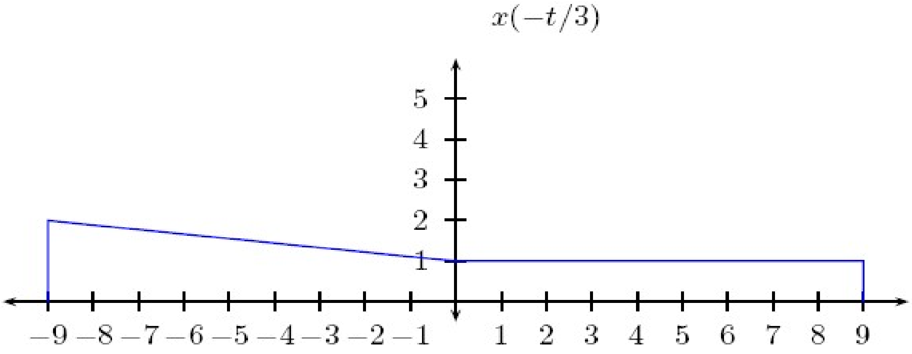
Chapter 2 solutions

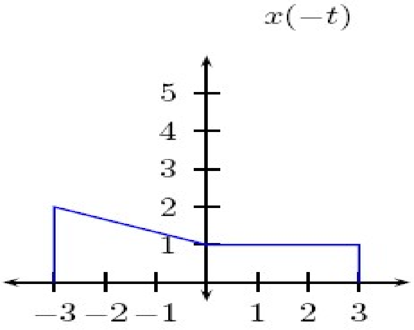
2.1

(a)

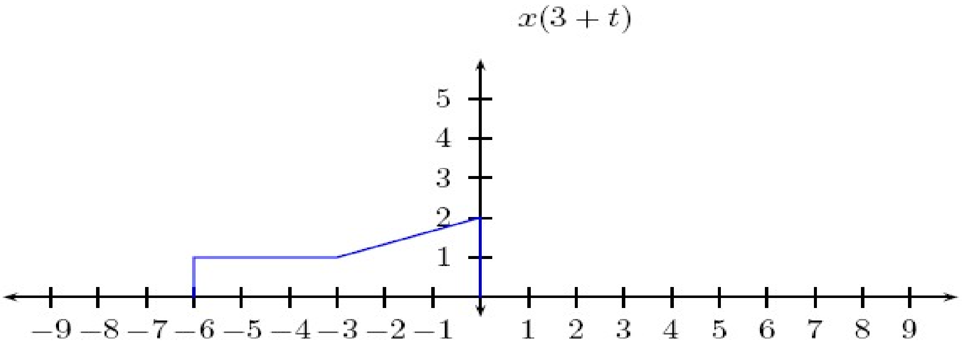
(i)



(ii)

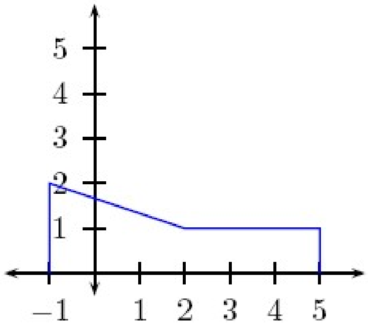


(iii)



(iv)

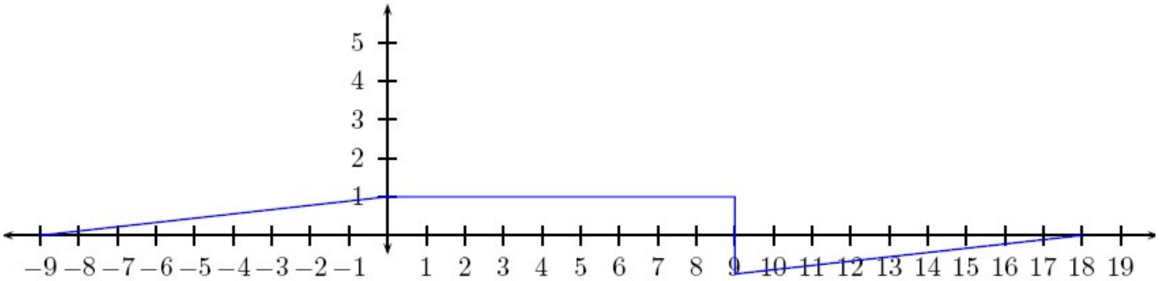
$x(2-t)$



(b)

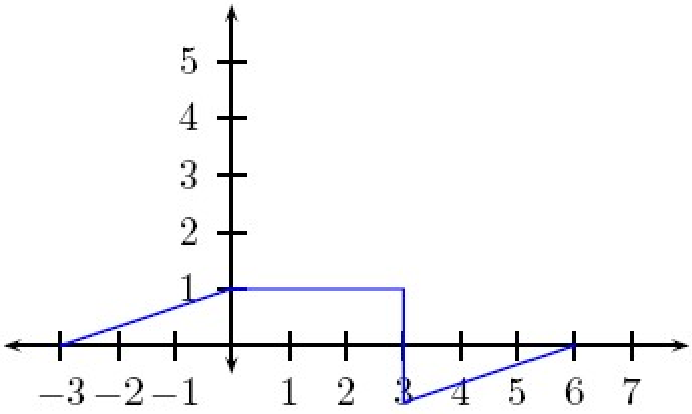
(i)

$x(-t/3)$

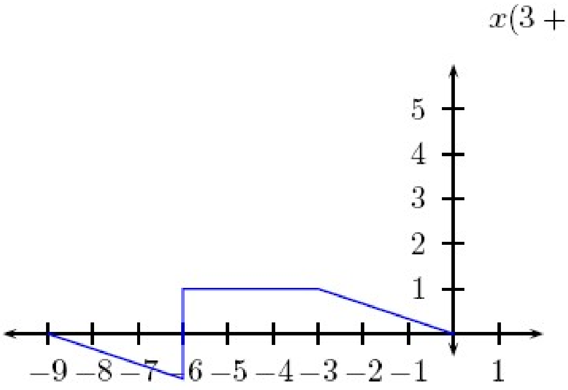


(ii)

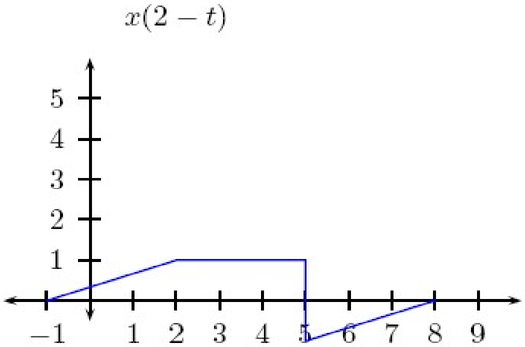
$x(-t)$

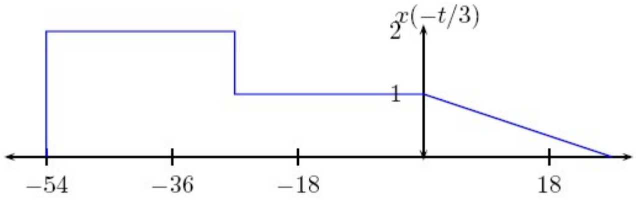


(iii)



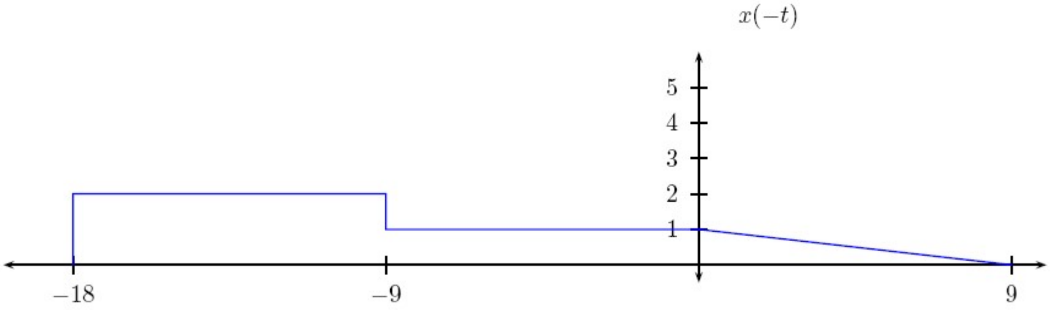
(iv)



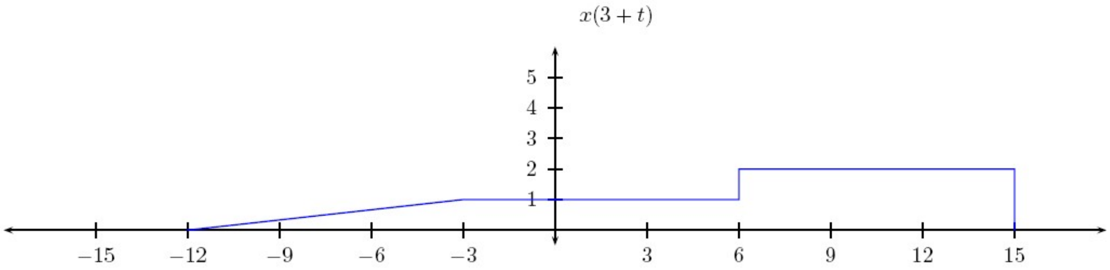


(c) (i)

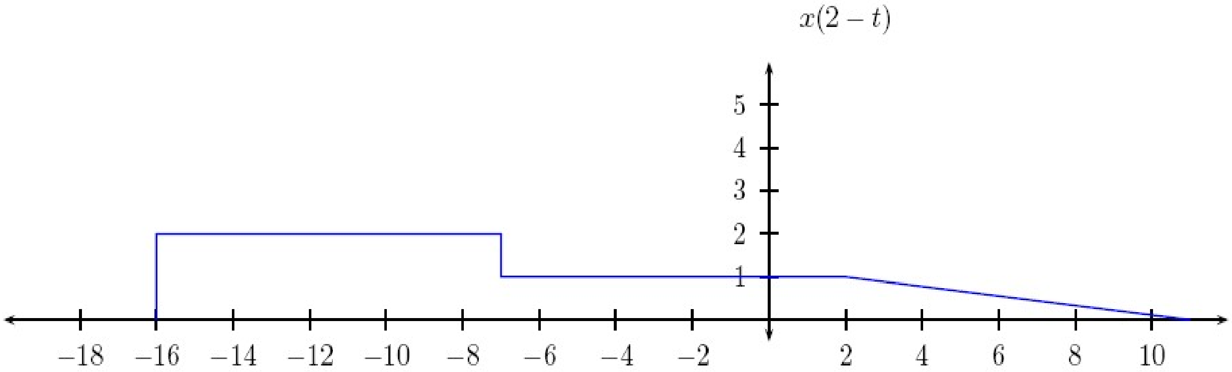
(ii)


$x(-t)$
5
4
3
2
1
−18
−9
9

(iii)

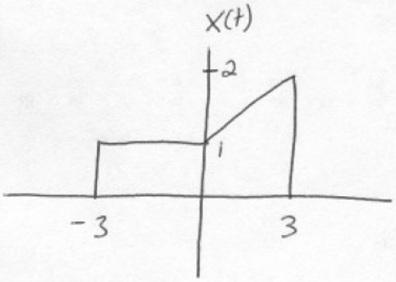

$x(3+t)$
5
4
3
2
1
−15
−12
−9
−6
−3
3
6
9
12
15

(iv)

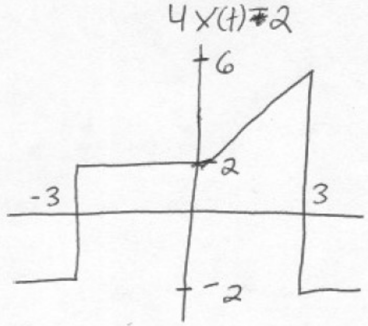

$x(2-t)$
5
4
3
2
1
−18
−16
−14
−12
−10
−8
−6
−4
−2
2
4
6
8
10

2.2

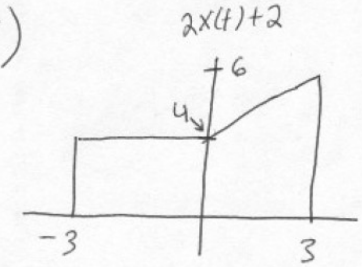
(a)



i)



ii)



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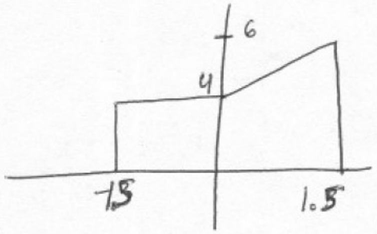
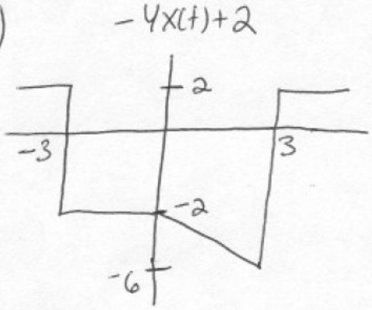
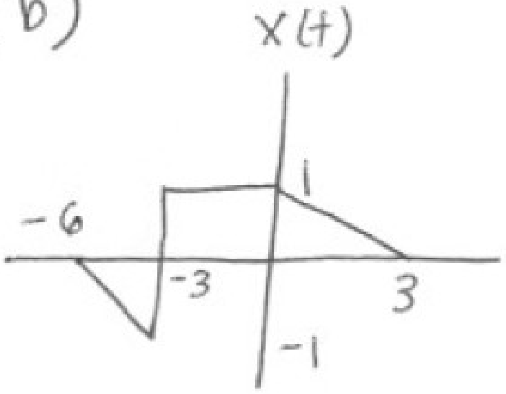


iv)

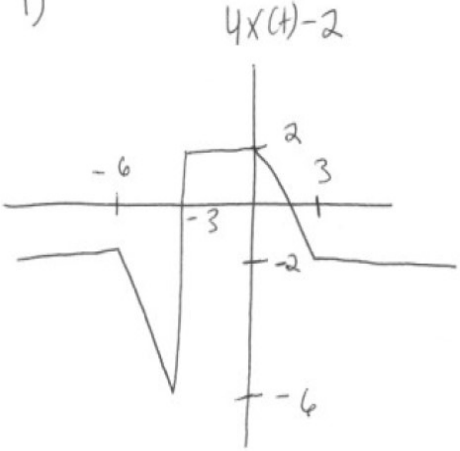


2.2

(b)



i)



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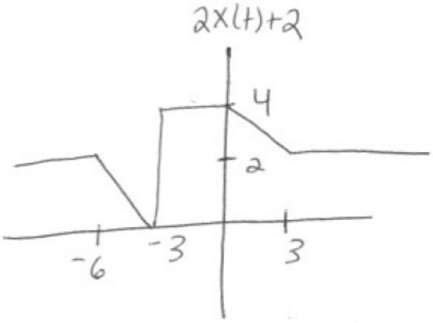

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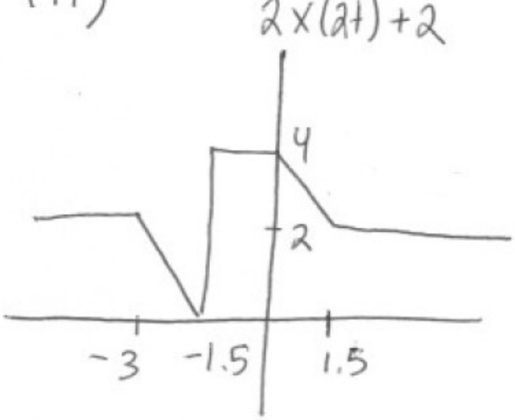

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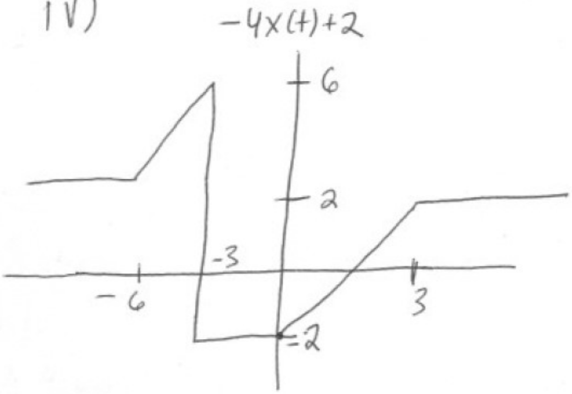
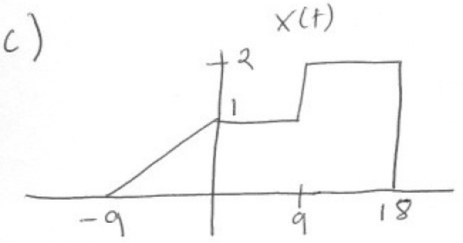
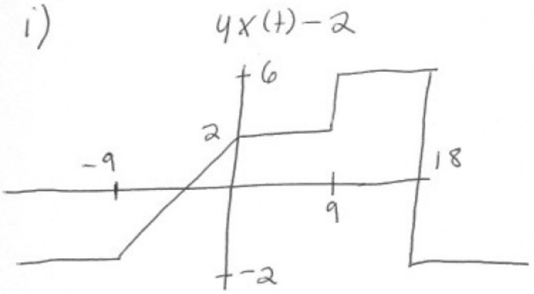


2.2

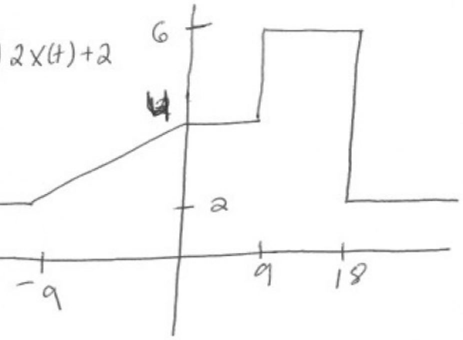
(c)



i)



ii)



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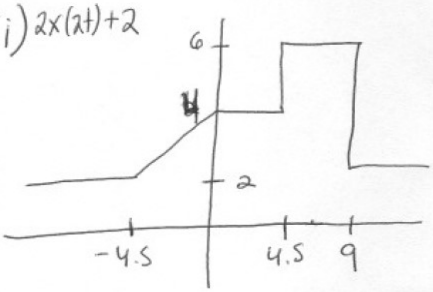

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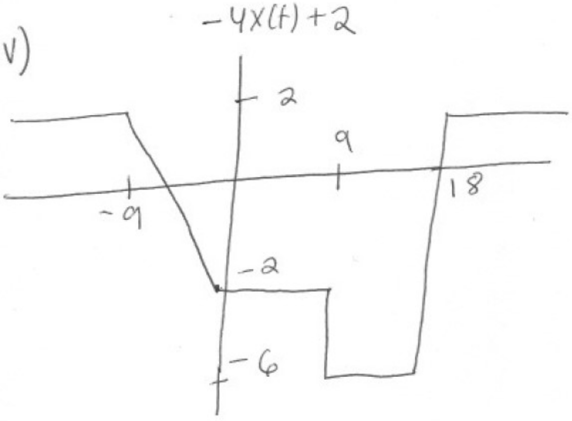
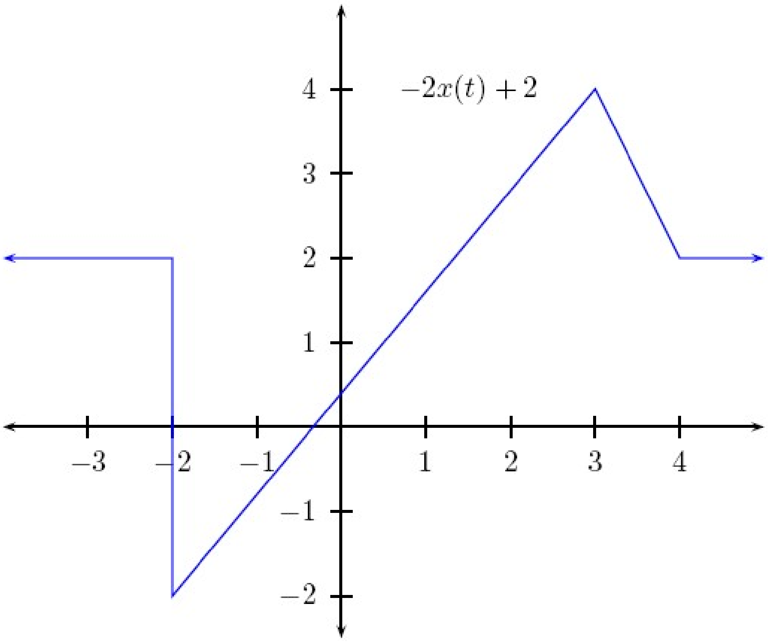
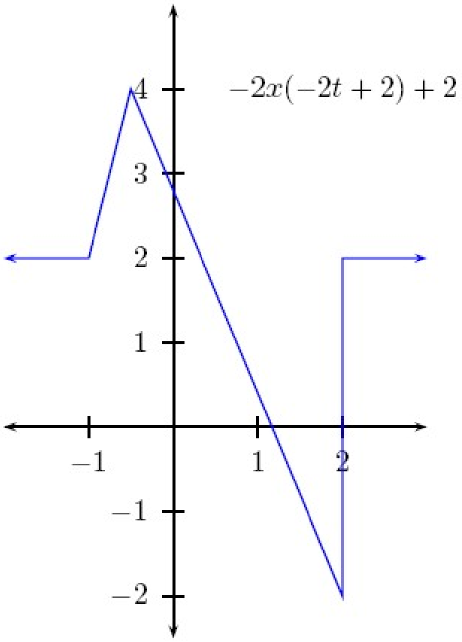


2.3





(a)

$y(t) = -2(x(-2t+2)) + 2$

(b)

t	$y(t)$	$-2t+2$	$-2(x(-2t-1))+2$
-0.5	4	3	4
-1	2	4	2
1	0.4	0	0.4

2.4

(a) $y(t) = -0.5(x(2t-4)) + 1.5$

(b)

t	y(t)	2t-4	-0.5(x(2t-4))+1.5
2	1.5	0	1.5
3	-1	2	-1
4.5	1.5	5	1.5

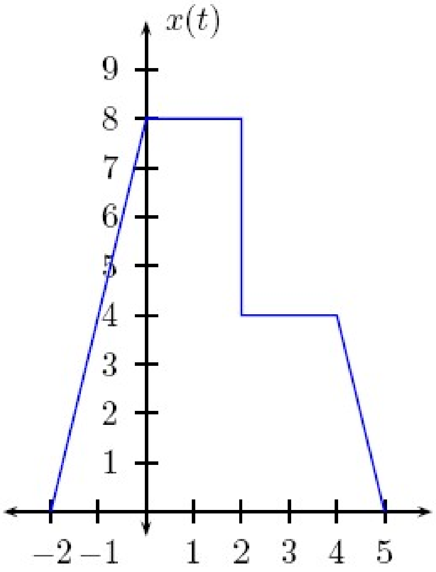
(c) $x(t) = -2y(\frac{t+4}{2}) + 3$

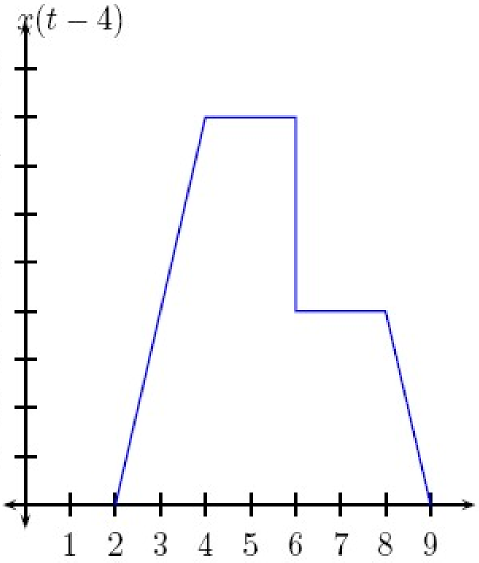
(d)

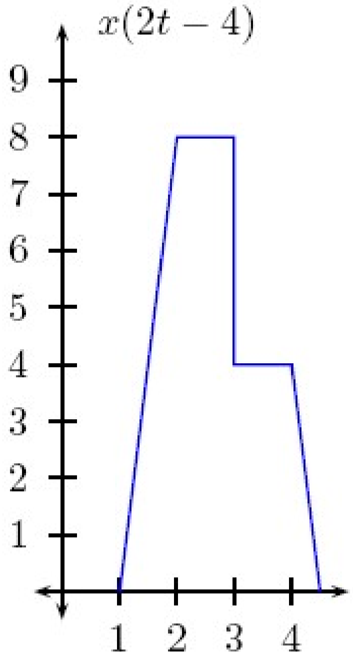
t	x(t)	$\frac{t+4}{2}$	$-2y(\frac{t+4}{2})+3$
0	0	2	0
4	-3	4	-3
5	0	4.5	0

2.5

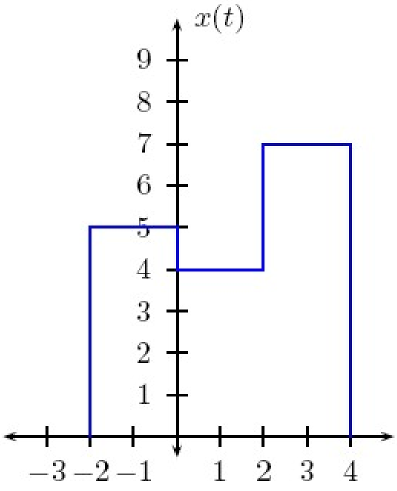
$$\begin{aligned} x(2t-4) &= 4[(2t-2)u(2t-2)-(2t-4)u(2t-4)-u(2t-6)-(2t-8)u(2t-8)-(2t-9)u(2t-9)] \\ &= 4[(2t-2)u(t-1)-(2t-4)u(t-2)-u(t-3)-(2t-8)u(t-4)-(2t-9)u(t-4.5)] \end{aligned}$$

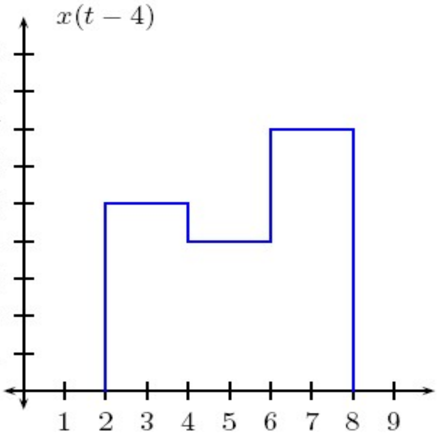


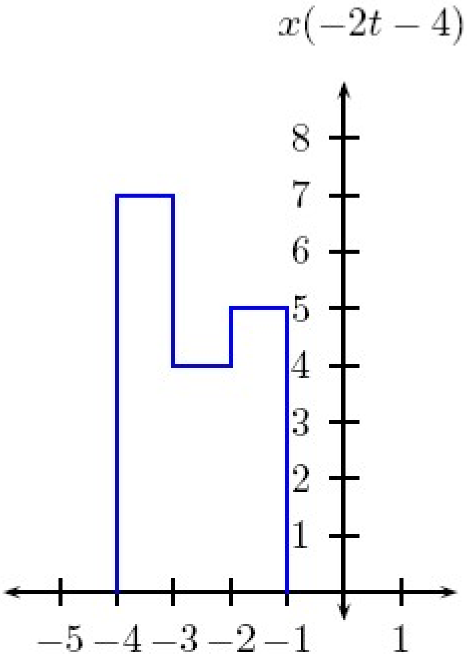




2.6



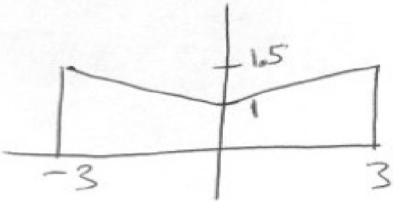




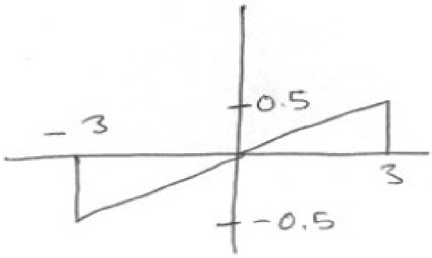
$$x(t) = 5u(-2t-2) - u(-2t-4) + 3u(-2t-6) - 7u(-2t-8)$$
$$= 5u(-(t+1)) - u(-(t+2)) + 3u(-(t+3)) - 7u(-(t+4))$$
$$\text{Or } x(t) = 7u(t+4) - 3u(t+3) + u(t+2) - 5u(t+1)$$

2.7

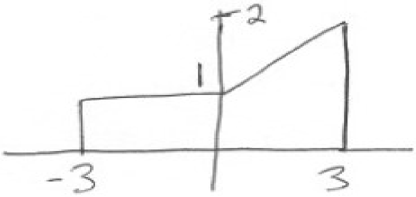
a) even



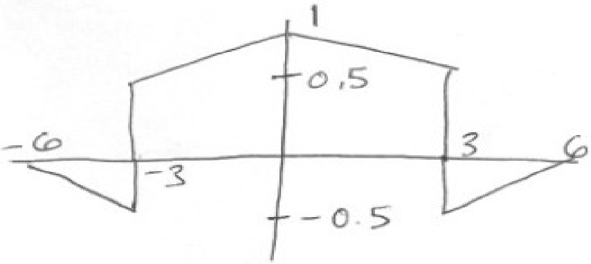
odd



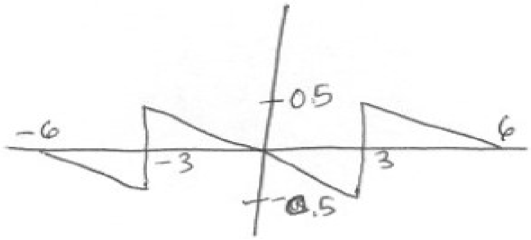
verify: even + odd:



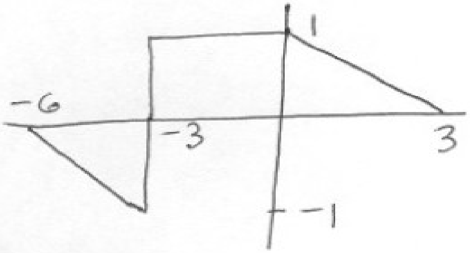
b) even



odd

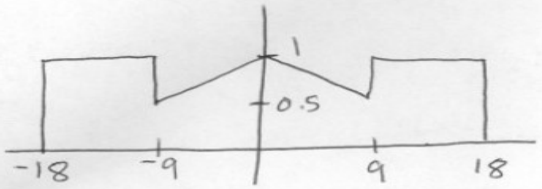


verify: even + odd:

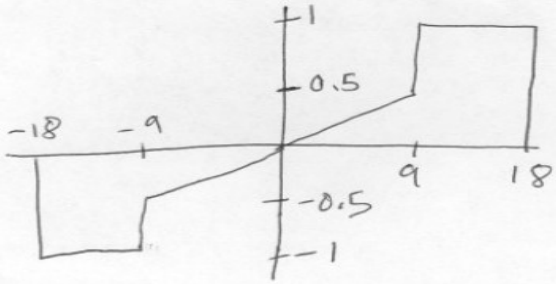


c)

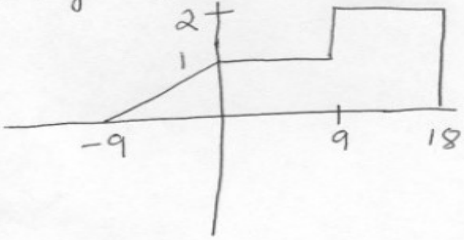
even



odd

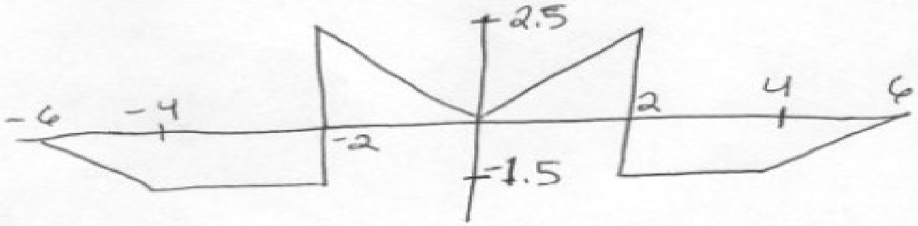


verify: even + odd:

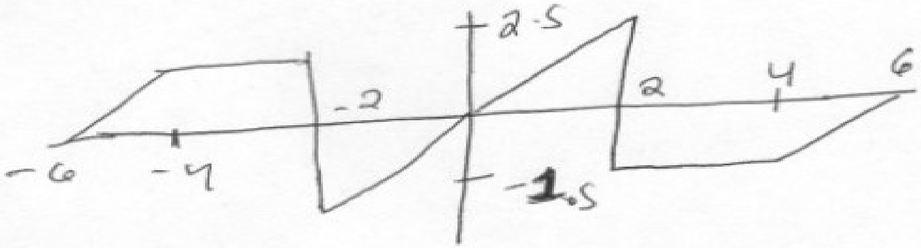


d)

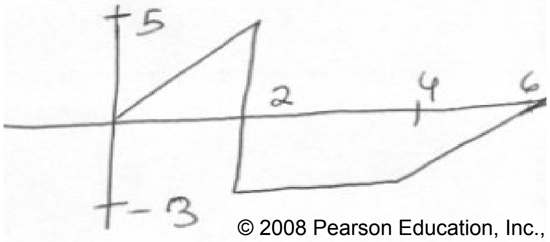
even



odd

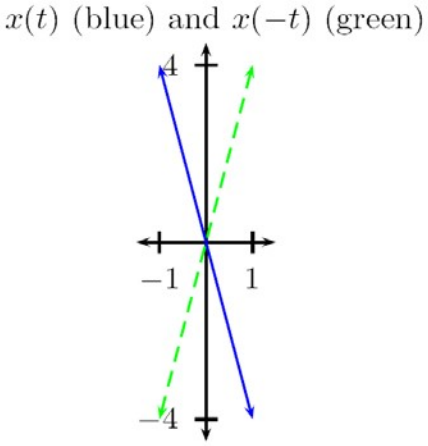


even + odd:

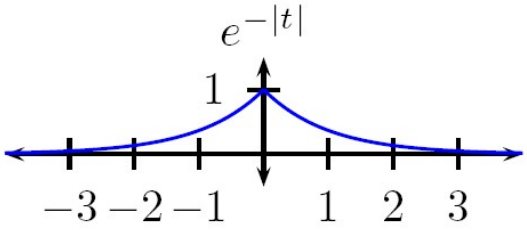


2.8

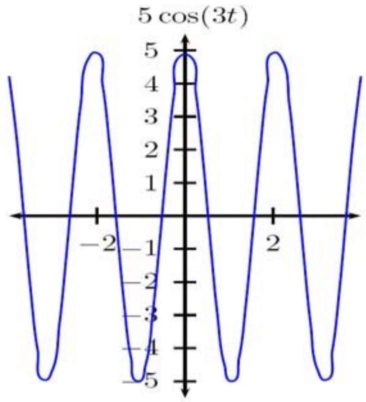
a) $-4t = -\left(-4(-t)\right)$ so it is odd.


$x(t)$ (blue) and $x(-t)$ (green)

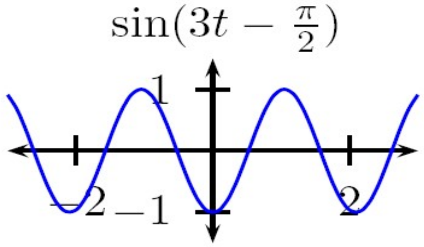
b) $e^{-|t|} = e^{-|-t|}$ so it is even ($|t| = |-t|$).



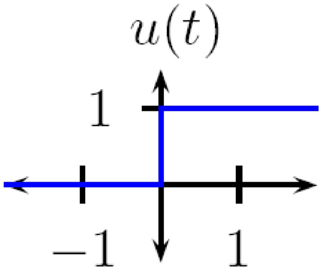
c) Since $\cos(t)$ is even, $5\cos(3t)$ is also even.



d) $\sin(3t - \frac{\pi}{2}) = -\cos(3t)$ which is even:



e) $u(t)$ is neither even nor odd; for example $u(3) = 1$ but $u(-3) = 0 \neq -u(3), \ \neq u(3)$.



2.9 (a) $\int_{-T}^{T} x_o(t) = \int_{-T}^{0} x_o(t)dt + \int_{0}^{T} x_o(t)dt$; $x_o(t) = -x_o(-t)$

$\therefore \int_{-T}^{0} x_o(t)dt = -\int_{-T}^{0} x_o(-t)dt \Big|_{t=-\tau} = \int_{-T}^{0} x_o(\tau)d\tau = -\int_{0}^{T} x_o(\tau)d\tau$

$\therefore \int_{-T}^{T} x_o(t)dt = 0$

(b) $\int_{-T}^{T} x(t)dt = \int_{-T}^{T} [x_e(t) + x_o(t)]dt = \int_{-T}^{T} x_e(t)dt$

and $A_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_e(t)d\tau$

(c) $x_o(0)=-x_o(-0)= -x_o(0)$. The only number with a=-a is a=0 so this implies $x_o(0)=0$.
$x(0)=x_e(0)+x_o(0)=x_e(0)$.

2.10

(a) Let $z(t)$ be the sum of two even functions $x_1(t)$ and $x_2(t)$. To show that $z(t)$ is even, we need to show that $z(t) = z(-t)$ for all t. This is easy to show, since $z(t) = x_1(t) + x_2(t)$ and $z(-t) = x_1(-t) + x_2(-t)$ (since to get $z(-t)$ we just plug in $-t$ everywhere for t, which amounts to just plugging in $-t$ in $x_1(t)$ and $x_2(t)$). Now since $x_1(t)$ and $x_2(t)$ are even, by definition $x_1(t) = x_1(-t)$ and $x_2(t) = x_2(-t)$ so $x_1(t) + x_2(t) = x_1(-t) + x_2(-t)$ so $z(t) = z(-t)$.

(b) Let $x_1(t)$ and $x_2(t)$ be two odd functions. Then $x_1(-t) + x_2(-t) = -x_1(t) + (-x_2(t)) = -(x_1(t) + x_2(t))$ which shows that $x_1(t) + x_2(t)$ is odd.

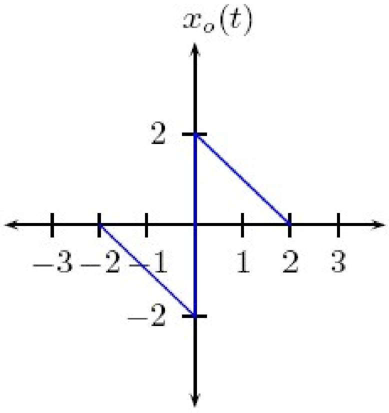
(c) Let $z(t) = x_1(t) + x_2(t)$ as in part a, where now $x_1(-t) = x_1(t)$ and $x_2(-t) = -x_2(t)$. We need to show that $z(t) \neq z(-t)$, $z(t) \neq -z(-t)$. Consider that $z(-t) = x_1(-t) + x_2(-t) = x_1(t) - x_2(t)$. In order to have $z(t)$ be even, we would therefore need to have $x_1(t) + x_2(t) = x_1(t) - x_2(t)$ for all t, which is equivalent to having $x_2(t) = -x_2(t)$ for all t, which is not possible for nonzero $x_2(t)$. Similarly, in order to have $z(t)$ be odd, we would need to have $z(t) = -z(t) \implies x_1(t) + x_2(t) = x_2(t) - x_1(t)$, which is not possible for nonzero $x_1(t)$. So the sum of an even and odd function must be neither even nor odd.

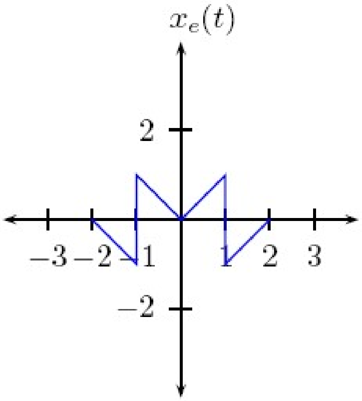
(d) Let $z(t) = x_1(t)x_2(t)$ where $x_1(t) = x_1(-t)$ and $x_2(t) = x_2(-t)$. Then $z(-t) = x_1(-t)x_2(-t) = x_1(t)x_2(t) = z(t)$ which shows that $z(t)$ is even.

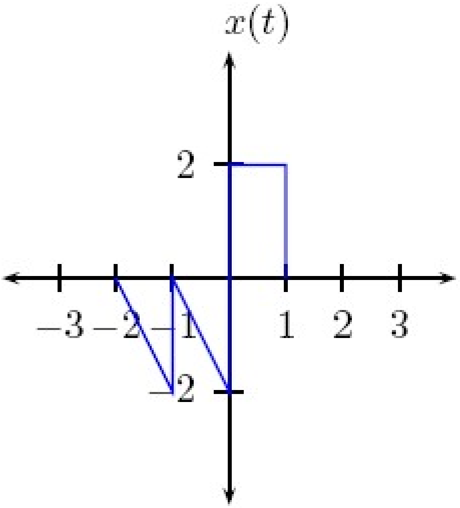
(e) Let $z(t) = x_1(t)x_2(t)$, where $x_1(t) = -x_1(-t)$ and $x_2(t) = -x_2(-t)$. Clearly $z(t)$ is even because $z(-t) = x_1(-t)x_2(-t) = (-x_1(t))\,(-x_2(t)) = x_1(t)x_2(t) = z(t)$, which is the definition of evenness.

(f) Let $z(t) = x_1(t)x_2(t)$, where $x_1(t) = -x_1(-t)$ and $x_2(t) = x_2(-t)$. Clearly $z(t)$ is odd because $z(-t) = x_1(-t)x_2(-t) = (-x_1(t))\,x_2(t) = -x_1(t)x_2(t) = -z(t)$, which is the definition of oddness.

2.11







The plot of $x_o(t)$ is determined by $x_o(-t) = -x_o(t)$, the plot of $x_e(t)$ is determined by $x_e(t) = x(t) - x_o(t)$, and the plot of $x(t)$ is determined by $x(t) = x_e(t) + x_o(t)$.

2.12

(a) $\sin(t) = \sin(t + n2\pi)$ for any integer n, so $7\sin(3t) = 7\sin(3t + n2\pi) = 7\sin\left(3(t + n\frac{2\pi}{3})\right)$; therefore $x(t)$ is periodic with fundamental period $T_0 = \frac{2\pi}{3}$ and fundamental frequency $\omega_0 = \frac{2\pi}{T_0} = 3$.

(b) $\sin(8(t + \frac{2\pi}{8}) + 30) = \sin(8t + 2\pi + 30) = \sin(8t + 30)$.
$\omega_0 = 8$ and $T_0 = \frac{2\pi}{8} = \frac{\pi}{4}$.

(c) $e^{jt} = \cos(t) + j\sin(t)$ is periodic with fundamental period 2π, so e^{j2t} is periodic with fundamental period $\frac{2\pi}{2} = \pi$, and fundamental frequency $\omega_0 = 2$.

(d) $\cos(t) = \cos(t + n2\pi)$ for any integer n, and $\sin(2t) = \sin(2(t + m\pi))$ for any integer m, so $\cos(t) + \sin(2t)$ will be periodic with period T_0 if $\cos(t) + \sin(2t) = \cos(t + T_0) + \sin(2(t + T_0))$. This will hold as long as $T_0 = n2\pi$ and $T_0 = m\pi$ for some integers n and m, and the fundamental period is the smallest value for which this holds, which is $T_0 = 2\pi$, with fundamental frequency $\omega_0 = 1$.

(e) $e^{j(5t+\pi)} = e^{j\pi}e^{j5t}$. So the phase shift of π just means a complex constant (constant with respect to time) out front and does not effect periodicity of the signal e^{j5t}, which has fundamental period $T_0 = \frac{2\pi}{5}$ and $\omega_0 = 5$.

(f) e^{-j10t} and e^{j15t} are both periodic with periods $\frac{\pi}{5}, \frac{2\pi}{15}$ and their sum is periodic with period $T_0 = LCM(\frac{\pi}{5}, \frac{2\pi}{15}) = \frac{2\pi}{5}$ and $\omega_0 = 5$:
$e^{-j10(t+\frac{2\pi}{5})} + e^{j15(t+\frac{2\pi}{5})} = e^{-j10t}e^{-j4\pi} + e^{j15t}e^{j6\pi}$ and since $e^{-j4\pi} = 1$ and $e^{j6\pi} = 1$ this $= e^{-j10t} + e^{j15t}$.

2.13

(a) periodic, $T_0 = 2\pi$, $\omega_0 = 1$
(b) periodic, $T_0 = \pi$, $\omega_0 = 2$
(c) not periodic since 1 and π do not have any common factors (the only factor of 1 is 1, but since π is irrational, it cannot be an integer times 1)
(d) periodic, $T_0 = 12$, $\omega_0 = \frac{\pi}{6}$

2.14

(a) periodic, $T_0 = \frac{\pi}{2}$, $\omega_0 = 4$
(b) periodic, $T_0 = \frac{\pi}{2}$, $\omega_0 = 4$
(c) not periodic, since 2π and 6 do not have a common factor
(d) periodic; $x_1(t)$ has period 2, $x_2(t)$ has period 1, and $x_3(t)$ has period $\frac{12}{5}$ so the sum has period $T_0 = LCM(2, 1, \frac{12}{5}) = 12$ and fundamental frequency $\omega_0 = \frac{\pi}{6}$.

2.15

(a) For $x_1(t) + x_2(t)$ to be periodic we need some number T such that $x_1(t+T) + x_2(t+T) = x_1(t) + x_2(t)$ for all t. This can only be true if $x_1(t+T) = x_1(t)$ and $x_2(t+T) = x_2(t)$, which can only be true if $T = k_1T_1$ and $T = k_2T_2$ (T is an integer multiple of both the periods). So we need there to be some integers k_1 and k_2 such that $k_1T_1 = k_2T_2 \implies \frac{T_1}{T_2} = \frac{k_2}{k_1}$.
(b) Put $\frac{k_2}{k_1}$ in its most reduced form $\frac{n}{m}$ by canceling any common terms in the numerator and denominator; then $T_0 = nT_2 = mT_1$.

2.16

Let $u = at$ so performing u substitution gives:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(at-b)\sin^2(t-4)dt &= \int_{-\infty}^{\infty} \delta(u-b)\sin^2(\tfrac{u}{a}-4)\tfrac{du}{a} \\ &= \sin^2(\tfrac{b}{a}-4)\tfrac{1}{a}\end{aligned}$$

2.17 By sifting property, y(t) = 1/2 x(2) + 1/2 x(-2)

2.18

(a) $x_1(t) = 2t\,u(t) - 4(t-1)\,u(t-1) + 2(t-2)\,u(t-2)$

(b) $t<0, \quad x_1(t)=0$ ✓

$0<t<1, \quad x_1(t) = 2t$ ✓

$1<t<2, \quad x_1(t) = 2t-4t+4 = 4-2t$ ✓

$2<t, \quad x_1(t) = 4-2t+2t-4=0$ ✓

(c) $x(t) = \sum_{k=-\infty}^{\infty} x_1(t-kT_0) = \sum_{k=-\infty}^{\infty} x_1(t-2k)$

2.19

(a) $x_1(t) = 5tu(t) - 5tu(t-1) + 5u(t-1) - 5u(t-3)$

(b)

$$\begin{aligned} t < \quad & 0, f(t) = 0-0+0-0=0 \\ 0<t< \quad & 1, f(t) = 5t-0+0=0=5t \\ 1<t< \quad & 3, f(t) = 5t-5t+5-0=5 \\ 3< \quad & t, f(t) = 5t-5t+5-5=0 \end{aligned}$$

(c) $x_2(t) = \sum_{k=-\infty}^{\infty} x_1(t-k4)$

2.20. (a) let $at=\tau$, ∴ $\int_{-\infty}^{\infty}\delta(at)\,dt = \int_{-\infty}^{\infty}\delta(\tau)\,\frac{d\tau}{a}$

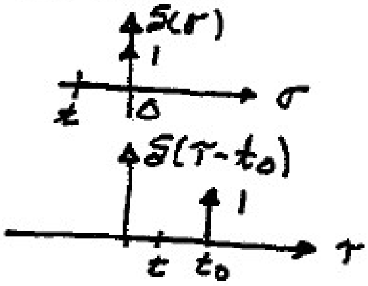
$$= \frac{1}{a}\int_{-\infty}^{\infty}\delta(\tau)\,d\tau \Rightarrow \underline{\delta(at) = \frac{1}{a}\delta(t), a>0}$$

For $a<0$, $at=\tau \Rightarrow -|a|t=\tau$, $dt = -\frac{d\tau}{|a|}$

∴ $\int_{-\infty}^{\infty}\delta(at)\,dt = \int_{\infty}^{-\infty}\delta(\tau)\,\frac{-d\tau}{|a|} = \frac{1}{|a|}\int_{-\infty}^{\infty}\delta(\tau)\,d\tau$

∴ $\underline{\delta(at) = \frac{1}{|a|}\delta(t)}$ for the general case.

(b) $\int_{-\infty}^{t}\delta(\sigma)\,d\sigma = u(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$



∴ $\int_{-\infty}^{t}\delta(\tau-t_0)\,d\tau = u(t-t_0)$

(c) $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$

(continued)…

2.20 (c)

Recall the rules about integrating delta functions: $\delta(t)$ is nonzero only at $t = 0$, so $x(t)\delta(t) = x(0)\delta(t)$, and $\int_{-\infty}^{\infty}\delta(t)dt = 1$, so $\int_{-\infty}^{\infty}x(t)\delta(t)dt = \int_{-\infty}^{\infty}x(0)\delta(t)dt = x(0)\int_{-\infty}^{\infty}\delta(t)dt = x(0)$. We can time-shift the delta function: $\delta(t-t_0)$ is nonzero only at $t = t_0$, so $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$ and $\int_{-\infty}^{\infty}x(t)\delta(t-t_0)dt = x(t_0)$.

i) $\int_{-\infty}^{\infty}\cos(2t)\delta(t)dt = \cos(2\cdot 0)\int_{-\infty}^{\infty}\delta(t)dt = 1$.

ii) $\delta(t-\frac{\pi}{4})$ is a time-shifted version of $\delta(t)$, and is nonzero only at $t = \frac{\pi}{4}$. So:

$$\begin{aligned}\int_{-\infty}^{\infty}\sin(2t)\delta(t-\frac{\pi}{4})dt = &\quad \textstyle\int_{-\infty}^{\infty}\sin(2\cdot\frac{\pi}{4})\delta(t-\frac{\pi}{4})dt \\ = &\quad \textstyle\sin(\frac{\pi}{2})\int_{-\infty}^{\infty}\delta(t-\frac{\pi}{4})dt = \sin(\frac{\pi}{2}) = 1\end{aligned}$$

iii) $\cos(2(t-\frac{\pi}{4}))\delta(t-\frac{\pi}{4}) = \cos\left(2(\frac{\pi}{4}-\frac{\pi}{4})\right)\delta(t-\frac{\pi}{4}) = 1\cdot\delta(t-\frac{\pi}{4})$, so the integral of this is 1.

iv) $\delta(t-2)$ is nonzero only at $t = 2$. Therefore $\int_{-\infty}^{\infty}\sin\left((t-1)\right)\delta(t-2)dt = \sin\left(2-1\right) = \sin(1) = 0.8414....$

v) $\delta(2t-4)$ is nonzero at $2t-4 = 0 \implies t = 2$. So:

$$\int_{-\infty}^{\infty}\sin\left(t-1\right)\delta(2t-4)dt = \sin\left(2-1\right)\int_{-\infty}^{\infty}\delta(2t-4)dt$$

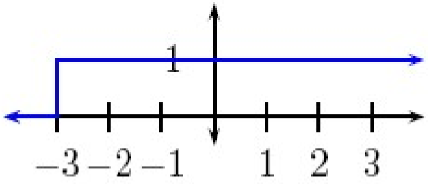
To figure out the integral, we can change variables—let $u = 2t$, so $dt = \frac{du}{2}$ and the $-\infty, \infty$ limits stay the same. This gives: $\int_{-\infty}^{\infty}\delta(2t-4)dt = \int_{-\infty}^{\infty}\delta(u-4)\frac{du}{2} = \frac{1}{2}$, so we get:

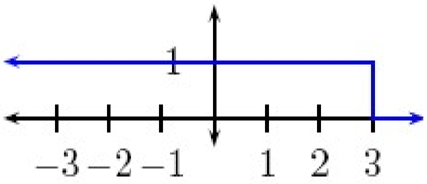
$$\int_{-\infty}^{\infty}\sin\left(t-1\right)\delta(2t-4)dt = 0.5\sin(1) = 0.4207...$$

2.21

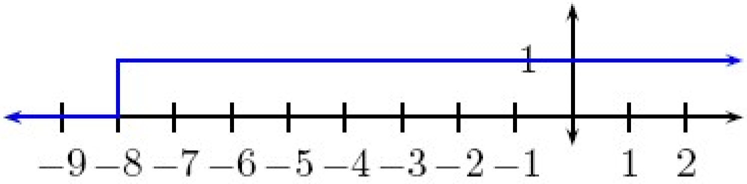
(a)$u(2t+6) = u(t+3)$

(b)$u(-2t+6) = u(-t+3)$

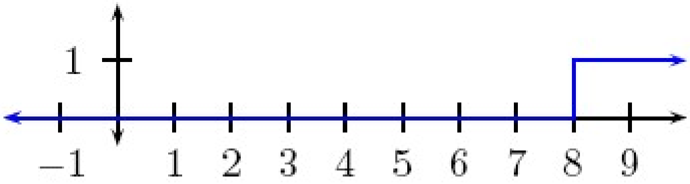




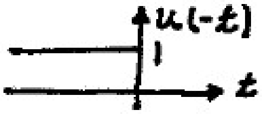
(c)$u(\frac{t}{4}+2) = u(t+8)$



(d)$u(\frac{t}{4}-2) = u(t-8)$

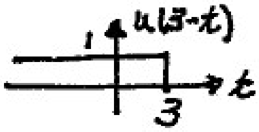


2.22 (a)



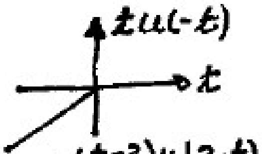
$u(-t) = 1 - u(t)$

(b)



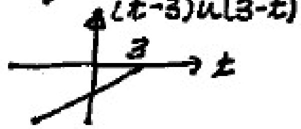
$u(3-t) = 1 - u(t-3)$

(c)



$tu(-t) = t[1 - u(t)]$

(d)



$(t-3)u(-t) = (t-3)[1 - u(t-3)]$

2.23 (a) $y_2(t) = T_2[T_1[x(t)]]$, $y_3(t) = T_3[T_1[x(t)]]$

$y(t) = T_2[T_1[x(t)]] + T_4\{T_3[T_1[x(t)]] + T_5[x(t)]$

(b) $y(t) = T_3\{T_2[T_1[x(t)]]\} + T_4\{T_2[T_1[x(t)]]\} + T_5[T_1[x(t)]]$

(c) $y(t) = T_2[T_1[x(t)] + T_4\{T_3[T_1[x(t)]] \times T_5[x(t)]\}$

(d) $y(t) = T_3\{T_2[T_1[x(t)]]\} \times T_4\{T_2[T_1[x(t)]]\} \times T_5[T_1[x(t)]]$

2.24 $y(t) = T_3[m(t) + T_1[x(t)]]$

$m(t) = T_2[x(t) - T_4[y(t)]]$

$\therefore \underline{y(t) = T_3\{T_2[x(t) - T_4[y(t)]] + T_1[x(t)]\}}$

2.25 $m(t) = T_1\{x(t) - T_4[y(t)]\} - T_3[y(t)]$

$y(t) = T_2[m(t)] = T_2\left[T_1\{x(t) - T_4[y(t)]\} - T_3[y(t)]\right]$

2.26

(a) (i) has memory; (ii) not invertible; (iii) stable; (iv) time invariant; (v) linear

(b) need $y(t_0)$ to only depend on $x(t)$ values as $t < t_0 \implies t_0 + 1 - \alpha < t_0 \implies$ causal for values of $\alpha \geq 1$.

2.27

2.27(a) system is: $y(t) = \cos(x(t-1))$

i) Not memoryless: $y(t)$ depends on $x(t-1)$.

ii) Not invertible: for a counterexample of two input signals that give the same output signal at all points, take any $x(t)$ and $x(t) + 2\pi$.

iii) Causal; output at time t does not depend on input at times greater than t.

iv) Stable: clearly $|y(t)| \leq 1$ for any values of the input.

v) Time invariant: $y_d(t) = cos(x(t-1-t_0))$ and $y(t-t_0) = cos(x(t-t_0-1))$.

vi) Not linear: for example, violates the scaling property because $ay(t) \neq cos(ax(t-1))$ (if we input a scaled version of the input $ax(t)$ we don't get the output scaled by the same amount $ay(t)$). This system also violates additivity, the other necessary property for a system to be linear.

2.27(b)

i) not memoryless (at time t_0 output depends on input at time $3t_0$)

ii) invertible ($x(t) = \frac{1}{3}y(\frac{t-3}{3})$)

iii) not causal ($3t_0 > t_0$ for $t_0 > 0$)

iv) stable

v) not time invariant ($x(t-t_0) \to 3x(3t-t_0+3)$ but $y(t-t_0) = 3x(3(t-t_0)+3) = 3x(3t-3t_0+3)$)

vi) linear

2.27(c) system is: $y(t) = \ln(x(t))$

i) Memoryless;

ii) Invertible: $x(t) = e^{(y(t))}$

iii) Causal;

iv) Not stable: for example, $y(t) = -\infty$ whenever $x(t) = 0$

v) Time invariant;

vi) Not linear: for example, violates additivity: $\ln(x_1(t) + x_2(t)) \neq \ln(x_1(t)) + \ln(x_2(t))$ in general. Scaling doesn't work either.

2.27(d) System is: $y(t) = e^{tx(t)}$

i) Memoryless;

ii) $x(t) = \frac{\ln(y(t))}{t}$ except when $t = 0$ (we can't get back the value of $x(0)$.) This system would therefore be considered noninvertible but it is mostly invertible.

iii) Causal;

iv) Not stable: for example, if $x(t) = c$ (some constant $c > 0$) then $y(t) = e^{tc}$ which goes to ∞ as $t \to \infty$ (we can't find any number K such that $e^{tc} < K$ for all t). not memoryless, invertible, not causal, stable, not time invariant, linear

v) Not time invariant: if the input is $x(t-t_0)$ we get $y_d(t) = e^{(tx(t-t_0))} \neq y(t-t_0) = e^{((t-t_0)x(t-t_0))}$

vi) Not linear: doesn't satisfy either necessary property.

2.27(e) System is: $y(t) = 7x(t) + 6$

This system is memoryless, invertible, causal, stable, time invariant, but NOT linear: if we input $x_1(t) + x_2(t)$ we get out $7(x_1(t) + x_2(t)) + 6$, while if we input $x_1(t)$ and $x_2(t)$ separately and add them, we get $y_1(t) + y_2(t) = 7(x_1(t)) + 6 + 7(x_2(t)) + 6$, so the system violates additivity. Also violates scaling. Note that to show a system is linear you need to show it satisfies both properties (which you can do by showing that $ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$), but to show that a system is NOT linear, you only need to show that it violates at least one of these properties.

2.27(f) System is: $y(t) = \int_{-\infty}^{t} x(5\tau)d\tau$

i), iii) Not memoryless, not causal: output at time t depends on both past values of $x(t)$ (because integrating from $-\infty$) and future values of t (because depends on $x(5t)$ and $5t > t$ for $t > 0$).

ii) invertible: $\frac{d}{dt}y(t) = x(5t) \implies x(t) = \frac{d}{dt}y(t)\mid_{t/5}$ (the function $y'(t)$ evaluated at $t/5$).

iv) Not stable: for instance, $x(t) = c$ (some constant) is a bounded input but the output is $y(t) = ct$, which goes to ∞ as t goes to ∞.

v) Not time-invariant: if the input is $x(t-t_0)$ we get $y_d(t) = \int_{-\infty}^{t} x(5\tau - t_0)d\tau$,but $y(t-t_0) = \int_{-\infty}^{t-t_0} x(5\tau)d\tau = \int_{-\infty}^{t} x(5(\tau - t_0))d\tau$.

vi) linear: if $x_1(t) \rightarrow y_1(t) = \int_{-\infty}^{t} x_1(5\tau)d\tau$ and $x_2(t) \rightarrow y_2(t) = \int_{-\infty}^{t} x_2(5\tau)d\tau$ then:

$$\begin{aligned} ax_1(t) + bx_2(t) \rightarrow \int_{-\infty}^{t} ax_1(5\tau) + bx_2(5\tau)d\tau &= a\int_{-\infty}^{t} x_1(5\tau)d\tau + b\int_{-\infty}^{t} x_2(5\tau)d\tau \\ &= ay_1(t) + by_2(t) \end{aligned}$$

2.27(g) System is: $y(t) = e^{-j\omega t}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$.

i), iii) Not memoryless, not causal: depends on $x(t)$ values at all t from $-\infty$ to ∞.

ii) Not invertible

iv) Not stable: say $\omega = 0$ and the input is a constant c; the output is infinite.

v) NOT time-invariant:

$$\begin{aligned} x(t-t_0) \rightarrow y_d(t) &= e^{-j\omega t}\int_{-\infty}^{\infty} x(\tau - t_0)e^{-j\omega\tau}d\tau \\ &= e^{-j\omega t}\int_{-\infty}^{\infty} x(u)e^{-j\omega(u+t_0)}du = e^{-j\omega(t+t_0)}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \end{aligned}$$

which comes from u-substitution, letting $u = t - t_0$. But $y(t-t_0) = e^{-j\omega(t-t_0)}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$ which is not equal to the above.

vi) Linear; the integral and multiplication by $e^{-j\omega t}$ are both linear operations.

2.27(h)

i) Not memoryless ($y(t)$ depends on input over last second)

ii) not invertible (for example, $x(t) = 0$ and $x(t) = \cos(2\pi t)$ have the same output signal

iii) causal

iv) stable

v) time invariant (since $x(t-t_0) \rightarrow \int_{t-1}^{t} x(\tau - t_0)d\tau = \int_{t-t_0-1}^{t-t_0} x(\tau)d\tau = y(t-t_0)$)

vi) linear

2.28

(a) $x_2(t) = 2u(t+1) - u(t) - u(t-1) = x_1(t) + 2x_1(t+1)$
so $y_2(t) = y_1(t) + 2y_1(t+1)$

(b) $x_1(t) = 2u(t-1) - u(t-2) - u(t-3)$ so $x_2(t) = x_1(t+2)$ and $y_2(t) = y_1(t+2)$

2.29

i) not memoryless unless $t_0=0$

ii) invertible: $x(t)=y(t+t_0)$

iii) If $t_0 \geq 0$ it is causal; otherwise not.

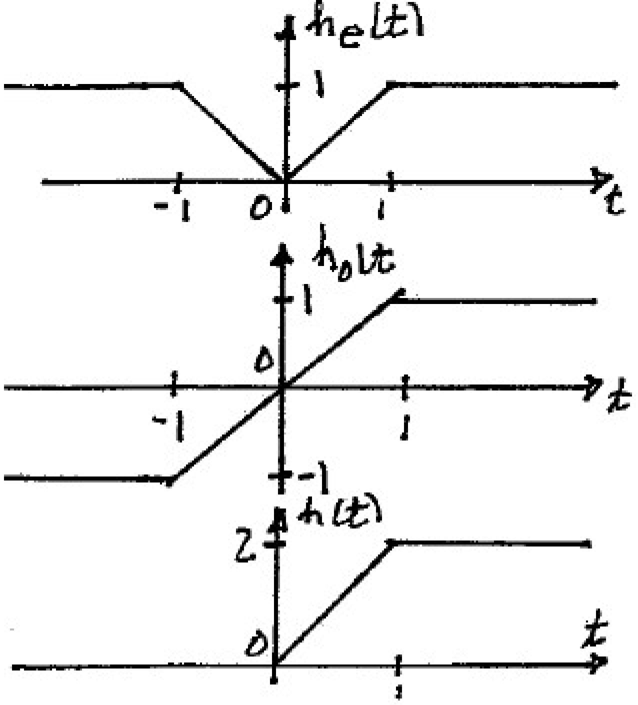
iv) stable; the output only takes value of the input so if the input is bounded the output will be too.

v) time invariant: let $y_d(t)$ be the output when $x(t-t_1)$ is the input. $x(t-t_1) \rightarrow y_d(t) = x(t-t_1-t_0)$ and $y(t-t_1) = x(t-t_1-t_0)$, so $y_d(t) = y(t-t_1)$.

vi) linear: scaling and adding two inputs $ax_1(t) + bx_2(t)$ gives output $ax_1(t-t_0) + bx_2(t-t_0)$, which is the same output we would get by putting $x_1(t)$ and $x_2(t)$ into the system separately and then scaling and adding the outputs.

2.30

$h_e(t) = t\,[u(t)-u(t-1)] + u(t-1)$



$h_e(t)$ even

$h_o(t) = h(t) - h_e(t)$

$\therefore h_o(t) = -h_e(t) \quad t<0$

and $h_o(t) = -h_o(-t)$

$\therefore h(t) = 2t\,u(t) - 2(t-1)u(t-1)$

$t<0,\ h(t) = 0$ ✓

$0<t<1,\ h(t) = 2t$ ✓

$2<t,\ h(t) = 2t-2t+2 = 2$ ✓

2.31

(a) (i) memoryless

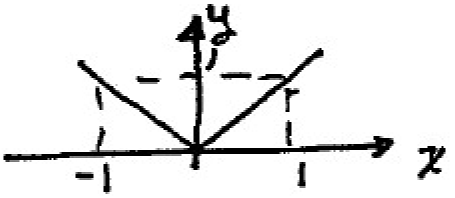
(ii) $y=1$ for $x=\pm 1$, not invertible

(iii) causal

(iv) stable

(v) time invariant

(vi) $|x_1+x_2| \neq |x_1|+|x_2|$ not linear



$y=|x|$

(b) (i) memoryless

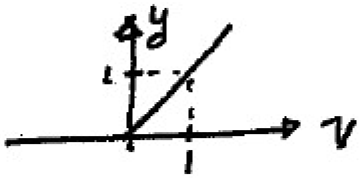
(ii) $y=0$ for $x \leq 0$, not invertible

(iii) causal

(iv) stable

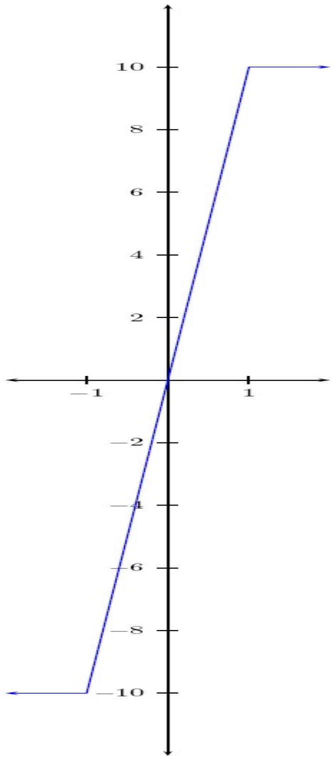
(v) time invariant

(vi) $y\big|_{\substack{x_1=1\\x_2=-1}} \neq y\big|_{x_1+x_2}$, not linear



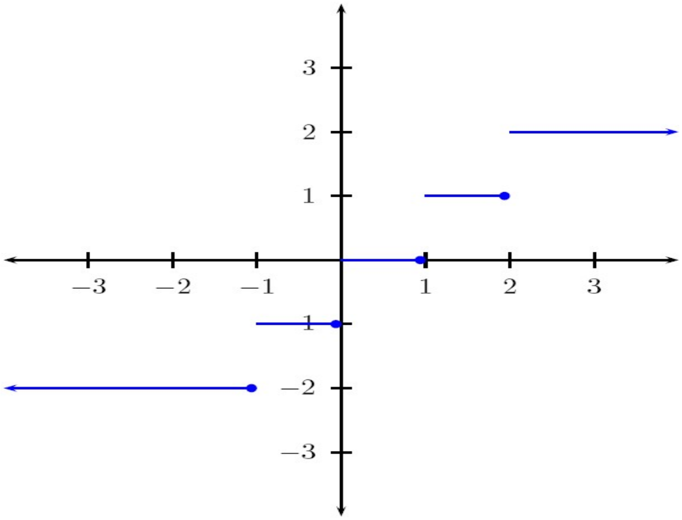
(parts c,d on next page)

(c)



The system is i) memoryless, ii) not invertible (output = 10 for all input values > 10, iii) causal, iv) stable ($|y(t)| \leq 10$ for any input), v) time invariant, vi) not linear (suppose $x(t) = 3$ then $y(t) = 3$ but $4x(t)$ has output $10 \neq 3(4) = 12$.

(d)



The system is i) memoryless, ii) not invertible (any input greater than 2 goes to the same output (2)), iii) causal, iv) stable, v) time invariant, vi) not linear ($x_1(t) = 2 \to 1$ and $x_2(t) = 1 \to 0$ but $x_1(t) + x_2(t) \to 2 \neq 1 + 0$.

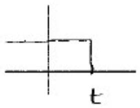
Chapter 3 Solutions

3.1 a) i $x(t) = u(t-2)$

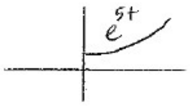
$$y(t) = u(t) * u(t-2) = \int_{2}^{t} d\tau = t-2 \;, \quad t > 2$$

$y(t) = 0 \;, \quad t < 2$

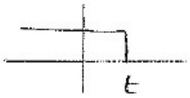
$\therefore \; y(t) = (t-2)u(t-2)$



ii $x(t) = e^{5t}u(t)$



$t < 0, \; y(t) = 0$



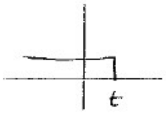
$$t \geq 0, \; y(t) = \int_{0}^{t} e^{5\tau} d\tau$$

$\therefore \; y(t) = 1/5\,(e^{5t} - 1)\,u(t)$

iii $x(t) = u(t)$

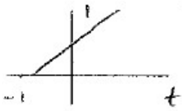
$t < 0, \; y(t) = 0$

$$t \geq 0, \; y(t) = \int_{0}^{t} d\tau = t$$



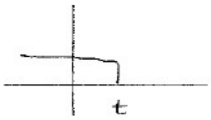
$\therefore \; y(t) = t\,u(t)$

iv $x(t) = (t+1)\,u(t+1)$



$t < -1, \; y(t) = 0$

$$t \geq -1, \; y(t) = \int_{-1}^{t} (\tau+1)\,d\tau = t^2/2 + t + 1/2$$



$\therefore \; y(t) = (t^2/2 + t + 1/2)\,u(t+1)$

3.1 b)

i)

$$\begin{aligned} y(t) = & -\int_0^t (t-\tau)d\tau = -(t^2 - \frac{t^2}{2}) = -\frac{t^2}{2}, t \geq 0 \\ = & \quad 0, t < 0 \\ = & \quad -\frac{t^2}{2}u(t) \end{aligned}$$

ii)

$$\begin{aligned} y(t) = & \int_0^t e^{-5\tau}d\tau = \frac{1}{5}(1 - e^{-5t}), t \geq 0 \\ = & \quad 0, t < 0 \\ = & \quad \frac{1}{5}(1 - e^{-5t})u(t) \end{aligned}$$

iii)

$$\begin{aligned} y(t) = & \int_1^t (\tau - 1)d\tau = \frac{t^2}{2} - t + \frac{1}{2}, t \geq 0 \\ = & \quad 0, t < 0 \\ = & \quad (\frac{t^2}{2} - t + \frac{1}{2})u(t) \end{aligned}$$

iv)

$$\begin{aligned} y(t) = u(t) * u(t) - u(t) * u(t-2) = & \int_0^t 1d\tau - \int_2^t 1d\tau = t - (t-2) = 2, t \geq 2 \\ = & \quad \int_0^t 1d\tau = t, 0 \leq t < 2 \\ = & \quad 0, t < 0 \\ = & \quad tu(t) + (2-t)u(t-2) \end{aligned}$$

3.1 c)

a-i

$$\begin{aligned} \int_{-\infty}^t u(\tau - 2)d\tau = & \int_2^t 1d\tau = t - 2, t \geq 2 \\ = & \quad 0, t < 0 \\ = & \quad (t-2)u(t-2) \end{aligned}$$

a-ii

$$\begin{aligned} \int_{-\infty}^t e^{5\tau}u(\tau)d\tau = & \int_0^t e^{5\tau}d\tau = \frac{1}{5}(e^{5t} - 1), t \geq 0 \\ = & \quad 0, t < 0 \\ = & \quad \frac{1}{5}(e^{5t} - 1)u(t) \end{aligned}$$

a-iii

$$\begin{aligned}\int_{-\infty}^{t} u(\tau)d\tau = & \int_0^t 1d\tau = t, t \geq 0 \\ = & \quad 0, t < 0 \\ = & \quad tu(t)\end{aligned}$$

a-iv

$$\begin{aligned}\int_{-\infty}^{t} (\tau+1)u(\tau+1)d\tau = & \int_{-1}^t (\tau+1)d\tau = \tfrac{t^2}{2} + t + \tfrac{1}{2}, t \geq -1 \\ = & \quad 0, t < -1 \\ = & \quad (\tfrac{t^2}{2} + t + \tfrac{1}{2})u(t)\end{aligned}$$

b-i

$$\begin{aligned}\int_{-\infty}^{t} (-\tau)u(\tau)d\tau = & \int_0^t -\tau d\tau = -\tfrac{t^2}{2}, t \geq 0 \\ = & \quad 0, t < 0 \\ = & \quad -\tfrac{t^2}{2}u(t)\end{aligned}$$

b-ii

$$\begin{aligned}\int_{-\infty}^{t} e^{-5\tau}u(\tau)d\tau = & \int_0^t e^{-5\tau}d\tau = \tfrac{1}{5}(1 - e^{-5t}), t \geq 0 \\ = & \quad 0, t < 0 \\ = & \quad \tfrac{1}{5}(1 - e^{-5t})u(t)\end{aligned}$$

b-iii

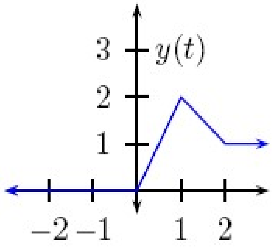
$$\begin{aligned}\int_{-\infty}^{t} (\tau-1)u(\tau-1)d\tau = & \int_1^t (\tau-1)d\tau = \tfrac{t^2}{2} - t + \tfrac{1}{2}, t \geq 1 \\ = & \quad 0, t < 1 \\ = & \quad (\tfrac{t^2}{2} - t + \tfrac{1}{2})u(t)\end{aligned}$$

b-iv

$$\begin{aligned}\int_{-\infty}^{t} (u(\tau) - u(\tau-2))d\tau = & \int_0^2 1d\tau = 2, t \geq 2 \\ = & \int_0^t 1d\tau = t, 0 \leq t < 2 \\ = & \quad 0, t < 0 \\ = & \quad tu(t) + (2-t)u(t-2)\end{aligned}$$

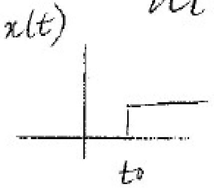
3.2

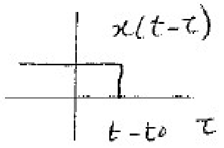
$$
\begin{aligned}
y(t) = \int_{-\infty}^{t} x(\tau)d\tau = & \quad 0, t < 0 \\
= & \quad 2t, 0 \leq t < 1 \\
= & \quad 2-(t-1), 1 \leq t < 2 \\
= & \quad 2-1 = 1, t \geq 2 \\
= & \quad 2t[u(t)-u(t-1)] + (3-t)[u(t-1)-u(t-2)] + u(t-2)
\end{aligned}
$$



3.3 $x(t) = u(t-t_0)$

$h(t) = u(t-t_1)$ $\quad t_1 > t_0$

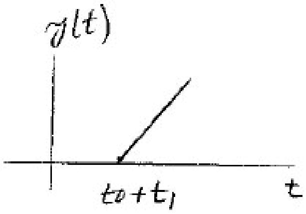




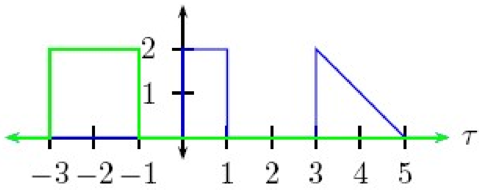
$y(t) = 0, \quad t - t_0 < t_1$

$y(t) = \int_{t_1}^{t-t_0} d\tau = t - t_0 - t_1 \quad , \; t - t_0 \geq t_1$

$\therefore \quad y(t) = (t - t_0 - t_1)u(t - t_0 - t_1)$

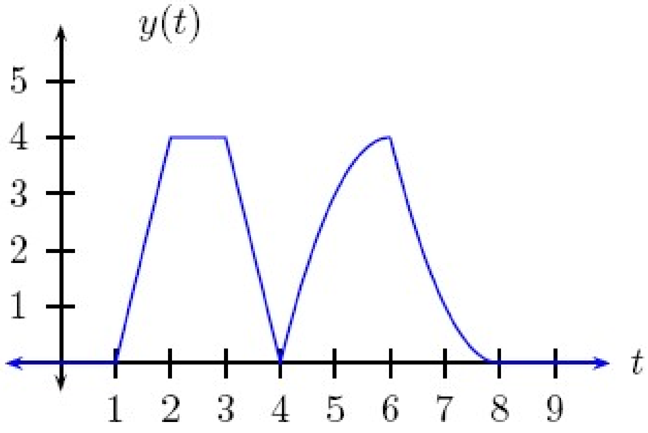


3.4


$h(\tau)$ (blue) and $x(0-\tau)$ (green)

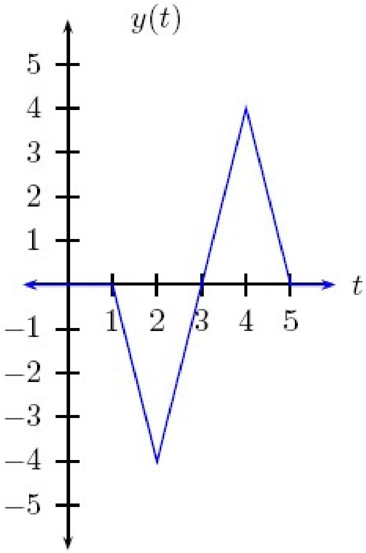
(a)

$$
\begin{aligned}
y(t) &= 0, t < 1 \\
&= \int_0^{t-1} 2(2)d\tau = 4(t-1), 1 \leq t < 2 \\
&= \int_0^{1} 2(2)d\tau = 4, 2 \leq t < 3 \\
&= \int_{t-3}^{1} 2(2)d\tau = 4(1-(t-3)) = 4(4-t), 3 \leq t < 4 \\
&= \int_3^{t-1} 2(-\tau+5)d\tau = -t^2 + 12t - 32, 4 \leq t < 6 \\
&= \int_{t-3}^{5} 2(-\tau+5)d\tau = t^2 - 16t + 64, 6 \leq t < 8 \\
&= 0, t > 8
\end{aligned}
$$



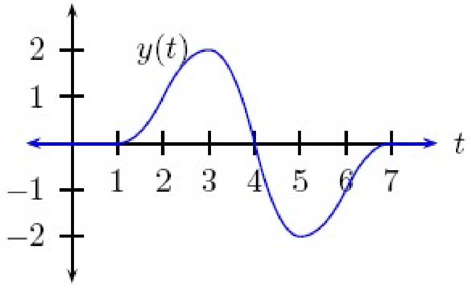
(b)

$$\begin{aligned}
y(t) = & \quad 0, t < 1 \\
= & \quad \int_0^{t-1} -2(2)d\tau = -4(t-1), 1 \le t < 2 \\
= & \quad -4 + \int_1^{t-1} 2(2)d\tau = -4 + 4(t-2) = 4t - 12, 2 \le t < 3 \\
= & \quad 4 + \int_{t-3}^{1} -2(2)d\tau = 4 - 4(4-t) = 4t - 12, 3 < t \le 4 \\
= & \quad \int_{t-3}^{2} 2(2)d\tau = 4(2-(t-3)) = -4t + 20, 4 \le t < 5 \\
= & \quad 0, t \ge 5
\end{aligned}$$



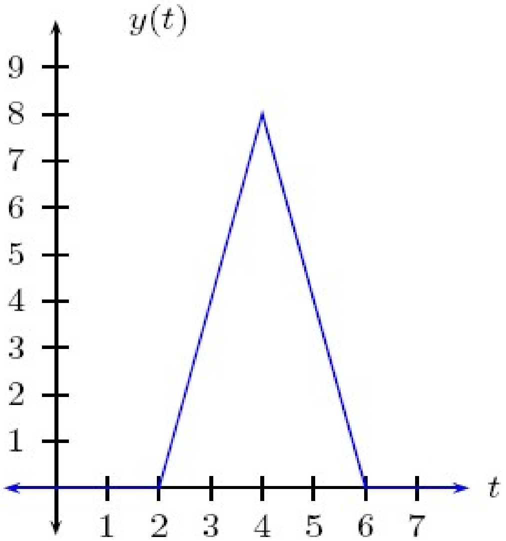
(c)

$$\begin{aligned}
y(t) = & \quad 0, t < 1 \\
= & \quad \int_0^{t-1} 2\tau d\tau = (t-1)^2, 1 \le t < 2 \\
= & \quad 1 + \int_1^{t-1} (-2\tau + 4)d\tau = -t^2 + 6t - 7, 2 \le t < 3 \\
= & \quad 2 - 2\int_0^{t-2} 2\tau d\tau = -2t^2 + 12t - 16, 3 \le t < 4 \\
= & \quad -2\int_1^{t-3} (-2\tau + 4)d\tau = 2t^2 - 20t + 48, 4 \le t < 5 \\
= & \quad -1 + \int_{t-3}^{3} (-2\tau + 4)d\tau = t^2 - 10t + 23, 5 \le t < 6 \\
= & \quad \int_{t-3}^{4} (2\tau - 8)d\tau = -t^2 + 14t - 49, 6 \le t < 7 \\
= & \quad 0, t \ge 7
\end{aligned}$$



(d)

$$
\begin{aligned}
y(t) &= 0, t < 2 \\
&= \int_1^{t-1} 2(2)d\tau = 4(t-2), 2 \le t < 4 \\
&= \int_{t-3}^{3} 2(2)d\tau = 4(6-t), 4 \le t < 6 \\
&= 0, t \ge 6
\end{aligned}
$$



3.5

(a)

$t = 0$:

$h(\tau)x(-\tau) = 0$ for all τ, so $y(0) = \int_{-\infty}^{\infty} h(\tau)x(-\tau)d\tau = 0$.

$t = 1$:

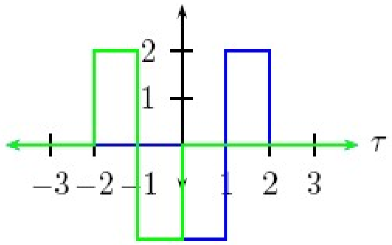
$h(\tau)x(1-\tau) = -2(-2) = 4$ for $0 \le \tau < 1$
and $= 0$ elsewhere,
so $y(1) = \int_{-\infty}^{\infty} h(\tau)x(1-\tau)d\tau = \int_0^1 4d\tau = 4$.

$t = 2$:

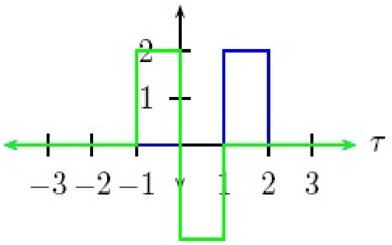
$h(\tau)x(2-\tau) = -2(2) = -4$ for $0 \le \tau < 2$
and $= 0$ elsewhere,
so $y(2) = \int_0^2 -4d\tau = -8$.

$t = 2.667$:

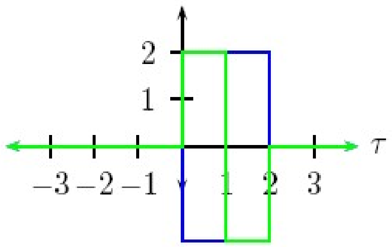
$h(\tau)x(2.667-\tau) = -2(2) = -4$ for $0.667 \le \tau < 1$,
$= 2(2) = 4$ for $1 \le \tau < 1.667$,
$= -4$ for $1.667 \le \tau < 2$,
and $= 0$ elsewhere.
Therefore $y(2.667) = (-4)(1-0.667) + 4(1.667-1) - 4(2-1.667) = -8(0.333) + 4(0.666) = 0$.



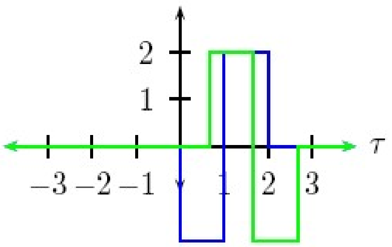
$h(\tau)$ (blue) and $x(-\tau)$ (green)



$h(\tau)$ (blue) and $x(1-\tau)$ (green)



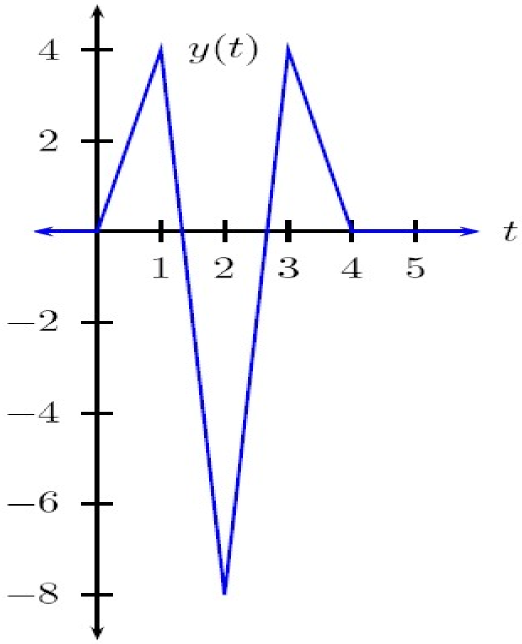
$h(\tau)$ (blue) and $x(2-\tau)$ (green)



$h(\tau)$ (blue) and $x(2.667-\tau)$ (green)

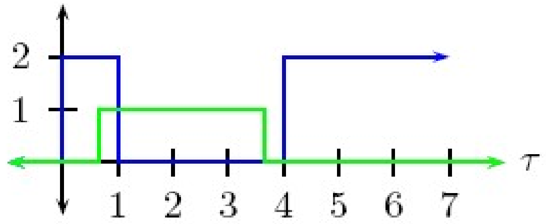
(b)

$$
\begin{aligned}
y(t) &= 0, t < 0 \\
&= \int_0^t -2(-2)d\tau = 4t, 0 \leq t < 1 \\
&= \int_0^{t-1} 2(-2)d\tau + \int_{t-1}^{1} -2(-2)d\tau + \int_1^t -2(2) = -8(t-1) + 4(2-t) = -12t + 16, 1 \leq t < 2 \\
&= \int_{t-2}^{1} 2(-2)d\tau + \int_1^{t-1} 2(2)d\tau + \int_{t-1}^{2} -2(2)d\tau = 12t - 32, 2 \leq t < 3 \\
&= \int_{t-2}^{2} 2(2)d\tau = 4(4-t) = 16 - 4t, 3 \leq t < 4 \\
&= 0, t \geq 4
\end{aligned}
$$



3.6

$x(\tau)$ (blue) and $h(t-\tau)$ (green), $4 < t < 5$



(a)

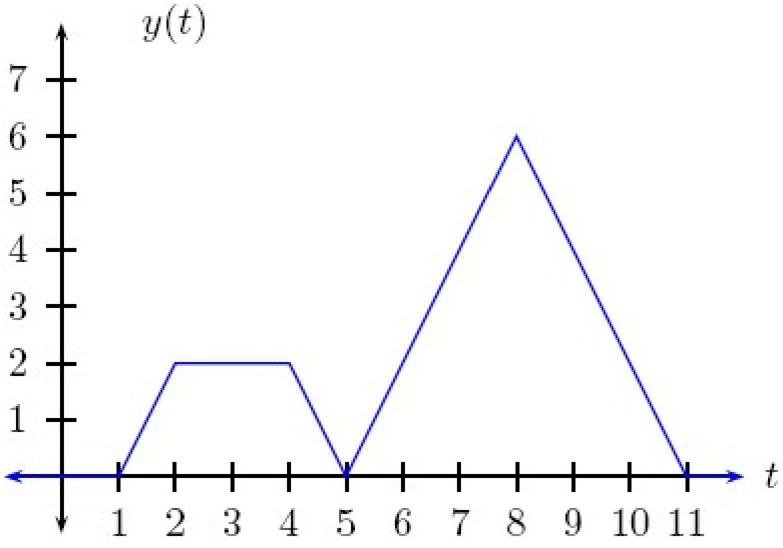
$$y(t) = \int_{t-4}^{1} 1(2)d\tau = -2t - 10, 4 \leq t \leq 5$$

(b) $y(t)$ is maximum when $t = 8$ (then $y(t) = (7-4)2 = 6$).

(c) $y(t) = 0$ when $t \leq 1$, $t = 5$, $t \geq 11$.

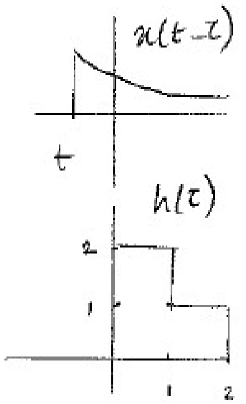
(d)

$$\begin{aligned} y(t) &= 0, t < 1 \\ &= \int_0^{t-1} 2(1)d\tau = 2t - 2, 1 \leq t < 2 \\ &= \int_0^1 2(1)d\tau = 2, 2 \leq t < 4 \\ &= \int_{t-4}^1 2(1)d\tau = -2t + 10, 4 \leq t < 5 \\ &= \int_4^{t-1} 2(1)d\tau = 2t - 10, 5 \leq t < 8 \\ &= \int_{t-4}^7 2(1)d\tau = -2t + 22, 8 \leq t < 11 \\ &= 0, t \geq 11 \end{aligned}$$



3.7

a) $x(t)=e^{t}u(-t)$



① $t>2$ no overlap $\therefore y(t)=0$

② $1\leqslant t\leqslant 2$ $y(t)=\int_t^2 e^{t-\tau}d\tau = e^{t}\int_t^2 e^{-\tau}d\tau$

$$y(t)=e^{t}\left[e^{-t}-e^{-2}\right] = 1-e^{t-2}$$

③ $0\leqslant t\leqslant 1$, $y(t)=2\int_t^1 e^{t-\tau}d\tau+\int_1^2 e^{t-\tau}d\tau = 2(1-e^{t-1})$

$$+e^{t}(e^{-1}-e^{-2}) = 2-e^{t-1}-e^{t-2}$$

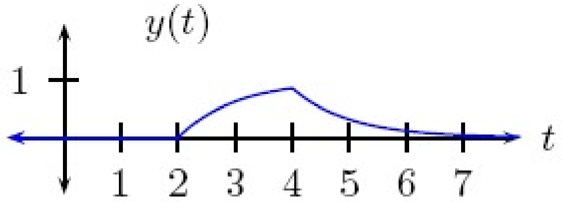
④ $t<0$, $y(t)=2\int_0^1 e^{t-\tau}d\tau+\int_1^2 e^{t-\tau}d\tau$

$$=2(e^{t}-e^{t-1})+e^{t}(e^{-1}-e^{-2}) = 2e^{t}-e^{t-1}-e^{t-2}$$

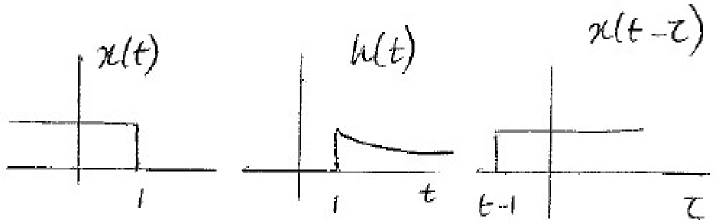
$$\therefore\ y(t)=(1-e^{t-2})[u(t-1)-u(t-2)]+(2-e^{t-1}-e^{t-2})\times$$
$$[u(t)-u(t-1)]+(2e^{t}-e^{t-1}-e^{t-2})u(-t)$$

(b)

$$\begin{aligned}
y(t) &= e^{-t}u(t) * [u(t-2)-u(t-4)] \\
&= 0, t<2 \\
&= \int_0^{t-2} e^{-\tau}d\tau = 1-e^{-(t-2)}, 2\leq t<4 \\
&= \int_{t-4}^{t-2} e^{-\tau}d\tau = e^{-(t-4)}-e^{-(t-2)}, t\geq 4
\end{aligned}$$



c)



① $t-1<1$ or $t<2$, $y(t)=\int_1^{\infty} e^{-\tau}d\tau = e^{-1}$

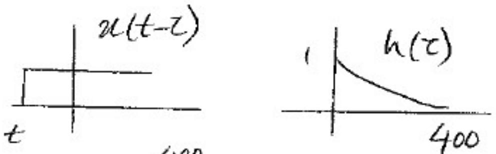
② $t-1>1$ or $t>2$ $y(t)=\int_{t-1}^{\infty} e^{-\tau}d\tau = -e^{-\tau}\Big|_{t-1}^{\infty} = e^{-(t-1)}$

$\therefore\ y(t)=e^{-1}u(2-t)+e^{-(t-1)}u(t-2)$

(d)

$$\begin{aligned} y(t) &= e^{-at}[u(t)-u(t-2)]*u(t-2) \\ &= 0, t<2 \\ &= \int_0^{t-2} e^{-a\tau}d\tau = \frac{1}{a}(1-e^{-a(t-2)}), 2\le t<4 \\ &= \int_0^{2} e^{-a\tau}d\tau = \frac{1}{a}(1-e^{-a2}), t\ge 4 \\ &= \frac{1}{a}(1-e^{-a(t-2)})[u(t-2)-u(t-4)]+\frac{1}{a}(1-e^{-a2})u(t-4) \end{aligned}$$

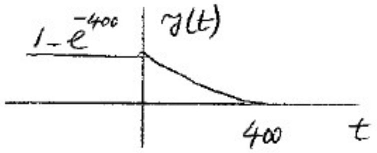
e) flip $x(t)$



① $t<0$, $y(t)=\int_0^{400} e^{-\tau}d\tau = -e^{-\tau}\Big|_0^{400} = 1-e^{-400}$

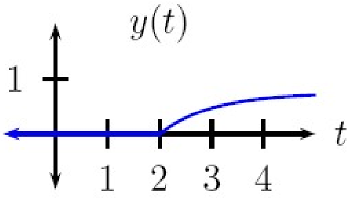
② $t>0$, $y(t)=\int_t^{400} e^{-\tau}d\tau = e^{-t}-e^{-400}$

③ $t>400$, $y(t)=0$



(f)

$$\begin{aligned} y(t) &= e^{-t}u(t-1) * 2u(t-1) \\ &= 0, t < 2 \\ &= \int_1^{t-1} 2e^{-\tau} d\tau = 2(e^{-1} - e^{-(t-1)}), t \geq 2 \\ &= 2(e^{-1} - e^{-(t-1)})u(t-2) \end{aligned}$$



3.8 $$[f(t) * g(t)] * h(t) = \int_{-\infty}^{\infty} h(t-s)\left[\int_{-\infty}^{\infty} f(s-\tau)g(\tau)d\tau\right]ds$$

$$= \int_{-\infty}^{\infty} g(\tau)\left[\int_{-\infty}^{\infty} h(t-s)f(s-\tau)ds\right]d\tau \quad , \text{let } s-\tau=\sigma$$

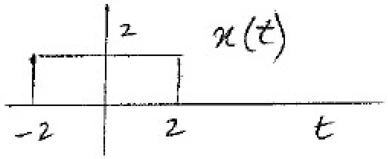
$$= \int_{-\infty}^{\infty} g(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau-\sigma)f(\sigma)d\sigma\right]d\tau \quad , \text{let } s=t-\sigma \quad ds=-d\sigma$$

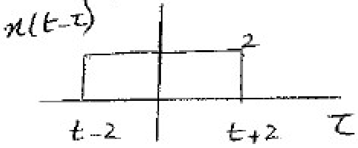
$$= \int_{-\infty}^{\infty} g(\tau)\left[\int_{-\infty}^{\infty} h(s-\tau)f(t-s)[-ds]\right]d\tau$$

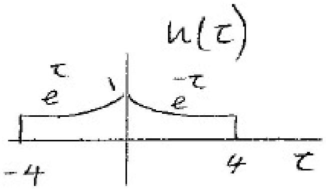
$$= \int_{-\infty}^{\infty} f(t-s)\left[\int_{-\infty}^{\infty} h(s-\tau)g(\tau)d\tau\right]ds$$

$$= f(t) * [g(t) * h(t)]$$

3.9 $x_1(t) = 2u(t+2) - 2u(t-2)$







① $t+2 < -4$, $t < -6$, $y(t) = 0$

② $-4 \leqslant t+2 \leqslant 0$, $-6 \leqslant t \leqslant -2$

$$y(t) = \int_{-4}^{t+2} 2e^{\tau} d\tau = 2\left[e^{t+2} - e^{-4}\right]$$

③ $0 \leqslant t+2 \leqslant 4$, $-2 \leqslant t \leqslant 2$

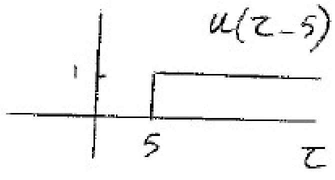
$$y(t) = 2\int_{t-2}^{0} e^{\tau} d\tau + 2\int_{0}^{t+2} e^{-\tau} d\tau = 2\left[1 - e^{t-2}\right] + 2\left[1 - e^{-(t+2)}\right]$$

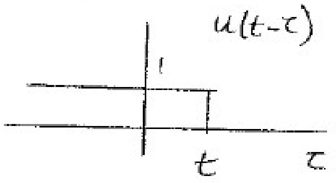
④ $0 \leqslant t-2 \leqslant 4$, $2 \leqslant t \leqslant 6$

$$y(t) = \int_{t-2}^{4} e^{-\tau} d\tau = 2\left[e^{-(t-2)} - e^{-4}\right]$$

⑤ $t \geqslant 6$, $y(t) = 0$

3.10

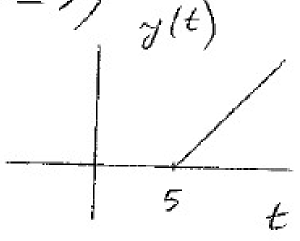




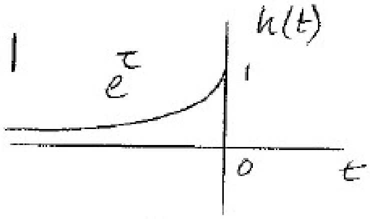
$t<5$, $y(t)=0$

$t \geq 5$, $y(t) = \int_5^t d\tau = (t-5)$

$\therefore$ $y(t) = (t-5)u(t-5)$

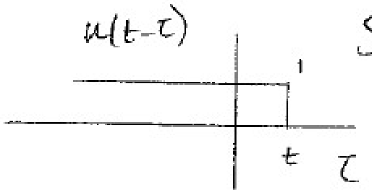


3.11



$x(t) = u(t+3) - u(t+2) + u(t-1) - u(t-2)$

Use superposition



$S(t) = u(t) * h(t)$

$t<0$, $y(t) = \int_{-\infty}^{t} e^{\tau} d\tau = e^{t}$

$t \geq 0$, $y(t) = \int_{-\infty}^{0} e^{\tau} d\tau = 1$

$\therefore y(t) = S(t+3) - S(t+2) + S(t-1) - S(t-2)$

3.12 a) $h(t) = h_1(t) * h_2(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) e^{-(t-\tau)} u(t-\tau) d\tau$

$= e^{-t} \int_0^t d\tau = t e^{-t} u(t)$

b) $h(t) = \delta(t) * \delta(t) = \int_{-\infty}^{\infty} \delta(\tau)\delta(t-\tau) d\tau = \delta(t)$

Parts c,d on next page→

3.12, continued

(c) $h(t) = \delta(t-2) * \delta(t-2) = \delta(t-2-2) = \delta(t-4)$

(d)

$$\begin{aligned}
(u(t-1)-u(t-5)) * (u(t-1)-u(t-5)) &= \quad 0, t < 2 \\
&= \quad \int_1^{t-1} 1(1)d\tau = t-2, 2 \le t < 6 \\
&= \quad \int_{t-5}^{5} 1(1)d\tau = 10-t, 6 \le t < 10 \\
&= \quad 0, t \ge 10 \\
&= \quad (t-2)[u(t-2)-u(t-6)] + (10-t)[u(t-6)-u(t-10)]
\end{aligned}$$

3.13

(a) Using a change of variables, let $u = t + \tau$, then:

$$z(t) = \int_{-\infty}^{\infty} x(-\tau + a)h(t+\tau)d\tau = \int_{-\infty}^{\infty} x(-u+t+a)h(t+u-t)du = \int_{-\infty}^{\infty} x(t+a-u)h(u)du = y(a+t)$$

(b) Using a change of variables, let $u = t + \tau$, and we see that:

$$w(t) = \int_{-\infty}^{\infty} x(t+\tau)h(b-\tau)d\tau = \int_{-\infty}^{\infty} x(u)h(b+t-u)du = y(b+t)$$

3.14

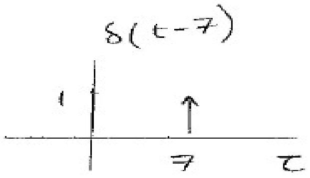
a) $x(t) = \delta(t) \longrightarrow y(t) = h(t)$

$y(t) = x(t-7)$

$h(t) = \delta(t-7)$

b) $y(t) = \int_{-\infty}^{t} x(\tau-7)d\tau$

$h(t) = \int_{-\infty}^{t} \delta(\tau-7)d\tau$

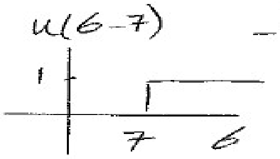


$t<7$, $h(t)=0$

$t>7$, $h(t)=1$ $\quad \therefore h(t) = u(t-7)$

c) $y(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{\sigma} x(\tau-7)d\tau\right]d\sigma$ let $x(t)=\delta(t)$

$h(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{\sigma} \delta(\tau-7)d\tau\right]d\sigma = \int_{-\infty} u(\sigma-7)d\sigma$



$t<7$, $h(t)=0$

$t \geq 7$, $h(t) = \int_{7}^{t} d\sigma = (t-7)$

$\therefore h(t) = (t-7)\,u(t-7)$

3.15 let $x(t-\tau)=\begin{cases}1 & h(\tau)>0\\ -1 & h(\tau)<0\end{cases}$ ∴ x is bounded

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

$$h(\tau)x(t-\tau)=\begin{cases}h(\tau), & h(\tau)>0\\ -h(\tau), & h(\tau)<0\end{cases}$$

$$\therefore h(\tau)x(t-\tau)=|h(\tau)|$$

$\therefore y(t)=\int_{-\infty}^{\infty}|h(\tau)|\,d\tau$ which is assumed unbounded

∴ system is not BIBO stable

3.16 a) $y_i(t)$ is the output of the ith system

$$y_1(t)=h_1(t)*x(t)$$
$$y_2(t)=h_2(t)*y_1(t)=h_1(t)*h_2(t)*x(t)$$
$$y_3(t)=h_1(t)*h_3(t)*x(t)$$
$$y_5(t)=h_5(t)*x(t)$$
$$y(t)=y_2(t)+y_4(t)$$
$$x(t)*[h_1(t)*h_2(t)+h_1(t)*h_3(t)*h_4(t)+h_4(t)*h_5(t)]$$

b) $h(t)=u(t)*5\delta(t)+u(t)*5\delta(t)*u(t)$
$+\,u(t)*e^{-2t}u(t)$

now $u(t)*e^{-2t}u(t)=\int_{-\infty}^{\infty}u(\tau)e^{-2(t-\tau)}u(t-\tau)\,d\tau$

$$=\int_0^t e^{-2(t-\tau)}\,d\tau=e^{-2t}\int_0^t e^{2\tau}\,d\tau=\tfrac{1}{2}(1-e^{-2t}]u(t)$$

$$\therefore h(t)=5u(t)+5tu(t)+\tfrac{1}{2}(1-e^{-2t}]u(t)$$

3.17 $y_i(t)$ is the output of the ith system

a) $y_2(t) = h_2(t) * [h_1(t) * x(t)] = h_1(t) * h_2(t) * x(t)$

$y_3(t) = h_3(t) * y_2(t) = h_1(t) * h_2(t) * h_3(t) * x(t)$

in a like manner : $y_4(t) = h_1(t) * h_2(t) * h_4(t) * x(t)$

$y_5(t) = h_1(t) * h_5(t) * x(t)$

$\therefore y(t) = (h_1(t) * h_2(t) * h_3(t) \quad h_1(t) * h_2(t) * h_4(t) + h_1(t) * h_5(t)] * x(t)$

b) $h(t) = 5\delta(t) * 5\delta(t) * u(t) + 5\delta(t) * 5\delta(t) * u(t) + 5\delta(t) * u(t) = 25u(t) + 25u(t) + 5u(t) = 55u(t)$

c) blocks 1 and 2 → gains of 5

blocks 3, 4, 5 → integrators

d) block 1 $- 5\delta(t)$

block 2 $- 25\delta(t)$

block 3 $- 25u(t)$

block 4 $- 25u(t)$

block 5 $- 5u(t)$

$\therefore y(t) = 55u(t)$

e) $\delta(t) * 55u(t) = 55u(t)$

3.18 $y(t)=h_1(t)*[x(t)-h_2(t)*y(t)]=h_1(t)*x(t)$
$-h_1(t)*h_2(t)*y(t)$

$$y(t)=u(t)*x(t)-u(t)*\delta(t)*y(t) = u(t)*x(t) - u(t)*y(t)$$

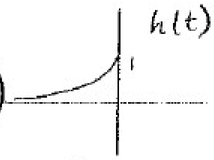
$$=\int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau - \int_{-\infty}^{\infty} y(\tau)u(t-\tau)d\tau$$

$$=\int_{-\infty}^{t} x(\tau)d\tau - \int_{-\infty}^{t} y(\tau)d\tau$$

by differentiating:

$$dy/dt = x(t)-y(t) \Rightarrow \frac{dy(t)}{dt}+y(t)=x(t)$$

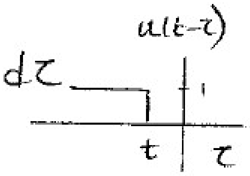
3.19 a)



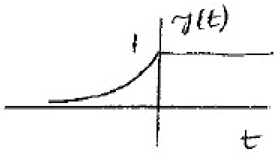
impulse response $\neq 0$ for $t<0$ ∴ non causal

b) $\int_{-\infty}^{\infty}|h(t)|dt = \int_{-\infty}^{0} e^{t}dt = e^{t}\Big|_{-\infty}^{0} = 1$ ∴ stable

c) $y(t)=\int_{-\infty}^{\infty} e^{\tau}u(-\tau)u(t-\tau)d\tau = \int_{-\infty}^{0} e^{\tau}u(t-\tau)d\tau$



$$\therefore y(t) = \begin{cases} \int_{-\infty}^{t} e^{\tau}d\tau = e^{\tau}\Big|_{-\infty}^{t} = e^{t}, & t<0 \\ \int_{-\infty}^{0} e^{\tau}d\tau = e^{\tau}\Big|_{-\infty}^{0} = 1, & t>0 \end{cases}$$



d) a) causal

b) $\int_{-\infty}^{\infty} h(t)dt = \int_{0}^{\infty} e^{t}dt = e^{t}\Big|_{0}^{\infty}$ ∴ unstable

c) $\int_{-\infty}^{\infty} e^{\tau}u(\tau)u(t-\tau)d\tau = \int_{0}^{t} e^{\tau}d\tau = (e^{t}-1)u(t)$

3.20

(a) Yes linear: $\cos(t)\left(ax_1(t)+bx_2(t)\right)=a\cos(t)x_1(t)+b\cos(t)x_2(t)$

(b) Not time invariant: $x(t-t_0)\rightarrow\cos(t)x(t-t_0)$, but $y(t-t_0)=\cos(t-t_0)x(t-t_0)\neq\cos(t)x(t-t_0)$

(c) $\delta(t)\rightarrow\cos(t)\delta(t)=1\delta(t)=\delta(t)$

(d) $\delta(t-\pi/2)\rightarrow\cos(t)\delta(t-\pi/2)=\cos(\pi/2)\delta(t-\pi/2)=0\delta(t)=0$. If the system were time-invariant than the response in part (d) would be part (c) delayed by $\pi/2$, but it is not.

3.21

(a) $h(t)=e^{-t}u(t-1)$: stable since $\int_{-\infty}^{\infty}|h(t)|dt$ is finite, causal since $h(t)=0$ for all $t<0$.

(b) $h(t)=e^{t-1}u(t-1)$: not stable; causal.

(c) $h(t)=e^{t}u(1-t)$: stable; not causal.

(d) $h(t)=e^{1-t}u(1-t)$: not stable; not causal.

(e) $h(t)=e^{t}\sin(-5t)u(-t)$: stable; not causal.

(f) $h(t)=e^{-t}\sin(5t)u(t)$: stable; causal.

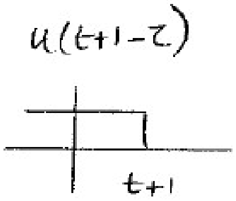
3.22 $y(t)=\int_0^t e^{-\tau}x_1(t-\tau)\,d\tau=\int_{-\infty}^{t} e^{-\tau}u(\tau)x_1(t-\tau)\,d\tau$

$=\int_{-\infty}^{\infty} e^{-\tau}u(\tau)x_1(t-\tau)u(t-\tau)\,d\tau$

a) $h(t)=e^{-t}u(t)$ $\quad x(t)=x_1(t)u(t)$

b) yes, $h(t)=0$ for $t<0$

c) $y(t)=\int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t+1-\tau)\,d\tau=\int_0^{t+1} e^{-\tau}d\tau$



$=-e^{-\tau}\Big|_0^{t+1}=\underline{\left[1-e^{-(t+1)}\right]u(t+1)}$

parts d,e next page→

3.22, continued

d) $h_t(t) = h(t) * \delta(t) - h(t) * \delta(t-1) * \delta(t)$

$$= [h(t) - h(t-1)] * \delta(t) = h(t) - h(t-1)$$

$$y(t) = \underline{e^{-t}u(t) - e^{-(t-1)}u(t-1)}$$

e) (i) $y(t) = y_c(t) - y_c(t)\Big|_{t=t+1}$

$$= \underline{[1 - e^{-(t+1)}]u(t+1) - [1 - e^{-t}]u(t)}$$

(ii) $y(t) = h(t) * u(t+1) = \int_{-\infty}^{\infty} u(t-\tau+1)[e^{-\tau}u(\tau) - e^{-(\tau-1)}u(\tau-1)]d\tau$

$$= \int_0^\infty e^{-\tau}u(t+1-\tau)d\tau - e^{1}\int_1^\infty e^{-\tau}u(t+1-\tau)d\tau = I_1 - I_2$$

$$I_1 = \int_0^{t+1} e^{-\tau}d\tau = -e^{-\tau}\Big|_0^{t+1} = [1 - e^{-(t+1)}]u(t+1)$$

$$I_2 = e^{1}\int_1^{t+1} e^{-\tau}d\tau = e^{1}(-e^{-\tau})\Big|_1^{t+1} = e^{1}(e^{-1} - e^{-(t+1)})u(t)$$

$$= (1 - e^{-t})u(t)$$

$\therefore \quad \underline{y(t) = [1 - e^{-(t+1)}]u(t+1) - [1 - e^{-t}]u(t)}$

3.23 a) $y(t)=\int_{-\infty}^{x} e^{-2(t-\tau)} x(\tau-1)\,d\tau$

(i) $h(t)=\int_{-\infty}^{t} e^{-2(t-\tau)}\delta(\tau-1)\,d\tau = \underline{e^{-2(t-1)}\,u(t-1)}$

(ii) $h(t)=0$ for $t<0$ ∴ Causal

(iii) $\int_{-\infty}^{\infty}\left|e^{-2(t-1)}u(t-1)\right|dt = \int_{1}^{\infty} e^{-2(t-1)}\,dt = e^{2}\left(e^{-2t}/_{-2}\right)\Big|_{1}^{\infty}$

$= e^{2}\left(e^{-2}/_{2}\right) = 1/2$ ∴ stable

b) $y(t)=\int_{-\infty}^{\infty} e^{-2(t-\tau)} x(\tau-1)\,d\tau$

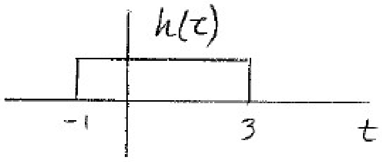
(i) $h(t)=\int_{-\infty}^{\infty} e^{-2(t-\tau)}\delta(\tau-1)\,d\tau = \underline{e^{-2(t-1)}}$

(ii) $h(t)\neq 0$, $t<0$ ∴ non causal

(iii) $\int_{-\infty}^{\infty}\left|e^{-2(t-1)}\right|dt = \int_{-\infty}^{\infty} e^{-2t}e^{2}\,dt = e^{2}\left(\frac{e^{-2t}}{-2}\right)\Big|_{-\infty}^{\infty}$

unbounded ∴ unstable

3.24

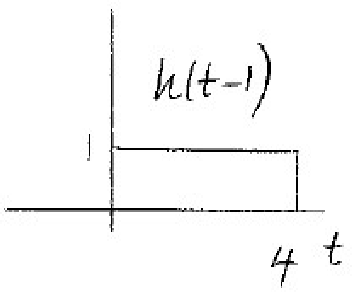


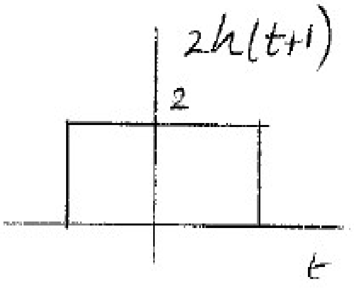
a) system is not causal

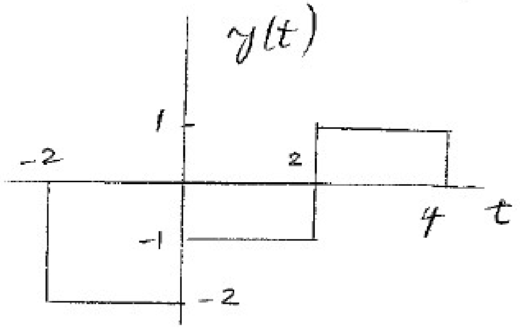
b) yes BIBO, stable - Integrates over a window of length.

c) $x(t)=\delta(t-1)-2\delta(t+1)$

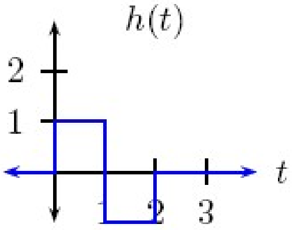
$y(t)=h(t)*x(t)=h(t-1)-2h(t+1)$





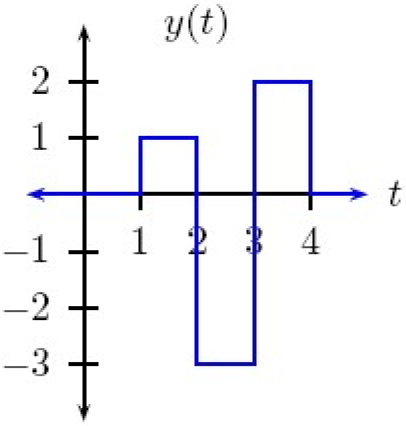


3.25

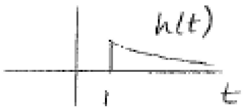


(a) Causal ($h(t) = 0$ for all $t < 0$).
(b) Stable ($\int_{-\infty}^{\infty} |h(t)|dt = 1(1) - 1(1) = 0$).
(c)

$$\begin{aligned} y(t) &= h(t) * \delta(t-1) - 2h(t) * \delta(t-2) \\ &= h(t-1) - 2h(t-2) \\ &= (u(t-1) - 2u(t-2) + u(t-3)) - 2\,(u(t-2) - 2u(t-3) + u(t-4)) \\ &= u(t-1) - 4u(t-2) + 5u(t-3) - 2u(t-4) \end{aligned}$$

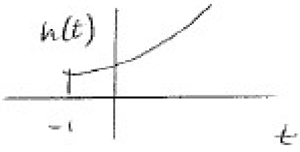


3.26



a) clearly system is causal since $h(t)=0,\ t<0$

b) $\int_{-\infty}^{\infty} |h(t)|dt = \int_{1}^{\infty} e^{-at} dt = 1/a\, e^{-a}$, $a>0$ ∴ stable



c) $h(t) = e^{-at} u(t+1)$, $a<0$

not causal since $h(t) \neq 0$, $-1 \leq t \leq 0$

$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-1}^{\infty} e^{-at} dt = -1/a\, e^{-at} \Big|_{-1}^{\infty} = \infty$ since $a<0$

∴ not stable

3.27

(i) Characteristic equation: $s + 3 = 0$, solution $s = -3$

$\implies y_c(t) = Ce^{-3t}u(t)$

Forced response of the form: $y_p(t) = Pu(t)$ where $\frac{dy_p(t)}{dt} + 3y_p(t) = 3u(t) \implies 0 + 3Pu(t) = 3u(t) \implies P = 1$

$y(t) = y_c(t) + y_p(t) = (Ce^{-3t} + 1)u(t)$

Need $y(0) = C + 1 = -1 \implies C = -2$

$\implies y(t) = (-2e^{-3t} + 1)u(t)$

This clearly satisfies the differential equation and initial conditions because

$\frac{dy(t)}{dt} + 3y(t) = 6e^{-3t} + 3(-2e^{-3t} + 1) = 3, t > 0$

$y(0) = -2e^{-3 \cdot 0} + 1 = -1$

(ii) Characteristic equation: $s + 3 = 0$, solution $s = -3$

$\implies y_c(t) = Ce^{-3t}u(t)$

Forced response of the form $y_p(t) = Pe^{-2t}u(t)$ where $\frac{dy_p(t)}{dt} + 3y_p(t) = 3e^{-2t}u(t) \implies (-2P + 3P)e^{-2t}u(t) = 3e^{-2t}u(t) \implies P = 3$

$y(t) = y_c(t) + y_p(t) = (Ce^{-3t} + 3e^{-2t})u(t)$

Need $y(0) = C + 3 = 2 \implies C = -1$

$\implies y(t) = (3e^{-2t} - e^{-3t})u(t)$

This clearly satisfies the differential equation and initial conditions since

$\frac{dy(t)}{dt} + 3y(t) = (-6e^{-2t} + 3e^{-3t}) + 3(3e^{-2t} - e^{-3t}) = 3e^{-2t}, t > 0$

$y(0) = 3e^{-2 \cdot 0} - e^{-3 \cdot 0} = 2$

Continued→

(iii) Characteristic equation: $s+3=0$, solution $s=-3$

$\implies y_c(t) = Ce^{-3t}u(t)$

Forced response of the form $y_p(t) = Pe^{2t}u(t)$ where $\frac{dy_p(t)}{dt} + 3y_p(t) = 3e^{2t}u(t) \implies (2P+3P)e^{2t}u(t) = 3e^{2t}u(t) \implies P = \frac{3}{5}$

$y(t) = y_c(t) + y_p(t) = (Ce^{-3t} + \frac{3}{5}e^{2t})u(t)$

Need $y(0) = C + \frac{3}{5} = 0 \implies C = -\frac{3}{5}$

$\implies y(t) = (-\frac{3}{5}e^{-3t} + \frac{3}{5}e^{2t})u(t)$

This clearly satisfies the differential equation and initial conditions since

$\frac{dy(t)}{dt} + 3y(t) = (\frac{9}{5}e^{-3t} + \frac{6}{5}e^{2t}) + (-\frac{9}{5}e^{-3t} + \frac{9}{5}e^{2t}) = 3e^{2t}, t > 0$

$y(0) = -\frac{3}{5}e^{-3\cdot 0} + \frac{3}{5}e^{-2\cdot 0} = 0$

(iv) Characteristic equation: $s+3=0$, solution $s=-3$

$\implies y_c(t) = Ce^{-3t}u(t)$

Forced response of the form $y_p(t) = (P_1\sin(3t) + P_2\cos(3t))u(t)$ where $\frac{dy_p(t)}{dt} + 3y_p(t) = \sin(3t)u(t) \implies 3P_1\cos(3t)u(t) - 3P_2\sin(3t)u(t) + 3(P_1\sin(3t) + P_2\cos(3t))u(t) = \sin(3t)u(t)$

$\implies P_1\cos(3t) + P_2\cos(3t) = 0 \implies P_1 = -P_2$

and $\implies -3P_2\sin(3t) + 3P_1\sin(3t) = \sin(3t) \implies 6P_1 = 1 \implies P_1 = \frac{1}{6}, P_2 = -\frac{1}{6}$

$y(t) = y_c(t) + y_p(t) = (Ce^{-3t} + \frac{1}{6}\sin(3t) - \frac{1}{6}\cos(3t))u(t)$

Need $y(0) = C - \frac{1}{6} = -1 \implies C = -\frac{5}{6}$

$\implies y(t) = (-\frac{5}{6}e^{-3t} + \frac{1}{6}\sin(3t) - \frac{1}{6}\cos(3t))u(t)$

This clearly satisfies the differential equation and initial conditions since

$\frac{dy(t)}{dt} + 3y(t) = (\frac{5}{2}e^{-3t} + \frac{1}{2}\cos(3t) + \frac{1}{2}\sin(3t)) + 3(-\frac{5}{6}e^{-3t} + \frac{1}{6}\sin(3t) - \frac{1}{6}\cos(3t)) = \sin(3t), t > 0$

$y(0) = -\frac{5}{6}e^{-3\cdot 0} + \frac{1}{6}\sin(0) - \frac{1}{6}\cos(0) = -1$

(v) Characteristic equation: $-0.7s + 1 = 0 \implies s - \frac{1}{0.7} = 0$, solution $s = \frac{1}{0.7} = 10/7$

$\implies y_c(t) = Ce^{t/0.7}u(t)$

Forced response of the form $y_p(t) = Pe^{3t}u(t)$ where $\frac{dy_p(t)}{dt} - \frac{10}{7}y_p(t) = \frac{-30}{7}e^{3t}u(t) \implies 3P - \frac{10}{7}P = \frac{-30}{7}$

Solving for P gives $P = -\frac{30}{11}$

$y(t) = y_c(t) + y_p(t) = (Ce^{t/0.7} - (\frac{30}{11})e^{3t})u(t)$

Need $y(0) = C - \frac{30}{11} = -1 \implies C = \frac{19}{11}$

$\implies y(t) = (\frac{19}{11}e^{t/0.7} - \frac{30}{11}e^{3t})u(t)$

This clearly satisfies the differential equation and initial conditions since

$\frac{dy(t)}{dt} - \frac{1}{0.7}y(t) = 2.47e^{t/0.7} - 3(\frac{30}{11})e^{3t} - 2.47e^{t/0.7} + \frac{300}{77}e^{3t} = \frac{-30}{7}e^{3t}$

$y(0) = \frac{19}{11} - \frac{30}{11} = -1$

3.28

(a) stable: roots are $s = -1, -2, -4$, and all are < 0 (on left side of s-plane).

(b) unstable: $s^2 + 1.5s - 1 = (s + 2)(s - 0.5)$, roots are $s = -2, 0.5$, and $0.5 > 0$ (on right side of s-plane).

(c) unstable: $s^2 + 10s = s(s + 10)$, roots are $s = 0, -10$, and $s = 0$ is on imaginary axis (not on left side).

(d) unstable: $s^3 + s^2 + 4s + 30 = (s - 1 - 3j)(s - 1 + 3j)(s + 3)$, roots are $1 + 3j, 1 - 3j, -3$ (roots can be found using `roots([1 1 4 30])` in MATLAB), and real part of $|1+3j|, |1-3j|$ is $1 > 0$ (so these roots are on right side of $s-$plane.)

3.29

(a) characteristic equation is $s^2 - 2.5s + 1 = (s - 2)(s - 0.5) = 0$; roots are $s = 2, 0.5$; modes are $e^{2t}, e^{0.5t}$. Unstable since roots > 0.

(b) characteristic equation $s^2 + 1.5s - 1 = (s + 2)(s - 0.5)0$, roots $s = -2, 0.5$; modes $e^{-2t}, e^{0.5t}$. Unstable since $0.5 > 0$.

(c) characteristic equation $s^2 + 9 = (s - 3j)(s + 3j) = 0$, roots $s = 3j, -3j$; modes e^{3jt}, e^{-3jt}. Unstable since real part of roots is 0 (roots lie on imaginary axis).

(d) characteristic equation $s^3 + s^2 + 4s + 3 = 0$, roots $s = -0.1 + 1.95j, -0.1 - 1.95j, -0.78$ found using `roots([1 1 4 3])` in MATLAB. Modes are $e^{(-0.1+1.95j)t}, e^{(-0.1-1.95j)t}, e^{-0.78t}$. Stable since roots have real part < 0 (are all to the left of imaginary axis.)

3.30

(a) Systems in 3.27(i), (ii), (iii), (iv) have system mode e^{-3t}.
3.27(v) has system mode $e^{t/0.7}$.

(b) For 3.27(i), (ii), (iii), (iv), time constant is $\tau = \frac{1}{3}$ sec. For (v), the system is unstable and the response doesn't decay (it grows).

(c) For 3.27(i), (ii), (iii), (iv), in approx. $\frac{4}{3} = 4\tau$ sec. For (v), the response grows to ∞.

(d) $H(s) = \frac{1}{s^2+1.5s-1}$, system modes are $e^{-2t}, e^{0.5t}$. For e^{-2t}, time constant is $\tau = \frac{1}{2}$ sec. For $e^{0.5t}$, grows to ∞ so no time constant. No constant output because of growing mode (output goes to ∞).

3.31

(a) Characteristic eqn. is

$0.04s^2 + 1 = 0$ or

$s^2 + 25 = 0.$

Roots are $s = 5j, -5j.$

Modes are $e^{5jt}, e^{-5jt}.$

(b) $y_c(t) = \frac{C}{2}e^{j\theta}e^{5jt} + \frac{C}{2}e^{-j\theta}e^{-5jt} = C\cos(5t+\theta)$ (where C is a real postive constant).

(c) The differential eqn. is: $\frac{d^2y}{dt^2} + 25y(t) = 25$

$y_p(t) = Pe^{-t}u(t)$, need $Pe^{-t} + 25Pe^{-t} = 25e^{-t} \implies P = \frac{25}{26}.$

So $y(t) = \left(C\cos(5t+\theta) + \frac{25}{26}e^{-t}\right)u(t)$

with $y(0) = C\cos(\theta) + \frac{25}{26} = 0$

and $y'(0) = -5C\sin(\theta) - \frac{25}{26} = 0.$

$\implies \tan(\theta) = \frac{1}{5}.$

$\implies \theta = \tan^{-1}(1/5) = 0.1974\ \ldots\ rad,$

$C = \frac{-5}{26\sin(\theta)} = -0.98\ \ldots\ .$

$y(t) = -0.98\cos(5t + 0.197) + \frac{25}{26}e^{-t}.$

(d) $\frac{d^2y}{dt^2} + 25y(t) = 25C\cos(5t+\theta) + \frac{25}{26}e^{-t} + 25(C\cos(5t+\theta) + \frac{25}{26}e^{-t}) = 25e^{-t}$

$y(0) = \frac{-5}{26\sin(\theta)}\cos(\theta) + \frac{25}{26} = \frac{-5}{26\tan(\theta)} + \frac{25}{26} = \frac{-5}{26(1/5)} + \frac{25}{26} = 0$

$y'(0) = -5\left(\frac{-5}{26\sin(\theta)}\right)\sin(\theta) - \frac{25}{26} = 0.$

3.32

.(a)(i) $x(t) = 4e^{(0)t} \Rightarrow \therefore s=0,\ H(0) = 5/4 = 1.25$

$y_{ss}(t) = H(0)x(t) = (1.25)(4) = \underline{5}$

(ii) $H(0) = 10/10 = 1,\ \therefore y_{ss}(t) = H(0)\,x(t) = (1)(4) = \underline{4}$

(b)(i) $s=3,\ H(3) = 5/7,\ y_{ss}(t) = H(3)x(t) = \underline{\frac{20}{7}e^{3t} = 2.857\,e^{3t}}$

(ii) $H(3) = \frac{6+10}{9+6+10} = \frac{16}{25},\ y_{ss}(t) = \left(\frac{16}{25}\right)(4e^{3t}) = \underline{\frac{64}{25}e^{3t} = 2.56\,e^{3t}}$

(c) (i) $s = j3,\ H(j3) = \frac{5}{4+j3} = 1\,\underline{\angle -36.87}^\circ$

$y_{ss}(t) = |H(j3)|\,4\cos(3t + \underline{\angle H(j3)}) = \underline{4\cos(3t - 36.8^\circ)}$

(ii) $H(j3) = \frac{10+j6}{-9+j6+10} = 1.917\,\underline{\angle -49.58}^\circ$

$\therefore y_{ss}(t) = (1.917)(4)\cos(3t - 49.58^\circ) = \underline{7.668\cos(3t - 49.56^\circ)}$

```
n=[0 2 10];
d=[1 2 10];
h=polyval(n,3*j)/polyval(d,3*j);
ymag=abs(h)
yphase=angle(h)*180/pi
```

(d) $s = j3$ - use part (c)

(i) $y_{ss}(t) = 4e^{j3t}$ (ii) $y_{ss}(t) = 7.668\,e^{j(3t - 49.56^\circ)}$

(e) from (c): (i) $y_{ss}(t) = 4\sin(3t - 36.8^\circ)$

(ii) $y_{ss}(t) = 7.668\sin(3t - 49.56^\circ)$

(f) $\sin 3t = \cos(3t - 90^\circ)$

$\therefore$ $y_{ss}(t)$ in (e) is that of (c) delayed by 90°.

(g) (i) $(s+4) = (s + \frac{1}{\tau}) \Rightarrow \tau = \frac{1}{4}\,s = \underline{0.25\,s}$

$s^2 + 2s + 10 = (s+1)^2 + 3^2 \Rightarrow s = -1 \pm j3,\ \therefore \tau = \frac{1}{1}\,s = \underline{1\,s}$

(ii) $\tau = 0.25s$, $t > 4\tau = 1\ s.$

$\tau = 1s$, , $t > 4\tau = 4s.$

3.33

(a) $H(j\omega) = \frac{5}{2}\underline{/-45^\circ} = \frac{K}{a+j\omega} = \frac{K}{a+j4}$, since $\omega = 4$

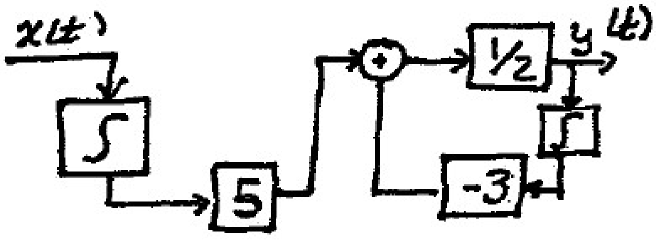
$\therefore\ a = \underline{4}$ to yield -45°, $\therefore |H(j4)| = 2.5 = \frac{K}{|4+j4|} = \frac{K}{4\sqrt{2}}$, $\therefore K = \underline{14.14}$

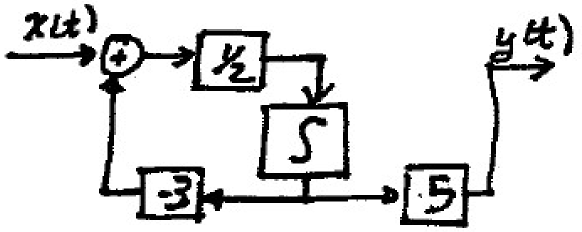
(b) $H(s) = \frac{14.14}{s+4}$;

```
n=[0 14.14];
d=[1 4];
h=polyval(n,4*j)/polyval(d,4*j);
ymag=abs(h)
yphase=angle(h)*180/pi
```

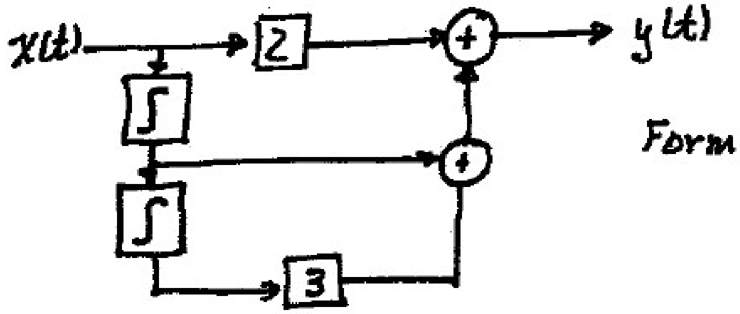
3.34

3.(a)



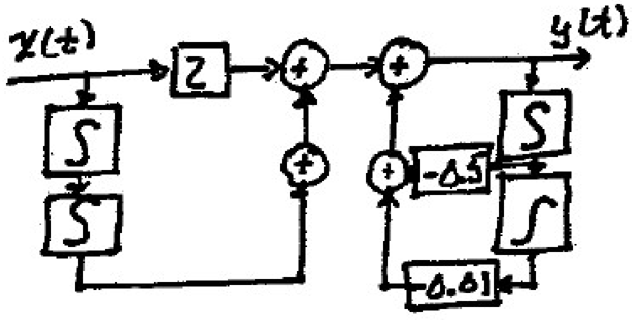


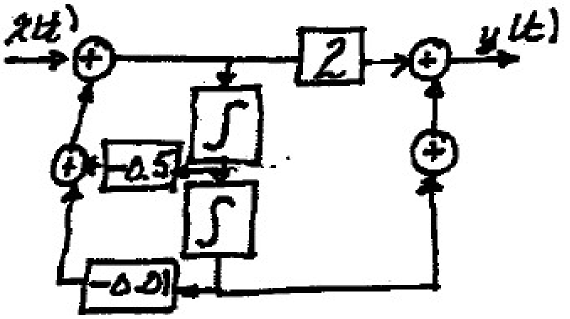
(b) $\frac{d}{dt}\left[\qquad\right] \Rightarrow \frac{d^2y}{dt^2} = 2\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x$



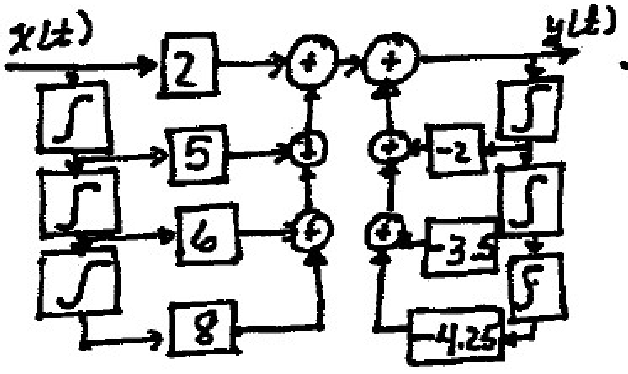
Forms I and II are same.

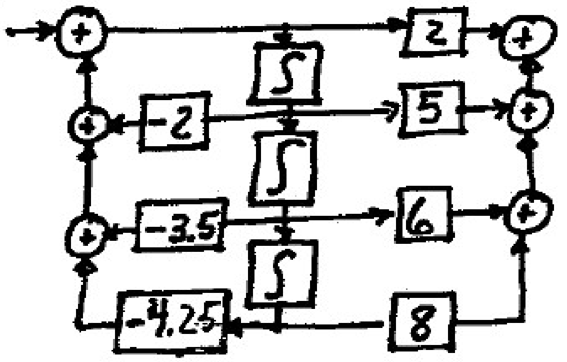
(c)





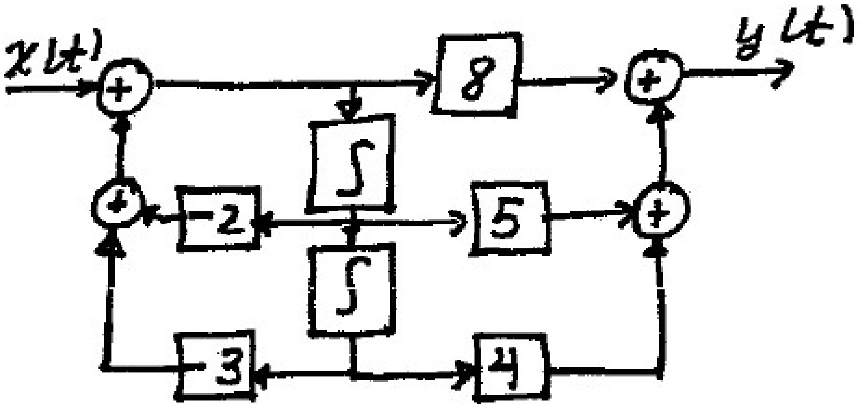
(d)





3.35

Redraw:



(b) ∴ Form II : (a) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 8\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x$

3.36

(a) $y_1(t) = x(t)\,H_1(s)$

$y_2(t) = y_1(t)\,H_2(s) = x(t)\,H_1(s)\,H_2(s) = x(t)\,H(s)$

$\therefore H(s) = \underline{H_1(s)\,H_2(s)}$

b) $y(t) = x(t)\,H_1(s) + x(t)\,H_2(s) = x(t)\left[H_1(s) + H_2(s)\right] = x(t)\,H(s)$

$\therefore H(s) = \underline{H_1(s) + H_2(s)}$

3.37

(a) $$y(t) = H_2(s)\left[H_1(s)x(t)\right] + H_4(s)\left[H_3(s)H_1(s)x(t) + H_5(s)x(t)\right]$$
$$= H(s)x(t)$$
$$\therefore H(s) = \underline{H_1(s)H_2(s) + H_1(s)H_3(s)H_4(s) + H_4(s)H_5(s)}$$

(b) $$y(t) = H_3(s)\left[H_2(s)\{H_1(s)x(t)\}\right] + H_4(s)\left[H_2(s)\{H_1(s)x(t)\}\right]$$
$$+ H_5(s)\left[x(t)H_1(s)\right] = H(s)x(t)$$
$$\therefore H(s) = \underline{H_1(s)H_2(s)H_3(s) + H_1(s)H_2(s)H_4(s) + H_1(s)H_5(s)}$$

(c) $$y(t) = H_1(s)\left[x(t) - H_2(s)y(t)\right] = H_1(s)x(t) - H_1(s)H_2(s)y(t)$$
$$\left[1 + H_1(s)H_2(s)\right]y(t) = H_1(s)x(t)$$
$$\therefore y(t) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}x(t) = H(s)x(t);\ \therefore\ H(s) = \underline{\frac{H_1(s)}{1 + H_1(s)H_2(s)}}$$

3.38

(a) $$y(t) = H_3(s)\left[H_1(s)x(t) + H_2(s)\{x(t) - H_4(s)y(t)\}\right]$$
$$= \left[H_1(s)H_3(s) + H_2(s)H_3(s)\right]x(t) - H_2(s)H_3(s)H_4(s)y(t)$$
$$\therefore y(t) = \underline{\frac{H_1(s)H_3(s) + H_2(s)H_3(s)}{1 + H_2(s)H_3(s)H_4(s)}}x(t) = H(s)x(t)$$

(b) $$y(t) = H_2(s)\left[H_1(s)\{x(t) - H_4 y(t)\} - H_3(s)y(t)\right]$$
$$= H_1(s)H_2(s)x(t) - \left[H_1(s)H_2(s)H_4(s) + H_2(s)H_3(s)\right]y(t)$$
$$\therefore y(t) = \frac{H_1(s)H_2(s)}{1 + H_1(s)H_2(s)H_4(s) + H_2(s)H_3(s)}x(t) = H(s)x(t)$$

Chapter 4 solutions

4·1 $\omega_0 = 2$, $T_0 = 2\pi/\omega_0 = \pi$

$$a)\ c_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi}(\cos 2t + 3\cos 4t)\,dt$$

$$= \frac{1}{\pi}\left(\frac{\sin 2t}{2} + 3/4 \sin 4t\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[1/2 \sin 2\pi + 3/4 \sin 4\pi - 1/2 \sin 0 - 3/4 \sin 0\right] = 0$$

$$c_K = \frac{1}{\pi}\int_0^{\pi} e^{-j2Kt}\left[\frac{e^{j2t}+e^{-j2t}}{2} + \frac{3(e^{j4t}+e^{-j4t})}{2}\right]dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[e^{j2(1-K)t} + e^{-j2(1+K)t} + 3e^{j2(2-K)t} + 3e^{-j2(2+K)t}\right]_0^{\pi}$$

$$= \begin{cases} \frac{1}{2\pi}\left[\frac{e^{j(1-K)\pi} - 1}{j2(1-K)}\right], & K = 1 \\ \frac{1}{2\pi}\left[\frac{3e^{j2(2-K)\pi} - 3}{j2(2-K)}\right], & K = 2 \end{cases}$$

$$\therefore c_1 = \lim_{K\to 1}\frac{1}{2\pi}\left[\frac{e^{j(1-K)\pi} - 1}{j2(1-K)}\right] = 1/2\pi\left[\frac{e^{j2\pi} - (j2\pi e^{-j2\pi k})}{j2(-1)}\right] = 1/2$$

$$\therefore c_2 = \lim_{K\to 2}\frac{1}{2\pi}\left[\frac{3e^{j2(2-K)} - 3}{j2(2-K)}\right] = \frac{1}{2\pi}\left[\frac{3e^{j4\pi}(-j2\pi e^{-j2K\pi})}{j2(-1)}\right] = 3/2$$

Alternatively, using Euler's formula:

$$\begin{aligned} f(t) = \cos(2t) + 3\cos(4t) =& \frac{1}{2}\left(e^{j2t} + e^{-j2t}\right) + \frac{3}{2}\left(e^{j4t} + e^{-j4t}\right) \\ =& \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{3}{2}e^{j2\omega_0 t} + \frac{3}{2}e^{-j2\omega_0 t} \\ & \Longrightarrow \end{aligned}$$

$$\begin{aligned} C_0 =& 0 \\ C_1 =& \frac{1}{2} \\ C_2 =& \frac{3}{2} \\ C_k =& 0, k \geq 3 \end{aligned}$$

4.2

(i) $x(t) = \sin(4t) + \cos(8t) + 7 + \cos(16t)$

(a) Exponential form: $\omega_0 = 4$

$$\begin{aligned} x(t) = & \quad \tfrac{1}{2j}e^{j4t} - \tfrac{1}{2j}e^{-j4t} + \tfrac{1}{2}e^{j8t} + \tfrac{1}{2}e^{-j8t} + 7e^{j\cdot 0} + \tfrac{1}{2}e^{j16t} + \tfrac{1}{2}e^{-j16t} \\ = & \quad 7 + (-0.5j)e^{j\omega_0 t} + (0.5j)e^{-j\omega_0 t} + (0.5)e^{j2\omega_0 t} + (0.5)e^{-j2\omega_0 t} + (0.5)e^{j4\omega_0 t} + (0.5)e^{-j4\omega_0 t} \end{aligned}$$

$$C_0 = 7$$

$$C_1 = -0.5j, C_{-1} = 0.5j$$

$$C_2 = 0.5, C_{-2} = 0.5$$

$$C_4 = 0.5, C_{-4} = 0.5$$

$$C_k = 0, k \neq 0, 1, -1, 2, -2, 4, -4$$

(b) Combined trigonometric form: $D_k = 2|C_k|, k > 0$

Since $\sin(4t) = \cos(4t - \pi/2)$

$$x(t) = \quad 7 + \cos(\omega_0 t - \pi/2) + \cos(2\omega_0 t) + \cos(4\omega_0 t)$$

$$D_0 = C_0 = 7$$

$$D_1 = 1, \theta_1 = -\pi/2$$

$$D_2 = 1, \theta_2 = 0$$

$$D_4 = 1, \theta_4 = 0$$

$$D_k = 0, k \neq 0, 1, 2, 4$$

(ii) $x(t) = \cos^2(t) = \frac{1}{2}[1 + \cos(2t)]$

(a) Exponential form: $\omega_0 = 2$

$$\frac{1}{2}[1 + \cos(2t)] = \quad \tfrac{1}{2} + \tfrac{1}{4}e^{j2t} + \tfrac{1}{4}e^{-j2t}$$

$$C_0 = \tfrac{1}{2}$$

$$C_1 = \tfrac{1}{4}, C_{-1} = \tfrac{1}{4}$$

$$C_k = 0, k \neq 0, 1, -1$$

Continued→

(b) Combined trigonometric: $D_k = 2|C_k|, k > 0$

$$D_0 = C_0 = \tfrac{1}{2},$$

$$D_1 = \tfrac{1}{2}, \theta_1 = 0$$

$$D_k = 0, k \neq 0, 1$$

(iii) $x(t) = \cos(t) + \sin(2t) + \cos(3t - \pi/3)$, $\omega_0 = 1$

(a) Exponential form:

$$x(t) = \tfrac{1}{2}e^{jt} + \tfrac{1}{2}e^{-jt} + \tfrac{1}{2j}e^{j2t} - \tfrac{1}{2j}e^{-j2t} + \tfrac{1}{2}e^{j3t} + \tfrac{1}{2}e^{-j3t}$$

$$C_1 = \tfrac{1}{2}, C_{-1} = \tfrac{1}{2}$$

$$C_2 = \tfrac{-j}{2}, C_{-2} = \tfrac{j}{2}$$

$$C_3 = \tfrac{1}{2}, C_{-3} = \tfrac{1}{2}$$

$$C_k = 0, k \neq 1, -1, 2, -2, 3, -3$$

(b) Trigonometric

$$x(t) = \cos(t) + \cos(2t - \pi/2) + \cos(3t - \pi/3), D_k = 2|C_k|, k > 0$$

$$D_0 = C_0 = 0,$$

$$D_1 = 1, \theta_1 = 0$$

$$D_2 = 1, \theta_2 = -\pi/2$$

$$D_3 = 1, \theta_3 = -\pi/3$$

$$D_k = 0, k > 3$$

(iv) $x(t) = 2\sin^2(2t) + \cos(4t) = (1 - \cos(4t)) + \cos(4t) = 1$

(a) Exponential: $C_0 = 1$, $C_k = 0, k \neq 1$

(b) Trigonometric: $D_0 = C_0 = 1$, $D_k = 0, k \neq 0$

(v) $x(t) = \cos(7t), \omega_0 = 7$

(a) Exponential: $C_1 = \frac{1}{2}$, $C_{-1} = \frac{1}{2}$, $C_k = 0, k \neq 1$

(b) Trigonometric: $D_1 = 1$, $\theta_1 = 0$,

$D_k = 0, k \neq 1$

(vi) $x(t) = 4\cos(t)\sin(4t) = 2\sin(5t) + 2\sin(3t), \omega_0 = 1$

(a) Exponential:

$$\begin{aligned} x(t) = \quad & -je^{j5t} + je^{-j5t} - je^{j3t} + je^{-j3t} \\ & C_3 = -j, C_{-3} = j \\ & C_5 = -j, C_{-5} = j \\ & C_k = 0, k \neq 3, -3, 5, -5 \end{aligned}$$

(b) Trigonometric:

$$\begin{aligned} x(t) = \quad & 2\cos(5t - \pi/2) + 2\cos(3t - \pi/2) \\ & D_3 = 2, \theta_3 = -\pi/2 \\ & D_5 = 2, \theta_5 = -\pi/2 \\ & D_k = 0, k \neq 3, 5 \end{aligned}$$

4.3

(a)

(i)

$x(t) = \cos(3t) + \sin(5t)$

$\omega_0 = 3 \quad T_0 = \frac{2\pi}{3} \quad \omega_1 = 5, \; T_1 = \frac{2\pi}{5} \quad \rightarrow T = 2\pi \quad \omega = 1$ ✓ yes.

(ii)

$x(t) = \cos(6t) + \sin(8t) + e^{j2t}$

$T_1 = \pi/3 \quad T_2 = \pi/4 \quad T_3 = \pi \longrightarrow T = \pi \quad \omega = 2$ ✓ yes

(iii)

aperiodic, NO

(iv)

$x(t) = \sin\left(\frac{\pi t}{6}\right) + \sin\left(\frac{\pi t}{3}\right)$

$T_1 = \frac{2\pi}{\pi/6} = 12 \qquad T_2 = \frac{2\pi}{\pi/3} = 6$

$T = 12, \; \omega = \pi/6$ ✓ Yes

(b)

(i) $\omega_0 = 1$, $x(t) = 0.5e^{j3t} + 0.5e^{-j3t} + \frac{1}{2j}e^{j5t} - \frac{1}{2j}e^{-j5t}$

$C_0 = 0$, $C_3 = C_{-3} = 0.5$, $C_5 = -0.5j$, $C_{-5} = 0.5j$.

(ii) $\omega_0 = 2$, $x(t) = e^{j2t} + \frac{1}{2}e^{j6t} + \frac{1}{2}e^{-j6t} + \frac{1}{2j}e^{j8t} - \frac{1}{2j}e^{-j8t}$

$C_0 = 0$, $C_1 = 1$, $C_{-1} = 0$, $C_3 = C_{-3} = \frac{1}{2}$, $C_4 = \frac{-j}{2}$, $C_{-4} = \frac{j}{2}$, $C_k = C_{-k} = 0, k > 4$

(iii) No Fourier Series

(iv) $x_3(t) = \sin(\frac{\pi}{6}t) + \sin(\frac{\pi}{3}t)$, $\omega_0 = \frac{\pi}{6}$

$x_3(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + \frac{1}{2j}e^{j2\omega_0 t} - \frac{1}{2j}e^{-j2\omega_0 t}$

$C_0 = 0$, $C_1 = \frac{-j}{2}$, $C_{-1} = \frac{j}{2}$, $C_2 = \frac{-j}{2}$, $C_{-2} = \frac{j}{2}$, $C_k = C_{-k} = 0, k > 2$

4·4 let $x(t)$ have a Fourier Series expansion:

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Then,

$$x(t-t_0)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 (t-t_0)} = \sum_{k=-\infty}^{\infty} \underbrace{\left[c_k e^{-jk\omega_0 t_0}\right]}_{\text{call this } \hat{c}_k} e^{jk\omega_0 t}$$

$$|\hat{c}_k| = \left|c_k e^{-jk\omega_0 t_0}\right| = |c_k|$$

$$\angle \hat{c}_k = \angle c_k e^{-jk\omega_0 t_0} = \angle c_k - k\omega_0 t_0$$

4·5

$$x(t)=A_0+\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t)+B_k \sin(k\omega_0 t)$$

$$= A_0 + \sum_{k=1}^{\infty} A_k\left[\frac{e^{jk\omega_0 t}+e^{-jk\omega_0 t}}{2}\right] + B_k\left[\frac{e^{jk\omega_0 t}-e^{-jk\omega_0 t}}{2j}\right]$$

$$= A_0 + \sum_{k=1}^{\infty}\left[\frac{A_k}{2}+\frac{B_k}{2j}\right]e^{jk\omega_0 t} + \left[\frac{A_k}{2}-\frac{B_k}{2j}\right]e^{-jk\omega_0 t}$$

Compare to $x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$$c_k=\begin{cases} A_0 & k=0 \\ \tfrac{1}{2}\left[A_k - jB_k\right], & k \geq 1 \\ \tfrac{1}{2}\left[A_k + jB_k\right], & k \leq -1 \end{cases}$$

4·6

a) $$\int_0^{2\pi} \sin^2(t)\,dt = \int_0^{2\pi} \frac{1}{2}\left[1-\cos 2t\right] dt$$

$$= \frac{1}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} = \pi$$

Continued→

b) $\int_0^{2\pi} \sin^2(2t)\,dt = \int_0^{2\pi} \frac{1}{2}\left[1 - \cos 4t\right] dt$

$$= \frac{1}{2}\left(t - \frac{1}{4}\sin 4t\right)\Big|_0^{2\pi} = \pi$$

c) $\int_0^{2\pi} \sin(t)\sin(2t)\,dt$

$$= \frac{1}{2}\int_0^{2\pi}\left[\cos t - \cos 3t\right] dt$$

$$= \frac{1}{2}\left(\sin t - \frac{1}{3}\sin 3t\right)\Big|_0^{2\pi}$$

$$= 0$$

(d) It illustrates the orthogonality of sinusoids, since it shows cases where if $f(t)$ and $g(t)$ are two sinusoids with $f(t) \neq g(t)$ then they are orthogonal over $[0, 2\pi]$ according to the definition of orthogonality in section 4.2. Complex exponentials are orthogonal over $[0, 2\pi]$ because $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$ and so

$$\begin{aligned}
\int_0^{2\pi} e^{j\omega_1 t} e^{-j\omega_0 t} dt &= \int_0^{2\pi}(\cos(\omega_1 t) + j\sin(\omega_1 t))(\cos(\omega_0 t) - j\sin(\omega_0 t))dt \\
&= \int_0^{2\pi}\left(\cos(\omega_1 t)\cos(\omega_0 t) + j\sin(\omega_1 t)\cos(\omega_0 t) - j\sin(\omega_0 t)\cos(\omega_1 t) + \sin(\omega_1 t)\sin(\omega_0 t)\right) dt \\
&= 0\ if\ \omega_0 \neq \omega_1
\end{aligned}$$

4.7. The integral of a sinusoid over an integer number of periods is zero. Orthogonal: $\int_a^b g(t)h(t)dt = 0$

(a) $\cos m\omega_0 t \cos n\omega_0 t$

$$= \frac{1}{2}\cos(m+n)\omega_0 t + \frac{1}{2}\cos(m-n)\omega_0 t$$

$$\therefore \frac{1}{2}\int_0^{T_0}[\cos(m+n)\omega_0 t + \cos(n-m)\omega_0 t]dt = 0, \quad m \neq n$$

$$= \frac{1}{2}\int_0^{T_0} dt = \frac{1}{2}t\Big]_0^{T_0}, \quad m = n \qquad \therefore \underline{m \neq n}$$

(b) $\cos m\omega_0 t \sin n\omega_0 t = \frac{1}{2}[\sin(m+n)\omega_0 t + \sin(n-m)\omega_0 t]$

$$\therefore \frac{1}{2}\int_0^{T_0}[\sin(m+n)\omega_0 t + \sin(n-m)\omega_0 t]dt = 0, \quad \underline{\text{all } m \neq n}$$

(c) $\sin m\omega_0 t \sin n\omega_0 t = \frac{1}{2}[\cos(m-n)\omega_0 t - \cos(m+n)\omega_0 t]$

$$\therefore \frac{1}{2}\int_0^{T_0}[\cos(m-n)\omega_0 t - \cos(m+n)\omega_0 t]dt = \begin{cases} 0, & m \neq n \\ \frac{T_0}{2}, & m = n \end{cases}$$

from(a) →

4.8. (a) $\underline{C_k = -j\frac{2X_0}{\pi k}}$, $\underline{k \text{ odd}}$ $\quad 2|C_k| = \frac{4X_0}{\pi k}$; $\theta_k = -90^\circ$

$$\therefore \underline{x(t) = \sum_{\substack{k=1 \\ k\text{ odd}}}^{\infty} \frac{4X_0}{\pi k}\cos(k\omega_0 t - 90^\circ)}$$

(b) $C_k = j\frac{X_0}{2\pi k}$, $2|C_k| = \frac{X_0}{\pi k}$, $\theta_k = 90^\circ$

$$\therefore \underline{x(t) = \frac{X_0}{2} + \sum_{k=1}^{\infty}\frac{X_0}{\pi k}\cos(k\omega_0 t + 90^\circ)}$$

(c) $C_k = -\frac{2X_0}{(\pi k)^2}$, k odd, $2|C_k| = \frac{4X_0}{(\pi k)^2}$; $\theta_k = 180^\circ$

$$\therefore \underline{x(t) = \frac{X_0}{2} + \sum_{\substack{k=1 \\ k\text{ odd}}}^{\infty}\frac{4X_0}{(\pi k)^2}\cos(k\omega_0 t + 180^\circ)}$$

(d) $C_k = \frac{wX_0}{T_0}\text{sinc}\frac{wk\omega_0}{2}$;

$$\underline{x(t) = \sum_{k=0}^{\infty}\frac{2wX_0}{T_0}\text{sinc}\left(\frac{wk\omega_0}{2}\right)\cos k\omega_0 t}$$

(e) $\underline{x(t) = \frac{2X_0}{\pi} + \sum_{k=1}^{\infty}\frac{4X_0}{\pi(4k^2-1)}\cos(k\omega_0 t + 180^\circ)}$

(f) $\underline{x(t) = \frac{X_0}{2}\cos(\omega_0 t - 90^\circ) + \sum_{\substack{k=0 \\ k\text{ even}}}^{\infty}\frac{2X_0}{\pi(k^2-1)}\cos(k\omega_0 t + 180^\circ)}$

(g) $x(t) = \sum_{k=0}^{\infty}\frac{2X_0}{T_0}\cos k\omega_0 t$

4.9. AppA used, with $e^{-jk\omega_0T_0} = e^{-jk2\pi}$

(a) $C_k = \frac{1}{T_0}\int_0^{T_0/2} X_0 e^{-jk\omega_0 t}dt - \frac{1}{T_0}\int_{T_0/2}^{T_0} X_0 e^{-jk\omega_0 t}$

$$= \frac{X_0}{-jk\omega_0T_0}\left[e^{-jk\omega_0 t}\Big|_0^{T_0/2} - e^{-jk\omega_0 t}\Big|_{T_0/2}^{T_0}\right]$$

$$= \frac{jX_0}{2\pi k}\left[e^{-jk\pi} - 1 - e^{-jk2\pi} + e^{-jk\pi}\right] = \underline{\begin{cases} 0\,; & k \text{ even} \\ -j\frac{2X_0}{\pi k}\,; & k \text{ odd}\end{cases}}$$

(b) $C_k = \frac{1}{T_0}\int_0^{T_0}\frac{X_0}{T_0} t e^{-jk\omega_0 t}dt = \frac{X_0}{T_0^2}\left[\frac{1}{(-jk\omega_0)^2}e^{-jk\omega_0 t}(-jk\omega_0 t-1)\right]_0^{T_0}$

$$= \frac{X_0}{-[k2\pi]^2}\left[e^{-jk2\pi}(-jk2\pi-1)-(-1)\right] = \frac{-X_0}{(2\pi k)^2}(-jk2\pi) = \underline{\frac{jX_0}{2\pi k}}$$

(c) $C_k = \frac{1}{T_0}\int_{-T_0/2}^{0} -\frac{2X_0}{T_0} t e^{-jk\omega_0 t}dt + \frac{1}{T_0}\int_0^{T_0/2}\frac{2X_0}{T_0}te^{-jk\omega_0 t}dt$

$$= \frac{2X_0}{T_0^2}\frac{1}{(-jk\omega_0)^2}\left[-e^{-jk\omega_0 t}(-jk\omega_0 t-1)\Big|_{-T_0/2}^{0} + e^{-jk\omega_0 t}(-jk\omega_0 t-1)\Big|_0^{T_0/2}\right]$$

$$= \frac{2X_0}{-(k2\pi)^2}\left[1 + e^{jk\pi}(jk\pi-1) + e^{-jk\pi}(-jk\pi-1) - (-1)\right]$$

Now, $e^{jk\pi} = e^{-jk\pi}$

$$\therefore C_k = \frac{2X_0}{-(k2\pi)^2}\left[-2e^{jk\pi}+2\right] = \underline{\begin{cases}\frac{-2X_0}{(\pi k)^2}\,; & k \text{ odd}\\ 0\,; & k\text{ even}\end{cases}}$$

(d) $C_k = \frac{1}{T_0}\int_{-w/2}^{w/2} X_0 e^{-jk\omega_0 t}dt = \frac{X_0}{-jk2\pi}e^{-jk\omega_0 t}\Big|_{-w/2}^{w/2}$

$$= \frac{X_0}{-jk2\pi}\left[e^{-jkw\omega_0/2} - e^{jkw\omega_0/2}\right] = \frac{X_0}{\pi k}\sin(kw\omega_0/2)$$

$$= \frac{X_0}{\pi k}\,\frac{kw\omega_0}{2}\,\frac{\sin(kw\omega_0/2)}{kw\omega_0/2} = \underline{\frac{wX_0}{T_0}\,\mathrm{sinc}(kw\omega_0/2)}$$

(e) $C_k = \frac{1}{T_0}\int_0^{T_0} X_0\sin(\frac{\omega_0}{2}t)e^{-jk\omega_0}dt$

$$= \frac{X_0}{T_0}\left[\frac{e^{-jk\omega_0 t}(-jk\omega_0 t\sin(\frac{\omega_0 t}{2}) - \frac{\omega_0}{2}\cos(\frac{\omega_0 t}{2}))}{-k^2\omega_0^2+\omega_0^2/4}\right]_0^{T_0}$$

$$= \frac{X_0}{T_0}\left[\frac{e^{-jk2\pi}(-jk2\pi\sin\pi - (\pi/T_0)\cos\pi) - (1)(-\frac{\pi}{T_0}\cos 0)}{-k^2\omega_0^2+\omega_0^2/4}\right]$$

4.9
(cont)

$$= \frac{4X_0}{T_0}\left[\frac{(1)\left(\frac{\pi}{T_0}\right)+\frac{\pi}{T_0}}{\omega_0^2(1-4k^2)}\right] = \frac{8X_0}{4\pi(1-4k^2)} = \underline{\frac{-2X_0}{\pi(4k^2-1)}}$$

(f) $$C_k = \frac{1}{T_0}\int_0^{T_0/2} X_0 \sin(\omega_0 t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{X_0}{T_0}\left[\frac{e^{-jk\omega_0 t}(-jk\omega_0 t \sin(\omega_0 t) - \omega_0 \cos(\omega_0 t))}{-k^2\omega_0^2+\omega_0^2}\right]_0^{T_0/2}$$

$$= \frac{X_0}{T_0}\left[\frac{e^{-j\pi k}(0-\omega_0\cos\pi)-(-\omega_0)}{\omega_0^2(1-k^2)}\right] = \frac{X_0}{2\pi}\,\frac{\left[e^{-jk\pi}+1\right]}{(1-k^2)}$$

$$= \underline{\frac{-X_0}{\pi(k^2-1)}, \ k \text{ even}}$$

$$C_1 = \lim_{k\to 1}\frac{X_0}{2\pi}\left[\frac{e^{-jk\pi}+1}{(1-k^2)}\right] = \frac{X_0}{2\pi}\left[\frac{-j\pi e^{-jk\pi}}{-2k}\right]_{k=1} = \underline{\frac{-jX_0}{4}}$$

$$\underline{C_k = 0 \quad , k \text{ odd and } k \neq 1}$$

(g) $$C_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} X_0\,\delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}X_0 e^{-j0} = \underline{\frac{X_0}{T_0}}$$

4.10

(a)

$$\begin{aligned}
T_0 &= 4, \omega_0 = \frac{\pi}{2}\\
C_k &= \frac{1}{4}\int_{-2}^{2} x_a(t)e^{-jk\omega_0 t}dt\\
&= \frac{1}{4}\int_{-1}^{0} 3e^{-jk\omega_0 t}dt + \frac{1}{4}\int_{0}^{1} -3e^{-jk\omega_0 t}dt\\
&= \frac{3}{4}\left(\frac{1}{jk\omega_0}\right)(e^{jk\omega_0}-1) - \frac{3}{4}\left(\frac{1}{jk\omega_0}\right)(1-e^{-jk\omega_0})\\
&= \frac{3}{4}\left(\frac{1}{jk\omega_0}\right)(e^{jk\omega_0}+e^{-jk\omega_0}-2)\\
&= \frac{3}{4}\,\frac{2\cos(k\omega_0)-2}{jk\omega_0}\\
&= \frac{-3j}{k\pi}\left(\cos\left(k\frac{\pi}{2}\right)-1\right)\\
C_0 &= \lim_{k\to 0} C_k = \lim_{k\to 0}\frac{-3j\pi/2\sin(k\frac{\pi}{2})}{\pi} = 0\\
C_0 &= \frac{1}{4}\left(\int_{-1}^{0} 3dt - \int_{0}^{1} 3dt\right) = 0
\end{aligned}$$

(b)

$$
\begin{aligned}
T_0 = 3, \omega_0 = & \frac{2\pi}{3} \\
C_k = & \frac{1}{3}\int_{-1}^{2} x_b(t) e^{-jk\omega_0 t} dt \\
= & \frac{1}{3}\int_{-1}^{0} 2e^{-jk\omega_0 t} dt + \frac{1}{3}\int_{0}^{1} 1e^{-jk\omega_0 t} dt \\
= & \frac{1}{3}(\frac{1}{jk\omega_0})(2e^{jk\omega_0} - 2 + 1 - e^{-jk\omega_0}) \\
= & \frac{1}{2\pi}(\frac{1}{jk})\left(2e^{jk\frac{2\pi}{3}} - 1 - e^{-jk\frac{2\pi}{3}}\right)
\end{aligned}
$$

$$
\begin{aligned}
C_0 = & \lim_{k\to 0} C_k = \lim_{k\to 0} \frac{1}{2\pi}(\frac{1}{j})(2j\frac{2\pi}{3}e^{jk\frac{2\pi}{3}} + j\frac{2\pi}{3}e^{-jk\frac{2\pi}{3}}) = (2+1)/3 = 1 \\
C_0 = & \frac{1}{3}\left(\int_{-1}^{0} 2dt + \int_{0}^{1} 1dt\right) = \frac{1}{3}(2+1) = 1
\end{aligned}
$$

(c)

$$
\begin{aligned}
T_0 = 2, \omega_0 = & \pi \\
C_k = & \frac{1}{2}\int_{0}^{1} 2te^{-jk\omega_0 t} dt \\
= & \frac{1}{(-jk\pi)^2}[e^{-jk\pi t}(-jk\pi t - 1)]_0^1 \\
= & \frac{-1}{k^2\pi^2}[e^{-jk\pi}(-jk\pi - 1) + 1] \\
= & \frac{1}{k^2\pi^2}[e^{-jk\pi}(jk\pi + 1) - 1] \\
= & \frac{1}{k^2\pi^2}[(-1)^k jk\pi + (-1)^k - 1] = \frac{j(-1)^k}{k\pi} + \frac{(-1)^k - 1}{k^2\pi^2} \\
C_0 = & \lim_{k\to 0} C_k = \lim_{k\to 0} \frac{1}{2k\pi^2}[-j\pi e^{-jk\pi}(jk\pi + 1) + j\pi e^{-jk\pi}] = \frac{1}{2} \\
C_0 = & \frac{1}{2}\int_{0}^{1} 2tdt = \frac{1}{2}
\end{aligned}
$$

(d) Note that over the nonzero part of the cycle, $x_d(t) = x_c(t) - 2$, so $C_k = C_k(\text{from part c}) - \frac{1}{2}\int_0^1 2e^{-jk\pi}dt$

$$\begin{aligned}
C_k =& \frac{1}{k^2\pi^2}[e^{-jk\pi}(jk\pi + 1) - 1] - \int_0^1 e^{-jk\pi}dt \\
=& \frac{1}{k^2\pi^2}[e^{-jk\pi}(jk\pi + 1) - 1] + \frac{j}{k\pi}(1 - e^{-jk\pi}) \\
=& \frac{1}{k^2\pi^2}[(-1)^k(jk\pi + 1) - 1] + \frac{j}{k\pi} - \frac{j(-1)^k}{k\pi} \\
=& \frac{(-1)^k}{k^2\pi^2} + \frac{-1}{k^2\pi^2} + \frac{j}{k\pi} \\
C_0 =& C_0(\text{from part c}) + \lim_{k\to 0}\frac{j}{\pi}(j\pi e^{-jk\pi}) \\
=& \frac{1}{2} - 1 = -\frac{1}{2} \\
C_0 =& \frac{1}{2}\int_0^1 (2t - 2)dt = -\frac{1}{2}
\end{aligned}$$

(e)

$$\begin{aligned}
T_0 =& 4, \omega_0 = \frac{\pi}{2} \\
C_k =& \frac{1}{4}\int_{-1}^{0} 2\cos(\frac{\pi}{2}t)e^{-jk\frac{\pi}{2}t}dt \\
=& \frac{1}{2}\frac{1}{-(k\frac{\pi}{2})^2 + (\frac{\pi}{2})^2}\left[e^{-jk\frac{\pi}{2}t}\left(-jk\frac{\pi}{2}\cos(\frac{\pi}{2}t) + \frac{\pi}{2}\sin(\frac{\pi}{2}t)\right)\right]_{-1}^{0} \\
=& \frac{2}{\pi^2(1-k^2)}\left[-jk\frac{\pi}{2} - e^{j\frac{\pi}{2}k}\left(-j\frac{\pi}{2}k(0 - \frac{\pi}{2}\sin(\frac{\pi}{2}))\right)\right] \\
=& \frac{1}{\pi(1-k^2)}\left[e^{j\frac{\pi}{2}k} - jk\right] \\
=& \frac{1}{\pi(1-k^2)}[j^k - jk] \\
C_1 =& \frac{1}{4} + \frac{j}{2\pi} \\
C_{-1} =& \frac{1}{4} - \frac{j}{2\pi} \\
C_0 =& \frac{1}{\pi}
\end{aligned}$$

(f)

$$\begin{aligned}
T_0 = 2&,\omega_0 = \pi \\
C_k =& \frac{1}{2}\int_1^2 \sin(\frac{\pi}{2}t)e^{-jk\pi t}dt \\
=& \frac{1}{2}\left[\frac{e^{-jk\pi t}}{(-jk\pi)^2 + (\frac{\pi}{2})^2}\left(-jk\pi\sin(\frac{\pi}{2}t) - \frac{\pi}{2}\cos(\frac{\pi}{2}t)\right)\right]_1^2 \\
=& \frac{1}{2}\left(\frac{1}{-(k\pi)^2 + (\frac{\pi}{2})^2}\right)\left[e^{-jk2\pi}(-jk\pi(0) - \frac{\pi}{2}(-1)) - e^{-jk\pi}(-jk\pi(1) - \frac{\pi}{2}(0))\right] \\
=& \frac{1}{2}\left(\frac{1}{-(k\pi)^2 + (\frac{\pi}{2})^2}\right)\left[\frac{\pi}{2} + jk\pi(-1)^k\right] \\
C_0 =& \frac{1}{\pi}
\end{aligned}$$

4.11

(a) entry 3 in table, with $X_0 = 4$, $\omega_0 = \frac{2\pi}{0.4\pi} = 5$, with

$$\begin{aligned}
C_0 =& 0, \\
C_k =& \frac{-2(4)}{(\pi k)^2}, k\ odd \\
C_k =& 0, k\ even
\end{aligned}$$

(b) entry 6 in table (rectangular wave), with a time delay ($\implies$ phase shift in C_k's) and a change in average value ($\implies$ change in C_0).

$$\begin{aligned}
\frac{T}{2} = 1,&T_0 = 3, \omega_0 = \frac{2\pi}{3}, X_0 = 15 \\
t_0(time\ delay) =& 2 \implies C_k = \hat{C}_k e^{-j2k\omega_0} \text{ where } \hat{C}_k = \frac{TX_0}{T_0}sinc(\frac{Tk\omega_0}{2}) \\
C_0 =& 2(10) - 5 = 15 \\
C_k =& 10sinc(\frac{2\pi k}{3})e^{-j2k\frac{2\pi}{3}}, k \neq 0
\end{aligned}$$

(c) entry 2 in table with $X_0 = 8$ and $T_0 = 0.2$.

$$\begin{aligned}
C_0 =& 0 \\
C_k =& \frac{j8}{2\pi k} = \frac{j4}{\pi k}, k \neq 0
\end{aligned}$$

(d) entry 3 advanced by 1 second, with $X_0 = 3$ and $T_0 = 4$, $\omega_0 = \frac{\pi}{2}$:

$$\begin{aligned} C_0 =& \frac{3}{2} \\ C_k =& \hat{C}_k e^{jk\omega_0}, \text{ where } \hat{C}_k = \frac{-2(3)}{(\pi k)^2} \\ =& \frac{-6}{(\pi k)^2} e^{jk\frac{\pi}{2}}, k\ odd \\ =& 0, k\ even \end{aligned}$$

(e) entry 4 with $T_0 = 2$ and $X_0 = 6$

$$\begin{aligned} C_0 =& \frac{12}{\pi} \\ C_k =& -\frac{12}{\pi(4k^2-1)}, k \neq 0 \end{aligned}$$

(f) entry 5 delayed by 1 second, with $X_0 = 8$, $T_0 = 4$.

$$\begin{aligned} C_0 =& \frac{8}{\pi} \\ C_k =& \hat{C}_k e^{-jk\omega_0} \text{ where } \hat{C}_k = \frac{-8}{\pi(k^2-1)}, k\ even; = -j2, k = 1; = j2, k = -1 \\ =& \frac{-8}{\pi(k^2-1)} e^{-jk\frac{\pi}{2}}, k\ even \\ =& -j2e^{-jk\frac{\pi}{2}} = -2, k = 1 \\ =& j2e^{-jk\frac{\pi}{2}} = -2, k = -1 \\ =& 0, k\ odd, k \neq -1, 1 \end{aligned}$$

4.12 (a) Only the value of ω_0 changes, the C_k's stay the same. Therefore: $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$, and from 4.11(a) $C_k = \frac{-8}{(\pi k)^2}$, k odd, and $C_k = 0$, k even .

(b) From 4.11(d)

$$\begin{aligned} C_0 =& \frac{3}{2} \\ C_k =& \frac{-6}{(\pi k)^2} e^{-jk\frac{\pi}{2}}, k\ odd \\ =& 0, k\ even \end{aligned}$$

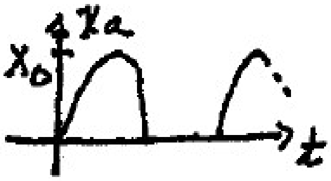
Continued→

(c) $\tau = -1$, $a_1 = 1$, $b_1 = \frac{4}{3}$
$x(t) = x_a(t) + \frac{4}{3}x_d(t+1) = 2 = A$

(d) $C_k = C_{ka} + C_{kb}e^{jk\omega_0(1)}$, where C_{ka} is the FS coeff for $x_a(t)$ and C_{kb} is the FS coeff for $x_b(t)$. The coefficients for both $x_a(t)$ and $x_d(t+1)$ are 0 when k even. For k odd:

$$\begin{aligned} C_k &= \frac{-8}{(\pi k)^2} + (4/3)\frac{-6}{(\pi k)^2}e^{jk\frac{\pi}{2}}e^{jk\frac{\pi}{2}} \\ &= \frac{-8}{(\pi k)^2} + \frac{-8}{(\pi k)^2}e^{jk\pi} \\ &= \frac{-8}{(\pi k)^2} + \frac{8}{(\pi k)^2} = 0 \end{aligned}$$

since $e^{jk\pi} = -1$ if k is odd.

4.13. (a) 

$C_0 = \frac{X_0}{\pi}$, $\underset{k\text{ even}}{C_{ka}} = \frac{-X_0}{\pi(k-1)}$; $\underset{k\text{ odd}}{C_{ka}} = 0$, $C_{1a} = -j\frac{X_0}{4}$

$$x_b(t) = x_a(\tau)\Big|_{\tau = t - \frac{T_0}{2}} = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0(t-\frac{T_0}{2})} = \sum_{k=-\infty}^{\infty} C_k e^{-j\omega_0 t} e^{-jk\omega_0\frac{T_0}{2}}$$

$$k\omega_0\frac{T_0}{2} = k\frac{2\pi}{T_0}\frac{T_0}{2} = k\pi \ ; \therefore e^{-jk\pi}\begin{cases} 1, & k\text{ even} \\ -1, & k\text{ odd}\end{cases}$$

$$\therefore \underset{k\text{ even}}{C_{kb}} = \frac{-X_0}{\pi(k^2-1)}, \quad C_{1b} = j\frac{X_0}{4}$$

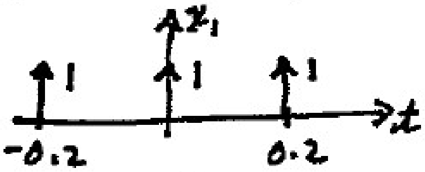
(b) For $x_1 = x_a + x_b$, from (a):

$$\underset{k\text{ even}}{C_{k1}} = \frac{-2X_0}{\pi(k_1^2-1)} \ ; \ \underset{k_1=1}{C_{k1}} = -j\frac{X_0}{4} + j\frac{X_0}{4} = 0, \quad C_0 = \frac{2X_0}{\pi}$$

Since k_1 is even, define $k = \frac{k_1}{2}$; then $k = 1, 2, 3, \ldots$

$$\therefore C_k = \frac{-2X_0}{\pi((2k)^2-1)} = \frac{-2X_0}{\pi(4k^2-1)} \checkmark \ ; \ C_0 = \frac{2X_0}{\pi} \checkmark$$

4.14. (a) $x(t) = x_1(t) + x_2(t)$, $\omega_0 = \frac{2\pi}{0.2} = 10\pi$



$$x_1(t) = \sum_{k=-\infty}^{\infty} \frac{1}{0.2} e^{jk10\pi t} = \sum_{k=-\infty}^{\infty} 5 e^{jk10\pi t}$$

$$x_2(t) = -x_1(t-0.1) = -\sum_{k=-\infty}^{\infty} 5 e^{jk10\pi(t-0.1)}$$

$$= -\sum_{k=-\infty}^{\infty} 5 e^{-jk\pi} e^{jk10\pi t}$$

$$\therefore x(t) = 5 \sum_{k=-\infty}^{\infty} (1 - e^{-jk\pi}) e^{jk10\pi t}$$

4.14(a) (cont)

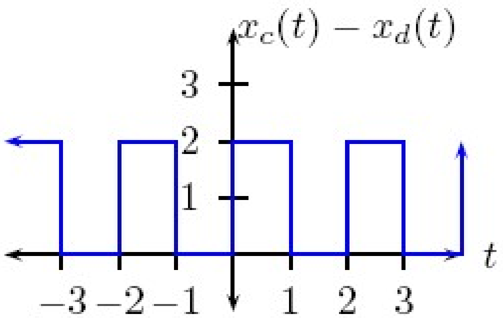
$$\therefore C_k = \underline{5(1 - e^{-jk\pi})} = \begin{cases} 10, & k = \pm 1, \pm 3, \dots \\ 0, & k = 0, \pm 2, \pm 4, \dots \end{cases}$$

(b)

$$C_k = \frac{1}{T_0} \int_{-T_0/4}^{3T_0/4} [\delta(t) - \delta(t-0.1)] e^{-jk\omega_0 t} dt$$

$$= 5[1 - e^{-jk10\pi(0.1)}] = \underline{5[1 - e^{-jk\pi}]}$$

4.15



$$T_0 = 2, \omega_0 = \pi$$

$$C_k = \frac{1}{2} \int_0^1 2e^{-jk\pi t} dt$$

$$= \frac{1}{jk\pi}(1 - e^{-jk\pi}) = \frac{j}{k\pi}((-1)^k - 1)$$

$$= -\frac{2j}{k\pi}, k \ odd$$

$$= 0, k \ even, k \neq 0$$

$$C_0 = \frac{1}{2}(2) = 1$$

Hence, for $k \neq 0$, the C_k's are the same as Example 4.2 with $V = 1$ (the signal is the same with $V = 1$, except for the offset of 1).

4.16. $C_k = \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt = \int_0^{T_0/2} x(t) e^{-jk\omega_0 t} dt + \int_{T/2}^{T_0} x(t) e^{-jk\omega_0 t} dt$

$= I_1 + I_2$

$I_2 = \int_{\frac{T_0}{2}}^{T_0} x(\tau) e^{-jk\omega_0 \tau} d\tau$: let $\tau = t_0 - T_0/2$

$\therefore I_2 = \int_0^{T_0/2} x(t - \frac{T_0}{2}) e^{-jk\omega_0 (t - T_0/2)} dt$ now $\frac{k\omega_0 T_0}{2} = \frac{k}{2}\left(\frac{2\pi}{T_0}\right) T_0 = k\pi$

$\therefore I_2 = -e^{jk\pi} \int_0^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt = -(-1)^k I_1$

$\therefore C_k = I_1 - (-1)^k I_1 = [1 - (-1)^k] = \begin{cases} 2I_1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$

4.17 Using property 6, $m-1$ is the order of the first derivative that has a discontinuity:

(a): $\frac{dx_a}{dt}$ discontinuous $\implies m-1 = 1$, $m = 2$

$|C_k| = \frac{8}{\pi^2}\frac{1}{k^2}$ (k odd) (check)

(b): $x_b(t)$ discontinuous $\implies m-1 = 0$, $m = 1$

$|C_k| = 10|sinc(\frac{2\pi k}{3})| = \frac{30|\sin(\frac{2\pi k}{3})|}{2\pi k}$, ($k > 0$) (check)

(c): $x_c(t)$ discontinuous $\implies m-1 = 0$, $m = 1$

$|C_k| = \frac{4}{\pi k}$ ($k > 0$) (check)

(d): $\frac{dx_d(t)}{dt}$ discontinuous $\implies m-1 = 1$, $m = 2$

$|C_k| = \frac{6}{\pi^2 k^2}$ (k odd) (check)

(e): $\frac{dx_e(t)}{dt}$ discontinuous $\implies m-1 = 1$, $m = 2$

$|C_k| = \frac{12}{\pi(4k^2-1)}$ (check)

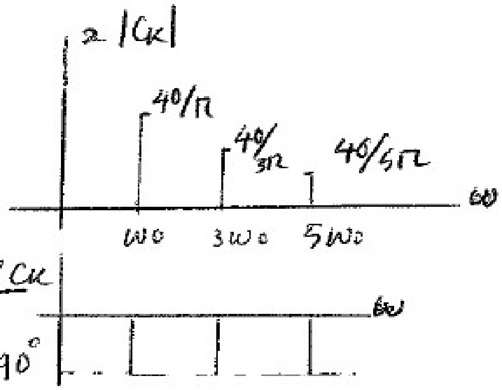
(f): $\frac{dx_f(t)}{dt}$ discontinuous $\implies m-1 = 1$, $m = 2$

$|C_k| = \frac{8}{\pi(k^2-1)}$, ($k$ even) (check)

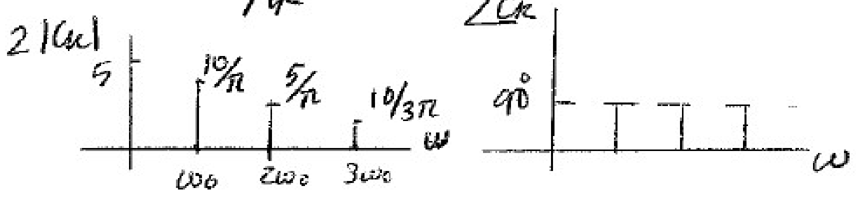
4.18

Will use combined trig form with $x_0 = 10$

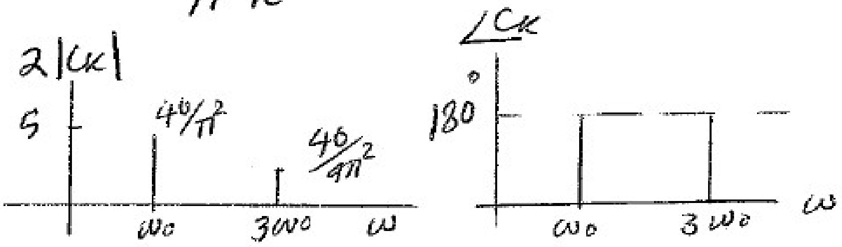
a) $2C_k = \frac{40}{\pi k} \angle -90^\circ$, k odd



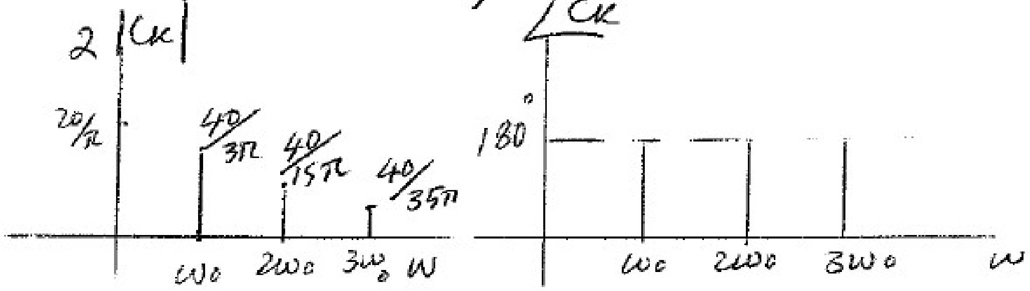
b) $2C_k = \frac{10}{\pi k} \angle 90^\circ$



c) $2C_k = \frac{-40}{\pi^2 k^2}$, k odd

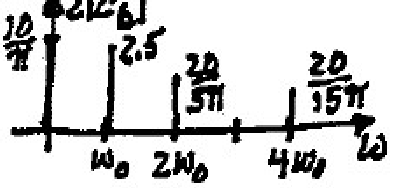


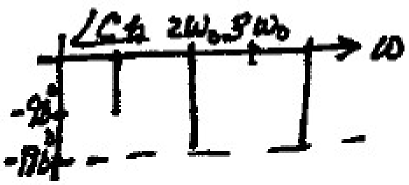
d) $2C_k = \frac{-40}{\pi(4k^2-1)}$



Continued→

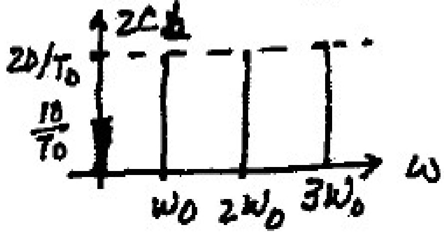
4.18.(e) $2C_k = \frac{-20}{\pi(k^2-1)}$, k even
(cont)
$C_1 = -j2.5$





(f) See Figure 4.13 with $\frac{2WX_0}{T_0} = \frac{20W}{T_0}$

(g) $2C_k = \frac{20}{T_0}$



4.19

The C_k's were found in problem 4.10.

(a)

$$\begin{aligned}
C_0 =& 0 \\
C_k =& \frac{-3j}{k\pi}(\cos(k\frac{\pi}{2}) - 1), k \neq 0 \\
2|C_k| =& \frac{6}{k\pi}|\cos(k\frac{\pi}{2}) - 1)|, k > 0 \\
\theta_k =& \frac{\pi}{2}
\end{aligned}$$

So the values of C_0 the first 4 harmonics in trigonometric form are given by:

$$\begin{aligned}
C_0 =& 0 \\
2|C_1| =& \frac{6}{k\pi} = \frac{6}{\pi} \\
2|C_2| =& \frac{12}{k\pi} = \frac{6}{\pi} \\
2|C_3| =& \frac{6}{k\pi} = \frac{2}{\pi} \\
2|C_4| =& 0
\end{aligned}$$

and $\theta_k = \frac{\pi}{2}$ for $k = 1, 2, 3$.

Continued→

4.19, continued

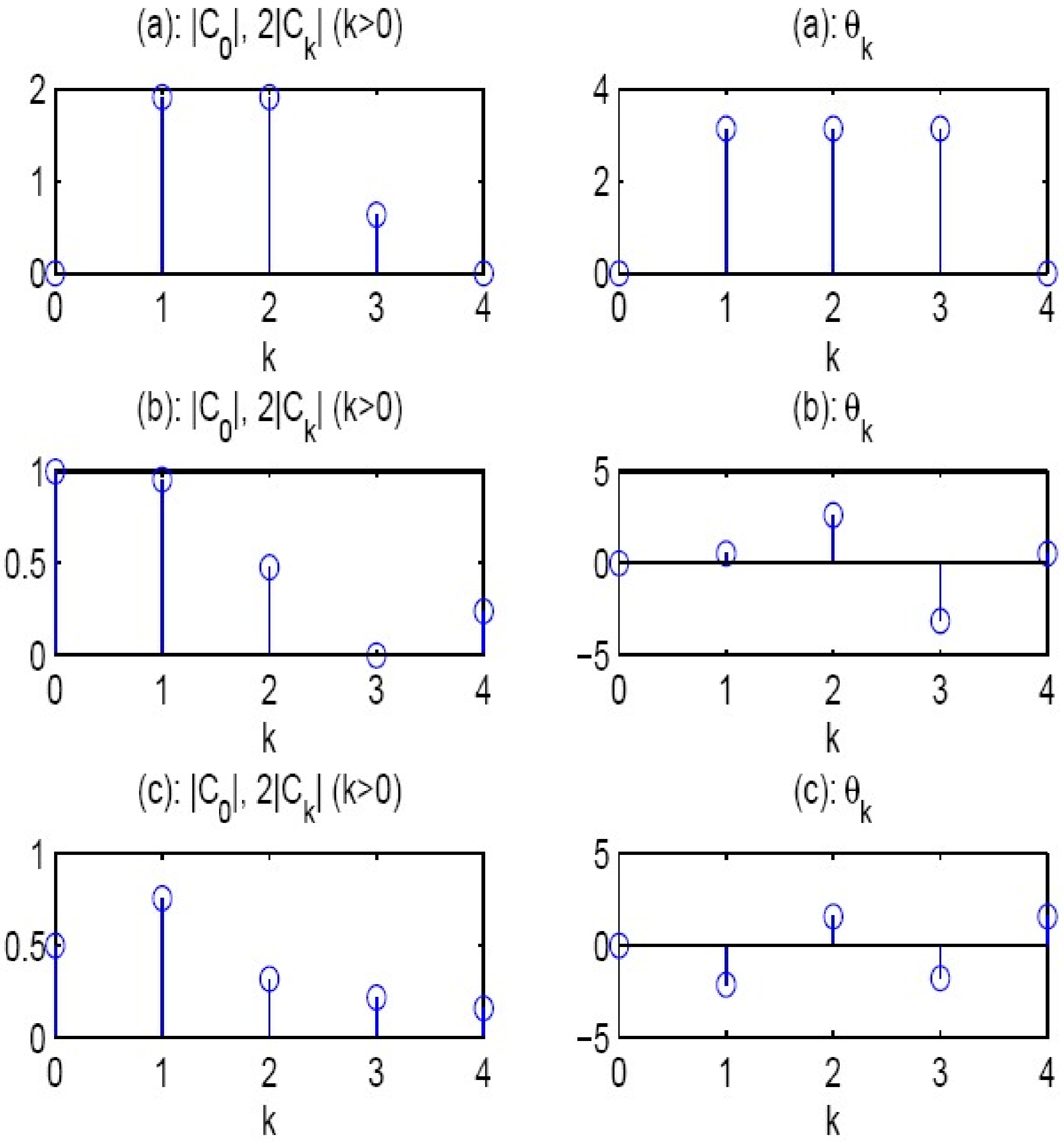


Figure 1: Fourier spectra for parts (a)-(c)

Continued→

(b)

$$\begin{aligned}
C_0 &= 1 \\
C_k &= \frac{1}{2\pi}\left(\frac{1}{jk}\right)\left(2e^{jk\frac{2\pi}{3}} - 1 - e^{-jk\frac{2\pi}{3}}\right) \\
C_1 &= \frac{1}{2\pi j}(-1.5 + j1.5\sqrt{3}) \\
2|C_1| &= 2\frac{3}{2\pi}, \theta_1 = \frac{4\pi}{6} \\
C_2 &= \frac{1}{4\pi j}(-1.5 - j1.5\sqrt{3}) \\
2|C_2| &= 2\frac{3}{4\pi}, \theta_2 = \frac{-4\pi}{6} \\
C_3 &= \frac{1}{6\pi j}(0) = 0 \\
2|C_3| &= 0 \\
C_4 &= \frac{1}{8\pi j}(-1.5 + j1.5\sqrt{3}) \\
2|C_4| &= 2\frac{3}{8\pi}, \theta_4 = \frac{4\pi}{6}
\end{aligned}$$

(c)

$$\begin{aligned}
C_0 &= \frac{1}{2} \\
C_k &= \frac{1}{k^2\pi^2}[e^{-jk\pi}(jk\pi + 1) - 1] \\
C_1 &= \frac{1}{\pi^2}[-2 - j\pi] \\
2|C_1| &= 0.7547, \theta_1 = -0.68\pi \\
C_2 &= \frac{j2\pi}{4\pi^2} = \frac{j}{2\pi} \\
2|C_2| &= 0.3183, \theta_2 = 0.5\pi \\
C_3 &= \frac{1}{9\pi^2}[-2 - 3j\pi] \\
2|C_3| &= 0.2169, \theta_3 = -0.5666\pi \\
C_4 &= \frac{j4\pi}{16\pi^2} = \frac{j}{4\pi} \\
2|C_4| &= 0.1592, \theta_4 = 0.5\pi
\end{aligned}$$

(d) $C_0 = -\frac{1}{2}$

C_k same as in part (c) for $k \neq 0$

Continued→

4.19, continued

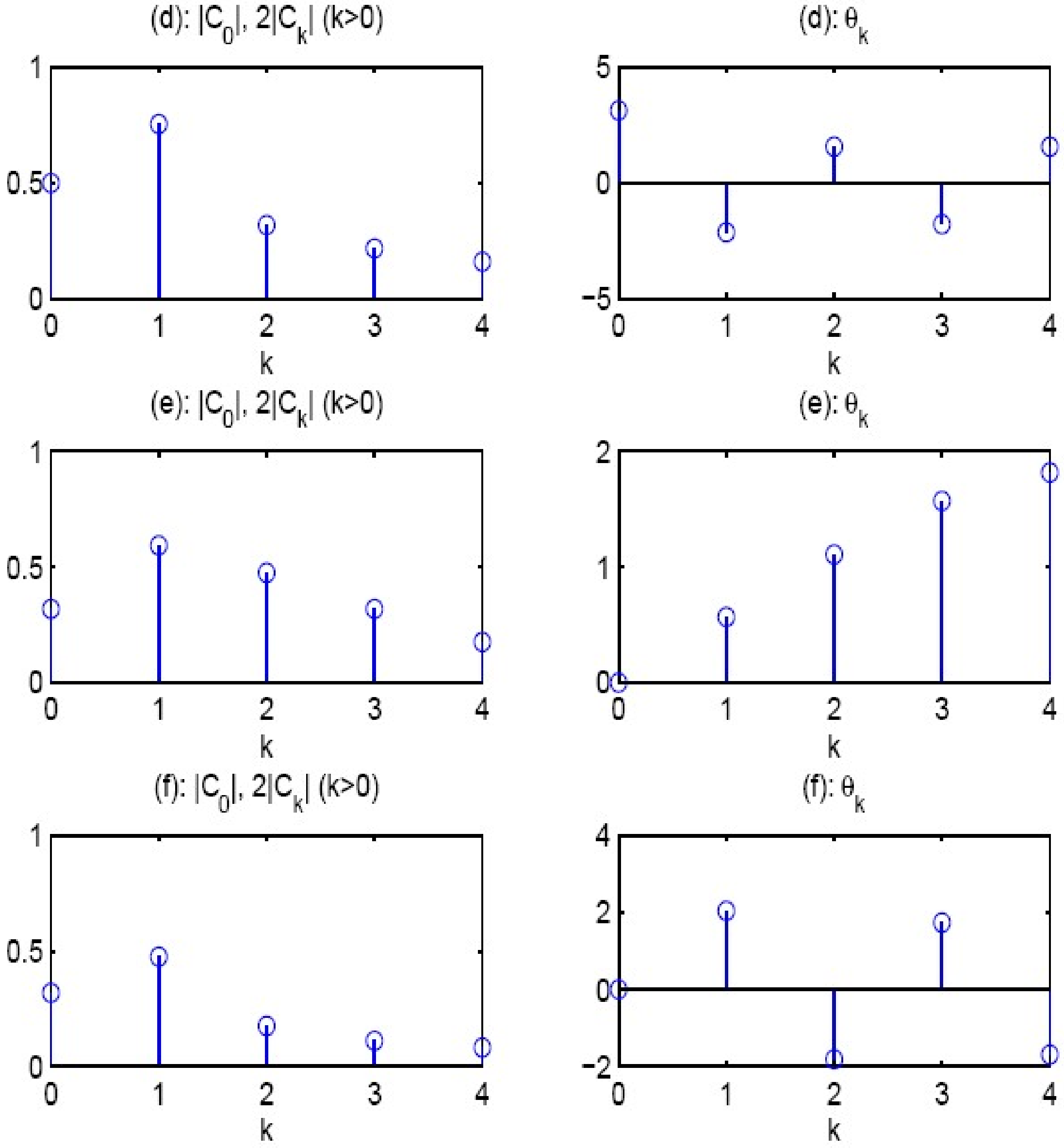


Figure 2: Fourier spectra for parts (d)-(f)

Continued→

4.19 continued

(e)

$$\begin{aligned}
C_0 =& \frac{1}{\pi} \\
C_1 =& \frac{1}{4} + \frac{j}{2\pi} \\
2|C_1| =& 0.5927, \theta_1 = 0.1805\pi \\
C_k =& \frac{1}{\pi(1-k^2)}[e^{j\frac{\pi}{2}k} - jk], k = 2, 3, 4 \\
C_2 =& \frac{1}{3\pi}(1+2j) \\
2|C_2| =& 0.4745, \theta_2 = 0.3524\pi \\
C_3 =& \frac{1}{8\pi}(4j) \\
2|C_3| =& 0.3183, \theta_3 = 0.5\pi \\
C_4 =& \frac{1}{15\pi}(-1+4j) \\
2|C_4| =& 0.1750.\theta_4 = 0.5780\pi
\end{aligned}$$

(f)

$$\begin{aligned}
C_0 =& \frac{1}{\pi} \\
C_k =& \frac{1}{2}\left(\frac{1}{-k^2\pi + \frac{1}{4}\pi}\right)\left[0.5e^{-jk2\pi} + jke^{-jk\pi}\right] \\
C_1 =& -\frac{2}{3\pi}(\frac{1}{2} - j) \\
2|C_1| =& 0.4745, \theta_1 = 0.6476\pi \\
C_2 =& -\frac{1}{7.5\pi}(\frac{1}{2} + 2j) \\
2|C_2| =& 0.1750, \theta_2 = -0.5780\pi \\
C_3 =& -\frac{1}{17.5\pi}(\frac{1}{2} - 3j) \\
2|C_3| =& 0.1106, \theta_3 = -0.5526\pi \\
C_4 =& -\frac{1}{31.5\pi}(\frac{1}{2} + 4j) \\
2|C_4| =& 0.0815, \theta_4 = -0.5396\pi
\end{aligned}$$

4.20

The C_k's were found in problem 4.11.

(a)

$$C_0 = 0$$
$$C_k = \frac{-8}{(\pi k)^2}$$
$$2|C_k| = \frac{16}{(\pi k)^2}, \theta_k = 0$$
$$2|C_1| = 0.8106$$
$$2|C_2| = 0.2026$$
$$2|C_3| = 0.0901$$
$$2|C_4| = 0.0507$$

(b)

$$C_0 = 15$$
$$C_k = 10 sinc(\frac{2\pi k}{3}) e^{-j2k\frac{2\pi}{3}}$$
$$2|C_k| = 20|sinc(\frac{2\pi k}{3})|$$
$$\theta_k = -\frac{4\pi}{3} + \pi, k = 2, 5, 8, 11...$$
$$\theta_k = -\frac{4\pi}{3}, k = 0, 1, 3, 4, 6, 7, 9, ...$$
$$2|C_1| = 8.2699, \theta_1 = 2.0944\ rad$$
$$2|C_2| = 4.1350, \theta_2 = -1.0472\ rad$$
$$2|C_3| = 0$$
$$2|C_4| = 2.0675, \theta_4 = 2.0944\ rad$$

Continued→

4.20, continued

(c)

$$
\begin{aligned}
C_0 &= 0 \\
C_k &= \frac{4j}{\pi k} \\
2|C_k| &= \frac{8}{\pi k}, \theta_k = \frac{\pi}{2} \\
2|C_1| &= 2.5465 \\
2|C_2| &= 1.2732 \\
2|C_3| &= 0.8488 \\
2|C_4| &= 0.6366
\end{aligned}
$$

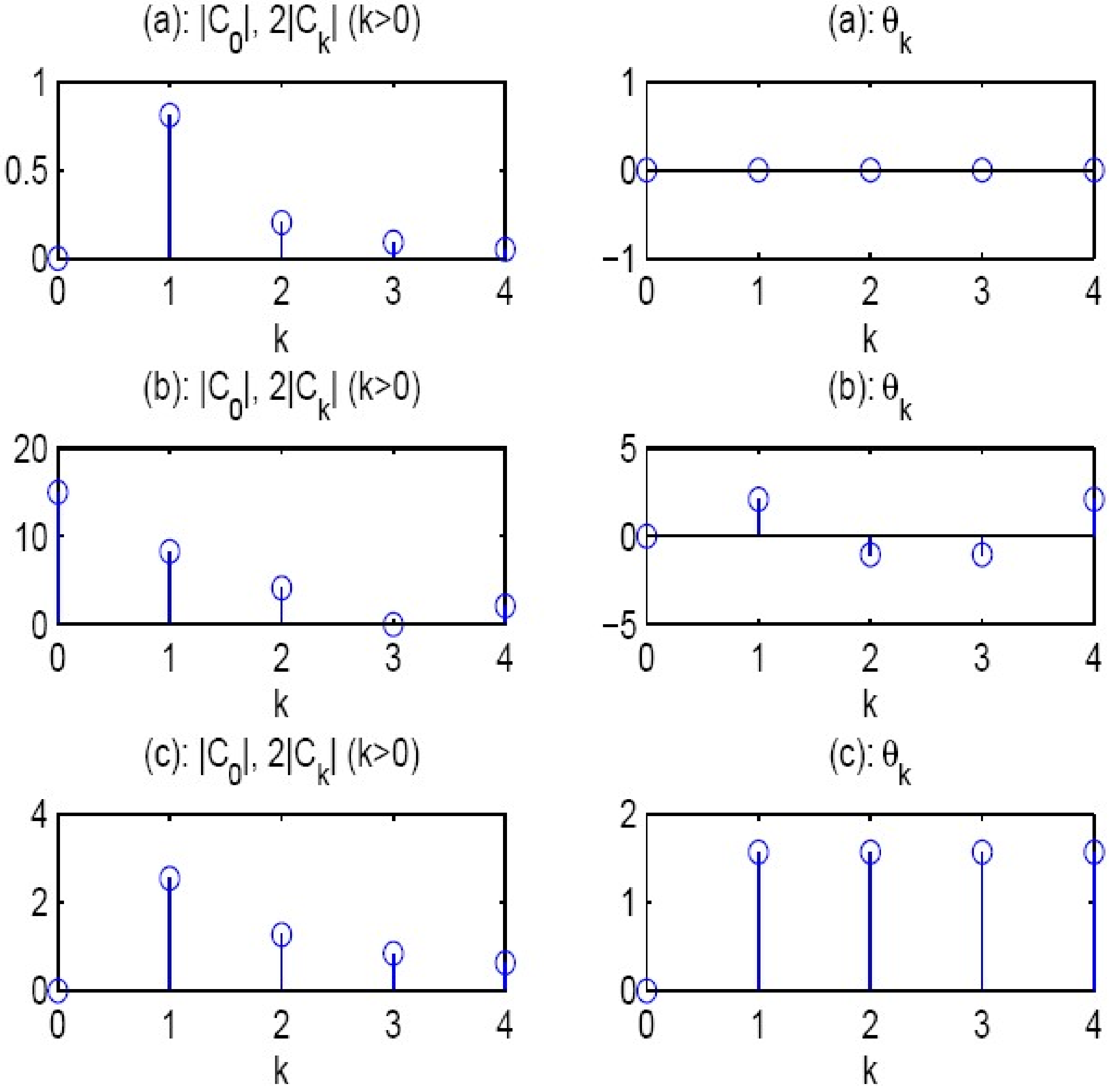


Figure 3: Fourier spectra for 4.20 (a)-(c)

4.21 $\omega_0 = \pi,\ c_0 = 2,\ c_1 = 1,\ c_3 = 1/2\, e^{j\pi/4},\ c_{-3} = 1/2\, e^{-j\pi/4}$

$$x(t) = 2 + e^{j\pi t} + 1/2\, e^{j\pi/4} e^{j3\pi t} + 1/2\, e^{-j\pi/4} e^{-j3\pi t}$$

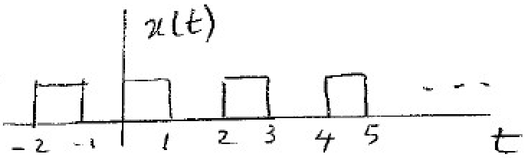
$$= 2 + e^{j\pi t} + \cos(3\pi t + \pi/4)$$

4.22

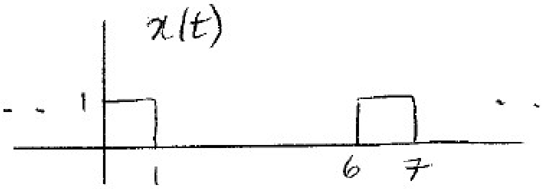
$$c_k = 1/2 \int_0^1 e^{-jk\omega_0 t}\, dt = 1/2\, \frac{1}{-jk\omega_0} e^{-jk\omega_0 t}\Big|_0^1$$

$$= \frac{1}{2jk\omega_0}\left[1 - e^{-jk\omega_0}\right],\ k \neq 0$$

$$c_0 = 1/2 \int_0^1 dt = 1/2$$



4.23



a) $T = 6 \quad f = 1/6$ & $\omega_0 = \frac{2\pi}{T} = \pi/3$

b) $c_0 = \frac{1}{T}\int_T x(t)\, dt = 1/6$

$$c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt = 1/6 \int_0^1 e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{jk\omega_0 T}\left(1 - e^{-jk\omega_0}\right),\ k \neq 0$$

This = 0 when $k \neq 0$, k a multiple of 6

4.24

$H(s) = \frac{10}{s+5}$, $\omega_0 = \frac{2\pi}{3}$, $T_0 = 3$

$H(0) = 10/5 = 2$, $H(j\omega_0) = \frac{10}{5+j2\pi/3} = 1.84\angle -22.7^\circ$

$H(j2\omega_0) = \frac{10}{5+j4\pi/3} = 1.533\angle -40^\circ$

$H(j3\omega_0) = \frac{10}{5+j2\pi} = 1.245\angle -51.5^\circ$

$c_{y_k} = H(jk\omega_0)c_{x_k}$

a) $x(t) = c_{x_0} = 0$, $c_{x_k} = -j\frac{2(20)}{\pi k} = \frac{40}{\pi k}\angle -90^\circ$, k odd

$c_{y_0} = 0$

$c_{y_1} = (1.84\angle -22.7)(12.72\angle -90^\circ) = 23.4\angle -112.7^\circ$

$c_{y_2} = 0$

$c_{y_3} = (1.245\angle -51.5^\circ)(4.24\angle -90^\circ) = 5.28\angle -141.5^\circ$

$y(t) = 46.8\cos(\tfrac{2}{3}\pi t - 112.7^\circ) + 10.56\cos(2\pi t - 141.5^\circ) + \cdots$

b)

```
w=[.66667*pi  2*pi]; n=[0 10]; d=[1 5];
h=freqs(n,d,w);
hmag=abs(h); hphase=angle(h)*180/pi;
[hmag' hphase']
```

c) (a) $C_{x0} = \frac{X_0}{2} = 10$; $C_{xk} = j\frac{20}{2\pi k}$

$C_{y0} = (20)(10) = \underline{20}$

$C_{y1} = (1.84\angle -22.7^\circ)(3.18\angle 90^\circ) = \underline{5.86\angle 67.3^\circ}$

$C_{y2} = (1.53\angle -40^\circ)(1.59\angle 90^\circ) = \underline{2.44\angle 50^\circ}$

$C_{y3} = (1.245\angle -51.5^\circ)(1.061\angle 90^\circ) = 1.32\angle 38.5^\circ$

$y(t) = \underline{20 + 11.72\cos(\tfrac{2}{3}\pi t + 67.3^\circ) + 4.88\cos(\tfrac{4}{3}\pi t + 50^\circ)}$

$\quad \underline{+ 2.64\cos(2\pi t + 38.5^\circ) + \cdots}$

(d) (a) $C_{x0} = 10$; $C_{xk} = -\frac{40}{\pi^2 k^2}$, k odd

$C_{y0} = 2(10) = \underline{20}$

$C_{y1} = (1.84\angle -22.7^\circ)(4.05\angle 180^\circ) = \underline{7.46\angle 157.3^\circ}$; $C_{y2} = \underline{0}$

$C_{y3} = (1.245\angle -51.5^\circ)(0.450\angle 180^\circ) = \underline{0.561\angle 128.5^\circ}$

$y(t) = \underline{20 + 14.92\cos(\tfrac{2}{3}\pi t + 157.3^\circ) + 1.122\cos(2\pi t + 128.5^\circ) + \cdots}$

(e) $C_{x0} = \frac{2(20)}{\pi} = \underline{12.73}$, $C_{xk} = \frac{-2X_0}{\pi(4k^2-1)} = -\frac{40}{\pi(4k^2-1)}$

$C_{y0} = (2)(12.73) = \underline{25.46}$

$C_{y1} = (1.84\angle -22.7^\circ)(4.244\angle 180^\circ) = \underline{7.81\angle 157.3^\circ}$

$C_{y2} = (1.53\angle -40^\circ)(0.849\angle 180^\circ) = \underline{1.30\angle 140^\circ}$

$C_{y3} = (1.245\angle -51.5^\circ)(0.364\angle 180^\circ) = \underline{0.453\angle 128.5^\circ}$

$y(t) = \underline{25.46 + 15.62\cos(\tfrac{2}{3}\pi t + 157.3^\circ) + 2.60\cos(\tfrac{4}{3}\pi t + 140^\circ)}$

$\quad \underline{+ 0.906\cos(2\pi t + 128.5^\circ) + \cdots}$

(f) $C_{x0} = 20/\pi = 6.367$; $C_{x1} = -j\frac{X_0}{4}$, $C_{x2} = -\frac{X_0}{3\pi}$, $C_{x3} = 0$

$C_{y0} = (2)(6.367) = \underline{12.73}$

$C_{y1} = (1.84\angle -22.7^\circ)(5\angle -90^\circ) = \underline{9.20\angle -112.7^\circ}$

$C_{y2} = (1.53\angle -40^\circ)(2.122\angle 180^\circ) = \underline{3.25\angle 140^\circ}$, $C_{y3} = \underline{0}$

$y(t) = \underline{12.73 + 18.4\cos(\tfrac{2}{3}t - 112.7^\circ) + 3.25\cos(\tfrac{4}{3}t + 140^\circ) + \cdots}$

(g) $C_{x0} = \frac{wX_0}{T_0} = \frac{(1)(20)}{3} = 6.67$; $C_k = \frac{wX_0}{T_0}\,\frac{\sin(\pi wk/T_0)}{\pi wk/T_0} = \frac{20}{\pi k}\sin k\pi/3$

$C_{y0} = (2)(6.67) = \underline{13.33}$

$C_{y1} = (1.84\angle -22.7^\circ)(5.51) = \underline{10.14\angle -22.7^\circ}$

$C_{y2} = (1.53\angle -40^\circ)(2.757) = \underline{4.22\angle -40^\circ}$; $C_{y3} = \underline{0}$

4. (cont) 24. $y(t) = \underline{13.3 + 20.28\cos(\frac{2}{3}\pi t - 22.7^\circ) + 8.44\cos(\frac{4}{3}\pi t - 40^\circ)}$

(h) $C_k = \frac{20}{3} = 6.67$

$C_{y0} = (2)(6.67) = \underline{13.33}$

$C_{y1} = (1.84\angle{-22.7^\circ})(6.67) = \underline{12.3\angle{-22.7^\circ}}$

$C_{y2} = (1.53\angle{-40^\circ})(6.67) = \underline{10.2\angle{-40^\circ}}$

$C_{y3} = (1.245\angle{-51.5^\circ})(6.67) = \underline{8.30\angle{-51.5^\circ}}$

$\therefore y(t) = \underline{13.33 + 24.6\cos(\frac{2}{3}\pi t - 22.7^\circ) + 20.4\cos(\frac{4}{3}\pi t - 40^\circ)}$
$\underline{+ 16.6\cos(2\pi t - 51.5^\circ)}$

4.25

(a) $T_0 = 1$, $\omega_0 = 2\pi$
$\frac{C_{1y}}{C_{1x}} = H(j1\omega_0) = H(j2\pi) = \frac{20}{j2\pi+4}$
$|\frac{C_{1y}}{C_{1x}}| = \frac{20}{\sqrt{4\pi^2+16}} = 2.6851$

(b)
$C_{1y} = H(j2\pi)C_{1x} = \frac{20}{4\pi+16}C_{1x}$
$C_{3y} = H(j6\pi)C_{3x} = \frac{20}{6\pi+16}C_{3x}$
$\frac{|C_{1y}|}{|C_{3y}|} = \frac{\sqrt{36\pi^2+16}}{\sqrt{4\pi^2+16}}\frac{|C_{1x}|}{|C_{3x}|}$

Note that from Table 4.3, $\frac{|C_{1x}|}{|C_{3x}|} = \frac{2X_0}{\pi}\frac{\pi 3}{2X_0} = 3$, so:
$\frac{|C_{1y}|}{|C_{3y}|} = 3\sqrt{\frac{36\pi^2+16}{4\pi^2+16}} = 7.7611$

(c)

```
>>omega_0=2*pi;
>>w=[omega_0*1, omega_0*3];
>>n=[0,20];
>>d=[1,4];
>>h=freqs(n,d,w); >>hmag=abs(h)
hmag=
2.6851    1.0379
>>3*hmag(1)/hmag(2)
ans=
7.7611
```

Continued→

4.25, continued

(d)

$\omega_0 = 20\pi,\ H(j\omega_0) = \frac{20}{4+j20\pi}$

$\frac{|C_{1y}|}{|C_{1x}|} = |H(j\omega_0)| = \frac{20}{\sqrt{16+400\pi^2}} = 0.318$

Same MATLAB as part (c) with `omega_0=20*`

(e)

$H(j3\omega_0) = \frac{20}{4+j60\pi}$

$3\frac{|C_{1y}|}{|C_{3y}|} = 3\sqrt{\frac{16+(60\pi)^2}{16+(20\pi)^2}} = 8.98$

(f)

$\omega_0 = 0.2\pi$

$H(j\omega_0) = \frac{20}{4+j0.2\pi}$

$\frac{|C_{1y}|}{|C_{1x}|} = |H(j\omega_0)| = \frac{20}{\sqrt{16+0.04\pi^2}} = 4.94$

(g)

$H(j3\omega_0) = \frac{20}{4+j0.6\pi}$

$\frac{|C_{1y}|}{|C_{3y}|} = 3\sqrt{\frac{16+(0.6\pi)^2}{16+(0.2\pi)^2}} = 3.28$

(h)

$\omega_0 = 0.2\pi$, ratio=4.94

$\omega_0 = 2\pi$, ratio=2.69

$\omega_0 = 20\pi$, ratio=0.318

The system is a low pass filter with a DC gain of 20/4=5. Most of the input at $\omega_0 = 0.2\pi$ gets through but most of the input at $\omega_0 = 20\pi$ gets filtered out.

(i)

$\omega_0 = 0.2\pi$, ratio=3.28

$\omega_0 = 2\pi$, ratio=7.76

$\omega_0 = 20\pi$, ratio=8.98

The ratio of harmonics of the input is 3, so this shows there is little effect at $\omega_0 = 2\pi$ but large effect at $\omega_0 = 20\pi$: most of the input in this case is filtered out.

4.26 $H(s) = \frac{1}{RCs+1} = \frac{1}{0.5s+1} = \frac{2}{s+2}$

(a) $\omega_0 = 1$, $H(j\omega_0) = \frac{2}{2+j1} = \frac{2}{2.236\angle 26.6^\circ} = 0.8944\angle -26.6^\circ$

$H(j3\omega_0) = \frac{2}{2+j3} = \frac{2}{3.606\angle 56.3^\circ} = 0.5547\angle -56.3^\circ$

$H(j5\omega_0) = \frac{2}{2+j5} = \frac{2}{5.385\angle 68.2^\circ} = 0.3714\angle -68.2^\circ$

$C_{kx} = -j\frac{20}{k\pi}$

$\therefore C_{1x} = -j\frac{20}{\pi}$; $C_{y1} = (0.8944\angle -26.6^\circ)(6.3662\angle -90^\circ) = 5.6939\angle -116.6^\circ$

$C_{3x} = -j\frac{20}{3\pi}$; $C_{3y} = (0.5547\angle -56.3^\circ)(2.1221\angle -90^\circ) = 1.1771\angle -146.3^\circ$

$C_{x5} = -j\frac{20}{5\pi}$; $C_{5y} = (0.3714\angle -68.2^\circ)(1.2732\angle -90^\circ) = 0.4729\angle -158.2^\circ$

$\therefore y_a(t) = 11.38\cos(t - 116.6^\circ) + 2.35\cos(3t - 146.3^\circ) + 0.95\cos(5t - 158.^\circ) + \cdots$

(b)

```
w=[1 3 5]; n=[0 2]; d=[1 2];
h=freqs(n,d,w);
hmag=abs(h); hphase=angle(h)*180/pi;
[hmag' hphase']
```

(c) $H(0) = 1 \therefore C_{y0} = H(0)C_{x0} = (1)(20) = 20$

$y_b(t) = \underline{20 + y_a(t)}$, $y_a(t)$ from (a)

(d) Yes, $|H(jk\omega_0)|$ decreases as k increases.

(e) $T_0 = \pi$, $\omega_0 = \frac{2\pi}{T_0} = 2$

(a) Since ω_0 is larger, the gain of the circuit is smaller. Hence the amplitude of the harmonics are smaller.

(c) The dc gain is unaffected. Hence the dc component in the output is unchanged.

4.27

$H(s) = \frac{Ls}{R+Ls} = \frac{s}{8+s}$

(a)

$$\begin{aligned}
C_{ky} =& H(jk\omega_0)C_{kx} = \frac{jk\omega_0}{8+jk\omega_0} \\
\omega_0 =& \frac{2\pi}{\pi} = 2 \\
C_{kx} =& \frac{-j2(10)}{\pi k} \\
C_{ky} =& \frac{-j20}{\pi k}\frac{j2k}{8+j2k} = \frac{20}{4\pi + j\pi k} \\
|C_{ky}| =& \frac{20}{\sqrt{16\pi^2 + k^2\pi^2}} \\
\theta_{ky} =& -\tan^{-1}(\frac{k}{4}) \\
|C_0| =& 0 \\
|C_1| =& 1.5440, \theta_1 = -0.2450 rad \\
|C_2| =& 1.4235, \theta_2 = -0.4636 rad \\
|C_3| =& 1.2732, \theta_3 = -0.6435 rad
\end{aligned}$$

(b)

```
>>w=[0,2,4,6];
>>n=[1,0];
>>d=[1,8];
>>h=freqs(n,d,w);
>>k=1:3;
>>Ckx=-j*20./(pi * k);
>>Ckx=[0,Ckx];
>>Cky=Ckx.*h;
>>magCky=abs(Cky)
magCky=0    1.5440    1.4235    1.2732
>>phCky=angle(Cky)
phCky=0    -0.2450    -0.4636    -0.6435
```

Continued→

4.27, continued

(c) This changes only the value of C_{0x} and therefore only the DC value C_{0y} of the output might change—however, since $H(0) = 0$ in this case, the DC value of the output does not actually change.

$C_{0x} = 20 \implies C_{0y} = 20H(0) = 0$

(d)No. The low frequencies get decreased in amplitude–in fact the DC component does not get through at all.

(e) This is a lower frequency square wave and so more of its energy will be attenuated by the filter. It will not change part c—the DC output is still 0.

4.28

$$y(t) = x(\tau)\Big|_{\tau = at+b} = x(at+b)$$

$$\therefore C_{kx} e^{jk\omega_0 \tau}\Big|_{\tau = at+b} = C_{kx} e^{jk\omega_0 (at+b)} = \left[C_{kx} e^{jk\omega_0 b}\right] e^{jk\omega_0 at}$$

$$\therefore \omega_{0y} = \frac{2\pi}{T_{0y}} = |a|\,\omega_{0x} = |a|\frac{2\pi}{T_{0x}}$$

$$\therefore \underline{T_{0y} = \frac{T_{0x}}{|a|}}$$ [a can be negative]

for a negative,

$$\therefore C_{kx} e^{jk\omega_0 \tau} \Rightarrow \left[C_{kx} e^{jk\omega_0 b}\right] e^{-jk|a|\omega_0 t}$$

Since $C_{-k} = C_k^*$

$$C_{ky} = \left[C_{kx} e^{jk\omega_0 b}\right]^*, \quad a<0$$

$$\therefore C_{ky} = \begin{cases} C_{kx} e^{jk\omega_0 b}, & a>0 \\ \left[C_{kx} e^{jk\omega_0 b}\right]^*, & a<0 \end{cases}$$

4.29. (a) $C_k = \frac{-2X_0}{\pi(4k^2-1)} = C_{-k}$

$$\therefore \left[C_k e^{jk\omega_0 t} + C_{-k} e^{-jk\omega_0 t}\right]_{t=-t} = C_{-k} e^{-jk\omega_0 t} + C_k e^{jk\omega_0 t}$$

$\therefore$ no change

(b) $y(t) = x(t - \frac{T_0}{2})$: $x(t) = \sum_k C_{kx} e^{jk\omega_0 t}$

$y(t) = \sum_k C_{kx} e^{jk\omega_0 (t - \frac{T_0}{2})} = \sum_k C_{kx} e^{-jk\omega_0 \frac{T_0}{2}} e^{jk\omega_0 t}$

Note $\omega_0 T_0 = 2\pi \implies$

$$C_{yk} = C_{xk} e^{-jk\omega_0 \frac{T_0}{2}} = C_{xk} e^{-jk\pi}$$

4.30 $h(t) = e^{-\alpha t} u(t)$

a) $\alpha > 0$

b) $x(t) = \sin(\omega_0 t) + \cos(3\omega_0 t) =$

$$\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) + \frac{1}{2}\left(e^{j3\omega_0 t} + e^{-j3\omega_0 t}\right)$$

$$H(S_k) = \int_{-\infty}^{\infty} h(\tau) e^{-S_k \tau} d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau} u(\tau) e^{-S_k \tau} d\tau$$

$$= \int_0^{\infty} e^{-(\alpha + S_k)\tau} d\tau = \frac{1}{\alpha + SK}$$

$$\phi_k(t) = e^{jK\omega_0 t} \quad \left(\psi_k(t) = \phi_k(t) * h(t)\right)$$

$$y(t) = \sum_k a_k H(S_k) e^{S_k t}$$

$$x(t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} + \frac{1}{2} e^{j3\omega_0 t} + \frac{1}{2} e^{-j3\omega_0 t}$$

$K=1$ $\quad K=-1$ $\quad K=3$ $\quad K=-3$

$$\therefore y(t) = \frac{1}{2j} \frac{1}{\alpha + j\omega_0} e^{j\omega_0 t} - \frac{1}{2j} \frac{1}{\alpha - j\omega_0} e^{-j\omega_0 t}$$

$$+ \frac{1}{2} e^{j3\omega_0 t} \frac{1}{\alpha + 3j\omega_0} + \frac{1}{2} e^{-j3\omega_0 t} \frac{1}{\alpha - 3j\omega_0}$$

4.31 $h(t) = \alpha e^{-\alpha t} u(t)$, $\alpha > 0$

a) $x(t) = \sin^2 2t = \frac{1}{2}(1 - \cos(4\omega_0 t))$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\left(e^{j4\omega_0 t} + e^{-j4\omega_0 t}\right)\right)$$

$$H(s_k) = \int_{-\infty}^{\infty} h(\tau) e^{-s_k \tau} d\tau = \int_{-\infty}^{\infty} \alpha e^{-\alpha\tau} u(\tau) e^{-s_k \tau} d\tau$$

$$= \int_{0}^{\infty} \alpha e^{-(\alpha + s_k)\tau} d\tau = \frac{\alpha}{\alpha + s_k}$$

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

$$x(t) = \frac{1}{2} - \frac{1}{4} e^{j4\omega_0 t} - \frac{1}{4} e^{-j4\omega_0 t}$$

$k = 0$ $\quad$ $k = 1$ $\quad$ $k = -1$

$$\therefore y(t) = \frac{1}{2} - \frac{1}{4} \frac{\alpha}{\alpha + j\omega_0} e^{j4t} - \frac{1}{4} \frac{\alpha}{\alpha - j\omega_0} e^{-j4t}$$

b) $x(t) = 1 + \cos t + \cos 8t$

$$= 1 + \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + \frac{1}{2}\left(e^{j8\omega_0 t} + e^{-j8\omega_0 t}\right)$$

$$y(t) = 1 + \frac{1}{2} \frac{\alpha}{\alpha + j\omega_0} e^{jt} + \frac{1}{2} \frac{\alpha}{\alpha - j\omega_0} e^{-jt} +$$

$$\frac{1}{2} \frac{\alpha}{\alpha + j8\omega_0} e^{j8t} + \frac{1}{2} \frac{\alpha}{\alpha - j8\omega_0} e^{-j8t}$$

4.32

$$x(t) = \sum_{k=1}^{\infty} \cos(kt) = 1/2 \sum_{k=-\infty}^{\infty} e^{jkt} - 1/2$$

$$H(jk) = \int_{0}^{\infty} e^{-at} e^{-jkt} dt = \frac{1}{a+jk}$$

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk) e^{jkt}$$

$$= \sum_{k=-\infty}^{\infty} 1/2 \frac{1}{a+jk} e^{jkt} - \frac{1}{(2a)}$$

Chapter 5 solutions

5.1

(a)

$$\begin{aligned} X(\omega) = \int_0^6 e^{-j\omega t} dt = & \frac{1}{-j\omega}\left(e^{-j\omega 6} - 1\right) \\ =& \frac{e^{-j\omega 3}}{j\omega}\left(e^{j\omega 3} - e^{-j\omega 3}\right) \\ =& \frac{e^{-j\omega 3}}{\omega} 2\sin(3\omega) \\ =& 6e^{-j\omega 3} sinc(3\omega) \end{aligned}$$

(b)

$$X(\omega) = \int_0^6 e^{-2t} e^{-j\omega t} dt = \frac{1}{2+j\omega}\left(1 - e^{-(2+j\omega)6}\right)$$

(c)

$$\begin{aligned} X(\omega) = \int_0^6 t e^{-j\omega t} dt = & \left[\frac{e^{-j\omega t}}{(-j\omega)^2}(-j\omega t - 1)\right]_0^6 \\ =& \frac{e^{-j\omega 6}}{-\omega^2}(-j\omega 6 - 1) + \frac{1}{\omega^2}(-1) \\ =& \frac{j6e^{-j\omega 6}}{\omega} + \frac{e^{-j\omega 6} - 1}{\omega^2} \\ =& \frac{j6e^{-j\omega 6}}{\omega} - \frac{2je^{-j\omega 3}}{\omega^2}\sin(\omega 3) \end{aligned}$$

(d)

$$\begin{aligned} X(\omega) = \int_{-3}^3 2\cos(9\pi t) e^{-j\omega t} dt = & \left[2\frac{e^{-j\omega t}}{(-j\omega)^2 + (9\pi)^2}\left(-j\omega\cos(9\pi t) + 9\pi\sin(9\pi t)\right)\right]_{-3}^3 \\ =& 2\left(\frac{e^{-j\omega 3}}{-\omega^2 + (9\pi)^2} j\omega - \frac{e^{j\omega 3}}{-\omega^2 + (9\pi)^2} j\omega\right) \\ =& \frac{2j\omega}{(9\pi)^2 - \omega^2}\left(e^{-j\omega 3} - e^{j\omega 3}\right) \\ =& \frac{4\omega}{(9\pi)^2 - \omega^2}\sin(3\omega) \end{aligned}$$

Note this is also equal to $3\left[sinc\left(3(\omega - 9\pi)\right) + sinc\left(3(\omega + 9\pi)\right)\right]$ (see 5.3 (d)).

5.2

(a)

$$F(\omega) = \int_0^{t_0} k e^{-bt} e^{-j\omega t} dt$$
$$k\left(\frac{1-e^{-(b+j\omega)t_0}}{b+j\omega}\right)$$

b) $f(t) = A\cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\omega_0 t} e^{j\phi} + \frac{A}{2} e^{-j\omega_0 t} e^{-j\phi}$

$$F(\omega) = \frac{Ae^{j\phi}}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt + \frac{Ae^{-j\phi}}{2}\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t} dt$$

aside : $\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$

$\Longrightarrow \mathcal{F}\left\{e^{j\omega_0 t}\right\} = 2\pi\delta(\omega-\omega_0)$

Similarly, $\mathcal{F}\left\{e^{-j\omega_0 t}\right\} = 2\pi\delta(\omega+\omega_0)$

Final answer :

$F(\omega) = A\pi e^{j\phi}\delta(\omega-\omega_0) + A\pi e^{-j\phi}\delta(\omega+\omega_0)$

Continued→

5.2, continued

(c)

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} e^{at}u(-t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{0} e^{a-j\omega}tdt \\ &= \frac{1}{a-j\omega} \end{aligned}$$

d) $F(\omega) = \int_{-\infty}^{\infty} C\delta(t+t_0)e^{-j\omega t}dt = Ce^{-j\omega(-t_0)} = Ce^{j\omega t_0}$

5.3

(a)

$$\begin{aligned} X(\omega) =& \mathcal{F}(u(t)) - \mathcal{F}(u(t-6)) \\ =& \pi\delta(\omega) + \frac{1}{j\omega} - (\pi\delta(\omega) + \frac{1}{j\omega})e^{-j6\omega} \end{aligned}$$

using Table 5.2 for $\mathcal{F}(u(t))$ and Table 5.1 (Time shift) to derive $\mathcal{F}(u(t-6))$. Noting that $\delta(\omega)e^{-j6\omega} = \delta(\omega)$ results in:

$$\begin{aligned} X(\omega) =& \frac{1}{j\omega}(1 - e^{-j6\omega}) \\ =& \frac{e^{-j\omega 3}}{j\omega}(e^{j\omega 3} - e^{-j\omega 3}) \\ =& \frac{e^{-j\omega 3}}{\omega} 2\sin(3\omega) = 6e^{-j\omega 3}sinc(3\omega) \end{aligned}$$

(b) Using the fact that:

$$e^{-2t}u(t) - e^{-2t}u(t-6) = e^{-2t}u(t) - e^{-12}e^{-2(t-6)}u(t-6)$$

and Table 5.2 for $\mathcal{F}(e^{-2t}u(t))$ and Table 5.2 for linearity and time shifting property results in:

$$\begin{aligned} \mathcal{F}(e^{-2(t-6)}u(t-6)) =& \frac{1}{2+j\omega}e^{-j6\omega} \\ X(\omega) =& \frac{1}{2+j\omega} - e^{-12}\frac{1}{2+j\omega}e^{-6j\omega} \\ =& \frac{1}{2+j\omega}\left(1 - e^{-6(2+j\omega)}\right) \end{aligned}$$

Continued→

5.3, continued

(c) From part (a), $u(t)-u(t-6) \leftrightarrow 6e^{-j\omega 3}sinc(3\omega)$.
Using the integration property in Table 5.1:

$$\begin{aligned}
t[u(t)-u(t-6)] &= \int_{-\infty}^{t}[u(\tau)-u(\tau-6)]d\tau - 6u(t-6) \\
&\leftrightarrow \frac{1}{j\omega}6e^{-j\omega 3}sinc(3\omega) + \pi 6\delta(\omega) - 6\left(\pi\delta(\omega)+\frac{1}{j\omega}\right)e^{-j\omega 6} \\
&= \frac{6}{j\omega}\left(e^{-j\omega 3}sinc(3\omega) - e^{-j\omega 6}\right) \\
&= \frac{j6e^{-j\omega 6}}{\omega} - \frac{2je^{-j\omega 3}}{\omega^2}\sin(\omega 3)
\end{aligned}$$

(d)

$$\begin{aligned}
\cos(9\pi t) &\leftrightarrow \pi(\delta(\omega-9\pi)+\delta(\omega+9\pi)) \\
u(t+3)-u(t-3) &\leftrightarrow 6sinc(3\omega)
\end{aligned}$$

Using multiplication/convolution property:

$$2\cos(9\pi t)[u(t+3)-u(t-3)] \leftrightarrow 2\left[\delta(\omega-9\pi)+\delta(\omega+9\pi)\right] * 3sinc(3\omega)$$

$$\begin{aligned}
X(\omega) &= 6\left[sinc(3(\omega-9\pi)) + sinc(3(\omega+9\pi))\right] \\
&= 6\left[\frac{\sin(3\omega-27\pi)}{3(\omega-9\pi)} + \frac{\sin(3\omega+27\pi)}{3(\omega+9\pi)}\right] \\
&= -2\sin(3\omega)\left[\frac{1}{\omega-9\pi}+\frac{1}{\omega+9\pi}\right] \\
&= 4\omega\frac{\sin(3\omega)}{(9\pi)^2-\omega^2}
\end{aligned}$$

5.4

a) $\mathcal{F}\left[af_1(t)+bf_2(t)\right] = \int_{-\infty}^{\infty}\left[af_1(t)+bf_2(t)\right]e^{-j\omega t}\,dt =$

$a\int_{-\infty}^{\infty} f_1(t)e^{-j\omega t} + b\int_{-\infty}^{\infty} f_2(t)e^{-j\omega t}\,dt = aF_1(\omega)+bF_2(\omega)$

b) time Shift

$\int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}\,dt$ let $u=t-t_0$

$= \int_{-\infty}^{\infty} f(u)e^{-j\omega(u+t_0)}\,du = e^{-j\omega t_0}\int_{-\infty}^{\infty} f(u)e^{-j\omega u}\,du$

$= F(\omega)e^{-j\omega t_0}$

c) Duality

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{+j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(a)e^{jat}\,da$

$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(a)e^{-ja\omega}\,da$, $2\pi f(-\omega) = \int_{-\infty}^{\infty} F(a)e^{-ja\omega}$

d) Frequency Shifting

$\int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}\,dt = F(\omega-\omega_0)$

e) Time Differentiation

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$

$\frac{d}{dt}f(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)j\omega e^{j\omega t}\,d\omega$

$\therefore \frac{d}{dt}f(t) \longleftrightarrow j\omega F(\omega)$

Continued→

5.4, continued

f) time Convolution

$$\int_{-\infty}^{\infty} x(t)*h(t)e^{-j\omega t}\,dt = \int\limits_{\substack{-\infty\\ dt}}^{\infty}\int\limits_{\substack{-\infty\\ d\tau}}^{\infty} x(\tau)h(t-\tau)\,d\tau\, e^{-j\omega t}\,dt$$

$$= \int\limits_{\substack{-\infty\\ dt}}^{\infty} x(\tau)\int\limits_{\substack{-\infty\\ d\tau}}^{\infty} h(t-\tau)e^{-j\omega t}\,dt\,d\tau \qquad \text{let } u = t-\tau$$

$$= \int\limits_{\substack{-\infty\\ d\tau}}^{\infty} x(\tau)\int\limits_{\substack{-\infty\\ du}}^{\infty} h(u)e^{-j\omega(u+\tau)}\,du\,d\tau =$$

$$\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau \int_{-\infty}^{\infty} h(u)e^{-j\omega u}\,du$$

$$= X(\omega)H(\omega)$$

g) prove the time scale property

$$\mathcal{F}[X(at)] = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt \qquad \text{let } u = at$$

$$du = a\,dt$$

$$= \int_{-\infty}^{\infty} x(u)e^{-j\omega\frac{u}{a}}\frac{du}{a}, \text{ if } a>0$$

$$= \frac{1}{a}X(\omega/a)$$

if $a<0$, then

$$= \int_{+\infty}^{-\infty} x(u)e^{-j\omega\frac{u}{a}}\frac{du}{a} = -\int_{-\infty}^{\infty} x(u)e^{\frac{-j\omega u}{a}}\frac{du}{a}$$

$$= \frac{-1}{a}X(\omega/a)$$

$$\therefore\ \mathcal{F}[x(at)] = \frac{1}{|a|}X(\omega/a)$$

Continued→

5.4, continued

(h) Time-multiplication property: want to show $f(t)g(t) \leftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega)$.

$$
\begin{aligned}
F(\omega) * G(\omega) = & \int_{-\infty}^{\infty} F(u)G(\omega - u)du \\
\mathcal{F}^{-1}[F(\omega) * G(\omega)] = & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} F(u)G(\omega - u)du\right]d\omega \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\left[\int_{-\infty}^{\infty} e^{-j\omega t}G(\omega - u)d\omega\right]du \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\left[\int_{-\infty}^{\infty} e^{-j(\omega+u)}G(\omega)d\omega\right]du \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-jut}du\int_{-\infty}^{\infty} G(\omega)e^{-j\omega t}d\omega \\
= & 2\pi\mathcal{F}^{-1}[F(\omega)]\cdot\mathcal{F}^{-1}[G(\omega)] \\
= & 2\pi f(t)g(t)
\end{aligned}
$$

5.5 $\mathcal{F}[\sin \omega_0 t] = \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

ⓐ Differentiation Property

$\frac{d}{dt} f(t) \longleftrightarrow j\omega F(\omega)$

$d/dt\,[\sin \omega_0 t] = \omega_0 \cos \omega_0 t$

$\omega_0 \cos \omega_0 t \longleftrightarrow \omega_0 \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

Show this is equal to $j\omega \mathcal{F}[\sin \omega_0 t] = \frac{j\omega\pi}{j}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$

$= \pi\omega[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

$= \pi\omega_0(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$, by Shifting property

Continued→

5.5, continued

ⓑ time shift property

$$\sin \omega_0 t = \cos(\omega_0 t - \pi/2) = \cos \omega_0 (t - \pi/2\omega_0)$$

$$f(t - t_0) \longleftrightarrow F(\omega) e^{-j\omega t_0}$$

$$\cos \omega_0 \left(t - \frac{\pi}{2\omega_0}\right) \longleftrightarrow \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]\, e^{\frac{-j\omega\pi}{2\omega_0}}$$

$$= \pi\delta(\omega+\omega_0) e^{\frac{j\omega_0\pi}{2\omega_0}} + \pi\delta(\omega-\omega_0) e^{\frac{-j\omega_0\pi}{2\omega_0}}$$

$$= \pi\,\delta(\omega+\omega_0) e^{j\pi/2} + \pi\delta(\omega-\omega_0) e^{-j\pi/2}$$

$$= \pi j\,\delta(\omega+\omega_0) - j\pi\,\delta(\omega-\omega_0) = \frac{\pi}{j}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

5.6 on next page

5.6 (a) $f(t) = Ae^{-\beta t}\cos(\omega_0 t)\,u(t) = f_1(t)\,f_2(t)$

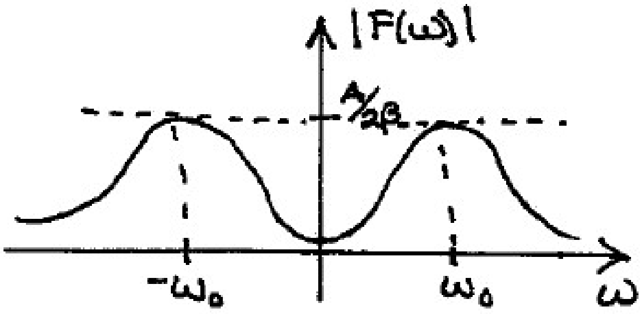
$f_1(t) = Ae^{-\beta t}u(t)$, $f_2(t) = \cos\omega_0 t$

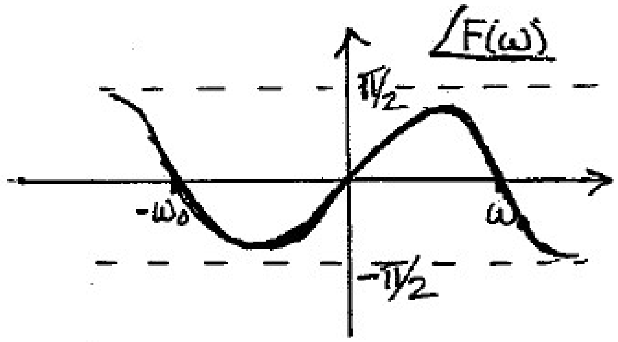
$F_1(\omega) = \frac{A}{\beta + j\omega}$, $F_2(\omega) = \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

use frequency convolution

$$F(\omega) = \frac{1}{2\pi}F_1(\omega) * F_2(\omega) = \frac{1}{2}\left(\frac{A}{\beta+j\omega}\right) * [\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

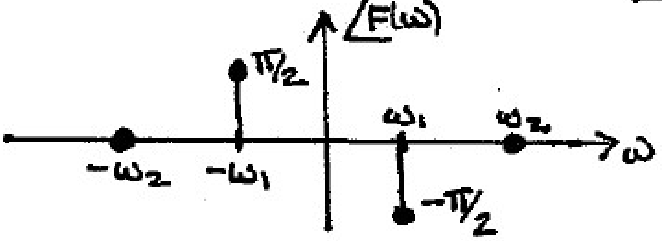
$$= \frac{A/2}{\beta + j(\omega-\omega_0)} + \frac{A/2}{\beta - j(\omega+\omega_0)}$$





(b) $f(t) = A\sin(\omega_1 t) + B\cos(\omega_2 t)$ ⇒ use the linearity Property: $F(\omega) = A\,\mathcal{F}\{\sin(\omega_1 t)\} + B\,\mathcal{F}\{\cos(\omega_2 t)\}$

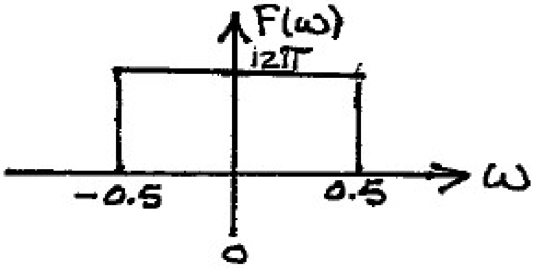
$$F(\omega) = \frac{A\pi}{j}[\delta(\omega-\omega_1)-\delta(\omega+\omega_1)] + B\pi[\delta(\omega-\omega_2)+\delta(\omega+\omega_2)]$$



(c) $f(t) = 6\,\mathrm{sinc}(0.5t)$, from Table 5.2 $\frac{\beta}{\pi}\mathrm{sinc}(\beta t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2\beta}\right)$

$\beta = 0.5 \quad \therefore\ 6 = 12\beta$

$F(\omega) = 12\pi\,\mathrm{rect}(\omega)$

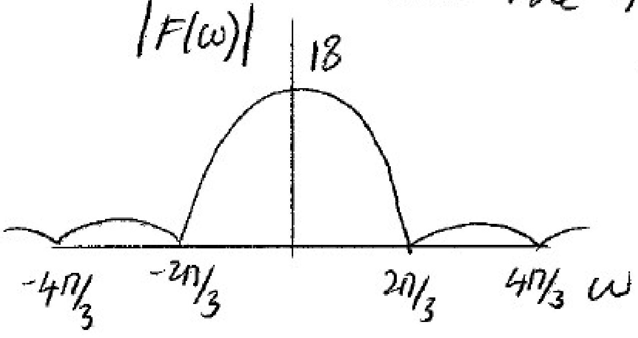


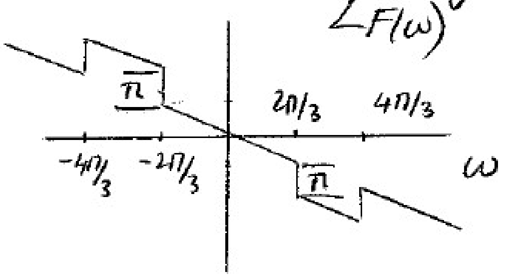
Continued→

5.6, continued

d) $f(t) = 6\,\text{rect}\left[\frac{(t-4)}{3}\right] \xleftrightarrow{\mathcal{F}} 18\,\text{sinc}\left(\frac{3\omega}{2}\right)e^{-j4\omega}$

use the time shift property





(e) From Table 5.2, $tri(t/T) \leftrightarrow Tsinc^2(T\omega/2)$, so using linearity and time shift properties:

$$4tri(\frac{t-4}{4}) \leftrightarrow 16sinc^2(2\omega)e^{-j4\omega}$$

See figure below for spectra plots.

(f) From Table 5.3, $sinc^2(Tt/2) \leftrightarrow \frac{2\pi}{T}tri(\omega/T)$, where here $T = 1/2$:

$$4sinc^2(t/4) \leftrightarrow 16\pi\ tri(2\omega)$$

See figure below for spectra plots.

(g)

$$\begin{aligned}
10\cos(100t) \leftrightarrow & 10\pi\left[\delta(\omega-100)+\delta(\omega+100)\right] \\
u(t)-u(t-1) \leftrightarrow & sinc(\omega/2)e^{-j\omega/2} \\
10\cos(100t)[u(t)-u(t-1)] \leftrightarrow & \frac{1}{2\pi}10\pi\left[\delta(\omega-100)+\delta(\omega+100)\right] * sinc(\omega/2)e^{-j\omega/2} \equiv F(\omega) \\
F(\omega) = & 5sinc\left(\frac{\omega-100}{2}\right)e^{-j(\omega-100)/2} + 5sinc\left(\frac{\omega+100}{2}\right)e^{-j(\omega+100)/2}
\end{aligned}$$

See figure below for spectra plots.

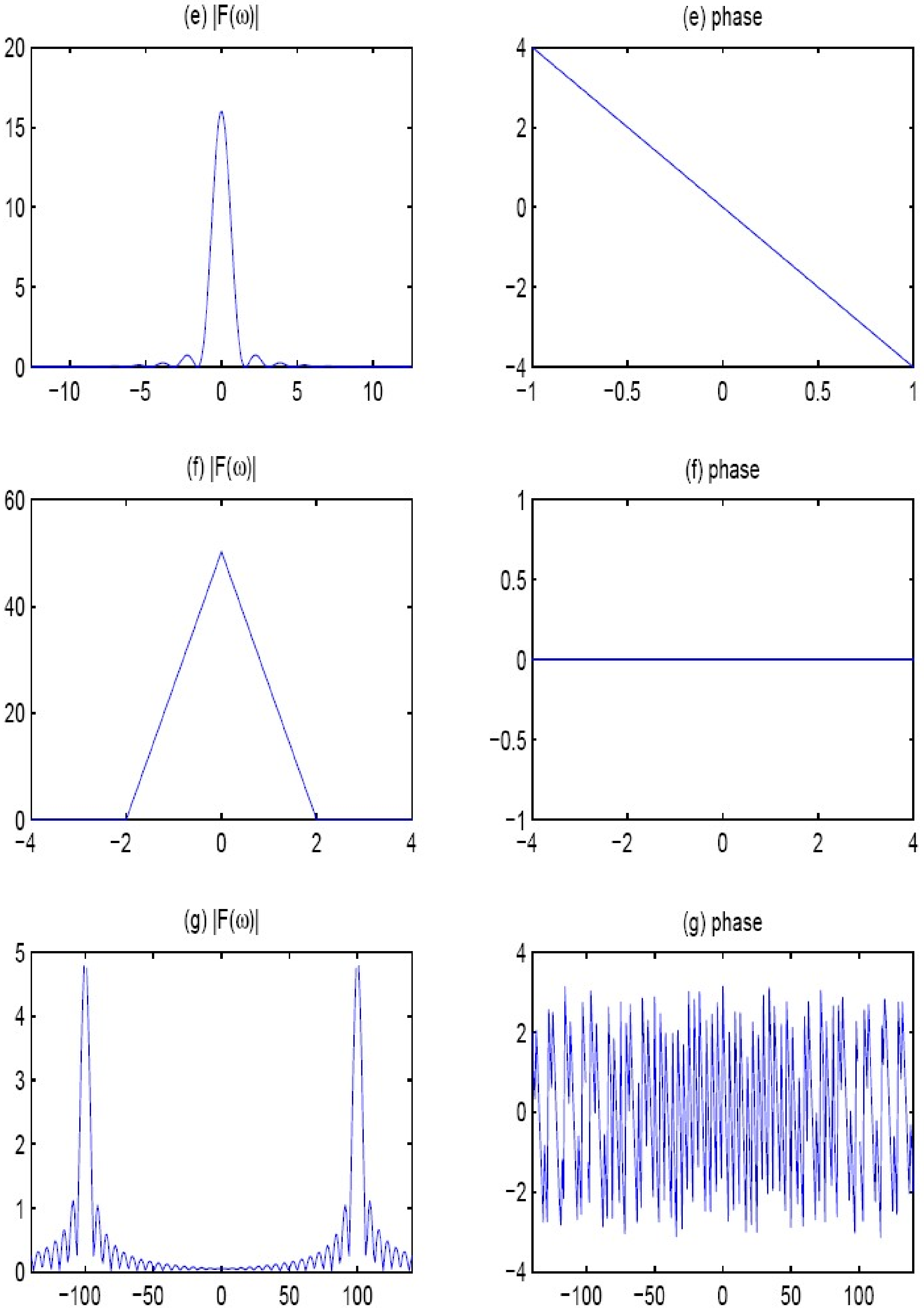


Figure 1: Spectra for 5.6(e),(f),(g)

5.7

Note the time axis is in units of ms.

$$
\begin{aligned}
g_4(t) =& rect(\frac{t}{0.01}) + rect(\frac{t}{0.02}) \\
G_4(\omega) =& 0.01sinc(0.005\omega) + 0.02sinc(0.01\omega) \\
g_5(t) =& 2.5rect(\frac{t}{0.01}) - 0.5rect(\frac{t}{0.02}) \\
G_5(\omega) =& 0.025sinc(0.005\omega) - 0.01sinc(0.01\omega) \\
g_6(t) =& 5g_4(t/10) \\
G_6(\omega) =& 50G_4(10\omega) = 0.5sinc(0.05\omega) + 1sinc(0.1\omega) \\
g_7(t) =& 10g_5(\frac{t-50}{5}) \\
G_7(\omega) =& 50G_5(5\omega)e^{-j50\omega} = 1.25sinc(0.025\omega)e^{-j50\omega} - 0.5sinc(0.05\omega)e^{-j50\omega}
\end{aligned}
$$

5.8 (a) use the derivate property

$$\frac{d}{dt}\left(e^{-|t|}\right) \overset{\mathcal{F}}{\longleftrightarrow} j\omega\left(\frac{2}{\omega^2+1}\right) = \frac{j2\omega}{\omega^2+1}$$

(b) $\frac{1}{2\pi(t^2+1)}$, from Table 5.1 $F(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi f(-\omega)$

$$\frac{1}{4\pi}\left(\frac{2}{t^2+1}\right) \overset{\mathcal{F}}{\longleftrightarrow} \left(\frac{1}{4\pi}\right) 2\pi e^{-|-\omega|} = \frac{1}{2} e^{-|\omega|}$$

(c) $$\frac{4\cos(2t)}{t^2+1} = \frac{2\left[e^{j2t} + e^{-j2t}\right]}{t^2+1} = \frac{2e^{j2t}}{t^2+1} + \frac{2e^{-j2t}}{t^2+1}$$

use frequency-shift and duality properties

$$F(\omega) = 2\pi\left[e^{-|\omega-2|} + e^{-|\omega+2|}\right]$$

5.9

(a) As in 5.7,

$$
\begin{aligned}
g_4(t) =& rect(\frac{t}{0.01}) + rect(\frac{t}{0.02}) \\
G_4(\omega) =& 0.01sinc(0.005\omega) + 0.02sinc(0.01\omega) \\
g_6(t) =& 5g_4(t/10) \\
G_6(\omega) =& 50G_4(10\omega) = 0.5sinc(0.05\omega) + 1sinc(0.1\omega)
\end{aligned}
$$

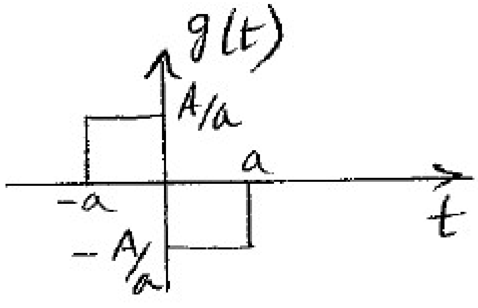
Continued→

5.9, continued

(b) $f(t)$ is the result of convolving two rectangular pulses, so its Fourier transform is the product of the transforms of the two pulses:

$$
\begin{aligned}
f(t) &= 2rect(\frac{t-0.5}{1}) * rect(\frac{t-1.5}{3}) \\
F(\omega) &= 2sinc(0.5\omega)e^{-j0.5\omega} \cdot 3sinc(1.5\omega)e^{-j1.5\omega} \\
g(t) &= f(2t) \\
G(\omega) &= 0.5F(0.5\omega) = 1sinc(0.25\omega)e^{-j0.25\omega} \cdot 1.5sinc(0.75\omega)e^{-j0.75\omega}
\end{aligned}
$$

5.10
(a)



let $g(t) = \frac{df(t)}{dt}$

$$g(t) = \frac{A}{a}\left[rect\left(\frac{t+a/2}{a}\right) - rect\left(\frac{t-a/2}{a}\right)\right]$$

Use the linearity property and the time Shift

$$G(\omega) = A\,Sinc\,(a\omega/2)\left[e^{ja\omega/2} - e^{-ja\omega/2}\right]$$

To find $F(\omega)$ use time integration property

$$F(\omega) = \frac{1}{j\omega}G(\omega) + \pi G(0)\delta(\omega)$$

$$G(0) = 0$$

$$\therefore \quad F(\omega) = aA\,sinc\left(\frac{a\omega}{2}\right)\left[\frac{e^{\frac{j\omega a}{2}} - e^{-\frac{j\omega a}{2}}}{2j\left(\frac{a\omega}{2}\right)}\right]$$

$$F(\omega) = aA\,Sinc^2\left(\frac{a\omega}{2}\right)$$

Continued→

5.10, continued

(b) Let $g(t) = \frac{d}{dt}f(t) = 2rect(\frac{t-0.5}{1}) - 2rect(\frac{t-3.5}{1})$

$$\begin{aligned} G(\omega) =& 2sinc(0.5\omega)e^{-j0.5\omega} - 2sinc(0.5\omega)e^{-j3.5\omega} \\ =& 2sinc(0.5\omega)e^{-j2\omega}[e^{j1.5\omega} - e^{-j1.5\omega}] \\ =& 4jsinc(0.5\omega)e^{-j2\omega}\sin(1.5\omega) \\ F(\omega) =& \frac{1}{j\omega}G(\omega) + \pi G(0)\delta(\omega) \\ G(0) =& 0 \\ F(\omega) =& 4(1.5)sinc(0.5\omega)e^{-j2\omega}sinc(1.5\omega) \\ =& 6sinc(0.5\omega)sinc(1.5\omega)e^{-j2\omega} \end{aligned}$$

5.11

(a)

$x(t) = \cos(t) + \sin(3t)$

$h(t) = \frac{.5\sin(2t)}{t} = sinc(2t) \longleftrightarrow \pi/2\, rect(\omega/4)$

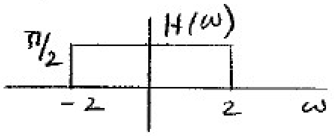
$X(\omega) = \pi[\delta(\omega-1) + \delta(\omega+1)] + \pi/j\,[\delta(\omega-3) - \delta(\omega+3)]$

$Y(\omega) = H(\omega)X(\omega)$

$Y(\omega) = \pi^2/2\,[\delta(\omega-1) + \delta(\omega+1)]$

impulses ∓3 will not pass the filter

$y(t) = \pi/2\cos(t)$



(b)

$$\begin{aligned} sinc(2\pi t) \leftrightarrow& 2\pi\frac{1}{4\pi}rect(\frac{\omega}{4\pi}) \\ H(\omega) =& 2\pi\frac{1}{4}rect(\omega/4) \\ Y(\omega) =& 2\pi\frac{1}{4}rect(\frac{\omega}{4}) \cdot \frac{1}{2}rect(\frac{\omega}{4\pi}) \\ =& \frac{\pi}{4}rect(\frac{\omega}{4}) \\ y(t) =& \frac{1}{2\pi}(\frac{\pi}{4})4sinc(2\omega) = \frac{1}{2}sinc(2\omega) \end{aligned}$$

5.12

(a)

(i) $H(\omega) = \frac{R/L}{j\omega + R/L} = \frac{10}{j\omega + 10}$

(ii) $|H(\omega)| = \frac{10}{\sqrt{\omega^2 + 100}}$
$\angle H(\omega) = -\tan^{-1}(\frac{\omega}{10})$. See figure (below) for magnitude and phase plots.

(iii) $h(t) = 10e^{-10t}u(t)$

(b)

(i) $H(\omega) = \frac{1}{j\omega + RC} = \frac{1}{j\omega + 1}$

(ii) $|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$
$\angle H(\omega) = -\tan^{-1}(\omega)$

(iii) $h(t) = e^{-t}u(t)$

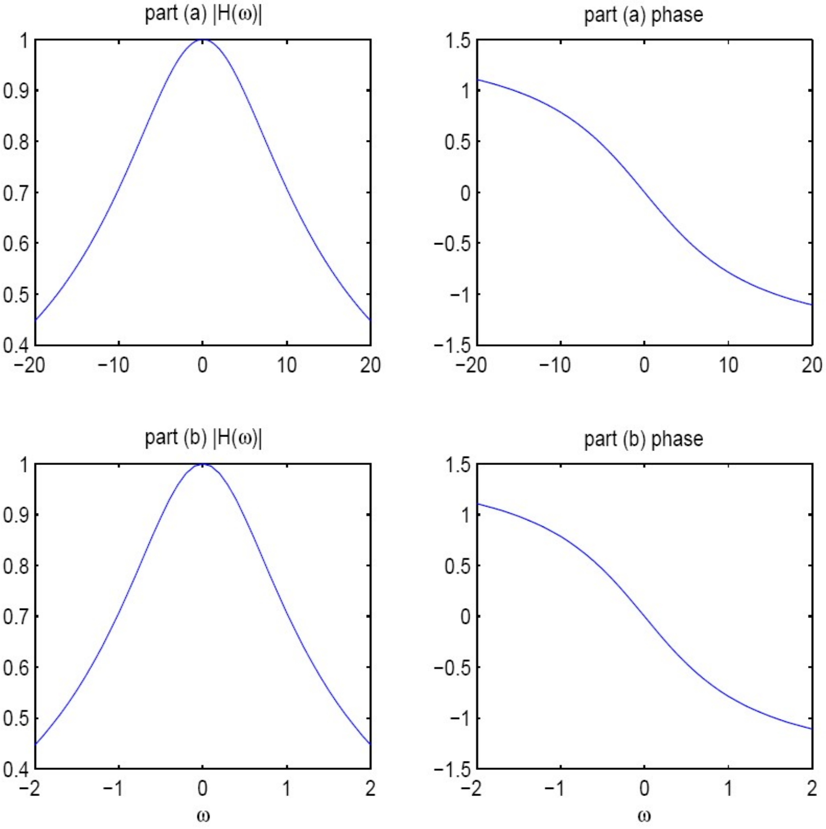


Figure 2: Magnitude and phase of frequency response for 5.12.

5.13 $F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$

$\mathcal{F}\{f(at\} = \int_{-\infty}^{\infty} f(at) e^{-j\omega t}\, dt$, let $\tau = at$

$d\tau = a\, dt$

$\mathcal{F}\{f(at)\} = \mathcal{F}\{f(\tau)\} = \int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau/a} \frac{1}{a}\, d\tau$

$= \frac{1}{a}\int_{-\infty}^{\infty} f(\tau) e^{-j\omega/a\,\tau}\, d\tau$

$\mathcal{F}\{f(at)\} = \frac{1}{a} F(\omega/a)$, $a>0$

5.14 a) $g_1(t) = 4\cos(100\pi t)\,\text{rect}(t/10^{-2}) = 2[e^{j100\pi t} + e^{-j100\pi t}] \times \text{rect}(t/10^{-2})$

$g_1(t) = 2e^{j100\pi t}\,\text{rect}(t/10^{-2}) + 2e^{-j100\pi t}\,\text{rect}(t/10^{-2})$

Use the frequency-shift property & linearity

$G_1(\omega) = 2\times10^{-2}[\text{sinc}(5\times10^{-3}(\omega+100\pi)) + \text{Sinc}(5\times10^{-3}(\omega-100\pi))]$

b) $g_2(t) = -1\, g_1(t - 5\times10^{-3})$, use the time-shift property

$G_2(\omega) = -.02\, e^{-j0.005\omega}[\text{sinc}(5\times10^{-3}(\omega+100\pi)) +$
$\text{Sinc}(5\times10^{-3}(\omega-100\pi))]$

c) $g_3(t) = g_1(10t + 5\times10^{-4})$, use the time transform

$G_3(\omega) = \frac{1}{10} G_1(\omega/10)\, e^{j5\times10^{-5}\omega}$

$G_3(\omega) = 2\times10^{-3}[\text{Sinc}(5\times10^{-4}(\omega+100\pi)) +$
$\text{Sinc}(5\times10^{-4}(\omega-100\pi))]\, e^{j5\times10^{-5}\omega}$

Continued→

5.14, continued

(d) Using the entry in Table 5.2 for $rect(t/T)\cos(\omega_0 t)$ with $T = 0.002$ and $\omega = 500\pi$:

$$-4rect(t/0.002)\cos(500\pi t) \leftrightarrow -4\frac{0.002}{2}\left[sinc\left((\omega-500\pi)0.001\right)+sinc\left((\omega+500\pi)0.001\right)\right]$$
$$= -0.004\left[sinc(0.001\omega-0.5\pi)+sinc(0.001\omega+0.5\pi)\right]$$

5.15

a) $G(\omega) = 5 rect(\omega/20)$

$\beta/\pi \, sinc(\beta t) \xleftrightarrow{F} rect(\omega/2\beta)$, $\beta = 10$

$g(t) = \frac{50}{\pi} sinc(10t)$

b) $G(\omega) = 5\cos\left(\frac{\pi\omega}{20}\right) rect(\omega/20)$

$= 2.5\left[e^{\frac{j\omega\pi}{20}} + e^{-\frac{j\omega\pi}{20}}\right] rect(\omega/20)$

$= 2.5 rect(\omega/20) e^{j\pi\omega/20} + 2.5 rect(\omega/20) e^{-j\omega\pi/20}$

Use the time shift & Linearity properties on the result of (a)

$g(t) = \frac{25}{\pi}\left[sinc(10t+5\pi) + sinc(10t-5\pi)\right]$

5.16

(a) $g(2t) \leftrightarrow 0.5G(0.5\omega) = \frac{j0.25\omega}{-0.25\omega^2+2.5j\omega+6}$

(b) $g(3t-6) = g(3(t-2)) \leftrightarrow \frac{1}{3}G(\frac{\omega}{3})e^{-j2\omega} = \frac{j\frac{1}{9}\omega}{-\frac{1}{9}\omega^2+\frac{5}{3}j\omega+6}e^{-j2\omega}$

(c) $\frac{dg(t)}{dt} \leftrightarrow j\omega G(\omega) = \frac{-\omega^2}{-\omega^2+5j\omega+6}$

(d) $g(-t) \leftrightarrow G(-\omega) = \frac{-j\omega}{-\omega^2-5j\omega+6}$

(e) $e^{-j100t}g(t) \leftrightarrow G(\omega+100) = \frac{j\omega+j100}{-\omega^2+\omega(5j-200)+500j+6-10000}$

(f) $\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j\omega}G(\omega) + \pi G(0)\delta(\omega) = \frac{1}{-\omega^2+5j\omega+6}$

5.17

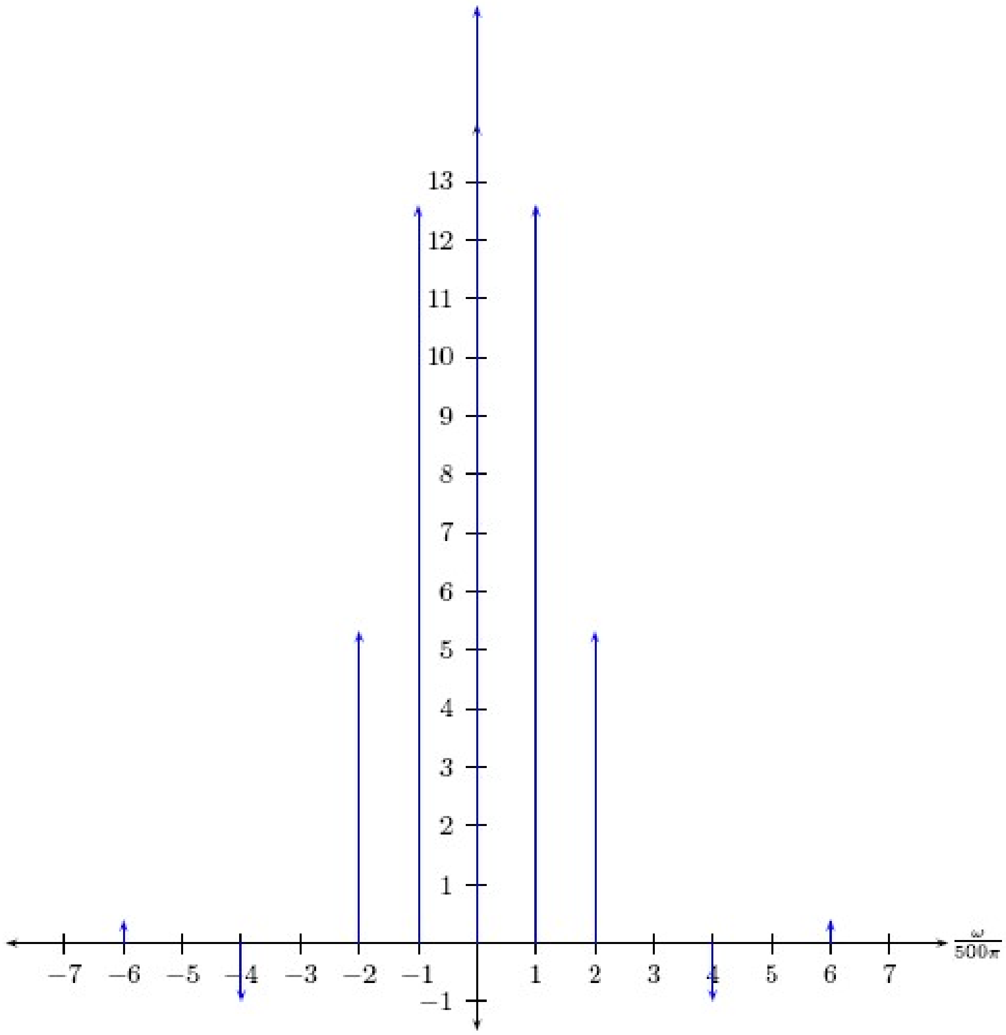
(a)

$$\begin{aligned}
f_1(t) =& g(t) * \sum_{n=-\infty}^{\infty} \delta(t - n0.004) \\
g(t) \equiv& 8\cos(500\pi t)rect(t/0.002) \\
F_1(\omega) =& G(\omega)500\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k500\pi) = 500\pi \sum_{k=-\infty}^{\infty} G(k500\pi)\delta(\omega - k500\pi) \\
G(\omega) =& 8(0.001)\left[sinc((\omega - 500\pi)0.001) + sinc((\omega + 500\pi)0.001)\right] \\
F_1(\omega) =& 4\pi \sum_{k=-\infty}^{\infty} \left[sinc(0.5\pi(k-1)) + sinc(0.5\pi(k+1)\right]\delta(\omega - k500\pi)
\end{aligned}$$

Noting that $sinc(0.5\pi(k-1)) = sinc(0.5\pi(k+1)) = 0$ when k is odd and $\neq \pm 1$:
$F_1(\omega) = 4\pi\delta(\omega - 1) + 4\pi\delta(\omega + 1) + 4\pi\sum_{k=-\infty}^{\infty}\left[sinc(0.5\pi(2k-1)) + sinc(0.5\pi(2k+1)\right]\delta(\omega - k500\pi)$

$$\begin{aligned}
F_1(0) =& 4(4) \\
F_1(500\pi) =& 4\pi \\
F_1(1000\pi) =& 4(-2/3 + 2) \\
F_1(1500\pi) =& F_1(2500\pi) = F_1(500\pi k) = 0, k \neq \pm 1, k\ odd \\
F_1(2000\pi) =& 4(-2/3 + 2/5) \\
F_1(3000\pi) =& 4(2/5 - 2/7)
\end{aligned}$$

Continued→



note the time axis is w/(500pi)

(b)

$$
\begin{aligned}
f_2(t) =& g(t) * \sum_{k=-\infty}^{\infty} \delta(t - n0.002) \\
F_2(\omega) =& G(\omega)1000\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k1000\pi) = 1000\pi \sum_{k=-\infty}^{\infty} G(k1000\pi)\delta(\omega - k1000\pi) \\
=& 8\pi \sum_{k=-\infty}^{\infty} \left[sinc(0.5\pi(2k-1)) + sinc(0.5\pi(2k+1)) \right] \delta(\omega - k1000\pi)
\end{aligned}
$$

The plot is identical to that in (a) except there are no impulses at $\omega = \pm 500\pi$ and all values are scaled by 2.

(c) The plots happen to be identical except for the impulses at $\omega = \pm 500\pi$ and the scaling by a factor of 2. However, note that in the frequency domain the impulses in (b) are twice as far apart as in (a), since T_0, the distance between impulses in the time domain, is half that in (a). However, every other impulse turns out to be zero in (a), except the $\pm k$ ones.

(d) If the period was halved the frequency spectra would have the same shape but would be expanded by a factor of 2 (the distance between impulses, in frequency, would double). (Also their amplitudes would be scaled by 2).

5.18

(a)

$$
\begin{aligned}
g(t) &= 10rect(t/2) \\
g_p(t) &= 10rect(t/2) * \sum_{n=-\infty}^{\infty} \delta(t-n4) \\
G_p(\omega) &= 20sinc(\omega) \cdot \frac{\pi}{2} \sum_{n=-\infty}^{\infty} \delta(\omega - n\frac{\pi}{2}) \\
&= 10\pi \sum_{n=-\infty}^{\infty} sinc(n\frac{\pi}{2})\delta(\omega - n\frac{\pi}{2}) \\
G_p(0) &= 10\pi \\
G_p(n\frac{\pi}{2}) &= 0, n\ even \\
G_p(\frac{\pi}{2}) &= 20 \\
G_p(\frac{3\pi}{2}) &= -20/3 \\
G_p(\frac{5\pi}{2}) &= 4 \\
G_p(\frac{7\pi}{2}) &= -20/7 \\
& etc
\end{aligned}
$$

See plot below.

(b) If the period was doubled the distance between impulses in the frequency domain would be halved. The spectrum would be compressed in frequency. It would also have slightly different values: $G(\omega) = 5\pi \sum_{n=-\infty}^{\infty} sinc(n\frac{\pi}{4})\delta(\omega - n\frac{\pi}{4})$. See plot below.

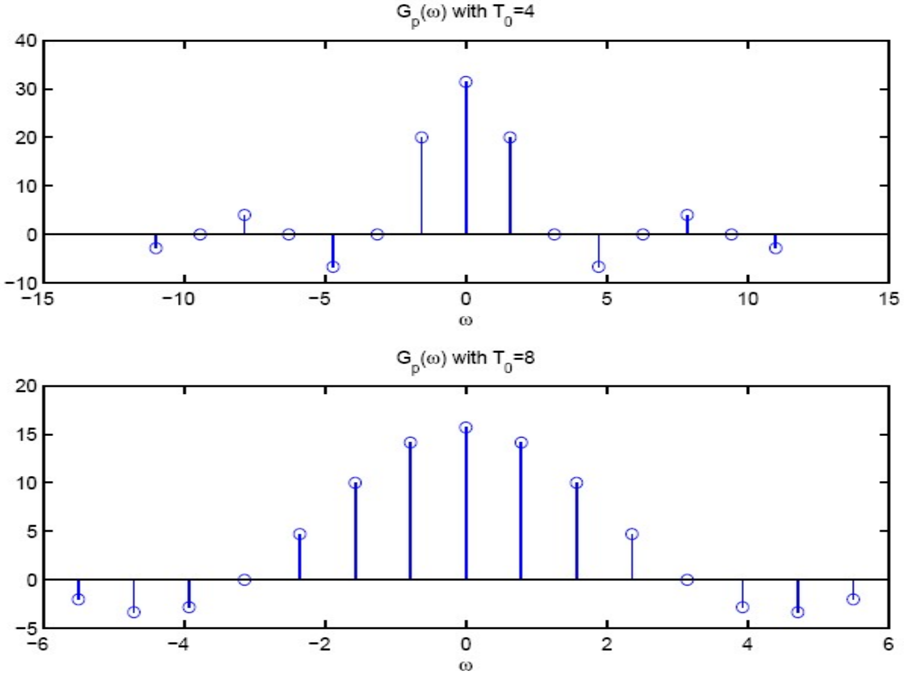


Figure 3: Plots for 5.18 a-b

5.19 a) Duality

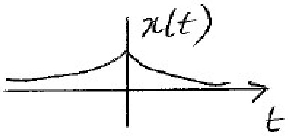
$$x(t) = \frac{1}{2\pi}\,\frac{1}{(a-jt)^2}$$

we know $\quad te^{-at}u(t) \longleftrightarrow \dfrac{1}{(a+j\omega)^2}$

$a > 0$

So $\quad \dfrac{1}{2\pi}\,\dfrac{1}{(a-jt)^2} \longleftrightarrow \omega e^{-a\omega}u(\omega)$

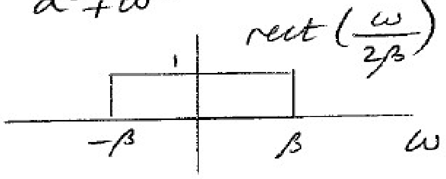
b)



$$X(\omega) = \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$

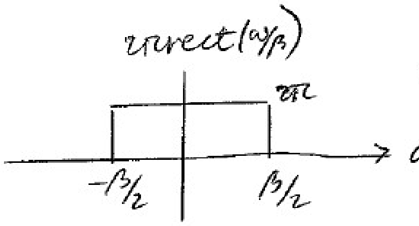
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

c) $\quad \beta/\pi\ \mathrm{Sinc}(\beta t) \longleftrightarrow$



By time scale, $f(at) \longleftrightarrow \dfrac{1}{|a|}F(\omega/a)$

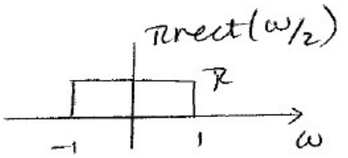
$$\therefore \beta\ \mathrm{Sinc}(\beta t/2) \longleftrightarrow \pi\,\frac{1}{1/2}\,\mathrm{rect}\left(\frac{\omega}{\frac{1}{2}\,2\beta}\right)$$



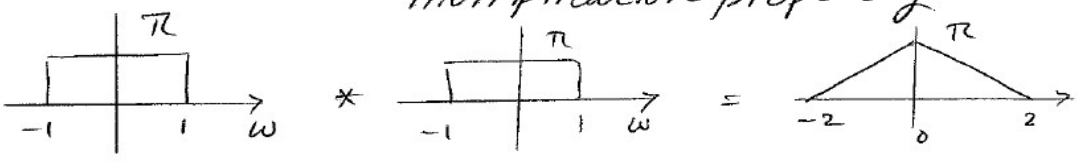
$$= 2\pi\,\mathrm{rect}(\omega/\beta)$$

d) $\quad F[\mathrm{Sinc}^2 t]$

$\mathcal{F}[\mathrm{Sinc}\ t] =$



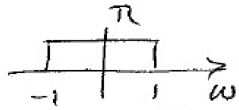
$\mathcal{F}[\mathrm{Sinc}^2 t] = \dfrac{1}{2\pi}\left(\pi\,\mathrm{rect}(\omega/2) * \pi\,\mathrm{rect}(\omega/2)\right)$ by multiplication property



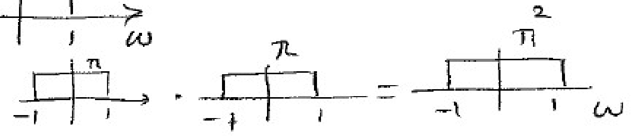
$$= \pi\,\mathrm{tri}(\omega/2)$$

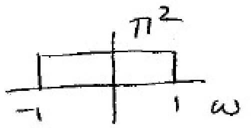
5.20

a) $\text{sinc}\, t \longleftrightarrow$



$\text{sinc}\, t * \text{sinc}\, t \longleftrightarrow$



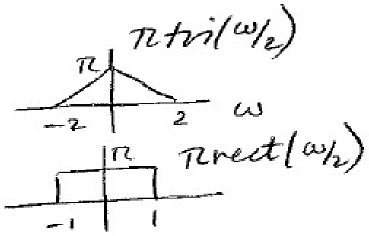


$\longleftrightarrow \pi\, \text{sinc}\, t$

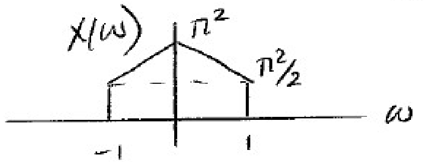
b) $\text{sinc}^2 t * \text{sinc}\, t$

$\text{sinc}^2 t \longleftrightarrow \pi\, \text{tri}(\omega/2)$

$\text{sinc}\, t \longleftrightarrow \pi\, \text{rect}(\omega/2)$

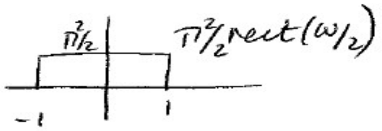


$\text{sinc}^2 t * \text{sinc}\, t \longleftrightarrow \pi\, \text{tri}(\omega/2) \cdot \pi\, \text{rect}(\omega/2) = X(\omega)$

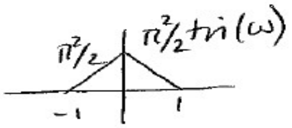


Now take the inverse Fourier transform of X(ω)

$X(\omega)$



+



$\therefore\ x(t) = \pi/2\ \text{sinc}\, t + \mathcal{F}^{-1}\left[\pi^2/2\ \text{tri}(\omega)\right]$

Since $\text{sinc}^2 t \longleftrightarrow \pi\, \text{tri}(\omega/2)$

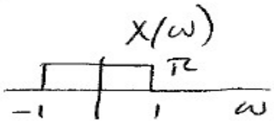
$\text{sinc}^2\, t/2 \longleftrightarrow 2\pi\, \text{tri}(\omega)$

And $\pi/4\ \text{sinc}^2(t/2) \longleftrightarrow \pi^2/2\ \text{tri}(\omega)$

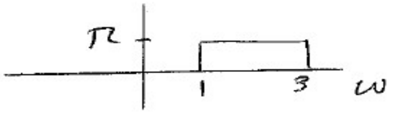
$\therefore\ x(t) = \pi/2\ \text{sinc}\, t + \pi/4\ \text{sinc}^2(t/2)$

c) $\text{sinc}\, t * e^{j2t}\, \text{sinc}\, t$

$x(t) = \text{sinc}\, t \longleftrightarrow$



$e^{j2t}\, \text{sinc}\, t \longleftrightarrow X(\omega-2)$



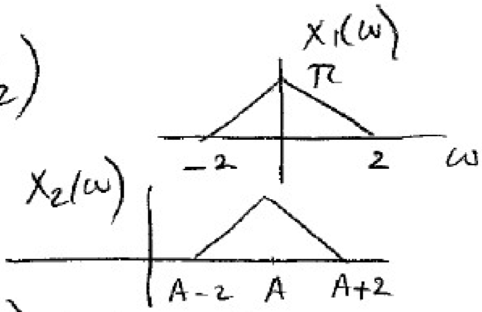
by modulation property

Multiply in frequency to get 0

$\therefore\ \text{sinc}\, t * e^{j2t}\, \text{sinc}\, t = 0$

5.21 $X_1(\omega) = \pi \, tri(\omega/2)$

$X_2(\omega) = X_1(\omega - A)$



$X_1(\omega) * X_2(\omega) \longleftrightarrow X_1(\omega) X_1(\omega - A)$

$x_1(t) * x_2(t)$ are nonzero for

$A - 2 < 2$ & $A + 2 > -2$

$\therefore$ the range is $-4 < A < 4$

5.22

(a)

$$
\begin{aligned}
v_1(t) =& \sin(50t) \\
V_1(\omega) =& \frac{\pi}{j}\left[\delta(\omega-50) - \delta(\omega+50)\right] \\
H(\omega) =& \frac{10}{10+j\omega} \\
V_2(\omega) =& V_1(\omega)H(\omega) = \frac{\pi}{j}\left[\frac{10}{10+j50}\delta(\omega-50) - \frac{10}{10-j50}\delta(\omega+50)\right] \\
v_2(t) =& \frac{\pi 10}{j(10+j50)}\frac{e^{j50t}}{2\pi} - \frac{\pi 10}{j(10-j50)}\frac{e^{-j50t}}{2\pi} \\
=& \frac{5}{j}\left[\frac{1}{10+j50}e^{j50t} - \frac{1}{10-j50}e^{-j50t}\right] \\
=& \frac{5}{j}\frac{1}{\sqrt{50^2+10^2}}\left(e^{j\theta}e^{j50t} - e^{-j\theta}e^{-j50t}\right), \quad \text{where } \theta = -\tan^{-1}(5/1) = -1.3734rad \\
=& 0.1961\sin(50t - 1.3734rad)
\end{aligned}
$$

(b)

$$
\begin{aligned}
H(\omega) =& \frac{1}{1+j\omega} \\
v_1(t) =& \sin(50t) \\
V_1(\omega) =& \frac{\pi}{j}\left[\delta(\omega-50) - \delta(\omega+50)\right] \\
V_2(\omega) =& V_1(\omega)H(\omega) = \frac{\pi}{j}\left[\frac{1}{1+j50}\delta(\omega-50) - \frac{1}{1-j50}\delta(\omega+50)\right] \\
v_2(t) =& \frac{1}{2j}\frac{1}{\sqrt{1+50^2}}\left[e^{j\theta}e^{j50t} - e^{-j\theta}e^{-j50t}\right], \quad \text{where } \theta = -\tan^{-1}(50/1) = -1.55rad \\
=& 0.02\sin(50t - 1.55rad)
\end{aligned}
$$

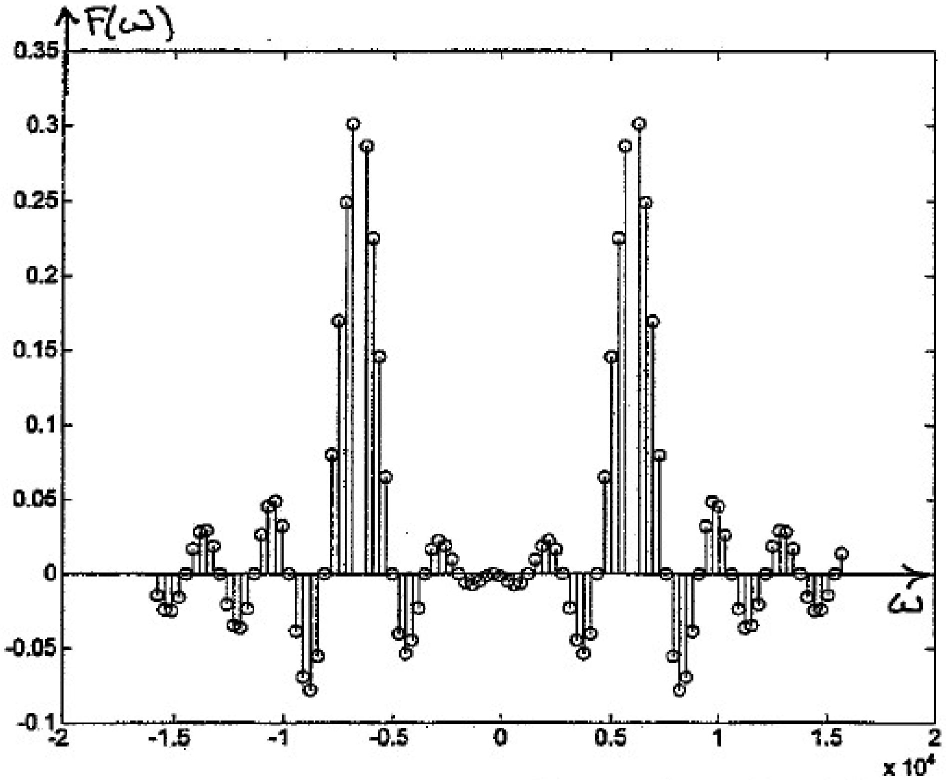
5.23 $f(t) = \sum_{n=-\infty} g(t-nT_0)$, $T_0 = 20(ms)$, $\omega_0 = 100\pi(rad/s)$

$$g(t) = 1\cos(2000\pi t)\,rect\left(t/_{2\times10^{-3}}\right)$$

$$G(\omega) = 1\times10^{-3}\left[sinc\left(10^{-3}(\omega-2000\pi)\right) + sinc\left(10^{-3}(\omega+2000\pi)\right)\right]$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} \omega_0\, G(n\omega_0)\, \delta(\omega - n\omega_0)$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi}{10}\left[sinc\left(\frac{2\pi}{10}(n-20)\right) + sinc\left(\frac{2\pi}{10}(n+20)\right)\right]\delta(\omega - n100\pi)$$



(b) If the frequency of the cosine was doubled, $g(t) = \cos(4000\pi t)rect(t/(2\times10^{-3}))$ so $G(\omega)$ is now the Fourier transform of $rect(t/(2\times10^{-3}))$ convolved with two deltas that are at $\pm4000\pi$ instead of $\pm2000\pi$. Therefore $G(\omega) = 1\times10^{-3}\left[sinc(10^{-3}(\omega-4000\pi)) + sinc(10^{-3}(\omega+4000\pi)\right]$.

(c) If the "off" time was halved, $g(t) = \cos(2000\pi t)rect(t/0.004)$ so $G(\omega)$ is now a narrower *sinc* convolved with two deltas at the same locations in frequency.
Therefore $G(\omega) = 2\times10^{-3}\left[sinc(2\times10^{-3}(\omega-2000\pi)) + sinc(2\times10^{-3}(\omega+2000\pi))\right]$.

5.24

$$X(\omega) = \sum_{-\infty}^{\infty} 2\pi C_k \, \delta(\omega - k\omega_0)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2\pi C_k \, \delta(\omega - k\omega_0) e^{j\omega t} \, d\omega$$

$$= \sum_{k=-\infty}^{\infty} C_k \int_{-\infty}^{\infty} \delta(\omega - k\omega_0) e^{j\omega t} \, d\omega$$

$$= \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$ by sifting property

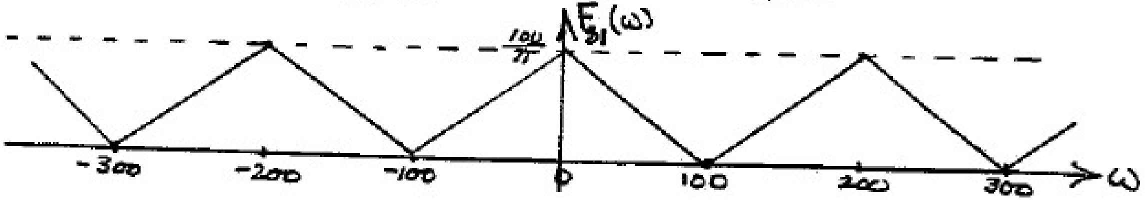
5.25.(a) THE SAMPLED SIGNAL CAN BE WRITTEN AS

$$f_{s1}(t) = f_1(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \quad T_s = \frac{2\pi}{\omega_s} = \frac{2\pi}{200} = \pi/100$$

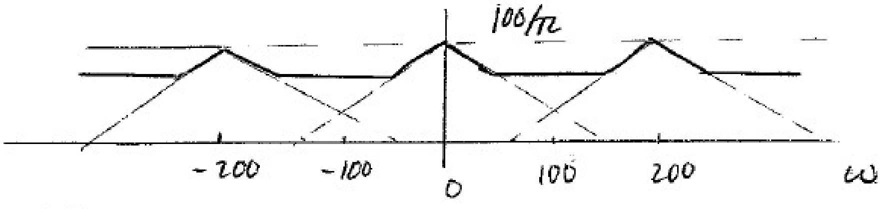
$$F_{s1}(\omega) = \frac{1}{2\pi} F_1(\omega) * \sum_{n=-\infty}^{\infty} \omega_s \delta(\omega - n\omega_s)$$

$$= \frac{\omega_s}{2\pi} \sum_{n=-\infty}^{\infty} F_1(\omega) * \delta(\omega - n\omega_s)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} F_1(\omega - n\omega_s) = \frac{100}{\pi} \sum_{n=-\infty}^{\infty} F_1(\omega - n\,200)$$



$$F_{S2}(\omega) = \frac{100}{\pi} \sum_{-\infty}^{\infty} F_2(\omega - n200)$$



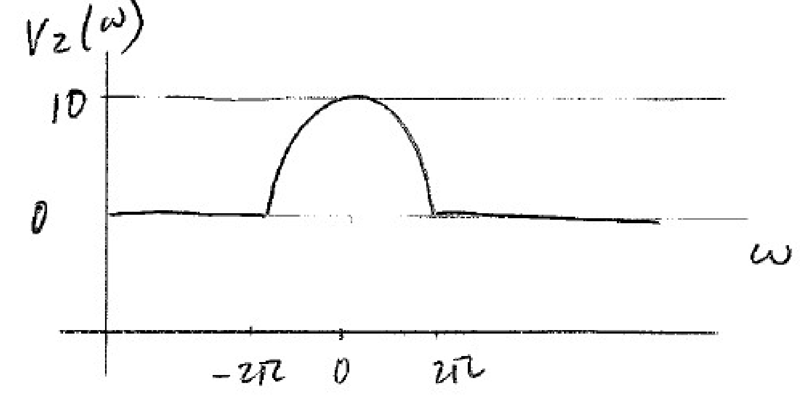
b) $\omega_s = 200$ (rad/s) is the Nyquist frequency for $f_1(t)$. $\omega_s \geqslant 300$ (rad/s) is necessary for proper sampling of $f_2(t)$.

5.26 $V_2(\omega) = H(\omega)V_1(\omega)$

$H(\omega) = rect(\omega/4\pi)$

$V_1(\omega) = \mathcal{F}\{10\, rect(t)\} = 10\, sinc(\omega/2)$

$$V_2(\omega) = \begin{cases} 10\, sinc(\omega/2), & |\omega| \leq 2\pi \\ 0, & |\omega| > 2\pi \end{cases}$$



5.27

$f(t) = e^{-t}u(t)$

$F(\omega) = \frac{1}{1+j\omega}$

$E_T = \int_{-\infty}^{\infty} |e^{-t}|^2 dt = \frac{1}{2}\ J$

Parseval's Theorem: $E_T = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi} 2 \int_0^{\infty} \frac{1}{1+\omega^2} d\omega$

$E_T = \frac{1}{\pi} tan^{-1}(\omega)|_0^{\infty} = \frac{1}{\pi} tan^{-1}(\infty) = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}\ J$

(a)

in the frequency band $-7 \leq \omega \leq 7$ (rad/s)

$E_7 = \frac{1}{\pi} tan^{-1}(\omega)\Big|_0^7 = \frac{1}{\pi}(tan^{-1}(7)) = .455\ J$

$E7/E_T \times 100\% = \frac{.455}{.5} \times 100\% = 91\%$

(b) in the frequency band $-1 \leq \omega \leq 1 (rad/s)$

$$E_1 = \frac{1}{\pi}\tan^{-1}(\omega)|_0^1 = 0.25J$$

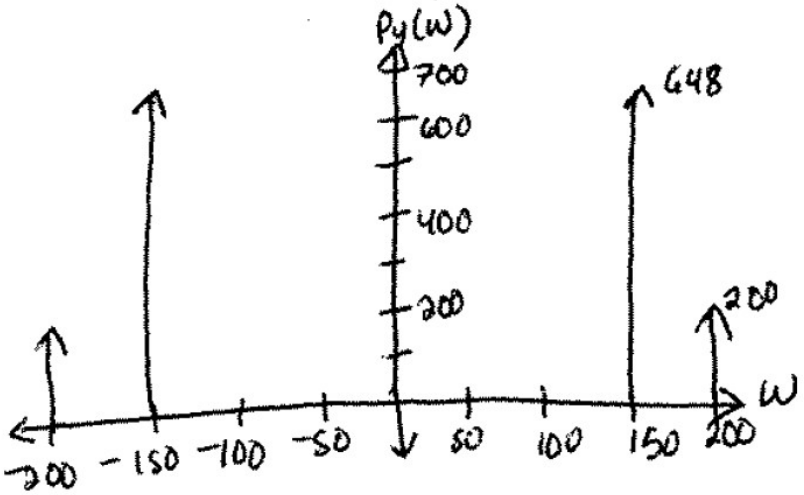
$$E_1/E_T \times 100\% = 50\%$$

5.28

(a) $P_y(\omega) = P_f(\omega)|H(\omega)|^2$
$H(150) = 8/200(150 - 100) + 16 = 18$
$H(200) = 8/200(200 - 100) + 16 = 20$

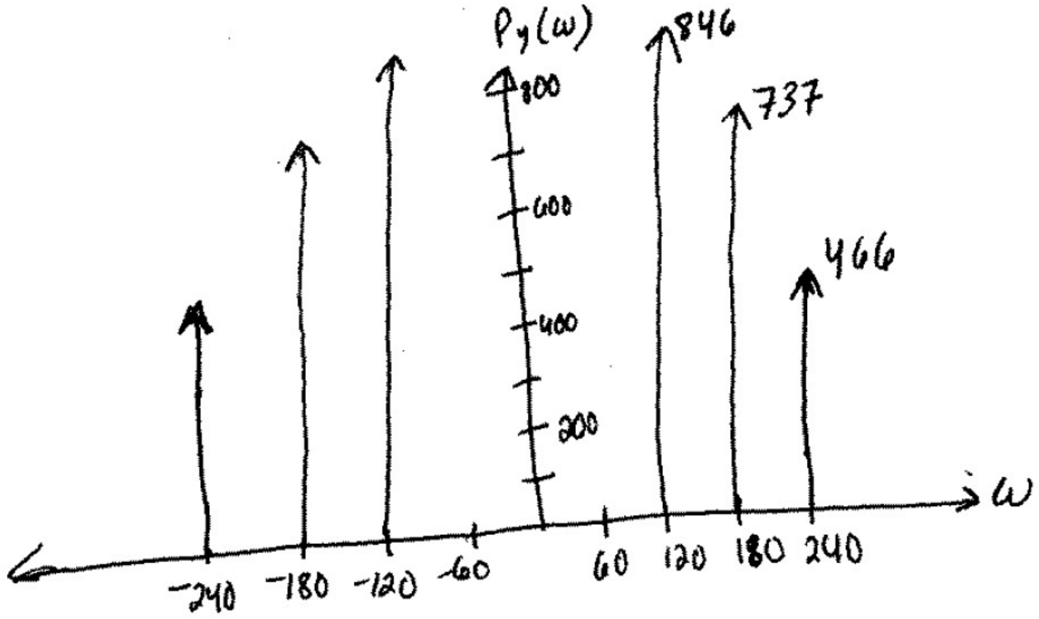
$$P_y(\omega) = (20^2)0.5\delta(\omega + 200) + (18^2)2\delta(\omega + 150) + (18^2)2\delta(\omega - 150) + (20^2)0.5\delta(\omega - 200)$$

$$P_y(\omega) = 200\delta(\omega + 200) + 648\delta(\omega + 150) + 648\delta(\omega - 150) + 200\delta(\omega - 200)$$



(b) $P_y(\omega) = P_f(\omega)|H(\omega)|^2$
$H(-240) = H(240) = 21.6$, $H(-180) = H(180) = 19.2$, $H(-120) = H(120) = 16.8$, $H(-60) = H(60) = 0$

$$P_y(\omega) = (21.6^2)[\delta(\omega+240)+\delta(\omega-240)]+(19.2^2)[2\delta(\omega+180)+2\delta(\omega-180)]+(16.8^2)[3\delta(\omega+120)+3\delta(\omega-120)]$$
$$P_y(\omega) = 466.56[\delta(\omega + 240) + \delta(\omega - 240)] + 737.28[\delta(\omega + 180) + \delta(\omega - 180)] + 846.72[\delta(\omega + 120) + \delta(\omega - 120)]$$

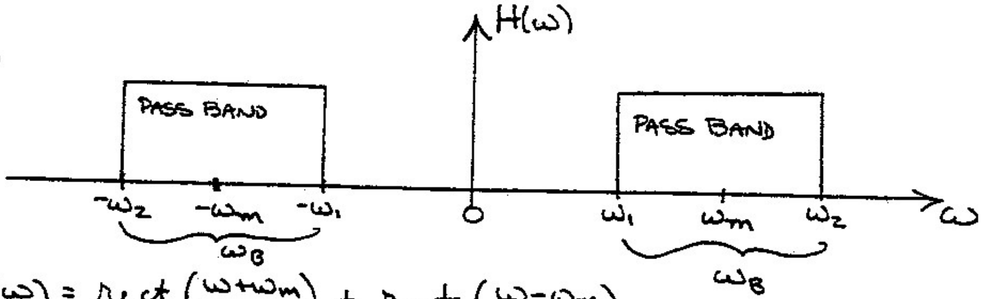


Chapter 6 solutions

6.1 $H(\omega) = 1 - \text{rect}(\omega/2\omega_c) \overset{\mathcal{F}}{\longleftrightarrow} \delta(t) - \frac{\omega_c}{\pi}\,\text{sinc}(\omega_c t) = h(t)$

h(t) is non-causal ∴ not physically realizable.

6.2.



$H(\omega) = \text{rect}\left(\frac{\omega+\omega_m}{2\omega_B}\right) + \text{rect}\left(\frac{\omega-\omega_m}{2\omega_B}\right)$

$h(t) = \mathcal{F}^{-1}\{H(\omega)\} = \frac{\omega_B}{2\pi}\underbrace{\text{sinc}\left(\frac{\omega_B t}{2}\right)}_{\text{NON-CAUSAL}}e^{j\omega_m t} + \frac{\omega_B}{2\pi}\underbrace{\text{sinc}\left(\frac{\omega_B t}{2}\right)e^{-j\omega_m t}}_{\text{NON CAUSAL}}$

6.3 For general T, $X(\omega) = 2\pi\sum_{k=-\infty}^{\infty} C_k\delta\left(\omega - k\frac{2\pi}{T}\right)$ where
$C_k = \frac{X_0}{2}sinc\frac{Tk\omega_0}{4} = \frac{1}{2}sinc(\frac{\pi k}{2})$.
Therefore $Y(\omega) = X(\omega)H(\omega) = 2\pi\sum_{k=-m}^{m} C_k\delta\left(\omega - k\frac{2\pi}{T}\right)$ where
m is such that $m\frac{2\pi}{T} \leq 180\pi$ but $(m+1)\frac{2\pi}{T} > 180\pi$. Using the fact that $\cos(\omega_0 t) \leftrightarrow \frac{1}{\pi}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$
and that $C_k = C_{-k}$ for this even signal, we'll have $y(t) = C_0 + \sum_{k=1}^{m} 2C_k\cos\left(k\frac{2\pi}{T}\right)$
Note that $C_0 = \frac{1}{2}$, $C_1 = C_{-1} = \frac{1}{\pi}$, $C_2 = C_{-2} = 0$, $C_3 = C_{-3} = -\frac{1}{3\pi}$, $C_4 = C_{-4} = 0$.

(a) $T = 0.040$, $\frac{2\pi}{T} = 50\pi$, $m = 3$, so $y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t) + \frac{2}{3\pi}\cos(150\pi t - \pi)$.

(b) $T = 0.025$, $\frac{2\pi}{T} = 80\pi$, $m = 2$, so $y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(80\pi t)$

(c) $T = 0.020$, $\frac{2\pi}{T} = 100\pi$, $m = 2$, so $y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(100\pi t)$

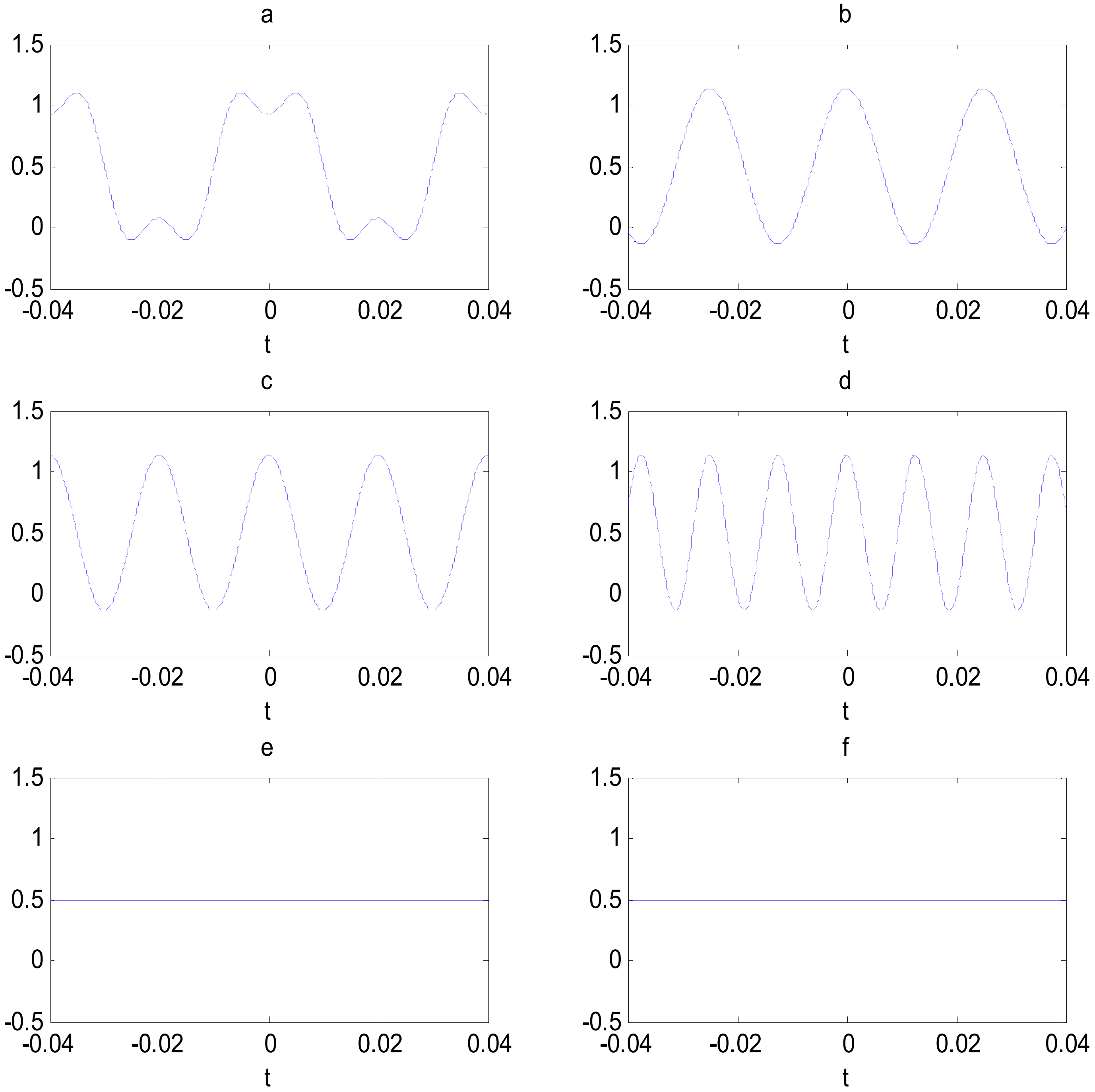
(d) $T = 0.0125$, $\frac{2\pi}{T} = 160\pi$, $m = 1$ so $y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(160\pi t)$

(e) $T = 0.010$, $\frac{2\pi}{T} = 200\pi$, $m = 0$, so $y(t) = \frac{1}{2}$

(f) $T = 0.00625$, $\frac{2\pi}{T} = 320\pi$, $m = 0$, so $y(t) = \frac{1}{2}$

See figures of output signals, next page→

6.3, continued

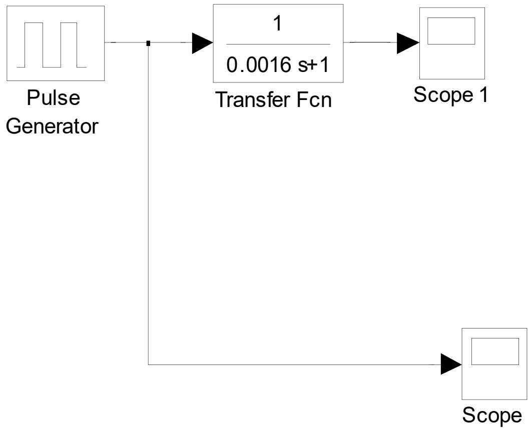


6.4 The SIMULINK model was set up using a Pulse Generator block, a Transfer Function block, and two scopes, following Example 6.6.

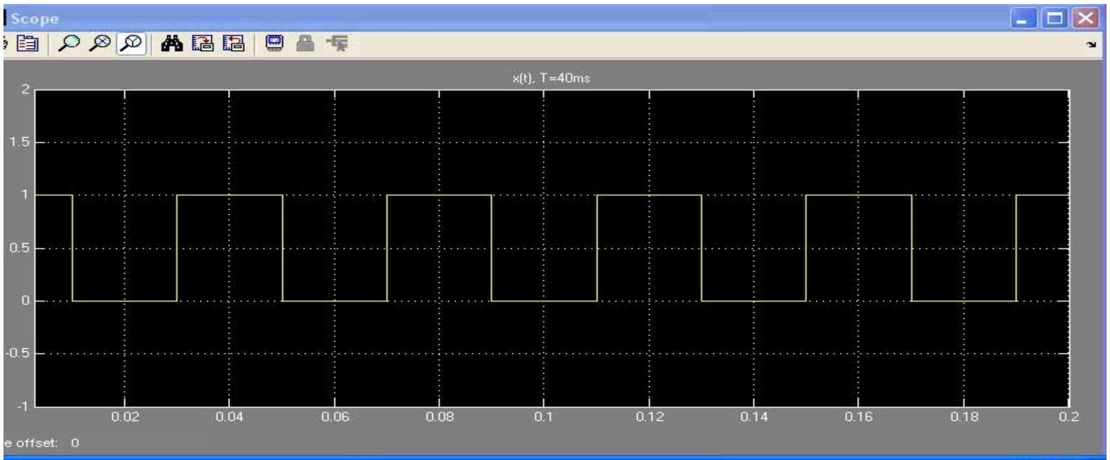
The parameters for the Pulse Generator were set at: Amplitude: 1; Period: 0.04 (for part (a)), Pulse Width: 50, and Phase Delay: −0.01 (one fourth of period).

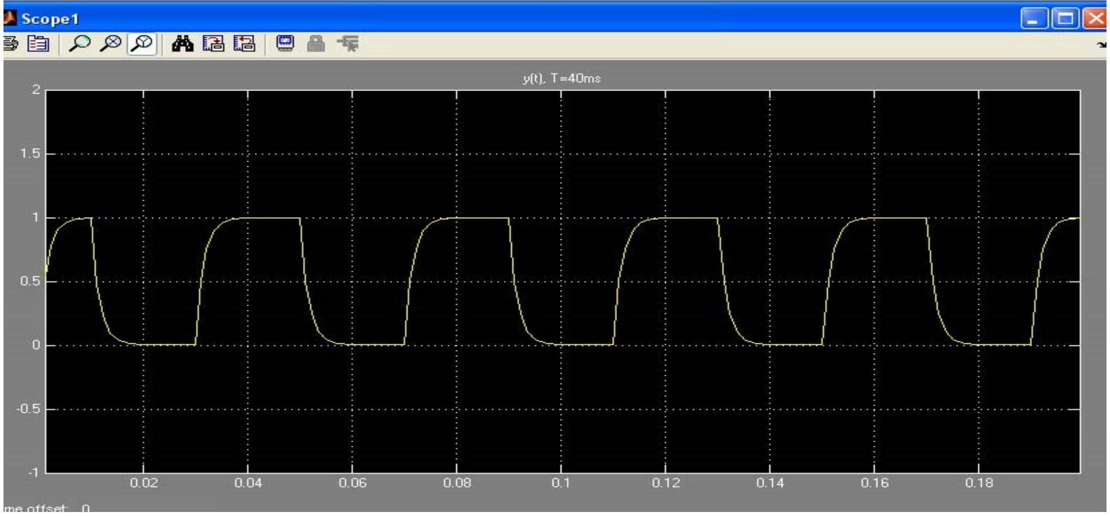
The parameters for the Transfer Function were found using `[B, A]= butter(1, 200*pi,‘s’)`, which gave `B=[0 628.3185]` and `A = [1.0000 629.3185]`. This transfer function is equivalent to `B=[0 1]` and `[0.0016 1]`, which were the coefficients entered into the Transfer Function numerator and denominator coefficient fields.

For parts (b)-(f), the period in the Pulse Generator was changed, and the Phase Delay was set to 1/4 the period.

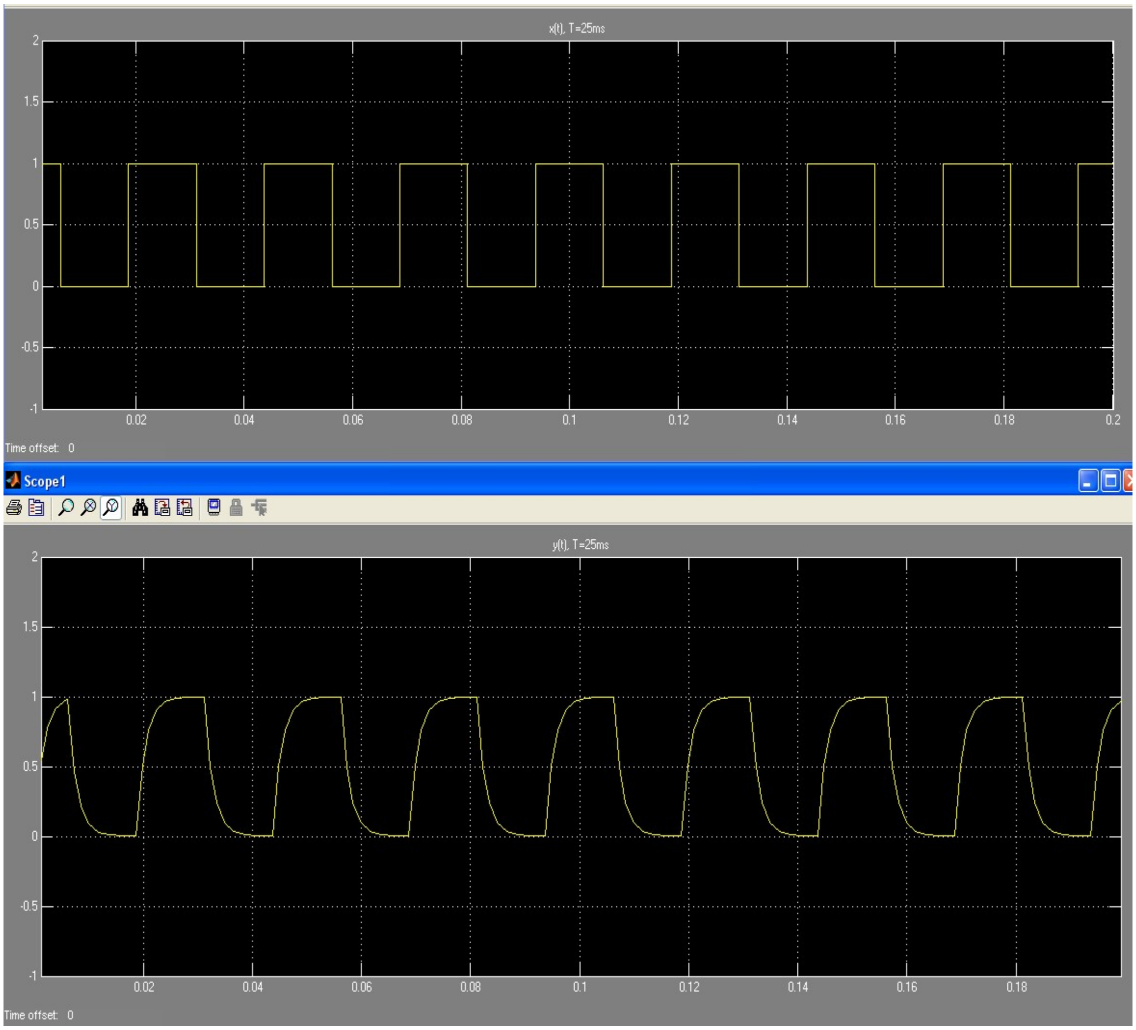


Part (a)

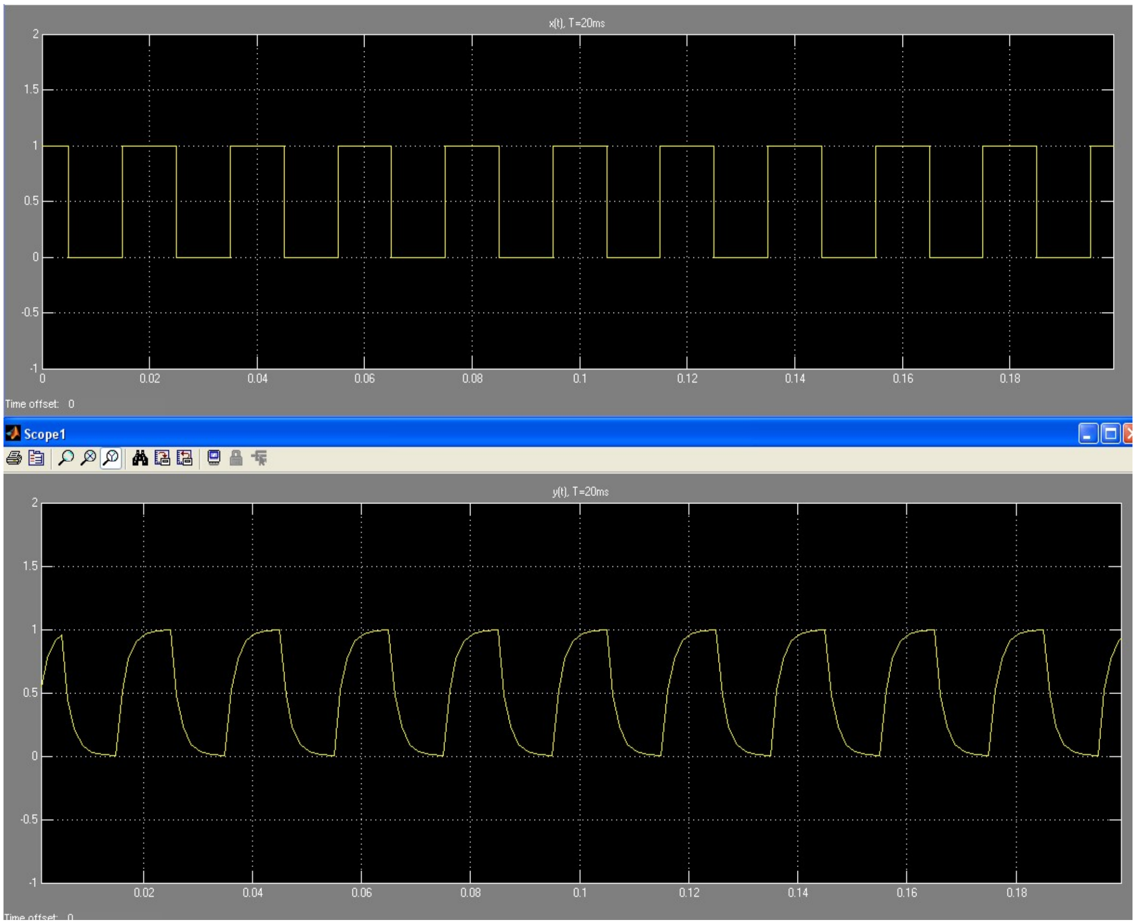




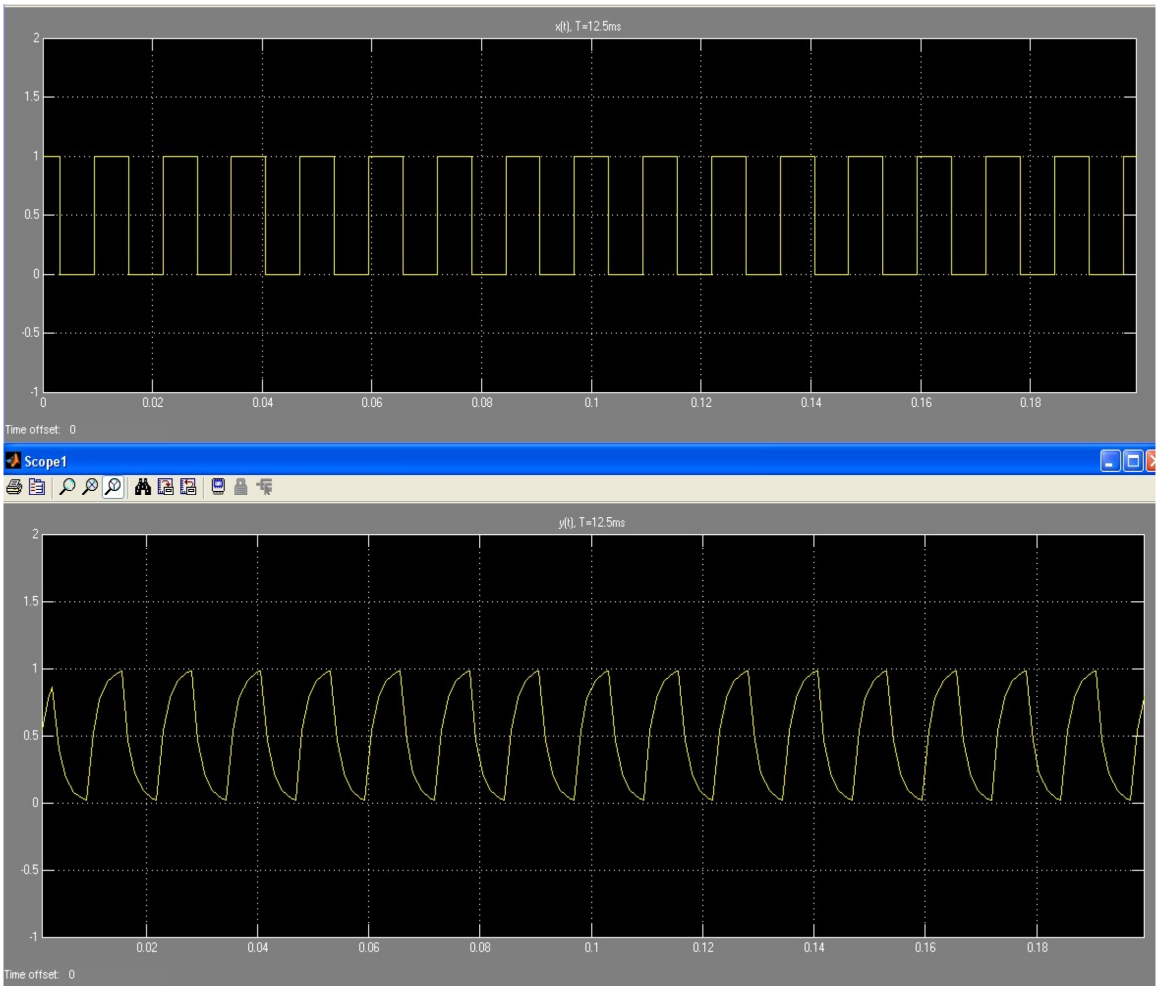
Part (b)


x(t), T=25ms
2
1.5
1
0.5
0
-0.5
-1
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
Time offset: 0
Scope1
y(t), T=25ms
2
1.5
1
0.5
0
-0.5
-1
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
Time offset: 0

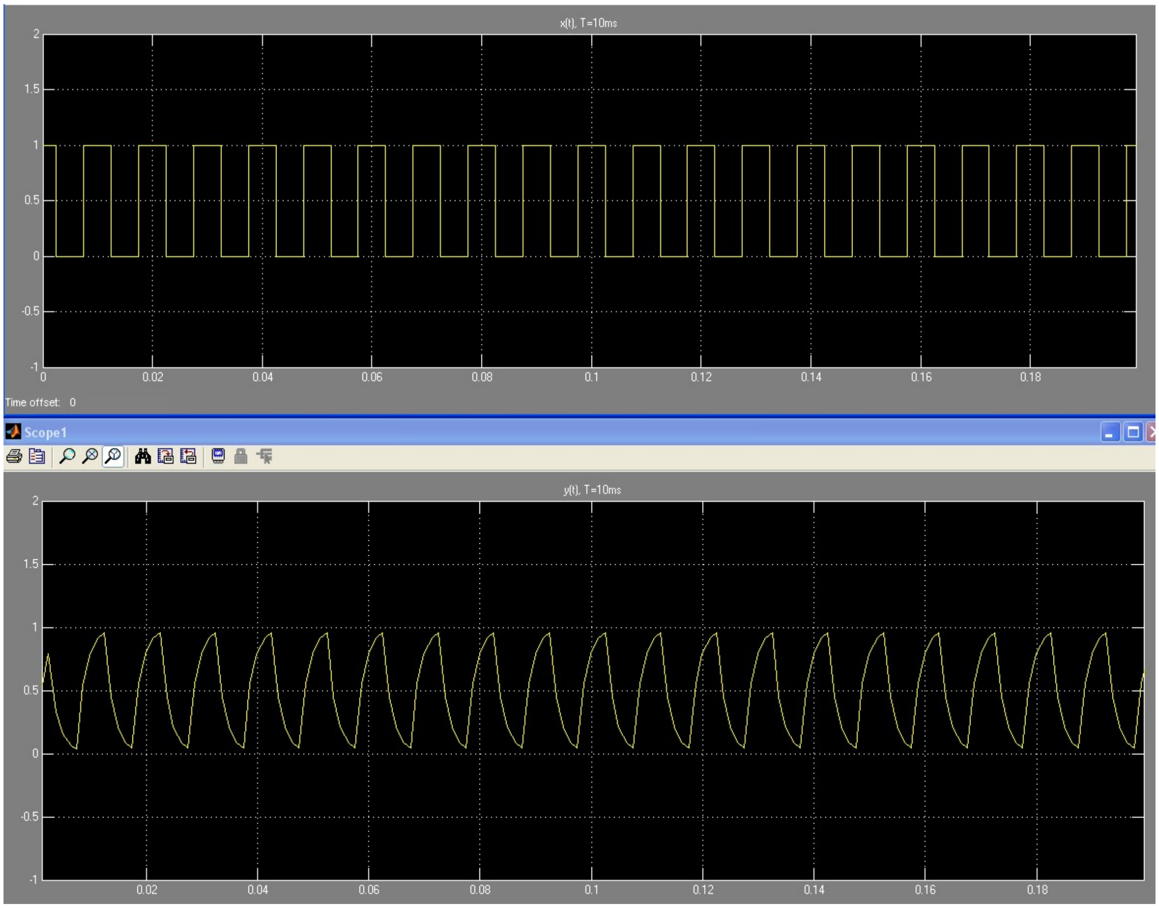
Part (c)


x(t), T=20ms
2
1.5
1
0.5
0
-0.5
-1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
Time offset: 0
Scope1
y(t), T=20ms
2
1.5
1
0.5
0
-0.5
-1
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
Time offset: 0

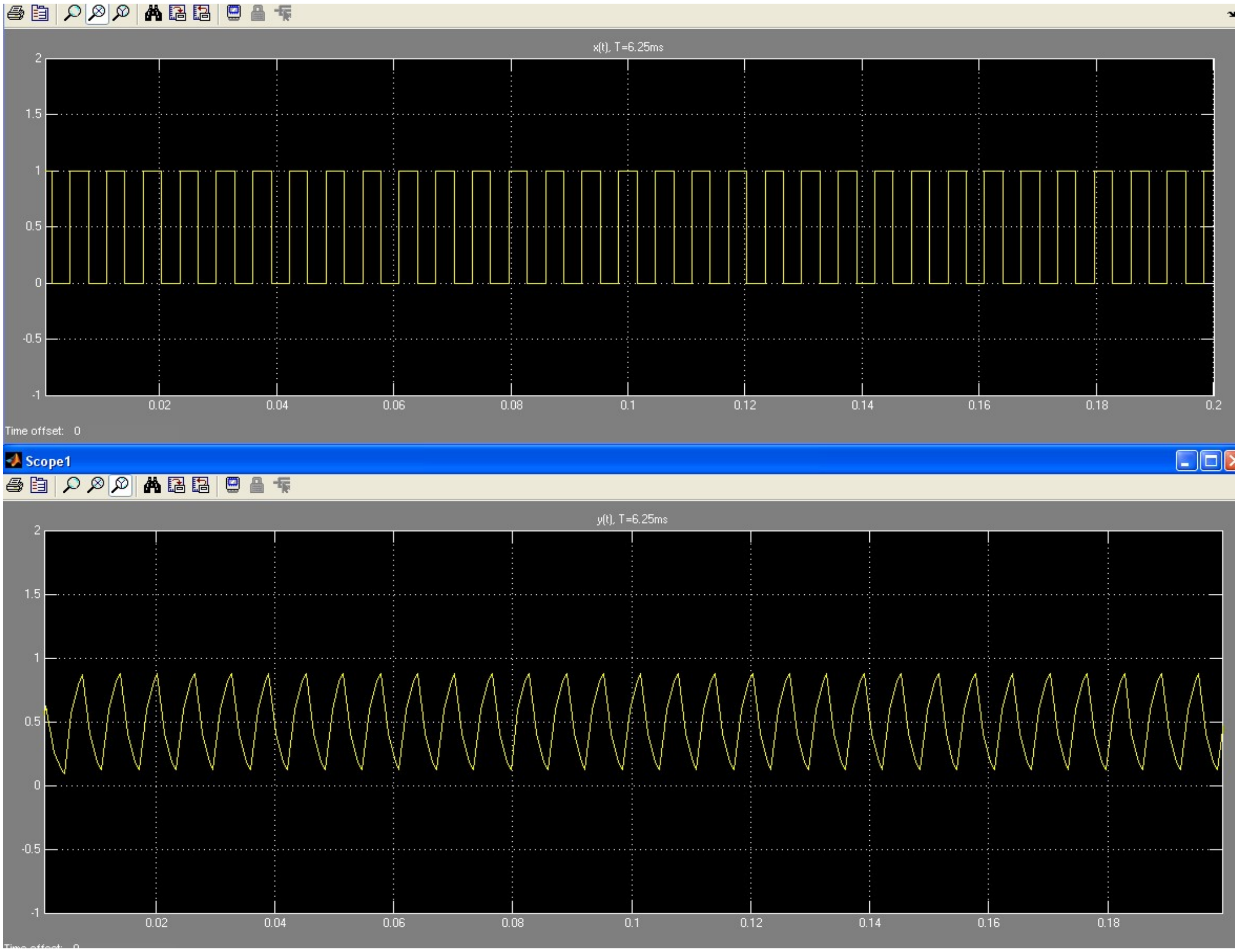
Part (d)


x(t), T=12.5ms
2
1.5
1
0.5
0
-0.5
-1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
Time offset: 0
Scope1
y(t), T=12.5ms
2
1.5
1
0.5
0
-0.5
-1
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
Time offset: 0

Part (e)


x(t), T=10ms
2
1.5
1
0.5
0
-0.5
-1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
Time offset: 0
Scope1
y(t), T=10ms
2
1.5
1
0.5
0
-0.5
-1
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18

Part (f)


x(t), T=6.25ms
2
1.5
1
0.5
0
-0.5
-1
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
Time offset: 0
Scope1
y(t), T=6.25ms
2
1.5
1
0.5
0
-0.5
-1
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18

6.5 $v_i(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$, $v_o(t) = Ri(t)$

$$V_i(\omega) = RI(\omega) + j\omega L I(\omega) + \frac{1}{j\omega C} I(\omega) + \frac{\pi}{C}\cancelto{0}{I(0)}\delta(\omega)$$

$$V_o(\omega) = RI(\omega)$$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{R}{R+j\omega L+\frac{1}{j\omega C}} = \frac{1}{1+j\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)}$$

$$H(\omega_m) = 1 \Rightarrow \frac{\omega_m L}{R} = \frac{1}{\omega_m RC} \Rightarrow \omega_m = \pm\frac{1}{\sqrt{LC}}$$

$$H(\omega_c) = \frac{1}{1\pm j1} \Rightarrow \frac{\omega_c L}{R} - \frac{1}{\omega_c RC} = \pm 1$$

$$\omega_{c_{1,2}} = \frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}$$

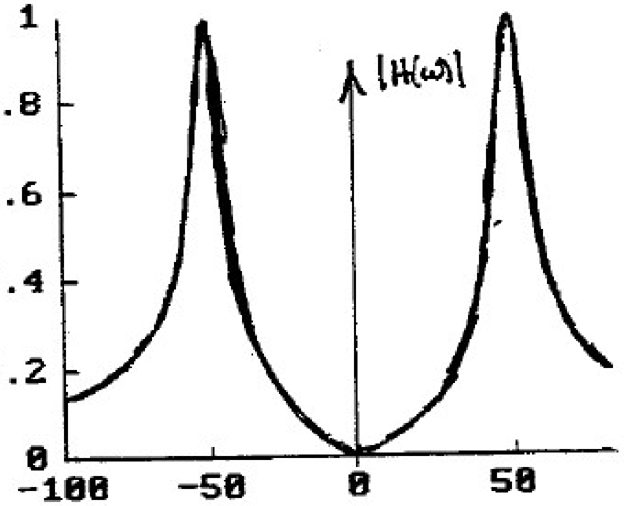
$$\omega_{c_{3,4}} = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}$$

THIS IS A BANDPASS FILTER.

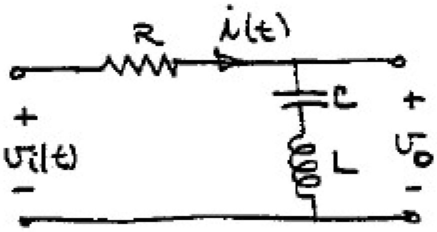
THE FIGURE SHOWS A PLOT OF $|H(\omega)|$ WHEN $R=1\Omega$

$L=0.1H$

$C=4\times10^{-3}F$



6.6



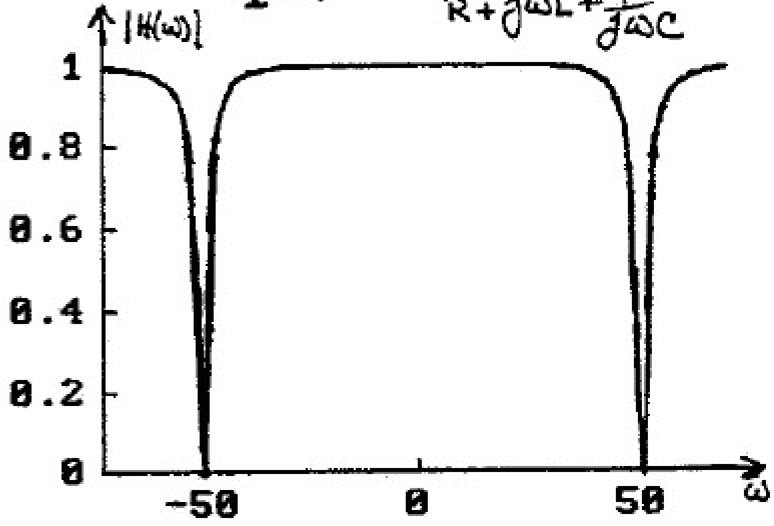
$$v_i(t) = Ri(t) + v_o(t)$$

$$v_o(t) = L\frac{di(t)}{dt} + \frac{1}{c}\int_{-\infty}^{t} i(\tau)d\tau$$

$$V_i(\omega) = RI(\omega) + j\omega L I(\omega) + \frac{1}{j\omega C}I(\omega) + \frac{\pi}{c}\cancelto{0}{I(0)}\delta(\omega)$$

$$V_o(\omega) = j\omega L + \frac{1}{j\omega C} + \frac{\pi}{c}\cancelto{0}{I(0)}\delta(\omega)$$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{j\omega L + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{1 + j\left(\frac{\omega RC}{1-\omega^2 LC}\right)}$$



THIS IS A BANDSTOP OR "NOTCH" FILTER FREQUENCY RESPONSE

6.7

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{\frac{1}{j\left(\frac{\sqrt{2}}{R_o\omega_c}\right)\omega}}{R_o + j\frac{R_o\omega}{\sqrt{2}\omega_c} + \frac{1}{j\frac{\sqrt{2}\omega}{R_o\omega_c}}} = \frac{1}{1 - \frac{\omega^2}{\omega_c^2} + j\frac{\sqrt{2}\omega}{\omega_c}}$$

$$|H(\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_c^2}\right)^2 + \frac{2\omega^2}{\omega_c^2}}} = \frac{1}{\sqrt{1 - \frac{2\omega^2}{\omega_c^2} + \frac{\omega^4}{\omega_c^4} + \frac{2\omega^2}{\omega_c^2}}}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^4}} = \frac{1}{\sqrt{1 + \left[\left(\frac{\omega}{\omega_1}\right)^2\right]^2}} \leftarrow$$

2nd ORDER BUTTERWORTH FREQUENCY RESPONSE FUNCTION

6.8 (a) $\omega_c = 2\pi \cdot 10\text{kHz}$, Assume $R = 1\text{k}\Omega$, then

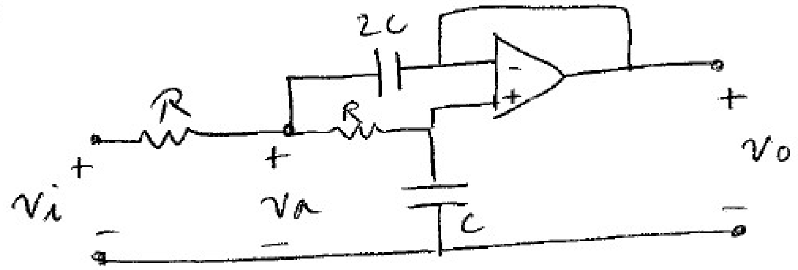
$$L = \frac{10^3}{(2\pi)(10{,}000)\sqrt{2}} = 0.0113\text{H} = 11.3\text{mH},\ C = \frac{\sqrt{2}}{\omega_c 1000} = 22.5\text{nF}$$

(b) $\omega_c = 2\pi \cdot 20\text{kHz}$, assuming $R = 1\text{k}\Omega$, then

$$L = \frac{10^3}{(2\pi)(20{,}000)\sqrt{2}} = 5.6\text{mH},\ C = \frac{\sqrt{2}}{\omega_c 1000} = 11.25\text{nF}$$

6.9

a)



$$\begin{bmatrix} 2/R + j\omega 2C & -1/R - j\omega 2C \\ -1/R & 1/R + j\omega C \end{bmatrix} \begin{bmatrix} V_a(\omega) \\ V_o(\omega) \end{bmatrix} = \begin{bmatrix} V_i(\omega)/R \\ 0 \end{bmatrix}$$

from KCL :

$$1/R\left(v_i(t) - v_a(t)\right) + 2C\frac{d}{dt}\left(v_o(t) - v_a(t)\right) + 1/R\left(v_o(t) - v_a(t)\right) = 0$$

$$1/R\left(v_o(t) - v_a(t)\right) + C\frac{dv_o(t)}{dt} = 0$$

Find Fourier Transform

$$1/R\left[V_i(\omega) - V_a(\omega)\right] + 2Cj\omega\left[V_o(\omega) - V_a(\omega)\right] + 1/R\left[V_o(\omega) - V_a(\omega)\right] = 0$$

$$1/R\left[V_o(\omega) - V_a(\omega)\right] + Cj\omega V_o(\omega) = 0$$

Continued→

6.9(a), continued

$$V_o(\omega) = \frac{\begin{vmatrix} \frac{2}{R} + j\omega 2C & \frac{V_i(\omega)}{R} \\ -\frac{1}{R} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{2}{R} + j\omega 2C & -\frac{1}{R} - j\omega 2C \\ -\frac{1}{R} & \frac{1}{R} + j\omega C \end{vmatrix}} = \frac{V_i(\omega)}{R^2\left(\frac{}{R^2} + \frac{j\omega 2C}{R} - \omega^2 2C^2\right)}$$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1}{1 - \omega^2 2R^2C^2 + j\omega 2RC}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + 4\omega^4 R^4 C^4}} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^4}}$$

2nd ORDER BUTTERWORTH FILTER

(b)

$$\omega_c = \frac{1}{\sqrt{2}\,RC} = \frac{1}{\sqrt{2}\,RC} = \frac{1}{\sqrt{2}(1000)(35\times 10^{-9})} = 20{,}203 \ (\text{rad/s})$$

(c)

20kHz$= 40,000\pi$ rad/sec. Want $\omega_c = \frac{1}{\sqrt{2}RC} = 40,000\pi$; letting $R = 1000$ gives $C = \frac{1}{\sqrt{2}(1000)(40{,}000\pi)} =$ 5.63nF. Therefore we can just replace the 35nF capacitor with a 5.63nF one.

6.10 $\omega_c = 2\pi \cdot 10,000$ rad/sec, and let $R_0 = 1000\Omega$.

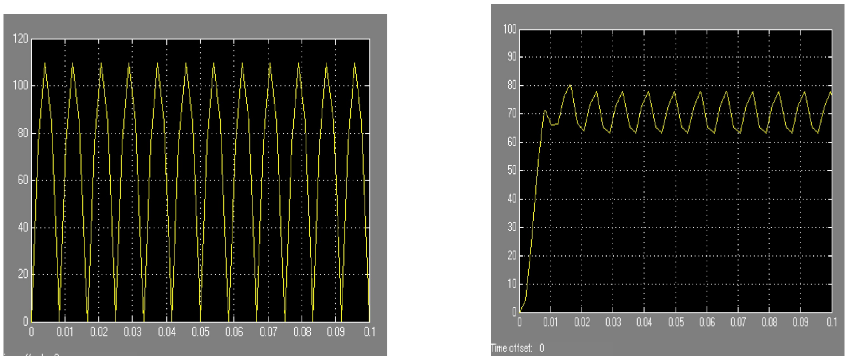
The Butterworth lowpass filter is in 6.12(a), with $L = \frac{1000}{2\pi(10{,}000)(\sqrt{2})} = 11.25$mH and $C = \frac{\sqrt{2}}{2\pi(10{,}000)(1000)} = 22.5$nF. The high-pass filter is constructed by interchanging the inductor and capacitor in the lowpass filter circuit in 6.12(a). The frequency response is then

$$H(\omega) = \frac{j\omega}{\sqrt{2}(2\pi)(10{,}000) + j(\omega - \frac{(2\pi\cdot 10{,}000)^2}{\omega})}$$

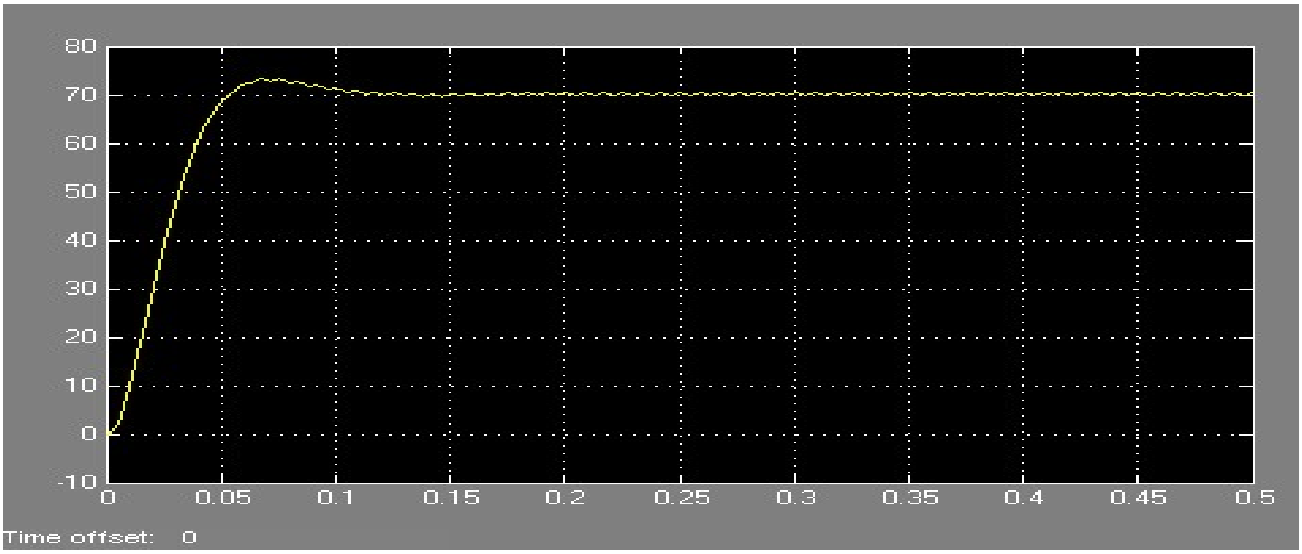
6.11

(a) (Note that you don't need the "Analog Butterworth LP Filter" block; just use a Transfer Function block with the coefficients derived from the '`butter(N, Wn, 's')`' command.)

We should select a cutoff frequency for the low-pass filter so that the oscillations in the signal are eliminated as much as possible. This doesn't specify a precise criterion, however. Here is the signal before and after filtering with a 2nd order Butterworth low-pass filter with $\omega_c = 100\pi$:

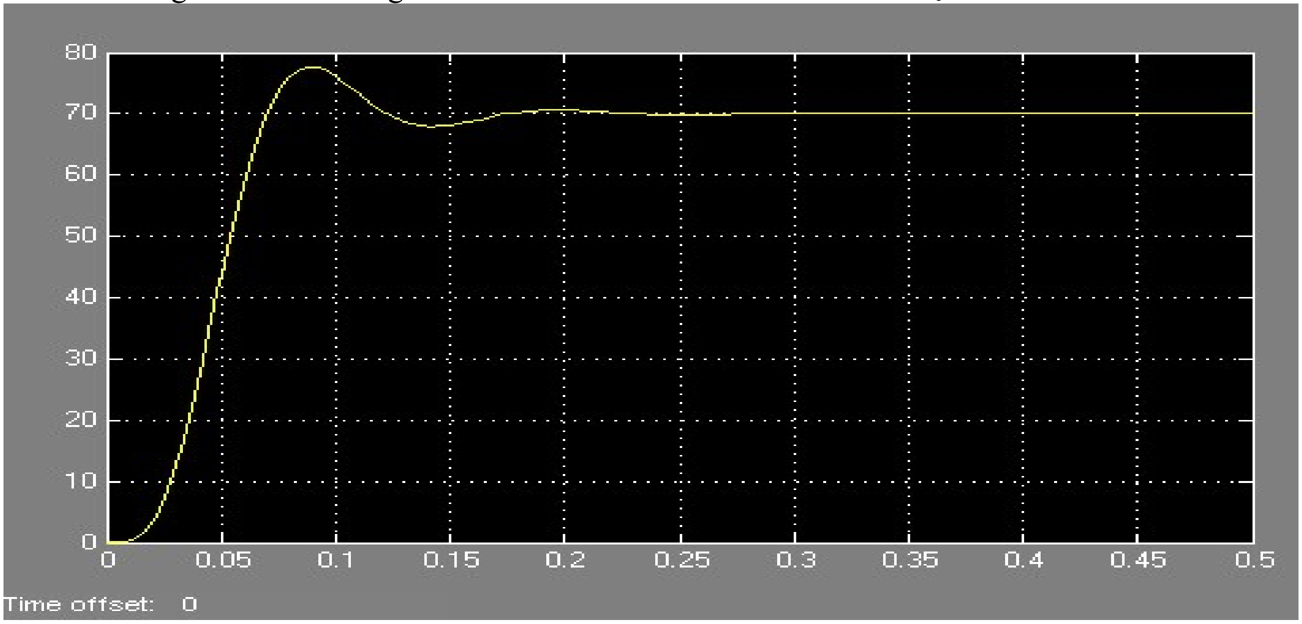


The next output plot uses $\omega_c = 20\pi$, giving a smoother result, although it takes longer to get there:



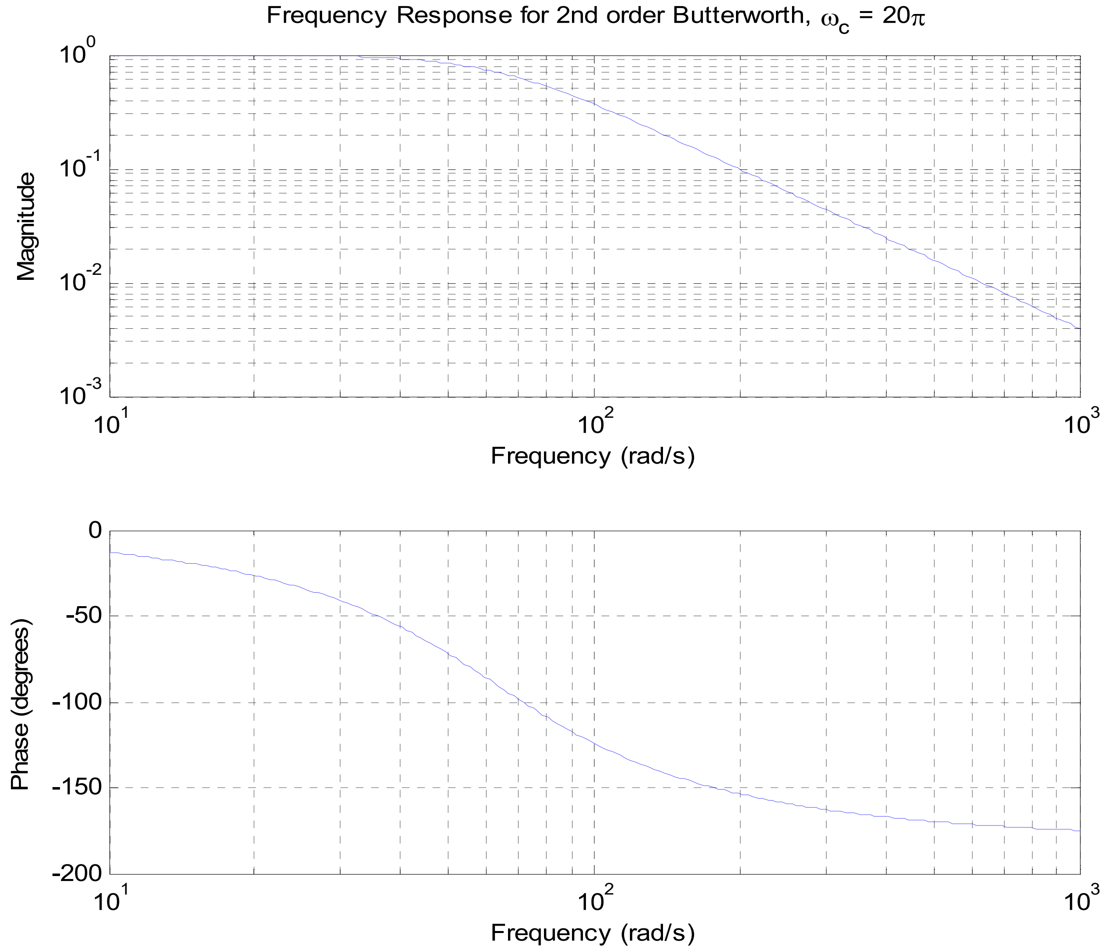
(b)

Here is the signal after filtering with a 4th order Butterworth filter with $\omega_c = 20\pi$:

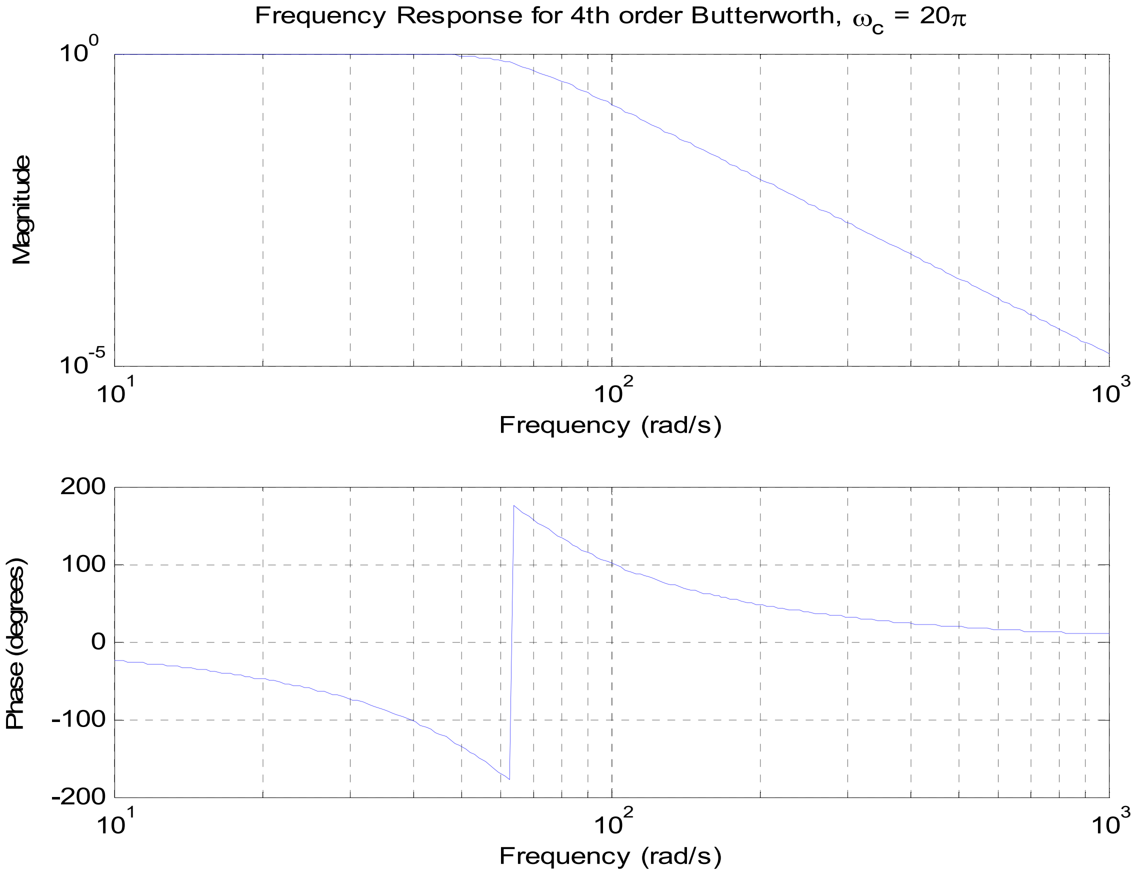


6.11, (c)

```
[b, a] = butter(2, 20*pi, ‘s’);
freqs(b, a);
```



```
[b, a] = butter(4, 20*pi, ‘s’);
freqs(b, a);
```



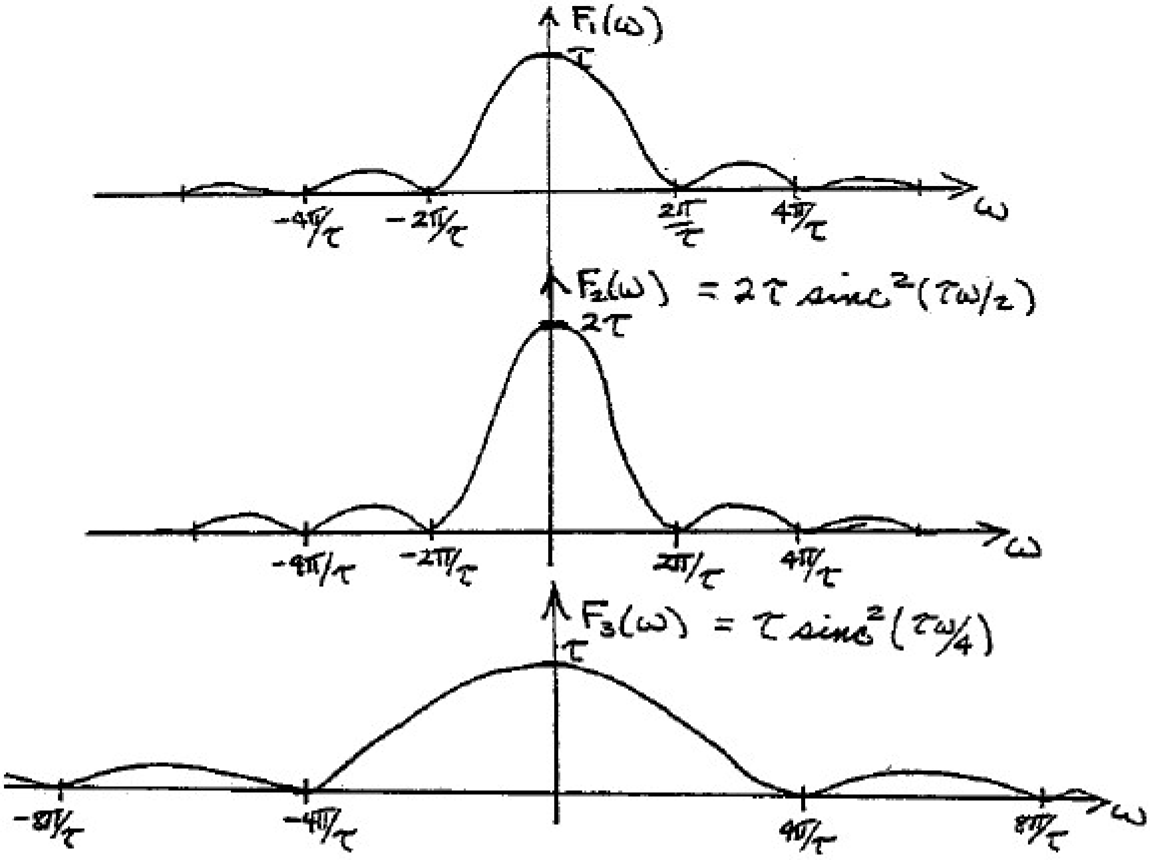
6.11, (d) For the 2nd order filter:

```
[b, a]=butter(2, 20*pi, 's');
h = freqs(b, a, [377:378]);
abs(h(1));
angle(h(1));
```

Gives: |H(377)| = 0.0278, θ(377) = -2.9.

For the 4th order filter: |H(377)| =7.715e-4 , θ(377) =0.44

6. 12(a) $f_1(t) = tri(t/\tau) \xleftrightarrow{\mathcal{F}} \tau\, sinc^2(\tau\omega/2) = F_1(\omega)$



(b) Shorter time duration results in wider bandwidth.

6.13

(a) Filter A is a high-pass filter since the DC component of the signal was removed and the high-frequency components remain

(b) Filter B is a low-pass filter since the signal was smoothed

6.14

(a) $V(\omega) = \frac{\pi}{j}[\delta(\omega-200) - \delta(\omega+200)]$

THE HIGHEST FREQUENCY COMPONENT IS $|\omega| = 200$ rad/s

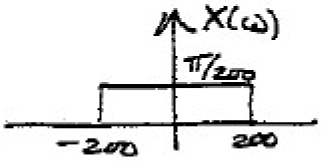
$\therefore\ \omega_s > 2\omega_M \Rightarrow \underline{\omega_s > 400 \text{ rad/s}}$

(b) $W(\omega) = \frac{\pi}{j}[\delta(\omega-100) - \delta(\omega+100)] - 4\pi[\delta(\omega-100\pi) + \delta(\omega+100\pi)]$
$+ 30\pi[\delta(\omega-200) + \delta(\omega+200)]$

$\Rightarrow \underline{\omega_s > 200\pi}$ rad/s

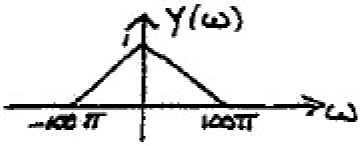
(c) $X(\omega) = \frac{\pi}{200} \text{rect}\left(\frac{\omega}{400}\right)$

$\underline{\omega_s > 2(200) = 400 \text{ rad/s}}$



(d) $Y(\omega) = \text{tri}(\omega/100\pi)$

$\underline{\omega_s > 200\pi}$

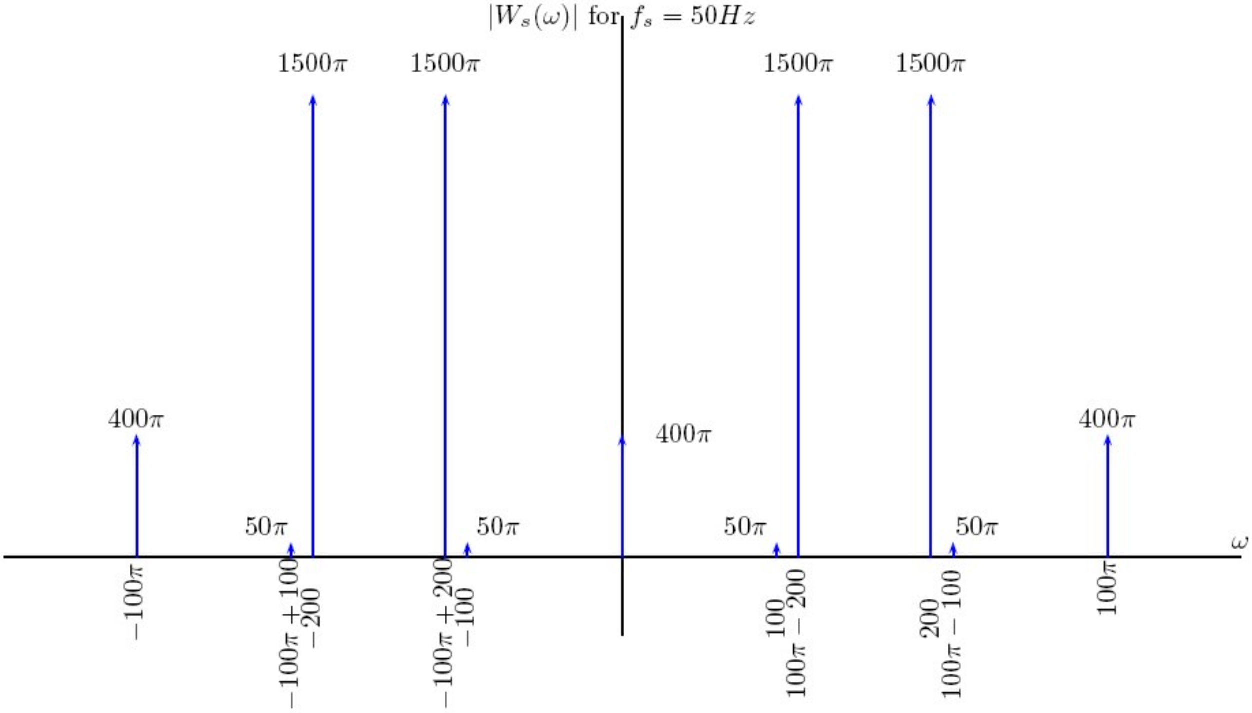


(e) the signal is not bandlimited; hence aliasing will occur at any sampling frequency. At higher frequencies, less aliasing will occur.

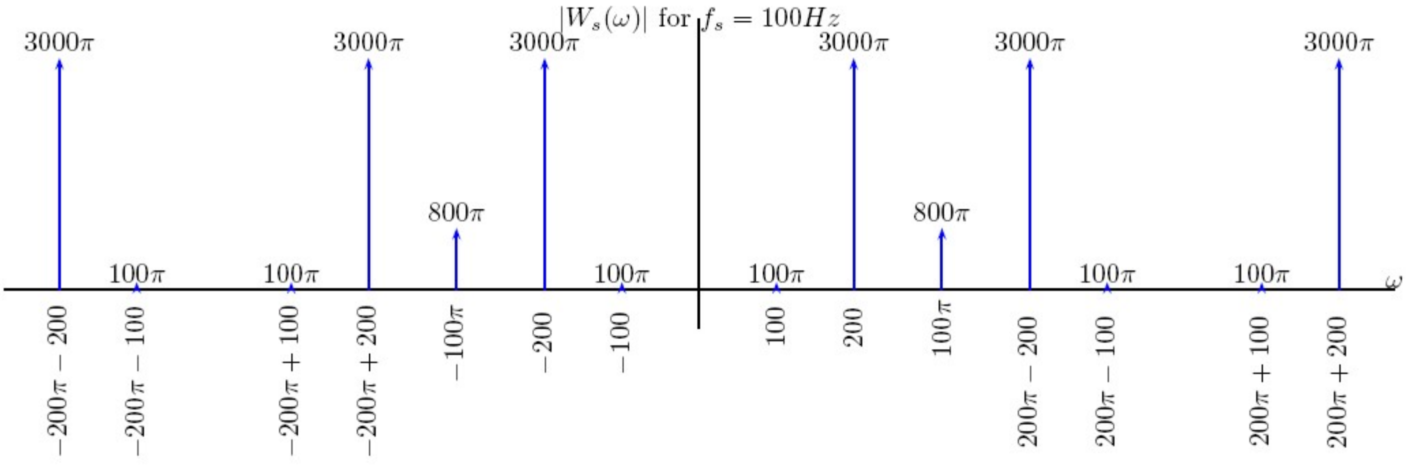
(f) same as (e): the signal is a sinc in the frequency domain, which is not bandlimited, so aliasing will occur at any sampling frequency. However, the width of the main lobe of the sinc is $\pm\frac{\pi}{10^{-3}}$, so sampling at least twice this ($\omega_s \geq 2000\pi$) will prevent aliasing of the main lobe (there will still be some aliasing of the smaller sidelobes).

6.15 (a) Frequency spectra:

(a) For $f_s = 50$Hz:



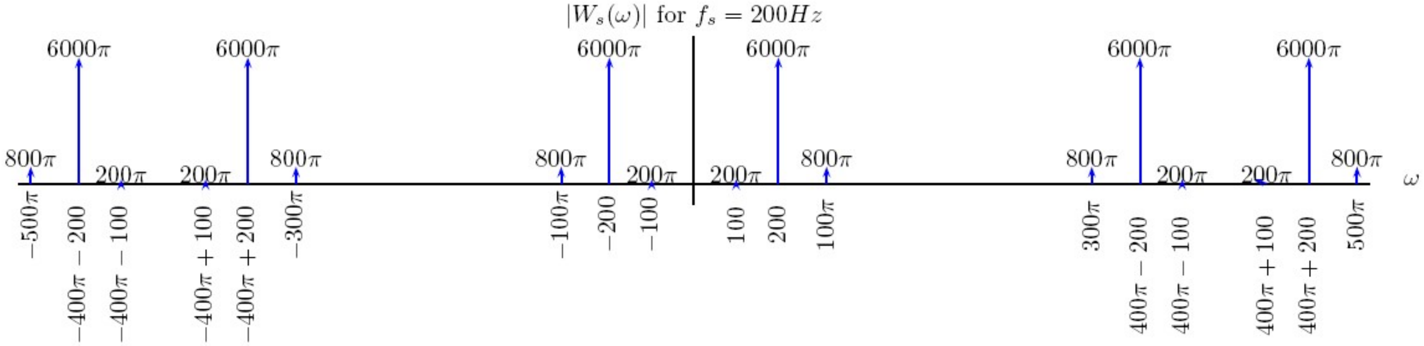
For $f_s = 100$Hz:



Continued→

6.15(a), continued

For $f_s = 200\text{Hz}$:



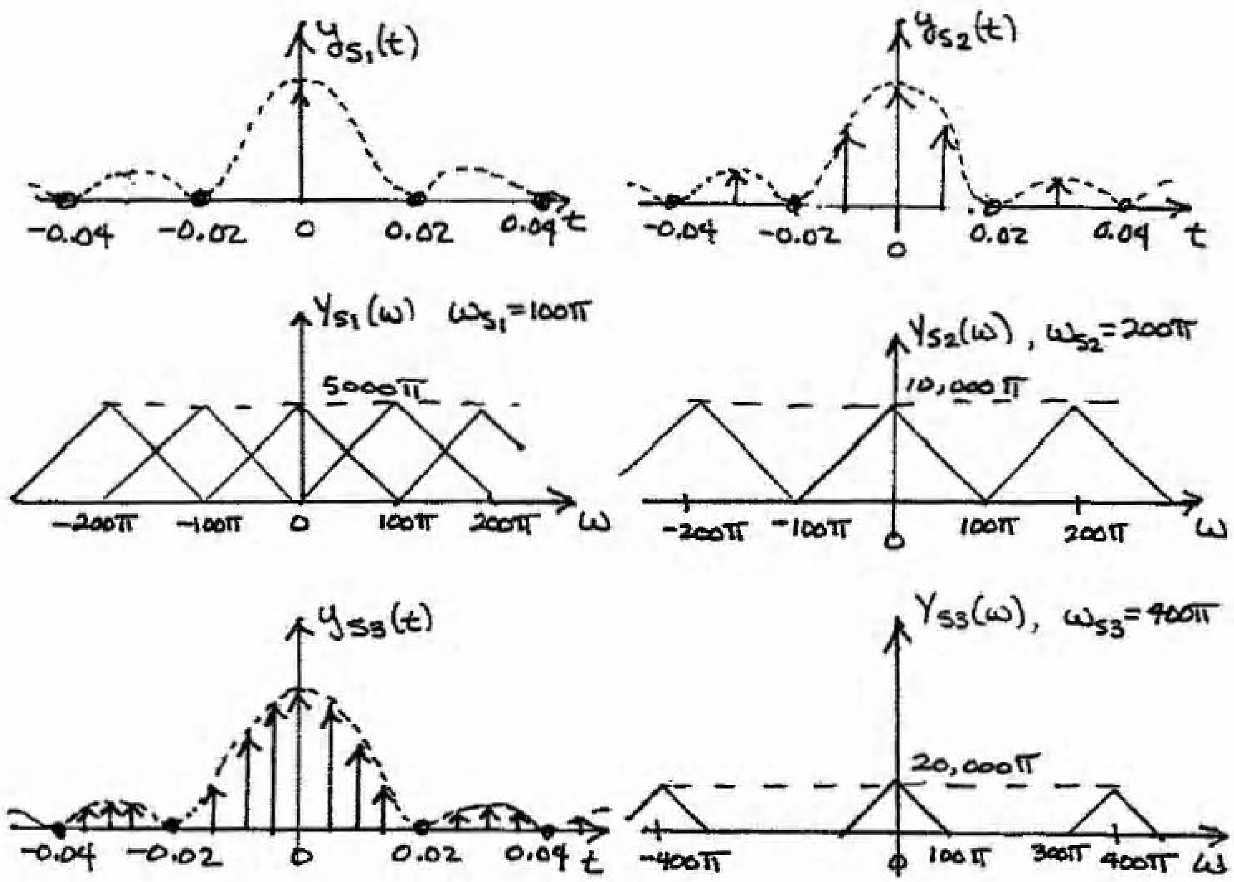
$|W_s(\omega)|$ for $f_s = 200Hz$

(b) The sampling frequency for this signal must be greater than 100 Hz. Therefore 50 Hz and 100 Hz are too low; the 200 Hz sampling frequency is suitable to avoid aliasing.

Continued→

6.15, continued

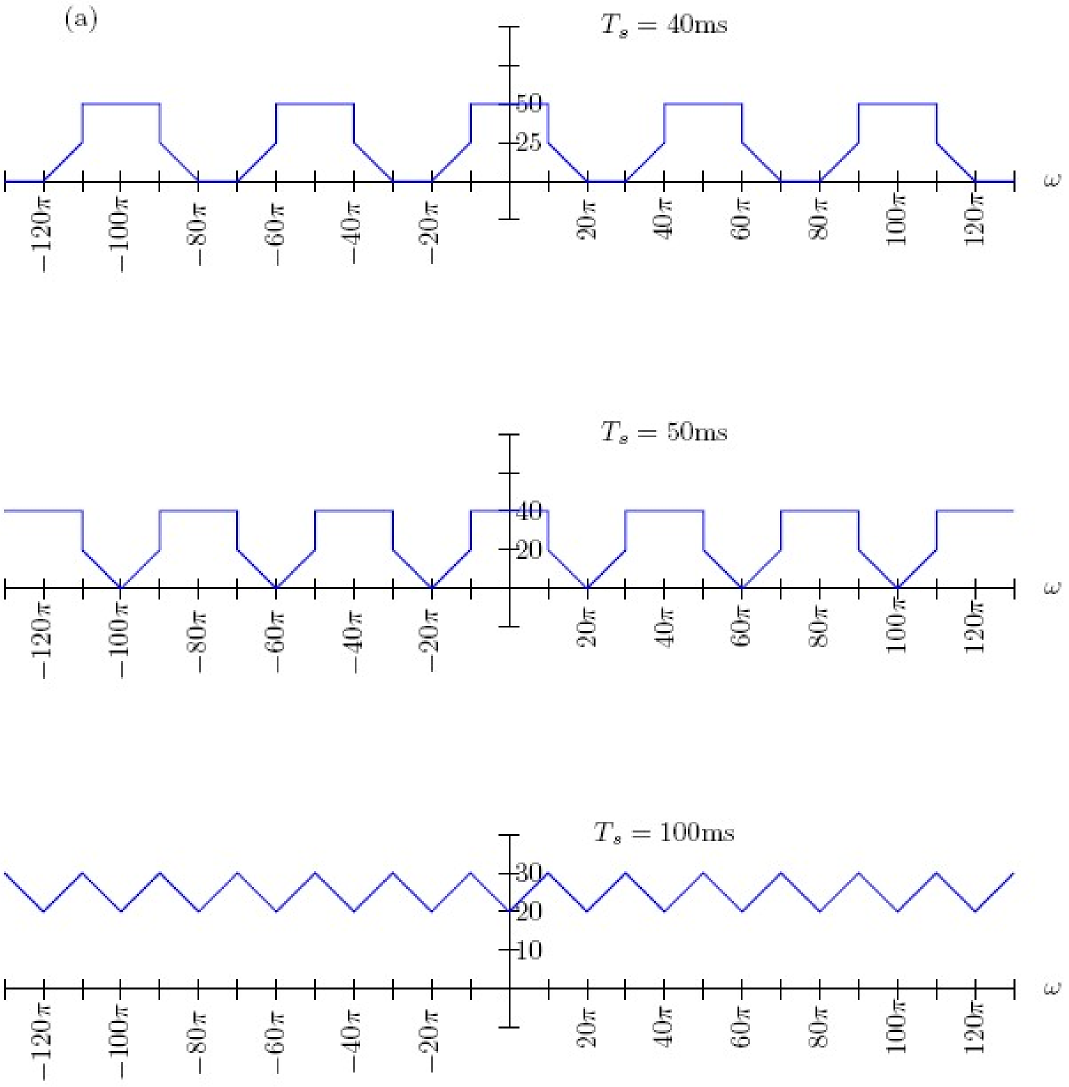
(c) (a)



(b) $f_s = 50Hz$ is not a suitable sampling frequency for this signal. $f_s = 50Hz$ is one-half the Nyquist rate for the signal. Aliasing is seen in the frequency spectrum.

$f_s = 100Hz$ is a satisfactory sampling frequency. This is the Nyquist rate for the signal.

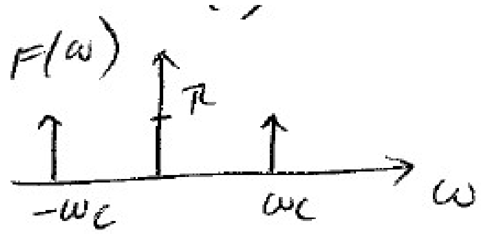
6.16



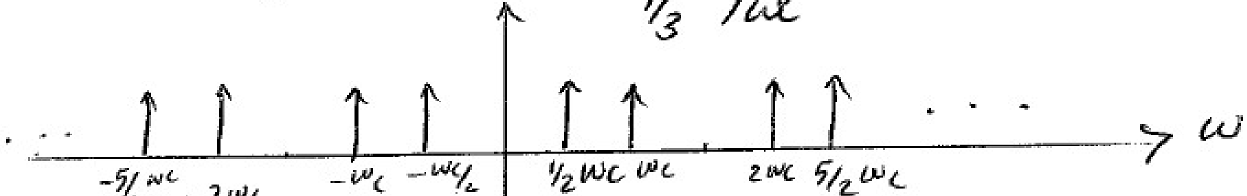
(b) Sampling frequencies of 50π and 40π rad/sec (sampling periods of 40ms and 50ms) are acceptable; sampling frequency of 20π rad/sec (sampling period of 100ms) is not, since it causes aliasing.

6.17

$f(t) = \cos\omega_c t$



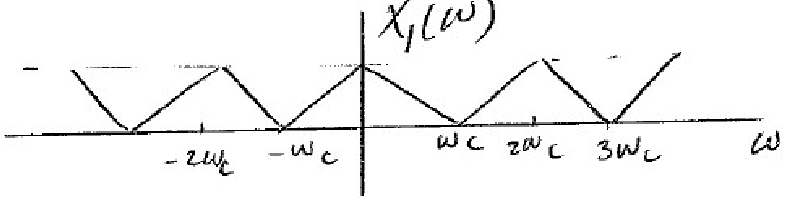
$T = 4/3\,\frac{\pi}{\omega_c}$, $\omega_s = \frac{2\pi}{T} = \frac{2\pi}{4/3\,\pi/\omega c} = 3/2\,\omega c$



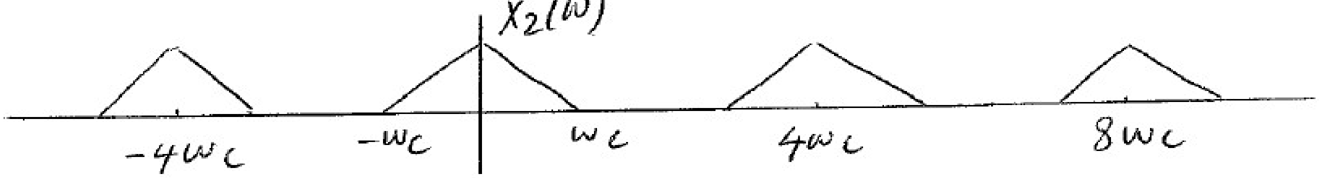
Sampled Spectrum

6.18

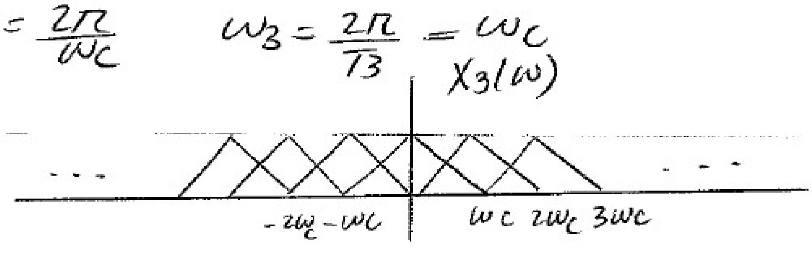
$T = \frac{\pi}{\omega_c}$ $\omega_1 = \frac{2\pi}{T_1} = 2\omega c$



$T_2 = \frac{\pi}{2\omega_c}$ $\omega_2 = \frac{2\pi}{T_2} = 4\omega_c$



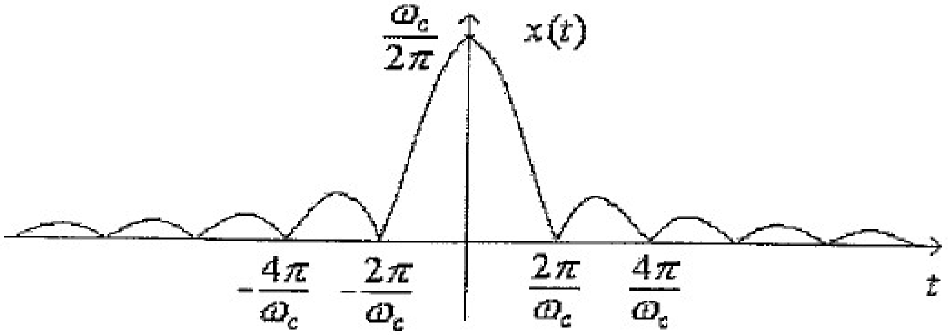
$T_3 = \frac{2\pi}{\omega_c}$ $\omega_3 = \frac{2\pi}{T_3} = \omega_c$



Aliasing example

6.19

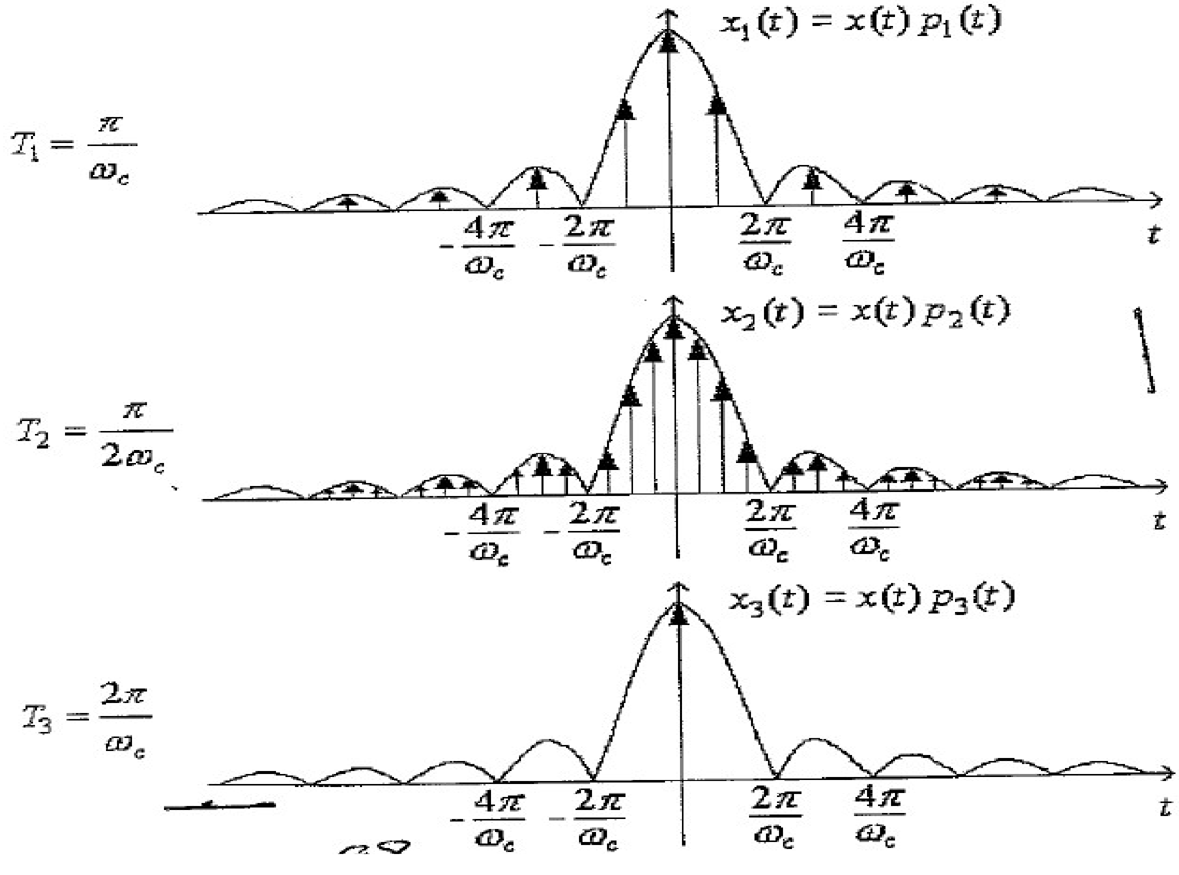
$$x(t) = \frac{\omega_c}{2\pi}\,\text{sinc}^2\left(\frac{\omega_c t}{2}\right)$$



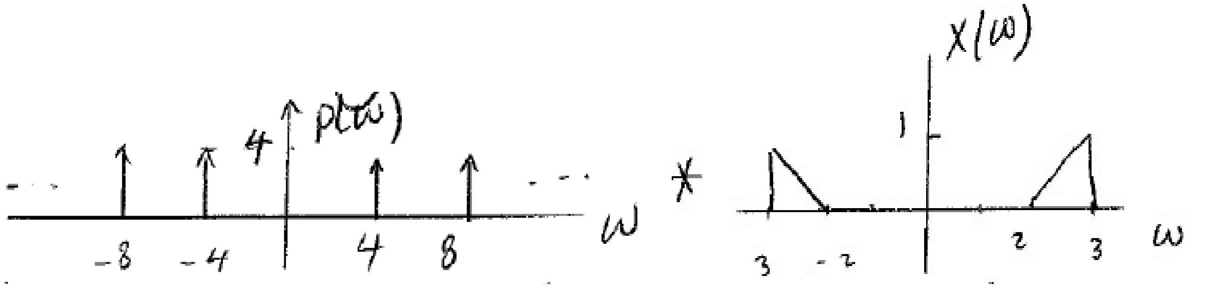
Draw the sampled signals using the sampling trains of the previous example

$$\left(T_1 = \frac{\pi}{\omega_c},\; T_2 = \frac{\pi}{2\omega_c},\; \text{and } T_3 = \frac{2\pi}{\omega_c} \right).$$

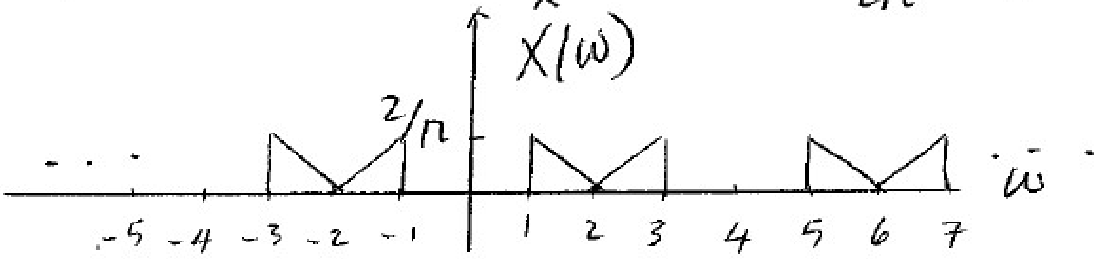
Notice how aliasing looks in the time domain.



6.20



$$\hat{x}(t) = x(t)\,P(t) \longleftrightarrow \hat{X}(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$



6.21

$\omega_0 = \frac{3\pi}{4}$ or $\omega_s \geq \frac{6\pi}{4} = \frac{3}{2}\pi$

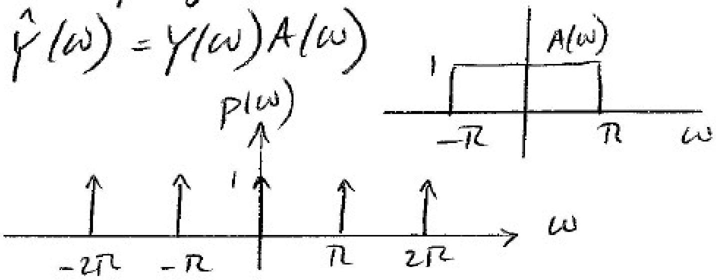
require $\omega_s \geq 2\omega_0$

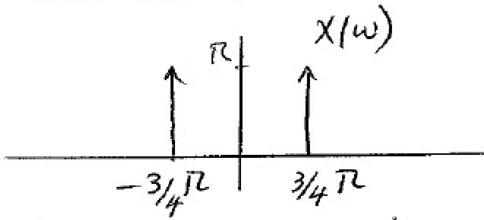
The given $r(t)$ has $T = 2$

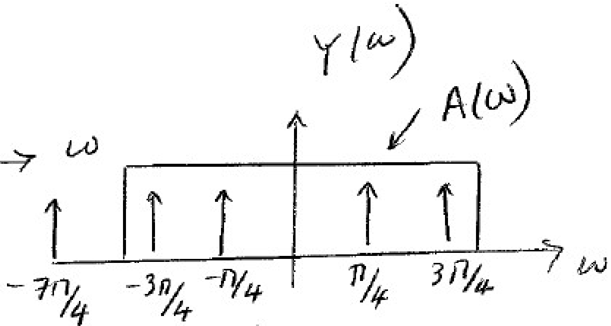
a) $\omega_s \geq \frac{3}{2}\pi \rightarrow 2\pi/T \geq \frac{3}{2}\pi$ or $T \leq 4/3$

$\therefore$ Sampling Therorem is violated

b) $\hat{Y}(\omega) = Y(\omega)A(\omega)$







$Y(\omega) = \frac{1}{2\pi} X(\omega) * p(\omega)$

only 4 impulses pass through $A(\omega)$

$\therefore \hat{y}(t) = \underbrace{\frac{1}{2\pi} \cos \pi/4\, t}_{\text{aliasing}} + 1/2\pi \cos \frac{3\pi}{4} t$

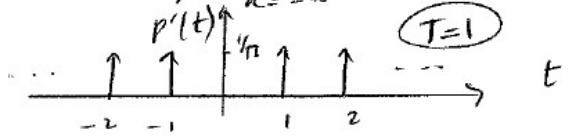
6.22

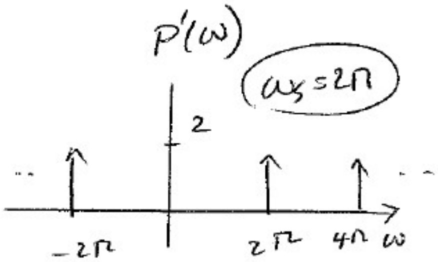
$x(t) = \cos\frac{2\pi}{4}t \qquad \omega_0 = \pi/2$

a) require : $\omega_s \geq 2\omega_0 = \pi$

$T_s = \frac{2\pi}{\omega_s} \quad \therefore \quad T_s \leq 2$

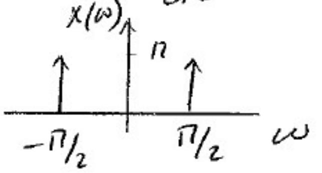
b) $p'(t) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\delta(t-k)$

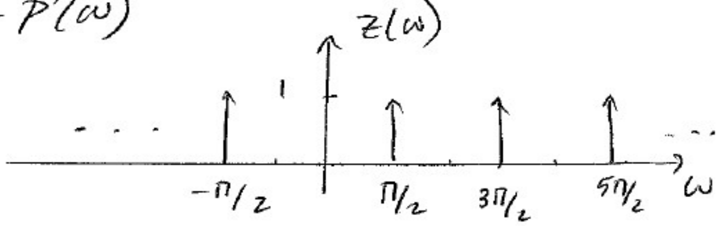




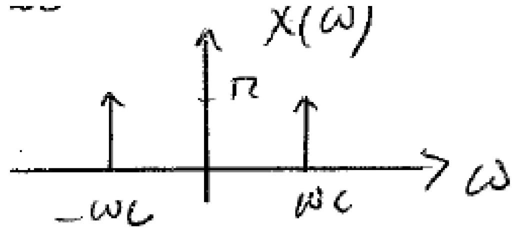
This satisfies sampling criterion of part a) so no aliating occurs

$Z(\omega) = \frac{1}{2\pi}X(\omega) * P'(\omega)$

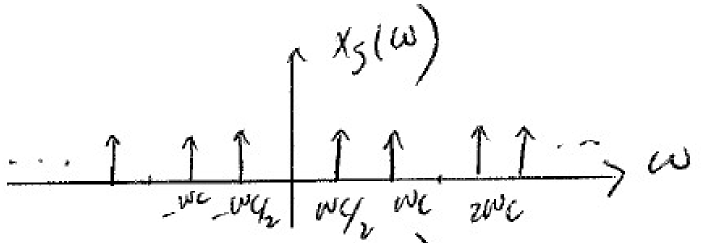


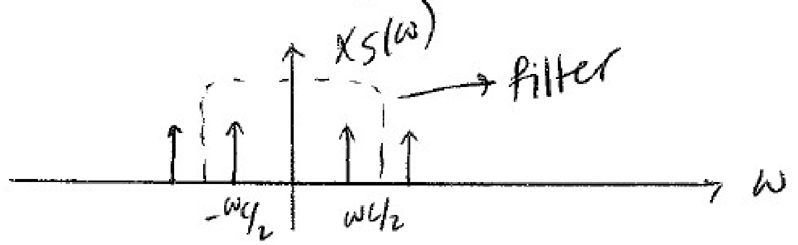


6.23 $\omega_s = 3/2\,\omega_c$



a)





(b)

$y(t) = \cos\frac{\omega_c}{2}t$ and aliating has occured

6.24

a) 40 Hz Sampled @ 60 Hz

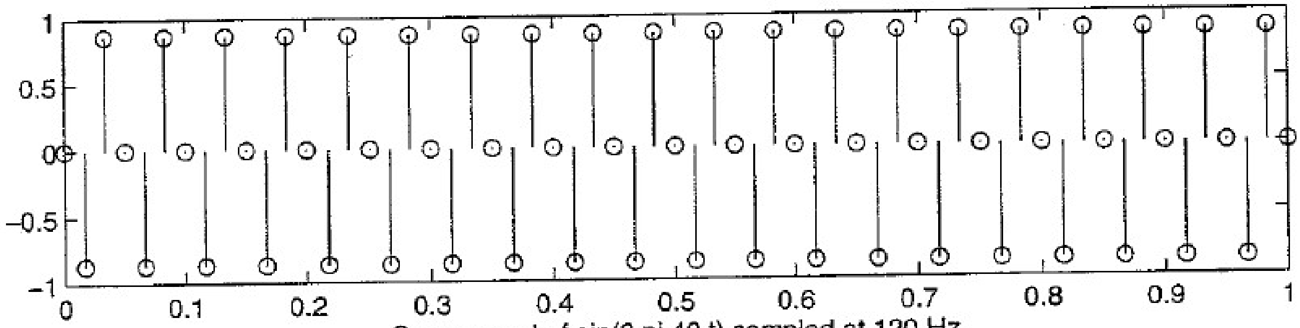
looks like 20 Hz due to aliasing

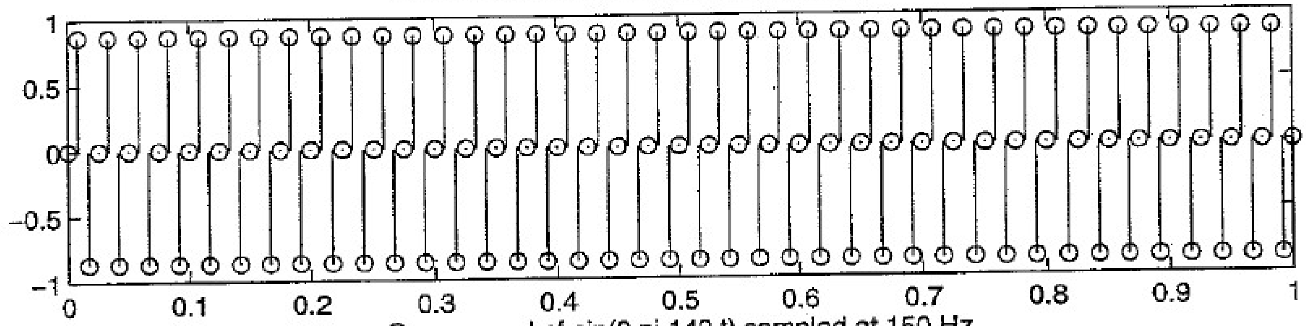
b) 40 Hz Sampled @ 120 Hz

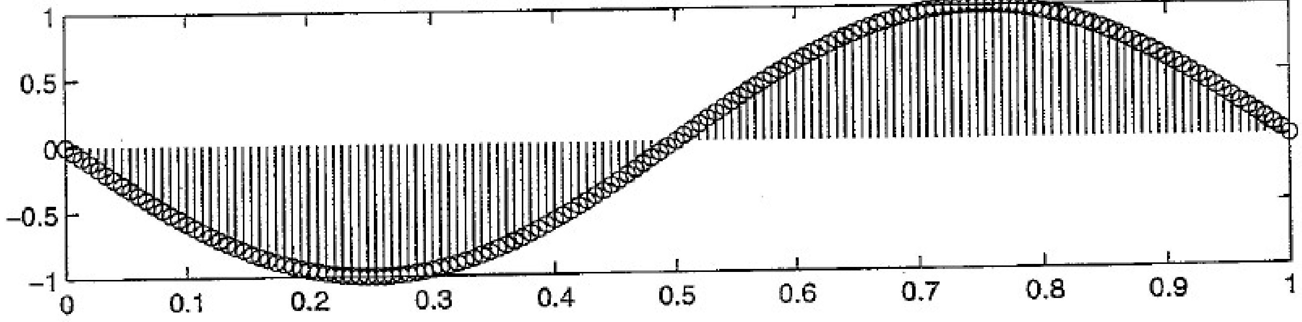
NO aliasing So looks like 40 Hz

c) 149 Hz Sampled @ 150 Hz

looks like 1Hz due to aliasing


One second of sin(2 pi 40 t) sampled at 60 Hz.


One second of sin(2 pi 40 t) sampled at 120 Hz.

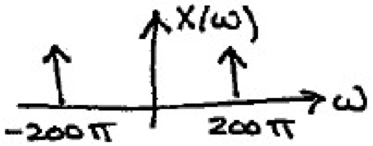

One second of sin(2 pi 149 t) sampled at 150 Hz.

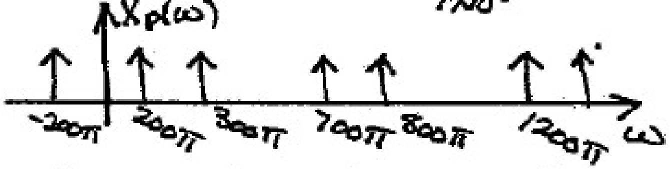
6.25

(a) $x(t)$ is bandlimited signal, so that its frequency components above some finite frequency, ω_m, are negligible. Then $\omega_s > 2\omega_m \Rightarrow T_s < \frac{\pi}{\omega_m}$

(b) To recover the original signal from $x_p(t)$, pass the signal through a lowpass filter so that all frequency components $|\omega| > \frac{\omega_s}{2}$ are eliminated,

(c) $x(t) = \cos(200\pi t)$, $T_s = 0.004s \Rightarrow \omega_s = \frac{2\pi}{4\times10^{-3}} = 500\pi$



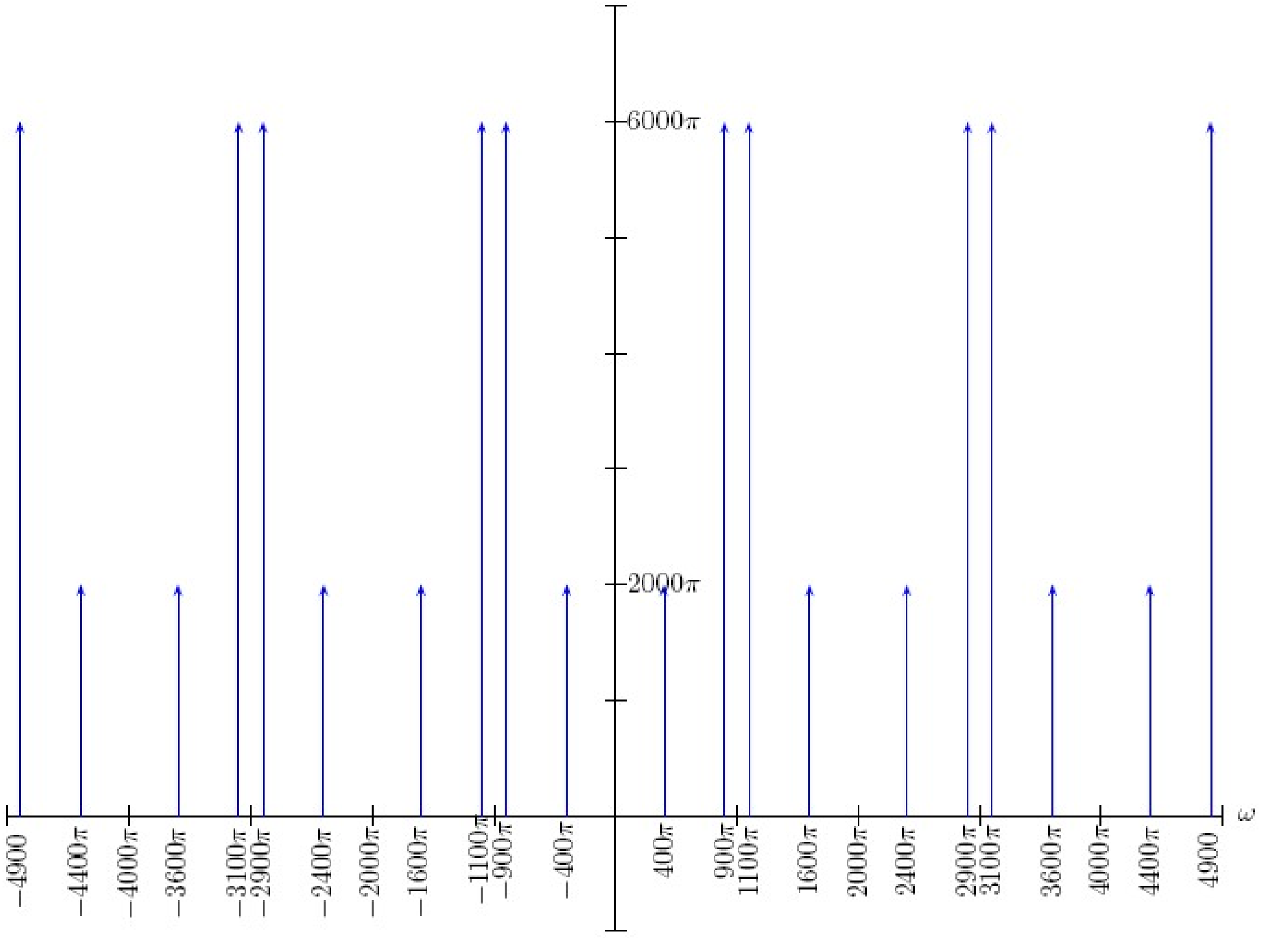


frequency Components less than 700 Hz (1400π rad/s) in $x_p(t)$ are: $\pm200\pi$, $\pm300\pi$, $\pm700\pi$, $\pm800\pi$, $\pm1200\pi$, $\pm1300\pi$

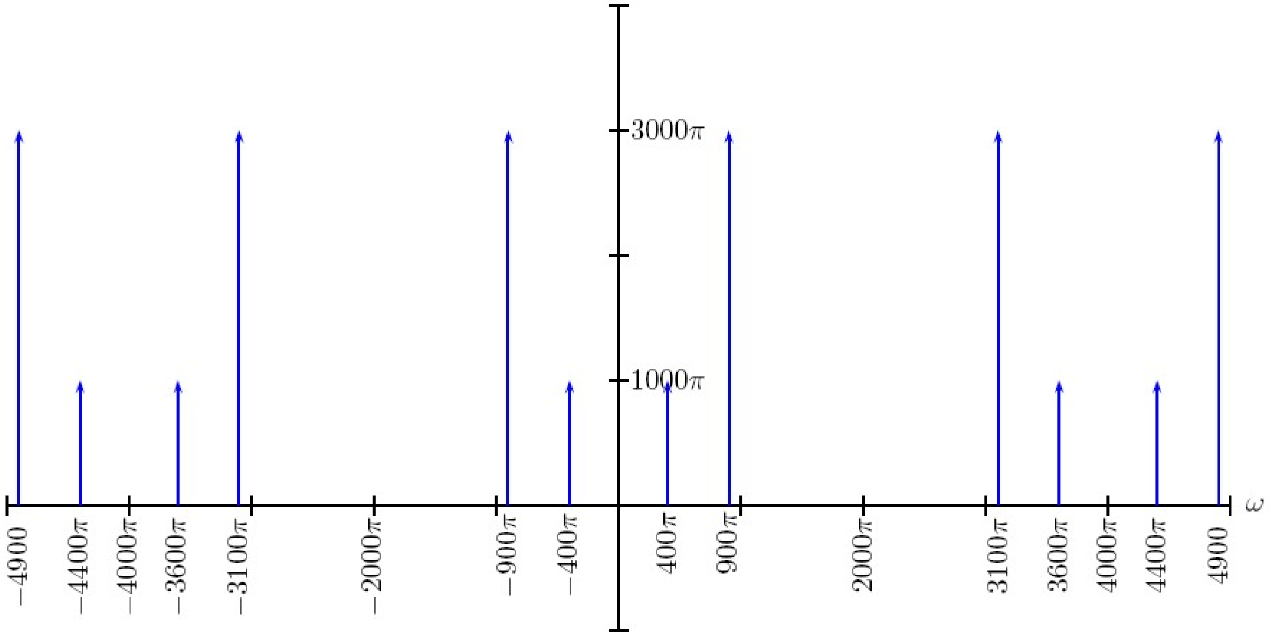
(d) $f_x = 100\text{Hz} \Rightarrow \omega_x = 300\pi \therefore x(t) = \cos(300\pi t)$

6.26

(a)


$T_s = 1\text{ms}$

(b)


$T_s = 0.5\text{ms}$

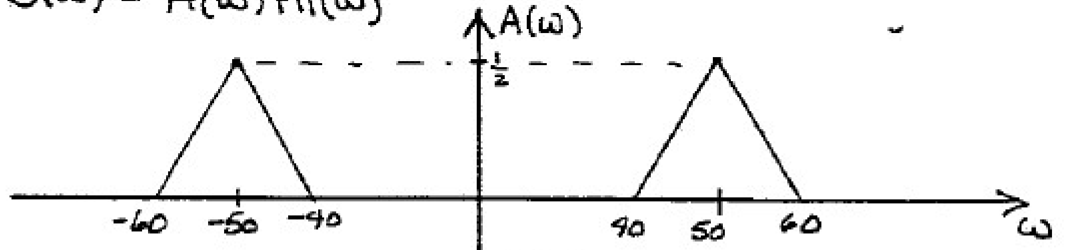
(c) The Nyquist rate for the signal is 1800π rad/sec = 900 Hz, so the sampling rate must be greater than this, or equivalently, the sampling period must be less than 1.11 ms.

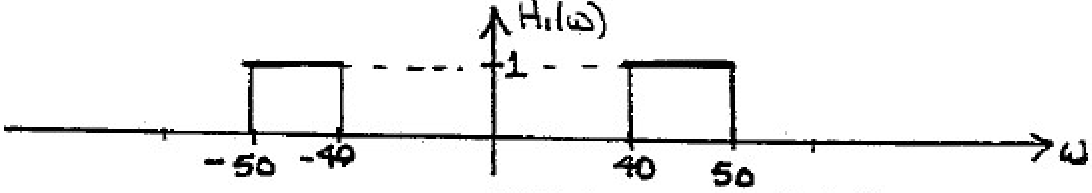
6.27

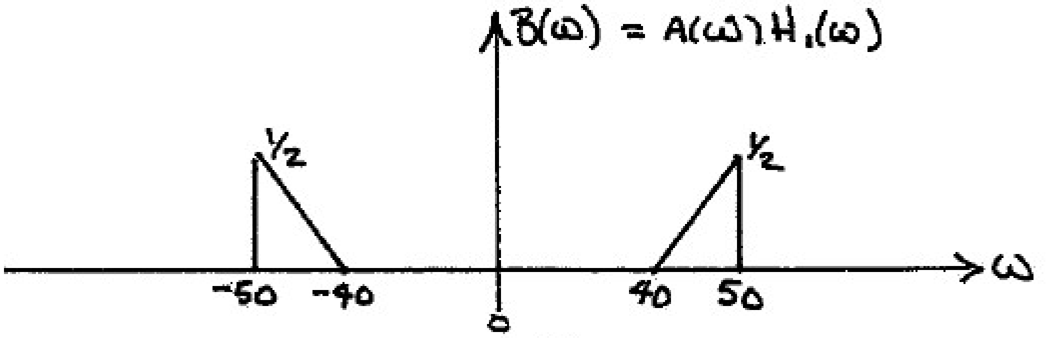
$a(t) = x(t)\cos 50t \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}X(\omega) * \pi[\delta(\omega-50)+\delta(\omega+50)]$

$A(\omega) = \frac{1}{2}X(\omega-50)+\frac{1}{2}X(\omega+50)$

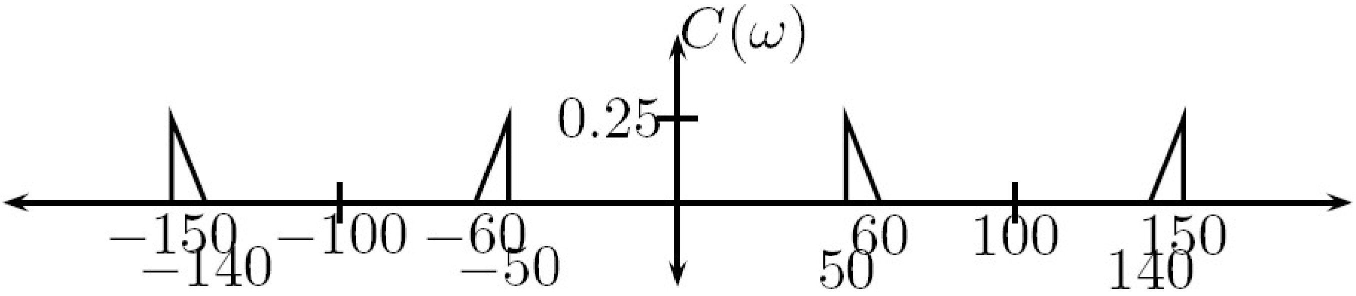
$B(\omega) = A(\omega)H_1(\omega)$

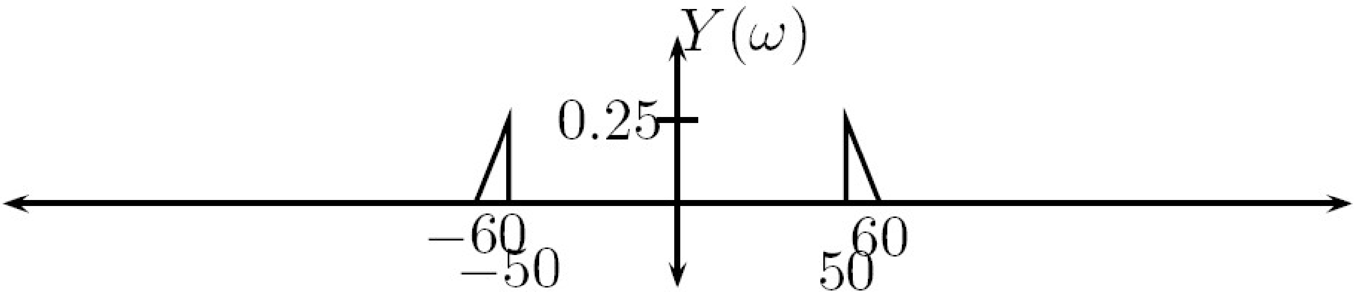






$c(t) = b(t)\cos(100t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}B(\omega-100)+\frac{1}{2}B(\omega+100)$





6.28

$$x(t) = m(t)C_1(t) = m(t)\cos(\omega_c t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[M(\omega-\omega_c)+M(\omega+\omega_c)\right]$$

$$y(t) = x(t)C_2(t) = x(t)\cos(\omega_c t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(\omega-\omega_c)+X(\omega+\omega_c)\right]$$

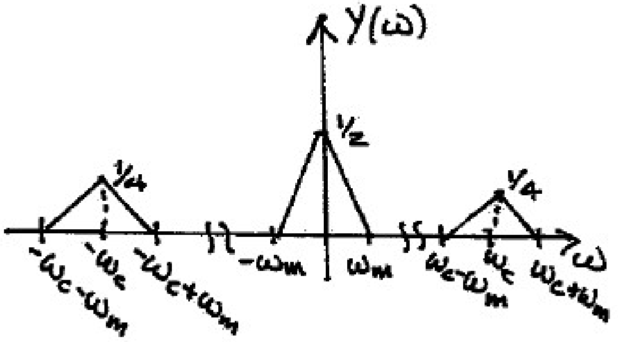
$$X(\omega+\omega_c) = \frac{1}{2}\left[M(\omega+2\omega_c)+M(\omega)\right]$$

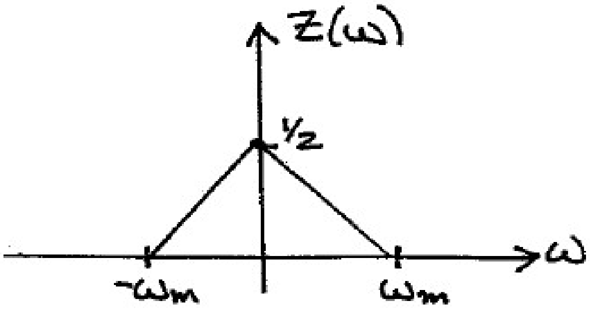
$$X(\omega-\omega_c) = \frac{1}{2}\left[M(\omega-2\omega_c)+M(\omega)\right]$$

$$\therefore Y(\omega) = \tfrac{1}{4}\left[2M(\omega)+M(\omega-2\omega_c)+M(\omega+2\omega_c)\right]$$

$$Z(\omega) = Y(\omega)H(\omega) = \begin{cases} Y(\omega), & |\omega| \le \omega_m \\ 0, & |\omega| > \omega_m \end{cases}$$

$$\therefore Z(\omega) = \frac{1}{2}M(\omega)$$





6.29

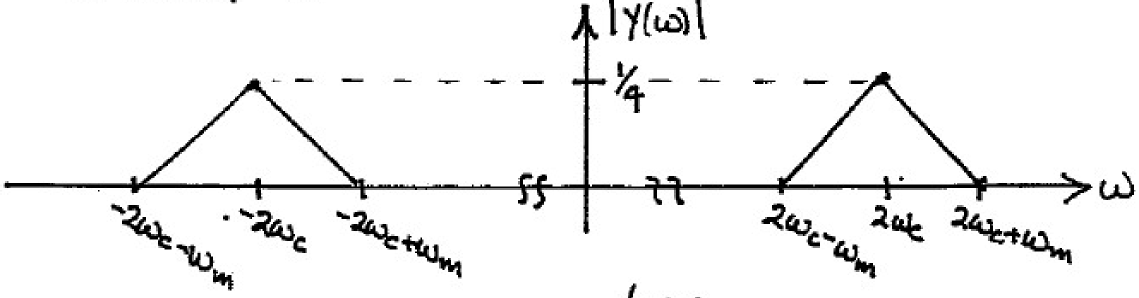
$$\text{from } 6.17,\ X(\omega) = \frac{1}{2}\left[M(\omega-\omega_c)+M(\omega+\omega_c)\right]$$

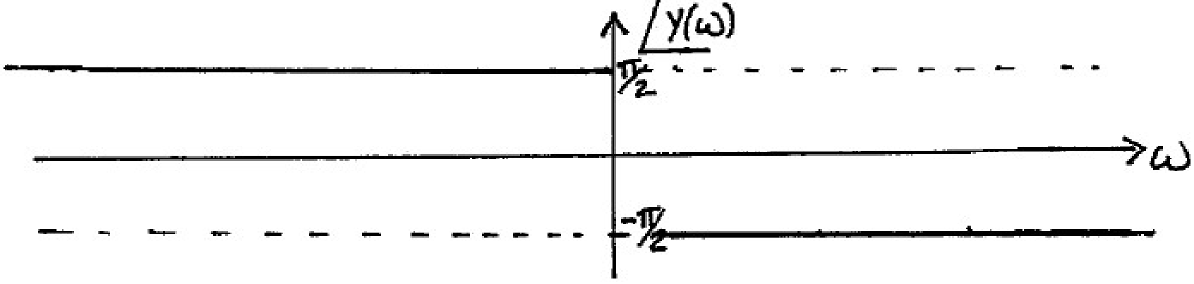
$$y(t) = x(t)\sin(\omega_c t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = \frac{1}{2j}\left[X(\omega-\omega_c)-X(\omega+\omega_c)\right]$$

$$Y(\omega) = \frac{1}{4j}\left[M(\omega-2\omega_c)+M(\omega+2\omega_c)\right]$$

$$Z(\omega) = Y(\omega)H(\omega) = \begin{cases} Y(\omega), & |\omega| \le \omega_m \\ 0, & |\omega| > \omega_m \end{cases}$$

$$\therefore Z(\omega) = 0$$





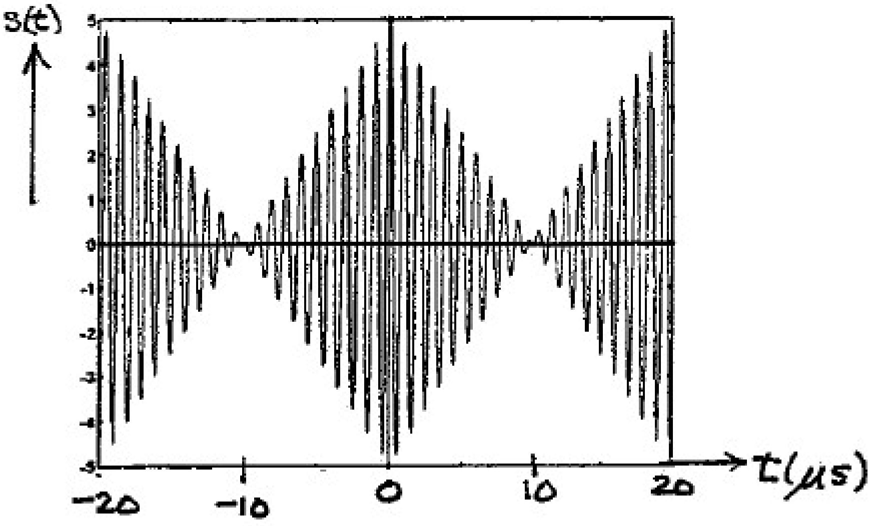
6.30 (a) $g_1(t) = \frac{1}{2}f_1(t) + \frac{1}{2}f_1(t)\cos(2\omega_c t) + \frac{1}{2}f_2(t)\sin(2\omega_c t)$

(b) $g_2(t) = \frac{1}{2}f_2(t) + \frac{1}{2}f_1(t)\sin(2\omega_c t) - \frac{1}{2}f_2(t)\cos(2\omega_c t)$

(c) $e_1(t) = \frac{1}{2}f_1(t)$ and $e_2(t) = \frac{1}{2}f_2(t)$

6.31

(a)



(b) $m(t) = -5 + \sum_{n=-\infty}^{\infty} 10\,\mathrm{tri}\left(\frac{t - n40\times10^{-6}}{40\times10^{-6}}\right) = -5 + \sum_{n=-\infty}^{\infty} g(t - nT_0)$

$$g(t) = 10\,\mathrm{tri}\left(\frac{t}{40\times10^{-6}}\right) \xleftrightarrow{\mathcal{F}} 4\times10^{-4}\,\mathrm{sinc}^2(10^{-5}\omega)$$

$$M(\omega) = -10\pi\,\delta(\omega) + \frac{2\pi}{40\times10^{-6}} \sum_{n=-\infty}^{\infty} 4\times10^{-4}\,\mathrm{sinc}^2\left(\frac{10^{-5} n\pi}{20\times10^{-6}}\right)\delta\left(\omega - \frac{n\pi}{20\times10^{-6}}\right)$$

$$= -10\pi\,\delta(\omega) + 20\pi \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2\left(\frac{n\pi}{2}\right)\delta\left(\omega - \frac{n\pi}{2\times10^{-5}}\right)$$

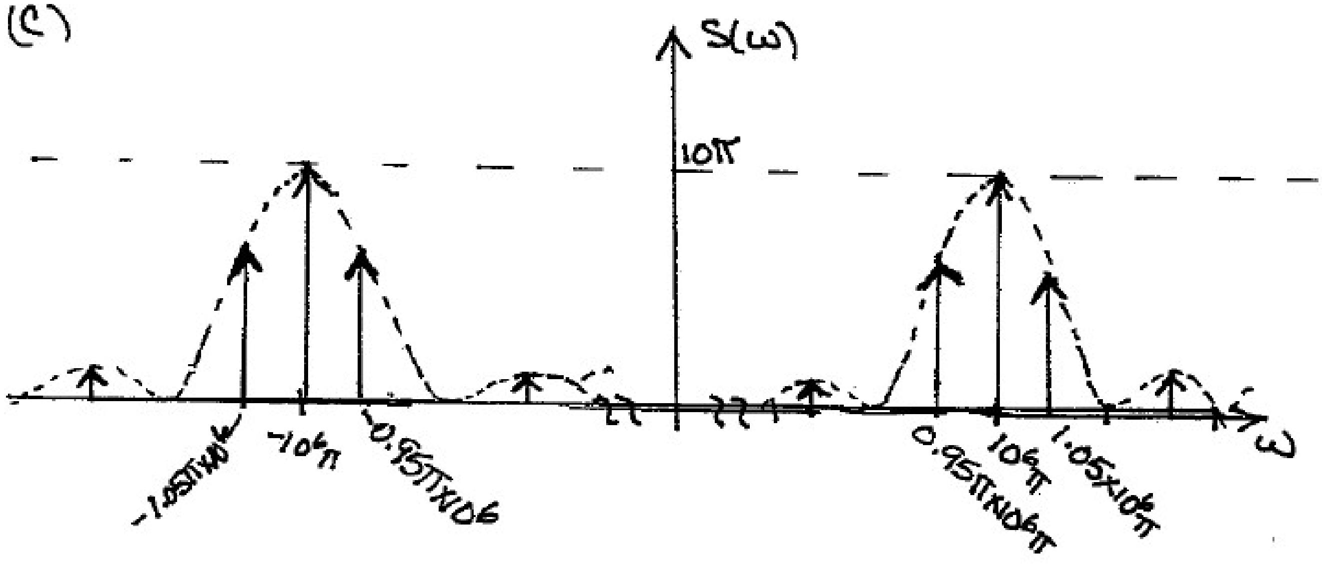
$$s(t) = m(t)\cos(10^6\pi t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} M(\omega) * \pi\left[\delta(\omega - 10^6\pi) + \delta(\omega - 10^6\pi)\right]$$

$$S(\omega) = \frac{1}{2} M(\omega - 10^6\pi) + \frac{1}{2} M(\omega + 10^6\pi)$$

$$= 5\pi\,\delta(\omega - 10^6\pi) + 10\pi \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2\left(\frac{n\pi}{2}\right)\delta\left(\omega - \left(1 + \frac{n}{20}\right)10^6\pi\right)$$

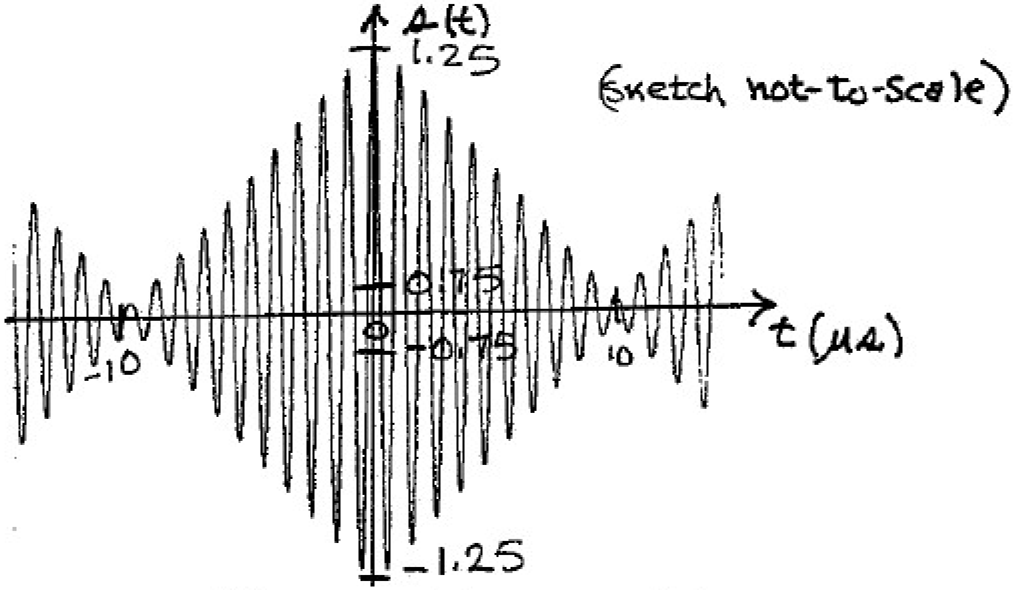
$$-5\pi\,\delta(\omega + 10^6\pi) + 10\pi \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2\left(\frac{n\pi}{2}\right)\delta\left(\omega + \left(1 - \frac{n}{20}\right)10^6\pi\right)$$

(c)



6.32

(a)



(Sketch not-to-Scale)

(b) $m(t) = -5 + \sum_{n=-\infty}^{\infty} 10\,\mathrm{tri}\left(\frac{t - n40\times10^{-6}}{20\times10^{-6}}\right)$

$m_2(t) = 1 + k_a m(t) = 1 - 5k_a + 10k_a \sum_{n=-\infty}^{\infty} \mathrm{tri}\left(\frac{t - n40\times10^{-6}}{20\times10^{-6}}\right)$

$M_2(\omega) = (1-5k_a)2\pi\delta(\omega) + 10\pi k_a \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2\left(\frac{n\pi}{2}\right)\delta\left(\omega - \frac{n\pi}{20\times10^{-6}}\right)$

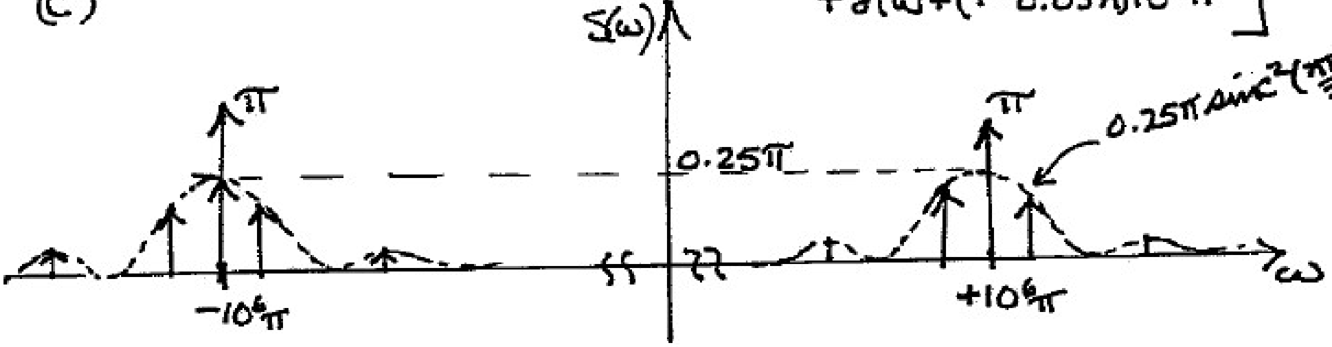
$s(t) = m_2(t)\cos 10^6\pi t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}M_2(\omega - 10^6\pi) + \frac{1}{2}M_2(\omega + 10^6\pi)$

$S(\omega) = 0.75\pi\left[\delta(\omega - 10^6\pi) + \delta(\omega + 10^6\pi)\right.$

$\qquad + 0.25\pi \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2\left(\frac{n\pi}{2}\right)\left[\delta(\omega - (1+0.05n)10^6\pi)\right.$

$\qquad\qquad \left. + \delta(\omega + (1-0.05n)10^6\pi\right]$

(c)



6.33

$$s(t) = m(t)\,p(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} M(\omega) * P(\omega) = S(\omega)$$

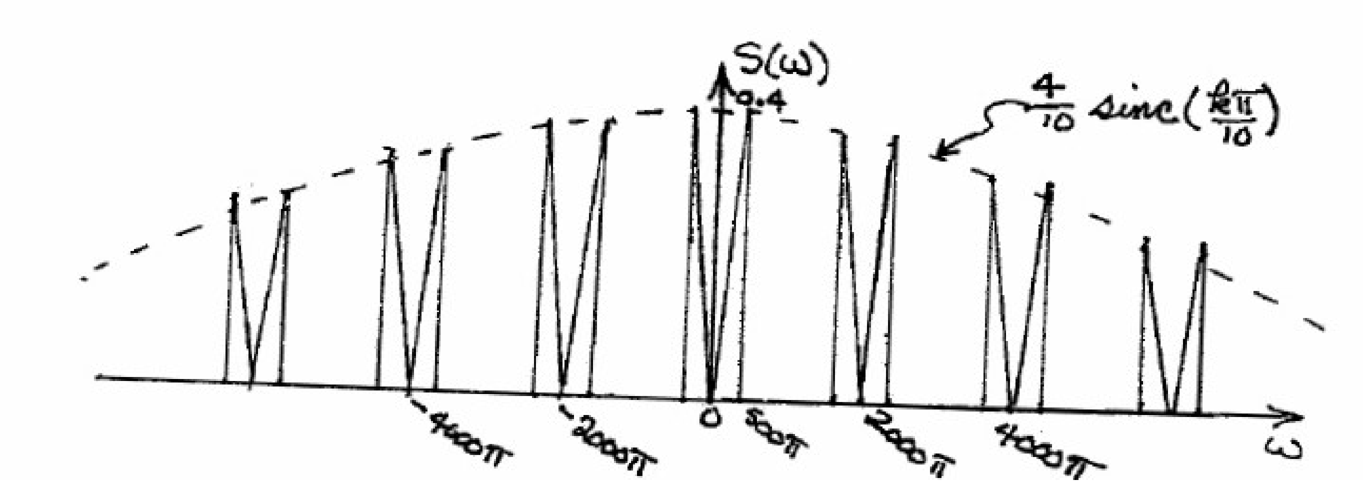
$$P(\omega) = \sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_c), \quad C_k = \frac{\Delta}{T_0}\,\mathrm{sinc}(k\omega_c\Delta/2) \quad (6.19)$$

$$C_k = \frac{1\times10^{-4}}{1\times10^{-3}}\,\mathrm{sinc}\left(k\left(\frac{2\pi}{10^{-3}}\right)\times\frac{10^{-4}}{2}\right) = \frac{1}{10}\,\mathrm{sinc}\left(k\pi/10\right)$$

$$P(\omega) = \frac{2\pi}{10}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k\pi}{10}\right)\delta(\omega - k\,2000\pi)$$

$$S(\omega) = \frac{1}{10} M(\omega) * \sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k\pi}{10}\right)\delta(\omega - k\,2000\pi)$$

$$S(\omega) = \frac{1}{10}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k\pi}{10}\right) M(\omega - k\,2000\pi)$$



6.34

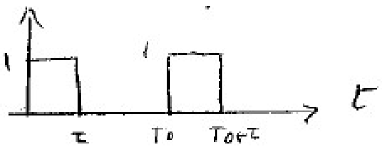
$f_s = 2.4\,\mathrm{MHz} \Rightarrow 2.4\times10^6$ PULSES/s.

(a) $\tau = 8\times10^{-6}$ (s/pulse) $\Rightarrow R_{max} = \frac{1}{\tau} \doteq 0.125\times10^6$ (pulses/s./SIGNAL)

$$\frac{2.4\times10^6 \text{ (PULSES/s.)}}{0.125\times10^6 \text{(PULSES/s/SIGNAL)}} = 19.2 \Rightarrow 19 \text{ SIGNALS CAN BE MULTIPLEXED}$$

(b) 1st null bandwidth of a rectangular pulse $= \frac{2\pi}{\tau}$

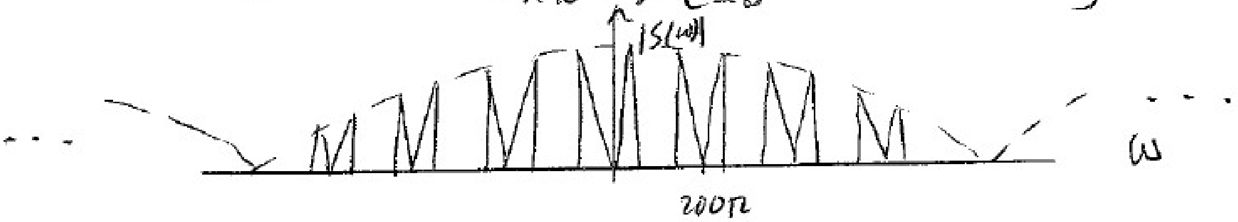
$$\omega_c = \frac{2\pi}{8\times10^{-6}} = 785.4 \text{(K-rad/s)}$$

6.35 

$$S(\omega) = \frac{\tau}{T_0} sinc\left(\frac{\omega\tau}{2}\right)\left[\sum_{-\infty}^{\infty} M(\omega - n\omega_s)\right] e^{\frac{-j\omega t}{2}}$$

$T_0 = 1.0$ ms , $\omega_s = \frac{2\pi}{T_0} = 2000\pi$ rad/sec , $\tau = .1$ ms

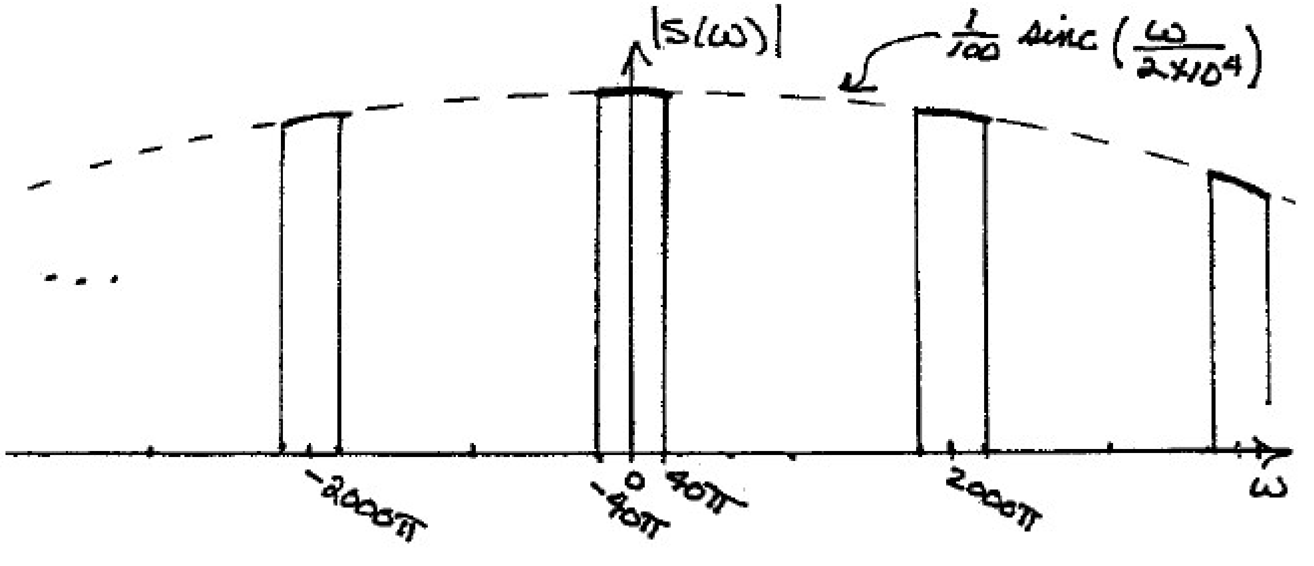
$$S(\omega) = 1/10\ sinc\left(\frac{\omega}{2\times10^4}\right)\left[\sum_{-\infty}^{\infty} M(\omega - 2000\ \pi n)\right] e^{-j\omega/2\times10^4}$$



6.36 $m(t) = 4\, sinc(40\pi t) \xleftrightarrow{\mathcal{F}} M(\omega) = \frac{1}{10} rect\left(\frac{\omega}{80\pi}\right)$

$$S(\omega) = \frac{\tau}{T_0} sinc\left(\frac{\omega\tau}{2}\right)\left[\sum_{n=-\infty}^{\infty} M(\omega - n\omega_s)\right] e^{-j\omega\tau/2}$$

$$= \frac{1}{10} sinc\left(\frac{\omega}{2\times10^4}\right)\left[\sum_{n=-\infty}^{\infty} M(\omega - n2000\pi)\right] e^{-j\frac{\omega}{2\times10^4}}$$



6.37

a) $s_0(t) = A\phi_0(t)$

$$\left.\begin{aligned} r_0 &= \int_0^T s_0(t)\phi_0(t)\,dt = A\int_0^T \phi_0^2(t)\,dt = A \\ r_1 &= \int_0^T s_0(t)\phi_1(t)\,dt = A\int_0^T \phi_0(t)\phi_1(t)\,dt = 0 \end{aligned}\right\}$$

means a 0 bit was sent

b) $s_1(t) = A\phi_1(t)$

$$\left.\begin{aligned} r_0 &= \int_0^T s_1(t)\phi_0(t)\,dt = A\int_0^T \phi_1(t)\phi_0(t)\,dt = 0 \\ r_1 &= \int_0^T s_1(t)\phi_1(t)\,dt = A\int_0^T \phi_1(t)^2\,dt = A \end{aligned}\right\}$$

means a 1 bit was sent

6.38

a) Digital 0 : $s(t) = -\phi(t)$

$$r = \int_0^T s(t)\phi(t)\,dt = -\int_0^T \phi^2(t)\,dt = -1$$

a digital <u>0</u> was sent

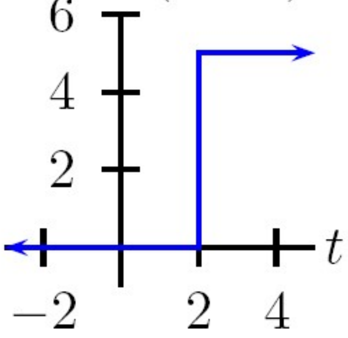
b) Digital 1 : $s(t) = \phi(t)$

$$r = \int_0^T s(t)\phi(t)\,dt = \int_0^T \phi^2(t)\,dt = 1$$

a digital <u>1</u> was sent

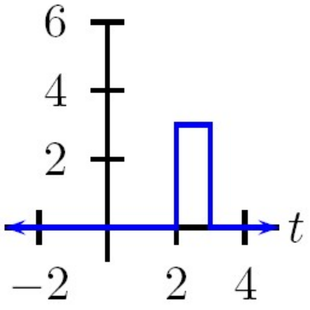
7.1 (a)

$5u(t-2)$



$$\mathcal{L}[5u(t-2)] = \int_2^\infty 5e^{-st}dt = \frac{5e^{-2s}}{s}$$

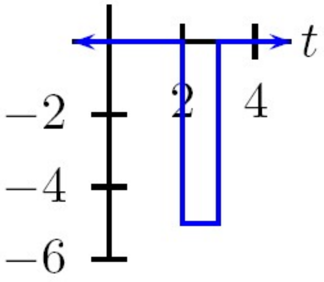
(b) $3[u(t-2)-u(t-3)]$



$$\mathcal{L}[3[u(t-2)-u(t-3)] = \int_2^3 3e^{-st}dt = \tfrac{3}{s}\left(e^{-2s}-e^{-3s}\right)$$

(c)

$-5u(t-2)u(3-t)$



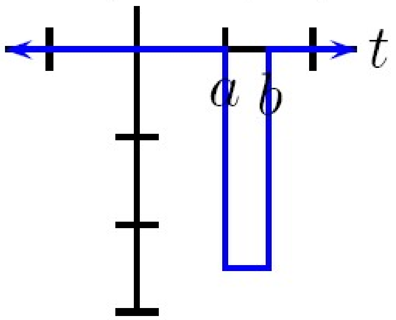
$$\mathcal{L}[-5u(t-2)u(3-t)] = \int_2^3 -5e^{-st}dt = \tfrac{-5}{s}\left(e^{-2s}-e^{-3s}\right)$$

Continued→

7.1, continued

(d)

$-5u(t-a)u(b-t)$



$$\mathcal{L}[-5u(t-a)u(b-t)] = \int_a^b -5e^{-st}dt = \frac{-5}{s}\left(e^{-as} - e^{-bs}\right)$$

7.2

a)

$$\mathcal{L}\left[t \sin bt\right] = \int_0^\infty t \sin bt\, e^{-st}\, dt = \int_0^\infty t\left[\frac{1}{2j}\left(e^{jbt} - e^{-jbt}\right)\right] e^{-st}\, dt$$

$$= \frac{+1}{2j}\int_0^\infty t\, e^{-(s-jb)t}\, dt - \frac{1}{2j}\int_0^\infty t\, e^{-(s+jb)t}\, dt$$

$$= \frac{1}{2j}\,\frac{e^{-(s-jb)t}}{(s-jb)^2}\left(\left((s-jb)t-1\right)\right|_0^\infty - \frac{1}{2j}\,\frac{e^{-(s+jb)t}}{(s+jb)^2}\left((s+jb)t-1\right)\Big|_0^\infty$$

$$= \frac{-1}{2j}\,\frac{(-1)}{(s-jb)^2} + \frac{1}{2j}\,\frac{(-1)}{(s+jb)^2} = \frac{1}{2j}\left[\frac{1}{(s-jb)^2} - \frac{1}{(s+jb)^2}\right]$$

$$= \frac{1}{2j}\,\frac{(s+jb)^2 - (s-jb)^2}{(s-jb)^2(s+jb)^2} = \boxed{\frac{2sb}{(s^2+b^2)^2}}$$

Continued→

7.2, continued

b) $\mathcal{L}[\cos bt] = \int_0^\infty \cos bt\, e^{-st} dt = \frac{1}{2}\int_0^\infty e^{jbt} e^{-st} dt +$

$\frac{1}{2}\int_0^\infty e^{-jbt} e^{-st} dt = \frac{1}{2}\int_0^\infty e^{-(s-jb)t} dt +$

$\frac{1}{2}\int_0^\infty e^{-(s+jb)t} dt = \frac{1}{2}\frac{1}{s-jb} + \frac{1}{2}\frac{1}{s+jb} = \frac{2s}{2(s^2+b^2)}$

$= \boxed{\frac{s}{s^2+b^2}}$

c) $F(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{(a-st)} dt = \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty = \boxed{\frac{1}{s-a}}$

d) $F(s) = \int_0^\infty t\, e^{at} e^{-st} dt = \int_0 t\, e^{(a-s)t} dt =$

$\frac{1}{(a-s)^2}\left[e^{(a-s)t}[at-st-1]\right]_0^\infty = \frac{1}{(a-s)^2}[0-(-1)] = \boxed{\frac{1}{(s-a)^2}}$

e) $\int u e^{u} du = e^{u}(u-1) + C$; $u = -st$

$\int_0^\infty t\, e^{-st} dt = \int_0^\infty \frac{-st}{-s} e^{-st} \frac{d(-st)}{-s} = \frac{1}{s^2} e^{-st}(st-1)\Big|_0^\infty$

$\therefore F(s) = 0 - \frac{1}{s^2}(-1) = \boxed{\frac{1}{s^2}}$, $Re(s) > 0$

f) $\int_0^\infty t\, e^{-(s+a)t} dt = \int_0^\infty \frac{-(s+a)t}{(-s+a)} e^{-(s+a)t} \frac{d[-(s+a)t]}{-(s+a)}$

$= \frac{1}{s+a} e^{-(s+a)t}[-(s+a)t-1]\Big|_0^\infty$

$\therefore F(s) = 0 - \frac{1}{(s+a)^2}(-1) = \boxed{\frac{1}{(s+a)^2}}$, $Re(s) > -a$

Continued→

7.2, continued

(g)

$$\mathcal{L}[\sin(bt)u(t)] = \int_0^\infty \sin(bt)e^{-st}dt = \left.\frac{e^{-st}(-s\sin(bt)-b\cos(bt))}{s^2+b^2}\right|_{t=0}^{\infty}$$

$$= 0 - \frac{-b}{s^2+b^2} = \frac{b}{s^2+b^2}$$

(h)

$$\mathcal{L}[e^{-at}\cos(bt)u(t)] = \int_0^\infty e^{-(a+s)t}\cos(bt)dt = \left.\frac{e^{-(a+s)t}(-(a+s)\cos(bt)+b\sin(bt))}{(a+s)^2+b^2}\right|_{t=0}^{\infty}$$

$$= 0 - \frac{-(a+s)}{(a+s)^2+b^2} = \frac{s+a}{(s+a)^2+b^2}$$

7.3 a) $f(t) = 5t\,u(t) - 5(t-2)u(t-2) - 15u(t-2) + 5u(t-4)$

b) $F(s) = \frac{5}{s^2} - \frac{5}{s^2}e^{-2s} - \frac{15}{s}e^{-2s} + \frac{5}{s}e^{-4s}$

7.4 a) $\omega = \frac{2\pi}{\pi} = 2$, $\therefore f(t) = 10\sin(2t)[u(t) - u(t-\pi)]$

b) $F(s) = \int_0^\pi 10\sin 2t\, e^{-st}dt = \left.\frac{10e^{-st}}{s^2+(2)^2}(-s\sin 2t - 2\cos 2t)\right|_0^\pi$

$= \frac{10}{s^2+4}\left[e^{-\pi s}(-2) - (-2)\right] = \frac{20(1-e^{-\pi s})}{s^2+4}$

c) $f(t) = 10\sin 2t\, u(t) - 10\sin[2(t-\pi)]u(t-\pi)$

$\therefore F(s) = \frac{20}{s^2+4} - \frac{20e^{-\pi s}}{s^2+4} = \frac{20(1-e^{-\pi s})}{s^2+4}$

7.5

(a)

$f(t) = \cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$

$\therefore\ F(s) = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$

(b) $\cos(bt)|_{b=aj} = \frac{e^{jbt}+e^{-jbt}}{2}|_{b=aj} = \frac{e^{-at}+e^{at}}{2} = \cosh(at)$

$\mathcal{L}[\cos(bt)]|_{b=aj} = \frac{s}{s^2+b^2}|_{b=aj} = \frac{s}{s^2-a^2}$

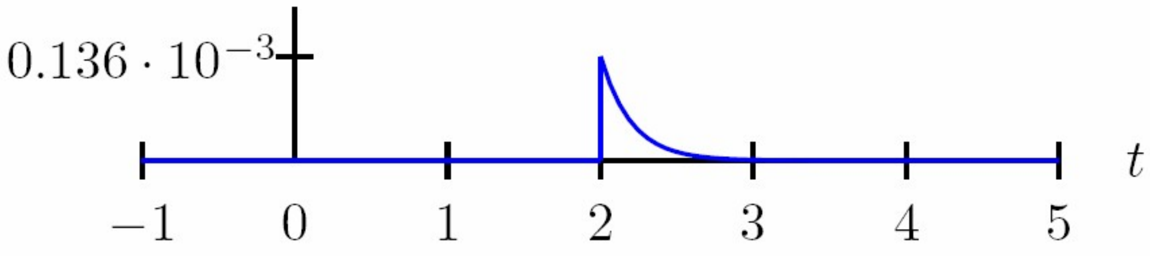
(c) $F(s) = \frac{a}{s^2-a^2}$

$\sin(bt)|_{b=aj} = j\sinh(at)$

$j\mathcal{L}[\sin(bt)]_{b=aj} = \frac{-jb}{s^2+b^2}|_{b=aj} = \frac{a}{s^2-a^2}$

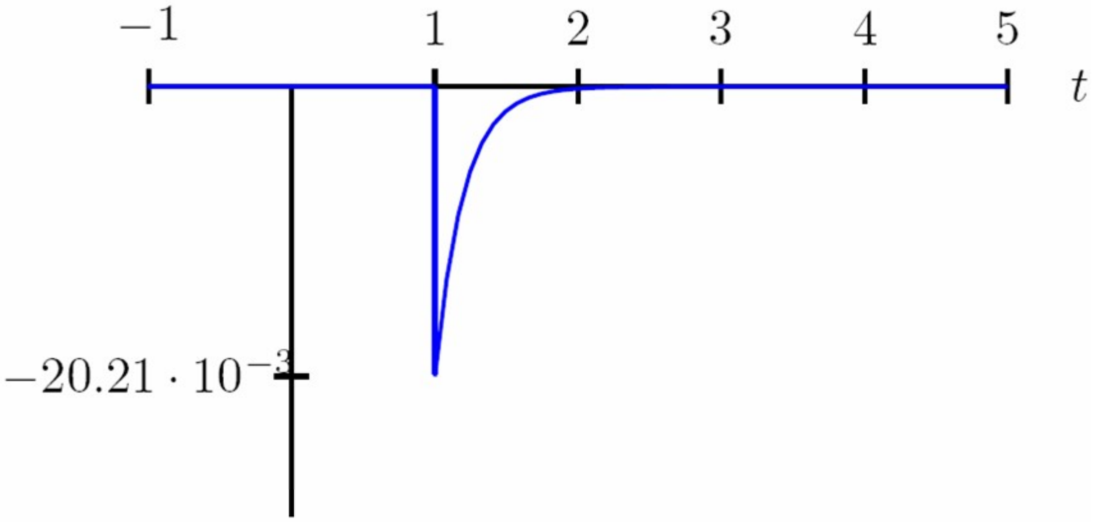
7.6 (i)

$3e^{-5t}u(t-2)$



(ii)

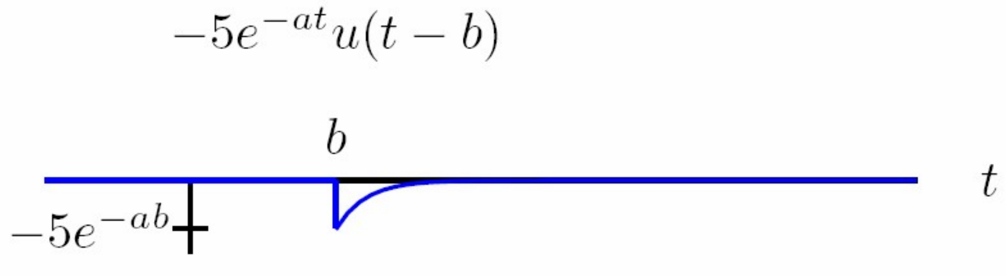
$-3e^{-5t}u(t-1)$



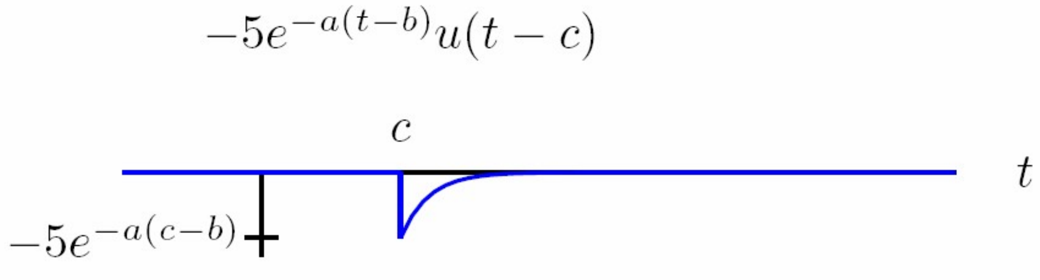
Continued→

7.6(a), continued

(iii)



(iv)



(b) (i)

$$\mathcal{L}[3e^{-5t}u(t-2)] = \int_2^\infty 3e^{-5t}e^{-st}dt = \frac{-3}{s+5}e^{-t(s+5)}\bigg|_2^\infty = \frac{3}{s+5}e^{-2(s+5)}$$

(ii)

$$\mathcal{L}[-3e^{-5t}u(t-1)] = \int_1^\infty -3e^{-5t}e^{-st}dt = \frac{3}{s+5}e^{-t(s+5)}\bigg|_1^\infty = \frac{-3}{s+5}e^{-1(s+5)}$$

(iii)

$$\mathcal{L}[-5e^{-at}u(t-b)] = \int_b^\infty -5e^{-at}e^{-st}dt = \frac{-5}{s+a}e^{-t(s+a)}\bigg|_b^\infty = \frac{-5}{s+a}e^{-b(s+a)}$$

(iv)

$$\mathcal{L}[-5e^{-a(t-b)}u(t-c)] = \int_c^\infty -5e^{-a(t-b)}e^{-st}dt = \frac{-5}{s+a}e^{-t(a+s)+ab}\bigg|_c^\infty = \frac{-5e^{ab}}{s+a}e^{-c(s+a)}$$

Continued→

7.6, continued

(c) (i)

$$\mathcal{L}[3e^{-5t}u(t-2)] = \mathcal{L}[3e^{-5(t-2)}e^{-10}u(t-2)] = \frac{3}{s+5}e^{-2s}e^{-10}$$

(ii)

$$\mathcal{L}[-3e^{-5t}u(t-1)] = \mathcal{L}[-3e^{-5(t-1)}e^{-5}u(t-1)] = \frac{-3}{s+5}e^{-s}e^{-5}$$

(iii)

$$\mathcal{L}[-5e^{-at}u(t-b)] = \mathcal{L}[-5e^{-a(t-b)}e^{-ab}u(t-b)] = \frac{-5}{s+a}e^{-bs}e^{-ab}$$

(iv)

$$\mathcal{L}[-5e^{-a(t-b)}u(t-c)] = \mathcal{L}[-5e^{-a(t-c-b)}e^{-ac}u(t-c)] = \frac{-5}{s+a}e^{-cs}e^{-ac}e^{ab}$$

(d) Results of (b) and (c) are equal.

7.7 (a)

$$\mathcal{L}[5u(t-2)u(3-t)] = \mathcal{L}[5[u(t-2)-u(t-3)]] = 5\frac{e^{-2s}}{s} - 5\frac{e^{-3s}}{s}$$

(b)

$$\mathcal{L}[3tu(t-2)] = \mathcal{L}[3(t-2)u(t-2)+6u(t-2)] = 3\frac{e^{-2s}}{s^2} + 6\frac{e^{-2s}}{s}$$

(c)

$$\mathcal{L}[3u(t-3)u(t-2)] = \mathcal{L}[3u(t-3)] = 3\frac{e^{-3s}}{s}$$

(d)

$$\begin{aligned}\mathcal{L}[3tu(t-1)-3tu(t-3)] = &\quad \mathcal{L}[3(t-1)u(t-1)-3(t-3)u(t-3)+3u(t-1)-9u(t-3)]\\ = &\quad 3\frac{e^{-s}}{s^2} - 3\frac{e^{-3s}}{s^2} + 3\frac{e^{-s}}{s} - 9\frac{e^{-3s}}{s}\end{aligned}$$

(e)

$$\begin{aligned}\mathcal{L}[3tu(t-a)-3tu(t-b)] = &\quad \mathcal{L}[3(t-a)u(t-a)-3(t-b)u(t-b)+3au(t-a)-3bu(t-b)]\\ = &\quad 3\frac{e^{-as}}{s^2} - 3\frac{e^{-bs}}{s^2} + 3a\frac{e^{-as}}{s} - 3b\frac{e^{-bs}}{s}\end{aligned}$$

Continued→

7.7, continued

(f)

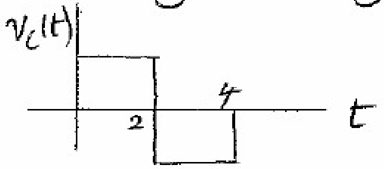
$$\mathcal{L}[2e^{-3t}u(t-5)] = \mathcal{L}[2e^{-15}e^{-3(t-5)}u(t-5)] = 2e^{-15}\frac{e^{-5s}}{s+3}$$

(g)

$$\mathcal{L}[2e^{-at}u(t-b)] = \mathcal{L}[2e^{-ab}e^{-a(t-b)}u(t-b)] = 2e^{-ab}\frac{e^{-bs}}{s+a}$$

7.8 a) $v(t) = \frac{5}{2} t u(t) - 5(t-2)u(t-2) + \frac{5}{2}(t-4)u(t-4)$

b) $V(S) = \frac{5/2}{S^2} - \frac{5e^{-2S}}{S^2} + \frac{5/2\, e^{-4S}}{S^2}$

c) 

$v_c(t) = \frac{5}{2}u(t) - 5u(t-2) + \frac{5}{2}u(t-4)$

d) $V_C(S) = \frac{1}{S}\left(\frac{5}{2} - 5e^{-2S} + \frac{5}{2}e^{-4S}\right)$

e) $\int_0^t v_c(\tau)\,d\tau = v(t) \quad \therefore \quad V(S) = \frac{1}{S}V_c(S)$

$\therefore \quad V(S) = \frac{1}{S^2}\left(\frac{5}{2} - 5e^{-2S} + \frac{5}{2}e^{-4S}\right)$ ✓

f) $v_c(t) = \frac{d\,v(t)}{dt}$; $V_c(S) = SV(s) - v(0^+)$

$\therefore \quad V_c(S) = S\left[\frac{1}{S^2}\left(\frac{5}{2} - 5e^{-2S} + \frac{5}{2}e^{-4S}\right)\right] - 0 =$

$\frac{1}{S}\left(\frac{5}{2} - 5e^{-2S} + \frac{5}{2}e^{-4S}\right)$

7.9 (a) (i)

$$v(0^+) = \lim_{s\to\infty} \frac{s^2}{(s+1)(s+2)} = 1$$

(ii)

$$\frac{s}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{2}{s+2}$$
$$v(t) = -e^{-t}u(t) + 2e^{-2t}u(t)$$
$$v(0^+) = 1$$

(b) (i)

$$\lim_{s\to 0} \frac{s^2}{(s+1)(s+2)} = 0$$

(ii)

$$\lim_{t\to\infty} -e^{-t}u(t) + 2e^{-2t}u(t) = 0$$

(c) `[r,p,k] = residue([0 1 0], [1 3 2])`

7.10 $V(s) = \frac{2S+1}{S^2+4} = \frac{2S}{S^2+4} + \frac{1}{S^2+4}$

a) (i) $v(0^+) = \lim_{S\to\infty} S\,V(S) = \lim_{S\to\infty} \frac{2S^2+S}{S^2+4} = 2$

(ii) $v(t) = [2\cos 2t + \frac{1}{2} \sin 2t]u(t)$,

$\therefore V(0^+) = 2 + \frac{1}{2}(0) = 2$ ✓

b) (i) $v(\infty) = \lim_{S\to 0} S\,V(S) = \lim_{S\to 0} \frac{2S^2+S}{S^2+4} = 0$ [in error]

(ii) $v(\infty) = \lim_{t\to\infty} (2\cos 2t + \frac{1}{2} \sin 2t) \Longrightarrow$ undefined

7.11 a) $\mathcal{L}[tu(t)] = \frac{-d}{ds}\mathcal{L}[u(t)] = \frac{-d}{ds}(1/s) = \underline{1/s^2}$

b) $\mathcal{L}[\cos bt] = \frac{s}{s^2+b^2}$

$\mathcal{L}[t\cos bt] = \frac{-d}{ds}\left[\frac{s}{s^2+b^2}\right] = \frac{-1}{s^2+b^2} + \frac{s \cdot 2s}{(s^2+b^2)^2}$

$= \underline{\frac{s^2-b^2}{(s^2+b^2)^2}}$

c) $\mathcal{L}[t\,t^{n-1}] = \frac{-d}{ds}\mathcal{L}[t^{n-1}] = \frac{-d}{ds}\left[\frac{(n-1)!}{s^n}\right] = \underline{\frac{n!}{s^{n+1}}}$

7.12 $f(t) = \frac{d}{dt}[\sin bt] = b\cos bt$

$F(s) = s\mathcal{L}[\sin bt] - \sin(0^+) = s\left[\frac{b}{s^2+b^2}\right] = b\mathcal{L}[\cos bt]$

$\therefore \mathcal{L}[\cos bt] = \underline{\frac{s}{s^2+b^2}}$

7.13

a) $F(s) = \frac{5}{s(s+2)} = \frac{2.5}{s} + \frac{-2.5}{s+2} \Rightarrow f(t) = 2.5(1-e^{-2t})u(t)$

b) $F(s) = \frac{s+3}{s(s+1)(s+2)} = \frac{1.5}{s} + \frac{-2}{s+1} + \frac{.5}{s+2} \Rightarrow f(t) = (1.5 - 2e^{-t}$
$+ .5e^{-2t})\,u(t)$

c) $F(s) = \frac{10(s+3)}{s^2+25} = \frac{K_1}{s+j5} + \frac{K_1^*}{s-j5}$; $K_1 = \frac{10(3+j5)}{-j5-j5}$

$\therefore K_1 = 5.831 \angle 149°$

Continued→

7.13, continued

d) $$F(s)=\frac{3}{s[(s+1)^2+(2^2)]}=\frac{3/5}{s}+\frac{k_1}{s+1+j2}+\frac{k_1^*}{s+1-j2}$$

$$k_1=\frac{3}{s(s+1-j2)}\Bigg|_{s=-1-j2}=\frac{3}{(-1-j2)(-j4)}=.335\angle{-153.4^\circ}$$

```
n=[0 0 5]; d=[1 2 0]; [r,p,k]=residue(n,d)
n=[0 0 1 3]; d=[1 3 2 0]; [r,p,k]= residue(n,d), pause
n=[0 10 30]; d=[1 0 25]; [r,p,k]=residue(n,d), pause
n=[0 0 0 3]; d=[1 2 5 0]; [r,p,k]=residue(n,d)
```

7.14 (a)

$$\begin{aligned}\frac{1}{s^2(s+1)} &= \frac{-1}{s}+\frac{1}{s^2}+\frac{1}{s+1}\\ f(t) &= -u(t)+tu(t)+e^{-t}u(t)\end{aligned}$$

Verify partial fraction exp. in MATLAB: `[r p k] = residue([0 0 0 1], [1 1 0 0])`

(b)

$$\begin{aligned}\frac{1}{s(s+1)^2} &= \frac{-1}{s+1}+\frac{-1}{(s+1)^2}+\frac{1}{s}\\ f(t) &= -e^{-t}u(t)-te^{-t}u(t)+u(t)\end{aligned}$$

Verify..`[r p k] = residue([0 0 0 1], [1 2 1 0])`

c) $$F(s)=\frac{1}{s^2(s^2+4)}=\frac{1/4}{s^2}+\frac{k_1}{s}+\frac{k_2}{s+j2}+\frac{k_2^*}{s-j2}$$

$$k_1=\frac{d}{ds}\left[\frac{1}{s^2+4}\right]_{s=0}=\frac{-2s}{(s^2+4)^2}\Bigg|_{s=0}=0$$

$$K_2=\frac{1}{s^2(s-j2)}\Bigg|_{s=-j2}=\frac{1}{(-4)(-j4)}=\frac{1}{16}\angle{-90^\circ}$$

Continued→

7.14, continued

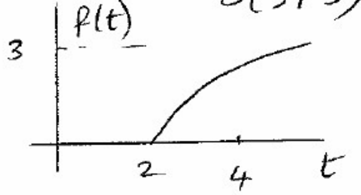
(d)

$$\begin{aligned}\frac{39}{(s+1)^2(s^2+4s+13)} &= \frac{-0.78}{s+1}+\frac{3.9}{(s+1)^2}+\frac{0.39+0.52j}{s+2-3j}+\frac{0.39-0.52j}{s+2+3j}\\ &= -0.78e^{-t}u(t)+3.9te^{-t}u(t)+(0.39+0.52j)e^{(-2+3j)t}u(t)+(0.39-0.52j)e^{(-2-3j)t}u(t)\\ &= -0.78e^{-t}u(t)+3.9te^{-t}u(t)+0.78e^{-2t}\cos(3t)u(t)-1.04e^{-2t}\sin(3t)u(t)\\ &= -0.78e^{-t}u(t)+3.9te^{-t}u(t)+1.3e^{-2t}\cos(3t+94.61^\circ)u(t)\end{aligned}$$

Verify: `[r p k] = residue([0 0 0 0 39], [1 6 22 30 13])`

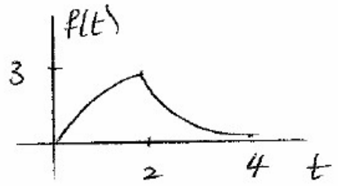
7.15

a) $F(s) = \dfrac{3e^{-2s}}{s(s+3)} = e^{-2s}\left(\dfrac{1}{s}+\dfrac{-1}{s+3}\right) \Rightarrow f(t) = (1-e^{-3(t-2)})u(t-2)$



$1-e^{-3(t-2)}\Big|_{t=4} = .0025$

b) $F(s) = \left(\dfrac{1}{s}+\dfrac{-1}{s+3}\right)(1-e^{-2s}) \Rightarrow f(t) = (1-e^{-3t})u(t)$
$-(1-e^{-3(t-2)})u(t-2)$



$\tau = \frac{1}{3}s$; $1-e^{-3t}\Big|_{t=2} = .0025$

7.16 (a)

$$\frac{s^{-2s}}{s(s+1)} = \quad e^{-2s}\left[\tfrac{1}{s}+\tfrac{-1}{s+1}\right]$$

$$f(t) = \quad [1-e^{-(t-2)}]u(t-2)$$

(b)

$$\frac{1-e^{-s}}{s(s+1)} = \quad \tfrac{1}{s}+\tfrac{-1}{s+1}+\tfrac{-e^{-s}}{s}+\tfrac{e^{-s}}{s+1}$$

$$f(t) = \quad [1-e^{-t}]u(t)-[1-e^{-(t-1)}]u(t-1)$$

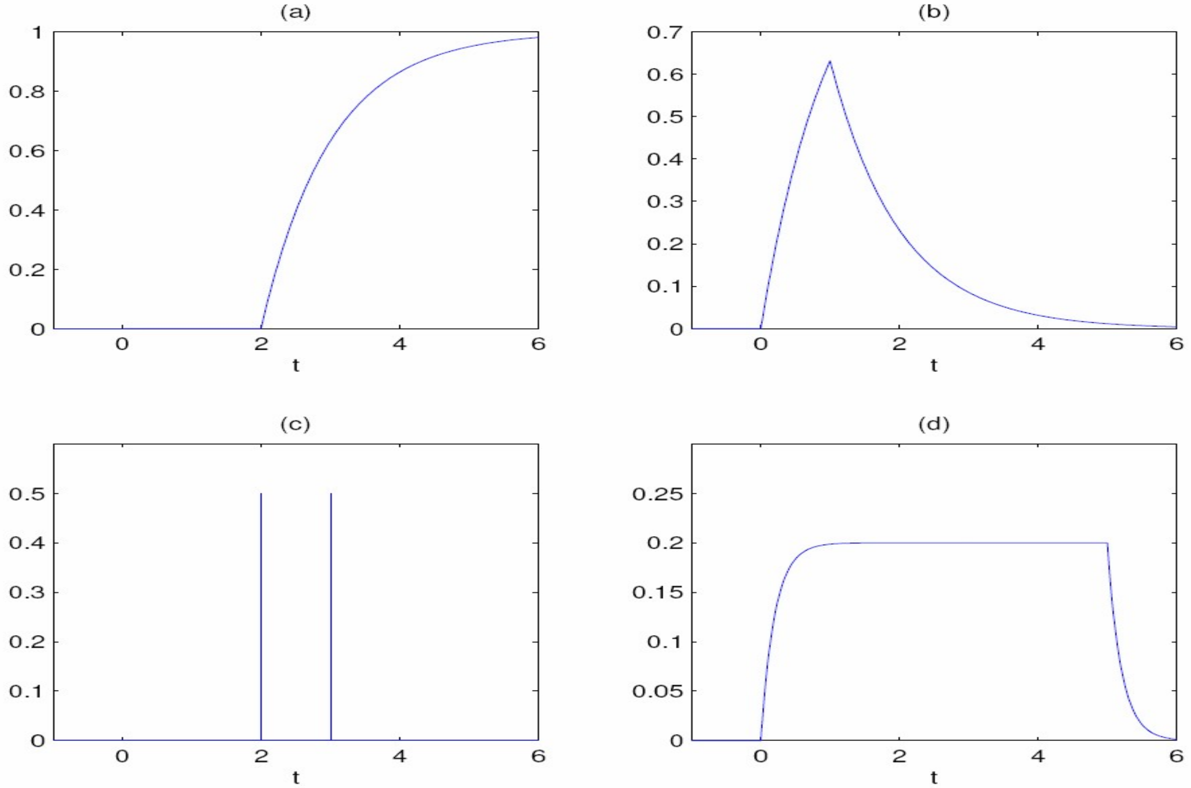
(c)

$$f(t) = \quad \tfrac{1}{2}[\delta(t-2)-\delta(t-3)]$$

(d)

$$\frac{1-e^{-5s}}{s(s+5)} = \quad \tfrac{0.2(1-e^{-5s})}{s}+\tfrac{-0.2(1-e^{-5s})}{s+5}$$

$$= \quad 0.2[u(t)-u(t-5)]-0.2[u(t)e^{-5t}-u(t-5)e^{-5(t-5)}]$$



7.17 (a) (i)

$$\begin{aligned} H(s) &= \frac{2}{s^2+5s+4} \\ &= \frac{2}{(s+4)(s+1)} \\ &= \frac{-2/3}{s+4} + \frac{2/3}{s+1} \\ h(t) &= (-2/3)e^{-4t}u(t) + (2/3)e^{-t}u(t) \end{aligned}$$

(ii)

$$\begin{aligned} H(s) &= \frac{2s+6}{s^2+5s+4} \\ &= \frac{2s+6}{(s+4)(s+1)} \\ &= \frac{2/3}{s+4} + \frac{4/3}{s+1} \\ h(t) &= (2/3)e^{-4t}u(t) + (4/3)e^{-t}u(t) \end{aligned}$$

(iii) Note the typo ($4\frac{d^2y(t)}{dt^2}$ should be $4\frac{dy(t)}{dt}$)

After correcting the typo:

$$\begin{aligned} H(s) &= \frac{6}{s^3+3s^2+4s+2} \\ &= \frac{6}{s+1} + \frac{-3}{s+1-j} + \frac{-3}{s+1+j} \\ h(t) &= 6e^{-t}u(t) + -3e^{(-1+j)t}u(t) + -3e^{(-1-j)t}u(t) \\ &= 6e^{-t}u(t) - 3e^{-t}[e^{jt} + e^{-jt}]u(t) \\ &= 6e^{-t}u(t) - 6e^{-t}\cos(t)u(t) \end{aligned}$$

On the other hand, if you neglected to correct the typo:

$$\begin{aligned} H(s) &= \frac{6}{s^3+7s^2+2} \\ &= \frac{0.1197}{s+7.0403} + \frac{-0.0598-0.7933j}{s-0.0202-0.5326j} + \frac{-0.0598+0.7933j}{s-0.0202+0.5326j} \\ h(t) &= 0.1197e^{-7.0403t}u(t) + (-0.0598-0.7933j)e^{(0.0202+0.5326j)t}u(t) + (-0.0598+0.7933j)e^{(0.0202-0.5326j)} \\ &= 0.1197e^{-7.0403t}u(t) - 0.1197e^{0.0202t}\cos(0.5326t)u(t) + 1.5865e^{0.0202t}\sin(0.5326t)u(t) \end{aligned}$$

Continued→

7.17, continued

(iv)

$$\begin{aligned} H(s) &= \tfrac{4s-8}{s^3-s^2+2} \\ &= \tfrac{1.2+0.4j}{s-1-j} + \tfrac{1.2-0.4j}{s-1+j} + \tfrac{-2.4}{s+1} \\ h(t) &= (1.2+0.4j)e^{(1+1j)t}u(t) + (1.2-0.4j)e^{(1-1j)t}u(t) + -2.4e^{-t}u(t) \\ &= 2.4e^t\cos(t)u(t) - 0.8e^t\sin(t)u(t) - 2.4e^{-t}u(t) \\ &= 2.5298e^t\cos(t+18.4349^\circ)u(t) - 2.4e^{-t}u(t) \end{aligned}$$

(b) (i)

$$\begin{aligned} s(t) &= \mathcal{L}^{-1}[H(s)\tfrac{1}{s}] = \mathcal{L}^{-1}[\tfrac{2}{s^3+5s^2+4s}] \\ \frac{2}{s^3+5s^2+4s} &= \tfrac{(1/6)}{s+4} + \tfrac{(-2/3)}{s+1} + \tfrac{(1/2)}{s} \\ s(t) &= \tfrac{1}{6}e^{-4t}u(t) - \tfrac{2}{3}e^{-t}u(t) + \tfrac{1}{2}u(t) \end{aligned}$$

(ii)

$$\begin{aligned} H(s)\frac{1}{s} &= \tfrac{2s+6}{s^3+5s^2+4s} \\ &= \tfrac{-1/6}{s+4} + \tfrac{-4/3}{s+1} + \tfrac{3/2}{s} \\ s(t) &= \tfrac{-1}{6}e^{-4t}u(t) - \tfrac{4}{3}e^{-t}u(t) + \tfrac{3}{2}u(t) \end{aligned}$$

Continued→

7.17(b), continued

(iii) (after correcting the typo)

$$\begin{aligned}
H(s)\frac{1}{s} &= \frac{6}{s^4+3s^3+4s^2+2s} \\
&= \frac{1.5+1.5j}{s+1-j} + \frac{1.5-1.5j}{s+1+j} + \frac{-6}{s+1} + \frac{3}{s} \\
s(t) &= (1.5+1.5j)e^{(-1+j)t}u(t) + (1.5-1.5j)e^{(-1-j)t}u(t) + -6e^{-t}u(t) + 3u(t) \\
&= 3e^{-t}\cos(t)u(t) - 3e^{-t}\sin(t)u(t) - 6e^{-t}u(t) + 3u(t) \\
&= 3\sqrt{2}e^{-t}\cos(t+45^\circ)u(t) - 6e^{-t}u(t) + 3u(t)
\end{aligned}$$

(iv)

$$\begin{aligned}
H(s)\frac{1}{s} &= \frac{4s-8}{s^4-s^3+2s} \\
&= \frac{0.8-0.4j}{s-1-j} + \frac{0.8+0.4j}{s-1+j} + \frac{2.4}{s+1} + \frac{-4}{s} \\
s(t) &= (0.8-0.4j)e^{(1+j)t}u(t) + (0.8+0.4j)e^{(1-1j)t}u(t) + 2.4e^{-t}u(t) - 4u(t) \\
&= 1.6e^{t}\cos(t)u(t) + 0.8e^{t}\sin(t)u(t) + 2.4e^{-t}u(t) - 4u(t) \\
&= 0.8\sqrt{5}e^{t}\cos(t-26.56^\circ)u(t) + 2.4e^{-t}u(t) - 4u(t)
\end{aligned}$$

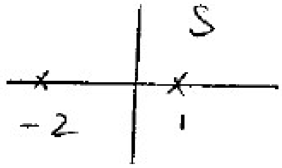
(c) Taking derivatives of the results in part (b) (and using $\frac{d}{dt}(f(t)u(t)) = f'(t)u(t) + \delta(t)f(0)$) gives the results in part (a).

(d) Partial fraction expansions were done using `[r p k] = residue(b, a)`. For example, for part (a)(i): `[r p k] = residue([0 0 2], [1 5 4])`. For part (a)(ii): `[r p k] = residue([0 2 6], [1 5 4])`.

7.18 (Note that these are just possible answers; any other answer that satisfies the conditions is correct)

(a)

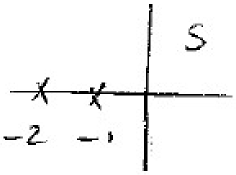
$H(s) = \dfrac{1}{(s-1)(s+2)}$



$, e^{t}, e^{-2t}$

(b)

$H(s) = \dfrac{1}{(s+1)(s+2)}$



$, e^{-t}, e^{-2t}$

(c)

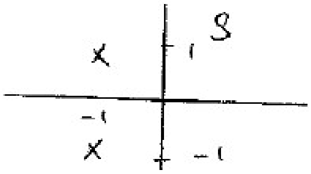
Same as (a)

continued→

7.18, continued

(d)

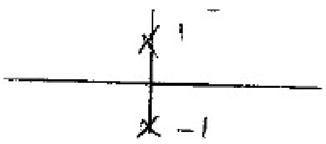
$H(S) = \frac{1}{(S+1)^2+1}$



$, e^{-t} \cos(t+\theta)$

(e)

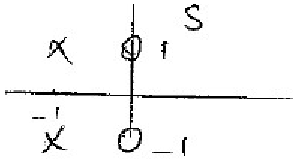
$H(S) = \frac{1}{S^2+1}$



$\cos(t+\theta)$

(f)

$H(S) = \frac{S^2+1}{S^2+2S+2}$



$h(t) = \delta(t) + Ce^{-t}\cos(t-\Theta)$

(g)

Same as (a)

7.19 (a) (i) stable

(ii) stable

(iii) stable

(iv) not stable

(b) (i) e^{-4t}, e^{-t}

(ii) e^{-4t}, e^{-t}

(iii) $e^{-t}, e^{(-1+j)t}, e^{(-1-j)t}$ or $e^{-t}\cos(t), e^{-t}$

(iv) $e^{(1+j)t}, e^{(1-j)t}, e^{-t}$ or $e^{t}\cos(t), e^{t}\sin(t), e^{-t}$, or $e^{t}\cos(t+\theta), e^{-t}$

(c) (i) $H_i(s) = \frac{s^2+5s+4}{2}$

(ii) $H_i(s) = \frac{s^2+5s+4}{2s+6}$

(iii) $H_i(s) = \frac{s^3+3s^2+4s+2}{6}$

(iv) $H_i(s) = \frac{s^3-s^2+2}{4s-8}$

7.20

(a)

$$Y(s) = \frac{1}{s+b} \qquad X(s) = \frac{1}{s+a}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+a}{s+b} = \frac{a}{s+b} + \frac{s}{s+b}$$

$$h(t) = a e^{-bt} u(t) + \frac{d}{dt}\left(e^{-bt} u(t)\right) = a e^{-bt} u(t) + -b e^{-bt} u(t) + e^{-bt} \delta(t)$$

$$\therefore\ h(t) = \delta(t) + (a-b) e^{-bt} u(t)$$

(b) We know that $h(t) = \frac{d}{dt}s(t)$ and here $s(t) = e^{-at}\cos(bt)u(t)$, so $h(t) = -ae^{-at}\cos(bt)u(t) - be^{-at}\sin(bt)u(t) + \delta(t)$.

We can also find the solution using $h(t) = \mathcal{L}^{-1}[H(s)]$ where $H(s) = \frac{Y(s)}{X(s)} = \frac{s(s+a)}{(s+a)^2+b^2}$.

7.21

(a)

$$\int_{-\infty}^{\infty} e^{-2t} u(t) e^{-st}\, dt = \int_{0}^{\infty} e^{-(2+s)t}\, dt = \frac{1}{s+2}, \quad Re(s) > -2$$

(b)

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2t}u(t-1)e^{-st}dt = &\quad \textstyle\int_1^\infty e^{-t(2+s)}dt \\ = &\quad \tfrac{1}{2+s}e^{-(s+2)}, ROC : Re(s) > -2\end{aligned}$$

(c)

$$-\int_{-\infty}^{\infty} e^{2t} u(-t) e^{-st}\, dt = -\int_{-\infty}^{0} e^{(2-s)t}\, dt = \frac{-1}{2-s} = \frac{1}{s-2}$$, ROC: Re(s) < 2

(d)

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2t}u(-t-1)e^{-st}dt = &\quad \textstyle\int_{-\infty}^{-1} e^{t(2-s)}dt \\ = &\quad \tfrac{1}{s-2}e^{s-2}, ROC : Re(s) < 2\end{aligned}$$

(e)

$$\int_{-\infty}^{\infty} e^{-2t} u(t+4) e^{-st}\, dt = \int_{-4}^{\infty} e^{-(s+2)t}\, dt = \frac{e^{(s+2)4}}{(s+2)}, \quad Re(s) > -2$$

Continued→

7.21, continued

(f)

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2t}u(-t+1)e^{-st}dt &= \int_{-\infty}^{1} e^{-t(2+s)}dt \\ &= \frac{-1}{2+s}e^{2+s}, ROC : Re(s) < -2\end{aligned}$$

7.22 Using known bilateral transforms of exponential signals:

(a)

$$F(s) = \frac{1}{s+10} - \frac{1}{s-5}, ROC : -10 < Re(s) < 5$$

(b) does not exist

(c) does not exist

(d)

$$F(s) = \frac{1}{s+10} - \frac{1}{s+5}, ROC : -10 < Re(s) < -5$$

7.23 (a)

$$\begin{aligned}F(s) &= \int_{-1}^{2} e^{5t}e^{-st}dt = \frac{1}{5-s}[e^{2(5-s)} - e^{-1(5-s)}] \\ &= \frac{1}{5-s}[e^{10-2s} - e^{s-5}], ROC : \text{all } s\end{aligned}$$

(b)

$$\begin{aligned}\mathcal{L}[e^{5t}(u(t+1) - u(t-2))] &= \mathcal{L}[e^{5t}u(t+1)] + \mathcal{L}[e^{5t}u(t-2)] \\ \mathcal{L}[e^{5t}u(t+1)] &= e^{-5}\mathcal{L}[e^{5(t+1)}u(t+1)] = e^{-5}\frac{1}{s-5}e^{s}, ROC : Re(s) > 5 \\ \mathcal{L}[e^{5t}u(t-2)] &= e^{10}\mathcal{L}[e^{5(t-2)}u(t-2)] = e^{10}\frac{1}{s-5}e^{-2s}, ROC : Re(s) > 5 \\ F(s) &= \frac{1}{s-5}[e^{s-5} - e^{-2s+10}] \\ &= \frac{1}{5-s}[e^{10-2s} - e^{s-5}]\end{aligned}$$

Note that the ROC of the sum is in general the intersection of the ROCs (in this case $Re(s) > 5$), but since we know it is a finite-duration signal, the ROC is in fact all s.

Continued→

7.23, continued

(c)

$$\begin{aligned}
\mathcal{L}[e^{5t}[u(2-t) - u(-1-t)] = &\quad \mathcal{L}[e^{5t}u(2-t) - e^{5t}u(-1-t)] \\
\mathcal{L}[e^{5t}u(2-t)] = &\quad e^{10}\mathcal{L}[e^{5(t-2)}u(-(t-2))] \\
= &\quad e^{10}\tfrac{1}{s-5}e^{-2s}, ROC : Re(s) < 5 \\
\mathcal{L}[e^{5t}u(-1-t)] = &\quad e^{-5}\mathcal{L}[e^{5(t+1)}u(-(t+1))] \\
= &\quad e^{-5}\tfrac{1}{s-5}e^{s}, ROC : Re(s) < 5 \\
F(s) = &\quad \tfrac{1}{5-s}[e^{10-2s} - e^{s-5}]
\end{aligned}$$

The intersection of the ROCs is $Re(s) < 5$, but since it is a finite signal, the ROC is all s.

7.24 (a) Left-sided function

$$F_b(s) = \frac{s+9}{s(s+1)} = \frac{9}{s} + \frac{-8}{s+1}$$

From (7.83), $f(t) = \underline{-9u(-t) + 8e^{-t}u(-t)}$

(b) Right-sided function

$f(t) = \underline{9u(t) - 8e^{-t}u(t)}$

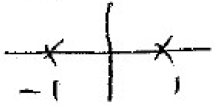
(c) $\frac{9}{s}$ left sided; $\frac{-8}{s+1}$ right sided

$\therefore f(t) = \underline{-9u(-t) - 8e^{-t}u(t)}$

(d) (a) $f(\infty) = 0$ (b) $f(\infty) = 9$ (c) $f(\infty) = 0$

7.25 $X(s) = \frac{s+3}{(s+1)(s-1)} = \frac{-1}{s+1} + \frac{2}{s-1}$

poles at $-1, 1$

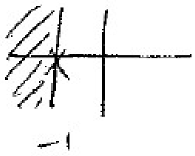


a) $Re(s) < -1$, $x(t) = e^{-t}u(-t) - 2e^{t}u(-t)$

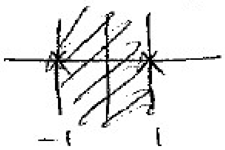
$-1 \leqslant Re(s) \leqslant 1$, $x(t) = -e^{-t}u(t) - 2e^{t}u(-t)$

$Re(s) \geqslant 1$, $x(t) = -e^{-t}u(t) + 2e^{t}u(t)$

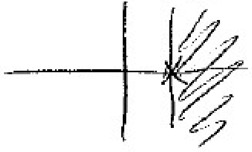
b) $Re(s) < -1$



$-1 \leqslant Re(s) \leqslant 1$



$Re(s) \geqslant 1$



c) for $Re(s) < -1$, $x(t)$ is noncausal

for $-1 \leqslant Re(s) \leqslant 1$, $x(t)$ is 2-sided

for $Re(s) \geqslant 1$, $x(t)$ is causal

d) for $Re(s) < -1$, $x(t)$ is Not BIBO Stable

for $-1 \leqslant Re(s) \leqslant 1$, $x(t)$ is BIBO Stable

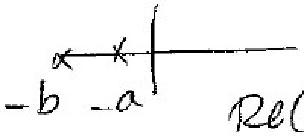
for $Re(s) \geqslant 1$, $x(t)$ is Not BIBO stable

e) for $Re(s) < -1$, Final value is 0

for $-1 \leqslant Re(s) \leqslant 1$, Final value is 0

for $Re(s) \geqslant 1$, Final value does not exist

7.26 a) $h(t)$ causal $\Rightarrow$ both functions are right-Sided

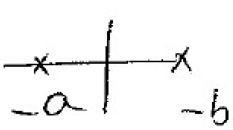


$Re(a) > 0$ & $Re(b) > 0$

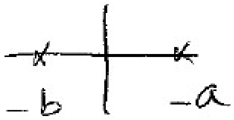
b) 2 sided $\Rightarrow$ one is left-Sided & one is right-Side

either $Re(b) < 0$ and $Re(a) > 0$

or $Re(a) < 0 < Re(b)$

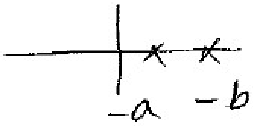


or



c) Both left-Sided

$Re(a) < 0$ & $Re(b) < 0$



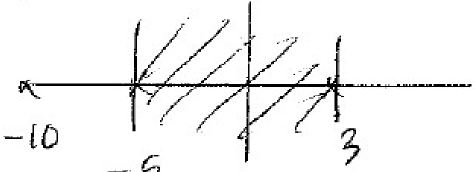
7.27

$$H(s) = \frac{s+1}{(s+4)(s+2)} = \frac{3/2}{s+4} + \frac{-1/2}{s+2}$$

$\frac{3/2}{s+4} \downarrow$ Right $\quad \frac{-1/2}{s+2} \downarrow$ Left

$$\therefore \; h(t) = \frac{3}{2} e^{-4t} u(t) + \frac{1}{2} e^{-2t} u(-t)$$

7.28 $H(s) = \dfrac{1}{(s+10)(s+5)(s-3)}$ poles at $-10, -5, 3$



Converges to the right of -10 & -5 $\Rightarrow \therefore$ these are right-Sided time functions

Converges to the left of 3 $\Rightarrow \therefore$ This is left-Sided time function

7.29

a) $x(t) = e^{5t}u(t)$, $X(s) = \frac{1}{s-5}$, $Re(s) > 5$

$h(t) = u(t)$, $H(s) = \frac{1}{s}$, $Re(s) > 0$

$Y(s) = H(s)X(s) = \frac{1}{s(s-5)}$, $Re(s) > 5$

$$Y(s) = \frac{-1/5}{s} + \frac{1/5}{s-5} \Longrightarrow y(t) = -\tfrac{1}{5}u(t) + \tfrac{1}{5}e^{5t}u(t)$$

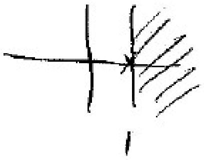
$$= \tfrac{1}{5}\left[e^{5t} - 1\right]u(t)$$

(b)

$$\begin{aligned}
X(s) &= \frac{1}{1+s} \\
H(s) &= \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-4s} \\
Y(s) &= H(s)X(s) = \frac{e^{-2s}}{(1+s)s} - \frac{e^{-4s}}{(1+s)s} \\
&= \frac{-e^{-2s}}{s+1} + \frac{e^{-2s}}{s} - \left[\frac{-e^{-4s}}{s+1} + \frac{e^{-4s}}{s}\right] \\
y(t) &= \left[1 - e^{-(t-2)}\right]u(t-2) - \left[1 - e^{-(t-4)}\right]u(t-4)
\end{aligned}$$

7.30 $h(t) = e^{t}u(t)$

a) $H(s) = \frac{1}{s-1}$, $Re(s) > 1$



NOT BIBO Stable

b) $w(t) = x(t) - Ay(t)$, $W(s) = X(s) - AY(s)$

$y(t) = w(t) * h(t)$, $Y(s) = W(s)H(s)$

Part (b) continued→

7.30(b), continued

$$\frac{Y(s)}{H(s)} = W(s) = X(s) - AY(s)$$

$$Y(s)\left[\frac{1}{H(s)} + A\right] = X(s) \quad , \quad \frac{Y(s)}{X(s)} = \frac{H(s)}{1+AH(s)}$$

c) For stability, examine $\dfrac{H(s)}{1+AH(s)} = \dfrac{\frac{1}{s-1}}{1+\frac{A}{s-1}} \quad \dfrac{\frac{1}{s-1}}{\frac{s+A-1}{s-1}}$

$$= \frac{1}{s+A-1}$$

As long as $A-1>0$, then the pole at $A-1$ will be in the left half-plane and the system will be stable.

$\therefore$ we require $A>1$

Chapter 8 Solutions

8.1. $L\frac{di}{dt} + Ri = v_i \Rightarrow \frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}v_i$, $v_R = Ri$

(a) $x_1 = i$, $u(t) = v_i$, $y = v_R$

$$\dot{x} = -\frac{R}{L}x + \frac{1}{L}u$$

$$y = Rx$$

(b) $x = v_R = Ri$, $i = \frac{1}{R}x$

$$\frac{1}{R}\dot{x} = -\frac{1}{L}x + \frac{1}{L}u \Rightarrow \begin{array}{l} \dot{x} = -\frac{R}{L}x + \frac{R}{L}u \\ y = x \end{array}$$

8.2.(a) $v_i = L\frac{di}{dt} + v_c \Rightarrow \frac{di}{dt} = -\frac{1}{L}v_c + \frac{1}{L}v_c$

$$v_c = \frac{1}{C}\int_0^t i\,d\tau \Rightarrow \frac{dv_c}{dt} = \frac{1}{C}i$$

$$\therefore \begin{bmatrix} di/dt \\ dv_c/dt \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} 1/L \\ 0 \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix}v_i \Rightarrow \dot{\underline{x}} = \begin{bmatrix} 0 & -1/L \\ 1/C & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix}u$$

$$v_c = [0 \quad 1]\begin{bmatrix} i \\ v_c \end{bmatrix} \qquad \Rightarrow y = [0 \quad 1]\underline{x}$$

(b) Same state equation, with

$$i = [1 \quad 0]\begin{bmatrix} i \\ v_c \end{bmatrix} \qquad \Rightarrow y = [1 \quad 0]\underline{x}$$

8.3 (a) Letting $x_1(t) = y(t)$:

$$\begin{aligned} \dot{x_1} = & \quad -ax_1 + bu \\ y = & \quad x_1 \end{aligned}$$

(b) Letting $x_1(t) = y(t)$:

$$\begin{aligned} \dot{x_1} = & \quad 2x_1 + 4u \\ y = & \quad x_1 \end{aligned}$$

(c) Letting $x_1(t) = y(t)$ and $x_2(t) = y\dot{(t)}$:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 9 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(d) Letting $x_1(t) = y(t)$ and $x_2(t) = y\dot{(t)}$:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(e) First correct the typo ($3y_1(t)$ in first equation should be $3\dot{y_1}(t)$).
Let $x_1(t) = y_1(t)$, $x_2(t) = y_2(t)$, and $x_3(t) = \dot{y_1}(t) = \dot{x_1}(t)$:

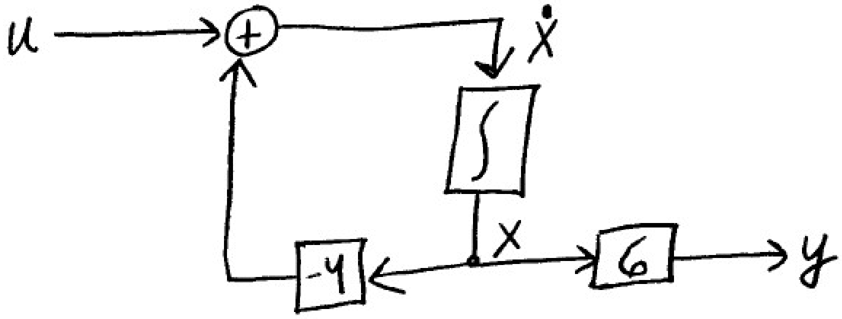
$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -4 & -2 & 0 \\ -6 & 9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(f) Letting $x_1(t) = y_1(t)$, $x_2(t) = y_2(t)$, and $x_3(t) = \dot{y_2}(t) = \dot{x_2}(t)$:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} -2 & -4 & 0 \\ 0 & 0 & 1 \\ 1 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

8.4

(a) $H(s) = \frac{6}{s+4}$



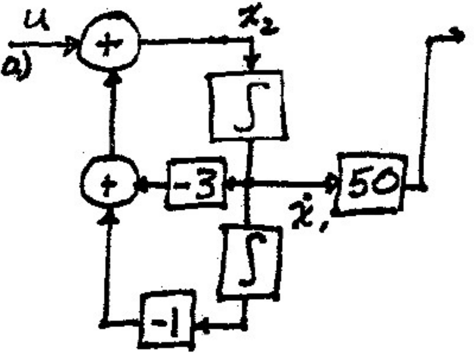
(b) $\dot{x} = -4x + u$

$y = 6x$

(c) $\frac{dy}{dt} + 4y(t) = 6u(t)$

(d)

(a)



(b) $\dot{\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \underset{\sim}{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 0 & 50 \end{bmatrix} \underset{\sim}{x}$

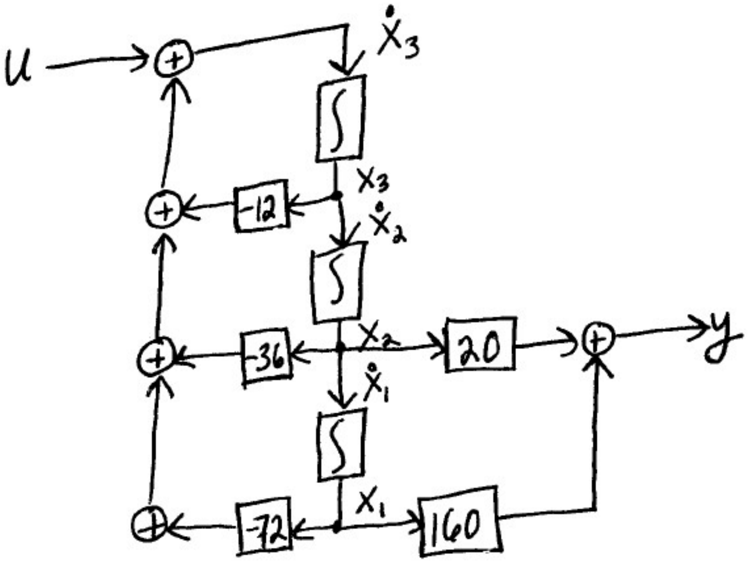
(c) $\ddot{y} + 3\dot{y} + y = 50\dot{u}$

Continued→

8.4, continued

(e) $H(s) = \dfrac{20s + 160}{s^3 + 12s^2 + 36s + 72}$

(a)

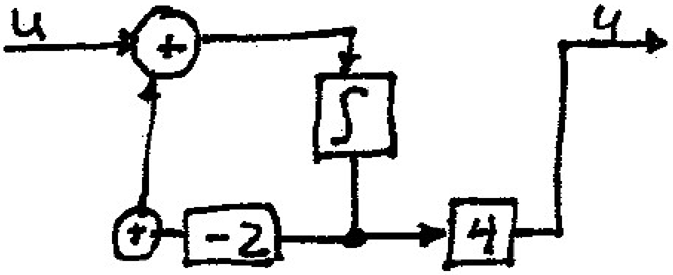


(b) $$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -72 & -36 & -12 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 160 & 20 & 0 \end{bmatrix} \underline{x}$$

(c) $$\frac{d^3y}{dt^3} + 12\frac{d^2y}{dt^2} + 36\frac{dy}{dt} + 72 = 20\frac{du}{dt} + 160u(t)$$

8.5. (a) $\dot{y} = -2y + 4u$



(b) $\dot{x} = -2x + u$

$y = 4x$

(c) $\frac{Y(s)}{U(s)} = \frac{4}{s+2}$

(d)

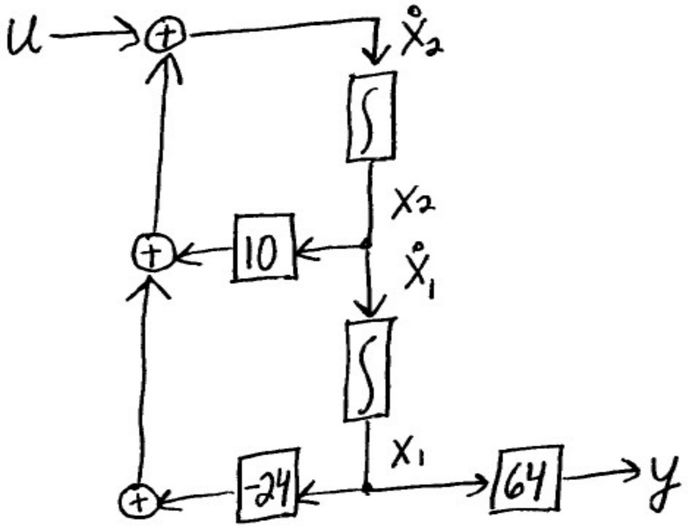
```
A=[-2];B=[1];C=[4];D=0;
[n,d]=ss2tf(A,B,C,D), pause,
A=[0 1;-12 8];B=[0;1];C=[40 0];D=0;
[n,d]=ss2tf(A,B,C,D), pause
A=[0 1 0;0 0 1;-15 -10 -20];B=[0;0;1];C=[50 0 0];D=0;
[n,d]=ss2tf(A,B,C,D)
```

Continued→

8.5, continued

(e) $\ddot{y}(t) - 10\,\dot{y}(t) + 24y(t) = 64u(t)$

(a)



(b)

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -24 & 10 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [64 \quad 0]\ \underline{x}$$

(c)

$$H(s) = \frac{64}{s^2 - 10s + 24}$$

(d)

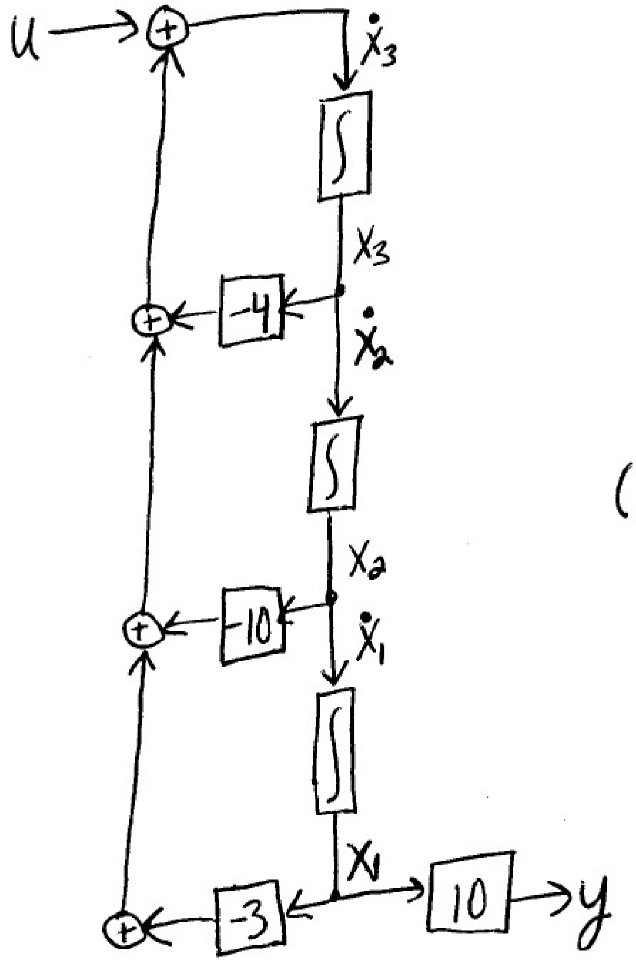
```
>> A=[0 1; -24 10]; B=[0; 1]; C=[64 0]; D=0;
>> [n d] = ss2tf(A, B, C, D)
```

Continued→

8.5 (f)

$$\dddot{y}(t) + 4\ddot{y}(t) + 10\dot{y}(t) + 3y(t) = 10\,u(t)$$

(a)



(b)

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -10 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [10\ 0\ 0]\,\underline{x}$$

(c) $$H(s) = \frac{10}{s^3 + 4s^2 + 10s + 3}$$

(d)

```
>> A=[0 1 0; 0 0 1; -3 -10 -4]; B=[0; 0; 1]; C=[10 0 0]; D=0;
>> [n d] = ss2tf(A, B, C, D)
```

8.6. (a) $\dot{x} = -3x + 6u$

$y = 4x$

(b)

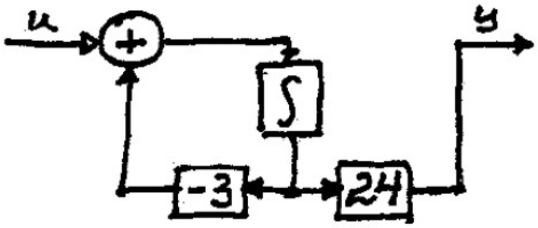
$$sI - A = s + 3$$

$$H(s) = C(sI - A)^{-1}B = 4\frac{1}{s+3}(6) = \frac{24}{s+3}$$

(c)

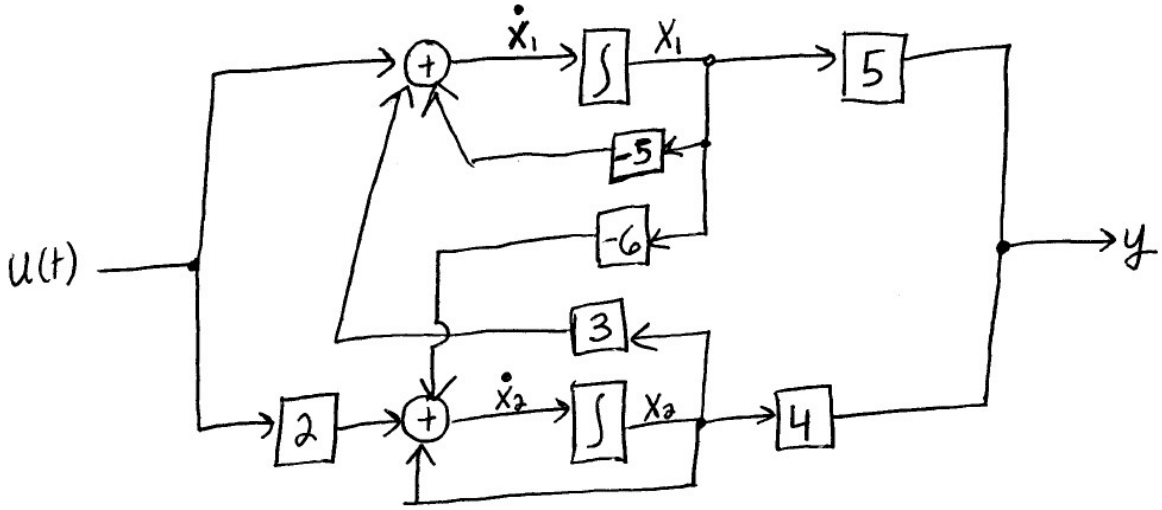
```
A=[-3];B=[6];C=[4];D=0;
[n,d]=ss2tf(A,B,C,D)
```

(d)



8.7

(a)



$\dot{x}_1 = -5x_1 + 3x_2 + u$

$\dot{x}_2 = -6x_1 + x_2 + 2u$

$$\Rightarrow \quad \underline{\dot{x}} = \begin{bmatrix} -5 & 3 \\ -6 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & 4 \end{bmatrix} \underline{x}$$

continued→

8.7 (b)

(following example 8.10)

calculation of resolvant matrix $(sI-A)^{-1}$:

$$sI-A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -5 & 3 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} s+5 & -3 \\ +6 & s-1 \end{bmatrix}$$

$$\text{adj}\,(sI-A) = \begin{bmatrix} s-1 & 3 \\ -6 & s+5 \end{bmatrix}$$

$$\det\,(sI-A) = (s+5)(s-1)-(-3)(6)$$
$$= s^2+4s+13$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s-1}{s^2+4s+13} & \frac{+3}{s^2+4s+13} \\ \frac{-6}{s^2+4s+13} & \frac{s+5}{s^2+4s+13} \end{bmatrix}$$

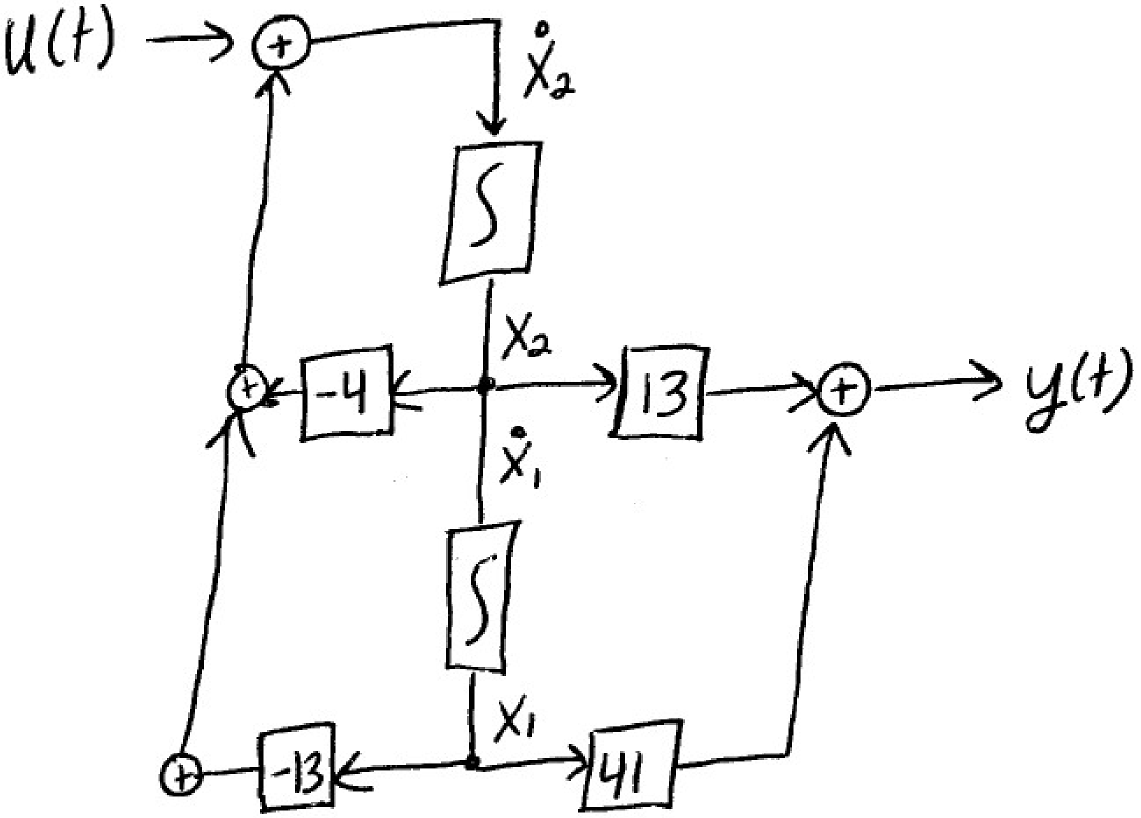
$$H(s) = C\,(sI-A)^{-1}B$$
$$= \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} \frac{s-1}{s^2+4s+13} & \frac{3}{s^2+4s+13} \\ \frac{-6}{s^2+4s+13} & \frac{s+5}{s^2+4s+13} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} \frac{s+5}{s^2+4s+13} \\ \frac{2s+4}{s^2+4s+13} \end{bmatrix} = \frac{13s+41}{s^2+4s+13}$$

continued→

8.7(c)

```
>> A=[-5 3; -6 1]; B=[1; 2]; C=[5 4]; D=0;
>> [n d] = ss2tf(A, B, C, D)
```

(d)



(e)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -13x_1 - 4x_2 + u$$

$$\Rightarrow \quad \underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -13 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [41 \quad 13] \, \underline{x}$$

Continued→

8.7(f)

$$sI - A = \begin{bmatrix} s & -1 \\ 13 & s+4 \end{bmatrix} \qquad adj(sI-A) = \begin{bmatrix} s+4 & 1 \\ -13 & s \end{bmatrix}$$

$$\det(sI-A) = s(s+4)+13 = s^2+4s+13$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s+4}{s^2+4s+13} & \frac{1}{s^2+4s+13} \\ \frac{-13}{s^2+4s+13} & \frac{s}{s^2+4s+13} \end{bmatrix}$$

$$H(s) = C\,(sI-A)^{-1}B = \begin{bmatrix} 41 & 13 \end{bmatrix} (sI-A)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 13 \end{bmatrix} \begin{bmatrix} \frac{1}{s^2+4s+13} \\ \frac{s}{s^2+4s+13} \end{bmatrix} = \frac{41+13s}{s^2+4s+13}$$

(g)

```
>> A=[0 1; -13 -4]; B=[0; 1]; C=[41 13]; D=0;
>> [n d] = ss2tf(A, B, C, D);
```

8.8

(a) $\dot{x}_1 = x_2$

$\dot{x}_2 = -5x_1 - 2x_2 + m$

$$\Rightarrow \quad \dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m$$

$$y = \begin{bmatrix} 3 & 4 \end{bmatrix} \underline{x}$$

(b) $$sI - A = \begin{bmatrix} s & -1 \\ 5 & s+2 \end{bmatrix} \qquad adj(sI-A) = \begin{bmatrix} s+2 & 1 \\ -5 & s \end{bmatrix}$$

$$\det(sI-A) = s(s+2)+5 \qquad (sI-A)^{-1} = \begin{bmatrix} \frac{s+2}{s^2+2s+5} & \frac{1}{s^2+2s+5} \\ \frac{-5}{s^2+2s+5} & \frac{s}{s^2+2s+5} \end{bmatrix}$$

$$= s^2+2s+5$$

$$H_p(s) = C(sI-A)^{-1}B = \begin{bmatrix} 3 & 4 \end{bmatrix}(sI-A)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{s^2+2s+5} \\ \frac{s}{s^2+2s+5} \end{bmatrix} = \frac{3+4s}{s^2+2s+5}$$

(c) $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y(t) = 4\frac{dm}{dt} + 3m(t)$$

Continued→

8.8, continued

(d)

$$\dot{x}_3 = -3x_3 + e$$

$$m(t) = 4x_3$$

(e) $$(sI-A)^{-1} = \frac{1}{s+3}$$

$$H_c(s) = C\,(sI-A)^{-1}B = 4 \cdot \frac{1}{s+3} \cdot 1 = \frac{4}{s+3}$$

(f) $$\frac{dm}{dt} + 3m(t) = 4e(t)$$

(g) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -5 & -2 & 4 \\ -3 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Continued→

8.8 (h)

$$sI-A=\begin{bmatrix} s & -1 & 0 \\ 5 & s+2 & -4 \\ 3 & 4 & s+3 \end{bmatrix}$$

It's easiest to find $(sI-A)^{-1}$ in MATLAB (or using a symbolic calculator)

```
>> s = sym('s');
>> M = [s -1 0; 5 s+2 -4; 3 4 s+3];
>> inv(M)
```

```
>> syms s;
>> M=[s -1 0; 5 s+2 -4; 3 4 s+3];
>> inv(M)
```

$$(sI-A)^{-1}=\begin{bmatrix} s^2+5s+22 & s+3 & 4 \\ -5s-27 & s^2+3s & 4s \\ -3s+14 & -4s-3 & s^2+2s+5 \end{bmatrix} \Big/ s^3+5s^2+27s+27$$

$$H(s)=C(sI-A)^{-1}B=\frac{12+16s}{s^3+5s^2+27s+27}$$

$C=[3\,4\,0]$

$B=\begin{bmatrix}0\\0\\1\end{bmatrix}$

(i) $$\frac{d^3y}{dt^3}+\frac{5d^2y}{dt^2}+27\frac{dy}{dt}+27y(t)=12u(t)+16\frac{du}{dt}$$

(j)

```
>> A=[0 1 0; -5 -2 4; -3 -4 -3]; B=[0; 0; 1]; C=[3 4 0]; D=0;
>> [n d] = ss2tf(A, B, C, D)
```

Continued→

(k) $$H_c(s) = \frac{4}{s+3} \qquad H_p(s) = \frac{3+4s}{s^2+2s+5}$$

$$H_c(s)H_p(s) = \frac{12+16s}{s^2+5s^2+11s+15}$$

$$H(s) = \frac{H_c(s)Hp(s)}{1+H_c(s)Hp(s)} = \frac{12+16s}{\left(1+\frac{12+16s}{s^3+5s+11s+15}\right)(s^3+5s^2+11s+15)}$$

$$= \frac{12+16s}{s^3+5s^2+27s+27}$$

8.9

parts (a) – (c) are the same as 8.8:

(a): $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m$

$y = [3 \;\; 4] \underline{x}$

(b) $sI - A = \begin{bmatrix} s & -1 \\ 5 & s+2 \end{bmatrix}$

$(sI-A)^{-1} = \begin{bmatrix} s+2 & 1 \\ -5 & s \end{bmatrix} \Big/ (s^2+2s+5)$

$H_p(s) = [3 \;\; 4] \begin{bmatrix} \frac{s+2}{s^2+2s+5} & \frac{1}{s^2+2s+5} \\ \frac{-5}{s^2+2s+5} & \frac{s}{s^2+2s+5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{3+4s}{s^2+2s+5}$

(c) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y(t) = 4\frac{dm}{dt} + 3m(t)$

(d) $m = 2e$

(e) $H_c(s) = 2$

(f) $m(t) = 2e(t)$

Continued→

8.9, continued

8.9 (g) $\dot{x}_1 = x_2$

$$\dot{x}_2 = 2(-1)(4)x_2 + 2(-1)(3)x_1 + (-5)x_1 + (-2)x_2 + 2u$$

$$= -11x_1 + -10x_2 + 2u$$

$\Rightarrow$

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -11 & -10 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 3 & 4 \end{bmatrix} \underline{x}$$

(h) $sI - A = \begin{bmatrix} s & -1 \\ 11 & s+10 \end{bmatrix}$ $\quad \text{adj}(sI - A) = \begin{bmatrix} s+10 & 1 \\ -11 & s \end{bmatrix}$

$$\det(sI - A) = s(s+10) + 11 = s^2 + 10s + 11$$

$$(sI - A)^{-1} = \begin{bmatrix} s+10 & 1 \\ -11 & s \end{bmatrix} \Big/ (s^2+10s+11)$$

$$H(s) = C(sI - A)^{-1}B = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{s+10}{s^2+10s+11} & \frac{1}{s^2+10s+11} \\ \frac{-11}{s^2+10s+11} & \frac{s}{s^2+10s+11} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

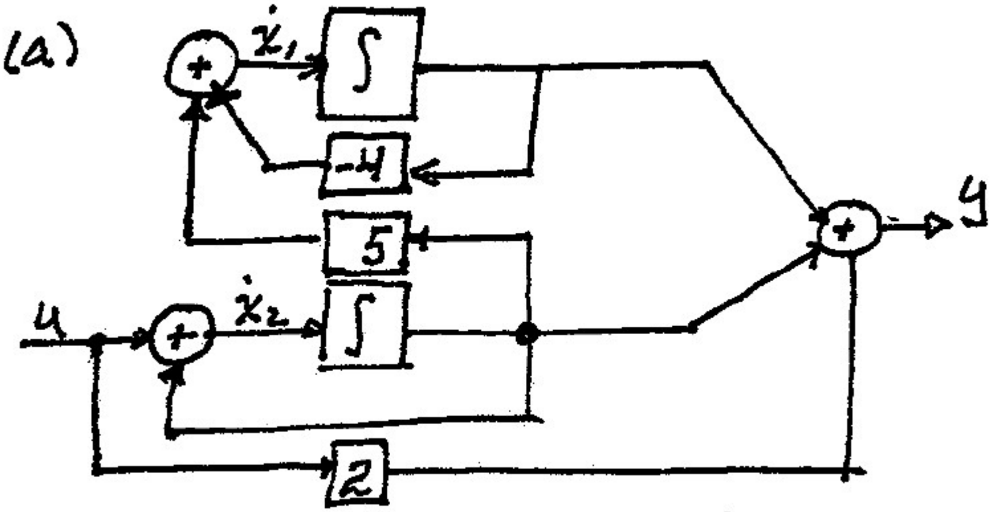
$$= \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{2}{s^2+10s+11} \\ \frac{2s}{s^2+10s+11} \end{bmatrix} = \frac{6 + 8s}{s^2 + 10s + 11}$$

(i) $\dfrac{d^2y}{dt^2} + 10\dfrac{dy}{dt} + 11y(t) = 6u(t) + 8\dfrac{du}{dt}$

(j)
```
>> A=[0 1; -11 -10]; B=[0; 2]; C=[3 4]; D=0;
>> [n d] = ss2tf(A, B, C, D);
```

(k) $$\frac{H_c(s)\,H_p(s)}{1 + H_c(s)\,H_p(s)} = \frac{6+8s}{\left(1 + \frac{6+8s}{s^2+2s+5}\right)(s^2+2s+5)} = \frac{6+8s}{s^2+10s+11}$$

8.10 (a)



(b) $|sI-A| = \begin{vmatrix} s+4 & -5 \\ 0 & s-1 \end{vmatrix} = s^2+3s-4$

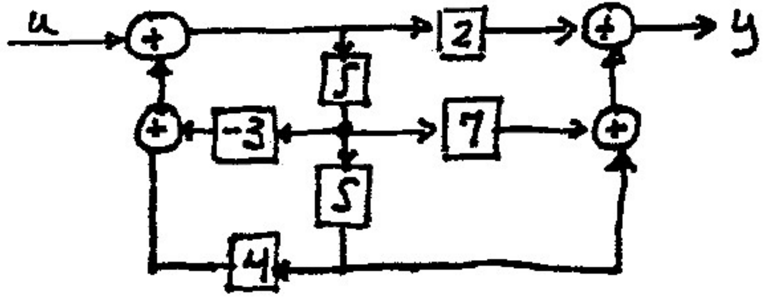
$$H(s) = C(sI-A)^{-1}B+D = \frac{1}{|sI-A|}[1 \quad 1]\begin{bmatrix} s-1 & 5 \\ 0 & s+4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}+2$$

$$= \frac{1}{|sI-A|}[1 \quad 1]\begin{bmatrix} 5 \\ s+4 \end{bmatrix}+2 = \frac{s+9}{s^2+3s-4}+2 = \frac{2s^2+7s+1}{s^2+3s-4}$$

(c)

```
A=[0 1;-4 -5];B=[0;1];C=[103 23];D=0;
[n,d]=ss2tf(A,B,C,D)
```

(d)



(e) $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$; $|sI-A| = \begin{vmatrix} s & -1 \\ -4 & s+3 \end{vmatrix} = s^2+3s-4$

$y = [9 \quad 1]\underline{x}$

(f) $H(s) = \frac{1}{|sI-A|}[9 \quad 1]\begin{bmatrix} s+3 & 1 \\ 4 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}+2 = \frac{1}{|sI-A|}[9 \quad 1]\begin{bmatrix} 1 \\ s \end{bmatrix}+2$

$$= \frac{s+9}{s^2+3s-4}+2 = \frac{2s^2+7s+1}{s^2+3s-4}$$

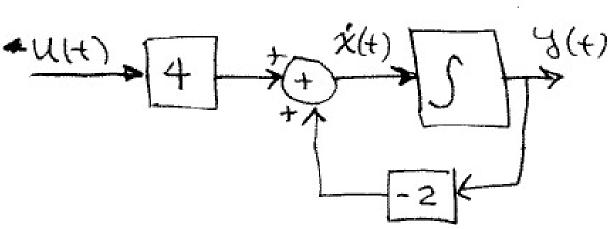
(d)

```
>> A=[0 1; 4 -3]; B=[0; 1]; C=[9 1]; D=0;
>> [n d] = ss2tf(A, B, C, D)
```

Continued→

8.10 (h) $\dot{x}(t) = -2x(t) + 4u(t)$

(a) $y(t) = x(t)$



$$sX(s) = -2X(s) + 4U(s)$$
$$Y(s) = X(s)$$

$$(s+2)X(s) = 4U(s)$$
$$X(s) = \frac{4}{s+2}U(s) \Rightarrow Y(s) = \frac{4}{s+2}U(s)$$
$$H(s) = \frac{4}{s+2}$$

or $A = -2 \Rightarrow sI - A = s+2$

$B = 4$, $C = 1$, $D = 0$

(b) $\frac{Y(s)}{U(s)} = C[sI-A]^{-1}B + D = 1(s+2)^{-1} \times 4 = \frac{4}{s+2}$

(c)

```
>> A=-2; B=4; C=1; D=0;
>> [n d] = ss2tf(A, B, C, D)
```

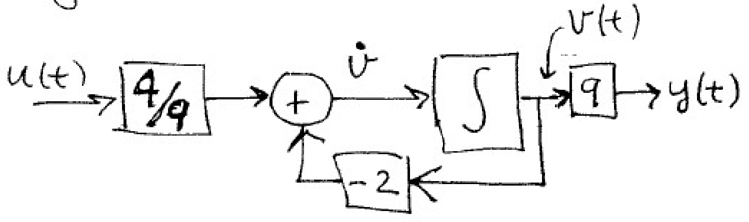
(d) let $P = 9$, $P^{-1} = 1/9 = Q$

$$v(t) = Qx(t) = 1/9\, x(t)$$

(8.60) $\dot{v}(t) = P^{-1}APv(t) + P^{-1}Bu(t)$

$$y(t) = CPv(t) + Du(t)$$

$$\therefore \underline{\dot{v}}(t) = \frac{1}{9}(-2)9v(t) + (1/9)(4)u(t)$$
$$y(t) = (1)(9)v(t)$$



(e) $\dot{v}(t) = -2v(t) + 4/9\, u(t)$

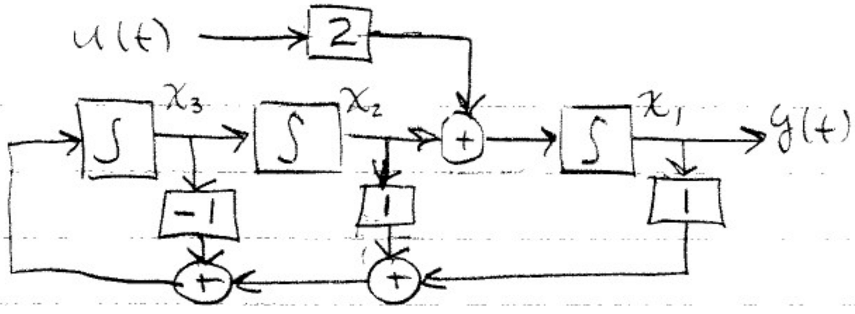
$$y(t) = 9v(t)$$

(f) $A_v = -2$, $B_v = 4/9$, $C_v = 9$, $D_v = 0$

$$\frac{Y(s)}{U(s)} = C_v(sI - A_v)^{-1}B_v + D_v$$
$$= \frac{9 \times 4/9}{s+2} = \frac{4}{s+2}$$

8.10(i)

(a)



(b) $H(s) = C\,[sI-A]^{-1}B + D$

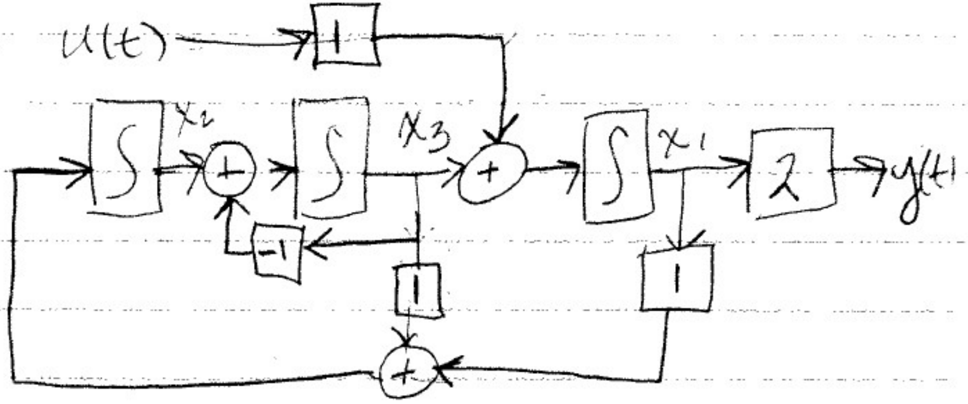
$$= [1\ 0\ 0]\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & -1 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{2(s^2+s-1)}{s^3+s^2-s-1}$$

(c)

```
>> A = [0 1 0; 0 0 1; 1 1 -1]; B = [2 0 0]; C=[1 0 0]; D=0;
>> [n d] = ss2tf(A, B, C, D)
```

(d)



(e) $$\dot{\underline{x}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = [2\ 0\ 0]\,\underline{x}$$

(f) $H(s) = C\,[sI-A]^{-1}B + D$

$$= [2\ 0\ 0]\begin{bmatrix} s & 0 & -1 \\ -1 & s & -1 \\ 0 & -1 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$H(s) = \frac{2(s^2+s-1)}{s^3+s^2-s-1}$$

8.11. (a) From Problem 8.1

$$\dot{x} = -\frac{R}{L}x + \frac{R}{L}u$$
$$y = x$$

(b) $H(s) = C(sI-A)^{-1}B = (1)\left(\frac{1}{s+R/L}\right)\left(\frac{R}{L}\right)$

$$= \underline{\frac{R/L}{s+R/L}}$$

(c) $$\frac{V_R(s)}{V_i(s)} = \frac{R}{sL+R} = \underline{\frac{R/L}{s+R/L}} = H(s)$$

8.12. (a) From Prob 8.1: $\dot{x} = -\frac{R}{L}x + \frac{1}{L}u$

$y = Rx$

(b) $H(s) = C(sI-A)^{-1}B = R\left(\frac{1}{s+R/L}\right)\frac{1}{L} = \frac{R/L}{s+R/L}$

(c) $\frac{V_R(s)}{V_i(s)} = \frac{R}{sL+R} = \frac{R/L}{s+R/L}$

8.13. (a) See problem 8.2.

(b) $|sI-A| = \begin{vmatrix} s & \frac{1}{L} \\ -\frac{1}{C} & s \end{vmatrix} = s^2 + \frac{1}{LC}$

$$H(s) = C(sI-A)^{-1}B = \frac{1}{|sI-A|}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s \end{bmatrix}\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} = \frac{1}{|sI-A|}\begin{bmatrix} \frac{1}{C} & s \end{bmatrix}\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

$$= \frac{\frac{1}{LC}}{s^2 + \frac{1}{LC}}$$

(c) $H(s) = \frac{\frac{1}{sC}}{sL + \frac{1}{sC}} = \frac{1}{s^2LC+1} = \frac{\frac{1}{LC}}{s^2+\frac{1}{LC}}$

8.14. (a) See Problem 8.2.

(b) From Problem 8.13(b)

$$H(s) = \frac{1}{|sI-A|}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -\frac{1}{L} \\ 1/C & s \end{bmatrix}\begin{bmatrix} 1/L \\ 0 \end{bmatrix} = \frac{1}{|sI-A|}\begin{bmatrix} s & -\frac{1}{L} \end{bmatrix}\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

$$= \frac{\frac{1}{L}s}{s^2+\frac{1}{LC}}$$

(c) $Z(s) = Ls + \frac{1}{Cs}$

$$\frac{I(s)}{V_i(s)} = \frac{1}{Zs} = \frac{1}{Ls+\frac{1}{Cs}} = \frac{Cs}{LCs^2+1} = \frac{\frac{1}{L}s}{s^2+\frac{1}{LC}}$$

8.15 (a) $\dot{x} = -3x + 6u$

$y = 4x$

(b) $\Phi(s) = (sI - A)^{-1} = \frac{1}{s+3}$; $\Phi(t) = \underline{e^{-3t}}$

(c) $y_c(t) = C\Phi(t)x(0) = \underline{8e^{-3t}}$, $t > 0$

(d) $X(s) = \Phi(s)BU(s) = \frac{1}{s+3} \cdot 6 \cdot \frac{1}{s} = \frac{6}{s(s+3)} = \frac{2}{s} + \frac{-2}{s+3}$

$\therefore x(t) = 2(1 - e^{-3t}), t > 0 \Rightarrow y_p(t) = 4x(t) = \underline{8(1 - e^{-3t})}, t > 0$

(e) From Problem 8.6, $H(s) = \frac{24}{s+3}$

$\therefore Y_p(s) = H(s) \cdot \frac{1}{s} = \frac{24}{s(s+3)} = \frac{8}{s} - \frac{8}{s+3} \Rightarrow y_p(t) = \underline{8(1 - e^{-3t})}, t > 0$

(f) $y(t) = y_c(t) + y_p(t) = 8e^{-3t} + 8 - 8e^{-3t} = \underline{8}, t > 0$

8.16

(a) [same as 8.7(a)]

$$\dot{x}_1 = -5x_1 + 3x_2 + u$$

$$\dot{x}_2 = -6x_1 + x_2 + 2u$$

$\Rightarrow$

$$\dot{\underline{x}} = \begin{bmatrix} -5 & 3 \\ -6 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = [5 \quad 4]\,\underline{x}$$

(b) $\Phi(s) = (sI - A)^{-1}$ was found in 8.7(b):

$$\Phi(s) = \begin{bmatrix} \dfrac{s-1}{s^2+4s+13} & \dfrac{3}{s^2+4s+13} \\[2ex] \dfrac{-6}{s^2+4s+13} & \dfrac{s+5}{s^2+4s+13} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{0.5+0.5j}{s+2-3j} + \dfrac{0.5-0.5j}{s+2+3j} & \dfrac{-0.5j}{s+2-3j} + \dfrac{0.5j}{s+2+3j} \\[3ex] \dfrac{j}{s+2-3j} + \dfrac{-j}{s+2+3j} & \dfrac{0.5-0.5j}{s+2-3j} + \dfrac{0.5+0.5j}{s+2+3j} \end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}[\Phi(s)] = \begin{bmatrix} e^{-2t}[\cos(3t) - \sin(3t)] & e^{-2t}\sin(3t) \\[2ex] -2e^{-2t}\sin(3t) & e^{-2t}[\cos(3t) + \sin(3t)] \end{bmatrix}$$

Continued→

8.16 (c) $u(t)=0$

$$\underline{X}(0) = [1 \;\; 0]^T$$

$$\underline{X}(s) = \Phi(s)\cdot\underline{X}(0) + \Phi(s)\cdot\underline{B}\cdot u(s)$$

From (b), $\Phi(s) = \begin{bmatrix} \frac{s-1}{s^2+4s+13} & \frac{3}{s^2+4s+13} \\ \frac{-6}{s^2+4s+13} & \frac{s+5}{s^2+4s+13} \end{bmatrix}$

$\underline{X}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u(s)=0$ since $u(t)=0$

$$\underline{X}(s) = \begin{bmatrix} \frac{s-1}{s^2+4s+13} & \frac{3}{s^2+4s+13} \\ \frac{-6}{s^2+4s+13} & \frac{s+5}{s^2+4s+13} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s^2+4s+13} \\ \frac{-6}{s^2+4s+13} \end{bmatrix}$$

$$\underline{X}(t) = \mathcal{L}^{-1}\left[\underline{X}(s)\right] = \begin{bmatrix} e^{-2t}\left[\cos(3t)-\sin(3t)\right] \\ -2e^{-2t}\sin(3t) \end{bmatrix}$$

(from the state trans. matrix $\mathcal{L}^{-1}[\Phi(s)]$ found in 8.16 (b))

$$y_c(t) = C\cdot\underline{X}(t) = [5 \;\; 4] \begin{bmatrix} e^{-2t}\left[\cos(3t)-\sin(3t)\right] \\ -2e^{-2t}\sin(3t) \end{bmatrix}$$

$$= 5e^{-2t}\cos(3t) - 13e^{-2t}\sin(3t), \; t>0$$

Continued→

8.16

(d) $\underline{X}(s) = \Phi(s) \cdot B \cdot U(s) = \Phi(s) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \frac{1}{s}$

$$= \begin{bmatrix} \frac{s+5}{s(s^2+4s+13)} \\ \frac{2s+4}{s(s^2+4s+13)} \end{bmatrix}$$

$$\underline{X}(t) = \begin{bmatrix} \frac{5}{13} - \frac{5}{13}e^{-2t}\cos(3t) + \frac{1}{13}e^{-2t}\sin(3t) \\ -\frac{4}{13}e^{-2t}\cos(3t) + \frac{6}{13}e^{-2t}\sin(3t) + \frac{4}{13} \end{bmatrix}$$

$y_p(t) = C \cdot \underline{X}(t) = [5 \ 4] \, \underline{X}(t)$

$$= \frac{41}{13} - \frac{41}{13}e^{-2t}\cos(3t) + \frac{29}{13}e^{-2t}\sin(3t), \ t > 0$$

Continued→

8.16 (e)

From 8.7(b), $H(s) = \dfrac{13s+41}{s^2+4s+13}$

$X(s) = \mathcal{L}[u(t)] = \frac{1}{s}$;

$Y(s) = H(s) \cdot \frac{1}{s} = \dfrac{13s+41}{s(s^2+4s+13)}$

$y_p(t) = \mathcal{L}^{-1}[Y(s)] = \frac{41}{13} - \frac{41}{13} e^{-2t} \cos(3t) + \frac{29}{13} e^{-2t} \sin(3t)$
$t > 0$

(Note: this could be done in MATLAB using:

```
>> syms s t;
>> ilaplace((13*s+41)/(s*(s^2+4*s+13)))
```
)

(f) $y(t) = y_c(t) + y_p(t)$, where $y_c(t)$ was found in part (c) and $y_p(t)$ was found in part (d)

$$y(t) = 5e^{-2t}\cos(3t) - 13e^{-2t}\sin(3t) + \frac{41}{13} - \frac{41}{13} e^{-2t} \cos(3t) + \frac{29}{13} e^{-2t} \sin(3t), \quad t > 0$$

$$= \frac{41}{13} + \frac{24}{13} e^{-2t} \cos(3t) - \frac{140}{13} e^{-2t} \sin(3t) \quad t > 0$$

8.17. (a) From Problem 8.10,

$$\Phi(s) = (sI-A)^{-1} = \begin{bmatrix} \frac{s-1}{(s-1)(s+4)} & \frac{5}{(s-1)(s+4)} \\ 0 & \frac{s+4}{(s-1)(s+4)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+4} & \frac{1}{s-1} + \frac{-1}{s+4} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

$$\therefore \Phi(t) = \begin{bmatrix} e^{-4t} & e^{t} - e^{-4t} \\ 0 & e^{t} \end{bmatrix}$$

$$x(t) = \Phi(t)\underline{x}(0) = \Phi(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-4t} \\ 0 \end{bmatrix}$$

$$\therefore y_c(t) = C\underline{x}(t) = [1 \quad 1]\underline{x}(t) = \underline{e^{-4t}}, \quad t > 0$$

(b) $$\Phi(t)BU(s) = \Phi(s)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s} = \begin{bmatrix} \frac{5}{s(s-1)(s+4)} \\ \frac{1}{s(s-1)} \end{bmatrix} = \begin{bmatrix} \frac{-5/4}{s} + \frac{1}{s-1} + \frac{1/4}{s+4} \\ -\frac{1}{s} + \frac{1}{s-1} \end{bmatrix}$$

$$\therefore \underline{x}(t) = \begin{bmatrix} -\frac{5}{4} + e^{t} + \frac{1}{4}e^{-4t} \\ -1 + e^{t} \end{bmatrix}$$

$$y(t) = [1 \quad 1]\underline{x}(t) + 2 = -\frac{9}{4} + 2e^{t} + \frac{1}{4}e^{-4t} + 2$$

$$= \underline{-\frac{1}{4} + 2e^{t} + \frac{1}{4}e^{-4t}}, \quad t > 0$$

(c) From Problem 8.10,

$$Y_p(s) = H(s)\cdot\frac{1}{s} = \frac{2s^2 + 7s + 1}{s(s-1)(s+4)} = \frac{-1/4}{s} + \frac{10/5}{s-1} + \frac{1/4}{s+4}$$

$$\therefore y_p(t) = \underline{-\frac{1}{4} + 2e^{t} + \frac{1}{4}e^{-4t}}, \quad t > 0$$

(d) $\ddot{y} + 3\dot{y} - 4y = 2\ddot{u} + 7\dot{u} + 1$

(e) $\dot{u} = \ddot{u} = 0, \quad \dot{y} = 2e^{t} - e^{-4t}$

$$\therefore (2e^{t} + 4e^{-4t}) + (6e^{t} - 3e^{-4t}) - (-1 + 8e^{t} + e^{-4t}) = 1$$

$$\therefore 1 = 1$$

(f) $y(t) = y_c(t) + y_p(t) = -\frac{1}{4} + 2e^{t} + \frac{5}{4}e^{-4t}, \quad t > 0$

$y(0) = C\underline{x}(0) + 2u(0) = [1 \quad 1]\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 = 3$ ✓

8.18. $(sI - A) = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}, \quad |sI - A| = s^2$

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix} \Rightarrow \Phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

8.19.(a) $(sI-A) = \begin{bmatrix} s & 0 \\ -1 & s \end{bmatrix}$, $|sI-A| = s^2$

$$\Phi(s) = (sI-A)^{-1} = \begin{bmatrix} \frac{1}{s} & 0 \\ \frac{1}{s^2} & \frac{1}{s} \end{bmatrix} \Rightarrow \Phi(t) = \underline{\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}}$$

(b) $\Phi(t) = I + At$; since $A^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\therefore \Phi(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}}$$

(c) $\underline{x}(t) = \Phi(t)\,\underline{x}(0) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2+t \end{bmatrix}$

$$y(t) = C\underline{x}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2+t \end{bmatrix} = \underline{2+t}\,, \; t > 0 \quad \checkmark$$

(d) $\dot{\underline{x}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\underline{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2+t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(e) $\underline{X}(s) = \Phi(s)BU(s) = \Phi(s)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s} = \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s^2} + \frac{1}{s^3} \end{bmatrix}$

$$\therefore x(t) = \begin{bmatrix} t \\ t + t^2/2 \end{bmatrix} \Rightarrow y(t) = C\underline{x}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\underline{x}(t) = \underline{t + \frac{t^2}{2}}\,, \; t > 0$$

(f) $H(s) = C(sI-A)^{-1}B = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s} & 0 \\ \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$H(s) = \frac{1}{s} + \frac{1}{s^2}$$

$$Y(s) = H(s)U(s) = \frac{1}{s^2} + \frac{1}{s^3} \Rightarrow y(t) = \underline{t + \frac{t^2}{2}}\,, \; t > 0$$

$$y_t(t) = \underline{2 + 2t + \frac{t^2}{2}}\,, \; t > 0$$

8.20 $\dot{x}(t) = -4x(t) + 8u(t)$

$y(t) = 2x(t)$

$A = -4, \quad B = 8, \quad C = 2$

(a) $\Phi(s) = [sI - A]^{-1}$, $[sI - A] = s+4$

$\therefore \Phi(s) = \frac{1}{s+4}$, $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = e^{-4t}$

(b) $\phi(t) = 1 + (-4)t + (-4)^2\frac{t^2}{2!} + (-4)^3\frac{t^3}{3!}$

$= e^{-4t}$ from (8.37)

(c) $x(t) = \phi(t)x(0)$ from (8.39)

$\therefore x(t) = \phi(t) = e^{-4t}$

$y(t) = 2x(t) = 2e^{-4t}$

(d) $\dot{x}(t) = -4x(t)$

for $x(t) = e^{-4t}$, $\frac{dx(t)}{dt} = -4e^{-4t} = Ax(t)$ ✓

(e) $X(s) = \Phi(s)\underline{x}(0) + \Phi(s)BU(s)$ (8.28)

$x(0) = 0$, $\therefore X(s) = \Phi(s)BU(s)$

for $u(t)$ = unit step function, $U(s) = 1/s$

$\Phi(s) = \mathcal{L}\{\Phi(t)\} = \frac{1}{s+4}$

$X(s) = \frac{8}{s(s+4)} = \frac{2}{s} - \frac{2}{s+4}$

$x(t) = (2 - 2e^{-4t})u(t)$

$y(t) = 2x(t) = 4(1 - e^{-4t})u(t)$

(f) $H(s) = C[sI-A]^{-1}B + D = 2\left[\frac{1}{s+4}\right]8$

$H(s) = \frac{16}{s+4}$

$Y(s) = \frac{1}{s}H(s) = \frac{16}{s(s+4)} = \frac{4}{s} - \frac{4}{s+4}$

$y(t) = 4(1 - e^{-4t})u(t)$

8.21. (a) From Problem 8.10(b), $H(s) = \dfrac{2s^2+7s+1}{s^2+3s-4}$

(b) Let $Q = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$, $\therefore P = Q^{-1} = \begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}$

$$A_v = P^{-1}AP = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-4 & 5\\ 0 & 1\end{bmatrix}P = \begin{bmatrix}-8 & 11\\ -4 & 6\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}-19 & 30\\ -10 & 16\end{bmatrix}$$

$$B_v = P^{-1}B = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}$$

$$C_v = CP = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix}; \quad D_v = D = 2$$

$$\therefore \dot{\underline{v}} = \begin{bmatrix}-19 & 30\\ -10 & 16\end{bmatrix}\underline{v} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\underline{v} + 2u$$

(c)

```
A=[-4 5;0 1];B=[0;1];C=[1 1];D=2;Q=[2 1;1 1];
P=inv(Q);
Av=Q*A*P
Bv=Q*B
Cv=C*P
Dv=D
pause
 [n,d]=ss2tf(Av,Bv,Cv,Dv)
```

(d) $|sI - A_v| = \begin{vmatrix}s+19 & -30\\ 10 & s-16\end{vmatrix} = s^2+3s-304+200 = s^2+3s-4$ ✓

$$C_v(sI-A_v)^{-1}B_v = \begin{bmatrix}0 & 1\end{bmatrix}\frac{1}{|sI-A_v|}\begin{bmatrix}s-16 & 30\\ -10 & s+19\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$= \frac{1}{|sI-A_v|}\begin{bmatrix}-10 & s+19\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{s+9}{s^2+3s-4}$$

$$\therefore H(s) = \frac{s+9}{s^2+3s-4} + 2 = \frac{2s^2+7s+1}{s^2+3s-4}$$ ✓

(e) See (c)

(f) $|sI-A| = |sI-A_v| = s^2+3s-4$ ✓ $= (s-1)(s+4) = (s-\lambda_1)(s-\lambda_2)$

$|A| = -4$; $|A_v| = -304+300 = -4 = \lambda_1\lambda_2 = (1)(-4) = -4$ ✓

$\text{tr}\,A = -4+1 = -3$; $\text{tr}\,A_v = -19+16 = -3$ ✓ $= \lambda_1+\lambda_2 = 1-4 = -3$

8.22 $\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

$y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} \underline{x}(t)$

(a) $H(s) = C[\underline{s}I - A]^{-1}B + D = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 5 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$H(s) = \begin{bmatrix} 2 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+4 & 1 \\ -5 & s \end{bmatrix}}{s^2+4s+5} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \frac{\begin{bmatrix} 1 \\ s \end{bmatrix}}{s^2+4s+5} = \frac{2}{s^2+4s+5}$$

Note: part (b) can be different for each student; parts (c)-(f) are self-checking.

8.23(a) $\dot{x}(t) = -4x(t) + 8u(t)$

$y(t) = 2x(t)$

$A = -4, \; B = 8, \; C = 2, \; D = 0$

$H(s) = C[sI - A]^{-1}B + D = 2\left[\frac{1}{s+4}\right]8$

$H(s) = \frac{16}{s+4}$

Note: part (b) can be different for each student; parts (c)-(g) are self-checking.

8.24

(a) From 8.10(i), $H(s) = \dfrac{2(s^2+s-1)}{s^3+s^2-s-1}$

(b) $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$A_v = P^{-1}AP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B_v = P^{-1}B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$C_v = CP = [1 \; 0 \; 0] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = [1 \; 1 \; 0]$$

$$\Rightarrow \quad \dot{\underline{V}} = A_v \cdot \underline{V} + B_v \cdot u \; , \quad y = C_v \cdot \underline{V}$$

$$\dot{\underline{V}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \underline{V} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u \; , \quad y = [1 \; 1 \; 0] \cdot \underline{V}$$

(c), (f)

```
>> A = [0 1 0; 0 0 1; 1 1 -1]; B=[2; 0; 0];C = [1 0 0]; D=0;
>> P = [1 1 0; 0 0 1; 1 0 0];
>> Q=inv(P)
>> Av = Q*A*P
>> Bv = Q*B
>> Cv = C*P
>> Dv = D
>> [n d] = ss2tf(Av, Bv, Cv, Dv)
```

(d) Show that $H(s)=C_v\,(sI-A)^{-1}\,B_v$ gives the same result as in part (a)

8.25. $C_v(sI-A_v)^{-1}B_v+D_v = CP(sI-P^{-1}AP)^{-1}P^{-1}B$

$= CP(sP^{-1}IP-P^{-1}AP)^{-1}P^{-1}B = CP(P^{-1}(sI-A)P)^{-1}P^{-1}B$

$= CPP^{-1}(sI-A)^{-1}PP^{-1}B = C(sI-A)B$, since $(AB)^{-1}=B^{-1}A^{-1}$

8.26

(a) $A=\begin{bmatrix}-4 & 5\\ 0 & 1\end{bmatrix}$ $|sI-A| = \begin{vmatrix} s+4 & -5 \\ 0 & s-1\end{vmatrix} = (s+4)(s-1)$

roots: $-4, 1$

not stable since root $1>0$

(b) e^{-4t}, e^{t}

(c) >>A = [-4 5; 0 1]; eig(A)

8.27 from Problem 8.22

(a) $CE = s^2+4s+5=0 = (s+2-j1)(s+2+j1)$

Eigenvalues: $s_1=-2+j1$, $s_2=-2-j1$

$Re\{s_1\}<0$, $Re\{s_2\}<0$

$\therefore$ System is stable

(b) System modes

$e^{(-2+j1)t} = e^{-2t}e^{jt}$ and $e^{(-2-j1)t} = e^{-2t}e^{-jt}$

(c)

```
>> A=[0 1; -5 -4];
>> eig(A)
```

8.28

(a) $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & -1\end{bmatrix}$ $|sI-A| = \begin{vmatrix} s & -1 & 0\\ 0 & s & -1\\ -1 & -1 & s+1\end{vmatrix} = s^3+s^2-s-1$

roots: $1, -1, -1$

not stable

(b) e^{t}, e^{-t}, te^{-t}

(c)
```
>>A = [0 1 0; 0 0 1; 1 1 -1];
>> eig(A)
```

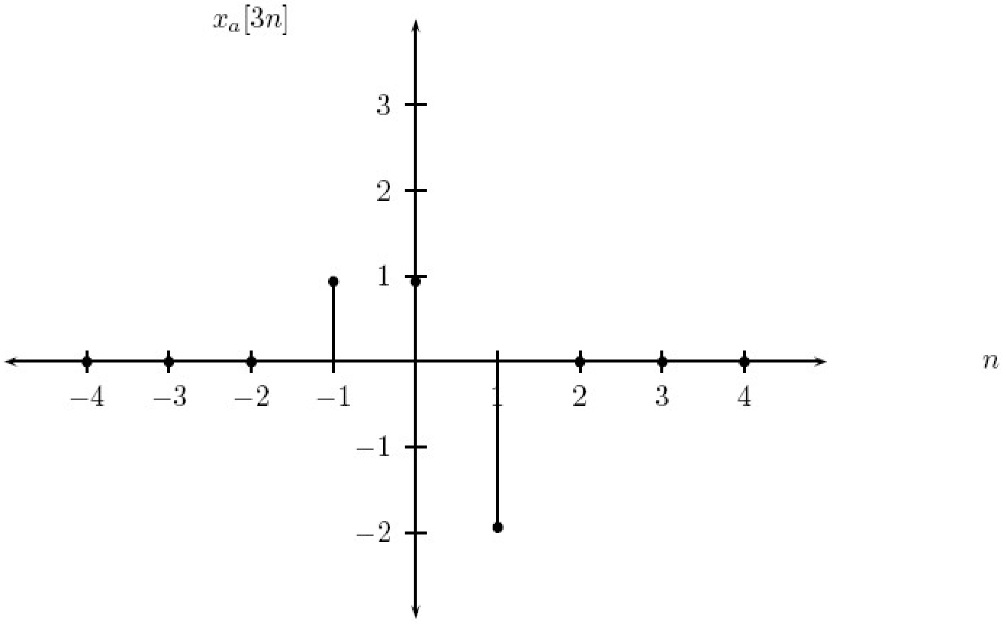
CHAPTER 9 solutions

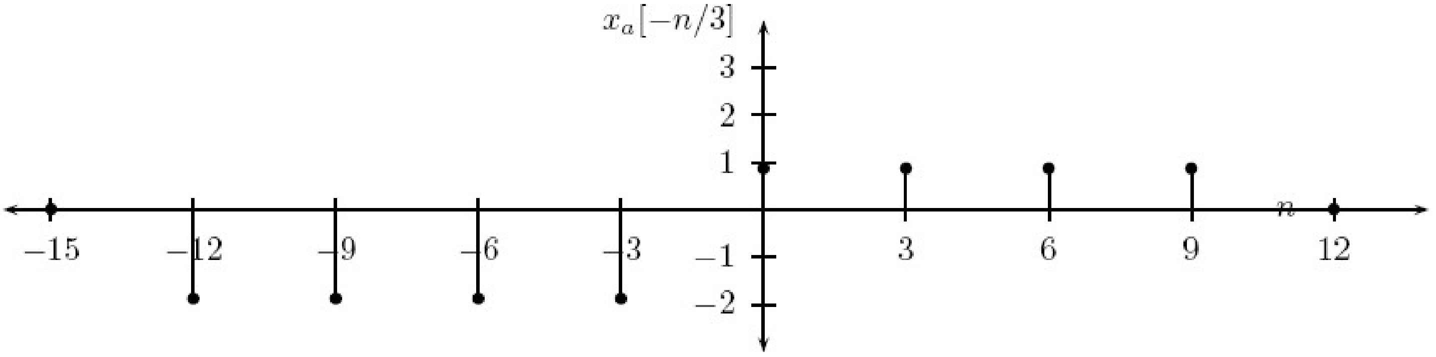
9.1 $x_1[n]$, $x_2[n]$ and $x_4[n]$ (parts (a), (b), and (d)) are all equal to the constant signal $x[n] = 1$ for all n. The one that is different is $x_3[n]$ (part (c)) which is equivalent to the signal

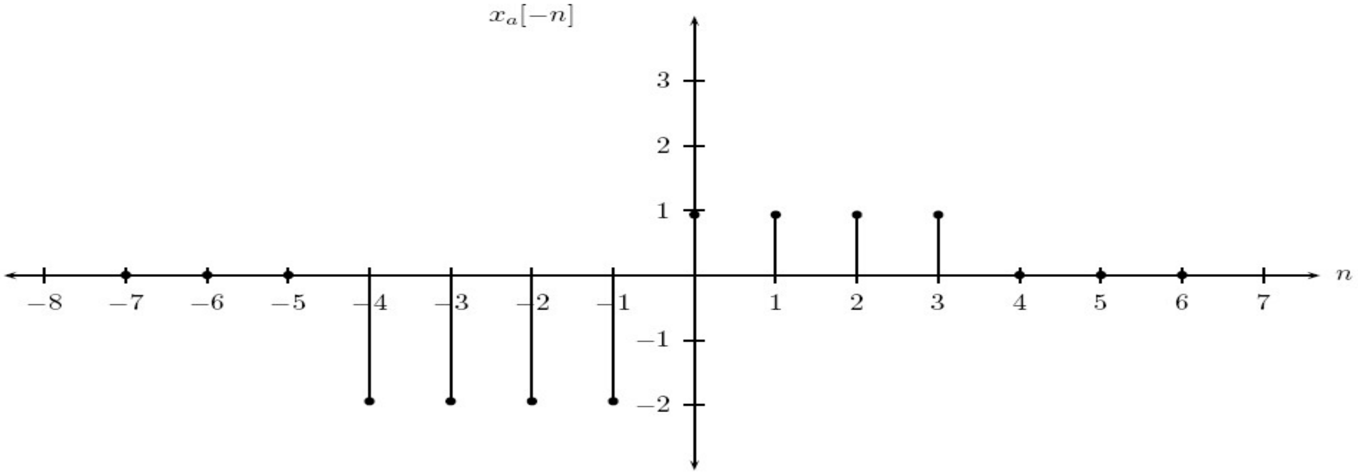
$$\begin{aligned} x[n] = & \quad 1, n \neq 0 \\ = & \quad 2, n = 0 \end{aligned}$$

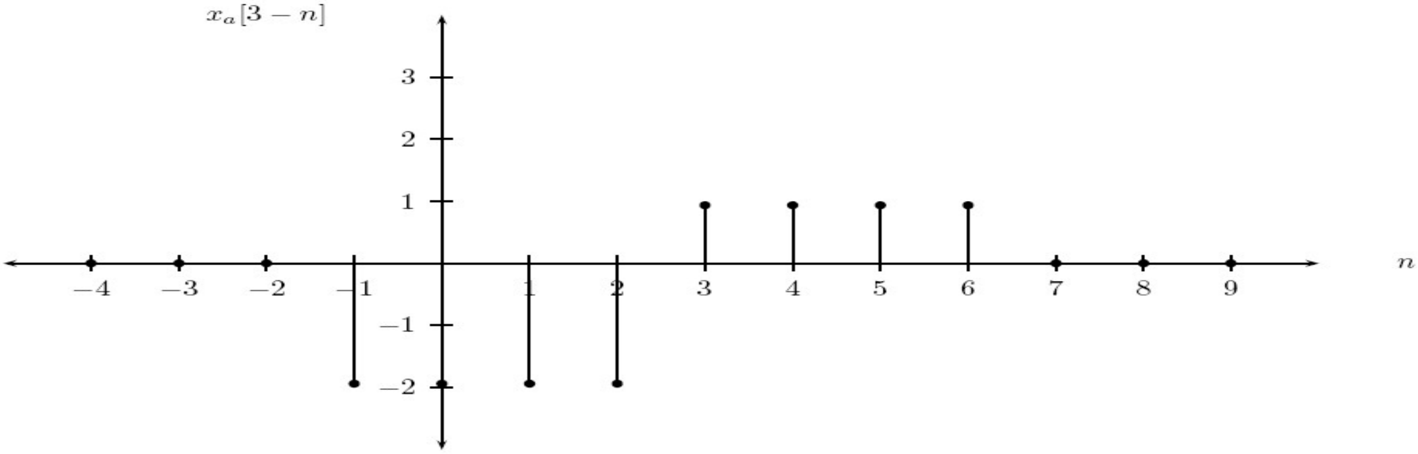
9.2

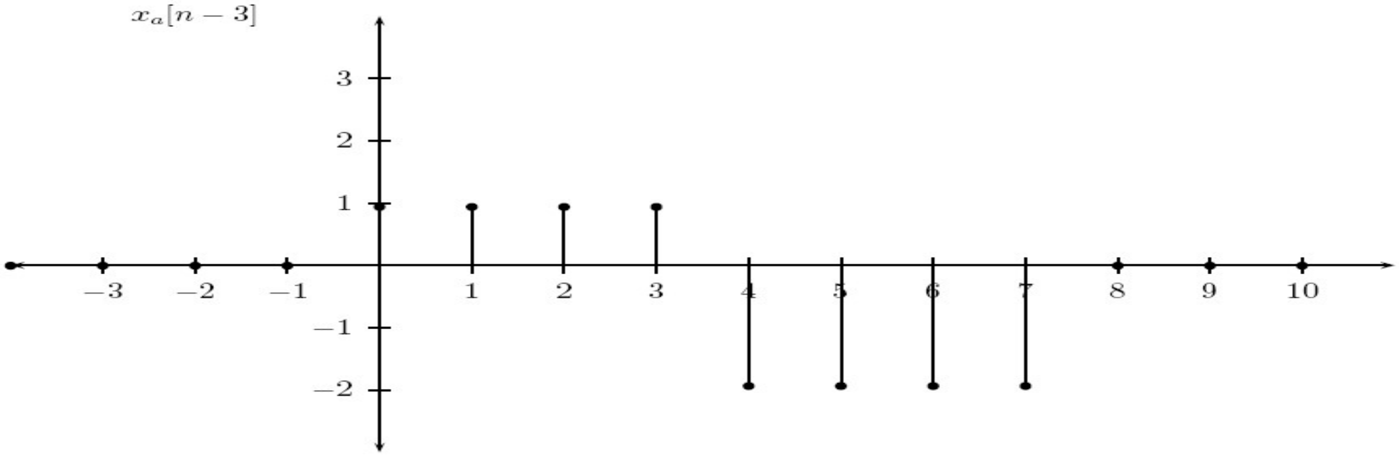
(a)

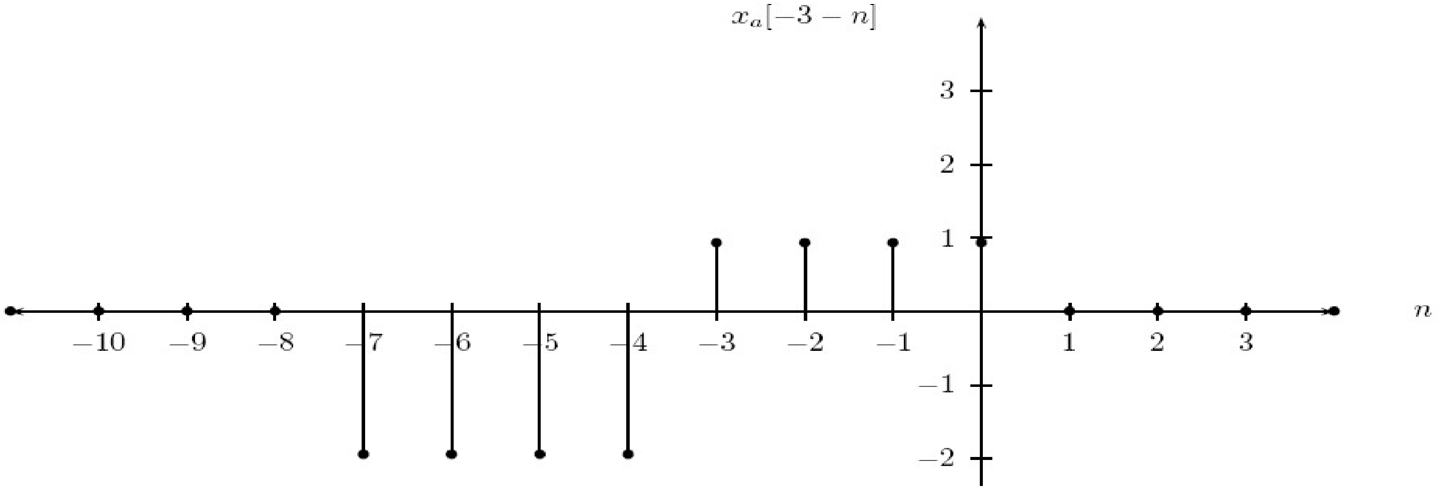




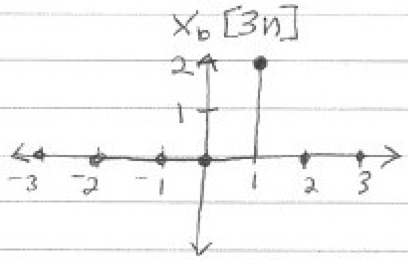

$x_a[-n]$
n
3
2
1
−1
−2
−8
−7
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6
7


$x_a[3-n]$
n
3
2
1
−1
−2
−4
−3
−2
−1
1
2
3
4
5
6
7
8
9

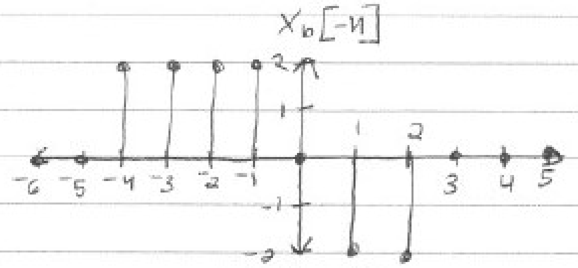

$x_a[n-3]$
n
3
2
1
−1
−2
−3
−2
−1
1
2
3
4
5
6
7
8
9
10


$x_a[-3-n]$
n
3
2
1
−1
−2
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
1
2
3

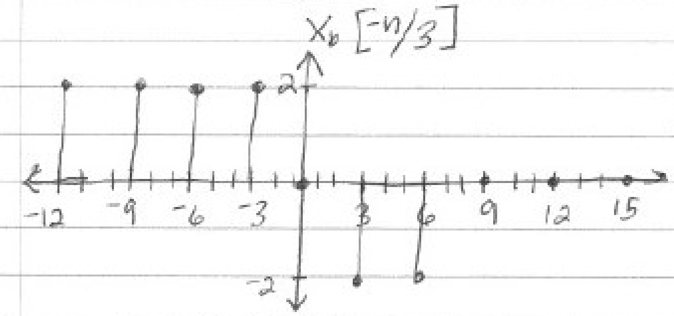
9.2 (b)



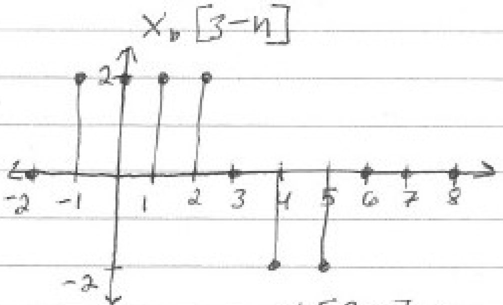
$x_b[3n]$
2
1
-3
-2
-1
1
2
3



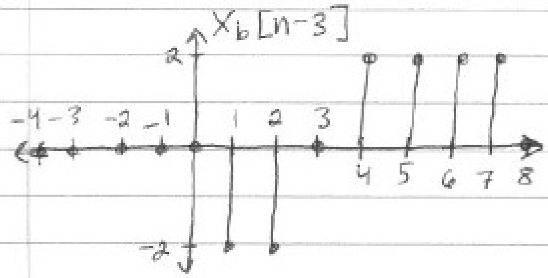
$x_b[-n]$
2
1
-1
-2
-6
-5
-4
-3
-2
-1
1
2
3
4
5



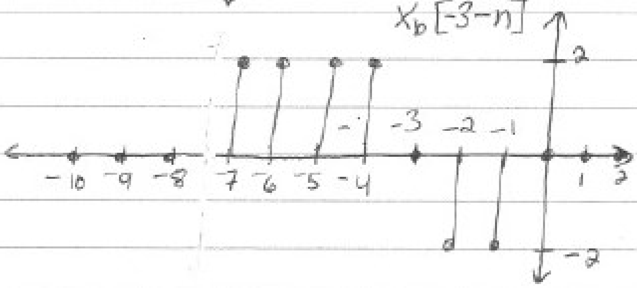
$x_b[-n/3]$
2
-2
-12
-9
-6
-3
3
6
9
12
15



$x_b[3-n]$
2
-2
-2
-1
1
2
3
4
5
6
7
8

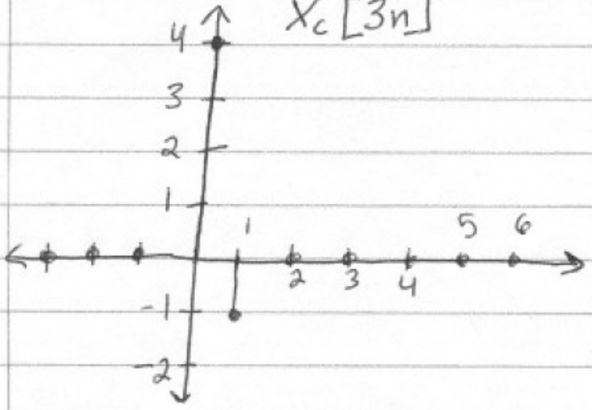


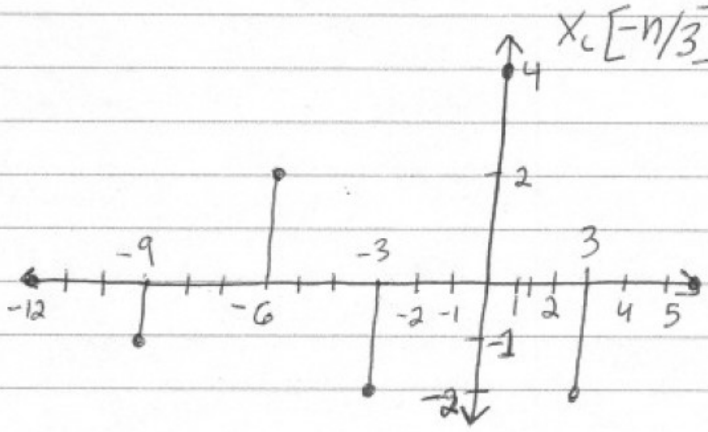
$x_b[n-3]$
2
-2
-4
-3
-2
-1
1
2
3
4
5
6
7
8

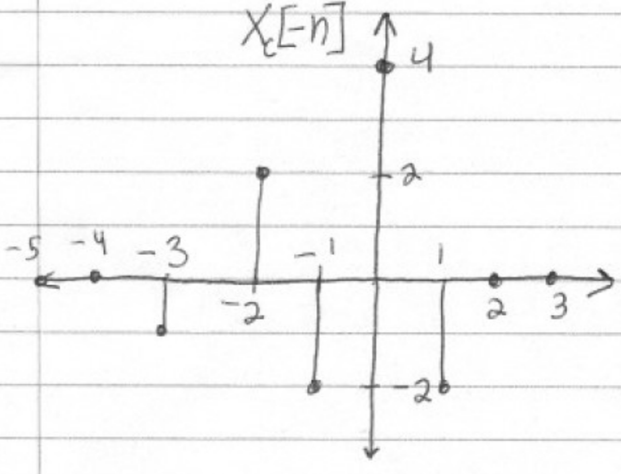


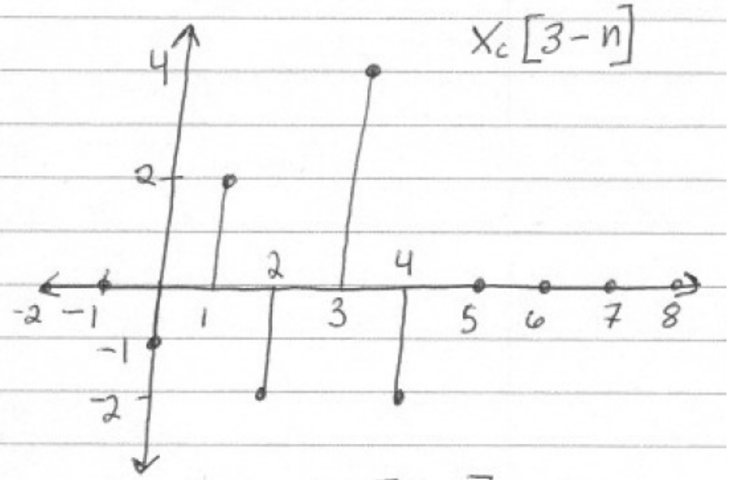
$x_b[-3-n]$
2
-2
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
1
2

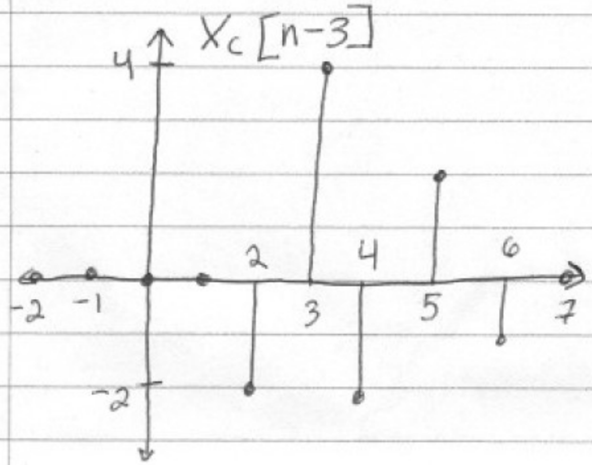
9.2 (c)

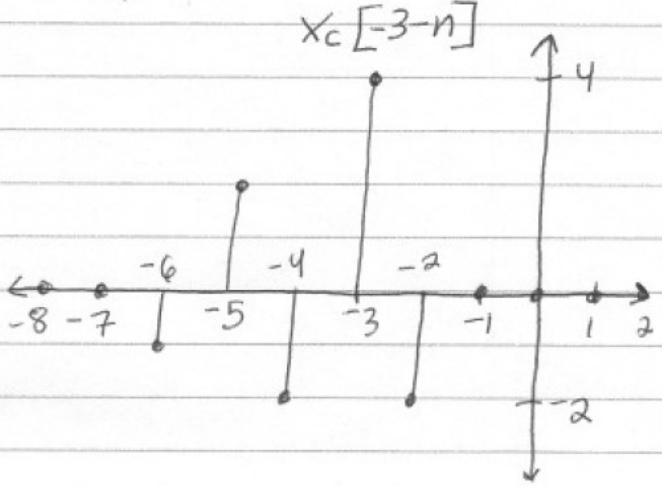

$X_c[3n]$
4
3
2
1
-1
-2
1
2
3
4
5
6


$X_c[-n/3]$
4
2
-1
-2
-12
-9
-6
-3
-2
-1
1
2
3
4
5

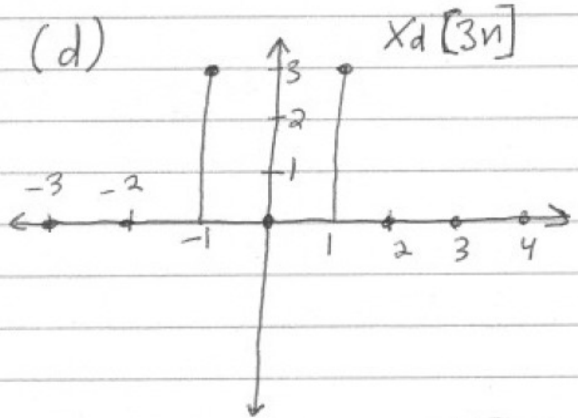

$X_c[-n]$
4
2
-2
-5
-4
-3
-2
-1
1
2
3

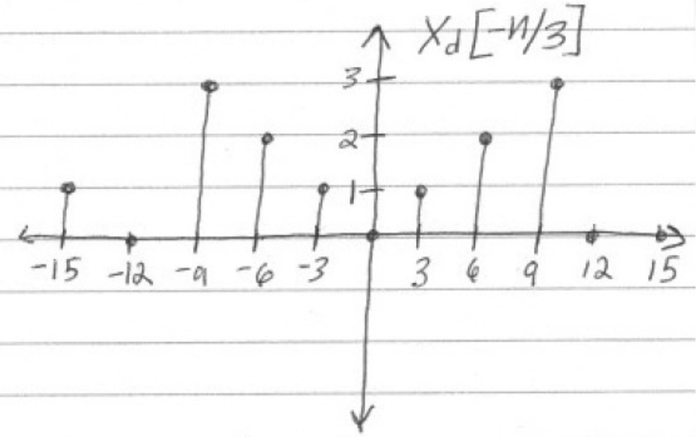

$X_c[3-n]$
4
2
-1
-2
-2
-1
1
2
3
4
5
6
7
8

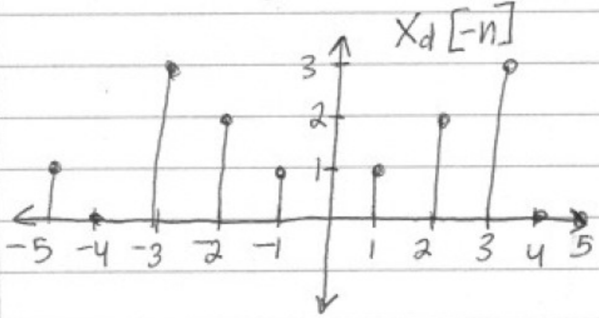

$X_c[n-3]$
4
-2
-2
-1
2
3
4
5
6
7

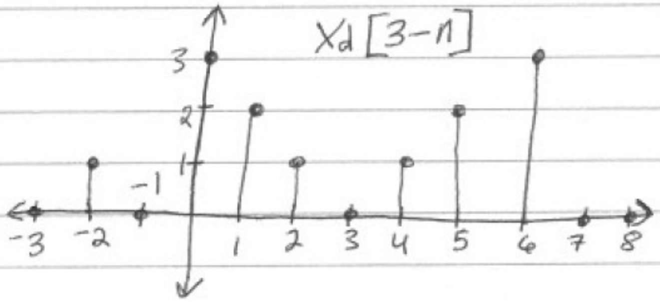

$X_c[-3-n]$
4
-2
-8
-7
-6
-5
-4
-3
-2
-1
1
2

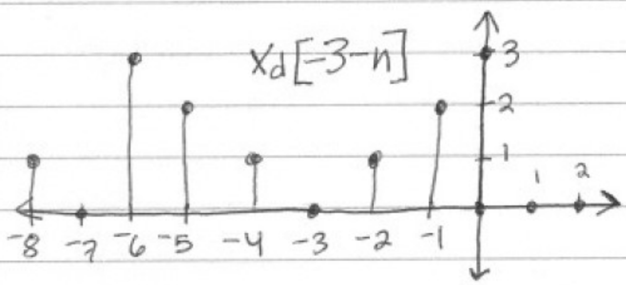
9.2 (d)

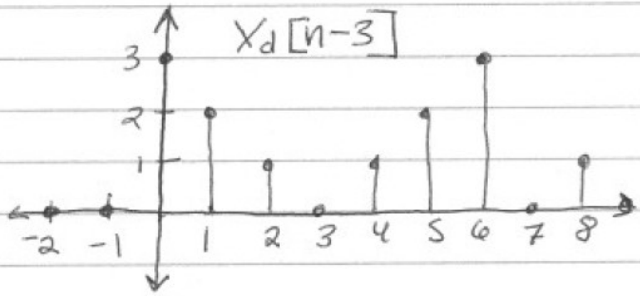

$x_d[3n]$


$x_d[-n/3]$

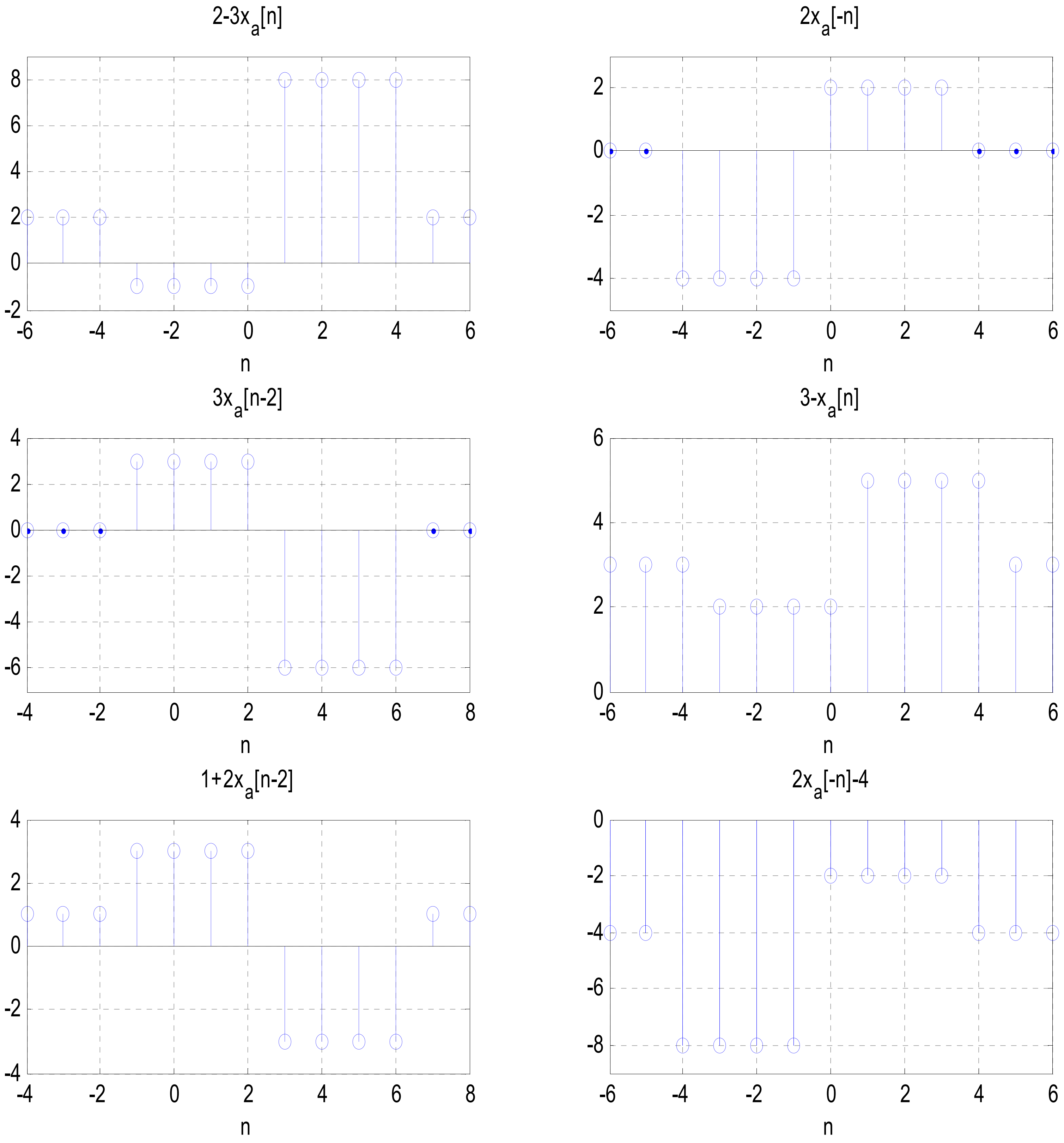

$x_d[-n]$


$x_d[3-n]$

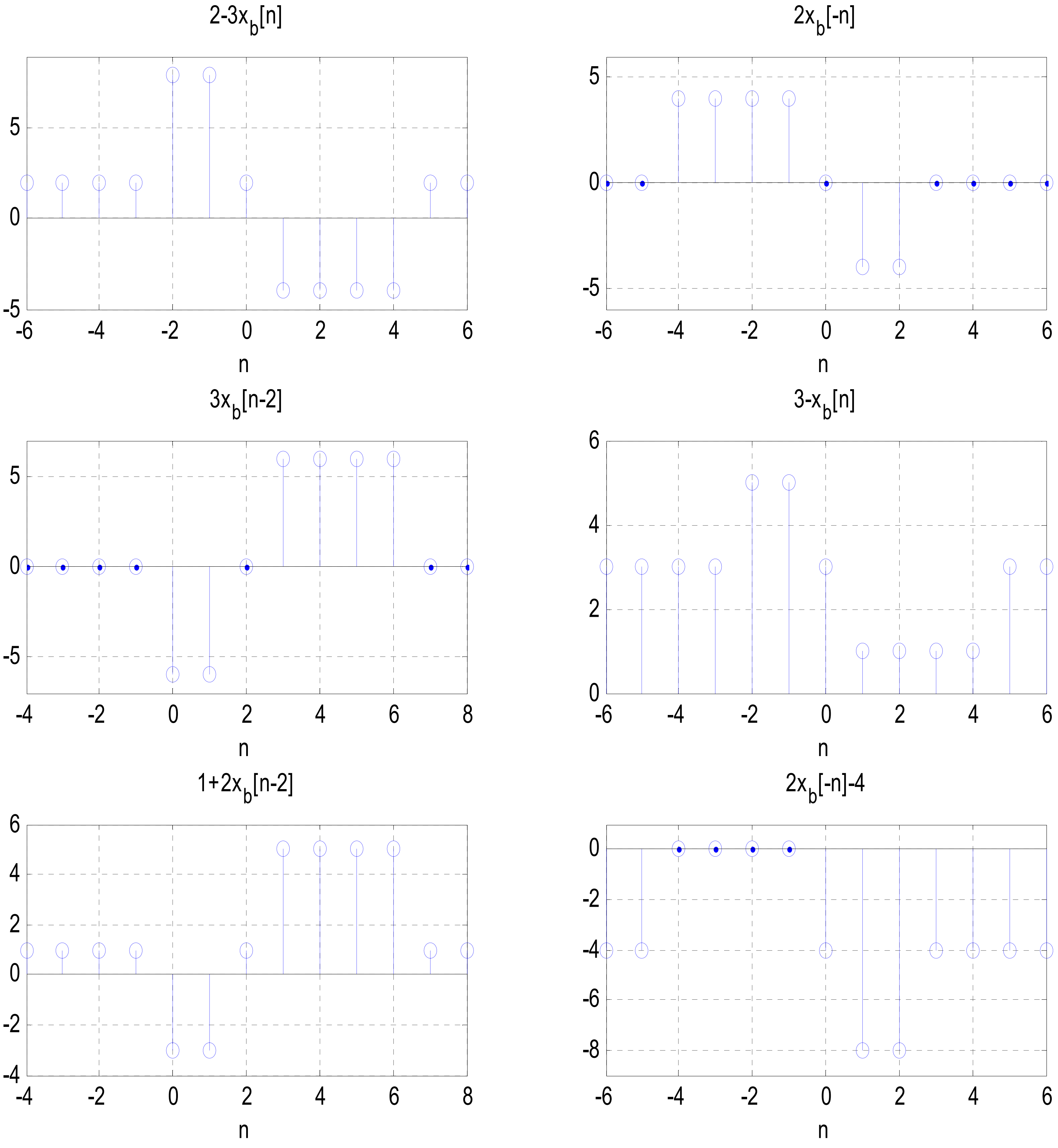

$x_d[-3-n]$


$x_d[n-3]$

9.3 (a)

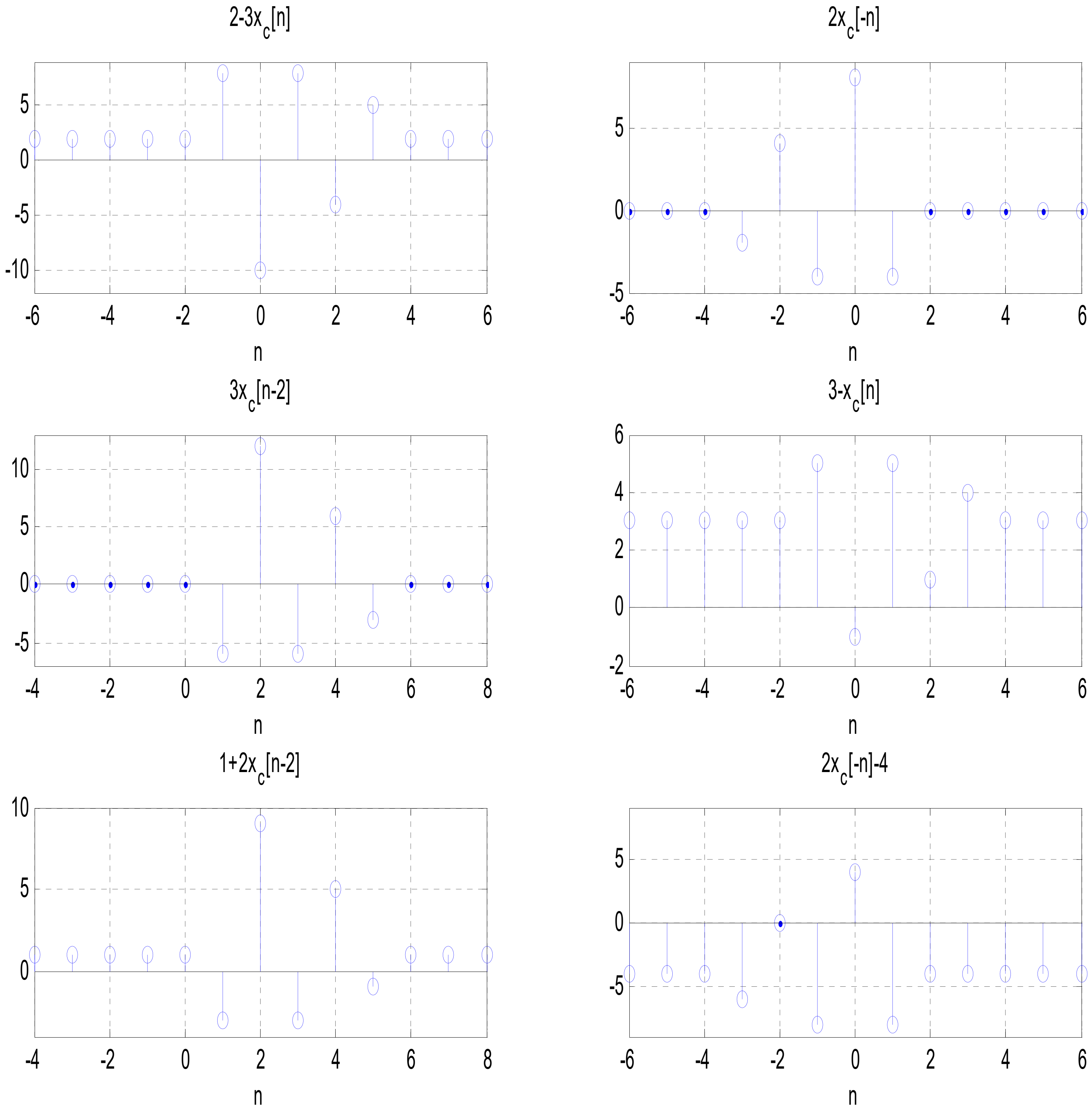

$2-3x_a[n]$
8
6
4
2
0
-2
-6
-4
-2
0
2
4
6
n
$2x_a[-n]$
2
0
-2
-4
-6
-4
-2
0
2
4
6
n
$3x_a[n-2]$
4
2
0
-2
-4
-6
-4
-2
0
2
4
6
8
n
$3-x_a[n]$
6
4
2
0
-6
-4
-2
0
2
4
6
n
$1+2x_a[n-2]$
4
2
0
-2
-4
-4
-2
0
2
4
6
8
n
$2x_a[-n]-4$
0
-2
-4
-6
-8
-6
-4
-2
0
2
4
6
n

9.3 (b)

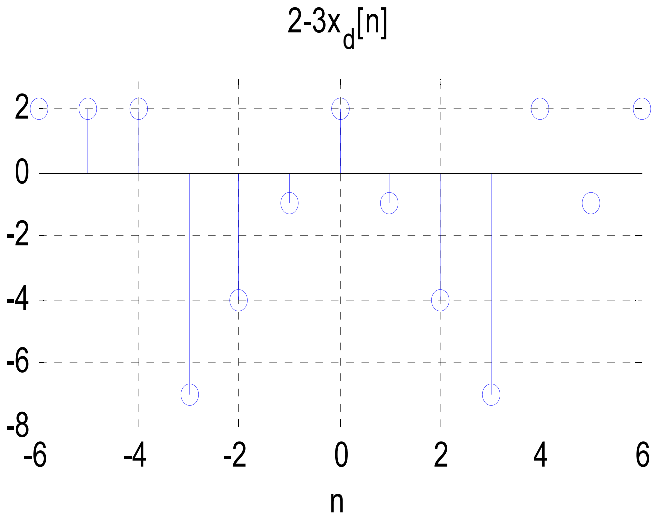


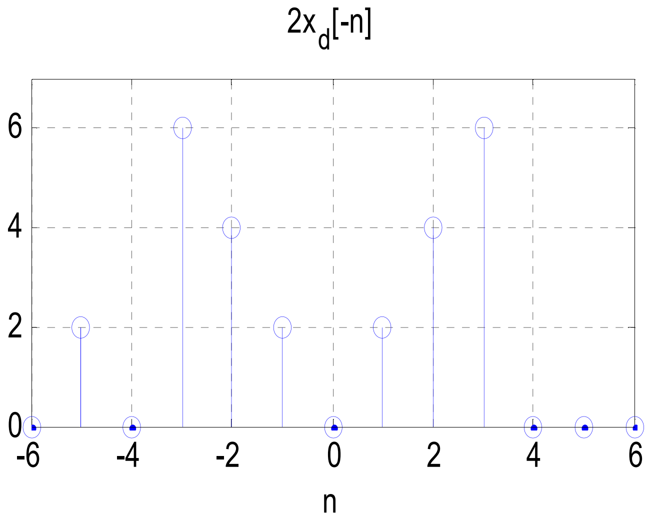
2-3x_b[n]
5
0
-5
-6 -4 -2 0 2 4 6
n
2x_b[-n]
5
0
-5
-6 -4 -2 0 2 4 6
n
3x_b[n-2]
5
0
-5
-4 -2 0 2 4 6 8
n
3-x_b[n]
6
4
2
0
-6 -4 -2 0 2 4 6
n
1+2x_b[n-2]
6
4
2
0
-2
-4
-4 -2 0 2 4 6 8
n
2x_b[-n]-4
0
-2
-4
-6
-8
-6 -4 -2 0 2 4 6
n

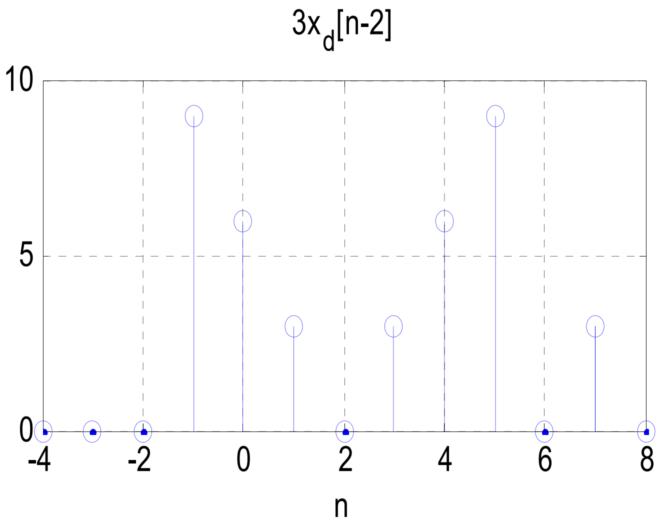
9.3 (c)

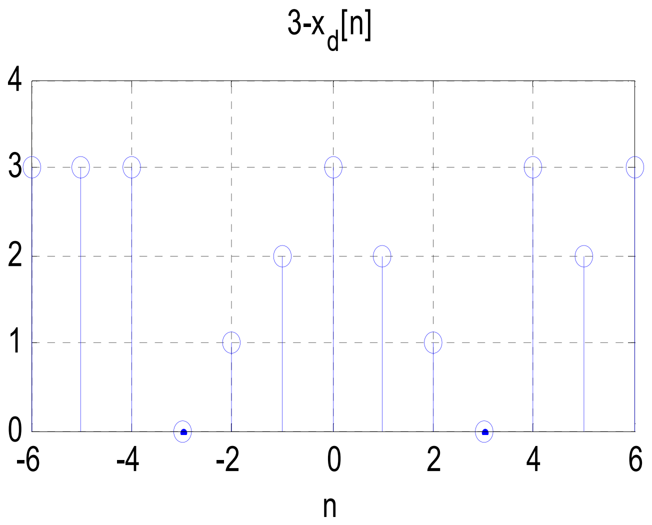


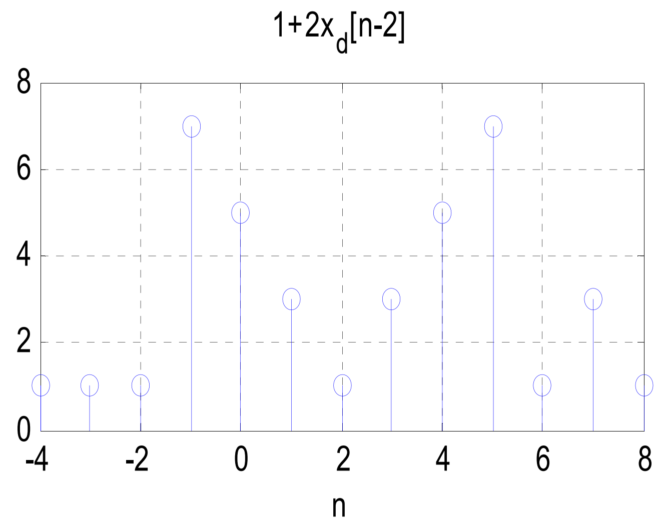
9.3 (d)

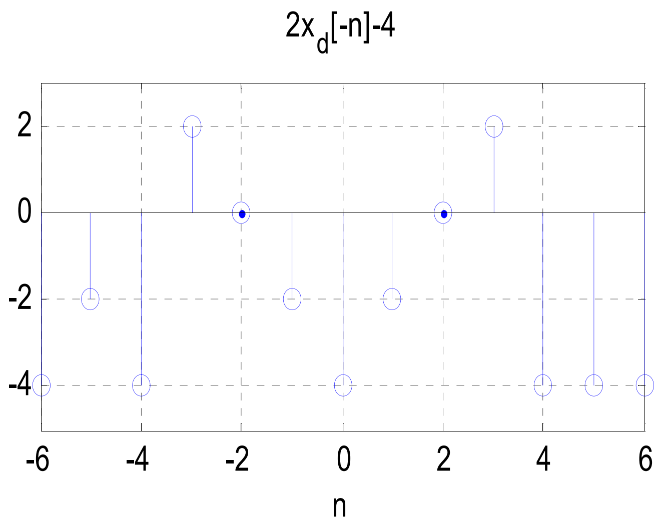

$2-3x_d[n]$


$2x_d[-n]$

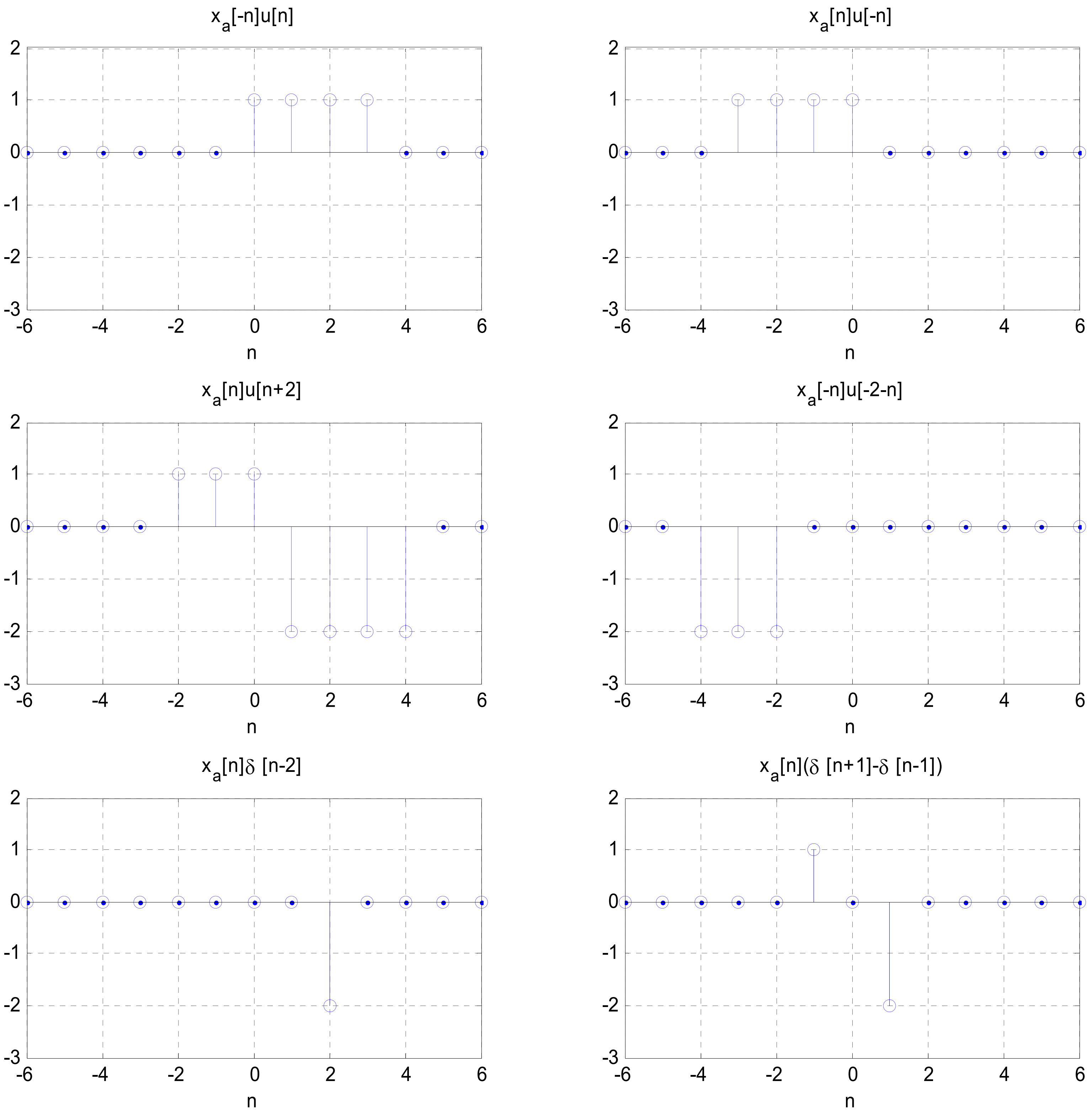

$3x_d[n-2]$


$3-x_d[n]$


$1+2x_d[n-2]$

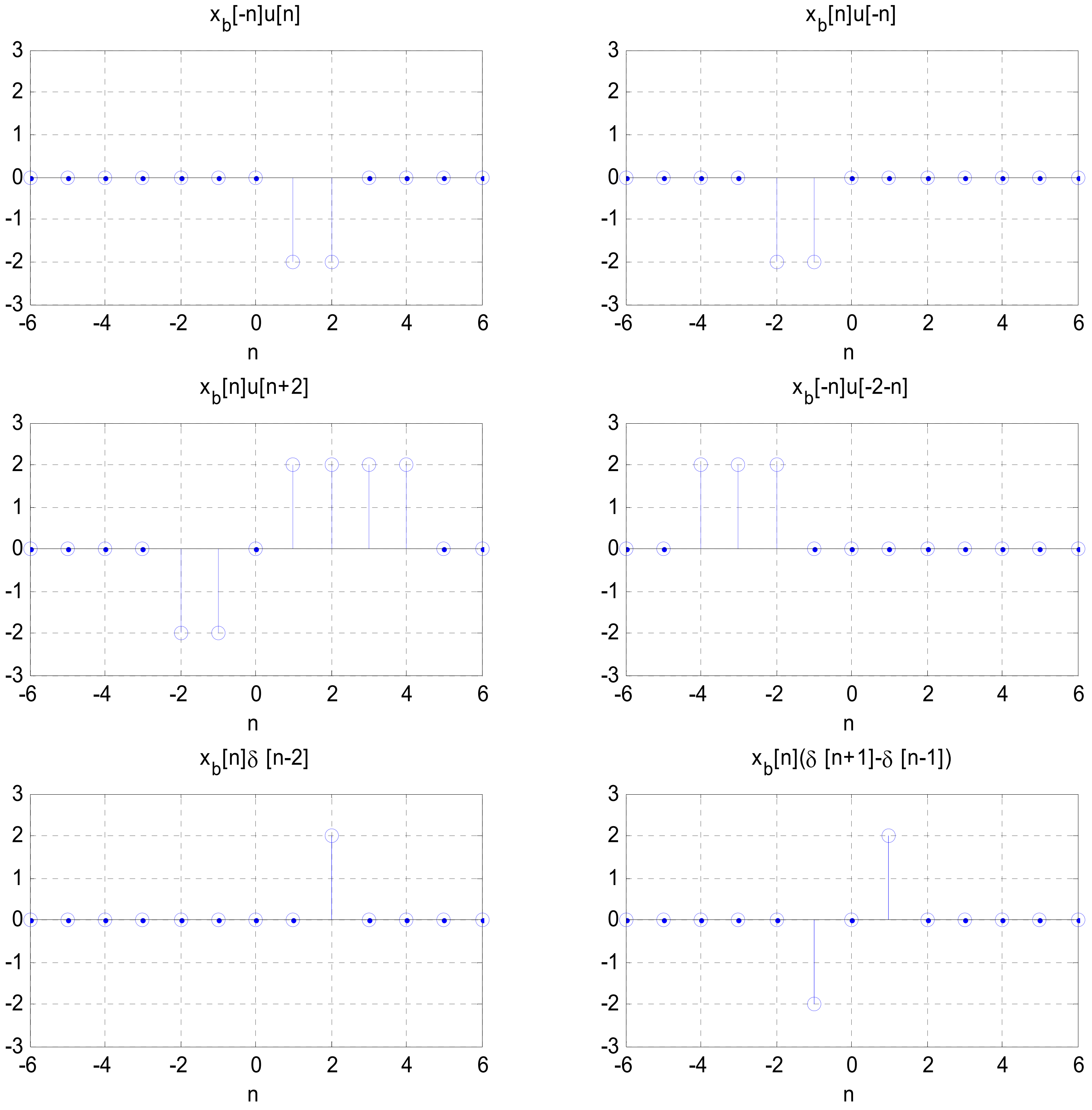

$2x_d[-n]-4$

9.4 (a)



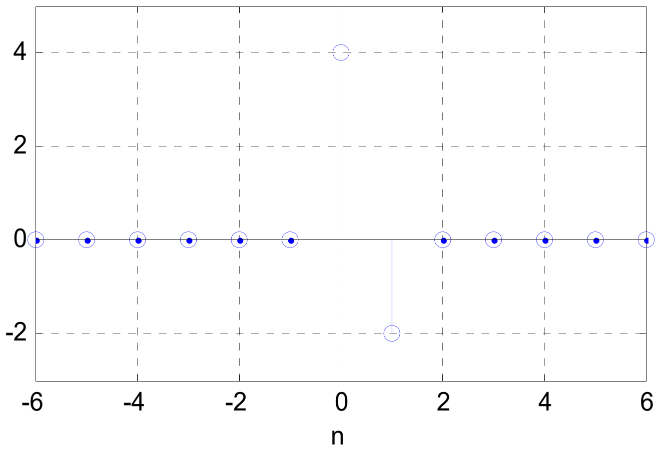
$x_a[-n]u[n]$
$x_a[n]u[-n]$
$x_a[n]u[n+2]$
$x_a[-n]u[-2-n]$
$x_a[n]\delta[n-2]$
$x_a[n](\delta[n+1]-\delta[n-1])$
2
1
0
-1
-2
-3
-6
-4
-2
0
2
4
6
n

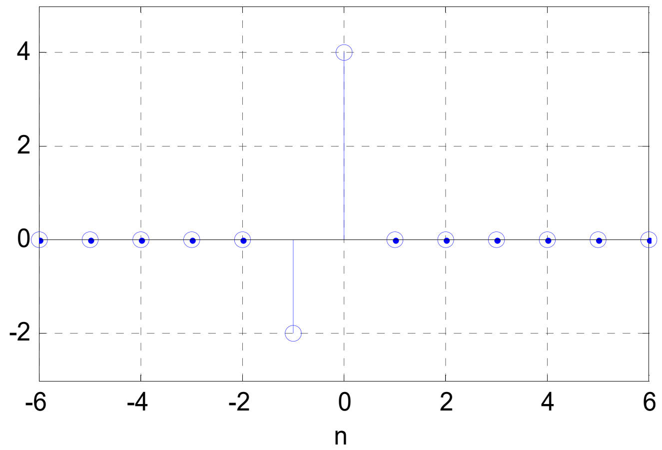
9.4 (b)

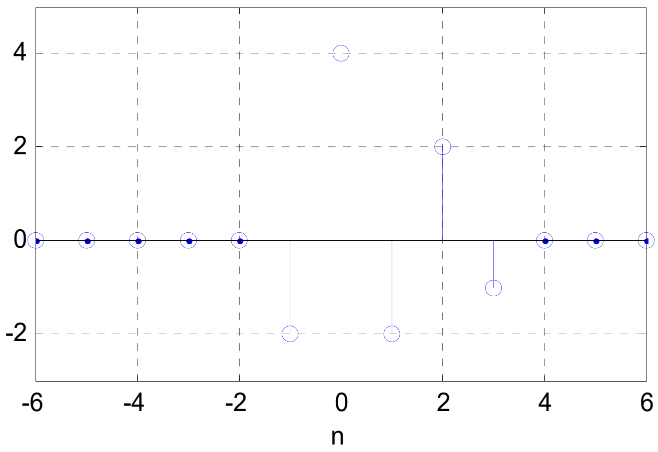


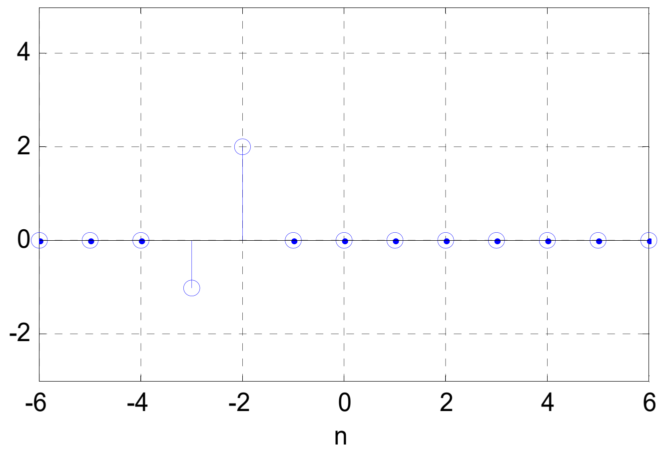
$x_b[-n]u[n]$
$x_b[n]u[-n]$
$x_b[n]u[n+2]$
$x_b[-n]u[-2-n]$
$x_b[n]\delta[n-2]$
$x_b[n](\delta[n+1]-\delta[n-1])$
3
2
1
0
-1
-2
-3
-6
-4
-2
0
2
4
6
n

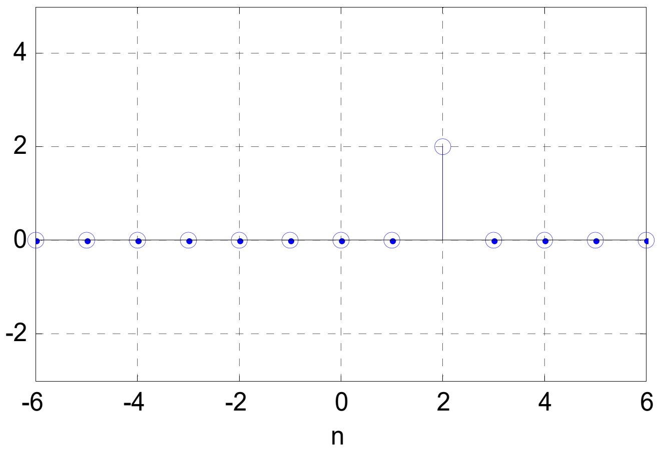
9.4 (c)

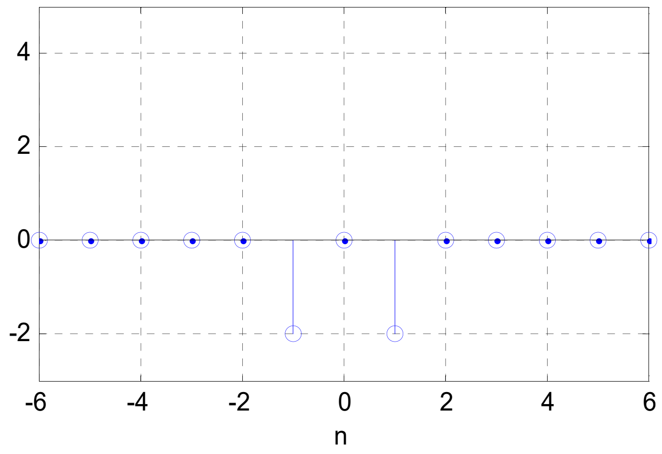

$x_c[-n]u[n]$


$x_c[n]u[-n]$

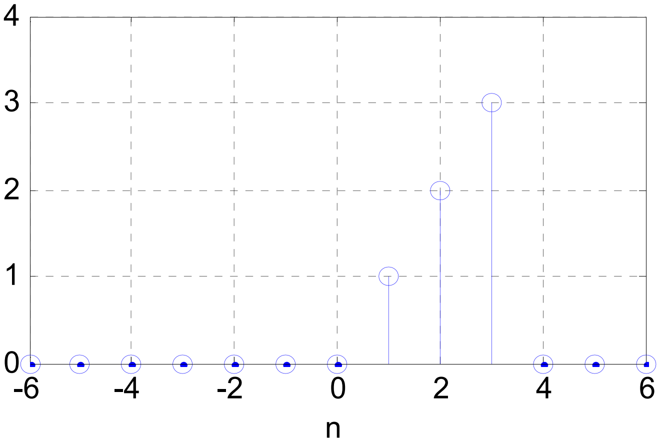

$x_c[n]u[n+2]$


$x_c[-n]u[-2-n]$

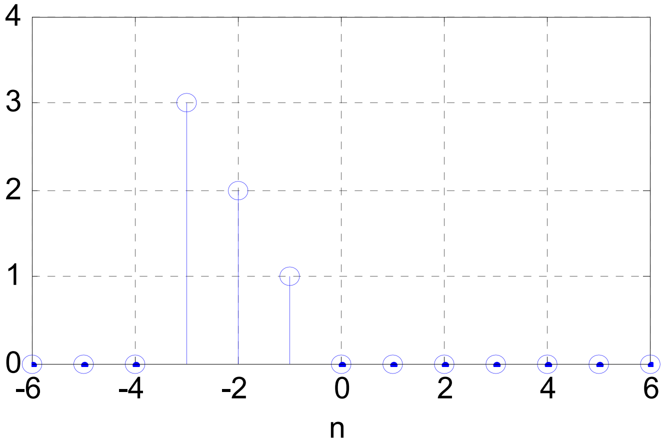

$x_c[n]\delta\ [n-2]$


$x_c[n](\delta\ [n+1]-\delta\ [n-1])$

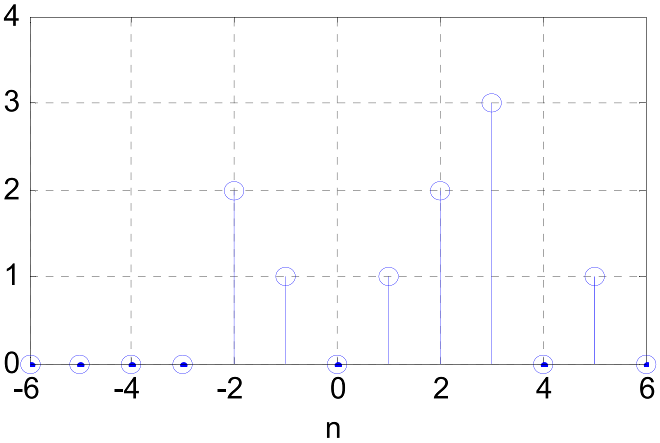
9.4 (d)



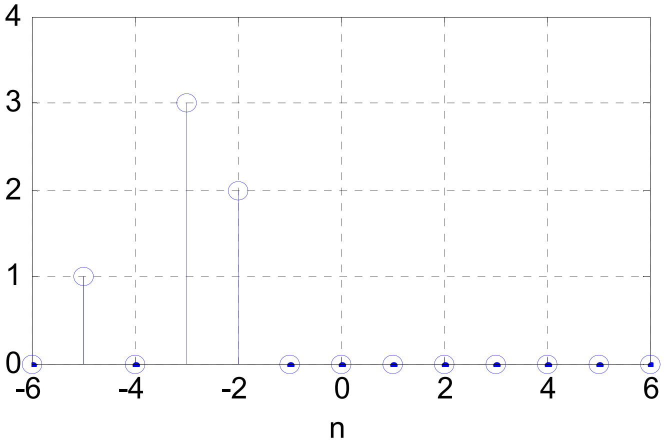
$x_d[-n]u[n]$



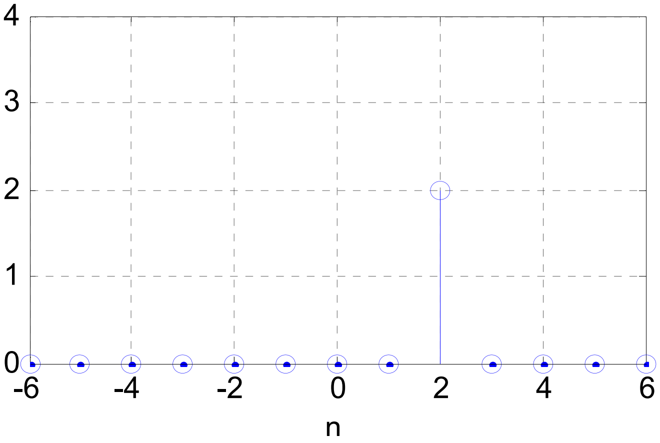
$x_d[n]u[-n]$



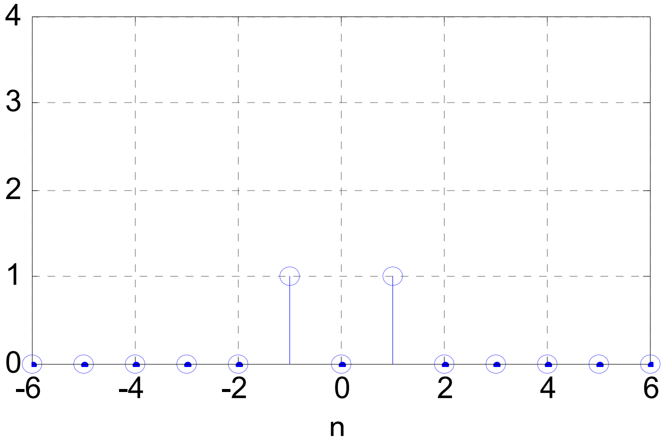
$x_d[n]u[n+2]$



$x_d[-n]u[-2-n]$



$x_d[n]\delta\ [n-2]$



$x_d[n](\delta\ [n+1]-\delta\ [n-1])$

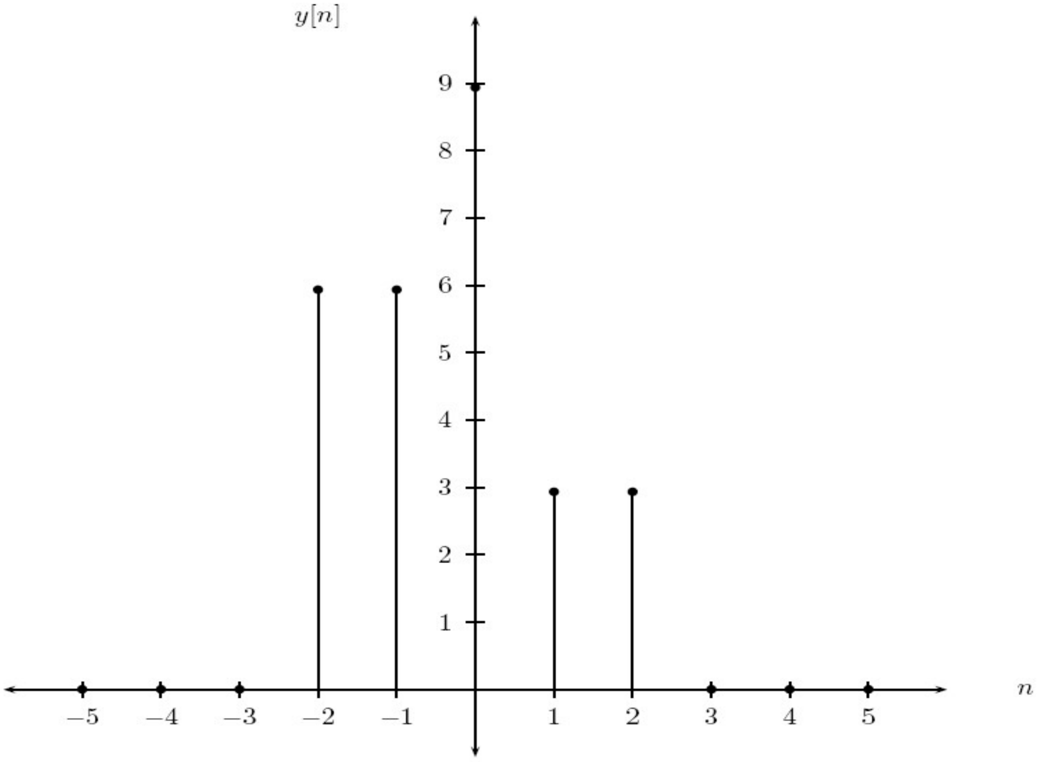
9.5

Replacing n with $-3n-1$ in $x[n]$ gives:

$$y[n] = 3(u[3n+1] - u[3n-7]) + 6(u[-3n] - u[-3n-7])$$

and using the facts that $u[3n+1] = u[n]$, $u[3n-7] = u[n-3]$, $u[-3n] = u[-n]$, and $u[-3n-7] = u[-n-3]$ gives:

$$y[n] = 3(u[n] - u[n-3]) + 6(u[-n] - u[-n-3])$$



9.6

a) $x_t[n] = Ax[an+n_0] + B$

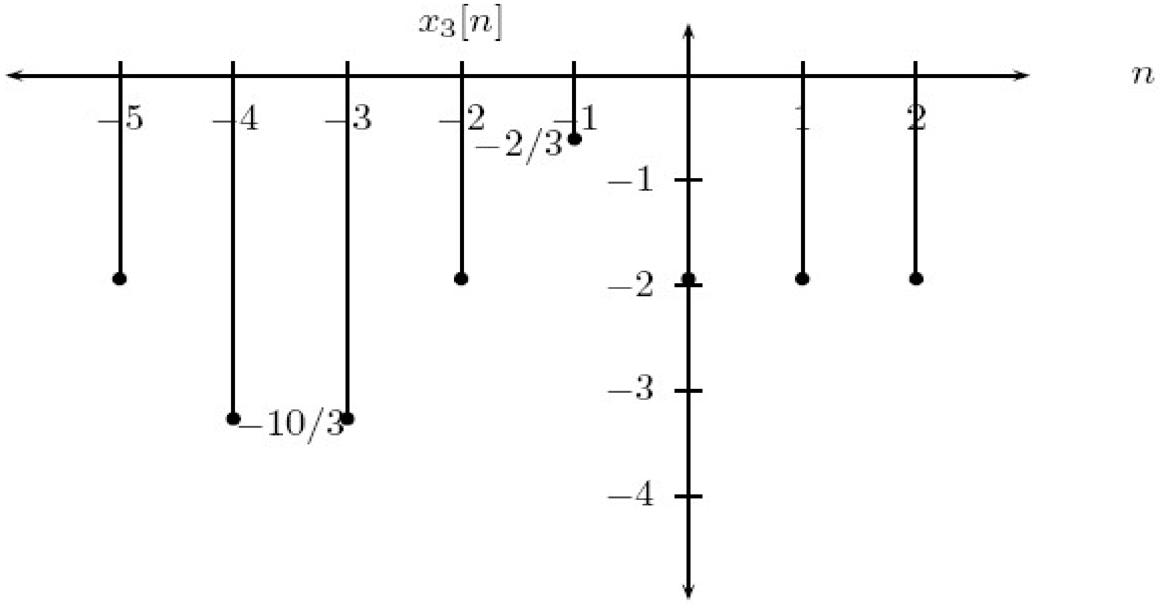
let $an+n_0 = m \Longrightarrow n = \frac{1}{a}m - \frac{1}{a}n_0$

$Ax[m] = x_t\left[\frac{1}{a}m - \frac{1}{a}n_0\right] - B$

let $m \leftarrow n$

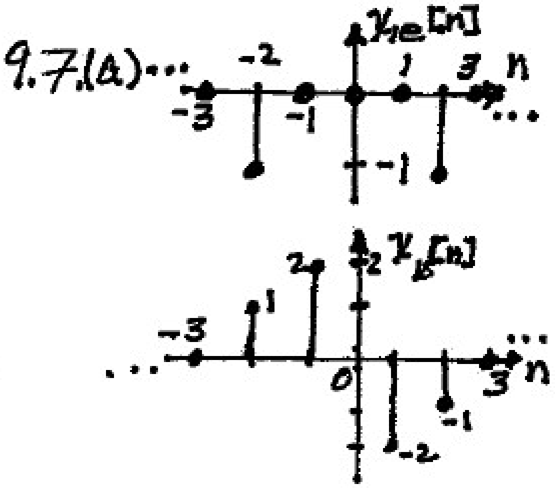
$x[n] = \frac{1}{A}x_t\left[\frac{1}{a}n - \frac{1}{a}n_0\right] - B/A$

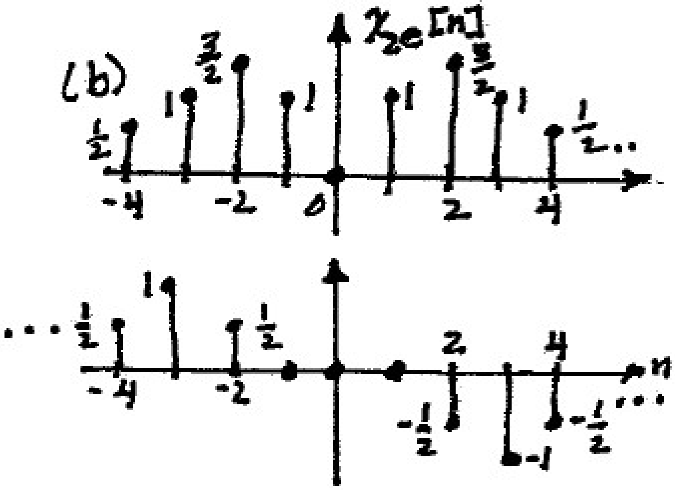
(b) $x_1[n] = 1.5x_3[-n-2] + 3 \Longrightarrow x_3[n] = \frac{2}{3}x_1[-n-2] - 2.$



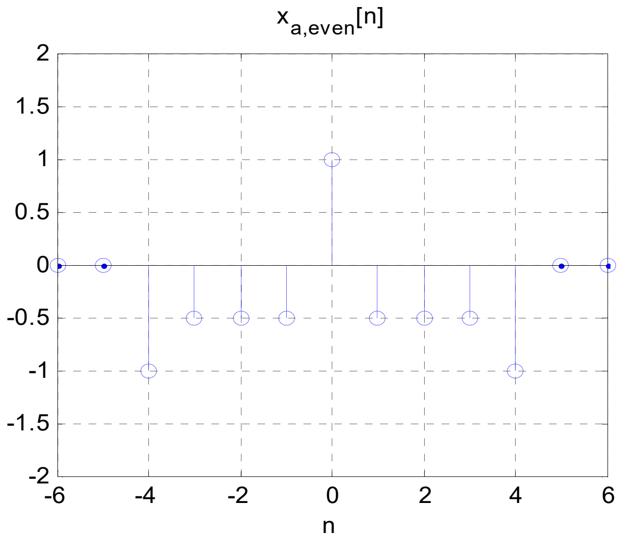
(c)

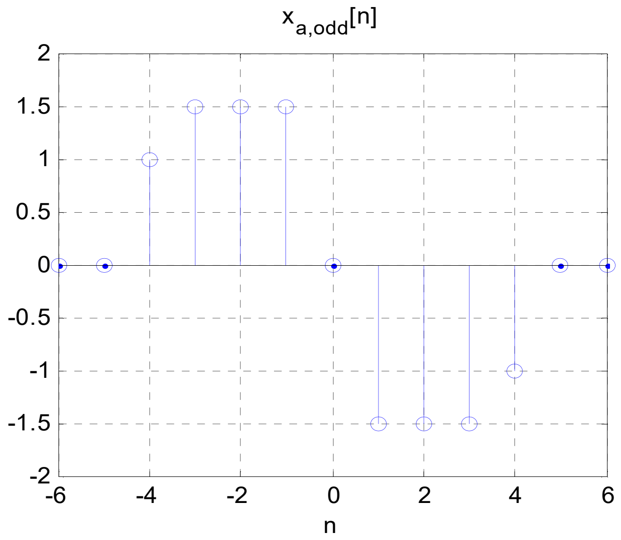
$$\begin{aligned}
x_1[0] &= \quad 1.5x_3[-2]+3 = 1.5(-2)+3 = 0\\
x_1[1] &= \; 1.5x_3[-3]+3 = 1.5(-\tfrac{10}{3})+3 = -2\\
x_1[2] &= \; 1.5x_3[-4]+3 = 1.5(-\tfrac{10}{3})+3 = -2
\end{aligned}$$

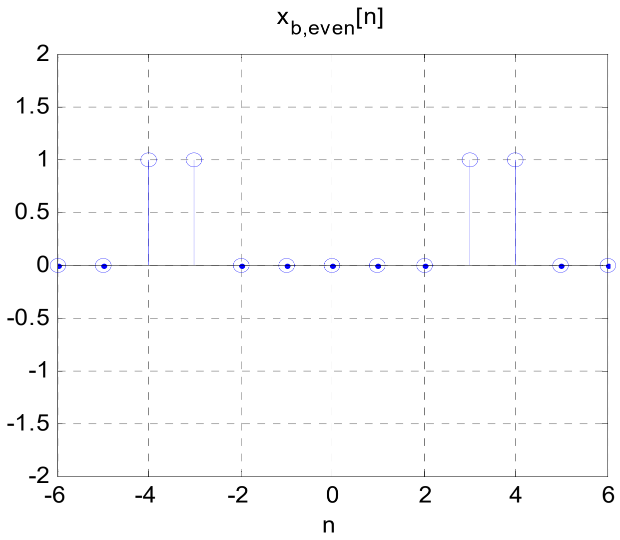


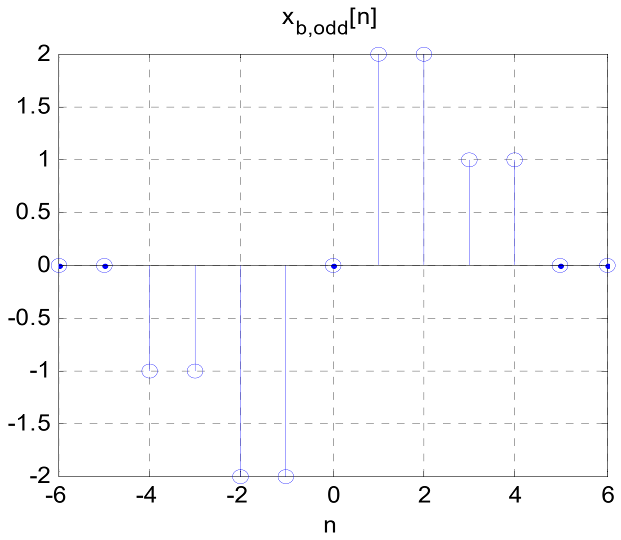


9.8



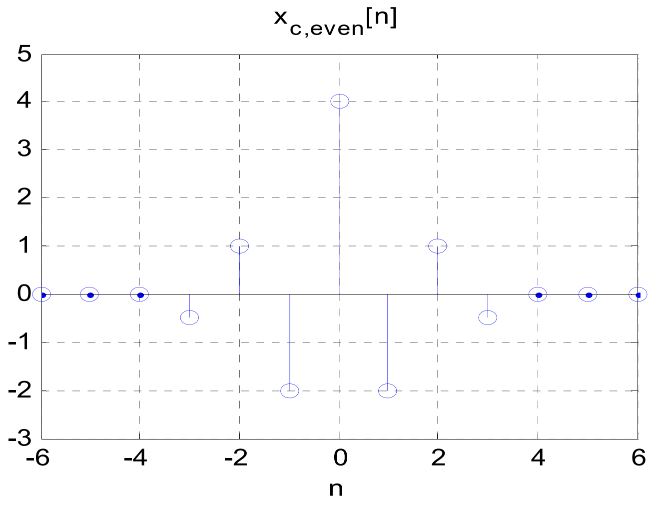


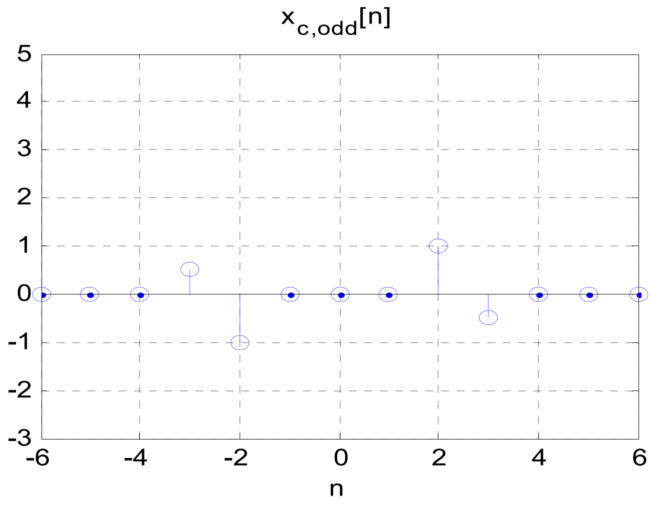


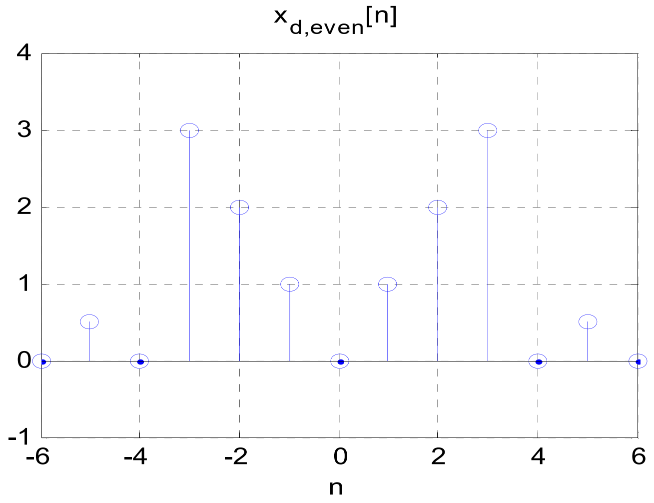


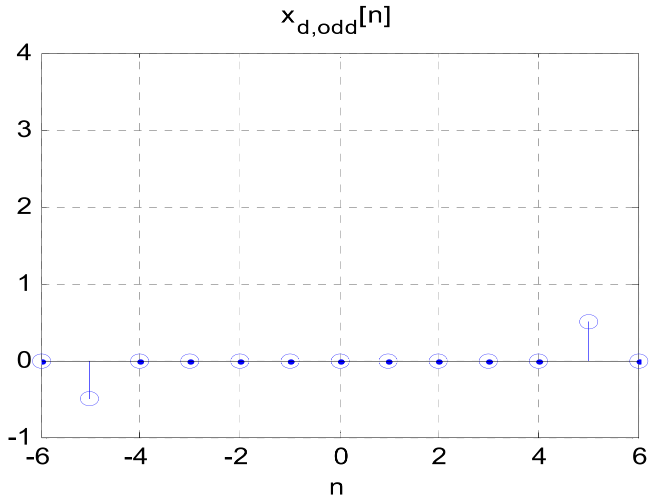
Continued→

9.8, continued









9.9

(a)

(i) $x[n] = 3u[n-2]$: $x[n] \neq x[-n]$, $x[n] \neq -x[-n]$. So this is neither even nor odd.

(ii) $x[n] = -n$, $x[-n] = n$, so $x[n] = -x[-n] \implies$ odd.

(iii) $x[n] = 0.2^{|n|} = 0.2^{|-n|} \implies$ even.

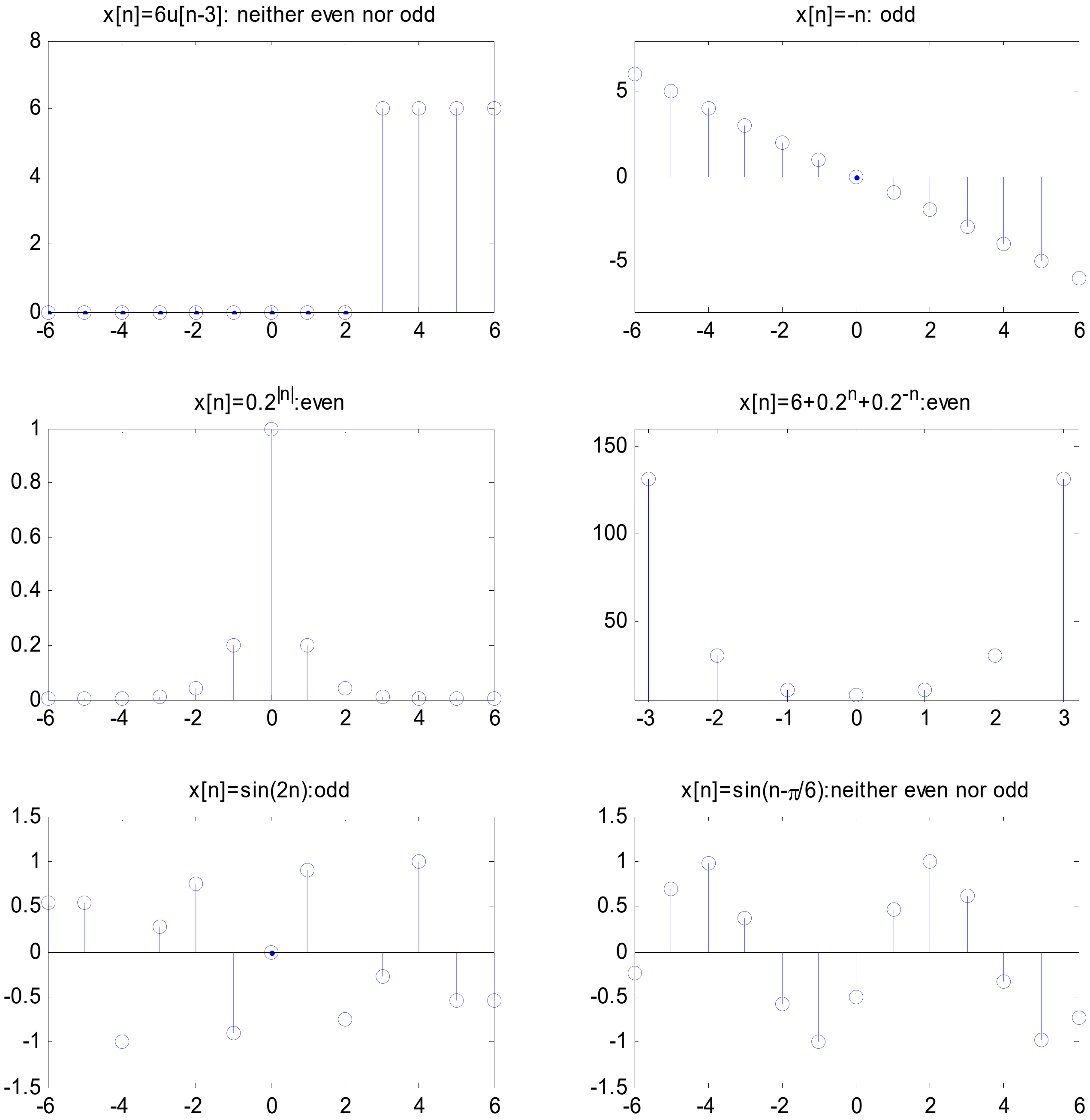
(iv) $x[n] = 6 + .2^n + .2^{-n} = x[-n] = 6 + .2^{-n} + .2^n \implies$ even.

(v) $\sin(2n) = -\sin(2(-n)) \implies$ odd.

(vi) $\sin(n - \pi/6) \neq \sin(-n - \pi/6), \neq -\sin(-n - \pi/6) \implies$ neither even nor odd.

Continued→

9.9, continued
(b)


x[n]=6u[n-3]: neither even nor odd

x[n]=-n: odd

$x[n]=0.2^{|n|}$:even

$x[n]=6+0.2^{n}+0.2^{-n}$:even

x[n]=sin(2n):odd

x[n]=sin(n-π/6):neither even nor odd

Continued→

9.9, continued

(c)

(i) $x_e[n] = \frac{x[n]+x[-n]}{2} = 1.5u[-n-2] + 1.5u[n-2]$,
$x_o[n] = \frac{x[n]-x[-n]}{2} = 1.5u[n-2] - 1.5u[-n-2]$. (plotted below)

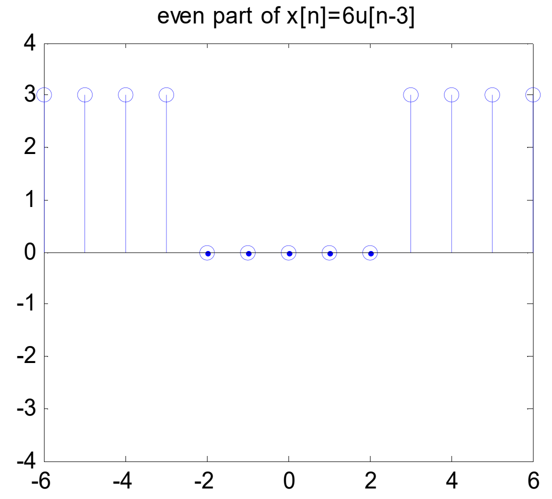
(ii) $x_o[n] = x[n] = -n$, $x_e[n] = 0$.

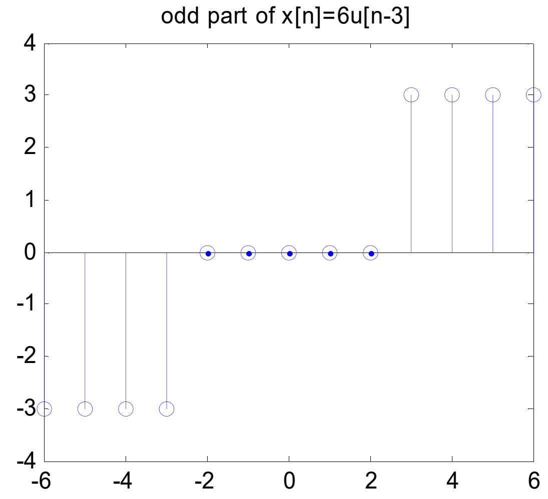
(iii) $x_e[n] = x[n] = .2^{|n|}$, $x_o[n] = 0$.

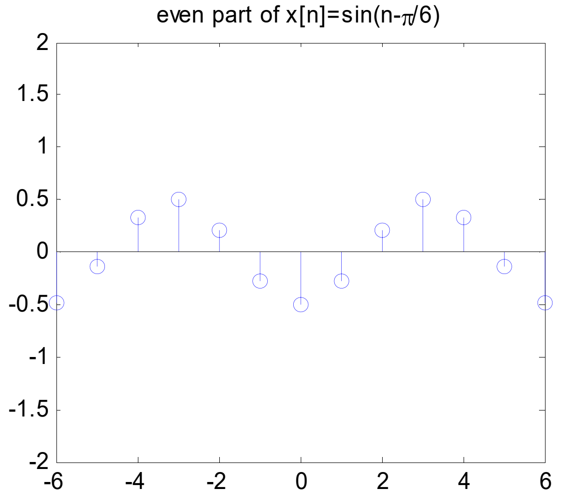
(iv) $x_e[n] = x[n] = 6 + .2^n + .2^{-n}$, $x_o[n] = 0$.

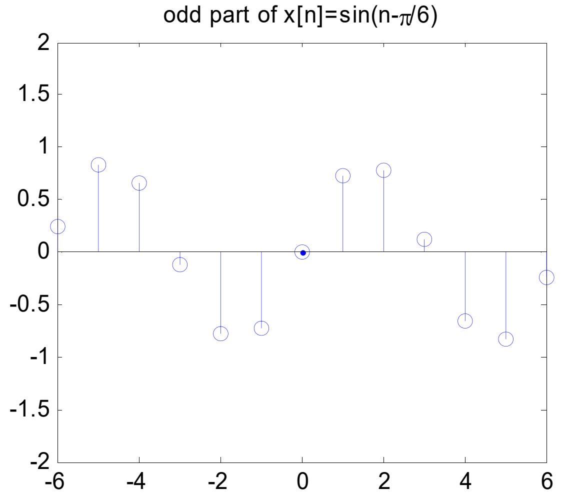
(v) $x_o[n] = x[n] = \sin(2n)$, $x_e[n] = 0$.

(vi) $x_e[n] = \frac{\sin(n-\pi/6)+\sin(-n-\pi/6)}{2} = \cos(n)\cos(2\pi/3) = \cos(n)(-0.5)$,
$x_o[n] = \frac{\sin(n-\pi/6)+\sin(n+\pi/6)}{2} = \sin(n)\cos(\pi/6) = \sin(n)(\sqrt{3}/2)$. (plotted below)

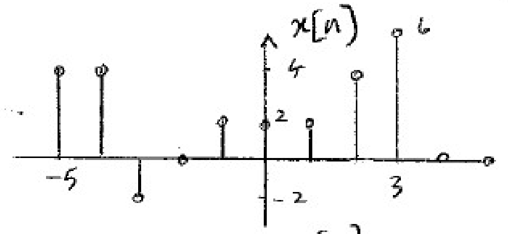

even part of x[n]=6u[n-3]


odd part of x[n]=6u[n-3]

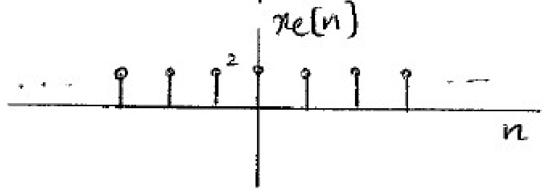

even part of x[n]=sin(n-π/6)


odd part of x[n]=sin(n-π/6)

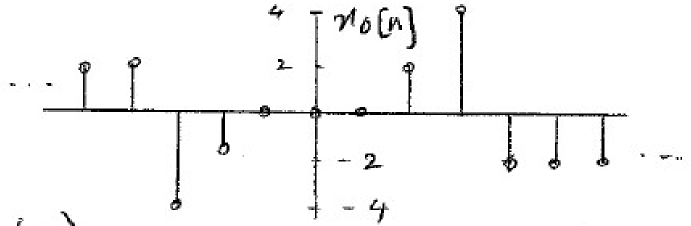
9.10



$x_e[n] = x_e$



$x_e[n] = x_e[-n]$



$x_o[n] = x[n] - x_e[n] \quad n \geq 0$
$x_o[n] = -x_o[-n] \quad n \leq 0$

(b) $x_o[0] = 0$ means that $x_e[0] = 0$ with no other changes.

9.11 (a) $x_o[n] = -x_o[-n] \Rightarrow x_o[0] = -x_o[0]$, $\therefore x_o[0] = 0$

$x_e[0] = x_o[0] - x_o[0] = x[0]$

(b) $\sum_{-\infty}^{\infty} x_o[n] = \sum_{-\infty}^{6} x_o[n] + \sum_{0}^{\infty} x[n] = \sum_{-\infty}^{0} -x_o[-n] + \sum_{0}^{\infty} x_o[n]$

9.11. (cont) Let $n \rightarrow -n$ in the first summation:

$\Rightarrow -\sum_{n=\infty}^{0} x[n] + \sum_{n=0}^{\infty} x[n] = 0$

(c) $\therefore \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{\infty} x_e[n] + \sum_{n=-\infty}^{\infty} x_o[n] = \sum_{n=-\infty}^{\infty} x_e[n]$, from (a)

9.11 (d) Similar to part c we can show that $\sum_{k=-n}^{n} x[k] = \sum_{k=-n}^{n} x_e[k]$ since $\sum_{k=-n}^{n} x_o[k] = 0$, but it is NOT true that $\sum_{k=n_1}^{n_2} x[n] = \sum_{k=n_1}^{n_2} x_e[n]$ in general if $n_1 \neq -n_2$.

9.12. (a) $x_t[n] = x_{e_1}[n] + x_{e_2}[n]$

$x_t[-n] = x_{e_1}[-n] + x_{e_2}[-n] = x_{e_1}[n] + x_{e_2}[n] = x_t[n]$, $\therefore$ $\underline{\text{even}}$

(b) $x_t[n] = x_{o_1}[n] + x_{o_2}[n]$

$x_t[-n] = x_{o_1}[-n] + x_{o_2}[-n] = -x_{o_1}[n] - x_{o_2}[n] = -x_t[n]$, $\therefore$ $\underline{\text{odd}}$

(c) $x_t[n] = x_e[n] + x_o[n]$

$x_t[-n] = x_e[-n] + x_o[-n] = x_e[n] - x_o[n]$, $\therefore$ $\underline{\text{neither}}$

(d) $x_t[n] = x_{e_1}[n] x_{e_2}[n]$

$x_t[-n] = x_{e_1}[-n] x_{e_2}[-n] = x_{e_1}[n] x_{e_2}[n] = x_t[n]$, $\therefore$ $\underline{\text{even}}$

(e) $x_t[n] = x_{o_1}[n] x_{o_2}[n]$

$x_t[-n] = x_{o_1}[-n] x_{o_2}[-n] = \left[-x_{o_1}[n]\right]\left[-x_{o_2}[n]\right] = x_t[n]$, $\therefore$ $\underline{\text{even}}$

(f) $x_t[n] = x_e[n] x_o[n]$

$x_t[-n] = x_e[-n] x_o[-n] = x_e[n]\left[-x_o[n]\right] = -x_t[n]$, $\therefore$ $\underline{\text{odd}}$

9.13

(a) $x_1[n] = \cos(\frac{2\pi n}{10})$: need $\frac{2\pi N_0}{10} = k2\pi$ for some integer k, $\Longrightarrow$ $N_0 = 10$, periodic.

(b) $x_2[n] = \sin(\frac{2\pi n}{25})$: need $\frac{2\pi}{25}N_0 = k2\pi \Longrightarrow N_0 = 25$, periodic.

(c) $x_3[n] = e^{j\frac{2\pi n}{20}}$, periodic, $N_0 = 20$.

(d) yes $x_1[n] + x_2[n] + x_3[n]$ is periodic with period $LCM(10, 25, 20) = 100$.

9.14. (a) $x[n+N] = e^{j5\pi(n+N)/7} = e^{j5\pi n/7}\, e^{j5\pi N/7} = e^{j5\pi n/7}\, e^{j2\pi k}$

$\therefore\ 5\pi N/7 = 2\pi k \Rightarrow N = \frac{14k}{5}\ ;\ k = 5\ ,\ \underline{N_0 = 14}$

(b) $x[n+N] = e^{j5n}\, e^{j5N}\quad \therefore 5N = 2\pi k$ <u>not periodic</u>

(c) $x[n+N] = e^{j2\pi n} e^{j2\pi N}\quad \therefore\ 2\pi N = 2\pi k,\ \underline{N_0 = 1}\ \ (x[n] = 1)$

(d) $x[n+N] = e^{j0.3n/\pi}\, e^{j\frac{0.3N}{\pi}}\ \therefore\ \frac{0.3N}{\pi} = 2\pi k\ \therefore$ <u>not periodic</u>

(e) $x[n+N] = \cos(3\pi n/7 + 3\pi N/7),\ \therefore \frac{3\pi N}{7} = 2\pi k,\ N = \frac{14k}{3}\ ,\ \underline{N_0 = 14}$

(f) $x[n+N] = e^{j0.3n} e^{j0.3N},\ \therefore 0.3N = 2\pi k,$ not periodic

(g) From parts (a) and (c), $e^{j5\pi n/7}$ has period N_0=14 and $e^{2\pi n}$ has period $N_0 = 1$ so their sum has period LCM(14,1)=14.

(h) From part (g), $e^{j5\pi n/7} + e^{j2\pi n}$ has period 14 and from part (e) $\cos(3\pi n/7)$ has period 14; so their sum has period LCM(14,14)=14.

(i) From part (f), the first term, $e^{j0.3n}$, is not periodic. So the sum $e^{j0.3n} + e^{j2\pi n}$ is not periodic.

9.15. $t = nT$, $x[n] = \cos(2\pi nT)$, $\omega_0 = 2\pi$, $\therefore T_0 = 1$

N_0 = # of samples in the fundamental period.

(a) (i) $x[n] = \cos(2\pi nT)$

$x[n+N_0] = \cos(2\pi n + 2\pi N_0)$, $\therefore 2\pi N_0 = 2\pi k \Rightarrow k = \underline{1}$

$\therefore$ periodic with $N_0 = \underline{1}$ (constant signal)

(ii) $x[n] = \cos(0.2\pi n) \overset{?}{=} \cos(0.2\pi n + 0.2\pi N_0)$

$\therefore 0.2\pi N_0 = 2\pi k \Rightarrow N_0 = \frac{2k}{0.2} \Rightarrow N_0 = \underline{10}, k = \underline{1}$, periodic

(iii) $x[n] = \cos(0.25\pi n) \overset{?}{=} \cos(0.25\pi n + 0.25\pi N_0)$

$\therefore 0.25\pi N_0 = 2\pi k \Rightarrow N_0 = \frac{2k}{0.25} \Rightarrow N_0 = \underline{8}, k = \underline{1}$ $\therefore$ periodic

(iv) $x[n] = \cos(0.26\pi n) \overset{?}{=} \cos(0.26\pi n + 0.26\pi N_0)$

$\therefore 0.26\pi N_0 = 2\pi k \Rightarrow N_0 = \frac{2k}{0.26} = \frac{200}{26}k = \frac{100}{13}k$

$\therefore N_0 = \underline{100}, k = \underline{13}$ periodic

(v) $x[n] = \cos(10\pi n) \overset{?}{=} \cos(10\pi n + 10\pi N_0)$

$\therefore 10\pi N_0 = 2\pi k$, $N_0 = \frac{k}{5} \Rightarrow N_0 = \underline{1}, k = \underline{5}$ $\therefore$ periodic (constant)

(vi) $x[n] = \cos(\frac{8}{3}\pi n) \overset{?}{=} \cos(\frac{8}{3}\pi n + \frac{8}{3}\pi N_0)$

$\therefore \frac{8}{3}\pi N_0 = 2\pi k \Rightarrow N_0 = \frac{6k}{8} \Rightarrow N_0 = \underline{3}, k = \underline{4}$ periodic

(b) (i) $k = \underline{1}$ (ii) $k = \underline{1}$ (iii) $k = \underline{1}$ (c) (i) $N_0 = 1$ (ii) $N_0 = 10$ (iii) $N_0 = 8$

(iv) $k = \underline{13}$ (v) $k = \underline{5}$ (vi) $k = \underline{4}$ (iv) $N_0 = 100$ (v) $N_0 = 1$ (vi) $N_0 = 3$

9.16 Want to find τ such that $e^{-t/\tau}|_{t=nT} = e^{-(nT)/\tau} = x[n]$. The sampling rate of 10Hz means $T = \frac{1}{10} = 0.1sec$.

(a) Need $e^{-n0.1/\tau} = 0.3^n \implies e^{-0.1/\tau} = 0.3 \implies -0.1/\tau = \log(0.3) \implies \tau = \frac{-0.1}{\log(0.3)} = 0.083$.

(b) Need $e^{-T/\tau} = 0.3 \implies$ same τ as (a).

To find ω need $\omega \cdot T = 1 \implies \omega = 1/0.1 = 10$.

(c) $(-0.3)^n = (0.3)^n(-1)^n = (0.3)^n \cos(\pi n)$. From (a), $\tau = 0.083$.

$\omega = \pi/0.1 = 10\pi$.

(d) Same τ and ω as part (b) because the sin instead of cos and the additional 1 just change the phase, not the frequency.

9.17

(a)

(i) $\cos(\pi n + \pi N_0) = \cos(\pi n + 2\pi) \implies N_0 = 2k$ for some integer k; $N_0 = 2$, periodic.

(ii) $-3\sin(0.01\pi n + 0.01\pi N_0) = -3\sin(0.01\pi n + 2\pi k) \implies 0.01N_0 = 2k \implies N_0 = 200$, periodic.

(iii) $\cos(3\pi(n + N_0)/2 + \pi) = \cos(3\pi n/2 + \pi + 3\pi N_0/2) \implies 3N_0/2 = 2k \implies k = 3, N_0 = 4$, periodic.

(iv) $\sin(3.15n + 3.15N_0) = \sin(3.15n + 2\pi k) \implies 3.15N_0 = 2\pi k \implies N/k = 2\pi/3.15$, not periodic since not rational.

(v) $1 + \cos(0.5\pi n + 0.5\pi N_0) = 1 + \cos(0.5\pi n + 2\pi k) \implies 0.5N_0 = 2k \implies N_0 = 4$, periodic.

(vi) $\sin(3.15\pi n + 3.15\pi N_0) = \sin(3.15\pi n + 2\pi k) \implies 3.15N_0 = 2k \implies N_0 = 2k/3.15 = 200k/315 = 40k/63 \implies k = 63, N_0 = 40$. periodic

(b) (i) $N_0 = 2$, (ii) $N_0 = 200$, (iii) $N_0 = 4$, (iv) not periodic, (v) $N_0 = 4$, (vi)$N_0 = 40$

9.18

(a)—C (alternating +/-5)

(b)—D (values 0,+5,0,-5 at n=0,1,2,3)

(c)—B (constant 3)

(d)—A (values +5,0,-5,0 at n=0,1,2,3)

9.19

(a) $x_a[n] = \delta[n+3] + \delta[n+2] + \delta[n+1] + \delta[n] - 2\delta[n-1] - 2\delta[n-2] - 2\delta[n-3] - 2\delta[n-4] = \sum_{k=-3}^{0} \delta[n-k] - 2\sum_{k=1}^{4} \delta[n-k]$

(b) $x_b[n] = -2\sum_{k=-2}^{-1} \delta[n-k] + 2\sum_{k=1}^{4} \delta[n-k]$

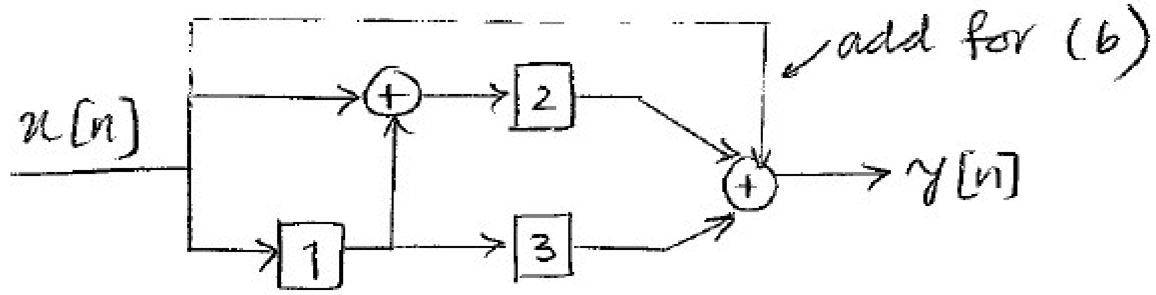
or $= -2\left(\delta[n+2] + \delta[n+1]\right) + 2\left(\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]\right)$

(c) $x_c[n] = -2\delta[n+1] + 4\delta[n] - 2\delta[n-1] + 2\delta[n-2] - \delta[n-3]$

(d) $x_d[n] = 3\delta[n+3] + 2\delta[n+2] + \delta[n+1] + \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + \delta[n-5]$

9.20

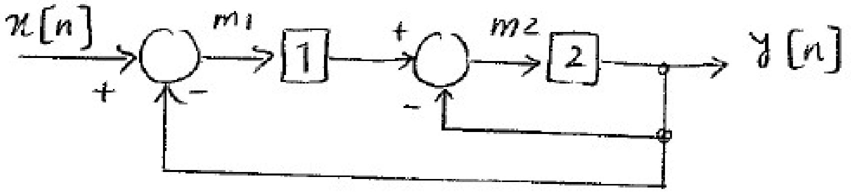
a&b)



9.21 a) $m[n] = T3\left[T2\{x[n] - T_4(y[n])\}\right]$

$y[n] = T_1(x(n]) + T3\left[T2\{x[n] - T_4(y(n])\}\right]$

b) $y[n] = T2(m_2(n]) = T2(T_1(x[n] - y(n] - y(n])$



$m_1[n] = x[n] - y[n]$

$m_2[n] = T_1(x[n] - y[n]) - y[n]$

9.22 a) $\therefore\ y[k] = y[k-1] + T/2\,[x[k] + x[k-1]]$

b)
```
y(1)=0 , T=.1 ;
for n=1:51
        y(n+1)=y(n)+T/2*(exp(-n*T)+exp((1-n)*T)) ;
end
y
```

c) Result : y = .9941

$\int_0^5 e^{-t}dt = e^{-t}\Big|_0^5 = 1 - e^{-5} = .9933$

9.23 (a)

(i) Not memoryless (depends on time $an + 1 \neq n$);

(ii) Not invertible because $y[0] = 0(x[1]) + 5$ we cannot get from $y[n]$ the value of $x[1]$

(iii) Not causal ($an + 1 > n$ so depends on input value at future time)

(iv) Not stable—for example, if $x[n] = 1$ is the input (a constant value 1), then $x[n]$ is bounded but the output is $y[n] = n + 5$ which goes to ∞ as $n \to \infty$

(v) Not time invariant: $x[n - n_0] \to nx[an + 1 - n_0] + 5$ but this $\neq y[n - n_0] = (n - n_0)x[a(n - n_0) + 1] + 5$

(vi) Not linear: $kx[n] \to nkx[an + 1] + 5$ but this $\neq ky[n] = k\,(nx[an + 1] + 5)$

(b)

(i) Not memoryless (depends on $-n + 2$)

(ii) Invertible: $x[n] = y[-n + 2]$

(iii) Not causal ($-n + 2 > n$ when $n \leq 0$)

(iv) Stable

(v) Not time invariant: $x[n - n_0] \to x[-n + 2 - n_0]$ but this $\neq y[n - n_0] = x[-(n - n_0) + 2] = x[-n + 2 + n_0]$

(vi) Linear: $k_1x_1[n] + k_2x_2[n] \to k_1y_1[n] + k_2y_2[n]$

Continued→

9.23, continued

(c)

(i) Memoryless

(ii) Not invertible (for example $x[n]$ and $x[n] + 2\pi$ are two inputs that have the same output for any $x[n]$)

(iii) Causal (memoryless implies causal)

(iv) Stable ($|\cos(x[n])| \leq 1$)

(v) Time invariant: $x[n - n_0] \rightarrow \cos(x[n - n_0]) = y[n - n_0]$

(vi) Not linear: $k_1 x_1[n] \rightarrow \cos(k_1 x_1[n]) \neq k_1 \cos(x_1[n]) = k_1 y[n]$

(d)

(i) Memoryless

(ii) Invertible: $x[n] = e^{y[n]}$

(iii) Causal

(iv) Not stable: if $x[n] = 0$ output is $-\infty$

(v) Time invariant

(vi) Not linear

(e)

(i) Memoryless

(ii) Not invertible: can't get back the value of $x[0]$ because it gets multiplied but 0, but can get back all other values.

(iii) Causal

(iv) Not stable (same reason as (d))

(v) Not time invariant: $x[n - n_0] \rightarrow log(nx[n - n_0])$ but $y[n - n_0] = log((n - n_0)x[n - n_0])$ (vi) Not linear

Continued→

9.23, continued

(f)

(i) Not memoryless (depends on $n-3$ input)

(ii) Invertible: $x[n] = (1/4)y[n+3] - 3/4$

(iii) Causal ($n-3 < n$ for all n)

(iv) Stable: if $|x[n]| < K$ then $|4x[n-3]+3| < 4K+3$

(v) Time invariant: $x[n-n_0] \rightarrow 4x[n-n_0-3]+3$ and $y[n-n_0] = 4x[n-n_0-3]+3$

(vi) Not linear: $x_1[n] + x_2[n] \rightarrow 4(x_1[n] + x_2[n]) + 3$ but $x_1[n] \rightarrow 4x_1[n] + 3$ and $x_2[n] \rightarrow 4x_2[n] + 3$ so $x_1[n] + x_2[n] \rightarrow 4(x_1[n] + x_2[n]) + 3 + 3$

9.24 $y[n] = 2y[n-1) - y[n-2] + x[n]$

a) has memory

b) $y[n-n_0] = 2y[n-n_0-1] - y[n-n_0-2] + x[n-n_0]$

$\therefore$ time invariant

c) $a_1 y_1[n] + a_2 y_2[n] - 2[a_1 y_1[n-1) + a_2 y_2[n-1]]$

$+ a_1 y_1[n-2] + a_2 y_2[n-2] = a_1 x_1(n] + a_2 x_2[n]$

$\therefore a_1\{y_1[n] - 2y_1[n-1] + y_1[n-2] - x_1[n]\} + a_2\{y_2[n]$

$- 2y_2[n-1] + y_2[n-2] - x_2[n]\} = 0$

$\Rightarrow 0 + 0 = 0$, $\therefore$ linear

9.25 a) $y[n] = \sum_{-n}^{n} x[k+a]$

(i) has memory

(ii) not invertible

(iii) not causal, whether or not it looks at future depends on $\underline{\underline{a}}$ & we don't know

Continued→

9.25, continued

(iv) stable

(v) Time varying (vi) linear

b) $y[n] = \frac{1}{2}\left[x[n] + x[n-1]\right]$

(i) has memory

(ii) $x[n] = 2y[n] - x[n-1]$ invertible

$= 2y[n] - 2y[n-1] + x[n-2] = 2y[n] - \cdots$

(iii) causal

(iv) Stable

(v) Time invariant

(vi) linear

c) (i) has memory

(ii) Invertible

(iii) Causal

(iv) Stable

(v) Time invariant

(vi) linear

9.26 $y[n] = K_n x[n]$ with $K_n = \left[\frac{n+2.5}{n+1.5}\right]^2$

as $n \to \infty$ & as $n \to -\infty$, $K_n \to 1$

$\therefore$ K_n is max for $n=-1$ & $|y[-1]| = 9|x[-1]|$

n	K_n
2	1.65
1	1.96
0	1.67
-1	9.0
-2	1
-3	.111

9.27

a) $y[n] = -3|x[n]|$

(i) memoryless

(ii) not invertible

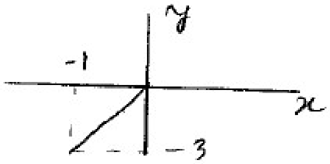
(iii) causal

(iv) Stable

(v) Time-Invariant

(vi) $|x_1| + |x_2| \neq |x_1 + x_2|$ ∴ Not linear

b) $y[n] = \begin{cases} 3x[n] & x<0 \\ 0 & x \geq 0 \end{cases}$



(i) memoryless

(ii) not invertible ; $y=0$, $x \geq 0$

(iii) causal

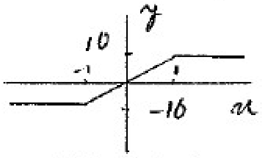
(iv) Stable

(v) time invariant

(vi) let $x_1=1$, $x_2=-1 \Rightarrow y_1=0$, $y_2=-1$

$$\therefore\ -1 = y[n]\Big|_{\substack{x_1=1 \\ x_2=-1}} \neq y[n]\Big|_{x=1-1=0} = 0$$ not linear

c)



(i) memoryless

(ii) $y=10$ for $x \geq 1$, not invertible

(iii) causal

Continued→

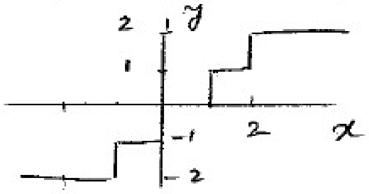
9.27, continued

(iv) Stable

(v) time-invariant

(vi) $x_1 = x_2 = 1 \Rightarrow y_1 = y_2 = 10$, $y\big|_{x=2} \neq 20$ ∴ not linear

d)



(i) memoryless

(ii) $y = 2$ for $x \geq 2$, not invertible

(iii) causal

(iv) Stable

(v) time-invariant

(vi) $y\big|_{x_1 = x_2 = 2} = 2 \neq y\big|_{x_1 = 2} + y\big|_{x_2 = 2}$ ∴ nonlinear

9.28

Causal system $\Longrightarrow$ $h[n] = 0, n < 0$. Must have $h_e[n] = h_e[-n]$ and $h_o[n] + h_e[n] = 0, n < 0$. This implies that the odd part for $n \leq 0$ is:

$$
\begin{aligned}
h_o[n] &= \quad 0, n = 0 \\
&= \ -3, n = -1 \\
&= \ -4, n = -2 \\
&= \ -1, n \geq 3
\end{aligned}
$$

continued→

9.28, continued

Adding the even and odd parts gives:

$$\begin{aligned} h[n] = \quad & 0, n \leq 0 \\ = \quad & 6, n = 1 \\ = \quad & 8, n = 2 \\ = \quad & 2, n \geq 3 \end{aligned}$$

So in other words, when we know that $h[n] = 0$ for $n < 0$ then $h[n] = 2h_e[n]$ for $n > 0$, $h[n] = 0$ for $n < 0$, and $h[0] = h_e[0]$.

9.29

Not memoryless (depends on previous inputs)

Causal—only depends on input values up to current time n

Linear: for two inputs $x_1[n]$ and $x_2[n]$ and their individual outputs $y_1[n]$ and $y_2[n]$,

$ax_1[n] + bx_2[n] \rightarrow \sum_{k=-\infty}^{n-1} k\left(ax_1[k+1] + bx_2[k+1]\right)$

$= a\sum_{k=-\infty}^{n-1} kx_1[k+1] + b\sum_{k=-\infty}^{n-1} kx_2[k+1] = ay_1[n] + by_2[n]$

Not time invariant: $x[n-n_0] \rightarrow \sum_{k=-\infty}^{n-1} kx[k+1-n_0] = \sum_{k=-\infty}^{n-n_0-1}(k+n_0)x[k+1]$ but $y[n-n_0] = \sum_{k=-\infty}^{n-n_0-1} kx[k+1]$.

Not stable: if $x[n]$ is a constant, $y[n] \rightarrow \infty$ as $n \rightarrow \infty$.

Chapter 10 Solutions

Chapter 10

10.1 $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ - replace k with $(n-k_1)$, n constant

$$\Rightarrow \sum_{k=-\infty}^{\infty} x[n-k_1]h[k_1] = \sum_{-\infty}^{\infty} h[k_1]x[n-k_1]$$

10.2 $g[n]*\delta[n] = \sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$

$$\delta[n-k] = \begin{cases} 1, & k=n \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \; g[n]*\delta[n] = g[n](1) = g[n]$$

10.3

(a) $y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k] = \sum_{k=1}^{6} h[5-k] = h[4]+h[3]+h[2]+h[1]+h[0]+h[-1] = 3\cdot 2 = 6$

(b) max is $h[1]+h[0]+h[-1]+h[-2] = 8$

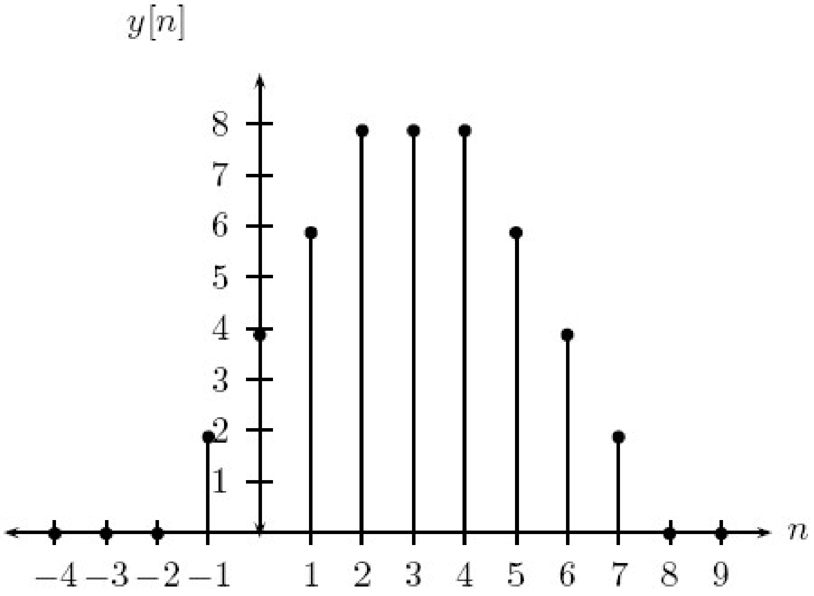
(c) max occurs at $n = 2, 3, 4$

(d)

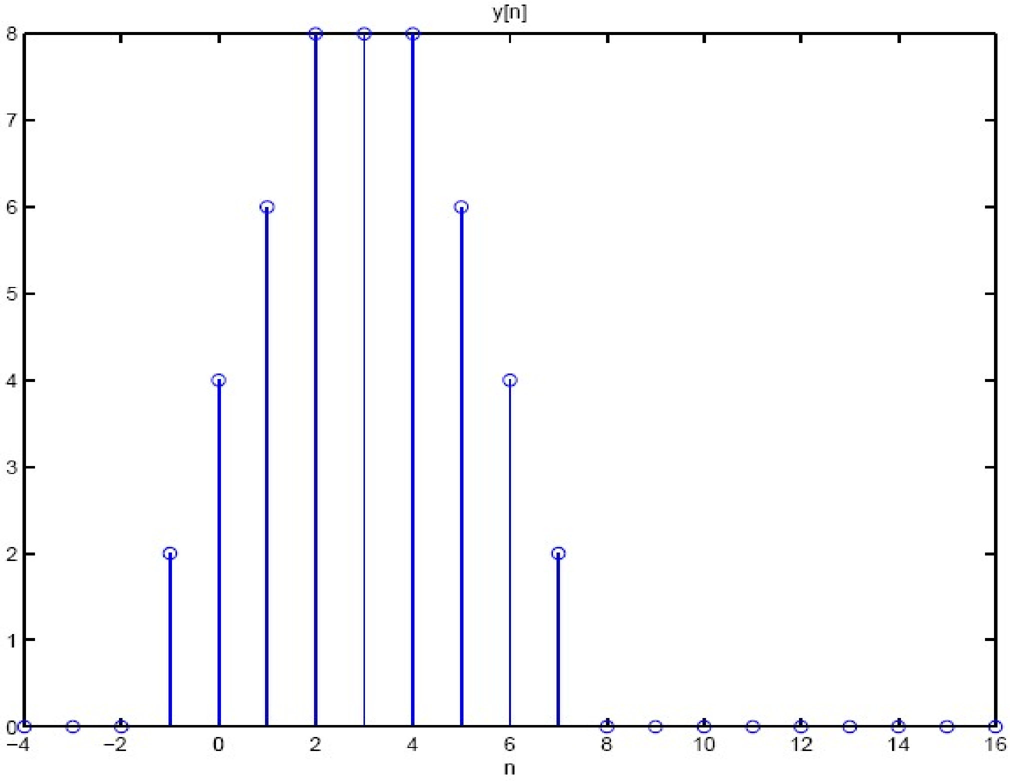
$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= 0, n \leq -2 \\
&= h[-2] = 2, n = -1 \\
&= h[-2] + h[-1] = 4, n = 0 \\
&= h[-2] + h[-1] + h[0] = 6, n = 1 \\
&= h[-2] + h[-1] + h[0] + h[1] = 8, n = 2, 3, 4 \\
&= h[-1] + h[0] + h[1] = 6, n = 5 \\
&= h[0] + h[1] = 4, n = 6 \\
&= h[1] = 2, n = 7 \\
&= 0, n \geq 8
\end{aligned}$$

Continued→

10.3d, continued



(e)
```
>>n=-2:8
>>x=[0,0,0,1,1,1,1,1,1,0,0];
>>h=[2,2,2,2,0,0,0,0,0,0,0];
>>y=conv(x,h);
>>stem((-2+-2):(8+8),y); title('y[n]'), xlabel('n')
```

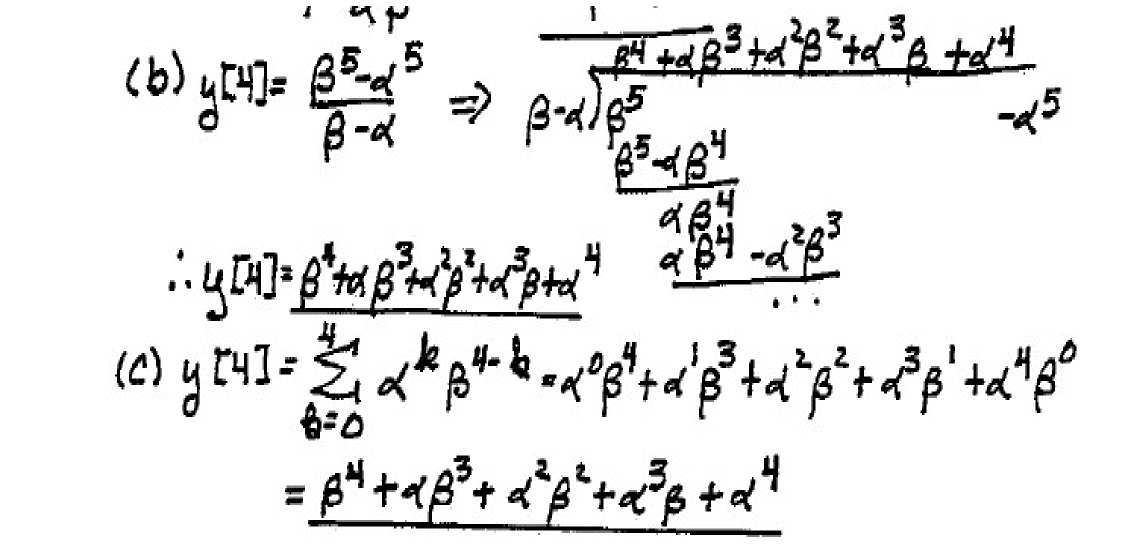


10.4

$h[n] = \alpha^n u[n]$, $x[n] = \beta^n u[n]$, $\alpha \neq \beta$

(a)

$$\begin{aligned} y[n] = &\quad \textstyle\sum_{k=-\infty}^{\infty} \alpha^k u[k]\beta^{n-k}u[n-k] = \left[\sum_{k=0}^{n} \alpha^k \beta^{n-k}\right] u[n] \\ = &\quad \beta^n \left[\textstyle\sum_{k=0}^{n} (\alpha\beta^{(-1)})^k\right] u[n] = \beta^n \left[\frac{1-\alpha^{(n+1)}\beta^{-(n+1)}}{1-\alpha\beta^{(-1)}}\right] u[n] \\ = &\quad \frac{\beta^n - \alpha^{(n+1)}\beta^{(-1)}}{1-\alpha\beta^{(-1)}} u[n] = \frac{\beta^{(n+1)}-\alpha^{(n+1)}}{\beta-\alpha} u[n] \end{aligned}$$



(b) $y[4] = \dfrac{\beta^5 - \alpha^5}{\beta - \alpha} \Rightarrow$

$$\begin{array}{r} \beta^4 + \alpha\beta^3 + \alpha^2\beta^2 + \alpha^3\beta + \alpha^4 \\ \beta - \alpha \,\big)\, \beta^5 \qquad\qquad\qquad\qquad -\alpha^5 \\ \underline{\beta^5 - \alpha\beta^4} \qquad\qquad\qquad\qquad \\ \alpha\beta^4 \qquad\qquad\qquad \\ \underline{\alpha\beta^4 - \alpha^2\beta^3} \qquad\qquad \\ \cdots \qquad\qquad \end{array}$$

$\therefore y[4] = \underline{\beta^4 + \alpha\beta^3 + \alpha^2\beta^2 + \alpha^3\beta + \alpha^4}$

(c) $y[4] = \sum_{k=0}^{4} \alpha^k \beta^{4-k} = \alpha^0\beta^4 + \alpha^1\beta^3 + \alpha^2\beta^2 + \alpha^3\beta^1 + \alpha^4\beta^0$

$= \underline{\beta^4 + \alpha\beta^3 + \alpha^2\beta^2 + \alpha^3\beta + \alpha^4}$

10.5

(a) $y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k] = 2\sum_{k=-1}^{3} h[4-k] = 2(h[5] + h[4] + h[3] + h[2] + h[1]) = 0$

(b) max is $2 \cdot 5 = 10$

(c) max occurs at $n = -3, -2$

(d)

Continued→

10.5(d), continued

$$
\begin{aligned}
y[n] &= 2\sum_{k=-1}^{3} h[n-k] \\
&= 0, n \leq -8 \\
&= h[-6] = 2, n = -7 \\
&= h[-6] + h[-5] = 4, n = -6 \\
&= h[-6] + h[-5] + h[-4] = 6, n = -5 \\
&= h[-6] + h[-5] + h[-4] + h[-3] = 8, n = -4 \\
&= h[-6] + h[-5] + h[-4] + h[-3] + h[-2] = 10, n = -3 \\
&= h[-5] + h[-4] + h[-3] + h[-2] + h[-1] = 10, n = -2 \\
&= h[-4] + h[-3] + h[-2] + h[-1] = 8, n = -1 \\
&= h[-3] + h[-2] + h[-1] = 6, n = 0 \\
&= h[-2] + h[-1] = 4, n = 1 \\
&= h[-1] = 2, n = 2
\end{aligned}
$$

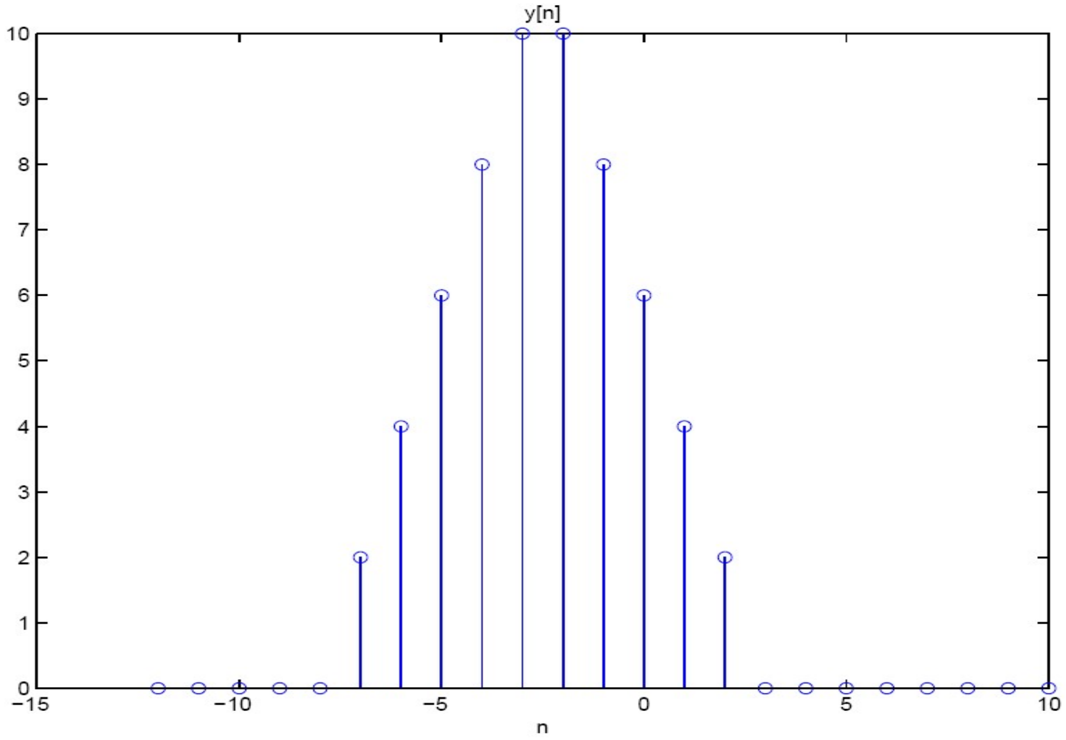
$= 0, n \geq 3$

(e)

```
>> n=-6:5;
>> x=[zeros(1,5),2*ones(1,5),0,0];
>> h=[ones(1,6),zeros(1,6)];
>> stem((-6+-6):(5+5),conv(x,h));
>> title(‘y[n]’); xlabel(‘n’);
```

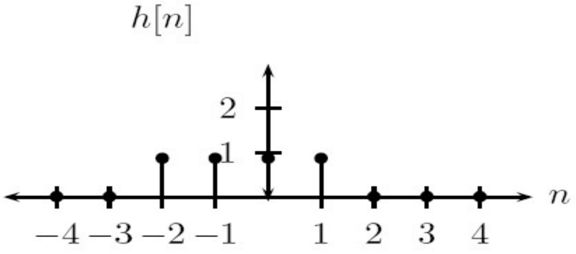
See plot next page→

10.5e plot



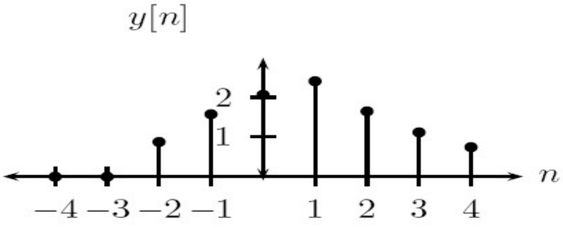
10.6

(a)



(b) $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$ and since $h[k] = 0$ outside of $k \in [-2, 1]$, we have:

$$\begin{aligned} y[n] = & \quad \sum_{k=-2}^{1} 1x[n-k] = \sum_{k=-2}^{1} (0.7)^{n-k}u[n-k] \\ = & \quad 0, n \leq -3 \\ = & \quad (0.7)^0 = 1, n = -2 \\ = & \quad (0.7)^1 + (0.7)^0 = 1.7, n = -1 \\ = & \quad (0.7)^2 + (0.7)^1 + 1 = 2.19, n = 0 \\ = & \quad (0.7)^{n+2} + (0.7)^{n+1} + (0.7)^n + (0.7)^{n-1}, n \geq 1 \end{aligned}$$



10.7 (a) $$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=0}^{1} x[n-k]+\sum_{k=4}^{5} x[n-k]$$

$$x[n-k]=u[n-k]$$

$$\therefore\ y[n]=u[n]+u[n-1]+u[n-4]+u[n-5]$$

$\therefore\ y[n]=0\ ,\ n<0$

$y[0]=1$

$y[1]=2$

$y[2]=2$

$y[3]=2$

$y[4]=3$

$y[n]=4\ ,\ n\geq 5$

(b) $$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=0}^{1}\left(u[n-k]-u[n-2-k]\right)+\sum_{k=4}^{5}(\quad)$$

$$=u[n]-u[n-2]+u[n-1]-u[n-3]$$
$$+u[n-4]-u[n-6]+u[n-5]-u[n-7]$$

$y[n]=0,\ n<5$

$y[0]=1$

$y[1]=2$

$y[2]=1$

$y[3]=0$

$y[4]=1$

$y[5]=2$

$y[6]=1$

$y[n]=0\quad n\geq 7$

(c)
```
x=[1 1 0 0 0 0];h=[1 1 0 0 1 1];
y=conv(x,h)
```

(d) $$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=0}^{1}\left(u[n-k]-u[n-6-k]\right)+\sum_{k=4}^{5}(\quad)$$

$$=u[n]-u[n-6]+u[n-1]-u[n-7]$$
$$+u[n-4]-u[n-10]+u[n-5]-u[n-11]$$

10.7 (cont) (d) $y[n]=0,\ n<0$

$y[0]=1$

$y[1]=2$

$y[2]=2$

$y[3]=2$

$y[4]=3$

$y[5]=4$

$y[6]=3$

$y[7]=2$

$y[8]=2$

$y[9]=2$

$y[10]=1$

$y[n]=0\ ,\ n\geq 11$

(e)
```
x=[1 1 1 1 1 1 0 0];h=[1 1 0 0 1 1]; y=conv(x,h)
```

(f) $$y[n]=\sum_{k=0}^{1}\left(u[n-k]-u[n-k-2]\right)=u[n]+u[n-1]-u[n-2]-u[n-3]$$

$\therefore\ y[n]=0\ ,\ n<0$

$y[0]=1$

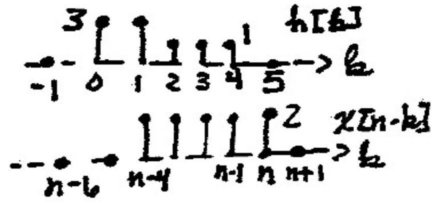
$y[1]=2$

$y[2]=1$

$y[n]=0\ ,\ n\geq 3$

(g)
```
x=[1 1 0 0];h=[1 1 0 0]; y=conv(x,h)
```

10.8(a)



$y[n]=0, n\leq 0$

$y[1]=6$

$y[2]=12$

$y[3]=14$

$y[4]=16$

$y[5]=18$

$y[6]=12$

$y[7]=6$

$y[8]=4$

$y[9]=2$

$y[n]=0, n\geq 10$

(b) $y[n]=0$, $n<0$; $y[n]=0$, $n\geq 8$

n	0	1	2	3	4	5	6	7
y[n]	2	0	2	0	0	-2	0	-2

(c) $y[n]=0$, $n<0$; $y[n]=0$, $n\geq 8$

n	0	1	2	3	4	5	6	7
y[n]	6	9	11	12	6	3	1	0

(d) $y[n]=0$, $n<0$; $y[n]=0$, $n\geq 8$

n	0	1	2	3	4	5	6	7
y[n]	3	0	1	0	-2	-1	0	-1

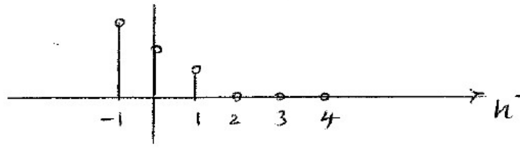
(e) $y[n]=0$, $n<0$; $y[n]=0$, $n\geq 8$

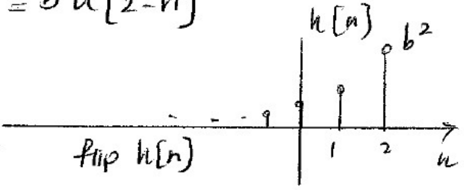
n	0	1	2	3	4	5	6	7
y[n]	-3	-6	-1	4	2	1	2	1

(f)
```
x=[2 2 2 2 2 0 0];h=[3 3 1 1 1 0 0]; y=conv(x,h),pause
x=[2 2 2 2 2 0 0];h=[1 -1 1 -1 0 0]; y=conv(x,h),pause
x=[2 2 2 2 2 0 0];h=[3 1.5 1 0.5 0 0]; y=conv(x,h),pause
x=[3 3 1 1 1 0 0];h=[1 -1 1 -1 0 0]; y=conv(x,h),pause
x=[3 3 1 1 1 0 0];h=[-1 -1 1 1 ]; y=conv(x,h)
```

10.9

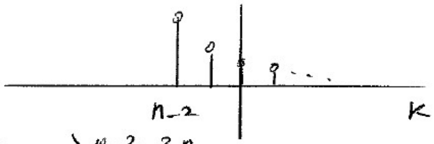
a) $x[n] = a^{-3n} u[1-n], \quad h[n] = b^{n} u[2-n]$





$n-2>1, \quad n>3, \quad y[n]=0$

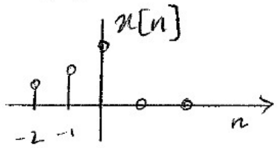
$n-2 \leqslant 1, \quad n \leqslant 3$

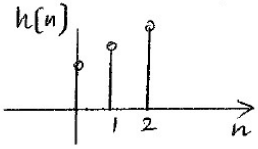


$$\sum_{k=n-2}^{1} b^{n-k} a^{-3k} = b^{n} \sum_{k=n-2}^{1} \left(\frac{1}{a^3 b}\right)^{k} = b^{n} \left(\frac{1}{a^3 b}\right)^{n-2} \sum_{0}^{3-n} \left(\frac{1}{a^3 b}\right)^{k}$$

$$= b^{n} \left(\frac{1}{a^3 b}\right)^{n-2} \left[\frac{1-\left(\frac{1}{a^3 b}\right)^{4-n}}{1-\frac{1}{a^3 b}}\right] = b^{2} \left(\frac{1}{a^3}\right)^{n-2} \times \left(\frac{a^3 b-(a^3 b)^{n-3}}{a^3 b-1}\right) u[3-n]$$

b) $x[n] = a^{n} u[-n], \quad h[n] = b^{n} u[n]$





$$n<0, \quad \sum_{-\infty}^{\infty} a^{k} b^{n-k} = b^{n} \sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{k} = b^{n} \sum_{-n}^{\infty} (b/a)^{k} = \frac{b^{n} (b/a)^{-n}}{1-b/a}$$

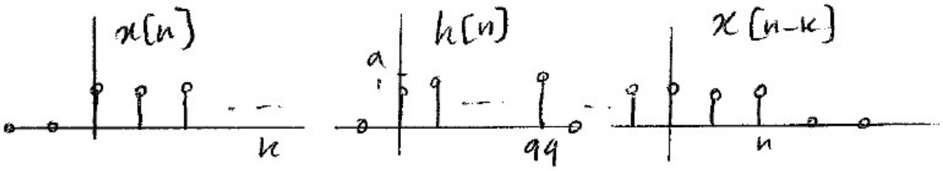
$$= \frac{a^{n}}{1-b/a}$$

$$\therefore y[n] = \frac{a^{n}}{1-b/a} u[-n-1] + \frac{b^{n}}{1-b/a} u(n]$$

$$= \frac{a^{n+1}}{a-b} u[-n-1] + \frac{a b^{n}}{a-b} u[n]$$

Continued→

c)



$n < 0, \quad y[n] = 0$

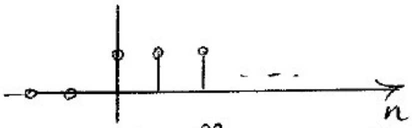
$0 \leqslant n \leqslant 99, \quad y[n] = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$

$n \geqslant 100, \quad y[n] = \sum_{0}^{99} a^k = \frac{1-a^{100}}{1-a}$

$$\therefore \quad y[n] = \left(\frac{1-a^{n+1}}{1-a}\right)\left(u[n]-u[n-100]\right) + \left(\frac{1-a^{100}}{1-a}\right)u(n-100]$$

d)

flip $x[n]$



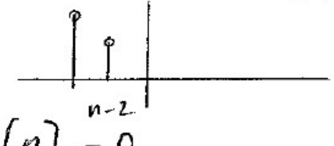
$n+2 < 1, \quad n < -1, \quad \overset{n+2}{y[n]} = \sum_{k=1}^{\infty} b^{-2k} = \sum_{k=1}^{\infty}\left(\frac{1}{b^2}\right)^k = \frac{(1/b^2)}{1-\frac{1}{b^2}}$

$n+2 \geqslant 1, \quad n \geqslant -1, \quad y[n] = \sum_{k=n+2}^{\infty}\left(\frac{1}{b^2}\right)^k = \frac{(1/b^2)^{n+2}}{1-1/b^2}$

$$\therefore \quad y[n] = \frac{1}{b^2-1}u[-n-2] + \frac{(1/b^2)^{n+2}}{1-1/b^2}u[n+1]$$

$$= \frac{1}{b^2-1}u[-n-2] + (1/b^2)^{n+1}\frac{1}{b^2-1}u[n+1]$$

e) flip $h[n]$



$n-2 < 0, \quad y[n] = 0$

$n-2 \geqslant 0, \quad n \geqslant 2, \quad y[n] = \sum_{k=0}^{n-2} a^{2k} b^{n-k} = b^n \sum_{k=0}^{n-2}\left(\frac{a^2}{b}\right)^k$

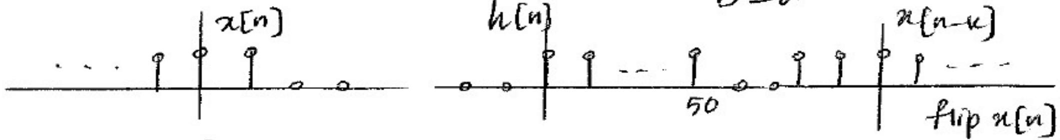
Continued→

10.9e, continued

$$= b^n \left[\frac{1 - \left(\frac{a^2}{b}\right)^{n-1}}{1 - a^2/b} \right] = b^n \left[\frac{b - b\left(a^2/b\right)^{n-1}}{b - a^2} \right]$$

$$= \left(\frac{b^{n+1} - b^2 (a^2)^{n-1}}{b - a^2} \right) \therefore y[n] = \left(\frac{b^{n+1} - b^2 (a^2)^{n-1}}{b - a^2} \right) u[n-2]$$

f)



$n-1 > 50, \quad y[n] = 0$

$$0 \leq n-1 \leq 50, \quad 1 \leq n \leq 51, \quad y[n] = \sum_{k=n-1}^{50} 1 = 50-(n-1)+1$$
$$= 52 - n$$

$n-1 < 0, \quad y[n] = 51$

$n < 1 \quad \therefore \quad y[n] = (52-n)\left(u[n-1] - u[n-52]\right) + 51\,u[-n]$

(g)

$$y[n] = \sum_{k=-\infty}^{\infty} a^{-k} u[-k+1] b^{n-k} u[n-k-1]$$

Note that $u[-k+1]$ is 0 if $k > 1$ and $u[n-k-1]$ is 0 if $k > n-1$. So the argument is nonzero only if both $k > 1$ and $k > n-1$. So if $n-1 > 1$ we sum to 1, otherwise to n-1. This gives:

If $n \geq 2$:

$$= b^n \sum_{k=-\infty}^{1} (ab)^{-k} = b^n \sum_{k=-1}^{\infty} (ab)^k$$
$$= b^n \left(\frac{1}{1-ab} + (ab)^{-1} \right) = \frac{b^n}{1-ab} + \frac{b^n}{ab}$$

(we know the sum converges because $|a| < 1$ and $|b| < 1 \implies |ab| < 1$.)

If $n < 2$:

$$= b^n \sum_{k=-\infty}^{n-1} (ab)^{-k} = b^n \sum_{k=-n+1}^{\infty} (ab)^k$$
$$= b^n \left(\frac{(ab)^{-n+1}}{1-ab} \right)$$

Therefore $y[n] = \left(b^n \frac{(ab)^{-n+1}}{1-ab} \right) u[1-n] + \left(\frac{b^n}{1-ab} + \frac{b^n}{ab} \right) u[n-2]$.

Continued→

10.9, continued

(h) $y[n] = \sum_{k=-\infty}^{\infty} b^k u[-k] a^{(n-k-3)} u[n-k-3]$. Since $u[-k] = 0$ when $k > 0$:

$$y[n] = \sum_{k=-\infty}^{0} b^k a^{(n-k-3)} u[n-k-3] = a^{n-3} \sum_{k=-\infty}^{0} (\tfrac{b}{a})^k u[n-k-3]$$

Since $u[n-k-3] = 0$ when $k > n-3$ the sum goes up to $min(0, n-3)$.

If $n > 3$ the sum is to 0:

$$= a^{(n-3)} \sum_{k=-\infty}^{0} (\tfrac{b}{a})^k = a^{(n-3)} \sum_{k=0}^{\infty} (\tfrac{a}{b})^k$$

$$= a^{(n-3)} \frac{1}{1-\frac{a}{b}}$$

as long as $|\frac{a}{b}| < 1$ (same as $|a| < |b|$.)

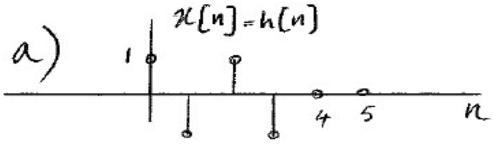
If $n \leq 3$ the sum is to $n-3$:

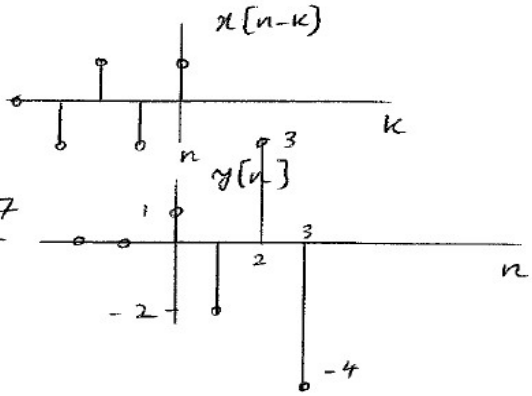
$$= a^{(n-3)} \sum_{k=-\infty}^{n-3} (\tfrac{b}{a})^k = a^{n-3} \sum_{k=-n+3}^{\infty} (\tfrac{a}{b})^k$$

$$= a^{n-3} \frac{(\frac{a}{b})^{(-n+3)}}{1-\frac{a}{b}}$$

Therefore $y[n] = \frac{a^{n-3}}{1-\frac{a}{b}} \left[(\frac{a}{b})^{-n+3} u[3-n] + 1u[n-4] \right]$.

10.10 a)





n	≤ -1	0	1	2	3	4	5	6	≥ 7
$y(n)$	0	1	-2	3	-4	3	-2	1	0

b) $y[0] = x[0]\, h[0] = (1)(1) = 1$

$y[1] = x[0]\, h[1] + h[0]\, x[1] = (1)(-1) + (1)(-1) = -2$

$y[2] = x[2]\, h[0] + x[1]\, h[1] + x[0]\, h[2] = 1 + 1 + 1 = 3$

$y[3] = x[3]\, h[0] + x[2]\, h[1] + x[1]\, h[2] + x[0]\, h[3] = -1 + (-1) + (-1) + (-1) = -4$

$y[4] = x[3]h[1] + x[2]h[2] + x[1]h[3]$
$= (-1)(-1) + (1)(1) + (-1)(-1) = 3$

Continued→

10.10b, continued

$$y[5] = x[3]h[2] + x[2]h[3] = (-1)(1) + (1)(-1) = -2$$

$$y[6] = x[3]h[3] = (-1)(-1) = 1$$

10.11 **(a)** The input gets convolved first with $h_1[n]$ and then with $h_2[n]$ so impulse response is $(\delta[n] * h_1[n]) * h_2[n] = h_1[n] * h_2[n]$ (because $\delta[n] * h[n] = h[n]$).

$$h_1[n] * h_2[n] = \sum_{k=-\infty}^{\infty}(0.6)^k u[k](0.6)^{n-k}u[n-k]$$

If $n < 0$ there are no nonzero terms in the sum and so it is 0.
If $n \geq 0$:

$$\begin{aligned} &= \sum_{k=0}^{n}(0.6)^k(0.6)^{n-k} \\ &= (0.6)^n \sum_{k=0}^{n} 1 \\ &= (0.6)^n(n+1) \end{aligned}$$

Therefore $h[n] = (0.6)^n(n+1)u[n]$.

(b)

$$h_1[n] * h_2[n] = \delta[n+2] * \delta[n+2] = \delta[n+4]$$

because $\delta[n+n_0] * x[n] = x[n+n_0]$ for any $x[n]$ and in this case we take $n_0 = 2$ and $x[n] = \delta[n+2]$ (giving $\delta[n+2] * x[n] = x[n+2] = \delta[n+4]$).

(c) $h_1[n] * h_2[n] = ... + h_1[-2]h_2[n+2] + h_1[-1]h_2[n+1] + h_1[0]h_2[n] + ...$, but only $h_1[-2] = 1$ (and $h_1[k] = 0$ for $k \neq -2$), so we have that

$$\begin{aligned} h_1[n] * h_2[n] = h_1[-2]h_2[n+2] &= 0, n \neq -4 \\ &= 1, n = -4 \end{aligned}$$

which is the definition of the function $\delta[n+4]$.

continued→

10.11, continued

(d)

$$\begin{aligned} h_1[n] * h_2[n] = &\quad \textstyle\sum_{k=-\infty}^{\infty} (u[k] - u[k-3]) (u[n-k] - u[n-k-3]) \\ = &\quad \textstyle\sum_{k=0}^{2} u[n-k] - u[n-k-3] \end{aligned}$$

$u[n-k] - u[n-k-3] = 1$ for $n - 3 < k \leq n$:

$$\begin{aligned} h_1[n] * h_2[n] = &\quad 0, n < 0, \\ = &\quad 1, n = 0 \\ = &\quad 2, n = 1 \\ = &\quad 3, n = 2 \\ = &\quad 2, n = 3 \\ = &\quad 1, n = 4 \\ = &\quad 0, n - 3 \geq 2 \implies n \geq 5 \end{aligned}$$

10.12 **(a)** Causal since $h[n] = 0$ for $n < 0$.

(b) Stable since $\sum_{n=-\infty}^{\infty} |(0.9)^n u[n]| = \sum_{n=0}^{\infty} 0.9^n = \frac{1}{1-0.9} < \infty$.

(c)

$$\begin{aligned} y[n] = u[n] * (0.9)^n u[n] = & \sum_{k=-\infty}^{\infty} u[k]0.9^{n-k}u[n-k] \\ = & \; u[n]\sum_{k=0}^{n} 0.9^{n-k} = 0.9^n \frac{1-0.9^{-(n+1)}}{1-0.9^{-1}} u[n] \\ = & \; \frac{0.9^n - \frac{1}{0.9}}{1-\frac{1}{0.9}} u[n] = \frac{1-0.9^{n+1}}{1-0.9} u[n] \end{aligned}$$

(d)

```
>>x=ones(1,100); % the more terms we include, the more accurate
>>n=0:99;
>>h=0.9.^n;
>>y=conv(x,h);
>>y(1:4) %index i corresponds to n=i-1 so this gives y[n] for n=0,1,2,3
ans =

1.0000 1.9000 2.7100 3.4390


>>y=(1-0.9.^(n+1))/(1-0.9); % analytical result
>>y(1:4)
ans =
1.0000 1.9000 2.7100 3.4390
```

Note that the signals $u[n]$ and $0.9^n u[n]$ go on forever so we had to truncate them in MATLAB. The more terms we include, the more accurate our result.

Continued→

10.12, continued

(e) Not causal since > 0 for some (all) $n < 0$.
Stable since $\sum_{k=-\infty}^{0} 3^n = \sum_{k=0}^{\infty} \frac{1}{3}^n = \frac{1}{1-\frac{1}{3}}$.

$$u[n] * (3)^n u[-n] = \quad \sum_{k=0}^{\infty} 3^{n-k} u[-(n-k)]$$

Note that $u[-n+k] = 0$ if $k < n$.
Therefore if $n \geq 0$ the sum starts at n:

$$\begin{aligned} &= 3^n \sum_{k=n}^{\infty} \frac{1}{3}^k = 3^n \frac{\frac{1}{3}^n}{1-\frac{1}{3}} \\ &= \frac{3}{2} \end{aligned}$$

If $n < 0$ the sum starts at 0:

$$\begin{aligned} &= 3^n \sum_{k=0}^{\infty} \frac{1}{3}^k = 3^n \frac{1}{1-\frac{1}{3}} \\ &= 3^n \left(\frac{3}{2}\right) \end{aligned}$$

So $y[n] = \frac{3}{2}u[n] + 3^n \frac{3}{2} u[-n-1]$.

In MATLAB:

```
>>x=[zeros(1,99),ones(1,100)];%u[n] from n=-99 to n=99
>>h=[3.^(-99:1:0),zeros(1,99)];%3.^n u[-n]
>>y=conv(x,h);
>>y(99+99+1:99+99+3+1) %99+99+1 corresponds to n=0

ans =
1.5000 1.5000 1.5000 1.5000
```

(f) Not causal, not stable, response to $u[n]$ is ∞.

(g) Causal, not stable, infinite response to $u[n]$.

10.13 $$f[n]*g[n]=\sum_{m=-\infty}^{\infty} f[m]\,g[n-m]=e[n]$$

$$f[n]*g[n]*h[n]=\sum_{k=-\infty}^{\infty} e[k]\,h[n-k]$$

$$=\sum_{k=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} f[m]\,g[k-m]\right]h[n-k]$$

$$=\sum_{m=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} g[k-m]\,h[n-k]\right]f[m] \quad \text{let } k-m=p \;\; \text{or } k=m+p$$

$$\therefore \Rightarrow \sum_{m=-\infty}^{\infty}\left[\sum_{p=\infty}^{\infty} g[p]\,h[n-m-p]\right]f[m] \quad \text{let } q=n-p \;\; \text{or } p=n-q$$

$$\Longrightarrow \sum_{m=-\infty}^{\infty}\left[\sum_{q=\infty}^{-\infty} g[n-q]\,h[q-m]\right]f[m]$$

$$=\sum_{q=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} f[m]\,h[q-m]\right]g[n-q]$$

$$=f[n]*h[n]*g[n]$$

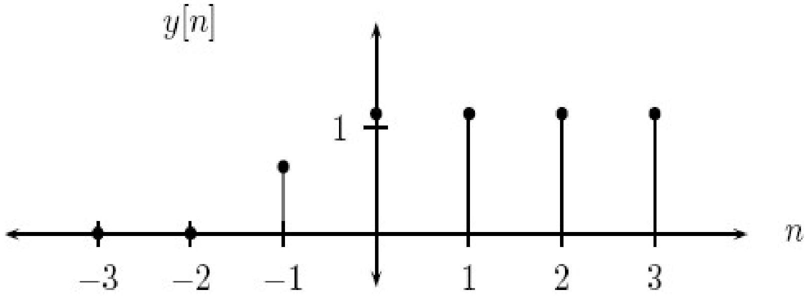
10.14

(a) $h[n]=0.5\delta[n-1]+0.7\delta[n]$.

(b) Yes causal—output only depends on past and present (or, simply note that $h[n]=0, n<0$).

(c) $x[n]=u[n+1]$,

$y[n]=0.5\,(u[n])+0.7\,(u[n+1])$



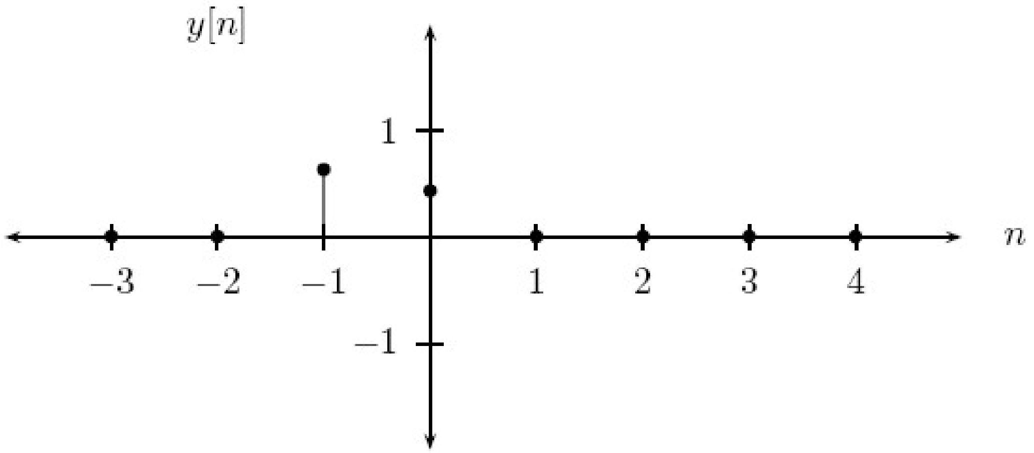
Continued→

10.14, continued

(d) Total response is $h[n] + \delta[n-1] * (-h[n]) = h[n] - h[n-1]$
$= 0.5\delta[n-1] + 0.7\delta[n] - (0.5\delta[n-2] + 0.7\delta[n-1])$
$= 0.7\delta[n] - 0.2\delta[n-1] - 0.5\delta[n-2]$

(e)

$$\begin{aligned} y[n] = h[n] * x[n] = &\quad 0.7x[n] - 0.2x[n-1] - 0.5x[n-2] \\ = &\quad 0.7u[n+1] - 0.2u[n] - 0.5u[n-1] \end{aligned}$$



10.15

(a) Yes linear: $ax_1[n] + bx_2[n] \to e^n\,(ax_1[n] + bx_2[n]) = ae^n x_1[n] + be^n x_2[n] = ay_1[n] + by_2[n]$

(b) Not time-invariant: $x[n-n_0] \to e^n x[n-n_0]$ but $y[n-n_0] = e^{n-n_0} x[n-n_0]$.

(c) $h[n] = e^n\delta[n] = e^0\delta[n] = \delta[n]$

(d) The response to $\delta[n-1]$ is $e^n\delta[n-1] = e^1\delta[n-1]$

(e) No it is not sufficient to describe a timevarying system completely by $h[n]$ because,as this case shows, the response to a delayed impulse might not be a delayed version of $h[n]$ but something else. Therefore we can't express the output of the system for any input just as a sum of weighted delayed $h[n]$ functions. However, it is sufficient to describe the system in terms of $h[n,m]$, the response of the system to $\delta[n-m]$.

10.16

(a) causal, unstable
(b) noncausal, unstable
(c) causal, unstable
(d) noncausal, unstable
(e) causal,stable
(f) causal, stable

10.17 $y[n] = \sum_0^\infty e^{-2k} x[n-k]$

a) let $x[n] = \delta[n]$

Then $h[n] = \sum_{k=0}^{\infty} e^{-2k} \delta[n-k] = \sum_0^\infty e^{-2n} \delta[n-k]$

b) Causal since $h[n]=0 \; , n<0$

c) Stable since $\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} e^{-2n} = \frac{1}{1-e^{-2}} < \infty$

d) $y[n] = \sum_{k=-\infty}^{n} e^{-2(n-k)} x[k-1]$

a) $h[n] = \sum_{k=-\infty}^{n} e^{-2(n-k)} \delta[k-1] = \begin{cases} 0 & , n<1 \\ e^{-2(n-1)} & , n \geq 1 \end{cases}$

$\therefore \; h[n] = e^{-2(n-1)} u[n-1]$

b) Causal, since $h[n]=0 \; , n<0$

c) $\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=1}^{\infty} e^{-2(n-1)} = e^{2} \sum_{k=1}^{\infty} e^{-2n} = \frac{e^{2} e^{-2}}{1-e^{-2}}$

$= \frac{1}{1-e^{-2}} < \infty \; \therefore$ stable

10.18

(a) $h[n] = \delta[n+7] + \delta[n-7]$

(b) $h[n] = \sum_{k=-\infty}^{n-3} \delta[k] + \sum_{k=n}^{\infty} \delta[k-2]$

If $n < 3$, $\sum_{k=-\infty}^{n-3} \delta[k] = 0$; if $n \geq 3$, $\sum_{k=-\infty}^{n-3} \delta[k] = 1$.

If $n > 2$, $\sum_{k=n}^{\infty} \delta[k-2] = 0$; if $n \leq 2$, $\sum_{k=n}^{\infty} \delta[k-2] = 1$.

Therefore $h[n] = u[n-3] + u[-n+2]$. We can show that convolving $h[n]$ with some input $x[n]$ is equivalent to the sum equation given for $y[n]$:

$$x[n] * u[n-3] = \sum_{k=-\infty}^{\infty} x[k]u[n-k-3] = \sum_{k=-\infty}^{n-3} x[k]x[n] * u[2-n] = \sum_{k=-\infty}^{\infty} x[k]u[2-(n-k)] = \sum_{n-2}^{\infty} x[k]$$

10.19

a) (i) $y[n] - 5/6\, y[n-1] = 2^n u[n]$, $y[-1] = 0$

$z - 5/6 = 0$ ∴ $y_c[n] = C(5/6)^n$

$y_p[n] = P(2)^n$

$P2^n - 5/6\, P2^{n-1} = 2^n$

$2P - 5/6\, P = 2 \longrightarrow 7/6\, P = 2 \Longrightarrow P = \frac{12}{7}$

$y[n] = C(5/6)^n + 12/7(2^n)$

$y[-1] = 0 = C(5/6)^{-1} + \frac{12}{7}(2^{-1})$ $\quad 6/5\, C + 6/7 = 0$

∴ $y[n] = -5/7(5/6)^n + (12/7)2^n$ $\quad$ ∴ $C = -5/7$

b) $y[-1] = -5/7(6/5) + (12/7)(1/2) = -6/7 + 6/7 = 0$ ✓

$n \geq 0$ $\quad y[n] - 5/6\, y[n-1] = -5/7(5/6)^n + 12/7\, 2^n + 5/6(5/7)(5/6)^{n-1}$

$-5/6(12/7)2^{n-1} = -5/7(5/6)^n + 12/7\, 2^n + 5/7(5/6)^n - 5/7\, 2^n$

$= 2^n$

Continued→

10.19, continued

(ii) $y_c[n] = C(.7)^n$

a) $y_p[n] = pe^{-n} \therefore pe^{-n} - .7pe^{-(n-1)} = pe^{-n}[1-.7e]$

$= pe^{-n}[-.903] = e^{-n}$

$\Longrightarrow p = -1.108$

$\therefore \quad y[n] = c(.7)^n - 1.108e^{-n}$

$y[-1] = 0 = \frac{c}{.7} - 1.108e \Longrightarrow \frac{c}{.7} = 3.012 \Longrightarrow C = 2.108$

$\therefore y[n] = -1.108e^{-n} + 2.108(.7)^n \quad , n \geq -1$

b) $y[-1] = -1.108e^{-1} + 2.108(.7)^{-1} = -3.012 + 3.01 = 0$ ✓

$y[n] - .7y[n-1] = -1.108e^{-n} + 2.108(.7)^n$

$-.7\left[-1.108e^{-(n-1)} + 2.108(.7)^{n-1}\right] = -1.108e^{-n}$

$+2.108(.7)^n + 2.108e^{-n} - 2.108(.7)^n = e^{-n}$ ✓

(iii) $y[n] + 3y[n-1] + 2y[n-2] = 3u[n]$

$y[-1] = 0 \, , \; y[-2] = 0$

$z^2 + 3z + z = (z+2)(z+1)$

$\therefore y_c[n] = c_1(-2)^n + c_2(-1)^n \qquad y_p[n] = P$

$\therefore \; p + 3p + 2p = 3 \Longrightarrow 6p = 3 \longrightarrow p = \frac{1}{2}$

$\therefore \; y[n] = \frac{1}{2} + c_1(-2)^n + c_2(-1)^n$

use initial conditions to solve for c_1 & c_2

$$\begin{aligned} y[-1] &= 0 = \tfrac{1}{2} + c_1\left(-\tfrac{1}{2}\right) + c_2(-1) \\ y[-2] &= 0 = \tfrac{1}{2} + c_1\left(\tfrac{1}{4}\right) + c_2 \end{aligned} \Longrightarrow \begin{aligned} c_1 &= 4 \\ c_2 &= -\tfrac{3}{2} \end{aligned}$$

$\therefore \; y[n] = \frac{1}{2} + 4(-2)^n - \frac{3}{2}(-1)^n$

b) $y[-1] = \frac{1}{2} + 4(-2)^{-1} - \frac{3}{2}(-1)^{-1} = \frac{1}{2} + \left(-\frac{4}{2}\right) + \frac{3}{2} = 0$ ✓

$y[-2] = \frac{1}{2} + 4(-2)^{-2} - \frac{3}{2}(-1)^{-2} = \frac{1}{2} + \frac{4}{4} - \frac{3}{2} = 0$ ✓

$y[n] + 3y[n-1] + 2y[n-2] = \frac{1}{2} + 4(-2)^n - \frac{3}{2}(-1)^n$

$+ \frac{3}{2} + 12(-2)^{n-1} - \frac{9}{2}(-1)^{n-1} + 1 + 8(-2)^{n-2}$

$- 3(-1)^{n-2} = 3$ ✓

10.20

(i) (a) mode is $(-0.6)^n$; (b) natural response is $y_c[n] = C(-0.6)^n$

(ii) (a) $z^2+1.5z-1 = (z-0.5)(z+2)$, modes are 0.5^n and $(-2)^n$; (b) natural response is $C_1(0.5)^n+C_2(-2)^n$

(iii) (a) $(z-j)(z+j) = 0$, modes are $(j)^n = e^{j\frac{\pi}{2}n}$ and $(-j)^n = e^{-j\frac{\pi}{2}n}$; (b) natural response (in real form) is $C\cos(\frac{\pi}{2}n+\beta)$.

(iv) (a) $(z-0.7)(z-3)(z+0.2) = 0$, modes $(0.7)^n$, 3^n, $(-0.2)^n$; (b) natural response is $C_1(0.7)^n + C_2 3^n + C_3(-0.2)^n$.

(v) (a) modes are 0.5^n, $n0.5^n$, and $n^2 0.5^n$; (b) natural response is $C_1 0.5^n + C_2 n 0.5^n + C_3 n^2 0.5^n$.

(vi) (a) modes are 0.5^n, 1.5^n, $(-0.7)^n$; (b) natural response is $C_1 0.5^n + C_2 1.5^n + C_3(-0.7)^n$.

10.21 Stable if all roots of characteristic eqn. are inside the unit circle:

(i) $z = -0.6$, stable;

(ii) $z = 0.5, 2$, unstable since 2 outside unit circle;

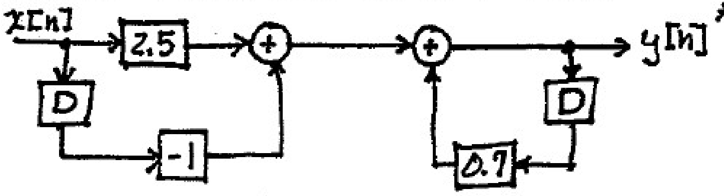
(iii) $z = \pm j$, unstable since $\pm j$ outside unit circle;

(iv) $z = 0.7, 3, -0.2$, unstable since 3 outside unit circle;

(v) $z = 0.5$, stable;

(vi) $z = 0.5, 1.5, -0.7$, unstable since 1.5 outside unit circle

*10.22. (a)



10.22. See b
page of this
chapter

(b) $y[n] = 0.7y[n-1] + 2.5x[n] - x[n-1]$

$y[0] = 0 + 2.5(1) - 0 = \underline{2.5}$

$y[1] = 0.7(2.5) + 0 - 1 = \underline{0.75}$

$y[2] = 0.7(0.75) + 0 - 0 = \underline{0.5250}$

$y[3] = 0.7(0.5250) = \underline{0.3675}$

$y[4] = 0.7(0.3675) = \underline{0.2573}$

(c) $w[0] = 2.5$ $\quad y[0] = \underline{2.5}$

$w[1] = 1$ $\quad y[1] = -1 + 0.7(2.5) = \underline{0.75}$

$w[2] = 0$ $\quad y[2] = 0.7(0.75) = \underline{0.5250}$

$w[3] = 0$ $\quad y[3] = 0.7(0.5250) = \underline{0.3675}$

$w[4] = 0$ $\quad y[4] = 0.7(0.3675) = \underline{0.2573}$

(d) $y[n] = h[n+2] - 3h[n] + 2h[n-1]$

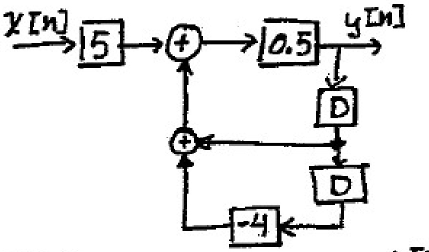
(e) $y[-3] = h[-1] - 3h[-3] + 2h[-4] = \underline{0}$

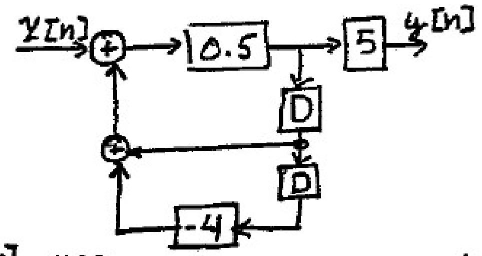
$y[-1] = h[1] - 0 + 0 = \underline{0.75}$

$y[1] = h[3] - 3h[1] + 2h[0]$

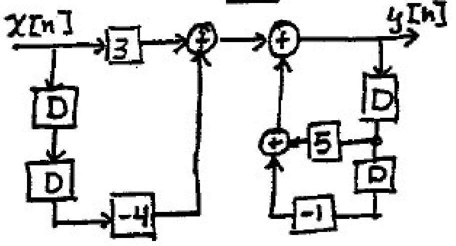
$= 0.3675 - 3(0.75) + 2(2.5) = \underline{3.118}$

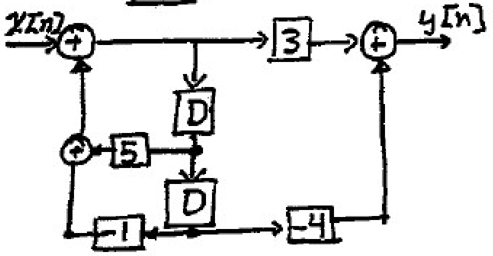
10.23 (a)



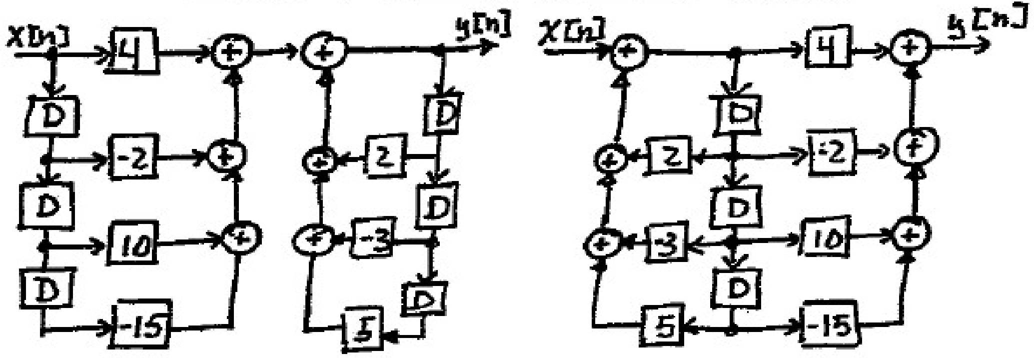


(b)

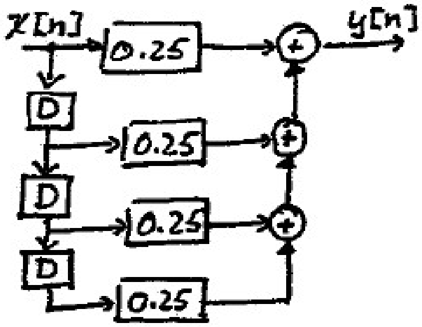




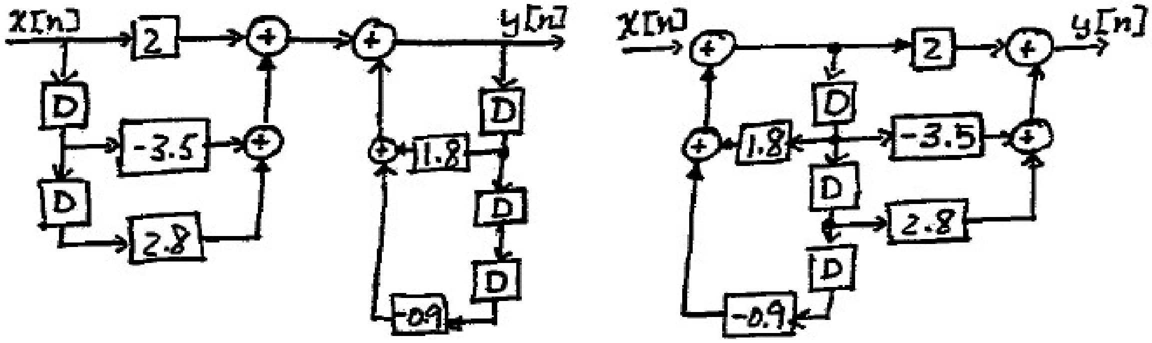
10.23.(c)
(cont)



(d)



(e)



10.24(a) $2z^2 - z + 4 = 2(z^2 - 0.5z + 2) = 2(z - z_1)(z - z_2)$

$\therefore (z_1 z_2) = 2$, and at least one root is greater than unity - <u>not stable</u>

(b) $(z^2 - 5z + 1) = (z - 4.79)(z - 0.21)$ <u>not stable</u>

(c) $z^3 - 2z^2 + 3z - 5 = (z - z_1)(z - z_2)(z - z_3)$

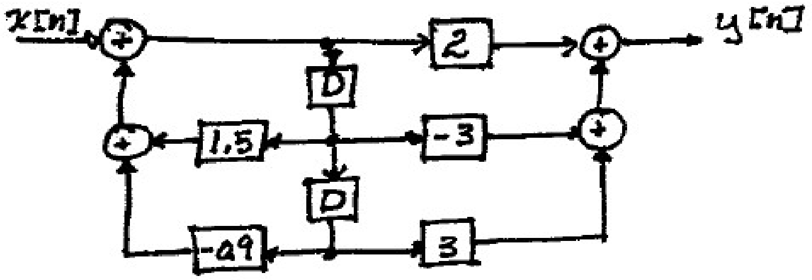
$\therefore (z_1 z_2 z_3) = 5$ - <u>not stable</u> (see (a))

(d) <u>stable</u> by inspection (no feedback)

(e) $z^2 - 1.8z + 0.9 = (z - 0.949\underline{/18.4^\circ})(z - 0.949\underline{/-18.4^\circ})$ <u>stable</u>

```
n=[2 -1 4];
roots(n)
pause
n=[1 -5 1];
roots(n)
pause
n=[1 -2 3 -5];
roots(n)
pause
n=[1 -1.8 .9];
roots(n)
```

10.25. (a)

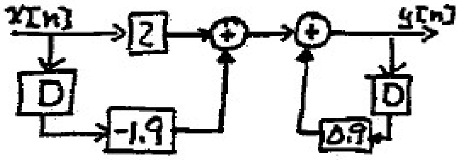


$$y[n] - 1.5y[n-1] + 0.9y[n-2] = 2x[n] - 3x[n-1] + 4x[n-2]$$

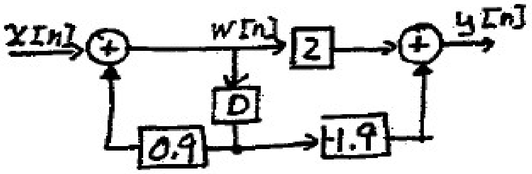
(b) <u>form II</u>

10.26. $y[n] - 0.9y[n-1] = 2x[n] - 1.9x[n-1]$

(a)



(b)



(c) $y[0] = 0.9(0) + 2 - 0 = \underline{2}$

$z - 0.9 = 0 \Rightarrow y_c[n] = C(0.9)^n$

$y_P[n] = P(0.8)^n \Rightarrow P(0.8)^n - \frac{0.9}{0.8}P(0.8)^n$

$= (P - 1.125P)(0.8)^n = (2 - 2.375)(0.8)^n \Rightarrow P = \underline{3}$

$\therefore y[n] = 3(0.8)^n + C(0.9)^n$

$y[0] = 2 = 3 + C \Rightarrow C = -1$ and $y[n] = \underline{3(0.8)^n - (0.9)^n}$

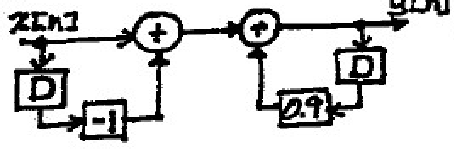
For example: $y[5] = 3(0.8)^5 - (0.9)^5 = 0.3926$ checks MATLAB

(d)

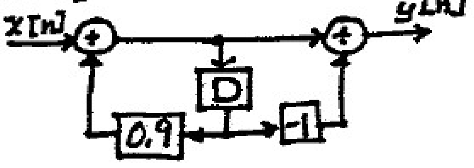
```
y(1)=2;
for n=1:5
  y(n+1)=.9*y(n)+2*((.8)^n)-1.9*((.8)^(n-1));
end
y
```

10.27.(a) $y[n] - 0.9y[n-1] = x[n] - x[n-1]$

(b)



(c)



(d) $x[n] = (0.7)^n u[n]$

(e) $z - 0.9 = 0 \Rightarrow y_c[n] = C(0.9)^n$; $y_P[n] = P(0.7)^n$

$P(0.7)^n - \frac{0.9}{0.7}P(0.7)^n = (0.7)^n - \frac{1}{0.7}(0.7)^n \Rightarrow \underline{P = 1.5}$

10.27 (e)
cont

$\therefore y[n] = C(.9)^n + (1.5)(.7)^n$

$y[0] = 0 = C + 1.5 \implies C = -1.5$

$\therefore y[n] = 1.5\left[(.7)^n - (.9)^n\right]$

$y[0] = 0$ $\quad y[2] = -.48$

$y[1] = -.3$ $\quad y[3] = -.579$

10.28

(a) $y[n] - 0.9y[n-1] = x[n] - x[n-1]$

(b) $y_p[n] = P(1)^n = P$; need $P - 0.9P = 1 - 1 = 0 \implies P = 0 \implies y_p[n] = 0.$

(c) $H(z) = \frac{1-z^{-1}}{1-0.9z^{-1}} = \frac{z-1}{z-0.9}$

(d) $Y(z) = H(z)X(z) = \frac{z-1}{z-0.9}\frac{z}{z-1} = \frac{z}{z-0.9}$ so $y[n] = (0.9)^n u[n]$ and $y_p[n] = \lim_{n\to\infty} y[n] = 0.$

(e) In the second statement, replace the statement `x(n)=0.7^(n-1)` with `x(n)=1` (or replace entire second line with `x=ones(1,6)`).

(f)
```
>>y(1)=0, x(1)=0; %first index corresponds to n=-1
>>for n=2:6; x(n)=1; end
>>for n=2:6 % indices 2-6 correspond to n=0 to 4
y(n)=0.9*y(n-1)+x(n)-x(n-1);
end
>>y

ans=
0 1.0000 0.9000 0.8100 0.7290 0.6561
>>n=0:4; 0.9.^n

ans=
1.0000 0.9000 0.8100 0.7290 0.6561
```
The result matches $y(n) = 0.9^n$ which is the natural response only (which decays to 0). There is no non-decaying particular response.

10.29 a) $y[n] - .7\,y[n-1] = x[n]$

$$Y(z) - .7\,z^{-1}Y(z) = X(z)$$

$$Y(z)\left[1 - .7\,z^{-1}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - .7z^{-1}} = \frac{z}{z - .7}$$

b) $x[n] = \cos(n)\,u[n] = \cos(\Omega n)\,u[n] \quad \therefore \Omega = 1$

$$\cos \Omega n \longrightarrow (1)\left|H(e^{j\Omega})\right| \cos(\Omega n + \theta 4)$$

$$e^{j\Omega}\Big|_{\Omega=1} = e^{j} = \cos 1 + j \sin 1 = .54 + .841\,j$$

$$\therefore H(e^{j}) = \frac{.54 + .841\,j}{.54 + j.841 - .7} = \frac{1 \angle 57.3^\circ}{.856 \angle 100.8^\circ} = 1.168 \angle -43.5^\circ$$

$$\therefore y_{ss}[n] = 1.168 \cos(n - 43.5^\circ)$$

d) $y_{ss}[n] - .7\,y_{ss}[n-1] = 1.168 \cos(n - 43.5^\circ)$

$$- .7(1.168)\cos(n - 43.5^\circ - 57.3^\circ)$$

$$= .847 \cos n + .804 \sin n + .153 \cos n -$$

$$.803 \sin \approx \cos n$$

10.30

(a) Need $|b| < 1$;

(b) $a^{-n}u[n] * b^n u[n+6] = \sum_{k=-\infty}^{\infty} a^{-k}u[k]b^{n-k}u[n-k+6] = \sum_{k=0}^{\infty} a^{-k}b^{n-k}u[n-k+6]$ $= u[n-6]\sum_{k=0}^{n-6} a^{-k}b^{n-k}$. Since this is a finite sum for a fixed n, there is no restriction or a,b for it to be finite.

(c) $a^n u[n-3] * u[-n-4] = \sum_{k=3}^{\infty} a^k u[-(n-k)-4]$ The term $u[-(n-k)-4] = 1$ when $k \geq n+4$. Therefore the sum starts at the value of k where both $k > n+4$ and $k > 0$: $= \sum_{k=max(3,n+4)}^{\infty} a^k$. The sum to ∞ requires $|a| < 1$ to converge.

(d) $a^n u[-n] * b^n u[-n-6] = \sum_{k=-\infty}^{0} a^k b^{n-k} u[-(n-k)-6]$ The term $u[-(n-k)-6] = u[k-(n+6)]$ is 0 if $k < n+6$, so the sum goes from $k = n+6$ to 0 or is 0 if $n+6 > 0$. So the sum is finite and will always converge for any a, b.

10.31 It is not linear, by the following reasoning: note that $x_3[n] = x_1[n] + x_2[n-1]$. A linear system must therefore satisfy $y_3[n] = y_1[n] + y_2[n-1]$ (because we know it's time invariant so that $x_2[n-1] \rightarrow y_2[n-1]$). But $y_1[n]+y_2[n-1] = 2\delta[n+1]+2\delta[n]+2\delta[n-1]+(2\delta[n-1] - 2\delta[n-2]) = 2\delta[n+1]+2\delta[n]+4\delta[n-1]-2\delta[n-2]$. This is not equal to $y_3[n]$ in this case, so the system must not be linear.

10.32 It is not linear; note that $x_2[n] = x_1[n+2] + x_1[n]$, and since the system is time-invariant it requires that input $x_1[n+2]$ has output $y_1[n+2]$. So if the system were linear that would imply that $x_1[n+2] + x_1[n] \rightarrow y_1[n+2] + y_1[n] = 2\delta[n] + 4\delta[n-1] + 2\delta[n-2] = 4\delta[n-3]$. However, this is not equal to $y_2[n]$ in this case so the system can't be linear.

Chapter 11 solutions

11.1

(a) $\sum_{n=0}^{\infty}(0.3)^n z^{-n} = \sum_{n=0}^{\infty}(\frac{0.3}{z})^n = \frac{1}{1-\frac{0.3}{z}} = \frac{z}{z-0.3}$ (with ROC $|z| > 0.3$) (can also get by using Table 11.1 or 2).

(b) $\sum_{n=0}^{\infty}(0.2^n+2(3)^n)z^{-n} = \sum_{n=0}^{\infty}(\frac{0.2}{z})^n+\sum_{n=0}^{\infty}2(\frac{3}{z})^n = \frac{z}{z-0.2}+2\frac{z}{z-3} = \frac{z(z-3)+2z(z-0.2)}{(z-0.2)(z-3)} = \frac{3z^2-3.4z}{z^2-3.2z+0.6}$ (with ROC $|z| > 3$) (can also get by using Table 11.1 or 2 and linearity of z-transform). (c) $\mathcal{Z}[3(e^{-.7})^n] = \frac{3z}{z-e^{-.7}}$ (from Table 11.1 or 2) with ROC $|z| > e^{-.7}$.

(d) $\mathcal{Z}[5(e^{-j0.3})^n] = \frac{5z}{z-e^{-j0.3}}$ (from Table 11.1 or 2) with ROC $|z| > e^{-j0.3}$.

(e) $\mathcal{Z}[5\cos(3n)] = 5\frac{z(z-\cos(3))}{z^2-2z\cos(3)+1}$ (from Table 11.2 entry 10) with ROC $|z| > 1$.

(f) $\mathcal{Z}[(e^{-.7})^n\sin(0.5n)] = \frac{e^{-.7}z\sin(0.5)}{z^2-2e^{-.7}z\cos(0.5)+e^{-1.4}}$ (from Table 11.2 entry 11) with ROC $|z| > e^{-.7}$.

11.2 $t = nT = .05n$

a) $2e^{-2t}$ or $z[2e^{-2(.05)n}] = z[2e^{-.1n}] = \frac{2z}{z-e^{-.1}}$

$$= \frac{2z}{z-.905}$$

b) $z[2e^{-.1n}+2e^{.05n}] = \frac{2z}{z-e^{-.1}} + \frac{2z}{z-e^{.05}} =$

$$\frac{2z}{z-.905} + \frac{2z}{z-1.05}$$

c) $z[2e^{-.2(.05)n}] = z[2e^{-.01n}] = \frac{2z}{z-e^{-.01}} = \frac{2z}{z-.99}$

d) $z[5e^{-.5j(.05)n}] = z[5e^{-.025jn}] = \frac{5z}{z-e^{-.025j}} = \frac{5z}{z-.9997+.025j}$

Continued→

11.2, continued

e) $z[5\cos(.05n)] = \frac{5z(z-\cos .05)}{z^2-2z\cos .05+1} = \frac{5z^2-4.99z}{z^2-1.998z+1}$

f) $z[5e^{-.05n}\cos(.05n)] = \frac{5z[z-(e^{-.05})\cos .05]}{z^2-2(e^{-.05})\cos .05\, z+(e^{-.05})^2}$

$= \frac{5z^2-4.75}{z^2-1.9z+.905}$

11.3 a) $x_a[nT] = e^{-5(.2)n} = e^{-n} = (e^{-1})^n = (.3679)^n$

b) $x_b[nT] = e^{-n} = (.3679)^n$

c) The value of the two signals are equal at each sample instant.

d) $e^{-anT} = (e^{-aT})^n = (e^{-1})^n \quad \therefore aT=1$

(i) $a=1/2,\ T=2$ (ii) $a=2,\ T=1/2$

11.4

(a) (i) $x = F(z)|_{z=1}$ where $F(z) = \mathcal{Z}[0.3^n] = \frac{z}{z-0.3}$ so $F(z)|_{z=1} = \frac{1}{1-0.3} = \frac{1}{0.7}$.
(ii) $x = F(z)|_{z=1}$ where $F(z) = \mathcal{Z}[0.3^n u[n-5]]$. Note that $F(z) = 0.3^5 z^{-5}\frac{z}{z-0.3}$ so $x = \frac{0.3^5}{1-0.3} = \frac{0.3^5}{0.7}$.

(b) $x = \mathcal{Z}[0.5^n\cos(0.1n)]|_{z=1} = \frac{z(z-0.5\cos(0.1))}{z^2-2(0.5)z\cos(0.1)+(0.5)^2}|_{z=1} = \frac{1-0.5\cos(0.1)}{1-\cos(0.1)+0.25} = 1.97.$

11.5

a) $$\mathcal{Z}[A\cos\Omega n] = \frac{Az(z-\cos\Omega)}{z^2-2\cos\Omega z+1} = \frac{3z(z-.6967)}{z^2-1.393z+1}$$

$\therefore A=\underline{3}$; $\cos\Omega = .6967 \Longrightarrow \Omega = 45.84^\circ = .8 \text{ rad} = \Omega$

b) $A=\underline{3}$; $\cos\Omega n = \cos(\omega T)n$, $\therefore \omega(.0001) = .8$

$\therefore \omega = 8000$

11.6

(a) $f[\infty] = \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z-2)} = \frac{1}{1-2} = -1.$

(b) $F(z) = \frac{z}{(z-1)(z-2)}$
We assume $f[n]$ is causal to get the inverse transform.
Partial fractions: $\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$
So $F(z) = \frac{-z}{z-1} + \frac{z}{z-2}$
Taking inverse transform: $f[n] = -u[n] + (2)^n u[n]$
$f[\infty] = \lim_{n\to\infty} -u[n] + (2)^n u[n] = -1 + \infty = \infty$

(c) The final value property doesn't apply when $\lim_{n\to\infty} f[n] = \infty$ (ie., when there is a pole outside the unit circle and $f[n]$ causal).

11.7

(a) Using property 5 in Table 11.4 (multiplication by n), $\mathcal{Z}[n3^n] = -z\frac{dF(z)}{dz} = -z\frac{-3}{(z-3)^2} = \frac{3z}{(z-3)^2}$

(b) Table 11.2 (entry 7) gives $\mathcal{Z}[n3^n] = \frac{3z}{(z-3)^2}$

11.8 a) $\mathcal{Z}[y[n-3]u[n-3]] = z^{-3}Y(z) = \frac{1}{z^3-3z^2+5z-9} = Y_1(z)$

b) $\mathcal{Z}[y[n+3]u[n]] = z^3[Y(z) - y[0] - y[1]z^{-1} - y[2]z^{-2}]$

$$\begin{array}{r|l} & 1+3z^{-1}+4z^{-2}+6z^{-3}+\cdots \\ \hline z^3-3z^2+5z-9 & z^3 \end{array}$$

$\therefore y[0]=1, \quad y[1]=3, \quad y[2]=4$

$$\therefore \mathcal{Z}[y[n+3]u[n]] = z^3\left[\frac{z^3}{z^3-3z^2+5z-9} - 1 - \frac{3}{z} - \frac{4}{z^2}\right]$$

$$= \frac{6z^3+7z^2+36z}{z^3-3z^2+5z-9} = Y_2(z)$$

c) $y[0]=1, \quad y[3]=6$ from(b)

$y_1[3]=1$, by inspection in a

$y_2[0]=6$, by inspection in b

d) $y_1[3] = y[n-3]u[n-3]\big|_{n=3} = y[0]$ ✓

$y_2[0] = y[n+3]u[n]\big|_{n=0} = y[3]$ ✓

11.9

(a) Time-scaling property: $\mathcal{Z}[f[n/7]] = F(z^7) = \frac{z^7}{z^7-a}$

(b) Time-shifting property: $\mathcal{Z}[f[n-7]u[n-7]] = z^{-7}\frac{z}{z-a} = \frac{z^{-6}}{z-a}$

This can be verified by $\sum_{k=7}^{\infty} a^{(n-7)}z^{-n} = a^{-7}\sum_{k=7}^{\infty}(\frac{a}{z})^n = a^{-7}\frac{(\frac{a}{z})^7}{1-\frac{a}{z}} = \frac{z^{-6}}{z-a}$

(c) $\mathcal{Z}[f[n+3]u[n]] = \mathcal{Z}[a^3a^nu[n]] = a^3\frac{z}{z-a}$

This is verified by $\sum_{k=0}^{\infty} a^{n+3}z^{-n} = \sum_{k=0}^{\infty} a^3(\frac{a}{z})^n = a^3\frac{1}{1-\frac{a}{z}} = a^3\frac{z}{z-a}$

(d) One method: $\mathcal{Z}[b^{2n}f[n]] = \mathcal{Z}[(ab^2)u[n]] = \frac{z}{z-ab^2}$ (using entry 6 in Table 11.2)

Another method: using complex shifting: $\mathcal{Z}[b^{2n}f[n]] = F(\frac{z}{b^2}) = \frac{\frac{z}{b^2}}{\frac{z}{b^2}-a} = \frac{z}{z-b^2a}$

11.10

(a)

(i) To get partial fractions for finding inverse transform: $\frac{X(z)}{z} = \frac{0.5z}{(z-1)(z-0.5)} = \frac{1}{z-1} - \frac{0.5}{z-0.5}$
so $X(z) = \frac{z}{z-1} - \frac{0.5z}{z-0.5}$
$x[n] = u[n] - 0.5^{n+1}u[n]$

(ii) $\frac{X(z)}{z} = \frac{0.5}{(z-1)(z-0.5)} = \frac{1}{z-1} - \frac{1}{z-0.5}$

Continued→

11.10a, continued

$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}$

$x[n] = u[n] - (0.5)^n u[n]$

(iii) $X(z) = \frac{1}{z-1} - \frac{1}{z-0.5} = z^{-1}\left(\frac{z}{z-1} - \frac{z}{z-0.5}\right)$ The z^{-1} implies a delayed version of the inverse transform of (ii): $x[n] = u[n-1] - (0.5)^{n-1}u[n-1]$

(iv) $\frac{X(z)}{z} = \frac{1}{(z-\frac{1}{2}-j\frac{\sqrt{3}}{2})(z-\frac{1}{2}+j\frac{\sqrt{3}}{2})}$

$= \frac{1}{j\sqrt{3}}\frac{1}{z-\frac{1}{2}-j\frac{\sqrt{3}}{2}} - \frac{1}{j\sqrt{3}}\frac{1}{z-\frac{1}{2}+j\frac{\sqrt{3}}{2}}$

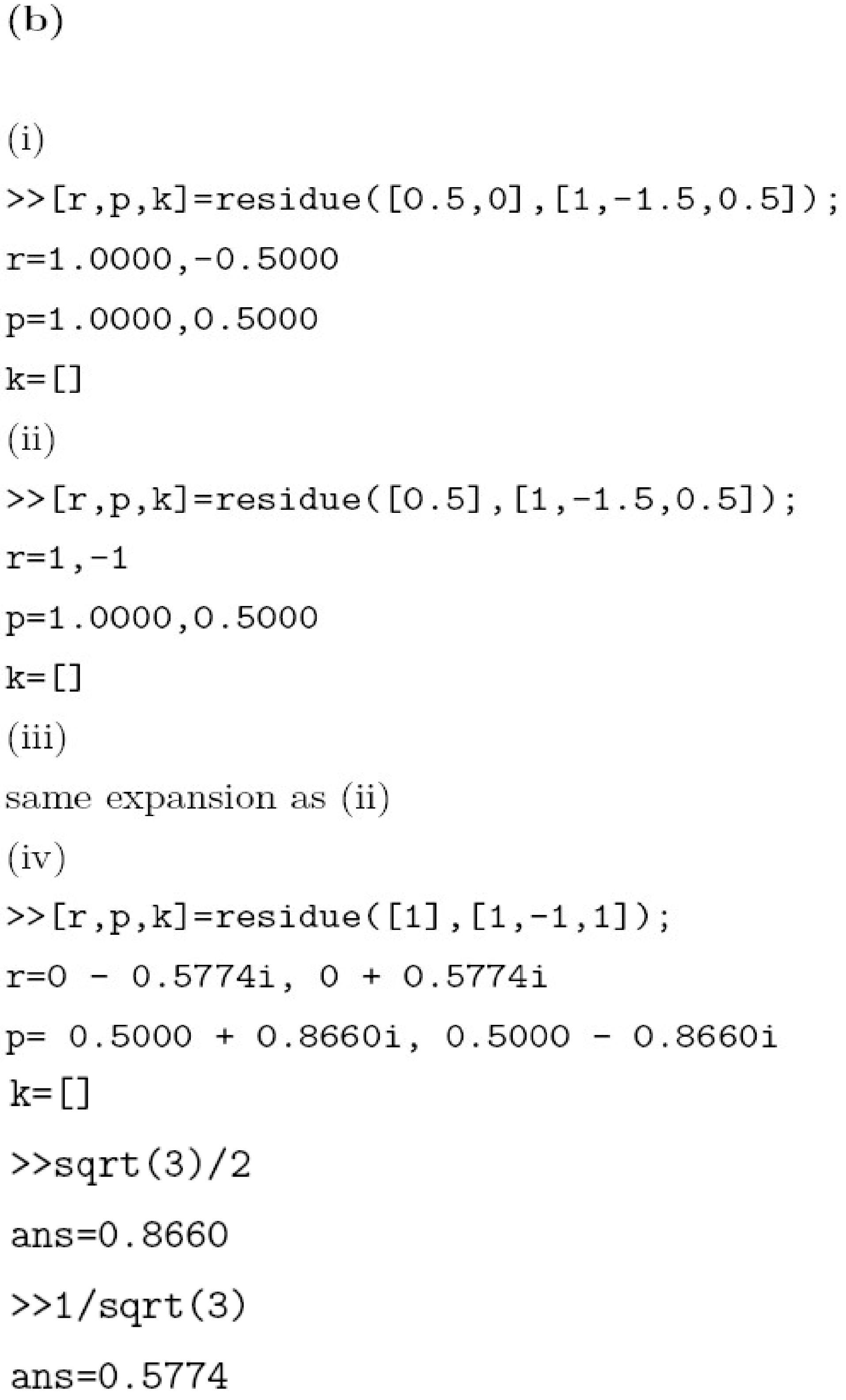
$x[n] = \frac{1}{j\sqrt{3}}(\frac{1}{2}+j\frac{\sqrt{3}}{2})^n u[n] + \frac{-1}{j\sqrt{3}}(\frac{1}{2}-j\frac{\sqrt{3}}{2})^n u[n]$

$= \frac{1}{j\sqrt{3}}(e^{j\pi/3})^n u[n] - \frac{1}{j\sqrt{3}}(e^{-j\pi/3})^n u[n]$

$= \frac{1}{j\sqrt{3}}(e^{jn\pi/3} - e^{-jn\pi/3})u[n] = \frac{2}{\sqrt{3}}\sin(\pi n/3)u[n]$

Continued→

11.10,continued



(b)

(i)

```
>>[r,p,k]=residue([0.5,0],[1,-1.5,0.5]);
r=1.0000,-0.5000
p=1.0000,0.5000
k=[]
```

(ii)

```
>>[r,p,k]=residue([0.5],[1,-1.5,0.5]);
r=1,-1
p=1.0000,0.5000
k=[]
```

(iii)

same expansion as (ii)

(iv)

```
>>[r,p,k]=residue([1],[1,-1,1]);
r=0 - 0.5774i, 0 + 0.5774i
p= 0.5000 + 0.8660i, 0.5000 - 0.8660i
k=[]
>>sqrt(3)/2
ans=0.8660
>>1/sqrt(3)
ans=0.5774
```

Continued→

11.10 continued

(c)

(i) First nonzero values are at $n = 0, 1, 2 : x[0] = 0.5, x[1] = 0.75, x[2] = 0.875$

(ii) First nonzero values are at $n = 1, 2, 3 : x[1] = 0.5, x[2] = 0.75, x[3] = 0.875$

(iii) First nonzero values are at $n = 2, 3, 4 : x[2] = 0.5, x[3] = 0.75, x[4] = 0.875$

(iv) First nonzero values are at $n = 1, 2, 4 : x[1] = \frac{2}{\sqrt{3}} \sin(\pi/3) = 1, x[2] = \frac{2}{\sqrt{3}} \sin(2\pi/3) = 1, x[3] = \frac{2}{\sqrt{3}} \sin(\pi) = 0$, $x[4] = \frac{2}{\sqrt{3}} \sin(4\pi/3) = -1$.

Continued→

11.10 continued

(d)

(i)

$$.5+.75z^{-1}+.875z^{-2}\ldots$$

$$z^2-1.5z+0.5 \,\big)\, .5z^2$$

$$-(.5z^2-.75z+.25)$$

$$.75z-.25$$

$$-(.75z-1.125+.375z^{-1})$$

$$.875-.375z^{-1}$$

(ii)

$$.5z^{-1}+.75z^{-2}+.875z^{-3}+\ldots$$

$$z^2-1.5z+0.5 \,\big)\, .5z$$

$$-(.5z-.75+.25z^{-1})$$

$$.75-.25z^{-1}$$

$$-(.75-1.125z^{-1}+.375z^{-2})$$

$$.875z^{-1}+.375z^{-2}$$

(iii)

$$0.5z^{-2}+0.75z^{-3}+.875z^{-4}+\ldots$$

$$z^2-1.5z+0.5 \,\big)\, 0.5$$

$$-(0.5-.75z^{-1}+.25z^{-2})$$

$$.75z^{-1}-.25z^{-2}$$

$$-(.75z^{-1}-1.125z^{-2}+.375z^{-3})$$

$$.875z^{-2}-.375z^{-3}$$

(iv)

$$z^{-1}+z^{-2}-z^{-4}+\ldots$$

$$z^2-z+1 \,\big)\, z$$

$$-(z-1+z^{-1})$$

$$1-z^{-1}$$

$$-(1-z^{-1}+z^{-2})$$

$$-z^{-2}$$

continued→

11.10 continued

(e)

(i) $x[\infty] = \lim_{z\to 1} \frac{0.5z^2}{z-0.5} = 1$

(ii) $x[\infty] = \lim_{z\to 1} \frac{0.5z}{z-0.5} = 1$

(iii) $x[\infty] = \lim_{z\to 1} \frac{0.5}{z-0.5} = 1$

(iv) $\lim_{n\to\infty} x[n]$ doesn't exist so final value property doesn't apply.

(f)

(i),(ii),(iii): $\lim_{n\to\infty} x[n] = \lim_{n\to\infty} u[n] = 1$

(iv), limit doesn't exist

(g)

(i) $x[0] = \lim_{z\to\infty} \frac{0.5z^2}{(z-1)(z-0.5)} = 0.5$

(ii) $x[0] = \lim_{z\to\infty} \frac{0.5z}{(z-1)(z-0.5)} = 0$

(iii) $x[0] = \lim_{z\to\infty} \frac{0.5}{(z-1)(z-0.5)} = 0$

(iv) $x[0] = \lim_{z\to\infty} \frac{z}{z^2-z+1} = 0$

(h)

From part (c): (i) $x[0] = 0.5$, (ii) $x[0] = 0$, (iii) $x[0] = 0$, (iv) $x[0] = 0$

11.11

(a) $x_2[n] = x_1[n-1]$,
$x_3[n] = x_2[n-1] = x_1[n-2]$

(b) See the solution to 11.10 (a), which shows that

$$\begin{aligned} x_1[n] &= \quad u[n] - 0.5^{n+1}u[n], \\ x_2[n] &= \quad u[n] - 0.5^{n}u[n], \\ x_3[n] &= u[n-1] - 0.5^{n-1}u[n-1] \end{aligned}$$

Clearly $x_1[n-1] = u[n-1] - 0.5^n u[n-1] = u[n] - 0.5^n u[n] = x_2[n]$ since $u[0] = 0.5^0 u[0] = 0$, and so then

$$x_1[n-2] = x_2[n-1] = u[n-1] - 0.5^{n-1}u[n-1]$$

(c) For MATLAB verifications of partial fraction expansion see soln. to 11.10 (b).

11.12 (a) $Y(z) = [1 - 1.5z^{-1} + 0.5z^{-2}] = X(z) \Rightarrow H(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$; $X(z) = z^{-1}$

$$\therefore \frac{Y(z)}{z} = \frac{1}{(z-1)(z-0.5)} = \frac{2}{z-1} + \frac{-2}{z-0.5} \Rightarrow y[n] = 2 - 2(0.5)^n$$

Continued→

11.12(b) (cont)

```
n=[0 0 1]; d=[1 -1.5 .5]; [r,p,k]=residue(n,d)
```

(c) $y[n]=2-2(0.5)^n \Rightarrow y[0]=\underline{0},\ y[1]=\underline{1},\ y[2]=\underline{1.5},$

$y[3]=\underline{1.75},\ y[4]=\underline{1.875}$

$y[n]=1.5y[n-1]-0.5y[n-2]+x[n]$

$y[0]=\ 0-0+0=\underline{0}$

$y[1]=\ 0-0+1=\underline{1}$

$y[2]=1.5(1)-0+0=\underline{1.5}$

$y[3]=1.5(1.5)-0.5(1)=\underline{1.75}$

$y[4]=1.5(1.75)-0.5(1.5)=\underline{1.875}$

(d)

```
x=[0 0 0 1 0 0 0 0]; y(1)=0; y(2)=0;
for n=3:7
   y(n)=1.5*y(n-1)-0.5*y(n-2)+x(n)
end
```

(e) $y[0]=\lim_{z\to\infty} Y(z)=\lim_{z\to\infty}\frac{z}{z^2}=\underline{0}$

(f) yes - $\lim_{z\to 1}(z-1)Y(z)=\lim_{z\to 1}\frac{z}{z-0.5}=\underline{2}$

11.13(a) $Y(z)-0.75z^{-1}Y(z)+0.125z^{-2}Y(z)=X(z)=1$

$\therefore \frac{Y(z)}{z}=\frac{z}{z^2-0.75z+0.125}=\frac{z}{(z-0.5)(z-0.25)}=\frac{2}{z-0.5}+\frac{-1}{z-0.25}$

$\therefore\ y[n]=\underline{(2(0.5)^n-(0.25)^n)u[n]}$

(c) $y[0]=\underline{1},\ y[1]=\underline{0.75},\ y[2]=\frac{2}{4}-\frac{1}{16}=\underline{0.4375}$

$y[3]=2(0.125)-\frac{1}{64}=\underline{0.2344},\ y[4]=\underline{0.1211}$

also $y[n]=0.75y[n-1]-0.125y[n-2]+x[n]$

$y[0]\ =0-0+1=\underline{1}$

$y[1]\ =0.75(1)-0+0=\underline{0.75}$

$y[2]=0.75(0.75)-0.125(1)+0=\underline{0.4375}$

$y[3]=0.75(0.4375)-0.125(0.75)=\underline{0.2344}$

$y[4]=0.75(0.2344)-0.125(0.4375)=\underline{0.1211}$

(e) $y[0]=\lim_{z\to\infty} Y(z)=1$

(f) Yes, $y[\infty]=\underline{0}$ from (a)

$y[\infty]=\lim_{z\to 1}(z-1)Y(z)=\lim_{z\to 1}\frac{z^2(z-1)}{(z-0.5)(z-0.25)}=\underline{0}$

11.13b,d next page→

11.13, continued

(b), (d)

```
n=[0 1 0];   d=[1 -.75 .125];
[r,p,k]=residue(n,d)
pause
x=[0 0 1 0 0 0 0 0];   y(1)=0;   y(2)=0;
for n=3:7
  y(n) = .75*y(n-1)-.125*y(n-2)+x(n);
end
y
```

11.14. (a) $Y(z) - z^{-1}Y(z) + 0.5z^{-2}Y(z) = X(z) = z^{-1}$

∴ $Y(z) = \frac{z}{z^2 - z + 0.5} = \frac{z}{z^2 - 2a\cos bz + a^2} = \frac{1}{a\sin b}\mathcal{Z}[a^n \sin bn]$

$a = \sqrt{0.5} = 0.707$, $\cos b = \frac{1}{2(0.707)} = 0.707$, $b = 45^0 = \frac{\pi}{4}$

∴ $y[n] = \frac{(0.707)^n}{0.707(0.707)} \sin\frac{\pi}{4}n = \underline{2(0.707)^n \sin(\frac{\pi}{4}n)\, u[n]}$

(c) $y[0] = \underline{0}$, $y[1] = \underline{1}$, $y[2] = \underline{1}$, $y[3] = \underline{0.5}$, $y[4] = \underline{0}$

also $y[n] = y[n-1] - 0.5y[n-2] + x[n]$

$y[0] = 0 - 0 + 0 = \underline{0}$

$y[1] = 0 - 0 + 1 = \underline{1}$

$y[2] = 1 - 0 + 0 = \underline{1}$

$y[3] = 1 - 0.5 + 0 = \underline{0.5}$

$y[4] = 0.5 - 0.5 + 0 = \underline{0}$

(e) $y[0] = \lim_{z\to\infty} Y(z) = \underline{0}$

(f) Yes, $y[\infty] = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}\frac{z(z-1)}{z^2 - z + 0.5} = \underline{0}$

(b), (d)

```
x=[0 0 0 1 0 0 0 0];   y(1)=0;   y(2)=0;
for n=3:7
  y(n) = y(n-1)-.5*y(n-2)+x(n);
end
```

11.15 a) $Y(z) = a z^{-1} X(z) + a z^{-1} Y(z)$

$[1 - a z^{-1}] Y(z) = a z^{-1} X(z)$

$\therefore \quad y[n] = a y[n-1] = a x[n-1]$

b) from a) $\dfrac{Y(z)}{X(z)} = \dfrac{a z^{-1}}{1 - a z^{-1}} = \dfrac{a}{z-a}$

c) pole at $z=1$ must be inside the unit circle

$\therefore \; |a| < 1$ or $-1 < a < 1$

d) $X(z) = 1$, $H(z) = \dfrac{a}{z-a} \Longrightarrow \dfrac{H(z)}{z} = \dfrac{a}{z(z-a)} = \dfrac{-1}{z} + \dfrac{1}{z-a}$

$$\therefore \; h[n] = \begin{cases} -1+1 = 0, & n = 0 \\ a^n, & n \geq 1 \end{cases} = a^n u[n-1]$$

Yes, this output is bounded only for $|a| < 1$

e) $Y(z) = H(z) X(z) = \dfrac{.5}{z-.5}\left(\dfrac{z}{z-1}\right)$

$\therefore \; \dfrac{Y(z)}{z} = \dfrac{1}{z-1} + \dfrac{-1}{z-.5} \Longrightarrow y[n] = 1 - .5^n, \; n \geq 0$

f) $\dfrac{Y(z)}{z} = \dfrac{-2}{z-1} + \dfrac{2}{z-2} \Longrightarrow y[n] = 2[1 - 2^n], \; n \geq 0$

↳ since $Y(z) = X(z) H(z)$

g)
```
n=[0 0 .5]; d=[1 -1.5 .5];
[r, p, k] =residue(n,d)
pause
n=[0 0 2]; d=[1 -3 2];
[r,p,k] =residue(n,d)
```

11.16

(a) Plugging in $\delta[n]$ fpr $x[n]$, since the input is nonzero only at $n=0$ we see that $h[n]=0$ until $n=0$; then $h[0]=\delta[0]=1$; then $h[n]=h[n-1]=1$ for $n>0$. Therefore $h[n]=u[n]$.

(b) $Y(z)=X(Z)H(Z)=\left(\frac{z}{z-0.5}\right)\left(\frac{z}{z-1}\right)=\frac{2z}{z-1}+\frac{-z}{z-0.5}$

$y[n]=2u[n]-(0.5)^n u[n]$, natural response: $y_c[n]=2u[n]$, forced response: $y_p[n]=-(0.5)^n u[n]$

(c) No, not BIBO stable because the system's pole $p=1$ lies on the unit circle. For example of a bounded input that gives unbounded output , the unit step function input has output $u[n]\sum_{k=0}^{n}u[k]=nu[n]\rightarrow\infty$.

11.17

(a) $h[n]=(0.5)^n u[n]$, $Y(z)=\frac{z}{z-0.5}\cdot\frac{z}{z-\frac{3}{2}}=\frac{-0.5z}{z-0.5}+\frac{1.5z}{z-\frac{3}{2}}$

$y[n]=-0.5(0.5)^n u[n]+1.5(1.5)^n u[n]=-(0.5)^{n+1}u[n]+(1.5)^{n+1}u[n]$, with forced response $y_p[n]=(1.5)^{n+1}u[n]$ and natural response $y_c[n]=-(0.5)^{n+1}u[n]$

(b) Bibo stable, since the pole $p=0.5$ is within the unit circle and the system is causal.

11.18

The general form of the z-transform of a system with this pole-zero diagram is $H(z) = \frac{Az^2}{(z-2)(z-3)} = \frac{k_1 z}{z-2} + \frac{k_2 z}{z-3}$ where A can be any constant (k_1, k_2 satify the partial fraction expansion).

For a DC gain of 1 we require $H(1) = 1$ which gives:

$$H(1) = \frac{A}{(-1)(-2)} = 1 \implies A = 2$$

Then

$$\begin{aligned} H(z)/z = \frac{2z}{(z-2)(z-3)} &= \tfrac{k_1}{z-2} + \tfrac{k_2}{z-3} \\ &= \tfrac{-4}{z-2} + \tfrac{6}{z-3} \\ H(z) = &\ \tfrac{-4z}{z-2} + \tfrac{6z}{z-3} \end{aligned}$$

The time functions will therefore be either:

(i) $h[n] = \left((-4)2^n + (6)3^n\right)u[n]$, with ROC $|z| > 3$

(ii) $h[n] = (-4)2^n u[n] + (-6)3^n u[-n-1]$, with ROC $2 < |z| < 3$

(iii) $h[n] = (4)2^n u[-n-1] + (-6)3^n u[-n-1]$, with ROC $|z| < 2$

(The z-transform does not exist for $h[n] = (4)2^n u[-n-1] + (-6)3^n u[n]$)

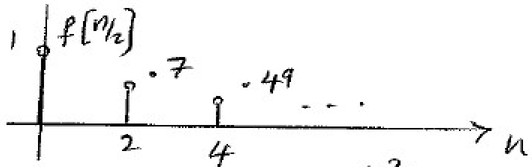
11.19

$$\mathcal{Z}\left[f[n/k]\right] = F(z^k)$$

(i) a) $F(z^2) = \frac{z^2}{z^2 - .7}$, $\therefore F(z) = \frac{z}{z - .7}$, $f[n] = (.7)^n$

$f[n/2] = (.7)^{n/2}$, $n = 0, 2, 4, \ldots$ $= 0$, otherwise

b)



c)

$$z^2 - .7 \,\overline{\smash{)}\, z^2}\quad 1 + .7z^{-2} + (.49)z^{-4} + \cdots$$

$$z^2 - .7$$

$$.7$$

$$.7 - (.49)z^{-2}$$

$$.49z^{-2}$$

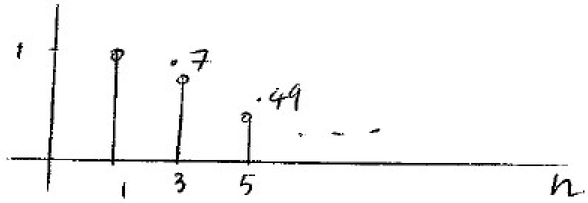
(ii) a) $\frac{z}{z^2 - .7} = F_1(z^2) = z^{-1}F(z^2) = z^{-1}\left[\frac{z^2}{z^2 - .7}\right]$

$\therefore F(z) = \frac{z}{z - .7}$, $f[n] = (.7)^n$

$\therefore f_1[n/2] = f\left[\frac{n-1}{2}\right]u[n-1] = (.7)^{\frac{n-1}{2}}u[n-1]$, $n = 1, 3, 5, \ldots$

$= 0$, otherwise

b)



c)

$$z^2 - .7 \,\overline{\smash{)}\, z}\quad z^{-1} + .7z^{-3} + .49z^{-5} + \cdots$$

$$z - .7z^{-1}$$

$$.7z^{-1} - (.49)z^{-3}$$

$$.7z^{-1}$$

$$(.49)z^{-3}$$

11.20

$$\begin{aligned} y[n] = &\quad x[n] * h[n] = (\delta[n] + 2\delta[n-1] + 3\delta[n-3]) * h[n] = h[n] + 2h[n-1] + 3h[n-3] \\ \sum_{n=-\infty}^{\infty} y[n] = &\quad \textstyle\sum_{n=-\infty}^{\infty} h[n] + \sum_{n=-\infty}^{\infty} 2h[n-1] + \sum_{n=-\infty}^{\infty} 3h[n-3] \\ = &\quad 7 + 2(7) + 3(7) = 6(7) = 42 \end{aligned}$$

11.21

a) $F(z) = \frac{z^{-9}}{z-a} = z^{-10} \frac{z}{z-a}$

$f[n] = a^{n-10} u[n-10]$

b) $F(z) = \frac{z^{-2}}{z-3} = z^{-3} \frac{z}{z-3}$

$f[n] = 3^{n-3} u[n-3]$

11.22

(a) $H(z) = \frac{z^3}{(z-1.1)^3}, |z| > 1.1$

(b) $H(z) = \frac{z^4}{(z-1.1)^3}, |z| < 1.1$

(c) $H(z) = \frac{z^4}{(z-0.9)^3}, |z| < 0.9$

(d) $H(z) = \frac{z^3}{(z-0.9)^3}, |z| > 0.9$

11.23

(a) (i) Not stable: pole on unit circle

(ii) Not stable: pole 2 outside unit circle

(iii) Not stable pole -2 outside unit circle

(iv) $z^3 - 1.6z^2 + 0.64z = z(z - 0.8)^2$ so there are poles at 0,0.8,0.8 $\Longrightarrow$ stable—all poles are within the unit circle

(v) $z^3 - 2z + 0.99z = z(z - .9)(z - 1.1)$ (which can be gotten from MATLAB using 'roots' or 'residue'). Unstable since $1.1 > 1$.

(b) For (i), there is a $Au[n]$ term in the impulse response so $u[n]$ will have the unbounded output $u[n] * Au[n] = Au[n]\sum_{k=0}^{n} u[k] = Anu[n]$. For (ii), (iii), and (v), there is an unbounded term in the natural response (due to the pole outside the unit circle) so for example $\delta[n]$ produces an unbounded output (the impulse responses are not bounded). Another example is $u[n]$.

(c)

(i) For $x[n] = u[n]$:

$$\begin{aligned} Y(z) = H(Z)X(z) = & \quad \tfrac{4(z-2)}{(z-1)(z-0.8)} \cdot \tfrac{z}{z-1} \\ = & \quad \tfrac{-20z}{(z-1)^2} + \tfrac{120z}{z-1} + \tfrac{-120z}{z-.8} \\ y[n] = & \quad -20nu[n] + 120u[n] - 120(0.8)^n u[n] \end{aligned}$$

(the partial fraction expansion can be done in MATLAB using `[r,p,k]=residue([4,-8],poly([1,1,0.8]))`). The unbounded term is $-20nu[n]$.

Continued→

11.23 (c), continued

(ii) For $x[n] = \delta[n]$:

$$\begin{aligned} Y(z) = H(z) = &\quad z^{-1}\frac{3(z+0.8)z}{z(z-0.8)(z-2)} \\ = &\quad z^{-1}\left(\frac{3.5z}{z-2} + \frac{-5z}{z-0.8} + \frac{1.5z}{z}\right) \\ y[n] = &\quad 3.5(2)^{n-1}u[n-1] - 5(0.8)^{n-1}u[n-1] + 1.5\delta[n-1] \end{aligned}$$

The partial fraction expansion of $H(z)$ we found in MATLAB using

`[r,p,k]=residue([0,0,3,2.4],poly([0,0.8,2]))`.

The unbounded term is $3.5(2)^{n-1}u[n-1]$.

(iii) For $x[n] = \delta[n]$

$$\begin{aligned} Y(Z) = H(Z) = &\quad z^{-1}\frac{3(z-0.8)z}{z(z+0.8)(z+2)} \\ = &\quad z^{-1}\left(\frac{-3.5z}{z+2} + \frac{5z}{z+.8} - \frac{1.5z}{z}\right) \\ y[n] = &\quad -3.5(-2)^{n-1}u[n-1] + 5(-0.8)^{n-1}u[n-1] - 1.5\delta[n-1] \end{aligned}$$

The partial fraction expansion of $H(z)$ we found in MATLAB using

`[r,p,k]=residue([0,0,3,-2.4],poly([0,-0.8,-2]))`.

The unbounded term is $-3.5(-2)^{n-1}u[n-1]$.

(v) For $x[n] = \delta[n]$

$$\begin{aligned} Y(z) = H(z) = &\quad z^{-1}\frac{(2z-1.5)z}{z^3-2z^2+0.99z} \\ = &\quad z^{-1}\left(\frac{3.1818z}{z-1.1} + \frac{-1.6667z}{z-.9} + \frac{1.5151z}{z}\right) \\ y[n] = &\quad 3.1818(1.1)^{n-1}u[n-1] - 1.66(0.9)^{n-1}u[n-1] + 1.52\delta[n-1] \end{aligned}$$

(got partial fractions using `[r,p,k]=residue([2,-1.5],[1,-2,0.99,0])`. The unbounded term is $3.1818(1.1)^{n-1}u[n-1]$.

11.24

(a) Poles are at $z = \pm 1$, zeros at $z = 0$. Bandstop, unstable.

(b) Poles are at $z = \pm 0.9j$, zeros at $z = 0$, bandpass, stable.

(c) Pole at $z = -1.1$, zero at $z = 0$, highpass, unstable.

(d) $\frac{z^2}{z^2-4.25z+1} = \frac{z^2}{(z-4)(z-1/4)}$, poles at $z = 4, z = 1/4$, zeros at $z = 0$, lowpass, unstable.

11.25

$$f[n] = a^n u[n] - b^{2n} u[-n-1]$$

a) $$F(z) = \underset{\substack{\uparrow \\ |z|>|a|}}{\frac{z}{z-a}} + \underset{\substack{\uparrow \\ |z|<|b^2|}}{\frac{z}{z-b^2}}$$

$$\therefore \quad |a| < |b^2|$$

b) $\frac{z}{z-a} + \frac{z}{z-b^2}$, $|a| < |z| < |b^2|$ or $|a| < |z| < b^2$

11.26

(a) $F(z) = \sum_{k=-\infty}^{\infty} 0.7^n u[n] z^{-n} = \sum_{k=0}^{\infty} 0.7^n z^{-n} = \frac{1}{1-0.7/z} = \frac{z}{z-0.7}$. ROC $|z| > 0.7$.

(b) $F(z) = \sum_{k=-\infty}^{\infty} 0.7^n u[n-7] z^{-n} = \sum_{k=7}^{\infty} 0.7^n z^{-n} = \frac{(\frac{0.7}{z})^7}{1-\frac{0.7}{z}} = 0.7^7 \frac{z^{-6}}{z-0.7}$. ROC $|z| > 0.7$.

(c) $F(z) = \sum_{k=-\infty}^{\infty} 0.7^n u[n+7] z^{-n} = \sum_{k=-7}^{\infty} 0.7^n z^{-n} = \sum_{k=0}^{\infty} 0.7^{n-7} z^{-(n-7)} = (\frac{0.7}{z})^{-7} \frac{z}{z-0.7} = 0.7^{-7} \frac{z^8}{z-0.7}$. ROC $|z| > 0.7$

(d) $F(z) = \sum_{k=-\infty}^{\infty} -0.7^n u[-n-1] z^{-n} = \sum_{k=-\infty}^{-1} -0.7^n z^{-n} = \sum_{k=1}^{\infty} -0.7^{-n} z^n = -\frac{z/0.7}{1-z/0.7} = \frac{z}{z-0.7}$, ROC $|z| < 0.7$

(e) $F(z) = \sum_{k=-\infty}^{\infty} (0.7)^{-n} u[n+7] z^{-n} = \sum_{k=-7}^{\infty} (0.7z)^{-n} = \sum_{k=0}^{\infty} (0.7z)^{-(n-7)} = (0.7z)^7 \frac{1}{1-(0.7z)^{-1}} = (0.7z)^7 \frac{z}{z-\frac{1}{.7}} = \frac{(0.7z)^8}{0.7z-1}$, ROC $|z| > \frac{1}{.7}$.

(f) $F(z) = \sum_{k=-\infty}^{\infty} (0.7)^n u[-n] z^{-n} = \sum_{k=0}^{\infty} 0.7^{-n} z^n = \frac{1}{1-\frac{z}{0.7}} = \frac{-0.7}{z-0.7}$, ROC $|z| < .7$.

11.27 $F_b(z) = \frac{0.6z}{(z-1)(z-0.6)} = z\left(\frac{3/2}{z-1} + \frac{-3/2}{z-0.6}\right)$

(a)

(i) $|z| < 0.6$: both leftsided, $f_b[n] = (3/2)\left(-u[-n-1] + 0.6^n u[-n-1]\right)$

(ii) $|z| > 1$: both rightsided, $f_b[n] = (3/2)\left(u[n] - 0.6^n u[n]\right)$

(iii) $0.6 < |z| < 1$: pole 1 term leftsided, pole 0.6 term rightsided,
$f_b[n] = (3/2)\left(-u[-n-1] - 0.6^n u[n]\right)$

(b)

(i) $f_b[\infty] = 0$

(ii) $f_b[\infty] = 3/2$

(iii) $f_b[\infty] = 0$

11.28

a) $F_b(z) = (1/2)^{-10} z^{10} + (1/2)^{-9} z^{9} + \cdots + 1 + (1/2) z + \cdots + (1/2)^{20} z^{20}$

$= (1/2\, z^{-1})^{-10} + (1/2\, z^{-1})^{-9} + \cdots + (1/2\, z^{-1})^{20}$

since: $\sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1} - a^{n_2+1}}{1-a}$

$\therefore F_b(z) = \frac{(1/2\, z^{-1})^{-10} - (1/2\, z^{-1})^{21}}{1 - 1/2\, z^{-1}}$

b) $(1/2)^{-10} z^{+10} + \cdots + (1/2)^{20} \frac{1}{z^{20}}$, $\therefore$ ROC: $|z| \neq 0$

c) $f_1[n] = (1/2)^n$, $-10 \leqslant n \leqslant 10$

from a), $F_{b_1}(z) = \frac{(1/2\, z^{-1})^{-10} - (1/2\, z^{-1})^{11}}{1 - 1/2\, z^{-1}}$, $|z| \neq 0$

$f_2[n] = (1/4)^n u[n-21] = (1/4)^{21} (1/4)^{n-21} u[n-21]$

$F_{b_2}(z) = (1/4)^{21} z^{-21} \frac{z}{z - 1/4} = \frac{(1/4)^{21}}{z^{20}(z - 1/4)}$, $|z| > 1/4$

$\therefore F_b(z) = F_{b_1}(z) + F_{b_2}(z)$, $|z| > 1/4$

d) $f_1[n] = (1/2)^n$, $-10 \leqslant n \leqslant 0$

$F_{b_1}(z) = 1 + (1/2)^{-1} z + (1/2)^{-10} z^{10} = 1 + (2z) + (2z)^2 + \cdots + (2z)^{10}$

$= \frac{1 - (2z)^{11}}{1 - 2z}$

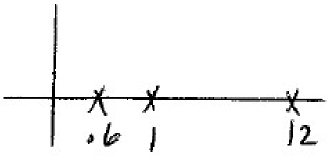
$f_2[n] = (1/4)^n$, $1 \leqslant n \leqslant 10$

$F_{b_2}(z) = \left(\frac{1}{4z}\right) + \left(\frac{1}{4z}\right)^2 + \cdots + \left(\frac{1}{4z}\right)^{10} = \frac{\frac{1}{4z} - \left(\frac{1}{4z}\right)^{11}}{1 - \frac{1}{4z}}$, $z \neq 0$

$\therefore F_b(z) = F_{b_1}(z) + F_{b_2}(z)$, $z \neq 0$

11.29

$$F(z) = \frac{3z}{z-1} + \frac{z}{z-12} - \frac{z}{z-.6}$$



a) $|z| < .6$, $.6 < |z| < 1$, $1 < |z| < 12$, $|z| > 12$

b) $|z| < .6$, $f[n] = -3u[-n-1] - (12)^n u[-n-1] + (.6)^n u[-n-1]$

$.6 < |z| < 1$, $f[n] = -(.6)^n u[n] - 3u[-n-1] - (12)^n u[-n-1]$

$1 < |z| < 12$, $f[n] = -(.6)^n u[n] + 3u(n] - (12)^n u[-n-1]$

$|z| > 12$, $f[n] = -(.6)^n u(n] + 3u[n] + (12)^n u[n]$

11.30

a) $Y_m(z) = Y(z^m)$

b) $X_m(z) = X(z^m)$

$H_m(z) = H(z^m)$

$\therefore\ Z[x_m[n] * h_m[n]) = X(z^m)\,H(z^m)$

Chapter 12 solutions

12.1 (a)

(i) $f(nT_s) = 8\cos[2\pi(.1n)] + 4\sin[4\pi(.1n)]$

$$f[n] = 8\cos[.2\pi n] + 4\sin[.4\pi n]$$

$$F(\Omega) = 8\pi\sum_{k=-\infty}^{\infty}\left[\delta(\Omega - .2\pi - 2\pi k) + \delta(\Omega + .2\pi - 2\pi k)\right]$$

$$- j4\pi\sum_{k=-\infty}^{\infty}\left[\delta(\Omega - .4\pi - 2\pi k) - \delta(\Omega + .4\pi - 2\pi k)\right]$$

(ii) $g[n] = 4\cos[.5\pi n]u[n]$

$$4\cos[.5\pi n] \longleftrightarrow 4\pi\sum_{k=-\infty}^{\infty}\left[\delta(\Omega - .5\pi - 2\pi k) + \delta(\Omega + .5\pi - 2\pi k)\right]$$

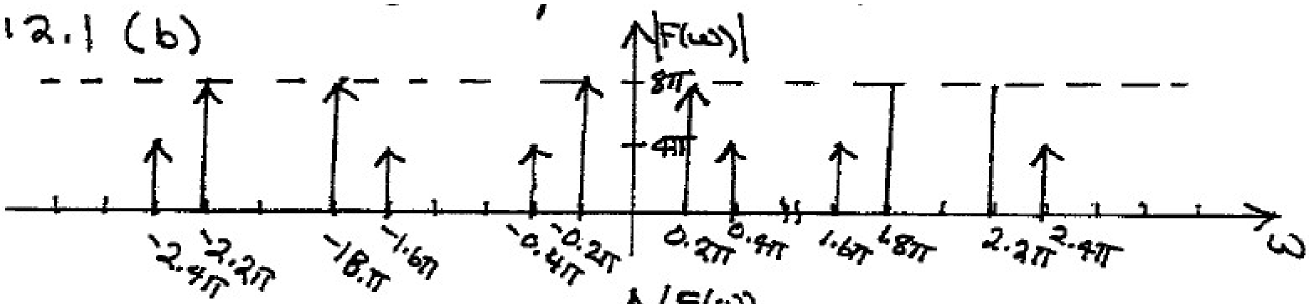
$$u[n] \longleftrightarrow \frac{1}{1 - e^{-j\Omega}} + \sum_{k=-\infty}^{\infty}\pi\delta(\Omega - 2\pi k)$$

$$x[n]y[n] \longleftrightarrow \frac{1}{2\pi}X(\Omega) * Y(\Omega)$$

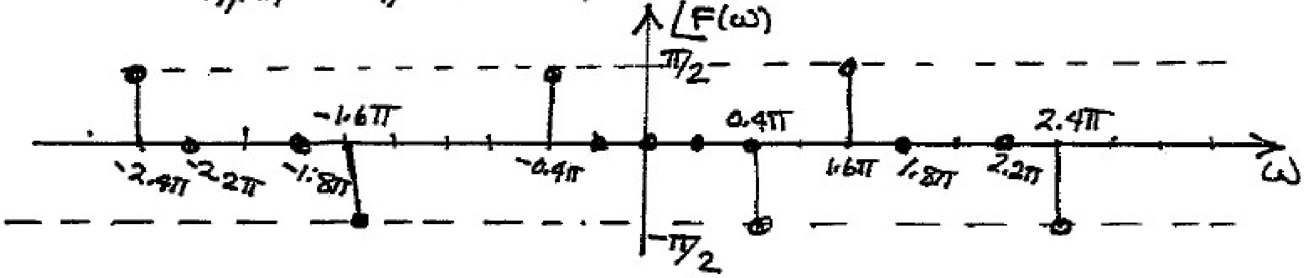
$$G(\Omega) = \frac{4e^{j2\Omega}}{1 + e^{j2\Omega}} + 2\pi\sum_{k=-\infty}^{\infty}\left[\delta(\Omega - .5\pi - 2\pi k) + \delta(\Omega + .5\pi - 2\pi k)\right]$$

part b) next page

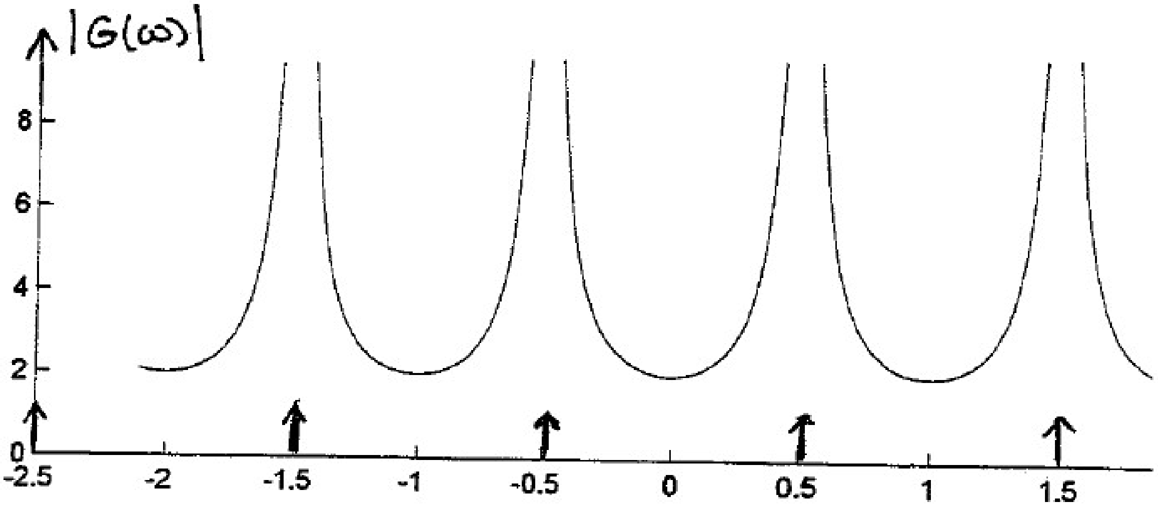
12.1 (b)



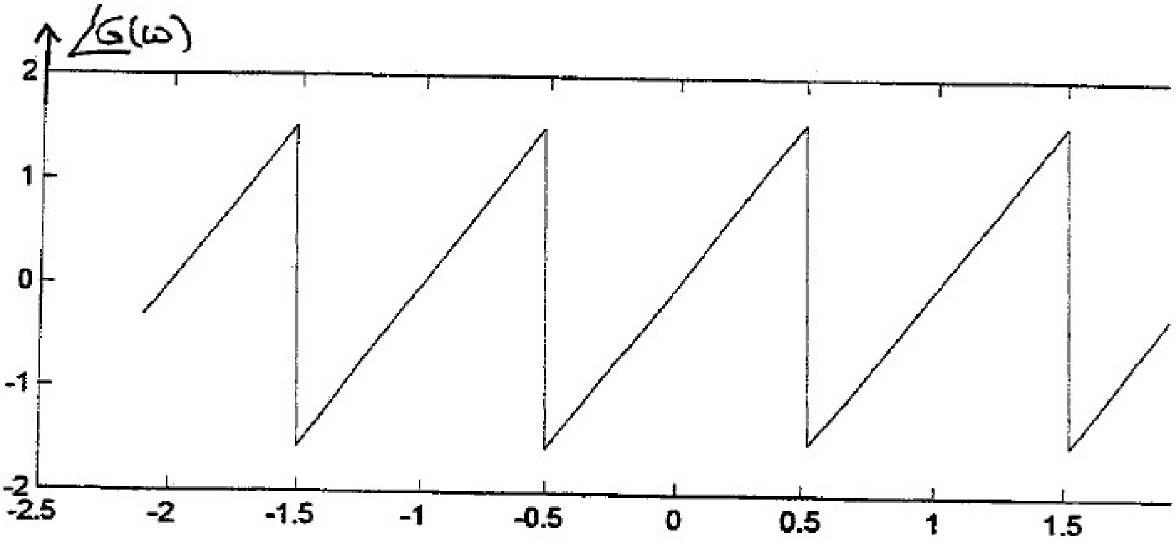
|F(ω)|
8π
4π
-2.4π
-2.2π
-1.8π
-1.6π
-0.4π
-0.2π
0.2π
0.4π
1.6π
1.8π
2.2π
2.4π
ω



∠F(ω)
π/2
-π/2
-1.6π
0.4π
2.4π
-2.4π
-2.2π
-1.8π
-0.4π
1.6π
1.8π
2.2π
ω



|G(ω)|
8
6
4
2
0
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5



∠G(ω)
2
1
0
-1
-2
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5

12.2 (a)

$$x[n] = \begin{cases} (.5)^n, & n \geq 0 \\ 0, & n < 0 \end{cases} ; \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega}$$

$$X(\Omega) = \sum_{n=0}^{\infty} (.5)^n e^{-jn\Omega} = \underbrace{1 + .5e^{-j\Omega} + (.5e^{-j\Omega})^2}_{\text{geometric series}} + (.5e^{-j\Omega})^3 + \cdots$$

$$X(\Omega) = \frac{1}{1 - .5e^{-j\Omega}}$$

(b) $y[n] = n(.5)^n u[n] \xleftrightarrow{DTFT} Y(\Omega) = \sum_{n=0}^{\infty} n(.5)^n e^{-jn\Omega}$

From TABLE 12.1 $\quad Y(\Omega) = \dfrac{.5e^{j\Omega}}{(e^{j\Omega} - .5)^2}$

(c) $v[n] = 2\,[u[n] - u[n-5]]$

$$V(\Omega) = \sum_{n=0}^{4} 2e^{-jn\Omega} = 2\left[1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega} + e^{-j4\Omega}\right]$$

$$= 2e^{-j2\Omega}\left[e^{j2\Omega} + e^{j\Omega} + 1 + e^{j\Omega} + e^{-j2\Omega}\right]$$

$$= 2e^{-j2\Omega}\left[1 + 2\cos\Omega + 2\cos 2\Omega\right]$$

or from TABLE 12.1 : $V(\omega) = 2\,\dfrac{\sin\left(\frac{5\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} e^{-j2\Omega}$

(WITH TIME-SHIFT PROPERTY)

(d) $w[n] = \text{rect}(n/4) + \text{rect}(n/10)$

$$W(\Omega) = \sum_{n=-5}^{5} 1e^{-jn\Omega} + \sum_{n=-2}^{2} 1e^{-jn\Omega} =$$

$$W(\Omega) = e^{j5\Omega} + e^{j4\Omega} + e^{j3\Omega} + 2e^{j2\Omega} + 2e^{j\Omega} + 2 + 2e^{-j\Omega} + 2e^{-j2\Omega} + e^{-j3\Omega} + e^{-j4\Omega} + e^{-j5\Omega}$$

or FROM TABLE 12.1

$$W(\Omega) = \frac{\sin\left(\frac{5\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} + \frac{\sin\left(\frac{11\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} \quad \text{or}$$

$$W(\Omega) = 2\cos 5\Omega + 2\cos 4\Omega + 2\cos 3\Omega + 4\cos 2\Omega + 4\cos\Omega + 2$$

12.3

Need to show that $\mathcal{DF}[ax_1[n] + bx_2[n]] = a\left(\mathcal{DF}[x_1[n]]\right) + b\left(\mathcal{DF}[x_2[n]]\right)$, where a, b are any constants and $x_1[n]$, $x_2[n]$ are two length-N signals.

$$\begin{aligned}
\mathcal{DF}[ax_1[n] + bx_2[n]] = & \quad \textstyle\sum_{n=0}^{N-1}(ax_1[n] + bx_2[n])e^{-j2\pi\frac{nk}{N}} = \sum_{n=0}^{N-1} ax_1[n]e^{-j2\pi\frac{nk}{N}} + bx_2[n]e^{-j2\pi\frac{nk}{N}} \\
= & \quad a\textstyle\sum_{n=0}^{N-1} x_1[n]e^{-j2\pi\frac{nk}{N}} + b\sum_{n=0}^{N-1} x_2[n]e^{-j2\pi\frac{nk}{N}} \\
= & \quad a\left(\mathcal{DF}[x_1[n]]\right) + b\left(\mathcal{DF}[x_2[n]]\right)
\end{aligned}$$

12.4 $\quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\Omega}, \quad \frac{d}{d\Omega}X(\Omega) = \frac{d}{d\Omega}\sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega}$

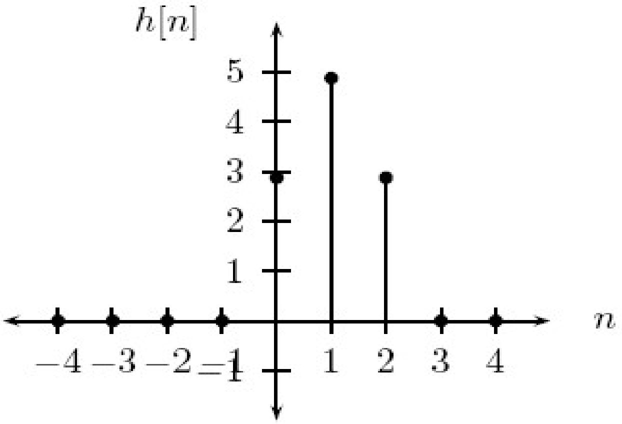
$$\frac{d}{d\Omega}X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\,\frac{d}{d\Omega}e^{-jn\Omega} = \sum_{n=-\infty}^{\infty} -jn\,x[n]\,e^{-jn\Omega}$$

$$j\frac{d}{d\Omega}X(\Omega) = \sum_{n=-\infty}^{\infty} (j)(-j)\,n\,x[n]\,e^{-jn\Omega} = \sum_{n=-\infty}^{\infty} n\,x[n]\,e^{-jn\Omega}$$

$$= DTF\{n x[n]\}$$

12.5

(a) Plugging $\delta[n]$ in for $x[n]$ gives: $h[n] = 3\delta[n] + 5\delta[n-1] + 3\delta[n-2]$.



(b) $H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} = 3 + 5e^{-j\Omega} + 3e^{-j2\Omega}$

(c) Yes linear phase: $h[n] = h[M-1-n]$ where in this case $M = 3$ ($h[0] = h[2]$),

$h[n] = e^{-j\Omega}(3e^{j\Omega} + 5 + 3e^{-j\Omega}) = e^{-j\Omega}(6\cos(\Omega) + 5)$

phase$=-\Omega$

12.6

(a) No this is an IIR filter with impulse response $h_1[n] = 0.7^n u[n]$ or $h_1[n] = -(0.7)^n u[-n-1]$

(b) Yes linear phase since $h_2[n] = h_2[M-1-n]$:

$$\begin{aligned} H_2(\Omega) = &\quad e^{-\frac{3}{2}j\Omega}(e^{\frac{3}{2}j\Omega} + e^{-\frac{3}{2}j\Omega}) + 3e^{-\frac{3}{2}\Omega}(e^{j\frac{1}{2}\Omega} + e^{-j\frac{1}{2}\Omega}) \\ = &\quad 2e^{-\frac{3}{2}j\Omega}(\cos(\tfrac{3}{2}\Omega) + 3\cos(\tfrac{1}{2}\Omega)) \\ phase = &\quad -\tfrac{3}{2}\Omega \end{aligned}$$

(c) Yes linear phase since $h_3[n] = h_3[M-1-n]$:

$$\begin{aligned} H_3(\Omega) = &\quad 2(e^{2j\Omega} + e^{-2j\Omega}) + 3(e^{j\Omega} + e^{-j\Omega}) + 7 \\ = &\quad 2(7 + 3\cos(\Omega) + 2\cos(2\Omega)) \\ phase = &\quad 0 \end{aligned}$$

(d) No symmetry conditions satisfied $\Longrightarrow$ nonlinear phase.

12.7 $X_0(\Omega) = 1 + e^{-2j\Omega} + e^{-4j\Omega}$

$$X(\Omega) = \frac{2\pi}{5} \sum_{k=-\infty}^{\infty} X_0\left(\frac{2\pi k}{5}\right) \delta\left(\Omega - \frac{2\pi k}{5}\right)$$

$$X_0(\Omega) = e^{-2j\Omega}\left(e^{j2\Omega} + 1 + e^{-2j\Omega}\right)$$

$$= e^{-2j\Omega}(1 + 2\cos 2\Omega) \quad \therefore \quad \angle X_0(\Omega) = -2\Omega$$

12.8 $y[n] = x[n/3]$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x[n/3] e^{-j\Omega n} \qquad \text{let } l = n/3$$

$$Y(\Omega) = \sum_{l=-\infty}^{\infty} x[l] e^{-j\Omega 3l} = X(3\Omega)$$

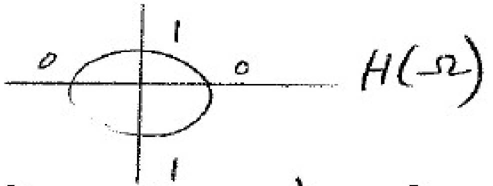
12.9 $x_0[n] = \text{idft } [4 \ 0 \ 4 \ 0]$

since $X[k] = X_0\left(\frac{2\pi k}{4}\right)$ for $N=4$

$$x_0[n] = 1/4\left(4 + 4e^{j\pi n}\right)$$

$$x_0[n] = [2 \ 0 \ 2 \ 0]$$

12.10



let $H[k] = H\left(\frac{2\pi k}{4}\right) = [0\ 1\ 0\ 1]$

$h[n]$ is simply IDFT of $H[k]$

$$h[n] = 1/4 \sum_{k=0}^{3} H[k] W_4^{-nk} = \frac{1}{4}\left[e^{\frac{j2\pi n}{4}} + e^{\frac{j6\pi n}{4}}\right]$$

$$h[n] = [1/2, 0, -1/2, 0]$$

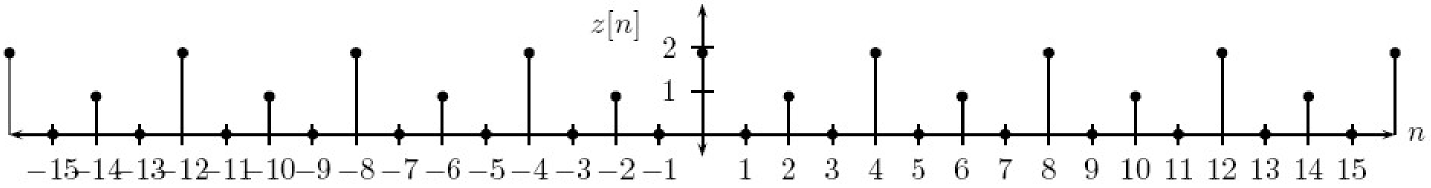
12.11

$$\begin{aligned} x[n] = &\quad \frac{1}{2\pi}\int_{\Omega=0}^{2\pi} X(\Omega) = \frac{1}{4}\int_{\Omega=0}^{2\pi}\left(6\delta(\Omega-\frac{2\pi}{4}) + 6\delta(\Omega-\frac{6\pi}{4})\right)e^{jn\Omega}d\Omega \\ = &\quad \frac{1}{4}\left(6e^{jn\frac{\pi}{2}} + 6e^{jn\frac{3\pi}{2}}\right) = 3/2e^{jn\frac{\pi}{2}} + 3/2e^{jn\frac{3\pi}{2}} \end{aligned}$$

This gives:

$$\begin{aligned} x[n] = &\quad 3, n = 0, 4, 8, ... \\ = &\quad 0, n = 1, 5, 9, ... \\ = &\quad -3, n = 2, 6, 10, ... \\ = &\quad 0, n = 3, 7, 11, ... \end{aligned}$$

12.12

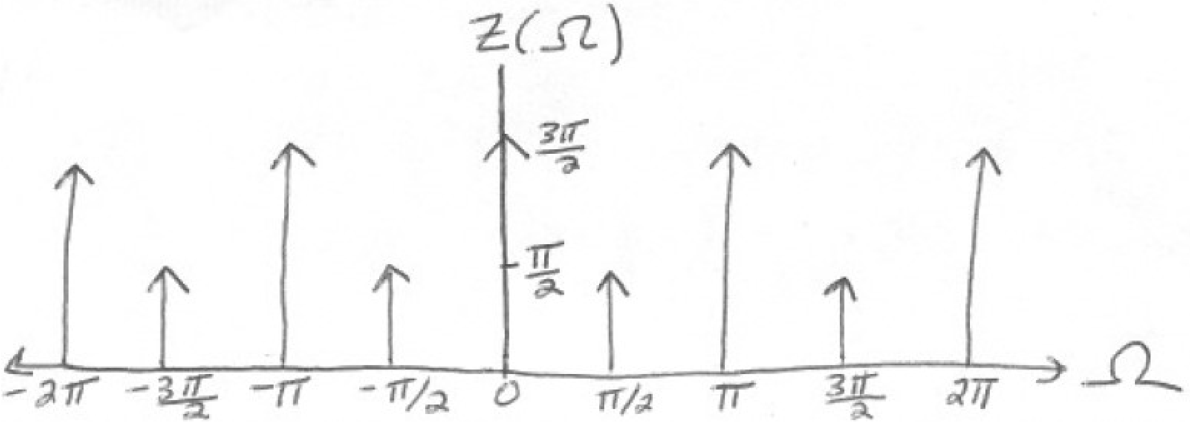


First consider the signal $z_0[n] = z[n]$ over $0 \leq n \leq 3$ and $z_0[n] = 0$ elsewhere. Then:

$$\begin{aligned} Z_0(\Omega) = & \quad \textstyle\sum_{n=0}^{3} z[n]e^{-jn\Omega} = 2e^{-j0\Omega} + 1e^{-j2\Omega} = 2 + e^{-j2\Omega} \\ Z(\Omega) = & \quad \tfrac{2\pi}{4}\textstyle\sum_{k=-\infty}^{\infty} Z_0(\tfrac{2\pi k}{4})\delta(\Omega - k\tfrac{2\pi}{4}) \\ = & \quad \tfrac{\pi}{2}\textstyle\sum_{k=-\infty}^{\infty}(2 + e^{-jk\pi})\delta(\Omega - k\tfrac{\pi}{2}) \end{aligned}$$

Note that $2 + e^{-jk\pi} = 1$ if k odd and $2 + e^{-jk\pi} = 3$ if k even. Therefore:

$$Z(\Omega) = \sum_{k=-\infty}^{\infty} \frac{\pi}{2}\delta(\Omega - (2k+1)\frac{\pi}{2}) + \frac{3\pi}{2}\delta(\Omega - (2k)\frac{\pi}{2})$$



12.13
(a)

, $T_s = 2\text{ms}$

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4}nk}, \quad k=0,1,2,3$$

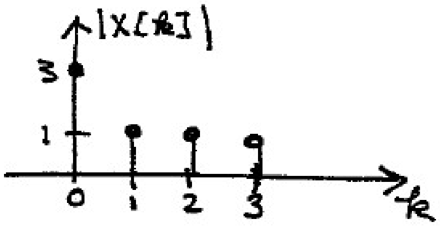
$$X[0] = 1+1+1+0 = 3$$

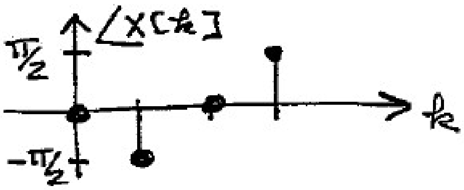
$$X[1] = 1+1e^{-j2\pi/4}+e^{-j\pi}+0 = e^{-j\pi/2} = -j1$$

$$X[2] = 1+e^{-j\pi}+e^{-j2\pi}+0 = 1$$

$$X[3] = 1+e^{-j3\pi/2}+e^{-j3\pi}+0 = e^{-j3\pi/2} = j1$$

$$\therefore X[k] = [3, -j1, 1, j1]$$





(b) MATLAB

```
>> x = [1 1 1 1 1 0 0 0];
>> X = fft(x);
>> stem(abs(X))
>> stem (angle(X))
```

12.13 c) Matlab problem

```
>> x = [1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0];
>> X = fft(x,16);
>> plot (abs(X))
>> plot (angle(X))
```

12.14

(a) Note that $x[n] = (0.5)^n$ over $n = 0, ..., 8$:

$$\begin{aligned} X[k] = & \sum_{n=0}^{7} x[n]e^{-j2\pi\frac{nk}{8}} = \sum_{n=0}^{7} 0.5^n e^{-j2\pi\frac{nk}{8}} \\ = & \frac{1-(0.5e^{-j2\pi\frac{k}{8}})^8}{1-0.5e^{-j2\pi\frac{k}{8}}}, k = 0, ..., 7 \end{aligned}$$

using the formula $\sum_{n=0}^{M-1} r^n = \frac{1-r^M}{1-r}, r \neq 1$, where in this case $r = 0.5e^{-j2\pi\frac{k}{8}}$.

(b)

```
>>xn=0.5.^[0:7];
>>Xk=fft(xn)
Xk= 1.9922      1.1861 - 0.6487i      0.7969 - 0.3984i      0.6889 - 0.1799i      0.6641      0.6889
+ 0.1799i      0.7969 + 0.3984i      1.1861 + 0.6487i
>>Xk=(1-(0.5*exp(-j*2*pi*[0:7]/8)).^8)./(1-0.5*exp(-j*2*pi*[0:7]/8))
Xk=1.9922      1.1861 - 0.6487i      0.7969 - 0.3984i      0.6889 - 0.1799i      0.6641 - 0.0000i
0.6889 + 0.1799i      0.7969 + 0.3984i      1.1861 + 0.6487i
```

(c)

$$\begin{aligned} X[k] = & \sum_{n=0}^{7} n(0.5)^n e^{-j2\pi\frac{nk}{8}} = \sum_{n=0}^{7} n(0.5e^{-j2\pi\frac{k}{8}})^n \\ = & a\frac{1-a^8-(8)a^7(1-a)}{(1-a)^2} \end{aligned}$$

where $a = 0.5e^{-j2\pi\frac{k}{8}}$. This comes from the formula $\sum_{k=0}^{n} ka^k = a\frac{d}{da}\sum_{k=0}^{n} a^k = a\frac{d}{da}\left(\frac{1-a^{n+1}}{1-a}\right)$.

(d)

```
>>xn=[0:7].*(0.5).^[0:7];
>>Xk=fft(xn)
Xk=1.9297   -0.2334 - 0.8758i    -0.3438 - 0.2266i    -0.2666 - 0.0633i    -0.2422   -0.2666

+ 0.0633i    -0.3438 + 0.2266i    -0.2334 + 0.8758i
>>a=0.5*exp(-j*2*pi*[0:7]/8);
>>Xk=a.*((1-a.^8)-8*a.^7.*(1-a))./(1-a).^2
Xk= 1.9297    -0.2334 - 0.8758i    -0.3438 - 0.2266i    -0.2666 - 0.0633i    -0.2422 - 0.0000i
-0.2666 + 0.0633i    -0.3438 + 0.2266i    -0.2334 + 0.8758i
```

12.15 a) $x[n] = [5.0, -4.05, 1.55, 1.55, -4.05, 5.0, -4.05, 1.55]$

$$X[k] = \sum_{k=0}^{7} x[n] e^{-j\frac{\pi}{4}nk}, \quad k = 0, 1, \cdots, 7$$

$$X[k] = [2.5, 2.65+j.81, 3.45+j2.14, 15.44+j11.98, -5.60, 15.44-j11.98, 3.45-j2.14, 2.65-j.81]$$

12.15(b) MATLAB

```
>> for n=1:8
   x(n)= 5*cos((n-1)*8*pi/10);
   end
>> x
>> X = ff(t)(x,8);
>> X
>> for n=1:8
   w(n) =(n-1)*2*pi*10/8;
   end
>> stem(w,abs(X));
>> stem(w,angle(X));
```

(c) $X(\omega) = \mathcal{F}\{5\cos(8\pi t)\} = 5\pi[\delta(\omega-8\pi)+\delta(\omega+8\pi)]$

$X(\omega) = 5\pi[\delta(\omega-25.137)+\delta(\omega+25.137)]$

It is seen that the DFT can be used to approximate the Fourier transform. The DFT results in this problem exhibit spectrum spreading.

12.16 The hanning window is given by eq. (12.58)

$$han[n] = [0, 0.1883, 0.6113, 0.9505, 0.9505, 0.6113, 0.1883, 0]$$

$$x_2[n] = han[n] * x[n] = [0, -0.7615, 0.9445, 1.4686, -3.8446, 3.0563, -0.7615, 0]$$

$$X_2[k] = [0.1015,\ 0.1068 - j0.0448,\ -4.0278 - j0.0826,\ 7.5829 + j3.3671 \\ -7.4252,\ 7.5829 - j3.3671,\ -4.0278 + j0.0826,\ 0.1068 + j0.0448]$$

THE FREQUENCY COMPONENTS OF $X_2[k]$ ARE AT

$$\omega[k] = \frac{2\pi k}{NT} = 2.5\pi k \quad , \quad k = 0, 1, \cdots 7$$

NOTICE THAT THERE IS A LARGE COMPONENT AT $k = 4$ OR $\omega[k] = 10\pi$ (rad/s) $= \omega_s/2$ BECAUSE OF SPECTRUM SPREADING – HOWEVER IT IS LESS THAN FOUND IN P 12.15.

The Hanning window generated by the "hanning(8)" command in MATLAB differs from that given by eq. (12.58). However, the result of using the MATLAB function is similar to the calculated results.

```
>> (generator "x" as in problem 12.15 (b))
>> xh = hanning(8)'.*x;
>>  Xh = fft(xh,8);
>> generate "W" as in problem 12.15 (b)
>> stem(w,abs(Xh)), axis([0,60,0,20])
```

12.17 $A[k]=\sum_{n=0}^{N-1}\left[\frac{1}{2}\left(e^{\frac{j2\pi kn}{N}}+e^{\frac{-j2\pi kn}{N}}\right)\right]\left[\frac{1}{2}\left(e^{\frac{j2\pi pn}{N}}+e^{\frac{-j2\pi pn}{N}}\right)\right]$

by orthogonality of exponentials,

$$=\frac{1}{4}N\delta[k+p]+\frac{1}{4}N\delta[k-p]+\frac{1}{4}N\delta[k-p]+\frac{1}{4}N\delta[k+p]$$

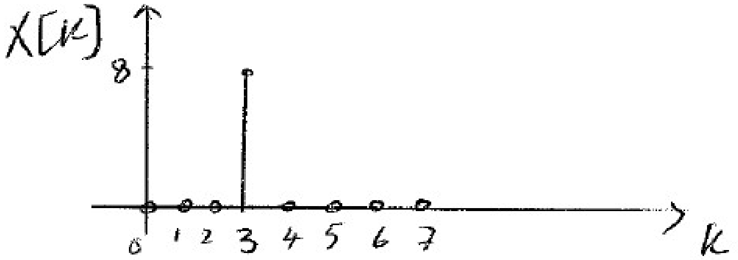
$$=\frac{N}{2}\left[\delta[k+p)+\delta[k-p]\right)$$

12.18

$x[n]=e^{\frac{j6\pi n}{8}}$, $N=8$

$$X[k]=\sum_{n=0}^{7}e^{\frac{j6\pi n}{8}}e^{\frac{-j2\pi nk}{8}}=\sum_{n=0}^{7}e^{\frac{j2\pi n(3-k)}{8}}$$

$=8\delta(k-3]$ by orthogonality of exponentials



12.19

(a) A ($x[n]$ has single frequency $-3/8$ which is equivalent to frequency $(8-3)/8$)

(b) C ($x[n]$ has single DC frequency)

(c) D ($x[n]$ has single frequency at 3/8;)

(d) B ($X[k]=\sum_{n=0}^{7}\delta[n]e^{-j2\pi\frac{kn}{8}}=1$ for all k)

12.20

$y[n]=x[n+1]=x[n-3]$

$$Y[k]=X[k]e^{\frac{j2\pi k}{4}}=X[k]e^{\frac{-3j2\pi k}{4}}=W^{3k}X[k]$$

$$=W^{-k}X[k]$$

12.21

(a)

$$F(\omega) = 3.5\pi\left[\delta(\omega-140)+\delta(\omega+140)+\delta(\omega-60)+\delta(\omega+60)\right]$$

the highest frequency component is 140(rad/s)

$\therefore \omega_s > 2\times 140$ (rad/s) $\Rightarrow \omega_s > 280$ rad/s

$T_s < \frac{2\pi}{\omega_s}$ $\therefore T_s < 22.4$ (ms)

(b) To have resolution of 1 rad/sec, at ω_s=300rad/sec, need 300 samples.

12.22

a) $\Delta\Omega = \frac{2\pi}{N} = \frac{2\pi}{1024}$.

$$\Delta\omega = \frac{\Delta\Omega}{T_s} = \frac{\frac{2\pi}{1024}}{\frac{1}{1024}} = 2\pi \text{ rad/sec}$$

b) Highest frequency allowed if aliasing can not occur is

ω_{max}

$$\omega_s = \frac{2\pi}{T_s} = \frac{2\pi}{\frac{1}{1024}} = 2048\pi$$

$\omega_s > 2\times\omega_{max} \Rightarrow \omega_{max} < 1024\pi$

12.23

(a) $X[k] = e^{j2\pi\frac{0}{4}} e^{-j2\pi\frac{0\cdot k}{4}} + e^{j2\pi\frac{1}{4}} e^{-j2\pi\frac{1\cdot k}{4}} + e^{j2\pi\frac{2}{4}} e^{-j2\pi\frac{2\cdot k}{4}} + e^{j2\pi\frac{3}{4}} e^{-j2\pi\frac{3\cdot k}{4}}$

$= 1 + e^{j\frac{\pi}{2}} e^{-jk\frac{\pi}{2}} + e^{j\pi} e^{-j\pi k} + e^{j\pi\frac{3}{2}} e^{-j\pi\frac{3k}{2}}$

$= 1 + j - 1 - j = 0, k = 0$

$= 1 + 1 + 1 + 1 = 4, k = 1$

$= 1 - j - 1 + 1 + j = 0, k = 2$

=1-1+1-1=0,k=3

$= [0, 4, 0, 0]$

$= 4\delta[n-1]$

(b) $H[k] = 2e^{-j2\pi\frac{0k}{4}} + 1e^{-j2\pi\frac{2k}{4}} = 2 + e^{-j\pi k}$

$= 2 + (-1)^k$

$= 3, k = 0$

$= 1, k = 1$

$= 3, k = 2$

$= 1, k = 3$

=[3,1,3,1] or

$= 3\delta[n] + \delta[n-1] + 3\delta[n-2] + \delta[n-3]$

(c) $X[k]H[k] = 0, k = 0, 2, 3$

$= 4(1) = 4, k = 1$

(d) $x[n] \bigotimes h[n] = \mathcal{DF}^{-1}(X[k]H[k]) = \frac{1}{4}\sum_{k=0}^{3}(X[k]H[k])e^{j\frac{2\pi nk}{4}}$

$= \frac{1}{4}4e^{j2\pi\frac{1n}{4}} = e^{j\pi n/2} = (j)^n = [1, j, -1, -j].$

12.24

(a) $x[n] = [-2, -1, 0, 2]$, $y[n] = [-1, -2, -1, -3]$

$x[n] * y[n] = [-2(-1), -2(-2) - 1(-1), -2(-1) - 1(-2) + 0(-1),$

$- 2(-3) - 1(-1) + 0(-2) + 2(-1), -1(-3) + 0(-1) + 2(-2), 0(-3) + 2(-1), 2(-3)]$

$= [2, 5, 4, 5, -1, -2, -6]$

(b) $x[n] \circledast y[n] = [-2(-1) - 1(-3) + 0(-1) + 2(-2), -2(-2) - 1(-1) + 0(-3) + 2(-1),$

$- 2(-1) - 1(-2) + 0(-1) + 2(-3), -2(-3) - 1(-1) + 0(-2) + 2(-1)]$

$= [1, 3, -2, 5]$

(c) $R_{xy}[n] = \sum_{k=0}^{3} x[k]y[n+k]$. We assume that the first element in the vector is at 0, so this works out to: $R_{xy}[n] = [-2, -4, -1, -2, 5, 5, 6]$ for $n = 0, 1, 2, 3, 4, 5, 6$

(d) $R_{yx}[n] = \sum_{k=0}^{3} x[n+k]y[k]$

$R_{yx}[n] = [6, 5, 5, -2, -1, -4, -2]$ for $n = 0, 1, 2, 3, 4, 5, 6$

(e) $R_{xx}[n] = \sum_{k=0}^{3} x[n+k]x[k]$

$R_{xx}[n] = [-4, -2, 2, 9, 2, -2, -4]$ for $n = 0, 1, 2, 3, 4, 5, 6$

(f) In MATLAB:

$x = [-2, -1, 0, 2];$

$y = [-1, -2, -1, -3];$

```
% linear convolution:
conv(x,y)
% circular convolution:
Xfft=fft(x);
Yfft=fft(y);
real(ifft(Xfft.*Yfft))
% Rxy:
conv([fliplr(x),zeros(1,3)], [zeros(1,3),y])
% Ryx:
conv([fliplr(y),zeros(1,3)], [zeros(1,3),x])
% Rxx:
conv([fliplr(x),zeros(1,3)], [zeros(1,3),x])
```

(Note that the linear convolution, and the correlations, could also be done in the frequency domain using `fft`).

12.25

The extended sequences must have $4+4-1=7$ elements: we just add 3 zeros onto the end of each and perform circular convolution. $x_z[n]=[-2,-1,0,2,0,0,0]$, $y_z[n]=[-1,-2,-1,-3,0,0,0]$
$x_z[n] \circledast y_z[n]=[2,5,4,5,-1,-2,-6]$

12.26

$X[K]=[12 \quad -2-2j \quad 0 \quad -2+2j]$

$H[K]=[2.3 \quad .51-.81j \quad .68 \quad .51+.81j]$

$y[n]=x[n] \circledast h[n]$

$Y[K]=X[K]H[K]=[27.6 \quad -2.64+.6i \quad 0 \quad -2.64-.6i]$

$y[n]=ifft(Y[K])=(5.58 \quad 6.6 \quad 8.22 \quad 7.2]$

$y[2]=8.22$

12.27

(a)

$$v[n] = x[n] * y[n], \quad V[k] \neq X[k]Y[k]$$

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} X[k]e^{j2\pi kn/4}, \quad n=0,1,2,3$$

$$y[n] = \frac{1}{4}\sum_{k=0}^{3} Y[k]e^{j2\pi kn/4}, \quad n=0,1,2,3$$

$$x[n] = [2,\ 6,\ 6,\ 8], \quad y[n] = [1,\ 3,\ 3,\ 1] = y[-n]$$

$$\left.\begin{array}{ccccccc} 0 & 0 & 2 & 6 & 6 & 8 & 0 \\ 0 & 1 & 3 & 3 & 1 & 0 & 0 \\ \hline \end{array}\right\}\text{ LINEAR CONVOLUTION, } p=2$$

$$v[2] = 0+0+6+18+6+0+0 \quad = \underline{30}$$

(b)

$$W[k] = X[k]Y[k] = [176,\ 12+j4,\ 0,\ 12-j4]$$

$$w[n] = \mathcal{DF}^{-1}\{W[k]\} = \frac{1}{4}\sum_{k=0}^{3} W[k]e^{j2\pi kn/4}, \quad n=0,1,2,3$$

$$w[2] = \frac{1}{4}\sum_{k=0}^{3} W[k]e^{j\pi n} \quad = \underline{38}$$

(c)

$$R_{xy} = x[n] * y[-n], \quad R_{xy}[2] = \begin{array}{c} 0\,0\,2\,6\,6\,8\,0 \\ 1\,3\,3\,1\,0\,0\,0 \\ \hline 0+0+6+6+0+0+0 = \underline{12} \end{array}$$

(d)

$$R_{yx} = x[-n] * y[n], \quad R_{yx}[2] = \begin{array}{c} 0\,0\,1\,3\,3\,1\,0 \\ 2\,6\,6\,8\,0\,0\,0 \\ \hline 0+0+6+24+0+0+0 = \underline{36} \end{array}$$

(e)

$$R_{xx} = x[n] * x[-n]$$

$$R_{xx}[2] = \begin{array}{cccccc} 0 & 0 & 2 & 6 & 6 & 8 \\ 2 & 6 & 6 & 8 & 0 & 0 \\ \hline \multicolumn{6}{c}{0+6+12+48+0+8} \end{array} \quad = \underline{60}$$

(f)

$$S_x[k] = \frac{1}{N} X[k]X^*[k]$$

$$= \frac{1}{4}[22 \quad -4+j2 \quad -6 \quad -4-j2][-22 \quad -4-j2 \quad -6 \quad -4+j2]$$

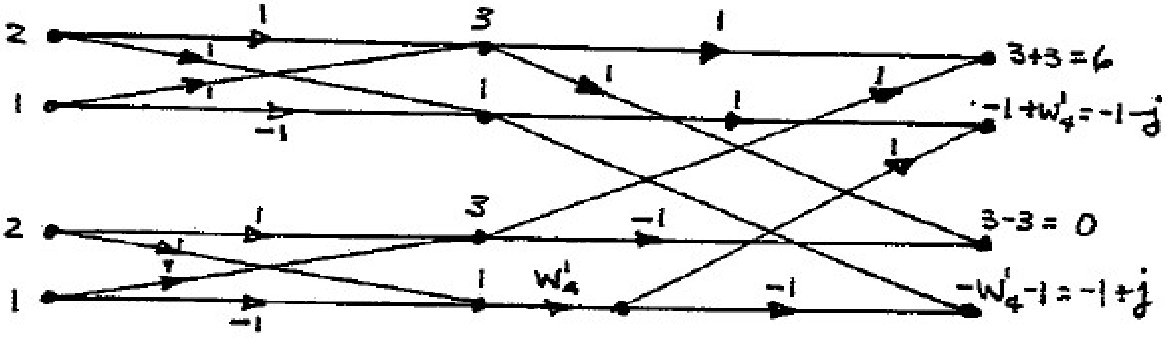
$$= \frac{1}{4}[(22)(22) \quad (-4+j2)(-4-j2) \quad (-6)(-6) \quad (-4-j2)(-4+j2)]$$

$$= \frac{1}{4}[484 \quad 20 \quad 36 \quad 20]$$

$$S_x[k] = \underline{[121 \quad 5 \quad 9 \quad 5]}$$

12.28

(a)

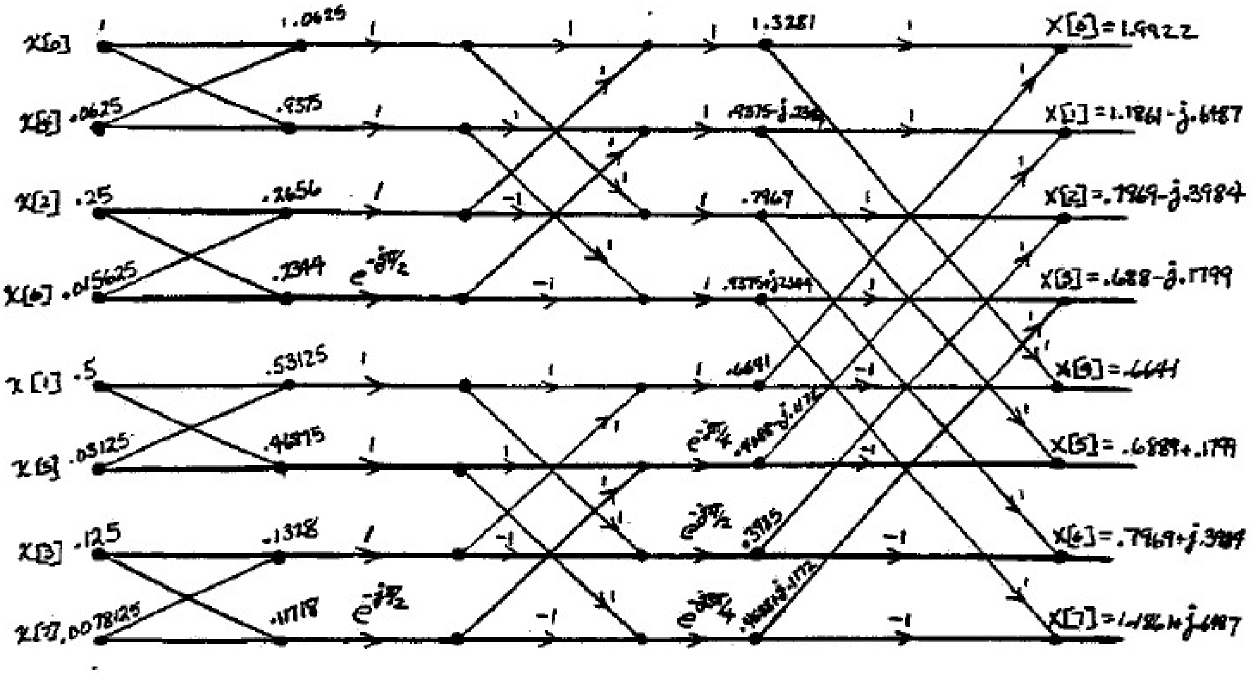


MATLAB

```
EDU» f=[1 2 2 1];
EDU» F=fft(f,4)
```

12.29

(a)



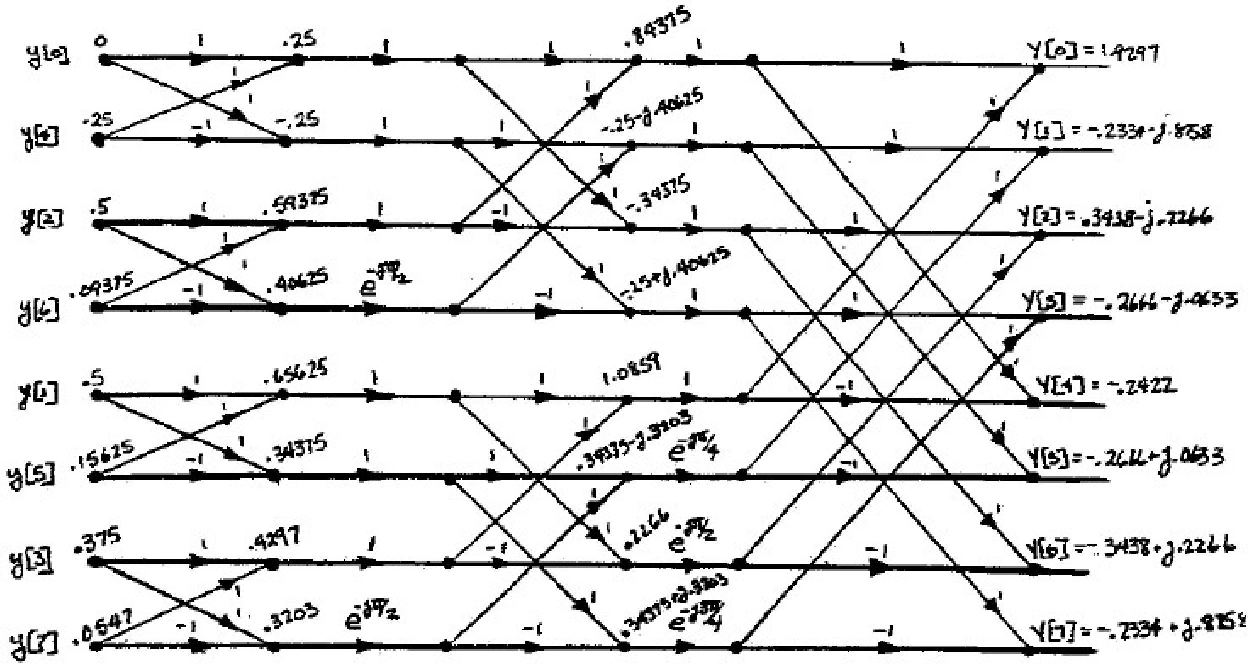
(b)

```
EDU» x=[1 0.5 0.25 0.125 0.0625 0.03125 0.03125/2 0.03125/4]

EDU» X=fft(x,8)
```

12.30

(a)



12.31

```
function compressimage(percentzero)


inputimage=imread('filename','pgm');
s=size(inputimage);
height=s(1);
width=s(2);

INPUTIMAGE=dct2(inputimage);

numbercoefficients=height*width*percentzero/100

side_percentzero=sqrt(numbercoefficients)

tpic=zeros(height,width);

for i=[1:round(side_percentzero)]

for j=[1:round(side_percentzero)]
    tpic(i,j)=INPUTIMAGE(i,j);
    end
end


iinputimage=idct2(tpic);
figure
imshow(iinputimage, [ 0 255])
```

CHAPTER 13

13.1. (a) $x[n] = y[n]$

$x[n+1] = 0.8y[n] + 1.9u[n]$

$y[n] = x[n]$

(b) Replace n with $n+2$

$y[n+2] + 0.8y[n] = u[n]$

$x_1[n] = y[n]$

$x_1[n+1] = y[n+1] = x_2[n]$;

$x_2[n+1] = y[n+2] = -0.8\,y[n] + u[n] = -0.8x_1[n] + u[n]$

$$\therefore\ \underline{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.8 & 0 \end{bmatrix} \underline{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[n]$$

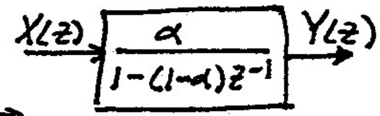
$$y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}[n]$$

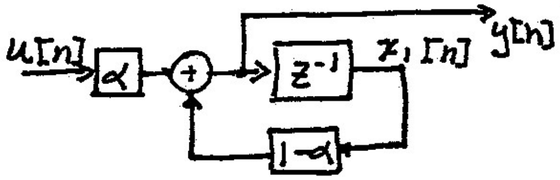
(c) 

$x[n] = y[n]$ $\therefore x[n+1] = u[n]$

$y[n] = x[n]$

13.2. (a) $zY(z) = (1-\alpha)Y(z) + \alpha z X(z)$

$\therefore\ Y(z) = \dfrac{\alpha z}{z-(1-\alpha)} X(z) \implies$ 

$\therefore$ 

$\therefore\ x_1[n+1] = (1-\alpha)x_1[n] + \alpha u[n]$

$y[n] = x_1[n+1] = (1-\alpha)x_1[n] + \alpha u[n]$

(b) (i) see above

(ii) $zX_1(z) = (1-\alpha)X_1(z) + \alpha U(z)$

$\therefore X_1(z) = \dfrac{\alpha}{z-(1-\alpha)} U(z)$

$\therefore Y(z) = (1-\alpha)X_1(z) + \alpha U(z) = \left[\dfrac{\alpha(1-\alpha)}{z-(1-\alpha)} + \alpha\right] U(z)$

$= \dfrac{\alpha z}{z-(1-\alpha)} U(z)$

13.3. (a) $x_1[n+1] = (1-\alpha)x_1[n] + (1-\alpha)Tx_2[n] + \alpha u[n]$

$x_2[n+1] = x_2[n] + \frac{\beta}{T}\left[-x_1[n] - Tx_2[n]\right] + \frac{\beta}{T}u[n]$

$y_1[n] = y[n] = x_1[n+1] = (1-\alpha)x_1[n] + (1-\alpha)Tx_2[n] + \alpha u[n]$

$y_2[n] = v[n] = x_2[n+1] = -\frac{\beta}{T}x_1[n] + (1-\beta)x_2[n] + \frac{\beta}{T}u[n]$

$$\therefore \underline{x}[n+1] = \begin{bmatrix} 1-\alpha & 1-\alpha \\ -\beta/T & 1-\beta \end{bmatrix}\underline{x}[n] + \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix}u[n]$$

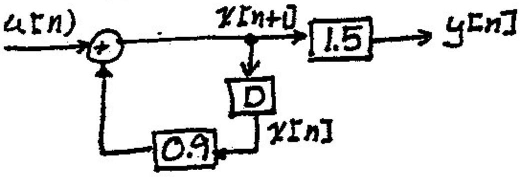
$$\underline{y}[n] = \begin{bmatrix} 1-\alpha & 1-\alpha \\ -\beta/T & 1-\beta \end{bmatrix}\underline{x}[n] + \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix}u[n]$$

(b) with $\beta=0$, input to $x_2[n+1]$ is zero, $\therefore x_2[n]=0$

$\therefore x_1[n+1] = (1-\alpha)x_1[n] + \alpha u[n]$

$y[n] = (1-\alpha)x_1[n] + \alpha u[n]$

13.4. (a) $y[n+1] - 0.9y[n] = 1.5u[n+1]$



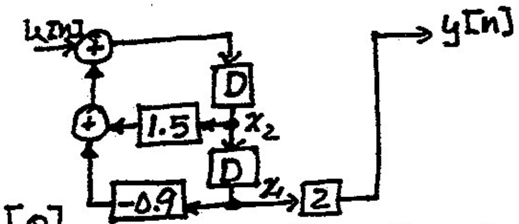
(b) $x[n+1] = 0.9x[n] + u[n]$

$y[n] = 1.5x[n+1] = 1.35x[n] + 1.5u[n]$

(c) $H(z) = \dfrac{1.5z}{z-0.9}$

(d)
```
A=[0.9];B=[1];C=[1.35];D=1.5;
[n,d]=ss2tf(A,B,C,D)
```

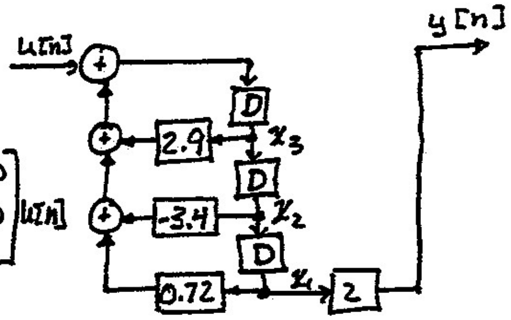
(e) (a) Form 2:



(b) $\underline{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.9 & 1.5 \end{bmatrix}\underline{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u[n]$; $y[n] = [2 \quad 0]\,\underline{x}[n]$

(c) $H(z) = \dfrac{2}{z^2-1.5z+0.9}$

(f) (a) Form 2:

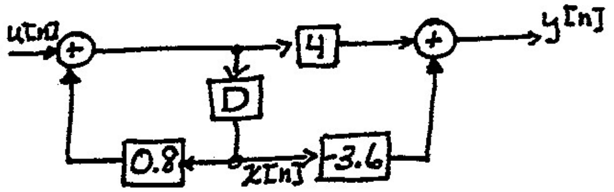


(b) $\underline{x}[n+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.72 & -3.4 & 2.9 \end{bmatrix}\underline{x}[n] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u[n]$

$y[n] = [2 \quad 0 \quad 0]\,\underline{x}[n]$

13.4. (c) $H(z) = \frac{z}{z^3-2.9z^2+3.4z-0.72}$

13.5. (a)



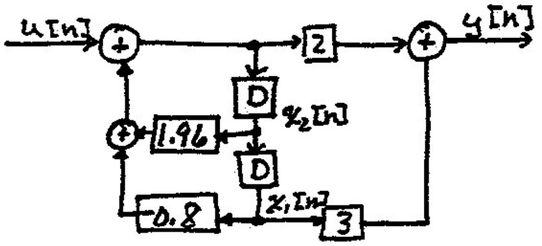
(b) $x[n+1] = 0.8x[n] + u[n]$

$y[n] = -0.4x[n] + 4u[n]$

(c) $y[n] = 0.8y[n-1] + 4u[n] - 3.6u[n-1]$

(d)
```
n=[4 -3.6];
d=[1 -0.8];
[A,B,C,D]=tf2ss(n,d)
```

(e) (a) Form 2:



(b)
$$\underline{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.8 & 1.96 \end{bmatrix}\underline{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u[n]$$

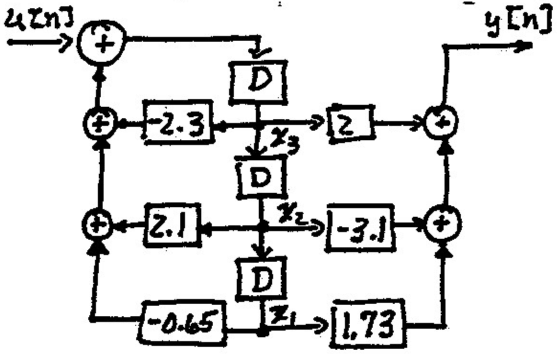
$$y[n] = 2x_2[n+1] + 3x_1[n] = [1.4 \quad 3.92]\underline{x}[n] + 2u[n]$$

(c) $y[n+2] - 1.96y[n+1] + 0.8y[n] = 2u[n+2] + 3u[n]$

(d)
$$\underline{x}[n+1] = \begin{bmatrix} 1.96 & -0.8 \\ 1 & 0 \end{bmatrix}\underline{x}[n] + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u[n]; \quad y[n] = [3.92 \quad 1.4]\underline{x}[n] + 2u[n]$$

Simulation diagram same as in (a), with x_1 & x_2 reversed.

(f) Form 2:
(a)



(b)
$$\underline{x}[n+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.65 & 2.1 & -2.3 \end{bmatrix}\underline{x}[n] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u[n]$$

$$y[n] = [1.73 \quad -3.1 \quad 2]\underline{x}[n]$$

13.5.(f) (cont) (c) $y[n+3]+2.3y[n+2]-2.1y[n+1]+0.65y[n]$

$$= 2u[n+2]-3.1u[n+1]+1.73u[n]$$

(d) MATLAB: $\underline{x}[n+1]=\begin{bmatrix}-2.3 & 2.1 & -0.65\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\underline{x}[n]+\begin{bmatrix}1\\0\\0\end{bmatrix}u[n]$

$$y[n]=[2 \quad -3.1 \quad 1.73]$$

Simulation diagram same as (a), with x_1 and x_3 reversed.

```
n=[2  -1.96];
d=[1  -.99];
[a,b,c,d]=tf2ss(n,d)
pause
n=[2 0 3];
d=[1 -1.96 .8];
[a,b,c,d]=tf2ss(n,d)
pause
n=[0 2 -3.1 1.73];
d=[1 2.3 -2.1 .65];
[a,b,c,d]=tf2ss(n,d)
```

13.6.(a) $x_1[n+1]=0.8x_1[n]+u[n]$

$$x_2[n+1]=1.6[2x_1[n+1]+2.2x_1[n]+0.9x_2[n]$$

$$=1.6[1.6x_1[n]+2u[n]+2.2x_1[n]+0.9x_2[n]$$

$$=6.08x_1[n]+0.9x_2[n]+3.2u[n]$$

$$y[n]=1.9x_2[n]$$

$$\therefore\ \underline{x}[n+1]=\begin{bmatrix}0.8 & 0\\ 6.08 & 0.9\end{bmatrix}\underline{x}[n]+\begin{bmatrix}1\\3.2\end{bmatrix}u[n]$$

$$y[n]=[0 \quad 1.9]\underline{x}[n]$$

(b) $(zI-A)=\begin{bmatrix}z-0.8 & 0\\ -6.08 & z-0.9\end{bmatrix}$; $|zI-A|=(z-0.8)(z-0.9)=\Delta(z)$

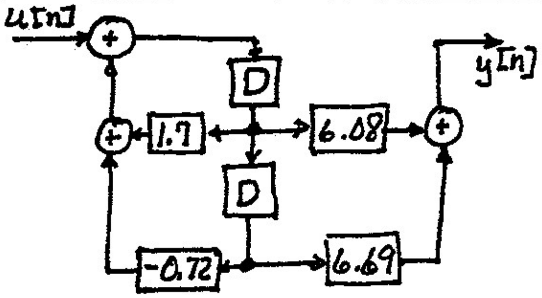
$$H(z)=C[zI-A]^{-1}B=\frac{1}{\Delta(z)}[0 \quad 1.9]\begin{bmatrix}z-0.9 & 0\\ 6.08 & z-0.8\end{bmatrix}\begin{bmatrix}1\\3.2\end{bmatrix}$$

$$=\frac{1}{\Delta(z)}[11.55 \quad 1.9z-1.52]\begin{bmatrix}1\\3.2\end{bmatrix}=\frac{6.08z+6.69}{(z-0.8)(z-0.9)}$$

(c)

```
A=[0.8 0;6.08 0.9];B=[1;3.2];C=[0 1.9];D=0;
[n,d]=ss2tf(A,B,C,D), pause
A=[0 1;-0.72 1.7];B=[0;1];C=[6.69 6.08];D=0;
[n,d]=ss2tf(A,B,C,D)
```

13.6 (d)
(cont)



(e) $\underline{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \underline{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[n]$

$y[n] = [6.69 \quad 6.08]\, \underline{x}[n]$

(f) $zI - A = \begin{bmatrix} z & -1 \\ 0.72 & z-1.7 \end{bmatrix}$; $|zI-A| = z^2 - 1.7z + 0.72 = \Delta(z)$

$$H(z) = C[zI-A]^{-1}B = [6.69 \quad 6.08] \frac{1}{\Delta(z)} \begin{bmatrix} z-1.7 & 1 \\ -0.72 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\Delta(z)}[6.69 \quad 6.08] \begin{bmatrix} 1 \\ z \end{bmatrix} = \frac{6.08z + 6.69}{z^2 - 1.7z + 0.72}$$

(g) See (c)

13.7 (a) $\underline{x}[n+1] = \begin{bmatrix} 0.8 & 1.5 \\ 2.3 & 0.7 \end{bmatrix} \underline{x}[n] + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u[n]$

$y[n] = [1.7 \quad 1.6]\, \underline{x}[n]$

(b) $zI - A = \begin{bmatrix} z-0.8 & -1.5 \\ -2.3 & z-0.7 \end{bmatrix}$; $|zI-A| = \Delta(z) = z^2 - 1.5z + 0.56 - 3.45$

$= z^2 - 1.5z - 2.89$

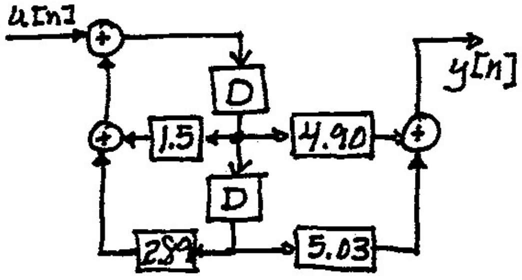
$$H(z) = C(zI-A)^{-1}B = [1.7 \quad 1.6] \frac{1}{\Delta(z)} \begin{bmatrix} z-0.7 & 1.5 \\ 2.3 & z-0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{\Delta(z)}[1.7 \quad 1.6] \begin{bmatrix} z+2.3 \\ 2z+0.7 \end{bmatrix} = \frac{4.09z + 5.03}{z^2 - 1.5z - 2.89}$$

(c)

```
A=[0.8 1.5;2.3 0.7];B=[1;2];C=[1.7 1.6];D=0;
[n,d]=ss2tf(A,B,C,D), pause
A=[0 1;2.89 1.5];B=[0;1];C=[5.03 4.90];D=0;
[n,d]=ss2tf(A,B,C,D)
```

(d)



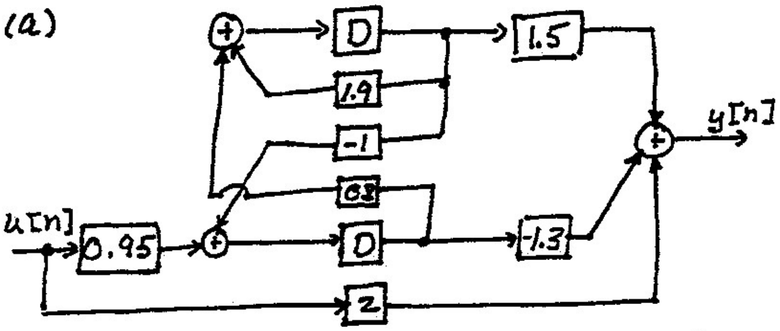
13.7.(e) (cont)

$$\underline{x}[n+1] = \begin{bmatrix} 0 & 1 \\ 2.89 & 1.5 \end{bmatrix}\underline{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u[n]$$

$$y[n] = [5.03 \quad 4.90]\underline{x}[n]$$

(f) See (c)

13.8.(a)



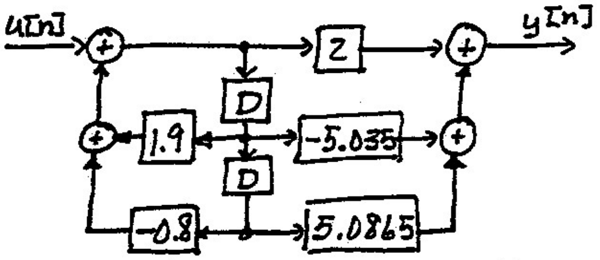
(b) $zI-A = \begin{bmatrix} z-1.9 & -0.8 \\ 1 & z \end{bmatrix}$, $|zI-A| = \Delta = z^2-1.9z+0.8$

$$H(z) = C(zI-A)^{-1}B+D = [1.5 \quad -1.3]\frac{1}{\Delta}\begin{bmatrix} z & 0.8 \\ -1 & z-1.9 \end{bmatrix}\begin{bmatrix} 0 \\ .95 \end{bmatrix} + 2$$

$$= [1.5 \quad -1.3]\frac{1}{\Delta}\begin{bmatrix} 0.76 \\ 0.95z-1.805 \end{bmatrix} + 2 = \frac{-1.235z+3.4865}{z^2-1.9z+0.8}+2$$

$$= \frac{2z^2-5.035z+5.0865}{z^2-1.9z+0.8}$$

(d)



(e) $$\underline{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.8 & 1.9 \end{bmatrix}\underline{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u[n]$$

$$y[n] = [5.0865-1.6]x_1[n] + [-5.035+3.6]x_2[n] + 2u[n]$$

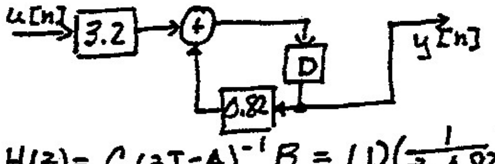
$$= [3.4865 \quad -1.435]\,\underline{x}[n] + 2u[n]$$

(f) $(zI-A) = \begin{bmatrix} z & -1 \\ 0.8 & z-1.9 \end{bmatrix}$, $|zI-A| = \Delta = z^2-1.9z+0.8$

$$H(z) = C(zI-A)^{-1}B = [3.4865 -1.435]\frac{1}{\Delta}\begin{bmatrix} z-1.9 & 1 \\ -0.8 & z \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2$$

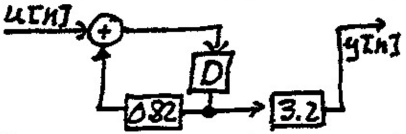
$$= [3.4865 -1.435]\frac{1}{\Delta}\begin{bmatrix} 1 \\ z \end{bmatrix} + 2 = \frac{-1.435z+3.4865}{z^2-1.9z+0.8}+2 = \frac{2z^2-5.035z+5.0865}{z^2-1.9z+0.8}$$

(h)(a)



(b) $$H(z) = C(zI-A)^{-1}B = (1)\left(\frac{1}{z-0.82}\right)(3.2) = \frac{3.2}{z-0.82}$$

13.8. (cont) (d)

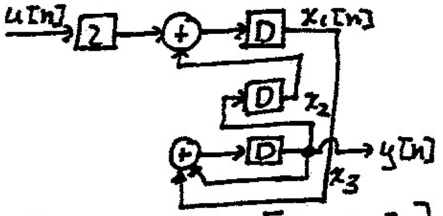


(e) $x[n+1]=0.82x[n]+u[n]$

$y[n]=3.2x[n]$

(f) $H(z)=C(zI-A)^{-1}B=(3.2)(z-0.82)^{-1}(1)=\dfrac{3.2}{z-0.82}$

(i) (a)



(b) $zI-A=\begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ -1 & 0 & z-1 \end{bmatrix}$

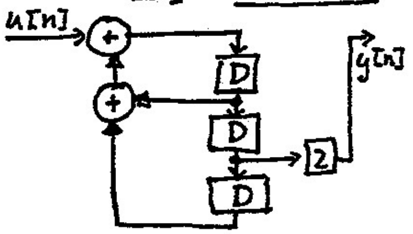
$|zI-A|=\Delta=z^3-z^2-1$

(b) cont. $\operatorname{cof}(zI-A)=\begin{bmatrix} \cdot & \cdot & z \\ \cdot & \cdot & 1 \\ \cdot & \cdot & z^2 \end{bmatrix}$

$\therefore H(z)=C(zI-A)^{-1}B=[0\ 0\ 1]\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ z & 1 & z^2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\frac{1}{\Delta}$

$=[z\ 1\ z^2]\frac{1}{\Delta}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}=\dfrac{2z}{z^3-z^2-1}$

(d) Form II:



(e) $\underline{x}[n+1]=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}x[n]+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u[n]$

$y[n]=[0\ 2\ 0]\underline{x}[n]$

(f) From (i)(b),

$H(z)=[0\ \ 2\ \ 0]\frac{1}{\Delta}\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ z & 1 & z^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}=[0\ \ 2\ \ 0]\begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}\frac{1}{\Delta}=\dfrac{2z}{z^3-z^2-1}$

MATLAB:
```
a=[1.9 0.8;-1 0]; b=[0; .95]; c=[1.5 -1.3]; d=2;
[n,d]=ss2tf(a,b,c,d)

a=[.82]; b=[3.2]; c=[1];
[n,d]=ss2tf(a,b,c,0)
```

13.9. (a) $\underline{x}[n+1]=\begin{bmatrix} 0 & 1 \\ -0.8 & 1.7 \end{bmatrix}\underline{x}[n]+\begin{bmatrix} 0 \\ 1 \end{bmatrix}m[n]$; $y[n]=[-1.3\ \ 1.5]\underline{x}[n]$

(b) $H_p(z)=\dfrac{1.5z-1.3}{z^2-1.7z+0.8}$

(c) $y[n+2]-1.7y[n+1]+0.8y[n]=1.5m[n+1]-1.3m[n]$

(d) $x_3[n+1]=0.98x_3[n]+e[n]$; $m[n]=2x_3[n]$

(e) $H_c(z)=\dfrac{2}{z-0.98}$

(f) $m[n+1]-0.98m[n]=2e[n]$

(g) $\underline{x}[n+1]=\begin{bmatrix} 0 & 1 & 0 \\ -0.8 & 1.7 & 2 \\ 1.3 & -1.5 & 0.98 \end{bmatrix}\underline{x}[n]+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u[n]$

$y[n]=[-1.3\ \ 1.5\ \ 0]\underline{x}[n]$

13.9.(h) $(zI-A) = \begin{bmatrix} z & -1 & 0 \\ 0.8 & z-1.7 & -2 \\ -1.3 & 1.5 & z-0.98 \end{bmatrix}$

$$|zI-A| = \Delta = z^3-2.68z^2+1.666z-2.6-[-3z-0.8z+0.748]$$

$$= z^3-2.68z^2+5.466z-3.384$$

$$\text{cof}(zI-A) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 2 & 2z & z^2-1.7z+0.8 \end{bmatrix}$$

$$H(z) = C(zI-A)^{-1}B = [-1.3 \quad 1.5 \quad 0]\frac{1}{\Delta}\begin{bmatrix} \cdot & \cdot & z \\ \cdot & \cdot & 2z \\ \cdot & \cdot & z^2-1.7z+0.8 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\Delta}[-1.3 \quad 1.5 \quad 0]\begin{bmatrix} z \\ 2z \\ 1 \end{bmatrix} = \underline{\frac{3z-2.6}{z^3-2.68z^2+5.466z-3.384}}$$

(i)
```
a=[0 1 0;-0.8 1.7 2;1.3 -1.5 .98];
b=[0; 0; 1]; c=[-1.3 1.5 0];
[n,d]=ss2tf(a,b,c,0)
```

(j) $$H = \frac{H_c H_p}{1+H_c H_p} = \frac{\left(\frac{z}{z-0.98}\right)\left(\frac{1.5z-1.3}{z^2-1.7z+0.8}\right)}{1+(\cdot)(\cdot)} = \underline{\frac{3z-2.6}{z^3-2.68z^2+5.466z-3.384}}$$

(k) $$y[n+3]-2.68y[n+2]+5.466y[n+1]-3.384y[n]$$
$$= 3u[n+1]-2.6u[n]$$

13.10.(a) From Problem 13.2: $x_1[n+1] = (1-\alpha)x_1[n]+\alpha u[n]$
$y[n] = (1-\alpha)x_1[n]+\alpha u[n]$

(b) $$H(z) = C(zI-A)^{-1}B+D = (1-\alpha)\frac{1}{z-(1-\alpha)}\alpha+\alpha$$

$$= \frac{\alpha(1-\alpha)+\alpha z-\alpha(1-\alpha)}{z-(1-\alpha)} = \underline{\frac{\alpha z}{z-(1-\alpha)}}$$

13.11.(a) From Prob. 13.3: $\underline{x}[n+1] = \begin{bmatrix} 1-\alpha & 1-\alpha \\ -\beta/T & 1-\beta \end{bmatrix}\underline{x}[n]+\begin{bmatrix} \alpha \\ \beta/T \end{bmatrix}u[n]$
$y[n] = [1-\alpha \quad 1-\alpha]\underline{x}[n]+\alpha u[n]$

(b) $zI-A = \begin{bmatrix} z-(1-\alpha) & -(1-\alpha) \\ \beta/T & z-(1-\beta) \end{bmatrix}$

$$|zI-A| = \Delta = z^2-(2-\alpha-\beta)z+(1-\alpha-\beta+\alpha\beta+\beta/T-\alpha\beta/T)$$

$$H(z) = C(zI-A)^{-1}B+D = [1-\alpha \quad 1-\alpha]\frac{1}{\Delta}\begin{bmatrix} z-(1-\beta) & 1-\alpha \\ -\beta/T & z-(1-\alpha) \end{bmatrix}\begin{bmatrix} \alpha \\ \beta/T \end{bmatrix}+\alpha$$

$$= \frac{(1-\alpha)}{\Delta}[z-(1-\beta)-\beta/T \quad z]\begin{bmatrix} \alpha \\ \beta/T \end{bmatrix}+\alpha$$

$$= \frac{1-\alpha}{\Delta}\left[\alpha z-\alpha(1-\beta)-\frac{\alpha\beta}{T}+\frac{\beta}{T}z\right]+\alpha = \frac{(1-\alpha)[(\alpha+\beta/T)z-\alpha(1-\beta-\beta/T)}{z^2-(2-\alpha-\beta)z+(1-\alpha-\beta+\alpha\beta+\frac{\beta}{T}-\frac{\alpha\beta}{T})}$$

(c) $$\beta=0,\ H(z) = \frac{(1-\alpha)[z\alpha-\alpha]}{z^2-(2-\alpha)z+(1-\alpha)}+\alpha = \frac{\alpha(1-\alpha)[z-1]+\alpha z^2-\alpha z-\alpha(1-\alpha)z+\alpha(1-\alpha)}{}$$

$$= \frac{\alpha z(z-1)}{(z-1)(z-(1-\alpha))} = \frac{\alpha z}{z-(1-\alpha)}$$

13.12. (a) From Prob. 13.6(a): $$\underline{x}[n+1]=\begin{bmatrix}0.8 & 0\\ 6.08 & 0.9\end{bmatrix}\underline{x}[n]+\begin{bmatrix}1\\ 3.2\end{bmatrix}u[n]$$
$$y[n]=[0 \quad 1.9]\underline{x}[n]$$

(b) From Prob 13.6(b):
$$z(zI-A)^{-1}=z\begin{bmatrix}\frac{z-0.9}{(z-0.8)(z-0.9)} & 0\\ \frac{6.08}{(z-0.8)(z-0.9)} & \frac{z-0.8}{(z-0.8)(z-0.9)}\end{bmatrix}=z\begin{bmatrix}\frac{1}{z-0.8} & 0\\ \frac{-60.8}{z-0.8}+\frac{6.08}{z-0.9} & \frac{1}{z-0.9}\end{bmatrix}$$
$$\therefore \Phi[n]=\begin{bmatrix}0.8^n & 0\\ 60.8[0.9^n-0.8^n] & 0.9^n\end{bmatrix}$$

(c) $$\underline{x}[n]=\Phi[n]\underline{x}[0]=\Phi[n]\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}0.8^n\\ 62.8(0.9)^n-60.8(0.8)^n\end{bmatrix}$$
$$y[n]=1.9x_2[n]=\underline{119.3(0.9)^n-115.5(0.8)^n},\ n\geq 0$$

(d) $$\underline{X}(z)=(zI-A)^{-1}BU(z)=\begin{bmatrix}\frac{1}{z-0.8} & 0\\ \frac{6.08}{(z-0.8)(z-0.9)} & \frac{1}{z-0.9}\end{bmatrix}\begin{bmatrix}1\\ 3.2\end{bmatrix}\frac{z}{z-1}$$
$$=z\begin{bmatrix}\frac{1}{(z-1)(z-0.8)}\\ \frac{6.08}{(z-0.8)(z-0.9)(z-1)}+\frac{3.2}{(z-1)(z-0.9)}\end{bmatrix}$$
$$=z\begin{bmatrix}\frac{5}{z-1}+\frac{-5}{z-0.8}\\ \frac{304}{z-1}+\frac{304}{z-0.8}+\frac{-608}{z-0.9}+\frac{32}{z-1}+\frac{-32}{z-1}\end{bmatrix}$$
$$=\begin{bmatrix}1-5(0.8)^n\\ 336+304(0.8)^n-640(0.9)^n\end{bmatrix}$$
$$\therefore y[n]=\underline{1.9x_2[n]=638.4+577.6(0.8)^n-1216(0.9)^n,\ n\geq 0}$$

(e) From Prob 13.6, $$H(z)=\frac{6.08z+6.69}{(z-0.8)(z-0.9)}$$
$$\frac{Y(z)}{z}=\frac{H(z)U(z)}{z}=\frac{6.08z+6.69}{(z-1)(z-0.8)(z-0.9)}=\frac{638.6}{z-1}+\frac{577.7}{z-0.8}+\frac{-1216.2}{z-0.9}$$
$$\therefore y[n]=\underline{638.5+577.7(0.8)^n-1216.2(0.9)^n}$$

(f) $$y[n]=638.5+462.2(0.8)^n-1{,}096.9(0.9)^n$$

13.13. (a) $$\underline{x}[n+1]=\begin{bmatrix}0.8 & 0\\ 0 & 0.7\end{bmatrix}\underline{x}[n]+\begin{bmatrix}1\\ 2\end{bmatrix}u[n],\quad y[n]=[1.7 \quad 1.6]\underline{x}[n]$$

(b) $$zI-A=\begin{bmatrix}z-0.8 & 0\\ 0 & z-0.7\end{bmatrix};\ |zI-A|=(z-0.8)(z-0.7)=\Delta$$
$$\Phi(z)=z(zI-A)^{-1}=\frac{z}{\Delta}\begin{bmatrix}z-0.7 & 0\\ 0 & z-0.8\end{bmatrix}=\begin{bmatrix}\frac{z}{z-0.8} & 0\\ 0 & \frac{z}{z-0.7}\end{bmatrix}$$

13.13.(b) (cont) $\Phi[n] = \mathcal{Z}^{-1}[\Phi(z)] = \begin{bmatrix} 0.8^n & 0 \\ 0 & 0.7^n \end{bmatrix}$

(c) $\underline{x}[n] = \Phi[n]\,\underline{x}[0] = \left[\ \downarrow\ \right]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.8^n \\ 2(0.7)^n \end{bmatrix}, \therefore y[n] = [1.7\ \ 1.6]\underline{x}[n]$

$= \underline{1.7(0.8)^n + 3.2(0.7)^n}$

(d) $\underline{X}(z) = (zI-A)^{-1}BU(z) = \begin{bmatrix} \frac{z}{z-0.8} & 0 \\ 0 & \frac{z}{z-0.7} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{z}{z-1}$

$$= \begin{bmatrix} \frac{z}{(z-1)(z-0.8)} \\ \frac{z}{(z-1)(z-0.7)} \end{bmatrix} = \begin{bmatrix} \frac{5z}{z-1} + \frac{-5z}{z-0.8} \\ \frac{6.67z}{z-1} + \frac{-6.67z}{z-0.7} \end{bmatrix} \Rightarrow \underline{x}[n] = \begin{bmatrix} 5(1-0.8^n) \\ 6.67(1-0.7^n) \end{bmatrix}$$

$\therefore y[n] = C\underline{x}[n] = [1.7\ \ 1.6][\cdot] = 8.5(1-0.8^n) + 10.67(1-0.7)^n$

(e) $H(z) = C(zI-A)^{-1}B = [1.7\ \ 1.6]\begin{bmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.7} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = [1.7\ \ 1.6]\begin{bmatrix} \frac{1}{z-0.8} \\ \frac{2}{z-0.7} \end{bmatrix}$

$\therefore H(z) = \frac{1.7}{z-0.8} + \frac{3.2}{z-0.7}$

$Y(z) = H(z)U(z) = \frac{1.7z}{(z-1)(z-0.8)} + \frac{3.2z}{(z-1)(z-0.8)} = \frac{8.5z}{z-1} + \frac{-8.5z}{z-0.8} + \frac{10.67z}{z-1} - \frac{10.67z}{z-0.7}$

$\therefore y[n] = 8.5(1-0.8^n) + 10.67(1-0.7)^n$

(f) $y[n] = 1.7(0.8)^n + 3.2(0.7)^n + 8.5 - 8.5(0.8)^n + 10.67 - 10.67(0.7)^n$

$= 19.17 - 6.8(0.8)^n - 7.47(0.7)^n$

(g) $y[0] = \underline{4.9}$ ✓, $y[2] = \underline{11.158}$ ✓

```
x1(1)=1; x2(1)=2;
for n=1:4
  y(n)=1.7*x1(n) + 1.6*x2(n);
  x1(n+1)=.8*x1(n)+0*x2(n)+1;
  x2(n+1)=0*x1(n)+.7*x2(n)+2;
end
y
```

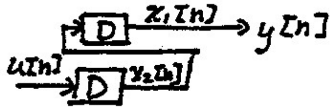
13.14.(a)(1) $zI-\Delta = \begin{bmatrix} z & -1 \\ 0 & z \end{bmatrix}, |zI-A|\ z^2, (zI-A)^{-1} = \begin{bmatrix} \frac{1}{z} & \frac{1}{z^2} \\ 0 & \frac{1}{z} \end{bmatrix}$

$\Phi[n] = \mathcal{Z}^{-1}(z(zI-A)^{-1}) = \mathcal{Z}^{-1}\begin{bmatrix} 1 & \frac{1}{z} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \delta[n] & \delta[n-1] \\ 0 & \delta[n] \end{bmatrix}$

(2) $\Phi[n] = A^n;\ \Phi(0) = I\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Phi[1] = A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\Phi[2] = A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore n \geq 2, \Phi[n] = \Phi[2]\Phi[n-2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \therefore \Phi[n] = \begin{bmatrix} \delta[n] & \delta[n-1] \\ 0 & \delta[n] \end{bmatrix}$

(b) 

∴ Realize by two cascaded delays.

13.15. (a) $zI-A = \begin{bmatrix} z & 0 \\ -1 & z \end{bmatrix}, |zI-A| = z^2, (zI-A)^{-1} = \begin{bmatrix} \frac{1}{z} & 0 \\ 1/z^2 & 1/z \end{bmatrix}$

]

13.15.(a) (CONT) $\therefore \Phi[n]\, \mathcal{Z}^{-1}\left[z(zI-A)^{-1}\right] = \mathcal{Z}^{-1}\begin{bmatrix} 1 & 0 \\ 1/z & 1 \end{bmatrix} = \underline{\begin{bmatrix} \delta[n] & 0 \\ \delta[n-1] & \delta[n] \end{bmatrix}}$

(b) $\Phi[n] = A^n$; $\Phi[0] = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Phi[1] = A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\Phi[2] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \therefore n \geq 2, \Phi[n] = \underline{0}$

$\therefore \Phi[n] = \underline{\begin{bmatrix} \delta[n] & 0 \\ \delta[n-1] & \delta[n] \end{bmatrix}}$

(d) $\underline{x}[1] = A\underline{x}[0] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underline{x}[2] = A\underline{x}[1] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\therefore \underline{x}[n] = \underline{0}$, $n \geq 2$

(c) $x[n] = \Phi[n]\, \underline{x}[0] = \begin{bmatrix} \delta[n] & 0 \\ \delta[n-1] & \delta[n] \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \delta[n] \\ 2\delta[n] + \delta[n-1] \end{bmatrix}$

$\therefore y[n] = [0 \quad 1]\underline{x}[n] = \underline{2\delta[n] + \delta[n-1]}$

(e) $y[0] = C\,x[0] = \underline{0}$

$\underline{x}[1] = A\underline{x}[0] + Bu[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $y[1] = x_2[1] = \underline{1}$

$\underline{x}[2] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $y[2] = x_2[2] = \underline{2}$

$\underline{x}[3] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $y[3] = \underline{2}$

$\underline{x}[4] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $y[4] = \underline{2}$

$$\therefore y[n] = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ 2, & n \geq 2 \end{cases}$$

(f) $\underline{X}(z) = (zI-A)^{-1}B\,U(z) = \begin{bmatrix} \frac{1}{z} & 0 \\ 1 & \frac{1}{z} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{z}{z-1} = \begin{bmatrix} \frac{1}{z} \\ \frac{1}{z^2} + \frac{1}{z} \end{bmatrix}\frac{z}{z-1} = \begin{bmatrix} \frac{1}{z-1} \\ \frac{1}{z(z-1)} + \frac{1}{z-1} \end{bmatrix}$

$Y(z) = C\underline{X}(z) = [0 \quad 1][\cdot] = \frac{1}{z(z-1)} + \frac{1}{z-1}$

$\therefore y[n] = \mathcal{Z}^{-1}\,Y(z) = \underline{u[n-2] + u[n-1]}$

(g) $H(z) = C(zI-A)^{-1}B = [0 \quad 1]\begin{bmatrix} \frac{1}{z} & 0 \\ \frac{1}{z^2} & \frac{1}{z} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{z^2} & \frac{1}{z} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{\frac{1}{z^2} + \frac{1}{z}}$

$\therefore Y(z) = H(z)U(z) = \left[\frac{1}{z^2} + \frac{1}{z}\right]\left[\frac{z}{z-1}\right] \Rightarrow y[n] = \underline{u[n-2] + u[n-1]}$

(h)

```
x1(1)=0; x2(1)=0;
for n=1:4
  y(n)=0*x1(n)  + 1*x2(n);
  x1(n+1)=0*x1(n)+0*x2(n)+1;
  x2(n+1)=1*x1(n)+0*x2(n)+1;
end
y
```

13.16. (a) $\Phi(z) = z(zI-A)^{-1} = z\frac{1}{z-0.95} = \frac{z}{z-0.95} \Rightarrow \Phi[n] = \underline{0.95^n}$

(b) $x[n] = \Phi[n]\,x[0] = 0.95^n$; $y[n] = C\,x[n] = \underline{3(0.95)^n}$

13.16 (c) $x[1] = 0.95\,x[0] = 0.95$ $\qquad x[3] = 0.95\,x[2] = (0.95)^3$

(cont) $x[2] = 0.95\,x[1] = (0.95)^2 \quad \therefore\ x[n] = \underline{(0.95)^n}$

(d) $X(z) = (zI-A)^{-1}BU(z) = \frac{1}{z-0.95}(1)\left(\frac{z}{z-1}\right)$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-0.95)} = \frac{20}{z-1} + \frac{-20}{z-0.95} \Rightarrow x[n] = \underline{20(1-0.95^n)},\ n \geq 0$$

$$y[n] = 3x[n] = \underline{60(1-0.95^n)},\ n \geq 0$$

(e) $H(z) = C(zI-A)^{-1}B = \frac{3}{z-0.95}$

$$\therefore\ \frac{Y(z)}{z} = \frac{3}{(z-1)(z-0.95)} = \frac{60}{z-1} + \frac{-60}{z-0.95} \Rightarrow y[n] = \underline{60(1-0.95^n)},\ n \geq 0$$

(f)

```
u=1;x(1)=0;
for n=1:5
   y=3*x(n)
   x(n+1)=0.95*x(n)+u;
end
```

13.17 (a) From Prob 13.16, $H(z) = \frac{1}{z} + \frac{1}{z^2} = \underline{\frac{z+1}{z^2}}$

(b) Let $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$$A_v = P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B_v = P^{-1}B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_v = CP = [0 \quad 1]\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = [1 \quad 2]$$

$$\therefore\ \underline{v}[n+1] = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\underline{v}[n] + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u[n]\ ;\ y[n] = [1 \quad 2]v[n]$$

(d) $zI - A_v = \begin{bmatrix} z+1 & 1 \\ -1 & z-1 \end{bmatrix}$; $|zI-A| = z^2 = \Delta$

$$H(z) = \underline{C}_v(zI-A_v)^{-1}B_v = [1 \quad 2]\begin{bmatrix} \frac{z-1}{z^2} & -1/z^2 \\ 1/z^2 & \frac{z+1}{z^2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1 \quad 2]\begin{bmatrix} \frac{z-1}{z^2} \\ 1/z^2 \end{bmatrix} = \underline{\frac{z+1}{z^2}}$$

(f) $\lambda_1 = \lambda_2 = 0$

(13.67) $|zI-A| = z^2 = |zI-A_v| = (z-0)(z-0)$

(13.68) $\det A = 0 = \det A_v = (0)(0)$

(13.69) $\operatorname{tr} A = 0 = \operatorname{tr} A_v = 0+0$

(c)(e)

```
a=[0 0;1 0]; b=[1;1]; c=[0 1]; d=0; q=[2 -1;-1 1];
p=inv(q);
av=q*a*p
bv=q*b
cv=c*p
pause
[n,d1]=ss2tf(av,bv,cv,d)
```

13.18. (a) From Prob. 13.8, $H(z) = \dfrac{2z^2-5.035z+5.0865}{z^2-1.9z+0.8}$

(b) Let $P=\begin{bmatrix}1&1\\1&2\end{bmatrix}$, $P^{-1}=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}$

$$A_v = P^{-1}AP = \begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}1.9&0.8\\-1&0\end{bmatrix}\begin{bmatrix}1&1\\1&2\end{bmatrix}$$

$$= \begin{bmatrix}4.8&1.6\\-2.9&-0.8\end{bmatrix}\begin{bmatrix}1&1\\1&2\end{bmatrix} = \begin{bmatrix}6.4&8\\-3.7&-4.5\end{bmatrix}$$

$$B_v = P^{-1}B = \begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}0\\0.95\end{bmatrix} = \begin{bmatrix}-0.95\\0.95\end{bmatrix}$$

$$C_v = CP = [1.5 \;\; -1.3]\begin{bmatrix}1&1\\1&2\end{bmatrix} = [0.2 \;\; -1.1]\text{ ; } D_v = D = 2$$

$$\therefore \underline{v}[n+1] = \begin{bmatrix}6.4&8\\-3.7&-4.5\end{bmatrix}\underline{v}[n] + \begin{bmatrix}-0.95\\0.95\end{bmatrix}u[n]$$

$$y[n] = [0.2 \;\; -1.1]\underline{v}[n] + 2$$

(d) $zI - A_v = \begin{bmatrix}z-6.4&-8\\3.7&z+4.5\end{bmatrix}$, $|zI-A| = z^2-1.9z+0.8$

$$H(z) = C_v(zI-A_v)^{-1}B_v + D_v = [0.2 \;\; -1.1]\frac{1}{\Delta}\begin{bmatrix}z+4.5&8\\-3.7&z-6.4\end{bmatrix}\begin{bmatrix}-0.95\\0.95\end{bmatrix}+2$$

$$= \frac{1}{\Delta}[0.2z+4.97 \;\; -1.1z+8.64]\begin{bmatrix}-0.95\\0.95\end{bmatrix}+2 = \frac{2z^2-5.035z+5.0865}{z^2-1.9z+0.8}$$

(f) (13.67) $|zI-A| = z^2-1.9z+0.8 = |zI-A_v| = \underline{(z-1.27)(z-0.63)}$

(13.68) $\det A = 0.8 = \det A_v = (1.27)(0.63)$

(13.69) $\operatorname{tr} A = 1.9 = \operatorname{tr} A_v = 1.27+0.63$

(c)(e)

```
a=[1.9 .8;-1 0]; b=[0;.95]; c=[1.5 -1.3]; d=2; q=[2 -1;-1 1];
p=inv(q);
av=q*a*p
bv=q*b
cv=c*p
pause
[n,d1]=ss2tf(av,bv,cv,d)
```

13.19. (a) From Prob. 13.18, C.E.: $z^2-1.9z+0.8=(z-1.27)(z-0.63)=0$

<u>not stable</u>

(b) modes: $(1.27)^n$, $(0.63)^n$

(c)
```
a=[1.9 0.8;-1 0];
eig(a)
```

13.20. (a)
```
a=[0 1 0;0 0 1;1 0 1];
eig(a)
```

From MATLAB, $z = 1.4656$, $0.826\underline{/\pm 106.4^\circ}$

∴ unstable

(b) modes: $(1.4656)^n$, $(-0.2328+j0.7926)^n$, $(-0.2328-j0.7926)^n$