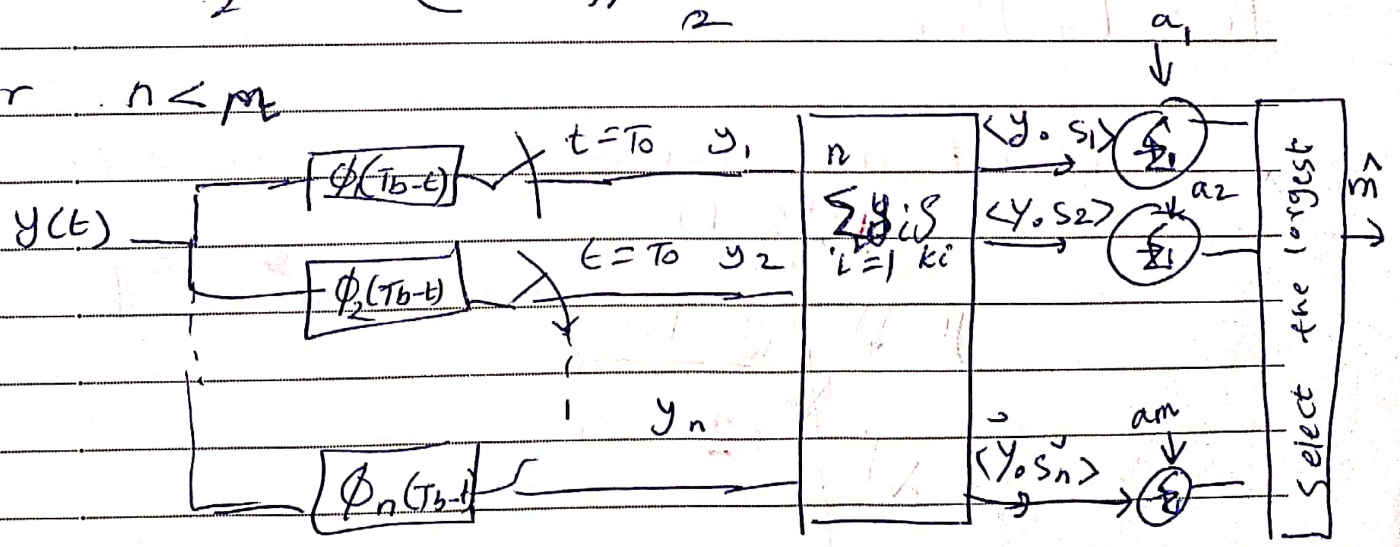
$N \ln(P(m_k)) - \|y - s_k\|^2$

→ decision function

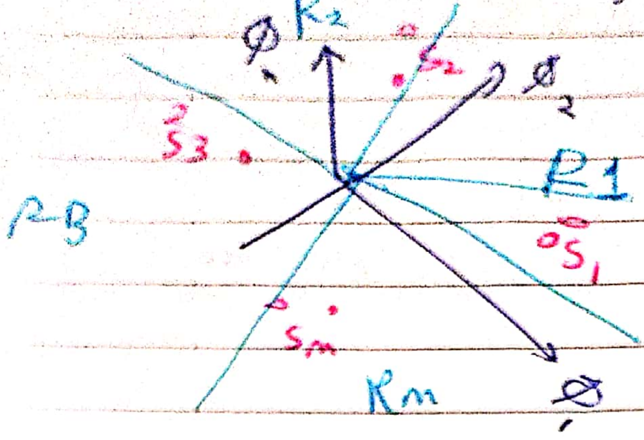
$b_k = a_k + \langle \vec{y}, \vec{s_k} \rangle$

$a_k = \frac{N}{2} \ln(P(m_k)) - \frac{E_k}{2}$

for $n < m$



(*) Decision Regions & Error probability.



P(C) ; the probability of correct reciption

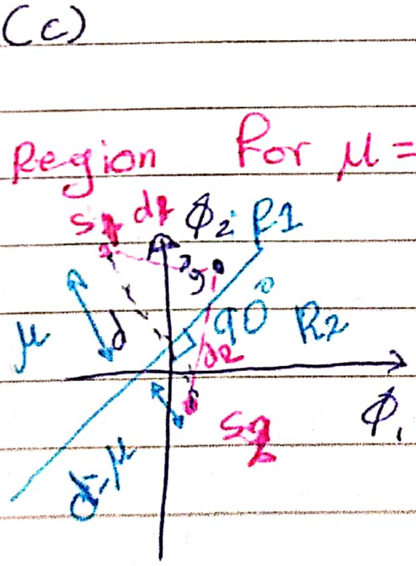
$$P(C/m_1)\,P(m_1) + P(C/m_2)\,P(m_2) ---- P(C/m_M)\,P(m_M)$$

$$= \sum_{i=1}^{M} \underline{P(C/m_i)}\,P(m_i)$$

such that $P(C/m_i) = P(\vec{y}/m_i \in R_i)$

$P_e = 1 - P(c)$

* decision Region for $\mu = 2$ and $n = 2$:



$d = \| \vec{s_2} - \vec{s_1} \|$

decision procedure

$$N\ln(P(m_1)) - \|\vec{y} - \vec{s}_1\|^2$$

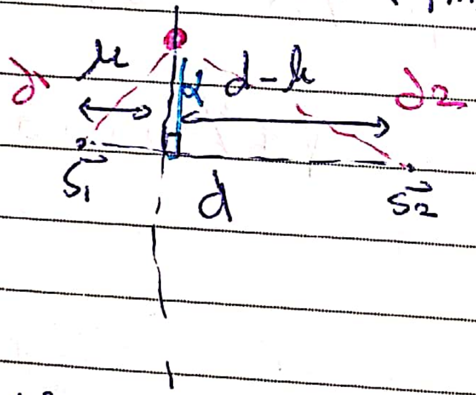
$$N\ln(P m_2) - \|y - s_2\|^2$$

$$\|y - s_1\|^2 - N\ln(P m_1) \underset{\hat{m} = m_2}{\overset{\hat{m} = m_1}{\lessgtr}} \|\vec{y} - \vec{s}_2\| - N\ln(P m_2)$$

$$d_1^2 - N\ln(P m_1) \lessgtr d_2^2 - N\ln(P m_2)$$

$$d_1^2 - d_2^2 \underset{m_2}{\overset{m_1}{\lessgtr}} N \ln\left(\frac{P m_1}{P m_2}\right)$$

if $(d_1^2 - d_2^2) = N \ln\left(\frac{P m_1}{P m_2}\right)$



$$d_1^2 = \alpha^2 + \mu^2 \qquad d_2^2 = \alpha^2 + (d-\mu)^2$$

$$d_1^2 - d_2^2 = \mu^2 - (d-\mu)^2$$

$$d_1^2 - d_2^2 = \cancel{\mu^2} - d^2 - \cancel{\mu^2} + 2\mu d$$

$$\boxed{d_1^2 - d_2^2 = 2\mu d - d^2} = N \ln\left(\frac{P m_1}{P m_2}\right)$$

$$\boxed{\mu = \frac{N \ln\left(\frac{P_{m_1}}{P_{m_2}}\right) + d^2}{2d}}$$

$$\boxed{\text{if } P(m_1) = P_{m_2} \qquad \mu = \cancel{N}\,\frac{d}{2}}$$

Ex:- Bpsk modulation :-

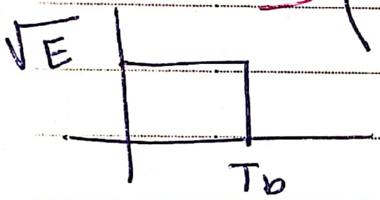
$$P_e = Q\sqrt{\frac{2E_b}{N}}$$

"1" = m_1 $\longrightarrow$ $S_1(t) = \sqrt{E^{q}/T_o}\ \cos(w_c t)$

"0" ← m_o $\longrightarrow$ $S_2(t)$ $\sqrt{2E^{q}/T_o}\ \cos(w_c t + 180)$

$\downarrow p(t)$

$\sqrt{E}\left(\sqrt{\frac{2}{T_o}}\cos(w_c t)\right)$ $\hookrightarrow 2$ $- \sqrt{2E^{q}/T_o}\ \cos(w c t)$.



design the reciever and find pe

Sol $S_1(t) \rightarrow \vec{S_1}$

$S_2(t) \rightarrow \vec{S_2}$

① find the basis functions $\phi_1(t)$ $\phi_2(t)$ -- $\phi_n(t)$

Use gram Schmidt procedure

$\phi_1(t) = \frac{S_1(t)}{\sqrt{E}} \quad = \quad \underline{\underline{E_{S1}}} = \int_0^{T_0} S_1^2(t)\,dt$

$= \int_0^{T_0} \frac{2E}{T_0} \left[\frac{1}{2} + \frac{1}{2} \cos 2w_c t \, dt \right)$

$= E + \text{zero} = \underline{\underline{E}} \qquad \boxed{E_{S1} = E}$

$$\boxed{\phi_1(t) = \frac{S_1(t)}{\sqrt{E}} = \sqrt{\frac{2}{T_0}} \cos(w_c t).}$$

$S_{11} = \sqrt{E}$

$\phi_2(t) = ??$

$g_2(t) = s_2(t) - \underline{\underline{S_{21}}} \, \phi_1(t) \qquad \int_0^{T_b} S_2(t) \, \phi_1(t) \, dt$

$= \int_0^{T_b} -\sqrt{\frac{2E}{T_0}} \cos(w_c t) * \sqrt{\frac{2}{T_0}} \cos(w_c t) \, dt$

$S_{21} = \int_0^{T_b} \frac{-2}{T_0} \sqrt{E} \cos^2(w_c t) = \int_0^{T_b} \frac{-2\sqrt{E}}{T_0} \left(\frac{1}{2} + \frac{1}{2} \cos(2w_c t) \, dt \right.$

$= = -\sqrt{E} + 0 = -\sqrt{E}$

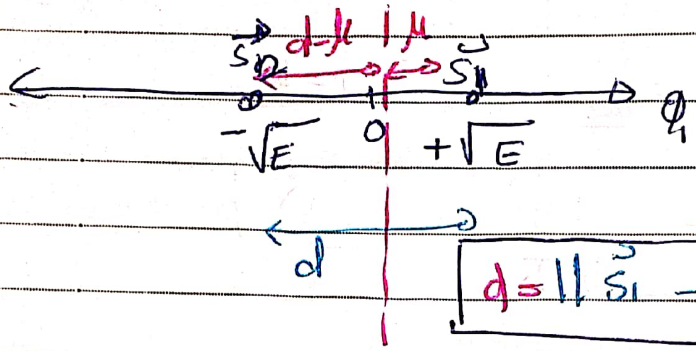
$\underline{g_2(t)} = -\sqrt{\frac{2E}{T_0}} \cos(w_c t) + \sqrt{E} \sqrt{\frac{2}{T_0}} \cos(w_c t) = 0$

So, $n=1$ $\phi_1(t) = \sqrt{\frac{2}{T_0}} \cos(w_c t)$

$S_{11} = \sqrt{E}$ $S_{21} = -\sqrt{E}$

$S_1(t) \longrightarrow \vec{S}_1 = (\sqrt{E})$

$S_2(t) \longrightarrow \vec{S}_2 = (-\sqrt{E})$



$$\boxed{d = \| \vec{S}_1 - \vec{S}_2 \| = 2\sqrt{E}}$$

$$\mu = \frac{N \ln\left(\frac{P_{m1}}{P_{m2}}\right) + d^2}{2d}$$

$$\mu = \frac{N \ln\left(\frac{P_{m1}}{P_{m2}}\right) + 4E}{4\sqrt{E}}$$

$$\mu = \frac{N}{4\sqrt{E}} \ln\left(\frac{P_{m1}}{P_{m2}}\right) + \sqrt{E}$$

$P_e = 1 - P(c)$

$P_e = 1 - \left[P(c|m_1) \cdot (P_{m1}) + P(c|m_2) \cdot P_{m2}\right]$

$P(c/m) - P(\vec{y}/m_1) \in R_1$

$\vec{y}/\vec{m}_1 = \vec{S}_1 + \vec{n}_1$

$\vec{y}/\vec{m}_1 = [y_1] = [\sqrt{E}] + [n_1]$

$y_1 = \sqrt{E} + n_1 \rightarrow N(0, N/2)$

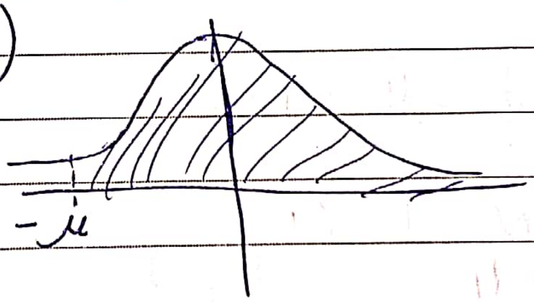
$\rightarrow N \sim (\sqrt{E}, N/2)$.

$P(\vec{c}/m_1) = P(y_1/m_1 \in R_1)$

$P(\vec{c}/m_1) = P(y_1/m_1 > a_0) = P(y_1/m_1 > \frac{d}{2}$ [illegible] $\mu)$

$= P(\sqrt{E} + n_1 > \frac{d}{2} - \mu) = P(\cancel{\sqrt{E}} + n_1 > \cancel{\sqrt{E}} - \mu)$

$= P(m > -\mu)$



$= 1 - P(n_1 > \mu)$

$= 1 - Q\left(\frac{\mu - 0}{\sqrt{N/2}}\right)$

$1 - Q\left(\frac{\mu}{\sqrt{N/2}}\right)$

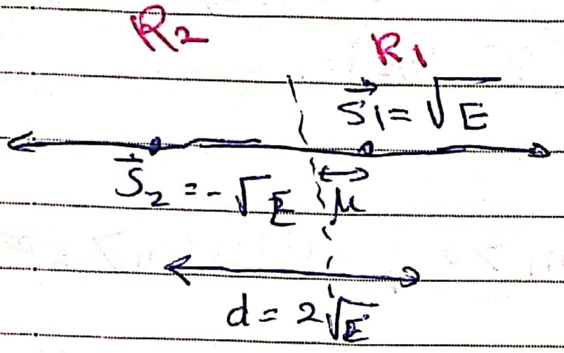
10/12/2019

* Ex:- BPSK $s_i(t) = \mp P(t)\sqrt{2E/T_b}\,\cos(w_c t)$

$m_1 = "1" \longrightarrow$ BPSK , $s_1(t) = \sqrt{\frac{2E}{T_b}}\cos(w_c t)$

$m_2 = "0" \longrightarrow$

$s_2(t) = -\sqrt{\frac{2E}{T_b}}\cos(w_c t$



$$\mu = \frac{N}{4\sqrt{E}} \ln\left(\frac{P_{m_1}}{P_{m_2}}\right) + \sqrt{E}$$

$$a_0 = \frac{d}{2} - \mu = \sqrt{E} - \mu$$

$$= \frac{N}{4\sqrt{E}} \ln\left(\frac{P_{m_2}}{P_{m_1}}\right)$$

$p(c) = p(c/m_1)p(m_1) + p(c/m_2)p(m_2)$

$p(c/m_1) = p(y/m_1 \in R_1) = P(\sqrt{E} + n > a_0)$

$p(\sqrt{E} + n > \sqrt{E} - \mu)$

$p(n > -m) = 1 - Q\left(\frac{\mu}{\sqrt{N/2}}\right)$

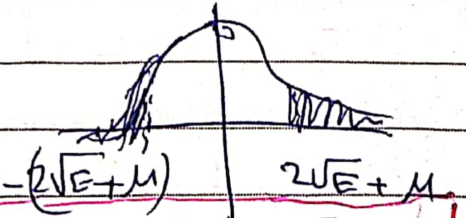
$P(c/m_2) = p(y/m_2 \in R_2) = P(-\sqrt{E} + n < \sqrt{E} - \mu)$

$= P(n < +2\sqrt{E} - \mu)$

$1 - p(n > 2\sqrt{E} - \mu) = 1 - Q\left(\frac{2\sqrt{E} - \mu}{\sqrt{N/2}}\right)$

$$P_e = 1 - P(c)$$

$$= 1 - \left[\left(P_{m_1} \left(1 - Q\left(\frac{\mu}{\sqrt{N/2}} \right) \right) + P_{m_2} \left(1 - Q\left(\frac{2\sqrt{E} - \mu}{\sqrt{N/2}} \right) \right) \right] \right.$$



$$\boxed{= 1 - P(m_1) - p(m_1)\, Q\left(\frac{\mu}{\sqrt{N/2}} \right) + P_{m_2} - P_{m_2}\, Q\left(\frac{2\sqrt{E} - \mu}{\sqrt{N/2}} \right)}$$

as aspecial case :- equal prob. $P(m_1) = p(m_2) = 0.5$

$\mu = \sqrt{E}$

$$P_e = \frac{1}{2} Q\left(\frac{\sqrt{E}}{\sqrt{N/2}} \right) + \frac{1}{2} Q\left(\frac{2\sqrt{E} - \sqrt{E}}{\sqrt{N/2}} \right)$$

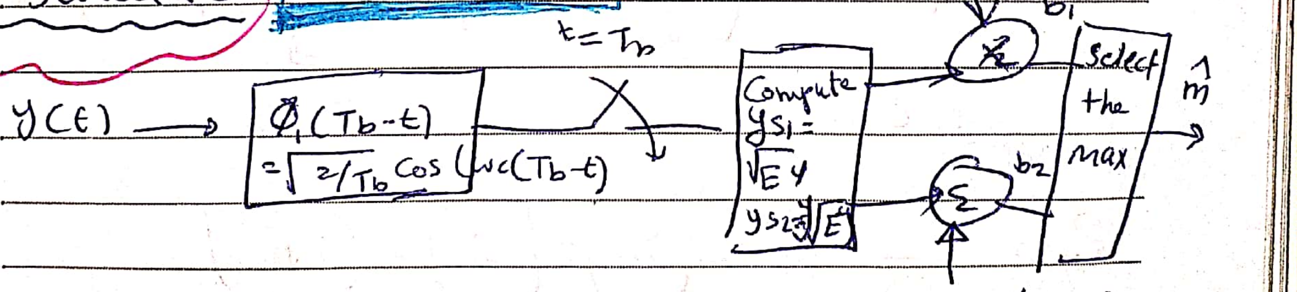
$$= Q \sqrt{\frac{2E}{N}} = Q\left(\sqrt{\frac{2E_b}{N}} \right)$$

avg powerper bit $E_b = p(m_1) E_{s1} + P(m_2) E_{s2}$

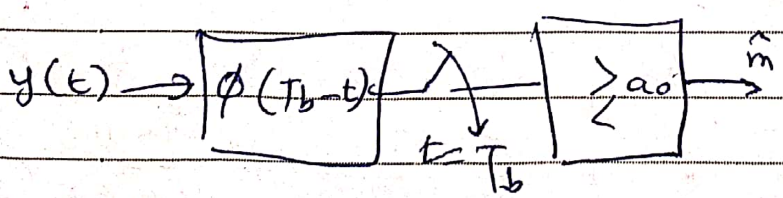
$$= \left(\frac{1}{2} + \frac{1}{2} \right) E = \underline{\underline{E}}$$

Rx structure $\boxed{n = 1 \quad \mu = 2}$

$a_1 = \frac{N}{2} \ln(P_{m_1}) - \frac{E_{s_1}}{2}$

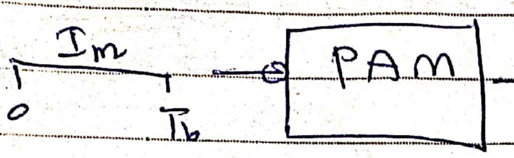


$a_2 = \frac{N}{2} \ln(P_{m_2}) - \frac{E_{s2}}{2}$



$$a_0 = \frac{N}{4\sqrt{E}} \ln\left(\frac{P_{m_2}}{P_{m_1}} \right)$$

Example M-ary Band pass PAM



$$s_i(t) = i\,p(t)\sqrt{\frac{2}{T_b}}\cos(\omega_c t)$$

$$= i\sqrt{\frac{2E}{T_b}}\cos(\omega_c t)$$

$$\boxed{i = \pm 1, \pm 3, \pm 5, -- \pm(M-1)}$$

$$E_{s_i} = \int_0^{T_b} s_i^2(t)\,dt = \int_0^{T_b} i^2 \frac{2E}{T_b}\left[\frac{1}{2} + \frac{1}{2}\cos(2\omega_c t)\,dt\right]$$

$$= i^2 E$$

$$\delta_1(t) = \sqrt{\frac{2E}{T_b}}\cos(\omega_c t)$$

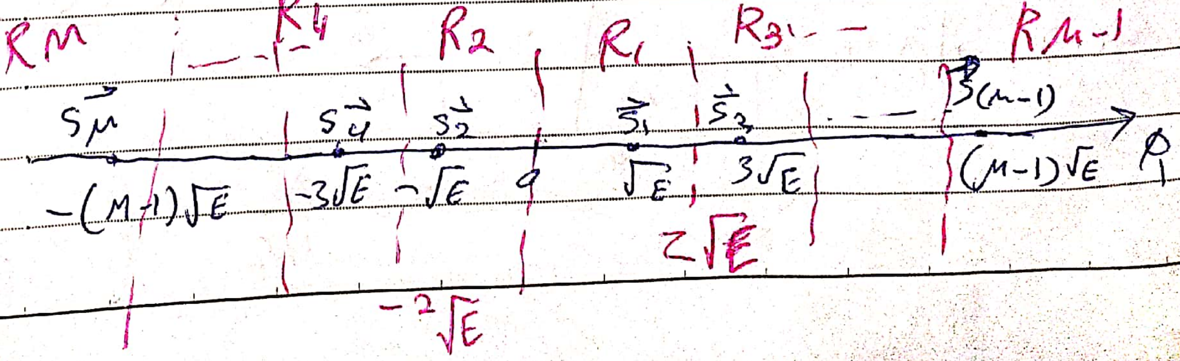
$$\phi_1(t) = \frac{s_i(t)}{\sqrt{E_{s_1}}} = \frac{s_i(t)}{\sqrt{E}} = \sqrt{\frac{2}{T_b}}\cos(\omega_c t)$$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{E_{g_2}}} = \frac{S_2(t) - S_{21}\,\phi_{21}\,\phi_1(t)}{\sqrt{E_{g_2}}}$$

$$= S_2(t) - \left[\int_0^{T_b} S_2(t)\,\phi_1(t)\,dt\right]\phi_1(t) \Big/ \sqrt{E_{g_2}}$$

= Zero

$$\vec{s_i}(t) \longrightarrow \vec{s_1} = [\delta_{i1}] = [S_i] = [i\sqrt{E}]$$



$$P(c) = \sum_{i=1}^{M} P(c|m_i)\,P_{m_i}$$

$$P_e = \sum_{i=1}^{M} P(e|m_i)\,p(m_i)$$

$$P(c|m_1) = P(y/m_1 \in R_1) = P(\sqrt{E}+n \in R_1) = P(0<\sqrt{E}+n<2\sqrt{E})$$

$$= P(-\sqrt{E} < n < \sqrt{E}) = 1-2P(n>\sqrt{E}) = 1-2Q\left(\frac{\sqrt{E}}{\sqrt{N/2}}\right)$$

$$= 1-2Q\left(\sqrt{\frac{2E}{N}}\right)$$

$$P(e|m_1) = 1-P(c|m_1) = \boxed{2Q\left(\sqrt{2E/N}\right)} \rightarrow M-2$$

$$= P(c|m_{M-1}) = P(y/m_{M-1} \in R_{M-1})$$

$$= P((M-1)\sqrt{E}+n > (M-2)\sqrt{E})$$

$$= P(n > (M-2)\sqrt{E} - (M-1)''$$

$$= P(n > -\sqrt{E})$$

$$= 1-P(n>\sqrt{E}) = 1-Q\left(\sqrt{\frac{E}{N/2}}\right)$$

$$= 1-Q\left(\sqrt{\frac{2E}{N}}\right) =$$

$$\Rightarrow \therefore P(e/m_{M-1}) = Q\left(\sqrt{\frac{2E}{N}}\right)$$

$$\therefore P_e = \frac{1}{M}\sum_{i=1}^{M}(P(e|m_i)) =$$

$$= \frac{1}{M}\left[(M-2)\, 2\, Q\left(\sqrt{\frac{2E}{N}}\right) + 2\, Q\left(\sqrt{\frac{2E}{N}}\right)\right.$$

$$= \frac{1}{M}\, Q\left(\sqrt{\frac{2E}{N}}\right)(2M-4+2)$$

$$= \boxed{\frac{2(M-1)}{M}\, Q\left(\sqrt{\frac{2E}{N}}\right)}$$

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M -ary PAM

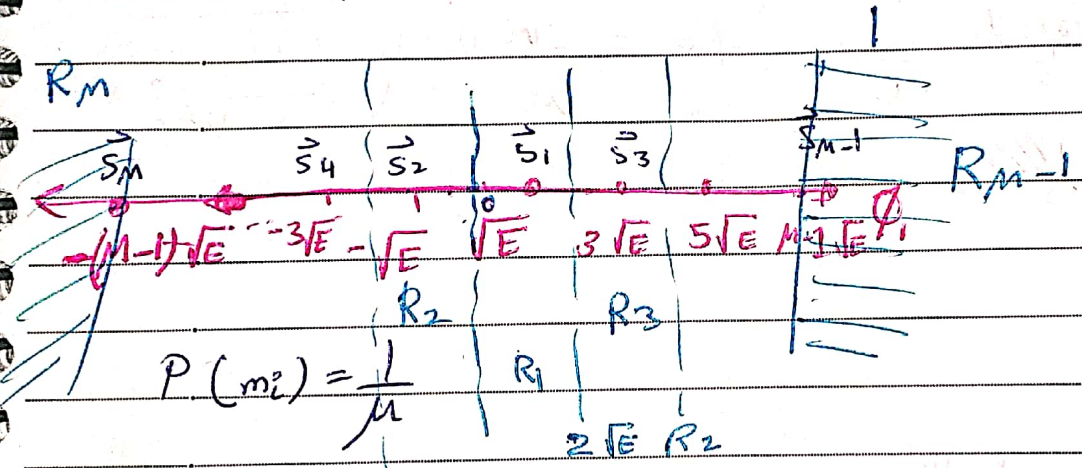
$$S_i(t) = i\sqrt{\frac{2E}{T_b}}\cos(w_c t) \qquad 0 \le t \le T_b$$

$$\phi_i(t) = \sqrt{\frac{2}{T_b}}\cos(w_c t)$$

$$S_i(t) \rightarrow \vec{S_i} = i\sqrt{E} \rightarrow S_i = i\sqrt{E}$$

$$i = \pm 1, \pm 3, \pm 5, \pm 7, \ldots, \pm (M-1)$$

$$E_{si} = \|\vec{s_i}\|^2 = i^2 E$$



$$P(m_i) = \frac{1}{M}$$

$$P_e = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2E}{N}}\right)$$; E(Energy for baseband pulses.

Eav : average Energy per symbol

$$E_{av} = \sum_{i=1}^{M} P(m_i)\, E_{si} = \sum_{i=1}^{M} \frac{E_{si}}{M}$$

$$= \frac{1}{M}\left[2E + 2(9E) + 2(25E) + \cdots\ 2(M-1)^2 E\right]$$

$$= \frac{2E}{M} \sum_{j=0}^{\frac{M-2}{2}} (2j+1)^2$$

$$= \frac{\cancel{2}E}{\cancel{M}}\left(\frac{\cancel{M}(M^2-1)}{\cancel{6}\;3}\right) = \frac{(M^2-1)E}{3}$$

↳ Energy per symbol

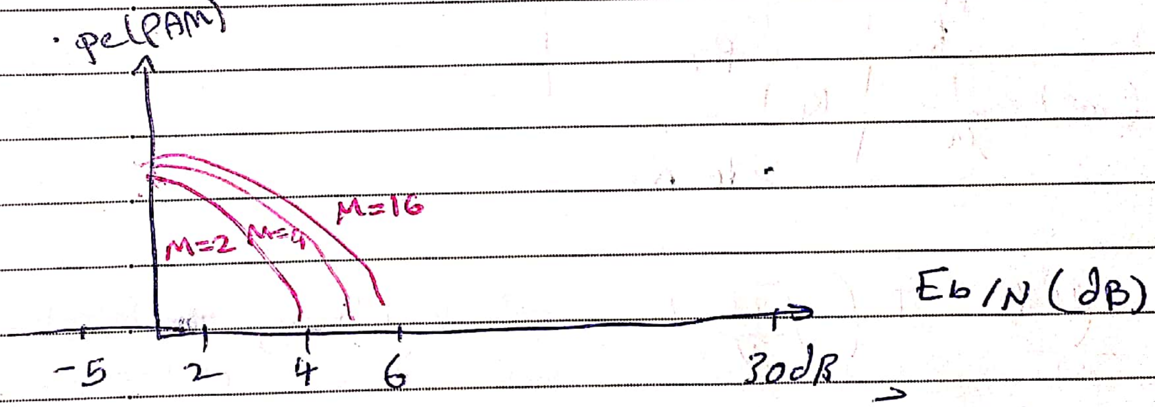
E_b = average energy per bit = $\frac{E_{av}}{\log_2 M}$ =

$\frac{M^2-1}{3 I_m} E$ $\quad I_m = \log_2 M$

$$E = \frac{3\log_2 M \; E_b}{(M^2-1)\cancel{E_b}}$$

PAM

$$P_e = \frac{2(M-1)}{M}\, Q\left(\sqrt{\frac{6\log_2 M}{(M^2-1)\cancel{E_b}} \cdot \frac{E_b}{N}}\right)$$



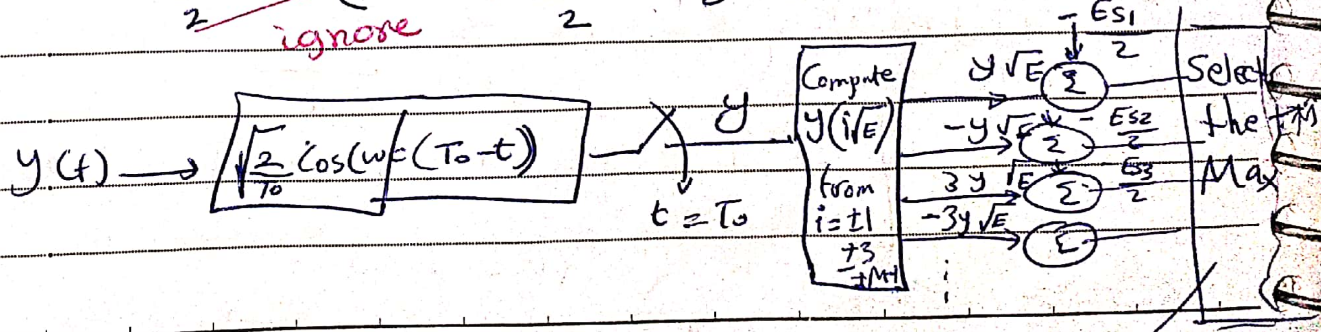
$\vec{y} = (y)$

$s_i = (i\sqrt{E})$

Rx ~~or~~ $b_i = a_i + \langle \vec{y}, \vec{s_1} \rangle$

$= \cancel{\frac{N}{2}\ln(P_m)} - \frac{E_{si}}{2} + y\,(i\sqrt{E})$

ignore



M-ary QAM

$$\phi_1(t) \qquad\qquad \phi_2(t)$$

$$m_i \rightarrow \boxed{M-QAM} \rightarrow s_i(t) = a_i \boxed{\sqrt{2/T_b}\ \cos(w_c t)} + b_i \boxed{\sqrt{\frac{2}{T_b}}\ \sin(w_c t)}.$$

$0 \longrightarrow T_b$

$\log_2 M$ bits

$i = 1, 2, \ldots, M$

$$a_i = \pm\frac{d}{2}, \pm\frac{3d}{2}, \pm\frac{5d}{2}, \ldots \pm (\sqrt{M}-1)\frac{d}{2}$$

$$b_i = \pm\frac{d}{2}, \pm\frac{3d}{2}, \pm\frac{5d}{2}, \ldots \pm (\sqrt{M}-1\ \frac{d}{2}).$$

$M = 2^{2m}$

2
4 ✓
8
16 ✓
64 ✓
128
256 ✓
?

$M = 4, 16, 64, 256, \ldots$

$$s_i(t) = \vec{s_i} = (s_{i1}, s_{i2})$$
$$= (a_i, b_i)$$

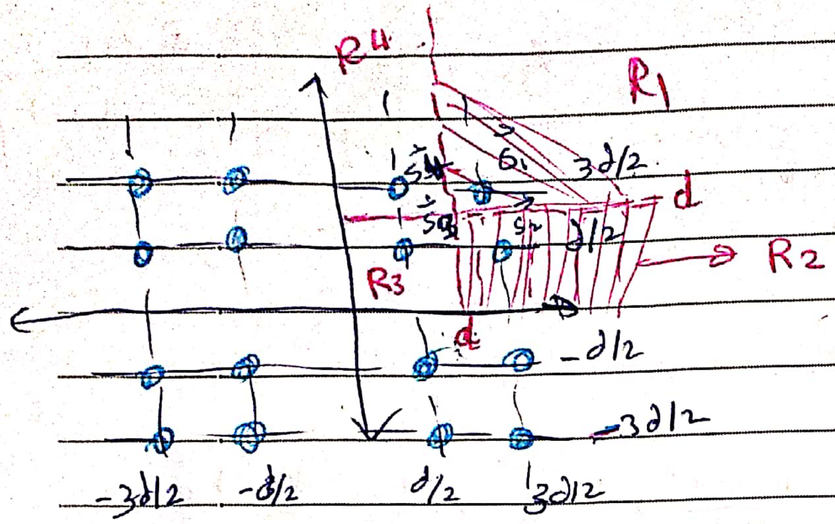
$$y(t) = s_i(t) + n(t)$$

$$\vec{y} = (y_1, y_2) = (s_{i1}, s_{i2}) + \underbrace{(n_1, n_2)}_{N \sim (0, \frac{N_0}{2})}$$

$$\Rightarrow y_1 = s_{i1} + n_1 \qquad y_2 = s_{i2} + n_2$$

Ex : Derive P_e^{16-QAM} and plot the Rx

$$a_i = \pm\frac{d}{2}, \pm\frac{3d}{2} \qquad b_i = \pm\frac{d}{2}, \pm\frac{3d}{2}$$



$p(c/m_1) = P(\vec{y}/m_1 \in R_1)$

$\vec{y}/m_1 = (s_{11}, s_{12}) + (n_1, n_2)$

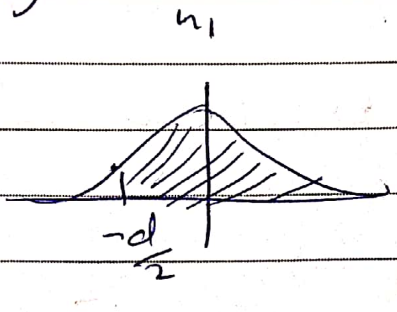
$= (3d/2, \frac{3d}{2}) + (n_1, n_2)$

$= (\underbrace{\frac{3d}{2} + n_1}_{y_1}, \underbrace{\frac{3d}{2} + n_2}_{y_2})$

$p(c/m_1) = P(\vec{y}/m_1 \in R_1) = P(y_1 > d \cap y_2 > d)$

$= P(y_1 > d) . P(y_2 > d)$

$P(\frac{3d}{2} + n_1 > d) . P(\frac{3d}{2} + n_2 > d)$



$= P(n_1 > d - \frac{3d}{2}) . P(n_2 > d - \frac{3d}{2})$

$P(n_1 > \frac{-d}{2}) . P(n_2 > \frac{-d}{2})$

$= [1 - P(n_1 > \frac{d}{2})] [1 - P(n_2 > \frac{d}{2})]$

$= \left[1 - Q\left(\frac{d/2}{\sqrt{N/2}}\right)\right]^2 = \left[1 - Q\left(\frac{d}{\sqrt{2N}}\right)\right]^2 = P^2$

$P(C/m_2) = P(\vec{y}/m_3 \in R_3)$

$P(\vec{s}_3 + \vec{n} \in R_3)$

$= P\left(\left(\underbrace{\frac{d}{2}+n_1}_{y_1}, \underbrace{\frac{d}{2}+n_2}_{y_2}\right) \in R_3\right)$

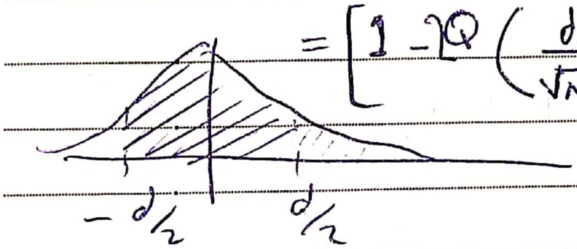
$= P(0 \leq y_1 < d \cap 0 < y_2 < d)$.

$= P(0 < y_1 < d) \cdot P(0 < y_2 < d)$

$= P\left(0 < \frac{d}{2} + n_1 < d\right) P\left(0 < \frac{d}{2} + n_2 < d\right)$

$P\left(-\frac{d}{2} < n_1 < \frac{d}{2}\right) P\left(-\frac{d}{2} < n_2 < \frac{d}{2}\right)$

$= \left[1 - 2Q\left(\frac{d/2}{\sqrt{N/2}}\right)\right]^2 = \left[1 - 2Q\left(\frac{d}{\sqrt{2N}}\right)\right]^2$



$= (2P-1)^2$

$P(C/m_2) = (\vec{y}/\vec{S_2} \in R_2)$

$\vec{y}/\vec{S_2} = \left(\underbrace{\frac{3d}{2}+n_1}_{y_1}, \underbrace{\frac{d}{2}+n_2}_{y_2}\right)$

$= P\left(\left(\frac{3d}{2}+n_1, \frac{d}{2}+n_2\right) \in R_2\right)$

$$p\left(\frac{3d}{2}+n_1 > d \cap 0 < \frac{d}{2}+n_2 < d\right)$$

$$P\left(n_1 > \frac{-d}{2}\right)\cdot P\left(\frac{-d}{2} < n_2 < \frac{d}{2}\right)$$

$$\left[1 - Q\left(\frac{d}{\sqrt{2N}}\right)\right]\left[1-2Q\left(\frac{d}{\sqrt{2N}}\right)\right]$$

$$= p\,(2p-1) = \underline{2p^2 - p}$$

$$p(c/m_4) = p(c/m_2) = 2p^2 - p$$

$$P(c) = \sum_{i=1}^{16} p(m_i)\,P(c/m_i) = \frac{1}{16}\sum_{i=1}^{16} p(c/m_i).$$

$$\frac{1}{16}\left[4P(c/m_1) + 4P(c/m_2) + 4P(c/m_3) + 4P(c/m_4)\right]$$

$$\frac{1}{4}\left(P^2 \cancel{c/m_i} + 2P(2P-1) + (2P-1)^2\right]$$

$$= \boxed{\frac{1}{4}\left(9P^2 - 6P + 1\right)} = \boxed{\left(\frac{3P-1}{2}\right)^2}$$

$$= \left[\frac{3}{2}\left[1 - Q\left(\frac{d}{\sqrt{2N}}\right)\right] - \frac{1}{2}\right]^2$$

$$= \left[\frac{3}{2} - \frac{3}{2}Q\left(\frac{d}{\sqrt{2N}}\right) - \frac{1}{2}\right]^2$$

$$\Rightarrow P_e^{16-QAM} = 1 - P_c^{16\,QAM} = \boxed{1 - \left[1 - \frac{3}{2} Q\left(\frac{d}{\sqrt{2N}}\right)\right]^2}$$

⇒ write P_e^{16-QAM} in terms of E_b/N:

E average energy per symbol $E_{av} = \sum_{i=1}^{16} P_{mi} E_{si}$

$$= \frac{1}{16} \sum_{i=1}^{16} E_{si}$$

$$E_{s1} = \|\vec{s_1}\|^2 = \left(\frac{3d}{2}\right)^2 + \left(\frac{3d}{2}\right)^2 = \frac{9d^2}{2}$$

$$E_{s2} = \|\vec{s_2}\|^2 = \left(\frac{3d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 = \frac{5d^2}{2} = E_{s4}$$

$$E_{s3} = \|s_3\|^2 = \left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 = \frac{d^2}{2}$$

$$\Rightarrow E_{av} = \frac{1}{16}\left[4E_{s1} + 4E_{s2} + 4E_{s3} + 4E_{s4}\right]$$

$$= \frac{1}{4}\left[\frac{9d^2}{2} + 5d^2 + \frac{d^2}{2}\right] = \frac{20d^2}{8} = \boxed{\frac{5d^2}{2}}$$

$$E_{av} \text{ Per bit} = E_b = \frac{E_{av}}{Im} = \boxed{\frac{5d^2}{8}}$$

$$d = \sqrt{\frac{8}{5} E_b}$$

$$P_e^{16-QAM} = \boxed{1 - \left[1 - \frac{3}{2} Q\sqrt{\frac{4E_b}{5N}}\right]^2}$$

$$\approx 2\left(\frac{3}{2} Q\left(\sqrt{\frac{4E_b}{5N}}\right)\right) = 3Q\left(\sqrt{\frac{4E_b}{5N}}\right)$$

compare with Bpsk $P_e^{BPSK} = Q\left(\sqrt{\frac{2E_b}{N}}\right)$

Performance of BPSK is better than that of 16-QAM

Spectral efficiency 4(bits/s)/HZ

BPSK 1(bits/s)/ HZ

E_{av} 16-QAM ⇒ 16 QAM $E_b = \frac{5}{2} E_b^{BPSK}$

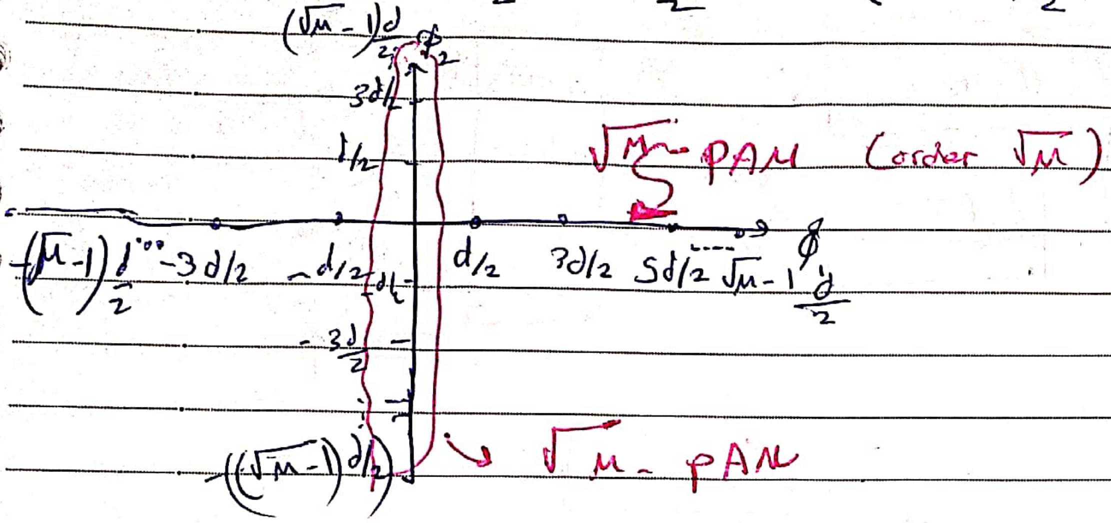
~~process~~ gives the same performance.

M-ary Rectangular QAM

$M = 2^k$, k is even

$$S_i(t) = a_i\sqrt{\frac{2}{T_0}}\cos(w_c t) + b_i\sqrt{\frac{2}{T_0}}\sin(w_c t)$$

$a_i = \pm\frac{d}{2}, \pm\frac{3d}{2}, \pm\frac{5d}{2}, \pm(\sqrt{M}-1)\frac{d}{2}$



$$P(C)^{M-QAM} = P_{(C)}^{\sqrt{M}-PAM} \cdot P_{(C)}^{\sqrt{M}-PAM}$$

$$= \left(P(C)^{\sqrt{M}-PAM}\right)^2 = P_{(C)}^{M-QAM} = \left(1 - P_e^{\sqrt{M}-PAM}\right)^2$$

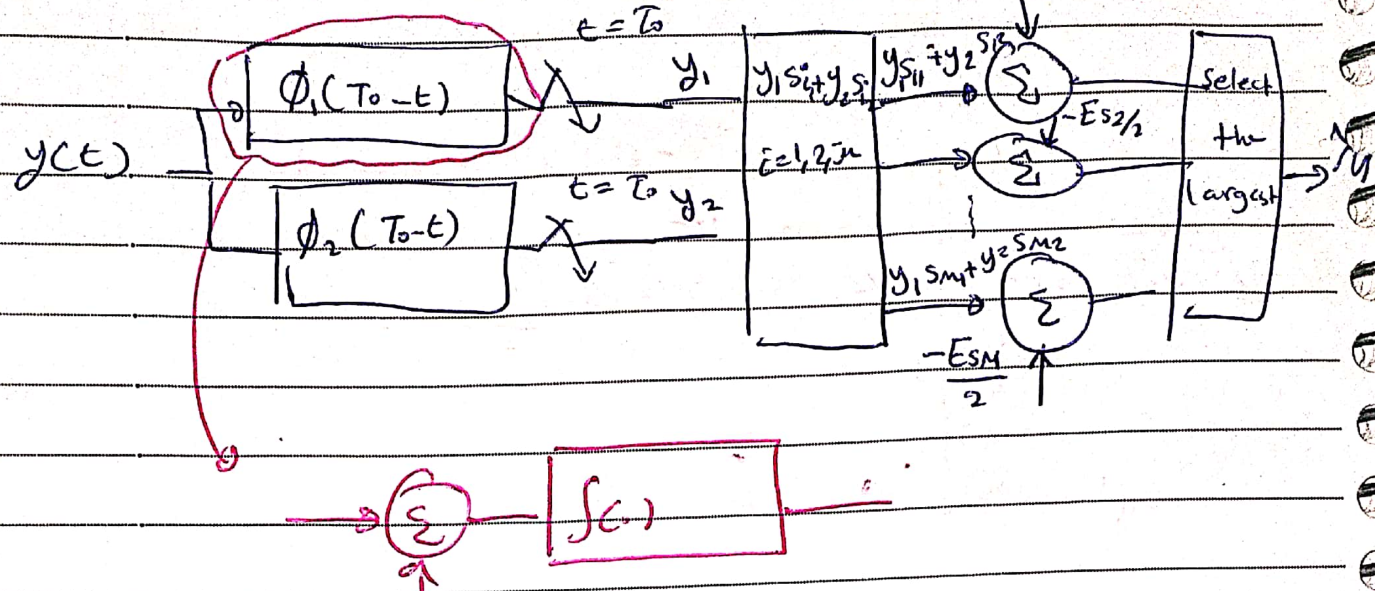
$$P_e^{M-QAM} = 1 - \left[1 - P_e^{\sqrt{M}-PAM}\right]^2 = 1 - \left[1 - \left(\frac{2(\sqrt{M}-1)}{\sqrt{M}}\right) Q\left(\sqrt{\frac{6\log_2\sqrt{M}}{M-1}\frac{E_b}{N}}\right)\right]^2$$

$M = 2^{\text{even}}$ 4, 16, 64, 256, 1024, —

for QAM

$$P_e^{16-QAM} = 1 - \left[1 - \frac{6}{4} Q\left(\sqrt{\frac{12}{15}\frac{E_b}{N}}\right)\right]^2$$ ✓

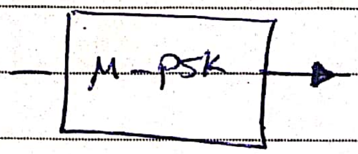
n = 2 ϕ_1 , ϕ_2



$\phi_1(t) = \sqrt{\frac{2}{T_0}} \cos(w_c t)$

الثلاثاء

M-ary PSK



$$s_i(t) = \sqrt{\frac{2E}{T_0}} \cos(w_c t + \theta_i)$$

$$\theta_i = \frac{2\pi}{M} (i-1)$$

$$i = 1, 2, \ldots, M$$

$$s_i(t) = \sqrt{E} \cos\theta_i \underbrace{\sqrt{\frac{2}{T_0}} \cos(w_c t)}_{\phi_1(t)} + \sqrt{E} \sin(\theta_i) \underbrace{\left(-\sqrt{\frac{2}{T_0}} \sin(w_c t\right.}_{\phi_2(t)}$$

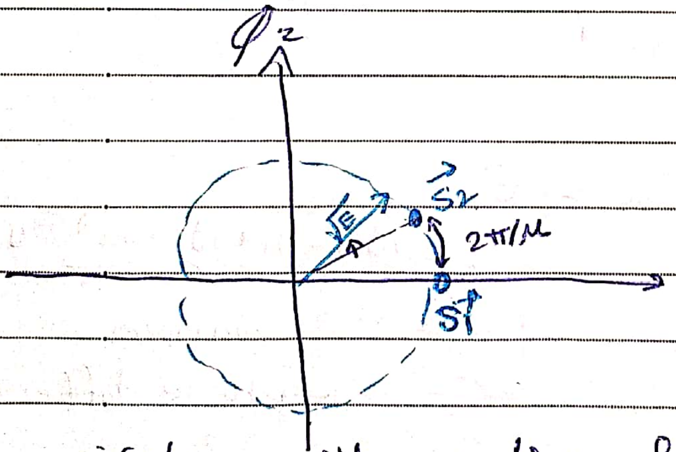
$$\vec{s_i} = \left(\sqrt{E}\cos(\theta_i), \sqrt{E} \sin(\theta_i)\right) \overset{\Delta}{=} \sqrt{E}\, e^{j\theta_i}$$

$$s_i = \sqrt{E}\cos\theta_i + j\sqrt{E}\sin\theta_i$$

$$E_{si} = \|\vec{s_i}\|^2 = E\cos^2(\theta_i) + E\sin^2\theta_i = E$$

$$E_{av} = E$$

$$E_b = \frac{E_{av}}{\log_2 M} = \frac{E_s}{\log_2 M} \qquad \theta_{i+1} - \theta_i = \frac{2\pi}{M}$$

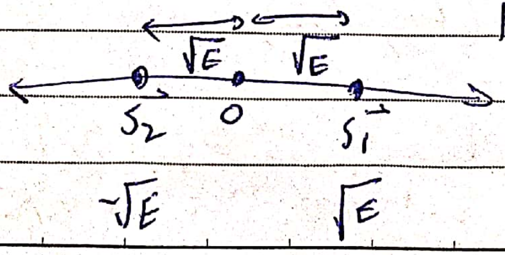


$\vec{s_1}$ $(\sqrt{E}, 0)$
$\vec{s_2}$ $(\sqrt{E}, \frac{2\pi}{M})$
$\vec{s_3}$ $(\sqrt{E}, \frac{4\pi}{M})$

circle with radius of $\sqrt{E}$

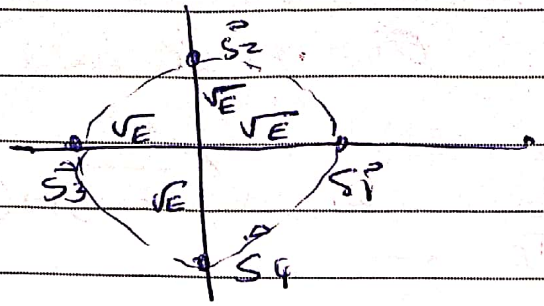
Ex = $\boxed{M = 2}$ $\theta_1 = 0$ $\theta_2 = \pi$

BPSK



$\cancel{F} \cdot Ex = M=4 \quad \theta i = \frac{\pi}{2}(i-1)$

$\theta_1 = 0 \quad \theta_2 = \pi/2 \quad \theta_3 = \pi \quad \theta 4 = 3\pi/2$



$\vec{s1} = (\sqrt{E} e^{j0}) = \sqrt{E} \quad \vec{s_2} = \sqrt{E} e^{j90} = j\sqrt{E} = (0, \sqrt{E})$

Reciever decision function 1

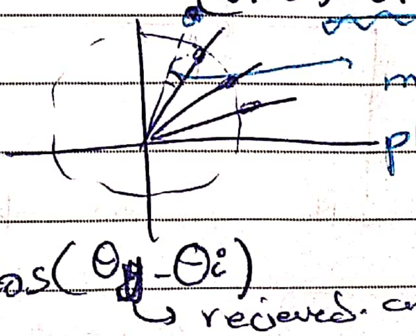
$bi = ai + \langle \vec{y}, \vec{si} \rangle$

$ai = \underbrace{\frac{N}{2} Ln(Pmi)}_{\text{Ignored}} \underbrace{\frac{Esi}{2}}_{\text{Ignored}}.$

$\vec{y} = (y_1, y_2)$

- $P(mi) = \frac{1}{M} \forall i \quad Esi = E$

$(y_1, y_2) = y_1 + jy_2 + \rho e^{j\theta_y}$



minimum phase difference

⇒ Desision function

$bi = \langle \vec{y}, \vec{si} \rangle$

select the Max $= \|\vec{y}\| \|\vec{s_i}\| \cos(\theta_y - \theta i)$ ↳ recieved. angle

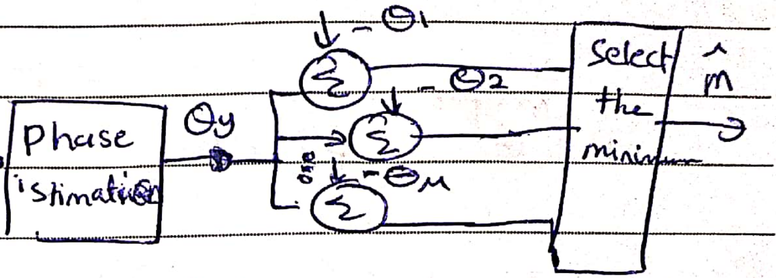
$= \|\vec{y}\| \sqrt{E} \cos(\theta y - \theta i)$

decision function $bi = \cos(\theta y - \theta i)$ select the max

⇒ $\boxed{b_i = |\Theta_y - \Theta_i|}$ ← select the minimum

Decision Rule: The Rx choose $\hat{m} = m_k$
if $|\Theta_y - \Theta_i|$
is minimum for $i = k$

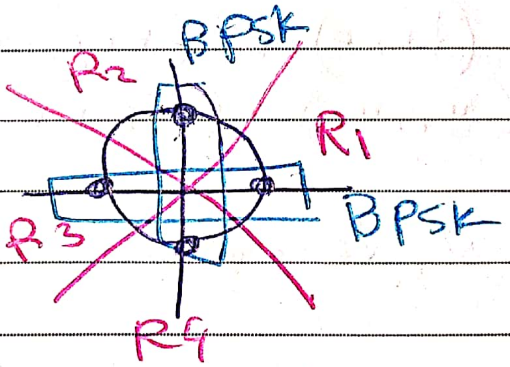
$y(t) = A\cos(w_c t + \Theta_y)$ →



Pe - performance.

$P_e^{BPSK} = Q\left(\sqrt{\frac{2E_b}{N}}\right)$ $E_b = E$

M = 4   QPSK

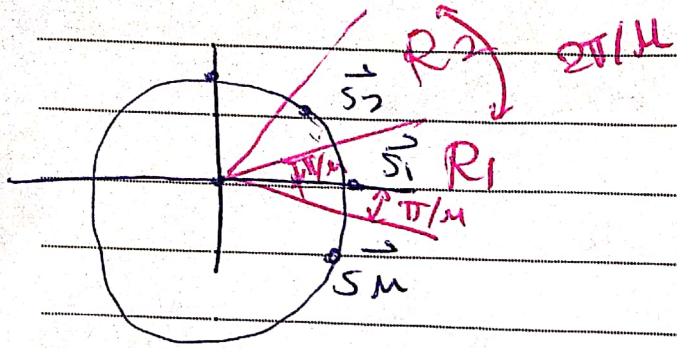


$P(c)^{QPSK} = P(c)^{Bpsk}_{I} \cdot P(c)^{QBpsk}_{Q}$

$= \left(P(c)^{BPSK}\right)^2 = \left[1 - Q\left(\sqrt{\frac{2E_b}{N}}\right)\right]^2$

$\boxed{P_e = 1 - \left[1 - Q\sqrt{\frac{2E_b}{N}}\right]^2}$

P_e^{M-psk} of $M>4$:-



$$P_e(C/m_i) = P(C/m_i) \quad i=1,\dots,M$$

$$P(C)_i = \sum_{i=1}^{M} P(m_i)P(C/m_i)$$

$$= P(c/m_i)$$

$$P_e^{M-psk} = 1 - P(C/m_i)$$

$$P(c/m_i) = P(\vec{y}/m_1 \in R_1)$$

$$\vec{y}/\vec{m_1} = y(y_1,y_2) = \vec{S_1} + \vec{n}$$

$$\vec{S_1} = (\sqrt{E},0) + (n_1, n_2)$$

$$\vec{y} = (y_1,y_2) = (\sqrt{E}+n_1, 0+n_2)$$

$$\vec{Y_1} \sim N(\sqrt{E}, N/2) \qquad y_2 \sim N(0, N/2)$$

$$P(c/m_1) = P((y_1,y_2) \in R_1)$$

$$P(0<y_1<\infty \cap y_1\tan\frac{\pi}{M} < y_2 < y_1\tan(\frac{\pi}{M}))$$

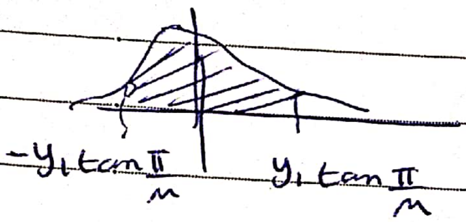
$$P\left[(0<y_1<\infty) \cap (-y_1\tan(\frac{-\pi}{M}) < y_2 < y_1\tan(\frac{\pi}{M}))\right]$$

$$= \iint f_{Y_1Y_2}(y_1,y_2)\, dy_1\, dy_2$$

$$R_1 = \iint f_{Y_1}(y_1) . f_{Y_2}(y_2)\, dy_1\, dy_2$$

$$\int_{-y_1\tan\frac{\pi}{M}}^{y_1\tan\frac{\pi}{M}}\int_{0}^{\infty} f_{y_2}(y_2)\, f_{y_1}(y_1)\, dy_2\, dy_1$$

$$= \int_{0}^{\infty} f_{y_1}(y_1) \int_{-y_1\tan\frac{\pi}{M}}^{y_1\tan\frac{\pi}{M}} \frac{1}{\sqrt{2\pi\frac{N}{2}}}\, e^{-(y_2)^2/\sqrt{(2N/2)}}$$



$$= 1 - 2Q\left(\frac{y_1\tan\left(\frac{\pi}{M}\right)}{\sqrt{N/2}}\right)$$

$$= \int_{0}^{\infty}\left[1 - 2Q\left(\frac{y_1\tan\left(\frac{\pi}{M}\right)}{\sqrt{N/2}}\right)\right] \cdot \frac{1}{\sqrt{2\pi(N/2)}}\, e^{-(y_1-\sqrt{E})^2/\sqrt{2N/2}}\, dy_1$$

let $X = y_1/\sqrt{N/2}$

$$P(c/m_1) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} 1 - 2Q\left(x\tan(\pi/M)\, e^{-\frac{1}{2}\left(X - \sqrt{\frac{2}{N}\log_2 M\, E_b}\right)}\right)$$

$$Pe = 1 - p(c/m)$$

M-psk 19/12/2019

$s_i(t) = \sqrt{\frac{2E}{T_0}} \cos(w_c t + \theta_i)$

$\theta_i = \frac{2\pi}{M}(i-1)$ $E_{si} = E$

$s_i(t) \longrightarrow \vec{s_i}\left(\sqrt{E}\cos\theta_i, \sqrt{E}\sin\theta_i\right)$

P_e :- ① B-psk $= Q\left(\sqrt{\frac{2E_b}{N}}\right)$

② Qpsk =

$1 - \left[1 - Q\left(\sqrt{\frac{2E_b}{N}}\right)\right]^2$

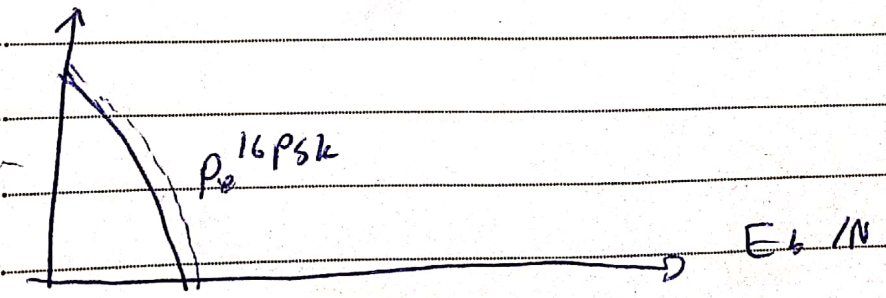
③ $M > 4$:- $P_e^{M\text{-ary}} =$

$$P_e^{M\text{-ary}} = 1 - \frac{1}{\sqrt{2\pi}} \int_0^\infty \left[1 - 2Q\left(x \tan\left(\frac{\pi}{M}\right)\right)\right] e^{-\frac{1}{2}\left[x - \sqrt{2\log_2 M \left(\frac{E_b}{N}\right)}\right]^2} dx$$

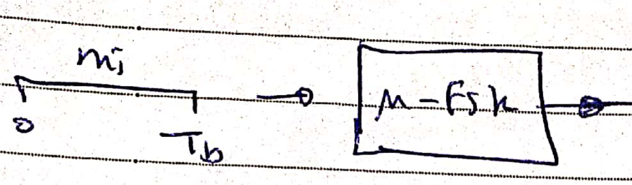
$$P_e^{M\text{-ary}} \simeq 2Q\left(\sqrt{2\log(M)\frac{E_b}{N}}\,\sin\left(\frac{\pi}{M}\right)\right)$$

as $M\uparrow$ $P_e^{M\text{-psk}}\uparrow$

For $\frac{E_b}{N} >> 1$, $M > 4$



⊛ M-ary orthogonal Fsk



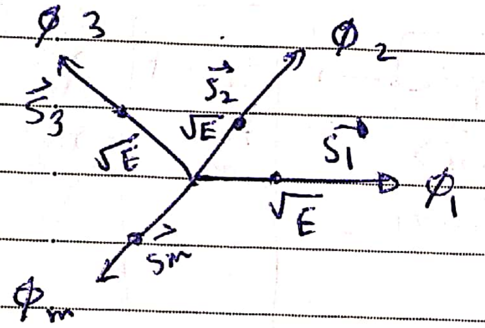
$s_i(t) = \sqrt{\frac{2E}{T_0}} \cos(2\pi f_i t)$

$f_i = f_c + (i-1)\delta f$

$\delta f = \frac{1}{2T}$

$s_i(t) = \sqrt{E} \sqrt{\frac{2}{T_0}} \cos(2\pi f_i t) \Rightarrow s_i(t) = \sqrt{E}\, \phi_i(t)$

$i = 1, 2, ---, m$



Each symbol has one component only.

* M-ary basis function

$s_i(t) = \vec{s_i} = (0, 0, ---, \sqrt{E}, 0, 0, --)$
↳ ith position

$\vec{S_1} = (\sqrt{E}, 0, 0, 0)$

$S_2 = (0, \sqrt{E}, 0, 0)$

$\vec{S_M} \; (0, 0, ----, \sqrt{E})$

* Energy each symbol

$E_{si} = ||s_i||^2 = E$. all has same energy

$E_{av} = E$

$E_b = \frac{E_{av}}{\log_2 M} = \frac{E}{\log_2 M}$

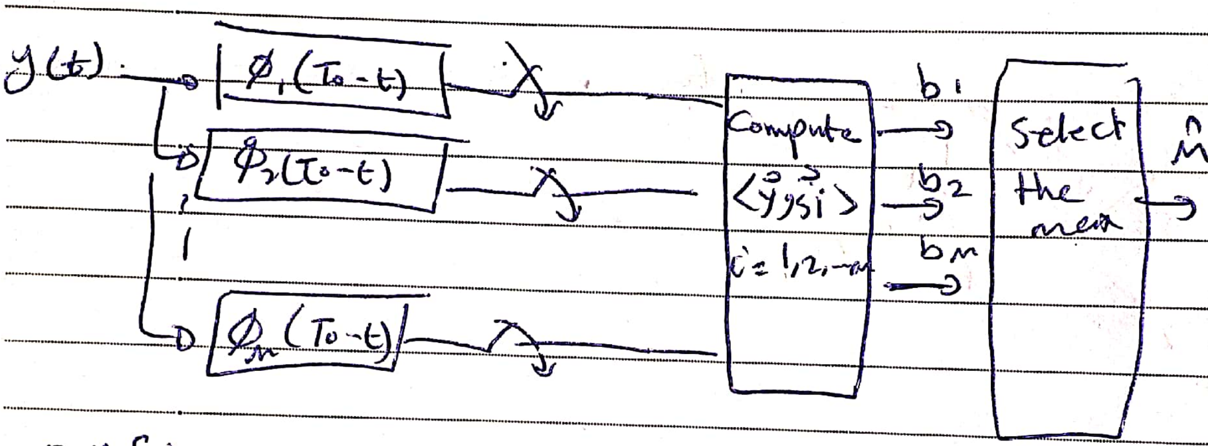
Rx of decision function

$$b_i = a_i + \langle y, \vec{s_i} \rangle$$

$$a_i = \frac{N}{2} \ln(p_{mi}) - \frac{E_{si}}{2}$$

$$= \frac{N}{2} \ln\left(\frac{1}{m}\right) - \frac{E}{2}$$

$b_i = \langle \vec{y}, \vec{s_i} \rangle$ ← select the max



$$P_e^{M\text{-}fsk} = 1 - P(C)$$

$$P_C = \sum_{i=1}^{M} P(m_i) P(c/m_i) = \frac{1}{M} \sum_{i=1}^{M} P(c/m_i)$$

because of symmetry $P(c/m_i) = P(c/m_1)$ for $i = (2, 3, \ldots, M)$

$$P(C) = P(C/m_1)$$

$$\boxed{P_c^{M\text{-}Fsk} = 1 - P(C/m_1)}$$

$$P(c/m_1) = pr(b_1 > b_2 \cap b_1 > b_3 \cap \ldots b_1 > b_m / m_1)$$

$$= \int_{b_1=-\infty}^{\infty} \int_{b_m=-\infty}^{b_1} \cdots \int_{b_3=-\infty}^{b_1} \int_{b_2=-\infty}^{b_1} f_{b_1, b_2, \ldots b_m}(b_1, b_2, \ldots b_m / m_1) \, dB_2 \ldots dB_m \, dB_1$$

if b_1, b_2

$$\boxed{P(b_1 > b_2) = \int_{b=-\infty}^{\infty}\int_{b_2=-\infty}^{b_1} f(b_1, b_2)\, db_2\, db_1}$$

$$b_i/m_1 = \langle \vec{y}/m_1, \vec{s_i} \rangle$$

$$\vec{y}/m_1 = \vec{s_1} + \vec{n} = (\sqrt{E}, 0, \dots 0) + (n_1, n_2, — n_m)$$

$$= (\sqrt{E} + n_1, n_2, n_3, — n_m).$$

$$b_i/m_1 = \langle \vec{y}/m_1, \vec{s_i})$$

$$= (\sqrt{E} + n_1, n_2, n_3 \_\_ n_m) \cdot (0, 0, 0, \sqrt{E}, 0, 0 — 0)$$

↑ ith position

$$b_i/m_1 = \begin{cases} E + \sqrt{E} n_1, & i = 1 \\ \sqrt{E} n_i, & i \neq 1 \end{cases} \qquad n_1 \sim N(0, \frac{N}{2})$$

1

$$b_i/m_1 \sim \begin{cases} N(E, \frac{EN}{2}), & i = 1 \\ N(0, \frac{EN}{2}, & i \neq 1 \end{cases}$$

$$f_{b_1, — b_m}(b_1, b_2, b_3 \dots b_m/m_1) = f_{b_1}(b_1/m_i) f(b_2/m_1) — f_{b_m}(b_m/m_1)$$

$$\int_{b_1=-\infty}^{\infty} f_{b_1}(b_1/m_1) \left[\prod_{i=2}^{M} \int_{b_i=-\infty}^{b_1} f_{b_i}(b_i/m_1)\, db_i \right] db_1$$

$$I = \int_{-\infty}^{b_i} N(0, \frac{EN}{2})\, db_i = 1 - Q\left(\frac{b_1 - 0}{\sqrt{EN/2}}\right)$$

$$\boxed{P(C/m_1) = \int_{b=-\infty}^{\infty} \frac{1}{\sqrt{\cancel{2\pi} \frac{EN}{\cancel{2}}}} e^{-(b_1 - E)^2/\cancel{2}EN/\cancel{2}} \left[1 - Q\left(\frac{b_1}{\sqrt{\frac{EN}{2}}}\right)\right]^{M-1} db_1}$$

$$P_e^{M-FSK} = 1 - P(C/m_1)$$

M ↑ Ped performance ↑

١٢/٢٢ الاثنين

let $y = b_1 / \sqrt{N E/2}$

$$P(C/m_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[y - \log_2(M)\left(\frac{E_b}{N}\right)\right]^2} \left[1 - Q(y)\right]^{M-1} dy$$

$$P_e^{M-FSK} = 1 - P(C/m)$$

$$\boxed{P_e^{M-FSK} \underset{\boxed{\approx}}{\le} (M-1)\, Q\left(\sqrt{\log_2(M)\left(\frac{E_b}{N}\right)}\right)}$$

using upper bound metho

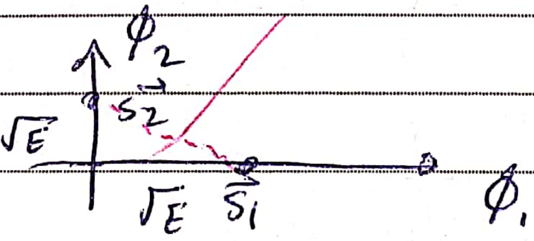
- As $M\uparrow$ $\quad P_e\downarrow$

- $M \rightarrow \infty$ $\quad P_e \rightarrow 0$ (error-free limit)
- Under the condition them $\frac{E_b}{N} > \ln(2) = 0.693$ (−1.6

$$\left(\frac{E_b}{N}\right)^{min} = 0.693 \quad ((\text{Shannon S/N R limit}))$$

B-FSK $\quad \vec{S}_1(t) = (\sqrt{E}, 0) \quad \boxed{M=2} \quad P(\vec{S}_1) = P(\vec{S}_2) = \frac{1}{2}$

$\vec{S}_2(t) = (0, \sqrt{E})$.



$d_{12} = (\sqrt{E})^2 + ((\sqrt{E})^2$

$d_{12} = \sqrt{2E}$

Note:-- $\quad P_e^{Binary} = Q\left(\frac{d_{12}}{\sqrt{2N_a}}\right)$

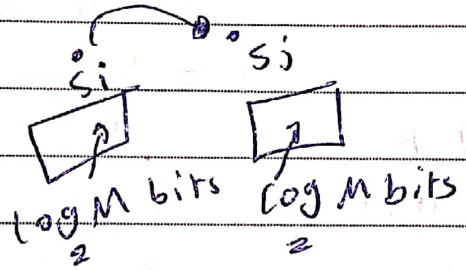
$$P^{B\text{-}FSK} = Q\left(\sqrt{\frac{(d_{12})^2}{2N}}\right) = Q\left(\sqrt{\frac{2E}{2N}}\right) = Q\left(\sqrt{\frac{E_b}{N}}\right)$$

Ex B-PAM P_e??

• Prob of Bit error P_b or BER

So far P_e ⇒ probability of symbol error
M-PAM, M-QAM, M-FSK, M-PSK

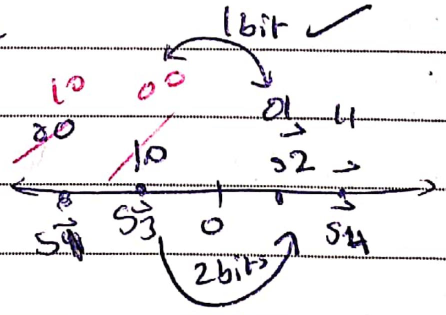
each symbol carries $\log_2 M$ bits.



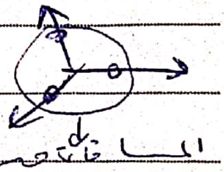
BER ↓ Gray code

Ex
4-PAM

$00 \rightarrow \vec{s}_1$
$01 \rightarrow \vec{s}_2$
$10 \rightarrow \vec{s}_3$
$11 \rightarrow \vec{s}_4$



FSK



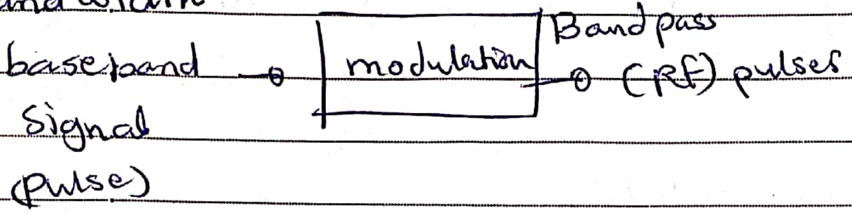
Gray code (PAM - PSK - QAM)

المسافات متساوية
ما بفيد Gray code .

$$P_b = \frac{P_e}{\log_2 M}$$ → PAM, psk, QAM with gray code

$$\boxed{P_b^{Fsk} = \frac{2^{* \log_2 M - 1}}{M-1} P_e^{Fsk}}$$

Bandwidth



assume raised cosine pulse shape

$$\boxed{B_{T\,\text{Base band}} = (1+r)\frac{R_s}{2}}$$

R_s → Symbol rate $= \frac{1}{T_s}$

$$T_s = \frac{1}{\log_2 M \, T_b} = \frac{R_b}{\log_2 M}$$

$$\boxed{B_{T\,RF\,\text{bandpass}} = (1+r) R_s}$$

↑ psk, PAM, QAM

$$B_w\,fsk = (2\Delta F) + 2 B_T^{\text{Base Band}}$$

$$\Delta F = \frac{F_M - F_1}{2}$$

$$= \frac{M-1}{2} \delta f$$

$$\delta f = \frac{1}{2 T_s}$$

Ex 10.5 ~~//~~ $R_b = 2.08 * 10^8$ bits/s

$P_b \le 10^{-6}$

use- Raised cosine pulse with r = 1 n

PSD for white Noise is = 10^{-8} W/HZ

$(\boxed{N/2}$

@ Base band with polar signaling

ⓑ 16 PAM ~~width~~ ⓒ 16-PSK

determine

Ⓐ transmission Bandwidth Ⓑ signal power

Base band polar

@ $B_T = (1+r)\frac{R_s}{2} = R_s = R_b$ $\boxed{2.08 * MHZ}$

Signal power

$P_{ave} = \frac{E_b}{T_b} = E_b R_b$

?

$BER = Q\left(\sqrt{\frac{2E_b}{N}}\right) = 10^{-6}$ $\frac{E_b}{N} = 11.35$

$E_b = 11.35 * (2) * 10^8$

$\boxed{P_{ave} = E_b R_b = 0.47 \text{ watt}}$

ⓑ $P^{bandpass} = 2B_T^{baseband} = \cancel{2}(1+r)\frac{R_s}{\cancel{2}} \rightarrow \frac{R_b}{4}$

$\frac{\cancel{2}(2.68 * 10^6)}{\cancel{4}} = \boxed{1.04 * MHZ}$

$P_b = \frac{P_e}{\log_2 16} =$ $P_e = 4 * 10^{-6}$

$$4 * 10^{-6} = \frac{2(16-1)}{16} Q\left(\sqrt{\frac{6*4}{16^2-1}\left(\frac{E_b}{N}\right)}\right)$$

then $E_b = 0.499 * 10^{-5}$.

$P_{av} = E_b R_b$

$= (4.99 * 10^{-5})(2.08 * 10^{8})$

$= \boxed{9.34 W}$

PSK

$P = 2.86$ watt

١٢/٢٤ اثلاثاء

Non-coherent detection

NC BFSK

"1" $\rightarrow s_1(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 t)$ $0 \leq t \leq T_b$

"0" $\rightarrow s_2(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_2 t)$

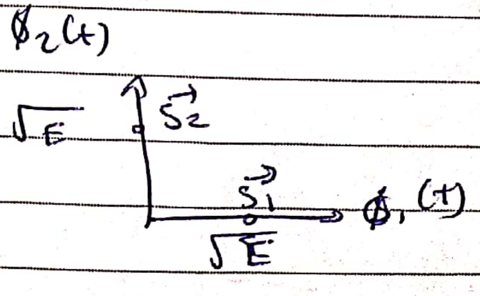
$f_2 - f_1 = \delta f = \frac{1}{2T_b}$

$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_1 t)$ $\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_2 t)$.

$E_{s1} = E_{s2} = E$

$s_1(t) \rightarrow \vec{s_1} = (\sqrt{E}, 0)$

$s_2(t) \rightarrow \vec{s_2} = (0, \sqrt{E})$



decision function

$$b_i := \langle \vec{y}, \vec{s_i} \rangle = \int_0^{T_b} y(t)\, s_i(t)\, dt$$

$$= y_1 s_{i1} + y_2 s_{i2} \quad *$$

① $\vec{y} / \text{"1"} = (\sqrt{E} + n_1, n_2)$

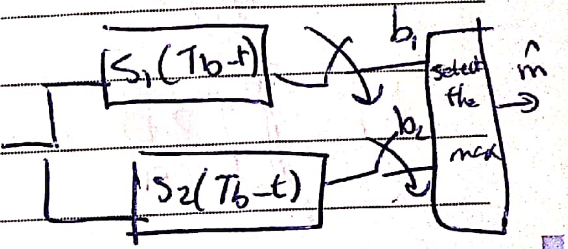
② $\vec{y} / \text{"0"} = (n_1, \sqrt{E} + n_2)$

$$b_1 = \begin{cases} E + \sqrt{E} n_1 , & \text{"1"} \\ \sqrt{E} n_1 , & \text{"0"} \end{cases}$$

Using * and ① & ②

$$b_2 = \begin{cases} \sqrt{E} n_2 , & \text{"1"} \\ E + \sqrt{E} n_2 , & \text{"0"} \end{cases}$$

$$y(t) = \sqrt{\frac{2E}{T_0}} \cos(2\pi f_i t) + n(t)$$



$$b_1 = \int_0^{T_b} y(t)\, s_1(t)\, dt$$

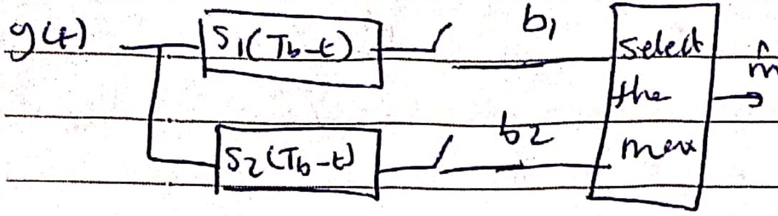
$$= \int_0^{T_b} \sqrt{\frac{2E}{T_b}} \cos(2\pi f_i t) + n(t) \left[\sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 t) dt \right.$$

$$= \begin{cases} E + \sqrt{E} n_1 , & i = 1 \\ 0 + \sqrt{E} n_1 , & i = 2 \end{cases}$$

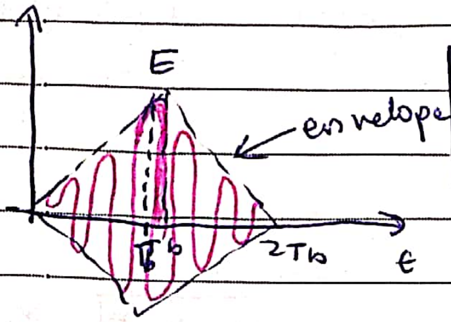
$$b_2 = \int_0^{T_b} y(t)\, s_2(t)\, dt = \int_0^{T_b} \sqrt{\frac{2E}{T_b}} \cos(2\pi f_i t) + n(t) \overset{S_2(t)}{\left[\sqrt{\frac{2E}{T_b}} \cos(2\pi f \right.}$$

$$= \begin{cases} \sqrt{E} n_2 , & i = 1 \\ \sqrt{E} n_2 + E , & i = 2 \end{cases}$$

$$y(t) = \sqrt{\frac{2E}{T_o}}\cos(2\pi f_i + \theta_{Rx}) + n(t)$$



θ_{Rx} is unknown at Rx

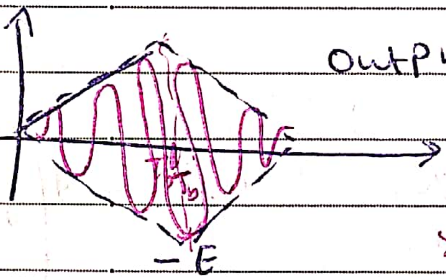


first matched filter output when $S_1(t)$ was transmitted

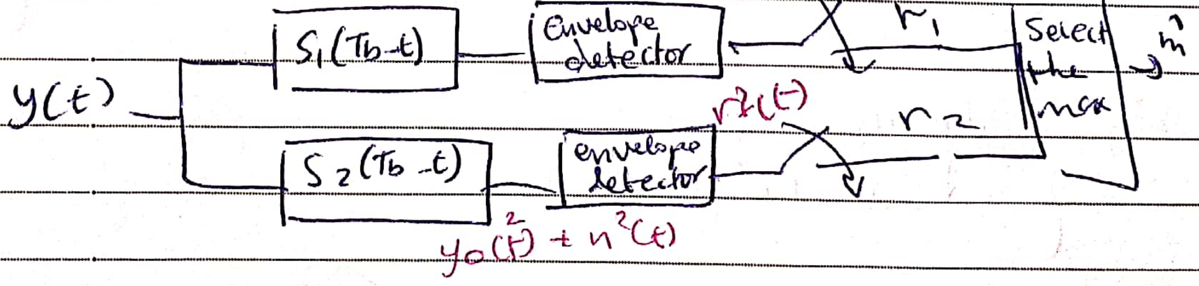
$$b_1 = \int_0^{T_b}\left[\sqrt{\frac{2E}{T_b}}\cos(2\pi f_i t + \theta_{Rx}) + n(t)\right]\left[\sqrt{\frac{2E\cos(2\pi f_i t)}{T_b}}\right.$$

$$= \begin{cases} E\cos\theta_{Rx} + \sqrt{E}n_1, & i=1 \\ \sqrt{E}n_2, & i=2 \end{cases}$$

$\theta_{Rx} = 180^\circ$



output of first matched filter when $S_1(t)$ was Txed and $\theta_{Rx} = 180^\circ$



$$y_0'(t) = \int_0^{T_b} y(t) * S_1(T_b - t)$$

$$= \int_0^{T_b} y(\tau)\, S_1(T_b - t + \tau) = \int_0^{T_b} y(\tau) \sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 (T_b - t + \tau))$$

$$= \cos(2\pi f_1 (T_b - t)) + 2\pi f_1 \tau$$

$$= \int_0^{T_b} y(\tau) \sqrt{\frac{2E}{T_b}} \Big[\cos(2\pi f_1 (T_b - t)) \cos(2\pi f_1 \tau) - \sin(2\pi f_1 (T_b - t)) \cdot \sin(2\pi f_1 \tau) \Big)$$

$$= \cos(2\pi f_1 (T_b - t)) \int_0^{T_b} (y(\tau) \sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 \tau)\, d\tau$$

$$- \sin(2\pi f_1 (T_b + t)) \int_0^{T_b} y(\tau) \sqrt{\frac{2E}{T_b}} \sin(2\pi f_1 \tau)\, d\tau$$

$$= \alpha \cos(\theta) - \beta \sin(\alpha)$$

$$= \sqrt{\alpha^2 + \beta^2 \cos(\theta + \tan^{-1}(\beta / \alpha))}$$

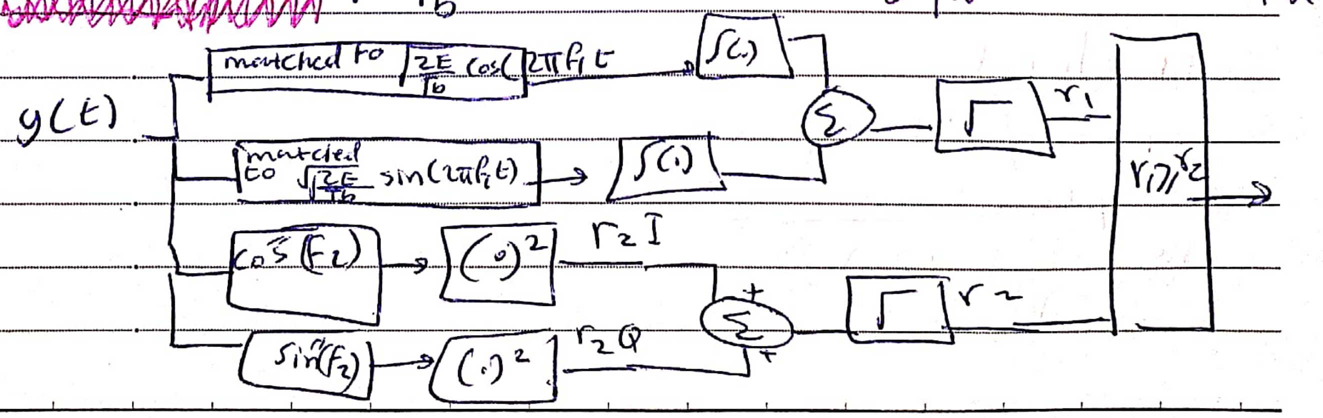
$$= \sqrt{\left[\int_0^{T_b} y(\tau) \sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 \tau \right]^2 + \left[\int_0^{T_b} y(\tau) \sin(2\pi f_1 t)\, d\tau \right]}$$

is envelope $\cos(2\pi f_1 (T_0 - t) + \tan^{-1}(\beta / \alpha))$

$$\tilde{r}(T_b) = \tilde{r}$$

$$y(\tau) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_i \cancel{\theta}) + n(t)$$

Squarelaw NC Rx



Non coherent BFSK.

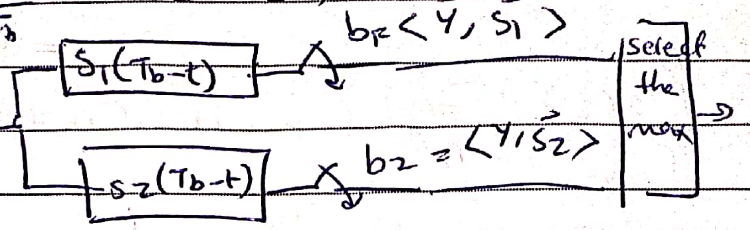
Coherent

$"1" \rightarrow s_1(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 t)$

$"0" \rightarrow s_2(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_2 t)$

$y(t) = s_k(t) + n(t)$

$k = 1, 2, \ldots$



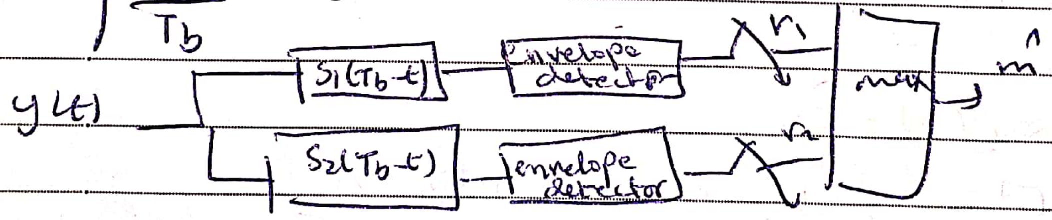
$$b_1 = \begin{cases} E + \sqrt{E}\, n_1 , & "1" \text{ txed} \\ 0 + \sqrt{E}\, n_1 , & "0" \text{ txed} \end{cases}$$

$$b_2 = \begin{cases} 0 + \sqrt{E}\, n_2 , & "1" \text{ txed} \\ E + \sqrt{E}\, n_2 , & "0" \text{ txed} \end{cases}$$

$f_2 - f_1 = \delta f = \frac{1}{T_b}$ orthogonality condition in coherent BFSK

NC-BFSK

$y(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_k + \theta^{Rx}) + n(t)$



$f_K = f_1$ or f_2

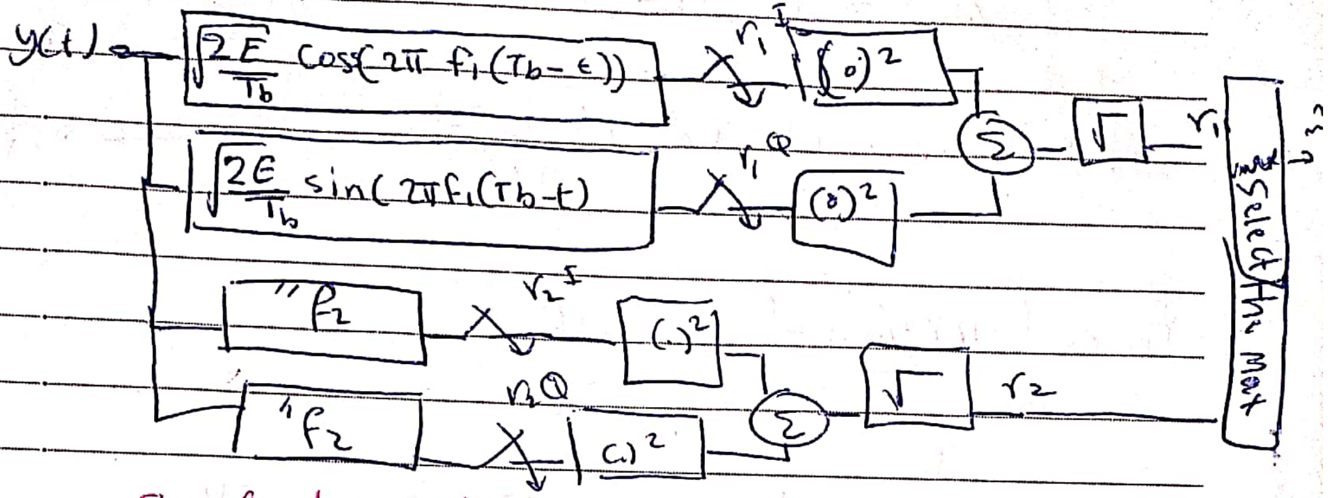
$r_1 = \sqrt{\underbrace{r_1^2 I}_{\text{Inphase}} + \underbrace{r_1^2 Q}_{\text{Quadrature}}}$

$r_2 = \sqrt{r_2^2 I + r_2^2 Q}$

$$r_i^{I} = \int_0^{T_b} y(t) \sqrt{\frac{2E}{T_b}} \cos(2\pi f_i t)\, dt$$

$i = 1, 2$

$$r_i^{Q} = \int_0^{T_b} y(t) \sqrt{\frac{2E}{T_b}} \sin(2\pi f_i t)\, dt$$



Square law NC Rx

if $s_i(t)$ was transmitted

$$r_1^I = \int_0^{T_b} \left[\sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 t + \theta^{Rx}) + n(t)\right] \left(\sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 t)\, dt\right)$$

$$= E \cos(\theta^{Rx}) + \sqrt{E}\, n_1^I \sim N\left(E\cos\theta\, \theta Rx, \frac{EN}{2}\right)$$

$$r_1^Q = E \sin\theta^{Rx} + \sqrt{E}\, n_1^Q \sim N\left(E\sin\theta, \frac{EN}{2}\right)$$

$$r_2^I = \int_0^{T_b} \left(\left[\sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 t + \theta^{Rx}) + n(t)\right] \sqrt{\frac{2E}{T_b}} \cos(2\pi f_2 t)\, dt\right.$$

$$= 0 + \sqrt{E}\, n^I \quad N \sim \left(0, E\frac{N}{2}\right)$$

$$r_2^Q = 0 + \sqrt{E}\, n^Q \quad N \sim \left(0, \frac{EN}{2}\right)$$

non coherent requires more bandwidth.

$$f_2 - f_1 = \frac{1}{T_b}$$

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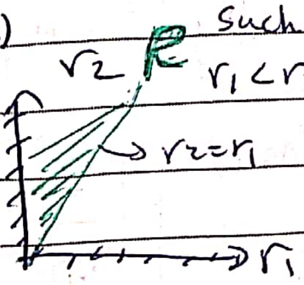
$$P_b = P(e/m_1)\,P(m_1) + P(e/m_2)\,P(m_2)$$

$$P(m_1) = P(m_2) = \frac{1}{2}$$

$$P(e/m_1) = P(e/m_2) \qquad P_b = P(e/m_1)$$

$$P(c/m_1) = P(r_1 < r_2/m_1)$$

$$= \int_{r_1=0}^{\infty}\int_{r_2=r_1}^{\infty} f(r_1, r_2/m_1)\, dr_2\, dr_1 \quad f_{r_1,r_2}(r_1, r_2)$$



Such that $r_1 < r_2$

$$f_{r_1,r_2}(r_1, r_2/m_1) = f_{r_1}(r_1/m_1)\, f_{r_2}(r_2/m_1)$$

$$r_1/m_1 = \sqrt{r_1^2 I/m_1 + r_1^2 Q/m_1}$$

$$\sim \sqrt{N^2\left(E\cos\theta^{Rx}, \frac{EN}{2}\right) + N^2\left(E\sin\theta^{Rx}, \frac{EN}{2}\right)}$$

Ricean R.V

$$f_{r_1}(r_1/m_1) = \frac{r_1}{\sigma_n^2}\, e^{-\left(\frac{r_1^2 + E}{2\sigma_n^2}\right)}\, I_0\left(\frac{r_1 E}{\sigma_n^2}\right)$$

Bissel Function

$$\sigma_n^2 = \frac{N}{2}$$

$$r_2/m_1 = \sqrt{r_2^2 I/m_1 + r_2^2 Q/m_1}$$

$$\sim \sqrt{N^2\left(0, \frac{EN}{2}\right) + N^2\left(0, \frac{EN}{2}\right)}$$

Rayleigh R.V

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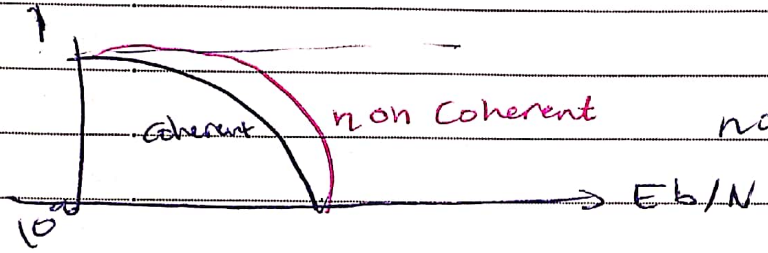
$$f_{r_2/m_1}(r_2) = \frac{r_2}{\sigma_n^2} e^{-r_2^2/2\sigma_n^2} \quad , r_2 > 0$$

$$\boxed{\begin{aligned} P(e/m_1) &= \frac{1}{2} e^{-\frac{1}{2}\left(\frac{E}{N}\right)} \\ &= \frac{1}{2} e^{-\frac{1}{2} Eb/N} \end{aligned}}$$

$$P_{b\,C}^{\text{coherent - BFSK}} = Q\left(\sqrt{\frac{E_b}{N}}\right)$$

$$P_b^{NC - BFSK} = \frac{1}{2} e^{-\frac{1}{2} Eb/N}$$



non-Coherent is preferable

non-Coherent $\xrightarrow{X}$ FSK QAM can't be used

Coherent → PAM