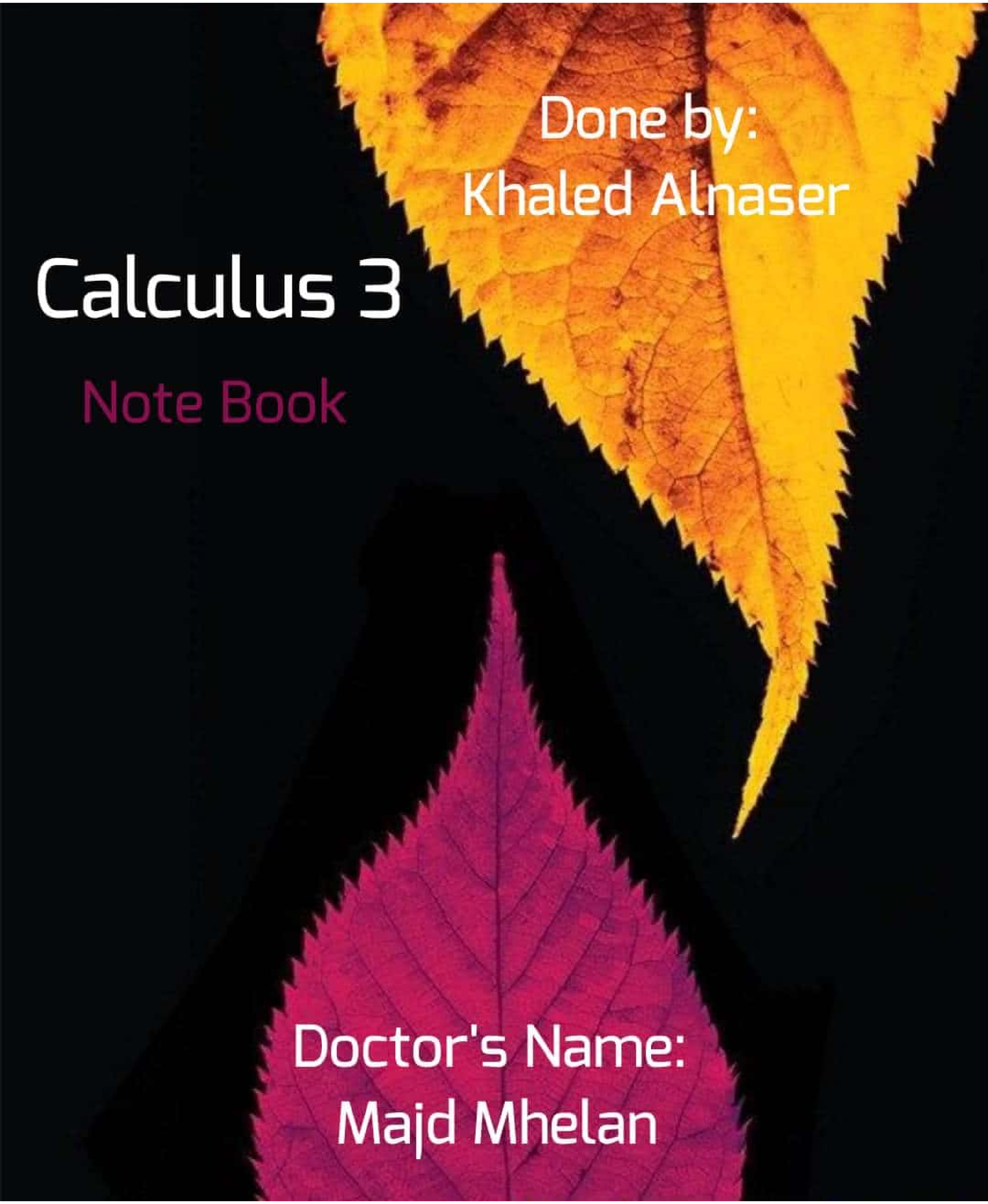


Done by:
Khaled Alnaser

Calculus 3

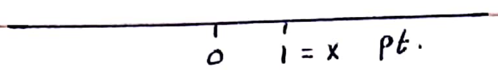
Note Book

Doctor's Name:
Majd Mhelan

Ch. 12 Vectors & the Geometry of Space

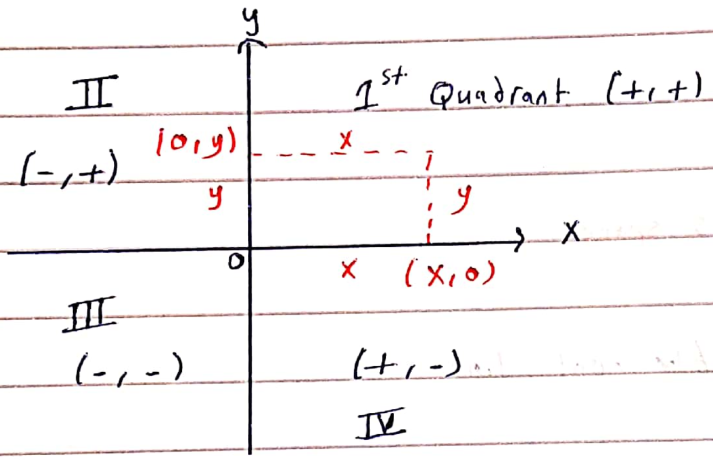
§ 12.1 The three dimensional Space $\equiv R^3$

1- Space $\equiv$ real line



2- Space:

R^2



3- Space: Consists of ① 3- Coordinate axes, x-axis, y-axis, z-axis

② 3 - Coordinate planes

i) xy - plane, eqn: $z = 0$

ii) xz - plane, eqn: $y = 0$

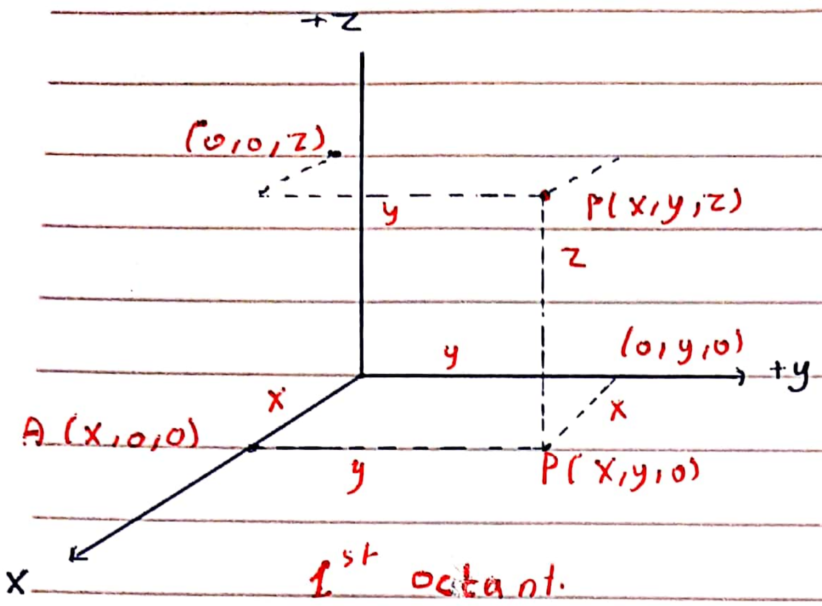
iii) yz - plane, eqn: $x = 0$

$c \equiv$ constant

$\boxed{i}$ $z = c \Rightarrow$ plane // xy - plane

$\boxed{ii}$ $y = c \Rightarrow$ plane // xz - plane

$\boxed{iii}$ $x = c \Rightarrow$ plane // yz - plane

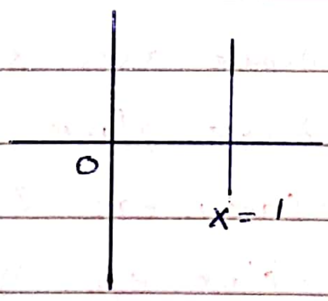


1st octant.

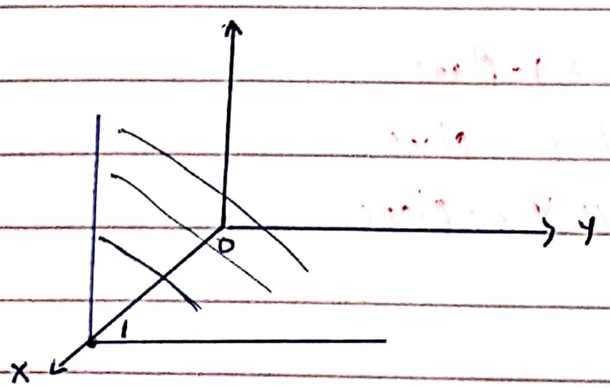
Ex. Identify $x=1$
in ① 1-space ② 2-space ③ 3-space

① $x=1$ in 1-space pt. on the real line

② $x=1$ in R^2 $\rightarrow$
V. line // y-axis

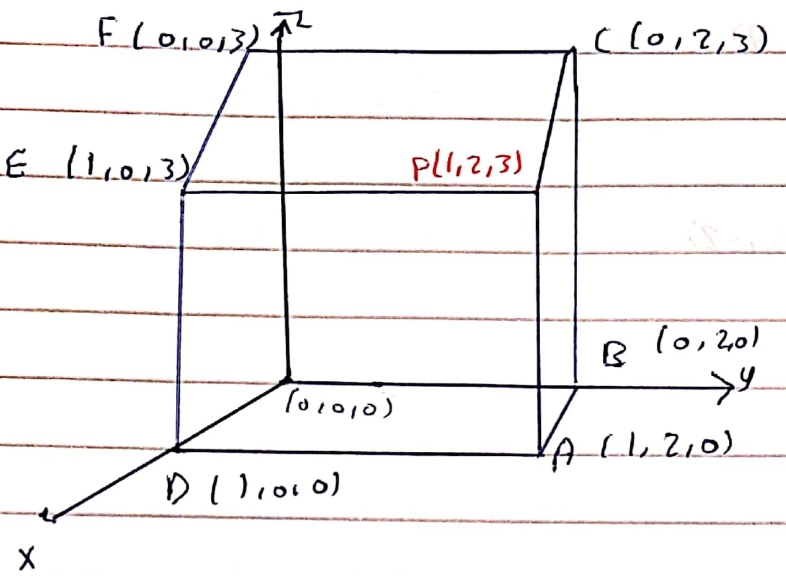


③ $x=1$ in R^3, plane // yz-plane



Scanned by CamScanner

Ex



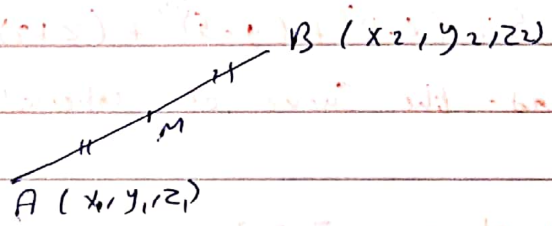
planes:

① ABCP // xz-plane → $y=2$

② PCFE // xy-plane → $z=3$

③ APED // yz-plane → $x=1$

* distance between 2-pt. in $\mathbb{R}^3$.



$$d(A,B) = \overline{AB} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

* Mid pt. (M)

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

* Sphere:

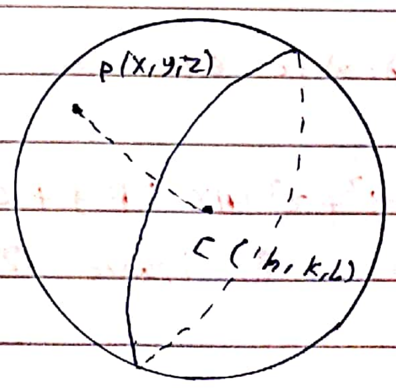
Def: The set of all pt.s $P(x,y,z)$ or in $\mathbb{R}^3$ that are at a constant distance from a given fixed pt.

rad $= r > 0$ center (h,k,l)

$$d(P,C) = r$$

$$\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r$$

$$\boxed{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2}$$



Sphere $C(h,k,l)$, rad $= r$

Scanned by CamScanner

Ex. Write an eqn of a sphere with end points of one of its diameters are pts. $A(-1,0,3)$ & $B(3,10,-7)$

→ Centre in the midpt.

$M = \left(\frac{-1+3}{2}, \frac{0+10}{2}, \frac{3-7}{2} \right) = C(1,5,-2)$

rad $= \overline{AC} = \sqrt{2^2+5^2+(-5)^2}$
$= \sqrt{54}$

$S: (x-1)^2 + (y-5)^2 + (z+2)^2 = 54$

Ex. $S: (x-1)^2 + (y-5)^2 + (z+2)^2 = 54$ ---- $(*)$

Find the Curve of intersection with:

① xy-plane : $z = 0$

$\xrightarrow[z=0]{(*)}$ $(x-1)^2 + (y-5)^2 + (0+2)^2 = 54$

$(x-1)^2 + (y-5)^2 = 50$

Circle $C(1,5,0)$, rad $= \sqrt{50}$

② Plane , $y = 50$

$\xrightarrow{(*)}$ $(x-1)^2 + (50-5)^2 + (z+2)^2 = 54$

$(x-1)^2 + (z+2)^2 = 54 - (45)^2 < 0$

No curve of int. , Solution set $= \phi$

* Quadratic eqn in x, y, z

$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$

Represents: ① Sphere ② Single Set ③ ϕ

Ex Identify or find the Sol. set.

$3x^2+3y^2+3z^2 = 10-6y+12z$

① $= -15 \xrightarrow{rad} = 0 \rightarrow$ single pt.

② $= -16 \xrightarrow{rad} = -1 \rightarrow \phi$

$3x^2+3y^2+6y+3z^2-12z = 10$

$3x^2+3(y^2+2y+1-1)+3(z^2-4z+2^2-2^2) = 10$

$3x^2+3(y+1)^2-3+3(z-2)^2-12 = 10$

$3x^2+3(y+1)^2+3(z-2)^2 = 10+15$ ----(*)

$3x^2+3(y+1)^2+3(z-2)^2 = 25$

$x^2+(y+1)^2+(z-2)^2 = 25/3$

Sphere, $C(0,-1,2)$, rad $= \sqrt{25/3}$

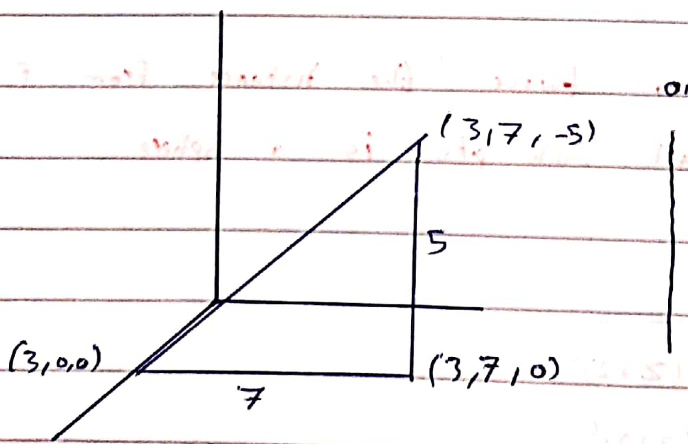
Ex. $A(3,7,-5)$

Find the distance from A to...

① xy-plane $= |-5| = 5$

② xz-plane $= |7| = 7$

③ x-axis: $d(P,+x) = \sqrt{(3-3)^2+7^2+(-5)^2} = \sqrt{7^2+5^2} =$



OR

$= \sqrt{5^2+7^2}$

④ y-axis $= \sqrt{3^2+5^2}$

* $S \equiv$ sphere:

$S: (x-h)^2 + (y-k)^2 + (z-L)^2 = r^2$

Ex. Write an eq. of a sphere centered at $C(2,-3,6)$ that touch

① xy-plane $\Rightarrow r = 6$

$S: (x-2)^2 + (y+3)^2 + (z-6)^2 = 6^2$

② xz-plane: $\Rightarrow r = |-3| = 3$

$S: (x-2)^2 + (y+3)^2 + (z-6)^2 = 3^2$

Ex. ∃ Shere $C(5,4,9)$

Find the largest sphere contained in the 1^{st} octant

$r = 4$ (الاصغر احداثاً)

$S: (x-5)^2 + (y-4)^2 + (z-9)^2 = 4^2$

Ex: Consider the pts. $P(x,y,z)$:

the distance from P to $A(-1,5,3)$ is twice the distance from P to $B(6,2,-2)$ Show that the set of all such pts. is a sphere

$d(P,A) = 2\, d(P,B)$

$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = 2\sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2}$

$(x+1)^2 + (y-5)^2 + (z-3)^2 = 4\,(x-6)^2 + (y-2)^2 + (z+2)^2$

$x^2 + \dots + y^2 + \dots + z^2 + \dots = 4x^2 + \dots + 4y^2 + \dots + 4z^2 + \dots$

$3x^2 + \dots + 3y^2 + \dots + 3z^2 + \dots = 0$

اكمال مربع $\vdots$

Sphere C __ , rad

(15) ? eqn of all pts. $P(x,y,z)$ that are equisdistant from A & B.

$d(P,A) = d(P,B)$

$\cancel{x^2} + 2x + \cdots + \cancel{y^2} - 10y + \cdots + \cancel{z^2} - 6z + \cdots = \cancel{x^2} - 12x + \cdots + \cancel{y^2} - 4y + \cdots + \cancel{z^2} + 4z + 4$

الثابت

$14x - 6y - 10z + D = 0$ (plane)

Ex. Describe the region in R^3

① $1 \leq x^2+y^2+z^2 \leq 4$, ② $z \leq 0$

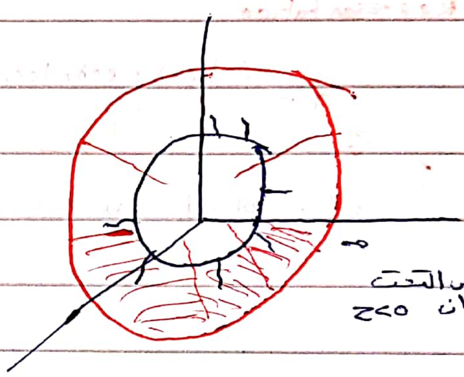
$z = 0$

① $1 \leq x^2+y^2+z^2$ (اكبر برا الدائرة)

$1 = x^2+y^2+z^2$

② $x^2+y^2+z^2 \leq 4$ (اصغر جوا الدائرة)

$x^2+y^2+z^2 = 4$



فقط النص التحت عشان $z<0$

§ 12.2 Vectors:

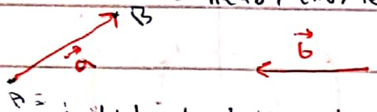
Def(1): Scalars: quantities with magnitude only, mass, length, distance, speed, times.

Def(2): Vectors: quantities with magnitude & direction

weight, position, displacement, velocity, acceleration, forces....

* Vector Representation:

B = head, end, terminal

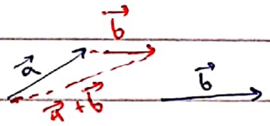
① arrows: 

A = initial pt, tail vector

$\vec{a} = \vec{AB}$

* $\|\vec{a}\| = |\vec{a}|$: length OR magnitude OR norm of $\vec{a}$.

* Vector add. (i) head - tail add; (ii) Parallelogram

(i) $\vec{a} + \vec{b}$ 

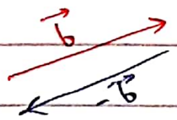
is vector,

with initial pt. as $\vec{a}$ & terminal pt ... as $\vec{b}$.

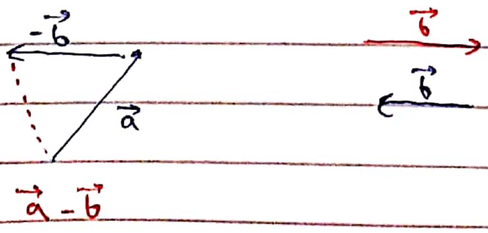
* $-\vec{b}$ is a vector,

with ① $\|-\vec{b}\| = \|\vec{b}\|$ مقداره

(ii) opposite direction of $\vec{b}$



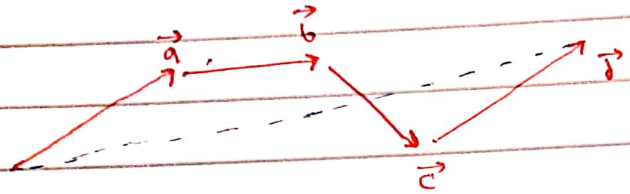
$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$



Ex.

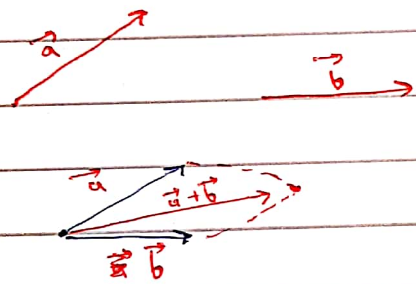
$\vec a + \vec b$

Ex

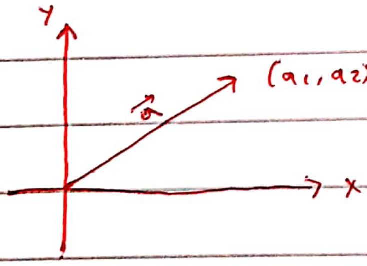


$\vec a + \vec b + \vec c + \vec d$

(ii) Parallelogram.



[2] Vector can be represented as ordered pairs or triples



$\vec a = (a_1, a_2) = \langle a_1, a_2 \rangle$

$\vec a = (a_1, a_2, a_3) = \langle a_1, a_2, a_3 \rangle$

$\equiv$ position vector with origion as its initial pt. & P_1 as terminal pt.

$\therefore \|\vec a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Scanned by CamScanner

* Vector Representation

[1] arrows [2] ordered pairs or triples

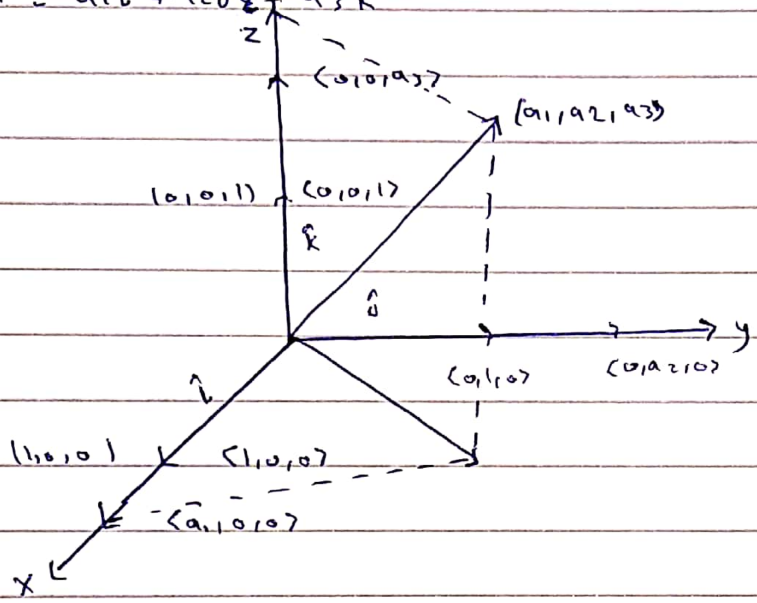
[3] linear combination of the three unit vectors $\hat{i}, \hat{j}, \hat{k}$.

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$

$\vec{a} = a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$

$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$



* norm or length or magnitude of $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Ex: $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ Find $\|\vec{a}\| = ?$

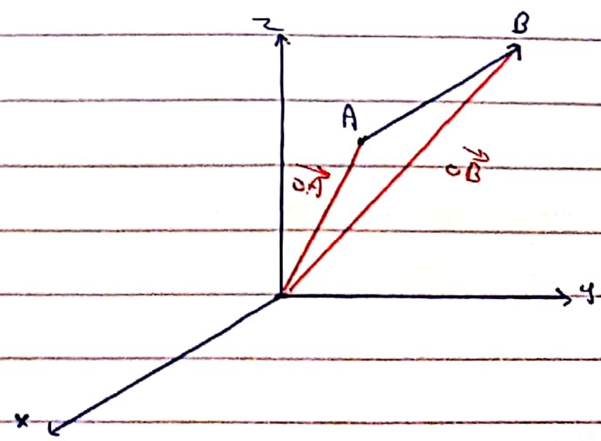
$\|\vec{a}\| = \sqrt{4 + 9 + 16}$

$= \sqrt{29}$

* Vector from a point $A(a_1, a_2, a_3)$ to $B(b_1, b_2, b_3)$

$\vec{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$

$= (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$



$\vec{OA} + \vec{AB} = \vec{OB}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= \langle b_1, b_2, b_3 \rangle - \langle a_1, a_2, a_3 \rangle$

$= \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$

Ex. $\vec{AB} = 2\hat{i} + 3\hat{j} - 4\hat{k}$

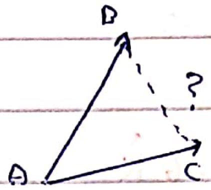
$\vec{AC} = -\hat{i} + \hat{j} + 2\hat{k}$

Find $\vec{CB} = ?$

$\vec{CB} = \vec{CA} + \vec{AB}$

$\vec{CB} = -\vec{AC} + \vec{AB}$

$= \vec{AD} - \vec{AC}$



$\vec{CD} = 3\hat{i} + 2\hat{j} - 6\hat{k}$

Scaler multiple of a vector.

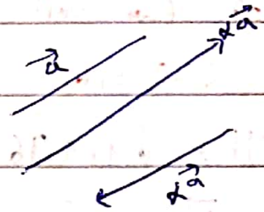
$\alpha \in \mathbb{R}$, $\vec{a}$ is a vector.

$\Rightarrow$ $\alpha\vec{a}$ is a vector;

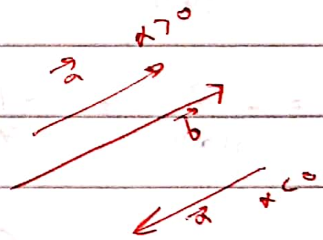
(i) $\|\alpha\vec{a}\| = |\alpha| \, \|\vec{a}\|$

(ii) direction ① $\alpha > 0$, in the same direction of $\vec{a}$.

② $\alpha < 0$, in the opposite direction of $\vec{a}$.



* Parallel vectors.



$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{b} = \alpha\vec{a}$, for some α

Ex. $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$

$\vec{b} = -4\hat{i} + 2\hat{j} - 6\hat{k}$

Is $\vec{b} \parallel \vec{a}$??

$\vec{b} = -2\vec{a}$

$\alpha = -2$ $\quad \vec{b} \parallel \vec{a}$

and it is in the ~~same~~ opposite direction of $\vec{a}$.

② $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$

$\vec{b} = -4t\hat{i} + L\hat{j} - 6\hat{k}$

Find t & L : $\vec{a} \parallel \vec{b}$

$\vec{a} \parallel \vec{b} \leftrightarrow \vec{b} = \alpha \vec{a}$ $\qquad \alpha = -2$

$\vec{b} = -2\vec{a}$

$-4t\hat{i} + L\hat{j} - 6\hat{k} = -4\hat{i} + 2\hat{j} - 6\hat{k}$

$-4t = -4 \rightarrow \boxed{t = 1}$

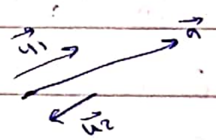
$\boxed{L = 2}$

$-6 = -6$

* $\dfrac{\vec{a}}{\|\vec{a}\|}$ unit vector in the direction of $\vec{a}$ $\Rightarrow$ $\hat{u}_{\vec{a}} = \dfrac{\vec{a}}{\|\vec{a}\|}$ $=$ $\boxed{\vec{a} = \|\vec{a}\| \, \hat{u}_{\vec{a}}}$

Ex. find two unit vectors parallel to

$\vec{a} = \langle -2, 3, 6 \rangle$



$\|\vec{a}\| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

$\vec{u}_1 = + \dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{1}{7} \langle -2, 3, 6 \rangle$ in the same dir. of $\vec{a}$

$\vec{u}_2 = - \dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{-1}{7} \langle -2, 3, 6 \rangle$ in the opposite dir. of $\vec{a}$

$= \left\langle \dfrac{2}{7}, \dfrac{-3}{7}, \dfrac{-6}{7} \right\rangle$

$\vec{a} = ??$

Ex. find avector in the same direction of the vector $\langle 6, -2, 3 \rangle$ but has length 4. $\qquad \langle 6, -2, 3 \rangle = \vec{b}$

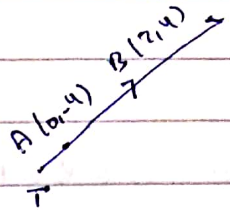
$\vec{u}_B = \dfrac{1}{7} \langle 6, -2, 3 \rangle \Rightarrow \hat{u}_{\vec{a}} = \hat{u}_{\vec{b}}$

من الجذر $\|\vec{b}\| = 7 \quad = \sqrt{36 + 4 + 9} = \sqrt{49} = \boxed{7}$

(*) $\vec{a} = \|\vec{a}\| \, \hat{u}_{\vec{a}}$

$= 4 \cdot \dfrac{1}{7} \langle 6, -2, 3 \rangle$

41 / 806 Find two unit vectors parallel to the tangent line to the parabola $y = x^2$ at $\overset{B}{(2,4)}$



$y' = 2x \big|_{x=2} = \boxed{4}$

$T_{line}: \quad y - 4 = 4(x-2)$

$x = 0 \qquad y - 4 = -8$

$\boxed{y = -4}$

$\overrightarrow{AB} \parallel T_{line}$

$$\boxed{\mp \frac{AB}{\|\overrightarrow{AB}\|}}$$

النهاية.
ناقص البداية $\overrightarrow{AB} = \langle 2-0, 4+4 \rangle$

$= \langle 2, 8 \rangle = \|\overrightarrow{AB}\| = \sqrt{2^2 + 8^2} = \sqrt{4+64} = \sqrt{68}$

$\hat{M}(1) = + \frac{1}{\sqrt{68}} \langle 2, 8 \rangle$

$\vec{M}(2) = - \frac{1}{\sqrt{68}} \langle 2, 8 \rangle$

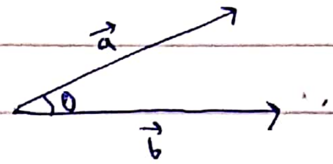
Scanned by CamScanner

§12.3 The dot (Scalar) product

$\vec{a}$ & $\vec{b}$ are two vectors in R^2 or R^3

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$



θ is the angle in between $0 \leq \theta \leq \pi$

Define $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta \in \mathbb{R}$ --- ①

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \in \mathbb{R}$ ---- ②

① & ② $\rightarrow$ $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$ ... ③

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)$$

Ex. $\vec{a} = 2\hat{i} - 3\hat{k}$

$\vec{b} = -\hat{i} + 2\hat{j} + 4\hat{k}$

Find the angle between $\vec{a}$ & $\vec{b}$

③ $\cos\theta = \dfrac{2(-1) + 0(2) - 3(4)}{\sqrt{4+9}\,\sqrt{1+4+16}}$

$\cos\theta = \dfrac{-14}{\sqrt{13(21)}}$

$\theta = \cos^{-1}\left(\dfrac{-14}{\sqrt{13(21)}}\right)$

* Properties :

① $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

② $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$

③ $L, m \in R$

$L\vec{a}\cdot m\vec{b} = (Lm)\,\vec{a}\cdot\vec{b}$

④ $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

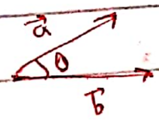
⑤ $\vec{a}\cdot\vec{b} = 0$, either $\vec{a} = 0$ oR $\vec{b} = 0$ oR $\theta = \pi/2$

⑥ if $\vec{a}$ is orthogonal to $\vec{b}$

$\vec{a} \perp \vec{b} \Rightarrow \vec{a}\cdot\vec{b} = 0$

Ex. $\|\vec{a}\| = 5$, $\|\vec{b}\| = 8$, $\theta = \frac{2\pi}{3}$

Find $\|2\vec{a} - 3\vec{b}\| = ?$



$$\|2\vec{a} - 3\vec{b}\|^2 = (2\vec{a} - 3\vec{b})(2\vec{a} - 3\vec{b})$$
$$= 4\vec{a}\cdot\vec{a} - 6\vec{a}\cdot\vec{b} - 6\vec{b}\cdot\vec{a} + 9\vec{b}\vec{b}$$
$$= 4\|\vec{a}\|^2 - 12\,\vec{a}\cdot\vec{b} + 9\|\vec{b}\|^2$$
$$= 4(5)^2 - 12\|\vec{a}\|\|\vec{b}\|\cos\frac{2\pi}{3} + 9(8)^2$$
$$= 100 - \overset{6}{12}(5)(8)\left(-\frac{1}{2}\right) + 9(64)$$
$$\|2\vec{a} - 3\vec{b}\|^2 = 100 + 240 + 576$$
$$\|2\vec{a} - 3\vec{b}\| = \sqrt{(\quad)}$$

Ex. If $\|\vec{a}\| = 0 \longrightarrow \vec{a} = \vec{0}$

i.e $\vec{a} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

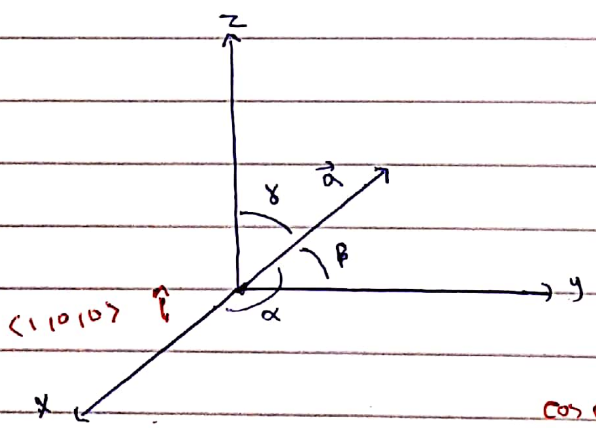
* Prof let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

if $\|\vec{a}\| = 0 \Rightarrow a_1 = 0 = a_2 = a_3$

$$\sqrt{a_1^2 + a_2^2 + a_3^2} = 0$$
$$a_1^2 + a_2^2 + a_3^2 = 0$$
$$\Rightarrow a_1^2 = 0 \ \& \ a_2^2 = 0 \ \& \ a_3^2 = 0$$
$$\rightarrow a_1 = 0 \ \& \ a_2 = 0 \ \& \ a_3 = 0$$

α, β, γ $\qquad$ $\cos\alpha, \cos\beta, \cos\gamma$

* Direction angles & the direction Cosines of a vector;



* $\vec{a} = \|\vec{a}\| \underbrace{(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})}$ $\rightarrow \|\,\|=1$

$\vec{u}_a$ = unit vector in the direct of $\vec{a}$.

* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

* $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\cos\alpha = \frac{\vec{a}\cdot\hat{i}}{\|\vec{a}\|\|\hat{i}\|} = \boxed{\frac{a_1}{\|\vec{a}\|}}$$

$$\cos\beta = \frac{a_2}{\|\vec{a}\|}$$

$$\cos\gamma = \frac{a_3}{\|\vec{a}\|}$$

direction Cosines

Ex. $\vec{a} = 2\hat{i} - 6\hat{j} + 3\hat{k}$

Find the direction of $\vec{a}$.

$\cos\alpha = \frac{a_1}{\|a\|}$; $\|\vec{a}\| = \sqrt{4+36+9} = \sqrt{49} = 7$

$\cos\alpha = \frac{2}{7} \rightarrow \alpha = \cos^{-1}\left(\frac{2}{7}\right)$

$\cos\beta = \frac{-6}{7} \rightarrow \beta = \cos^{-1}\left(\frac{-6}{7}\right)$

$\cos\gamma = \frac{3}{7} \rightarrow \gamma = \cos^{-1}\left(\frac{3}{7}\right)$

Ex. ① Can these angles $45°, 60°, 30°$ be direction angels of any vector?

$\cos^2 45 + \cos^2 60 + \cos^2 30 = 1$??

$\frac{1}{2} + \frac{1}{4} + \frac{3}{4} = 1$

$1\frac{1}{2} \neq 1$ No

② $\alpha = 30$, $\beta = 60$, find $\gamma = ??$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\frac{3}{4} + \frac{1}{4} + \cos^2\gamma = 1$

$1 + \cos^2\gamma = 1$

$\cos^2\gamma = 0$

$\boxed{\gamma = 90}$

③ $\alpha = 45$, $\beta = 60$ find $\gamma = ??$

$\cos^2 45 + \cos^2 60 + \cos^2\gamma = 1$

$\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$

$\cos^2\gamma = \frac{1}{4}$

$\cos\gamma = \pm\frac{1}{2}$

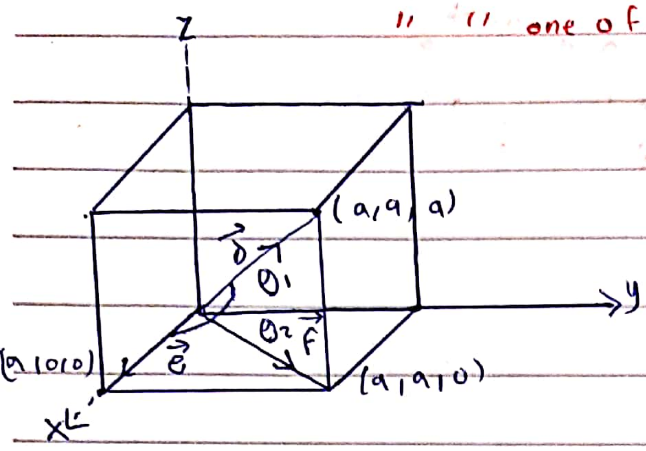
$\cos\gamma = \frac{1}{2} \rightarrow \gamma = 60°$ acute

$\cos\gamma = -\frac{1}{2} \rightarrow \gamma = 120°$ obtuse منفرجة

Scanned by CamScanner

Ex. θ_1 angle between a diagonal of a cube &

" " one of its edges.



$\vec{d} = \langle a,a,a \rangle$

$\vec{e} = \langle a,0,0 \rangle$

$$\cos\theta_1 = \frac{\vec{d}\cdot\vec{e}}{\|\vec{d}\|\,\|\vec{e}\|}$$

$$= \frac{a^2}{\sqrt{a^2+a^2+a^2}\,a}$$

$$= \frac{a^2}{\sqrt{3a^2}\,a} = \frac{a^2}{\sqrt{3}\,aa}$$

$$\cos\theta_1 = \frac{1}{\sqrt{3}} \rightarrow \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

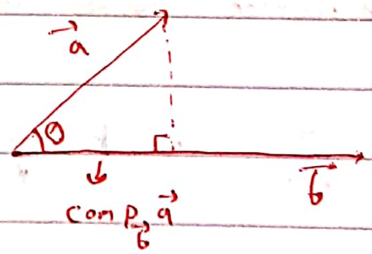
diagonal of face

$\vec{f} = \langle a,a,0 \rangle$ $\quad\quad$ $\vec{e} = \langle a,0,0 \rangle$

$\cos\theta_2 = ??$

* Scalar Projection of $\vec{a}$ onto $\vec{b}$

$$\equiv \text{Comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$$



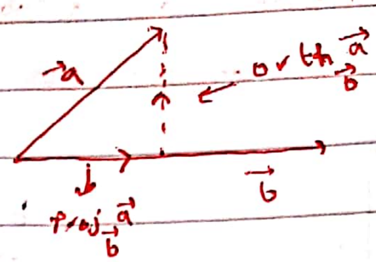
Proof $\quad \text{Comp}_{\vec{b}}\,\vec{a} = \|\vec{a}\|\cos\theta$

$$= \|\vec{a}\|\cos\theta\,\frac{\|\vec{b}\|}{\|\vec{b}\|}$$

$$\boxed{\text{Comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}}$$

* Vector projection of $\vec{a}$ onto $\vec{b}$

$$\equiv \text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|b\|} \cdot \frac{\vec{b}}{\|\vec{b}\|}$$



$$\boxed{\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b}}$$

* proj. of orthogonal to $\vec{b}$

$$\equiv \text{orth}_{\vec{b}}\vec{a} = \vec{a} - \text{proj}_{\vec{b}}\vec{a}$$

Ex. $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$\vec{b} = 3\hat{i} + 6\hat{j} - 4\hat{k}$

Find ① the angle between $\vec{a}$ & $\vec{b}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{6-18-16}{\sqrt{4+9+16}\,\sqrt{9+36+16}}$$

$$\theta = \cos^{-1}(\quad) \qquad = \frac{-28}{\sqrt{29(61)}}$$

② Find the direction cosines of $\vec{a}$

$$\cos\alpha = \frac{a_1}{\|\vec{a}\|} = \frac{2}{\sqrt{29}}, \quad \cos\beta = \frac{-3}{\sqrt{29}}, \quad \cos\gamma = \frac{4}{\sqrt{29}}$$

③ $\text{comp}_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} = \dfrac{-28}{\sqrt{61}}$

④ $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} = \dfrac{-28}{\sqrt{29}}$

⑤ $\text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\cdot\vec{b}$

$$= \frac{-28}{61}\vec{b}$$

$$= -\frac{84}{61}\hat{i} - \frac{168}{61}\hat{j} - \frac{112}{61}\hat{k}$$

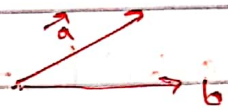
⑥ $\text{orth}_{\vec{b}}\vec{a} = \vec{a} - \text{proj}_{\vec{b}}\vec{a}$

$$= \langle 2,-3,4\rangle - \langle \frac{84}{61} + \frac{168}{61} + \frac{112}{61} \rangle$$

$$\text{orth}_{\vec{b}}\vec{a} = \langle \frac{206}{61}, \frac{-15}{61}, \frac{356}{61} \rangle$$

⑦ $\text{proj}_{\vec{b}}\vec{a} + \text{orth}_{b}\vec{a} = \vec{a}$

§ 12.4 The cross (vector) product



$\vec{a}$ & $\vec{b}$ are two vectors in $\mathbb{R}^3$

$\Rightarrow$ $\vec{a}\times\vec{b}$ is a vector ;

مقدار ① $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\| \sin\theta$

الاتجاه ② $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ & $\vec{b}$

i.e $\vec{a}\times\vec{b}$ is " to the plane that contains $\vec{a}$ & $\vec{b}$

$\Rightarrow$ ① $\vec{a}\cdot\vec{a}\times\vec{b} = 0$, since $\vec{a}\times\vec{b} \perp \vec{a}$

② $\vec{b}\cdot\vec{a}\times\vec{b} = 0$, " $\vec{a}\times\vec{b} \perp \vec{b}$

$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}_{2\times2}$ matrix

$\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}_{2\times2} = 1(3) - 2(-1)$

$= 5 \in \mathbb{R}$

Ex. $\begin{array}{ccc} + & - & + \end{array}$
$\begin{vmatrix} 2 & 3 & 4 \\ -1 & 0 & 5 \\ 6 & 2 & 1 \end{vmatrix}_{3\times3} = 2 \begin{vmatrix} 0 & 5 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} -1 & 5 \\ 6 & 1 \end{vmatrix} + 4 \begin{vmatrix} -1 & 0 \\ 6 & 2 \end{vmatrix}$

$= 2(-10) - 3(-1-30) + 4(-2-0)$

$= -20 + 93 - 8 \in R$

* $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

* Properties:

① $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

② $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

③ $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

④ $\vec{a} \times \vec{a} = 0$

⑤ $\vec{a} \parallel \vec{b} \Rightarrow \vec{a} \times \vec{b} = 0$

⑥ $L, m \in R$

$L\vec{a} \times m\vec{b} = (Lm)\,\vec{a} \times \vec{b}$

Ex. $\vec{a} = 2\hat{i} - 4\hat{k}$

$\vec{b} = -3\hat{i} + 2\hat{j} + 5\hat{k}$.

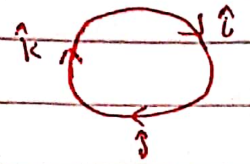
Find $2\vec{a} \times -3\vec{b} = -6\,\vec{a} \times \vec{b}$

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -4 \\ -3 & 2 & 5 \end{vmatrix}$

$= -8\hat{i} + 2\hat{j} + 4\hat{k}$

$= -6 \langle 8, 2, 4 \rangle$

$= \langle -48, -12, -24 \rangle$



$\hat{i} \times \hat{j} = k$

$\hat{j} \times \hat{i} = -\hat{k}$

$k \times \hat{j} = -\hat{i}$

$\hat{j} \times \hat{j} = 0$

$$\hat{i} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 0\hat{i} - 0\hat{j} + \hat{k} = \boxed{\hat{k}}$$

* $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

Ex. Give an example.

$\hat{i} \times (\hat{i} \times \hat{j}) \neq (\hat{i} \times \hat{i}) \times \hat{j}$

$\hat{i} \times \hat{k} \quad = \quad 0 \times \hat{j}$

$-\hat{j} \quad \neq \quad 0$

$\boxed{*}$ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Ex $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ & $\vec{c} = 3\hat{i} + 4\hat{j} + \hat{k}$

Find $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$= (6 - 4 + 3)\vec{b} - (2 - 2 - 3)\vec{c}$

$= 5\vec{b} + 3\vec{c}$

$= \langle 5, 10, -5 \rangle + \langle 9, 12, 3 \rangle$

$= \langle 14, 22, -2 \rangle$

$= 14\hat{i} + 22\hat{j} - 2\hat{k}$

?? $(\vec{a} \times \vec{b}) \times \vec{c} = -[\vec{c} \times (\vec{a} \times \vec{b})]$

$= -[(\vec{c} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}]$

Ex. $\vec{a}, \vec{b}$ & $\vec{c}$ are three vectors:

$\|\vec{a}\| = 2$, $\|\vec{b}\| = 4$, θ = angle between $\vec{A}$ & $\vec{b}$ is $\pi/3$

$\vec{c} = 2\hat{i} - 3\hat{j} + \hat{k}$, & $\vec{a} \perp \vec{c}$.

Find $\vec{a} \times (\vec{b} \times \vec{c}) = (\overset{0}{\vec{a} \cdot \vec{c}})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$= 0\vec{b} - \|\vec{a}\| \, \|\vec{b}\| \cos\frac{\pi}{3}$

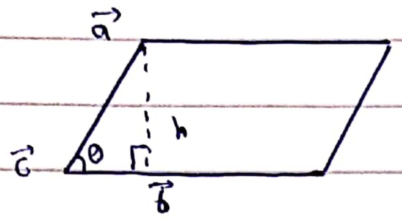
Scanned by CamScanner

$= -\cancel{2}\,(4)\,(\frac{1}{\cancel{2}})\,\vec{c}$

$= -4\vec{c}$

$= -8\hat{i} + 12\hat{j} - 4\hat{k}$

* Geometrical meaning of $\|\vec{a}\times\vec{b}\|$ (المطلق) is the area of the parallelogram (متوازي الاضلاع) with sides $\vec{a}$ & $\vec{b}$.



area ▱ $\overset{?}{=} \|\vec{a}\times\vec{b}\|$

area ▱ $=$ (base)(height)

$= \|\vec{b}\|\, h$

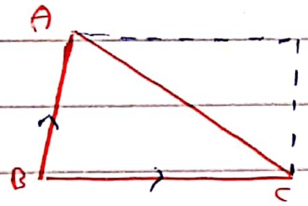
$= \|\vec{b}\|\,\|\vec{a}\|\,\sin\theta$

$= \|\vec{b}\times\vec{a}\|$

$= \|\vec{a}\times\vec{b}\|$

$h = \|\vec{a}\|\sin\theta$

Ex: Find the area of the triangle with vertices $A(2,0,1)$, $B(-1,1,0)$, $C(4,5,1)$



area $\Delta = \frac{1}{2}$ area ▱

$= \frac{1}{2}\|\vec{BA}\times\vec{BC}\|$

$= \frac{1}{2}\sqrt{25+4+(17)^2}$

الاحداثيات النهاية ناقص البداية

$$\vec{BA}\times\vec{BC} = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 5 & 4 & 1 \end{vmatrix} \end{array}$$

$= -5\hat{i} + 2\hat{j} + 17\hat{k}$

Ex. Find two unit vectors orthogonal (بناتج الـ cross بطلع العامودي) to the two vectors $\underbrace{2\hat{i}+\hat{j}-\hat{k}}_{\vec{a}}$ & $\underbrace{-3\hat{i}+2\hat{j}+3\hat{k}}_{\vec{b}}$

$\mp \dfrac{\vec{a}\times\vec{b}}{\|\vec{a}\times\vec{b}\|}$

$$\vec{a}\times\vec{b} = \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ -3 & 2 & 3 \end{vmatrix}$$

$$= \langle 5, -3, 7 \rangle$$

$$= \pm \frac{1}{\sqrt{25+9+49}} \langle 5, -3, 7 \rangle$$

* Triple Scalar product: (Cross بعدين dot)

$$\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Ex: $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$

$\vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$

$\vec{c} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

Find $\vec{a}\cdot\vec{b}\times\vec{c} =$

ما حطينا $\hat{i}, \hat{j}, \hat{k}$
لأنه ضرب dot.

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 2 & -1 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= (4-2) - 1(7) - 2(-10)$$

$$2 - 7 + 20$$

$$= \boxed{15}$$

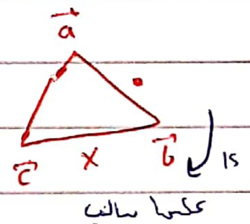
Ex ② $\vec{b}\times\vec{c}\cdot\vec{a} = \vec{a}\cdot\vec{b}\times\vec{c} = \boxed{15}$

③ $\vec{b}\cdot\vec{a}\times\vec{c} = -15$

④ $2\vec{c}\cdot 3\vec{b}\times\vec{a} = 6(\vec{c}\cdot\vec{b}\times\vec{a})$

$= 6(-15) = -90$

⑤ $\vec{c}\cdot\vec{b}\times\vec{c} = 0$



* $|\vec{a}\cdot\vec{b}\times\vec{c}|$ is the volume of the parallele piped with adjacent Sides $\vec{a}$ & $\vec{b}$ & $\vec{c}$

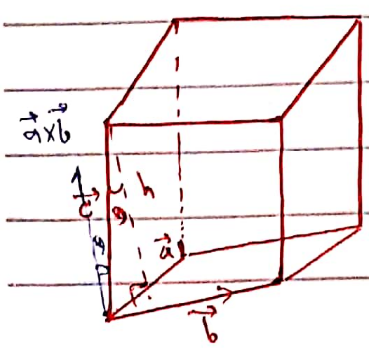
$V = (\text{area base})\ \text{height}$

$= \|\vec{a}\times\vec{b}\|\, h$

$= \|\vec{a}\times\vec{b}\|\ \|\vec{c}\|\cos\theta$

$= |\vec{a}\times\vec{b}\cdot\vec{c}|$

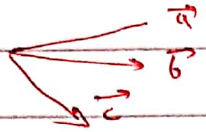
$= |\vec{a}\cdot\vec{b}\times c|$



Scanned by CamScanner

تعني الـ plane

* if $\vec{a} \cdot \vec{b} \times \vec{c} = 0 \iff \vec{a}, \vec{b}, \& \vec{c}$ are Coplaner



Ex. Are these pts. Coplaner or Not.
$A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$ & $D(3,6,-4)$

$? \; \overline{AB} \cdot \overline{AC} \times \overline{AD} \overset{?}{=} 0$

$$\begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = ?\, 0$$

$2(6+6) + 4(-24+4) + 4(12+2) \overset{?}{=} 0$

$24 - 80 + 56 = 0$

$80 - 80 = 0$

Yes

Ex. Show that $|\bar{a} \times \bar{b}|^2 = |a|^2 |b|^2 - (\bar{a} \cdot \bar{b})^2$

$|\bar{a}|^2 |b|^2 \sin^2\theta \overset{?}{=} |\bar{a}|^2 |\bar{b}|^2 - |a|^2 |b|^2 \cos^2\theta$

$= |\bar{a}|^2 |\bar{b}|^2 (1 - \cos^2\theta)$

$|\bar{a}|^2 |\bar{b}|^2 \sin^2\theta = |\bar{a}|^2 |b|^2 \sin^2\theta$

Ex. if $\bar{a} \cdot \bar{b} = \sqrt{3}$ & $\bar{a} \times \bar{b} = \langle 1,2,2 \rangle$
Find the angle between $\bar{a}$ & $\bar{b}$

$\bar{a} \cdot \bar{b} = \sqrt{3} \longrightarrow |\bar{a}|\,|\bar{b}| \cos\theta = \sqrt{3}$ --- ①

$\bar{a} \times \bar{b} = \langle 1,2,2 \rangle \longrightarrow \|\bar{a} \times \bar{b}\| = \sqrt{1+4+4} = 3$

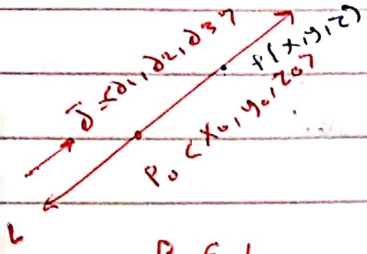
$\|\bar{a}\|\,\|\bar{b}\| \sin\theta = 3$ --- ②

$\frac{2}{1}$ $\quad \tan\theta = \frac{3}{\sqrt{3}} = \sqrt{3}$

$\theta = \frac{\pi}{3}$

§ 12.5 Line & planes :

* Lines



$P_0 \in L$

$\vec{v} \parallel L$

$\overrightarrow{P_0P} \parallel \vec{v}$ يوازي

$\overrightarrow{P_0P} = t\,\vec{v}$ $\{t \in \mathbb{R}\}$

(I) Eqn. of line in vector notation

$$\boxed{(x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k} = t\,(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})}$$

(II) Parametric eqn. of line

$$\Leftrightarrow \quad x - x_0 = t\,v_1 \rightarrow \quad x = x_0 + t\,v_1$$
$$y - y_0 = t\,v_2 \rightarrow \quad y = y_0 + t\,v_2$$
$$z - z_0 = t\,v_3 \rightarrow \quad z = z_0 + t\,v_3$$

(III) Symmetric eqn of line.

$$\boxed{\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}}$$

Ex Write Parametric eqns. of line passes throught, $P_0(-2, 1, 3)$ & Parallel to $2\hat{i} - 3\hat{j} + 4\hat{k} = \vec{v}$

$L:$ $x = -2 + 2t$, $y = 1 - 3t$, $z = 3 + 4t$

(ii) Give a pt. on L other than P_0.

$t = 1 \rightarrow$ $x = 0$, $y = -2$, $z = 7$ $\quad (0, -2, 7)$

(iii) Is this pt. $(-4, 4, -1)$ on L or not (لازم نفس t)

$-4 = -2 + 2t \rightarrow \boxed{t = -1}$

$4 = 1 - 3t \rightarrow \boxed{t = -1}$

$t = -1 \rightarrow z = 3 - 4 = -1 = z$ ✓

yes

Ex. Write parametric eqns. of line ($L = ?$) passes throught $P_0(2,-1,3)$ & Parallel to line

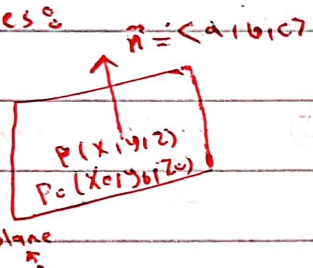
$$L_1: \frac{2x-1}{6} = \frac{1-3y}{12} = -z$$

$$L \parallel L_1 \iff \vec{d}_L \overset{\parallel}{=} \vec{d}_{L_1} = \left\langle \frac{6}{2}, \frac{12}{-3}, \frac{1}{-1} \right\rangle$$ المقام عن المعاملات

$$= \langle 3, -4, -1 \rangle$$

$$L: \; x = 2+3t, \; y = -1-4t, \; z = 3-t$$

* Planes:



$$\vec{n} \perp \overline{P_0P}$$

$$\vec{n} \cdot \overline{P_0P} = 0$$

$P_0 \in \pi$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$ eqn. of plane

$\vec{n} \perp \pi$

$$\overline{P_0P} = \langle x-x_0, y-y_0, z-z_0 \rangle$$

$$ax + by + cz + \overbrace{(-ax_0 - by_0 - cz_0)}^{d} = 0$$

$$\boxed{ax + by + cz + d = 0}$$

Scanned by CamScanner

$\pi \equiv$ plane

$P_0(x_0, y_0, z_0) \in \pi$

$\bar{n} = \langle a, b, c \rangle$

$\bar{n} \perp \pi$

$\pi: a(x - x_0) + b(y - y_0) + c(z - z_0)$

$ax + by + cz + d = 0$

$d = -ax_0 - by_0 - cz_0$

Ex: Write an eqn. of the plane containing the pt. $P_0(-1, 2, 3)$ of perpendicular to the line

$\pi = ??$

$$\frac{x-1}{2} = \frac{2-y}{1} = \frac{-z}{3}$$

$$\bar{n} = \bar{d}_L = \langle 2, \tfrac{+1}{-1}, \tfrac{3}{-1} \rangle$$

$$\bar{n} = \langle 2, -1, -3 \rangle$$

$$\pi: 2(x+1) - 1(y-2) - 3(z-3) = 0$$

$$\pi: 2x - y - 3z + 13 \cancel{8} = 0$$

Find a pt on π:

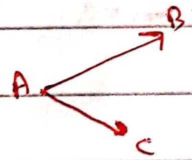
$x = 0, \ z = 0 \rightarrow y = ?$

$-y + 13 = 0$

$\boxed{y = 13}$

$(0, 13, 0) \in \pi$

Ex: Write an eqn of the plane containing the three pts:
$A(2,1,3)$, $B(0,-1,1)$, $C(1,2,0)$



$$\bar{n} = \overline{AB} \times \overline{AC}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -2 & -2 \\ -1 & 1 & -3 \end{vmatrix}$$

$$\bar{n} = 8\hat{i} - 4\hat{j} - 4\hat{k}$$

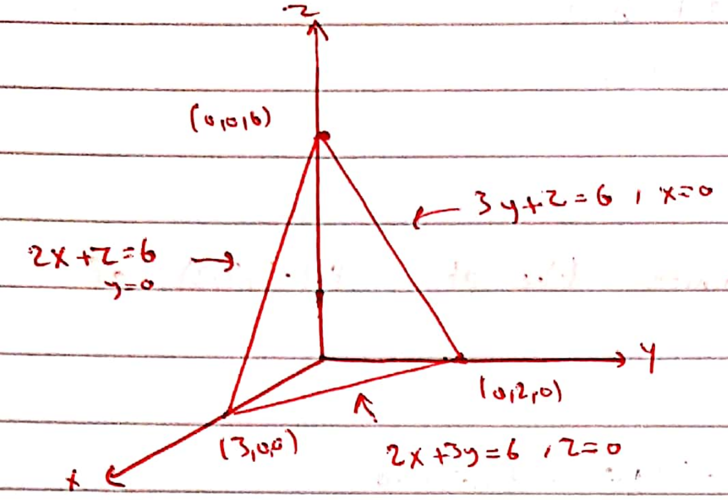
$$\pi: 8(x-2) - 4(y-1) - 4(z-3) = 0$$

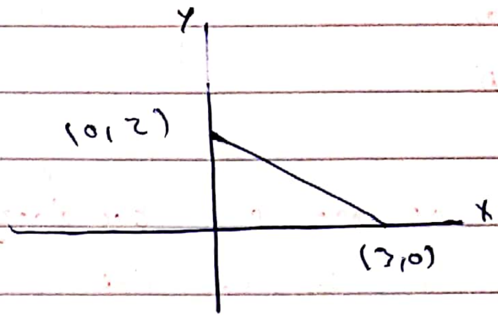
Scanned by CamScanner

Ex: $\pi: 2x+3y+z=6$.

لايجاد تقاطع Plane مع Coplane نصفر

أي معادلة لخط مستقيم بـ 2space دايما Symmetrical





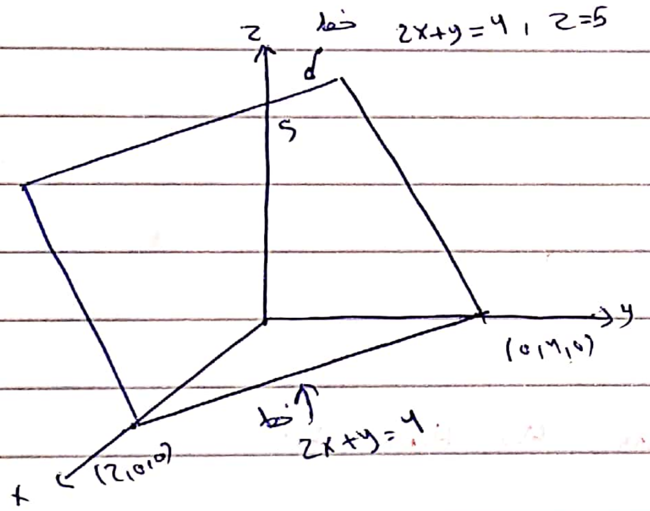
$$\frac{x}{3}+\frac{y}{2}=1$$

$$\frac{2x+3y}{6}=1$$

$$2x+3y=6$$

$$L: \frac{2x}{1}, \frac{6-3y}{1}, z=0$$

Ex: $\pi: 2x+y=4$



Ex. Find Parametric eqns. of line of intersection of the two planes:

$L=??$

$\pi_1: x+2y+z=1$

$\pi_2: 2x-y-z=3$

Find two pts. on Line of int.

P_1 & P_2

$x=0 \xrightarrow{\pi_1} 2y+z=1$

$\xrightarrow{\pi_2} -y-z=3$

$y=4$ ⟩ $P(0,+4,-7) \in L$

$z=-3-y$

$-3-4$

$z=\boxed{-7}$

Scanned by CamScanner

or $y=0 \xrightarrow{\pi_1} x+z=1$

$\xrightarrow{\pi_2} 2x-z=3$

$3x=4$

$x=\frac{4}{3}$

$z=1-x$

$z=1-\frac{4}{3}$

$\boxed{z=-\frac{1}{3}}$ $\therefore P_2\left(\frac{4}{3},0,-\frac{1}{3}\right)$

$\therefore \vec{d}_L \parallel \overrightarrow{P_1P_2} = \langle \frac{4}{3}, -4, \frac{20}{3} \rangle$

OR $\vec{d}_L = \langle 4, -12, 20 \rangle$

$L:$ $x = 0 + 4t = 4t$

$y = 4 - 12t$

$z = -7 + 20t$

* Direction of line of int of the two planes π_1 & π_2

$$\vec{d}_L = \bar{n}_1 \times \bar{n}_2$$

Proof:

$\bar{n}_1 \times \bar{n}_2 \perp \bar{n}_1$ & $\bar{n}_1 \perp \pi_1 \Longrightarrow \bar{n}_1 \times \bar{n}_2 \parallel \pi_1$

& $\bar{n}_1 \times \bar{n}_2 \perp \bar{n}_2$ & $\bar{n}_2 \perp \pi_2 \Longrightarrow \bar{n}_1 \times \bar{n}_2 \parallel \pi_2$

$\therefore \bar{n}_1 \times \bar{n}_2 \parallel \pi_1$ & π_2

$\therefore \bar{n}_1 \times \bar{n}_2 \parallel$ line of int. of π_1 & π_2

$L = ??$

Ex. Write parametric eqns. of the line passes through pt. $(-1,2,3)$ & is parallel to the two planes

$\pi_1: 2x - y = 4 - 2z \rightarrow 2x - y + 2z = 4$

$\pi_2: x + 2y + 3z = 5$

$\therefore \vec{\partial}_L \parallel$ Line of int. of π_1 & π_2

$\therefore \vec{\partial}_L \parallel \vec{n}_1$ & $\vec{n}_2$

$\therefore \partial_L = \vec{n}_1 \times \vec{n}_2$

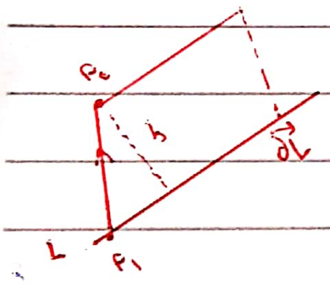
$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \langle -7, -4, 5 \rangle$$

$$= \vec{\partial}_L$$

$L: \quad x = -1 - 7t, \quad y = 2 - 4t, \quad z = 3 + 5t$

* Distance between a pt & a line.



$h = \partial(P_0, L) = \dfrac{\text{area} \,\square}{\text{base}}$

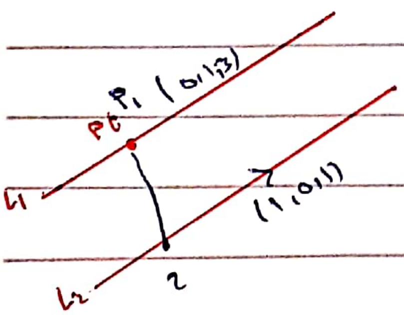
$$\partial(P_0, L) = \frac{\| \vec{P_1 P_0} \times \vec{\partial}_L \|}{\| \vec{\partial}_L \|}$$

area $\square$ = base . h

Ex: $L_1: \quad x = 2t, \quad y = 1 - 4t, \quad z = 3 + 6t$ متوازيين اقسم الكرسي العيز

$L_2: \quad x = 1 - s, \quad y = 2s, \quad z = 1 + 3s$

$L_1 \parallel L_2$: Since $\vec{\partial}_{L_1} = -2\, \vec{\partial}_{L_2}$, find distance between L_1 & L_2



$$\partial(L_1, L_2) = \partial(P_1, L_2)$$

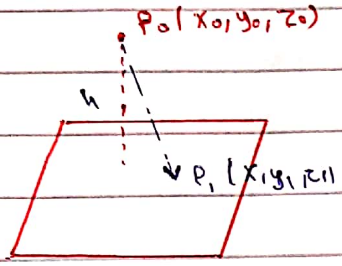
$$= \frac{\| \vec{P_1 P_2} \times \vec{\partial}_{L_2} \|}{\| \vec{\partial}_{L_2} \|}$$

$$\vec{P_1 P_2} \times \vec{\partial}_{L_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -2 \\ -1 & 2 & -3 \end{vmatrix}$$

$= \langle 7, +5, 1 \rangle$

$$= \frac{\sqrt{49+25+1}}{\sqrt{1+4+9}}$$

* Distance between a pt. of a plane:



$$\boxed{d(P_0, \pi) = \frac{|a x_0 + b y_0 + c z_0 + d|}{\sqrt{a^2+b^2+c^2}}}$$

$\pi: ax + by + cz + d = 0$

$$h = \text{Comp}_{\vec{n}}\, \vec{P_0P_1}$$

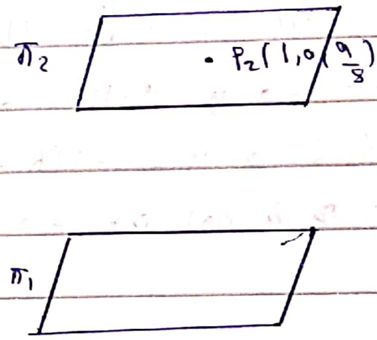
$$= \frac{|\vec{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|}$$

Ex: $\pi_1: 2x+3y-4z=1 \Rightarrow 2x+3y-4z-1=0$

$\pi_2: -4x-6y+8z=5$

distance between π_1 & π_2

$\pi_1 \parallel \pi_2$, Since $\pi_2 = -2\pi_1$



$-4+8z=5$

$8z=9$

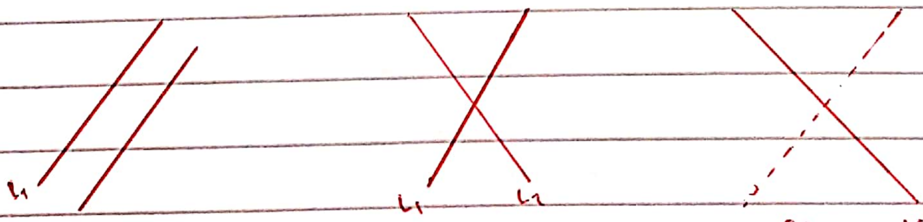
$z=\frac{9}{8}$

$$d(\pi_1, \pi_2) = d(P_2, \pi_1)$$

$$= \frac{|2(1)+3(0)-4(\frac{9}{8})-1|}{\sqrt{4+9+16}}$$

$$= \frac{|-\frac{7}{2}|}{\sqrt{29}} = \frac{\frac{7}{2}}{\sqrt{29}}$$

* in $\mathbb{R}^3$ Line are



Parallel — intersected — Skew متخالفين

$\underbrace{\text{Parallel} \qquad \text{intersected}}$

There is a plane containing L_1 & L_2

Not parallel & not intersected

No plane containing two skew lines.

Ex: L_1 : $x = 3+2t$, $y = 4-t$, $z = 1+3t$

L_2 : $x = 1+4s$, $y = 3-2s$, $z = 4+5s$

are these lines parallel , int or skew

① ? parallel No , $\vec{v}_{L_1} \times \vec{v}_{L_2} \neq 0$

② ? int.

$x = x \rightarrow 3+2t = 1+4s$ } → Same z

$y = x \rightarrow (4-t = 3-2s) \times 2$

$11 = 7$!!

$x = x \rightarrow 3(3+2t = 1+4s)$

& } ? Same y

$z = z \rightarrow -2(1+3t = 4+5s)$

$7 = -5 + 2s$

$12 = 2s \rightarrow \boxed{s=6}$

$s = 6 \rightarrow 3+2t = 1+4(6)$

$2t = 22 \rightarrow \boxed{t=11}$

L_1 ; $t = 11 \rightarrow y = 4-11 = -7$ } not Same y

L_2 ; $s = 6 \rightarrow y = 3-2(6) = -9$ ∴ Not int.

Not Parallel & not int. → Skew

Ex: Find the point at which this line

$L: x = 2+3t\ ,\ y = -4t\ ,\ z = 5+t$

intersects the plane:

$\pi: 4x + 5y - 2z = 18$

$4(2+3t) + 5(-4t) - 2(5+t) = 18$

$12t - 20t - 2t + 8 - 10 = 18$

$-10t = 20$

$\boxed{t = -2} \xrightarrow{L} \quad x = 2-6 = -4$

$y = -4(-2) = 8$

$z = 5-2 = 3$

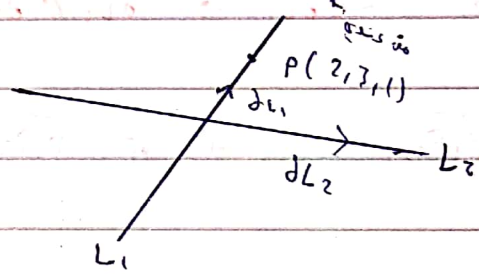
$P_0(-4, 8, 3)$

Ex: $L_1: x-2 = \dfrac{y-3}{-2} = \dfrac{z-1}{-3}$

$L_2: x-3 = \dfrac{y+4}{3} = \dfrac{z-2}{-7}$

Find an eqn. of the plane ($\pi = ?$) containing L_1 & L_2.

? parallel or ? intersected



$\bar{d}_{L_1} \times \bar{d}_{L_2} \neq 0 \rightarrow$ Not parallel $\Rightarrow$ int.

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 1 & 3 & -7 \end{vmatrix} = \langle 23, +4, 5 \rangle \neq 0$$

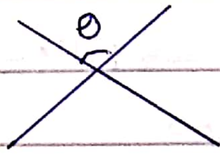
$n = \bar{d}_{L_1} \times \bar{d}_{L_2}$

$= \langle 23, 4, 5 \rangle$

$\pi: 23(x-2) + 4(y-3) + 5(z-1) = 0$

② Find the pt. of int.

③ angle between L_1 & L_2

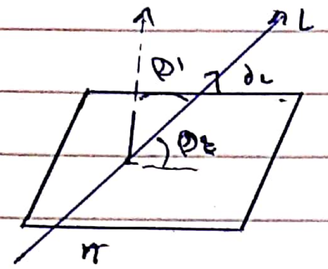


$$\cos\theta = \frac{\bar{d}_{L_1} \cdot \bar{d}_{L_2}}{\|d_{L_1}\|\,\|d_{L_2}\|}$$

* angle between two planes π_1 & π_2

= acute angle between $\bar{n}_1$ & $\bar{n}_2$

$$\cos\theta = \frac{|\bar{n}_1 \cdot \bar{n}_2|}{\|n_1\| \cdot \|n_2\|}$$



angle between a Line & a plane $= \theta$

$$\theta = \frac{\pi}{2} - \theta_1$$

θ_1 = acute angle, L & $\bar{n}$

δ_L & $\bar{n}$

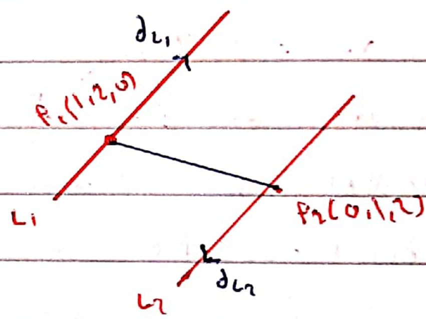
$$\cos\theta_1 = \frac{|\delta_L \cdot \bar{n}|}{\|\delta_L\| \, \|n\|}$$

Ex. Find eqn. plane containing the two lines

L_1: $x = 1+t$, $y = 2-3t$, $z = 2t$

L_2: $x = -2s$, $y = 1+6s$, $z = 2-4s$

$\bar{\delta}_{L2} = -2\bar{\delta}_{L1} \rightarrow \bar{\delta}_{L1} \parallel \bar{\delta}_{L2} \rightarrow L_1 \parallel L_2$



$\bar{n} = \overline{P_1P_2} \times \bar{\delta}_{L1}$ oR $\bar{n} = \overline{P_1P_2} \times \bar{\delta}_{L2}$

Ex: L_1: $x = 1+t$, $y = -2+3t$, $z = 4-t$

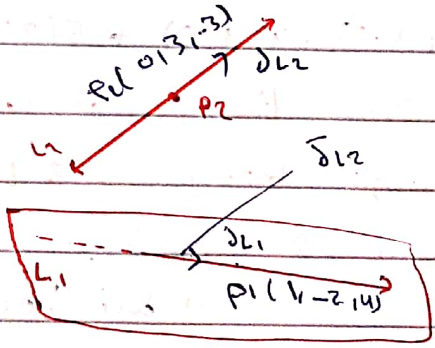
L_2: $x = 2s$, $y = 3+s$, $z = -3+4s$

Find the distance between L_1 & L_2

is ? parallel ? skew

? $\bar{\delta}_{L1} \times \bar{\delta}_{L2} \neq 0$ not parallel

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \langle 13, -6, -5 \rangle \neq 0 \quad \Rightarrow \boxed{\text{Skew}}$$



? $d(L_2, L_1) = d(L_2, \pi_1)$

$= d(P_2, \pi_1)$

(I) π_1 containing L_1 & is Parallel to L_2

تعامد 2planes ال Normal الهم يوازي الاخر

ما يكون plane عامودي ع 2planes و متعامد مع خط التقاطع بينهم

① ? π_1, $P_1 \in L_1 \rightarrow P_1 \in \pi_1$

$\bar{n} = \bar{d}_{L_1} \times \bar{d}_{L_2} \perp \pi_1$

$\pi_1: 13(x-1) - 6(y+2) - 5(z-4) = 0$

$d(L_1, L_2) = \cdots = d(P_2, \pi_1)$

$$= \frac{|13(0-1) - 6(3+2) - 5(-3-4)|}{\sqrt{13^2 + (-6)^2 + (5)^2}}$$

$\pi = ??$

Ex: Find the plane passes through $A(0,-2,5)$ & $B(-1,3,1)$ & is perpendicular to the plane π_1: $2z = 5x + 4y$. $\rightarrow \bar{n}_1 = \langle 5, 4, -2 \rangle$

$A \in \pi$

$\boxed{\begin{matrix} \overline{AB} \parallel \pi \\ \bar{n}_1 \parallel \pi \end{matrix}} \rightarrow \overline{AB} \times \bar{n}_1 = \bar{n}$

$$\bar{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 5 & -4 \\ 5 & 4 & -2 \end{vmatrix} = \langle 6, -22, -29 \rangle$$

$\pi: 6(x-0) - 22(y+2) - 29(z-5) = 0$

Ex: Find the plane π ∋ passes pt. $A(1,5,1)$ & in perpendicular ~~the~~ to

① the planes π_1: $2x + y - 2z = 2$

& π_2: $x + 4z = 4$

$\left.\begin{matrix} \pi \perp \pi_1 \\ \pi \perp \pi_2 \end{matrix}\right\} \rightarrow \pi \perp$ line of int. of π_1 & π_2

$\pi \perp \bar{d}_i = \bar{n}$ اتجاه خط التقاطع

$\bar{n}_1 \times \bar{n}_2$

$\bar{n} = \bar{n}_1 \times \bar{n}_2 \perp \pi$

$A \in \pi$

② ? parametric eqn. of $\overset{L?}{\text{Line}}$ passes the $A(1,5,1)$ & parallel to the Two planes π_1 & π_2

$A \in L$

$d_L \parallel$ line of int. of π_1 & π_2

$\bar{d}_1 = \bar{n}_1 \times \bar{n}_2$

$\pi = ?$

Ex. Find the plane that passes through the line of int of the two plane $\pi_1 : x - z = 1$

$\pi_2 : y + 2z = 3$

& is perpendicular to the plane; $\pi_3 : x + y - 2z = 1$?

$A \in$ line of int. π_1 & π_2

$A \in \pi$

& d line of int. $\parallel \pi$

$\bar{n}_1 \times \bar{n}_2 \parallel \pi$ (1)

$$\bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

$= \langle 1, -2, 1 \rangle$

$A \in \pi$; نقطة تقاطع

π_1 & π_2

$z = 0 \xrightarrow{\pi_1} x = 1$

$\xrightarrow{\pi_2} y = 3$

$\therefore A(1, 3, 0) \in \pi$

$\pi \perp \pi_3 \rightarrow$ $\boxed{\bar{n}_3 \parallel \pi}$ (2)

& $\rightarrow$ $\boxed{\bar{n}_1 \times \bar{n}_2 \parallel \pi}$

$\bar{n}_3 \times (\bar{n}_1 \times \bar{n}_2) \perp \pi$

$\pi = ?$

Ex: Find an eqn. of plane that consists of all pts. that are equidistant from the two pts. $A(1, 0, -2)$ & $B(3, 4, 0)$

$M\left(\frac{1+3}{2}, \frac{4+0}{2}, \frac{-2+0}{2}\right)$

$M(2, 2, -1) \in \pi$

$A \quad M \quad B$

Mid pt $\in \pi$ $\quad \overline{AB}$ or $\overline{MB}$ or $\overline{MA}$

$= \bar{n}$

§ 12.6 Cylinders & Quadric Sufaces

* Cylinder,

Any eqn. in R^3 containing two variables of out of three is a cylinder with the missing variable as an axis (//)

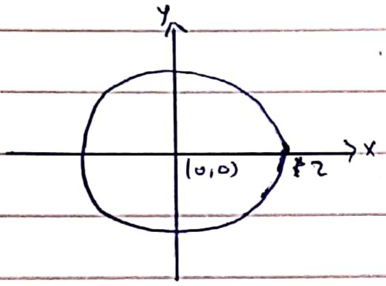
Ex Identify & graph.

$x^2+y^2=4$

① 2-Space.

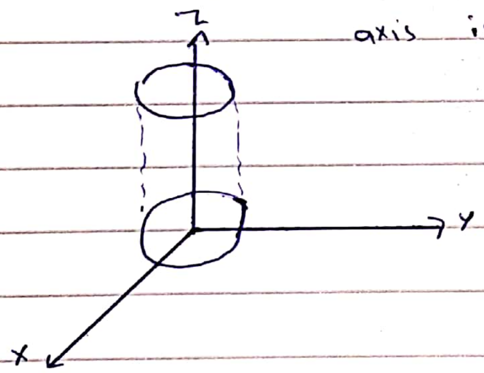
② 3- Space.

① 2Space: $x^2+y^2=4$, circle



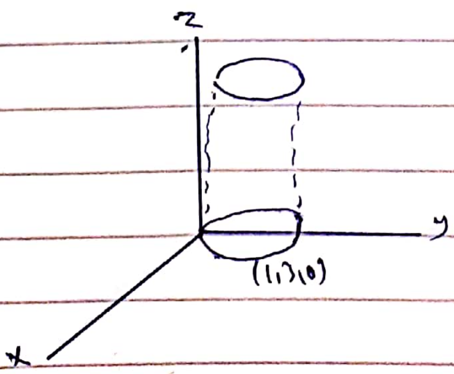
② 3 Space: Cylinde

axis is z-axis



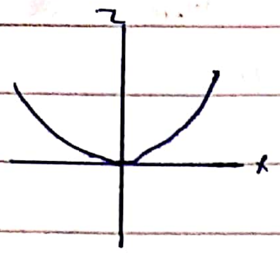
Ex: $(x-1)^2+(y-3)^2=4$ in R^3

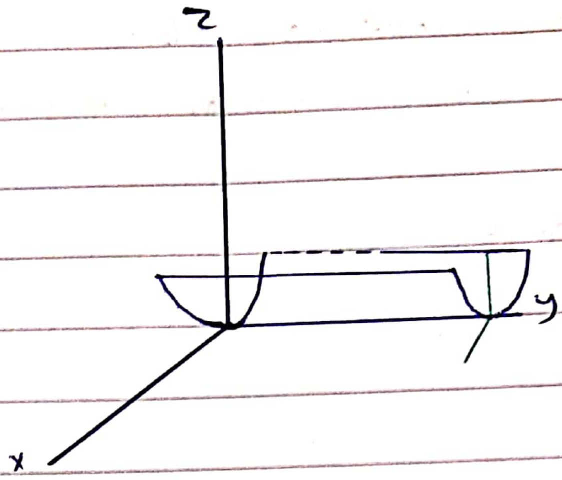
cylinder axis (//) z



Ex $z = x^2$

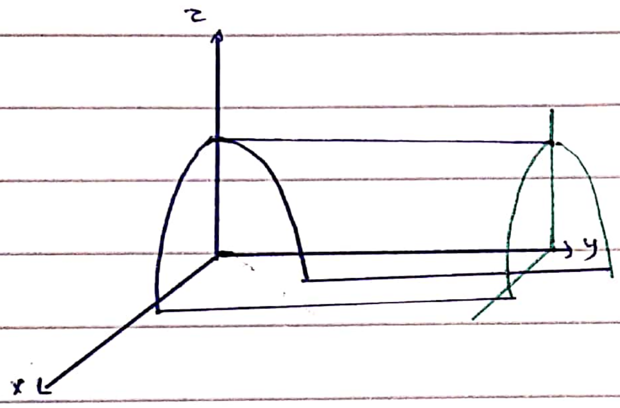
Cylinder axis is y-axis





Ex $z = 4 - x^2$

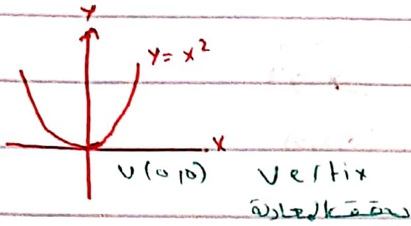
Cylinder // y-axis



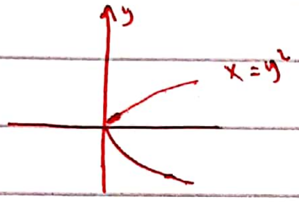
* Conic Section:

1 Parabola

$y = x^2$

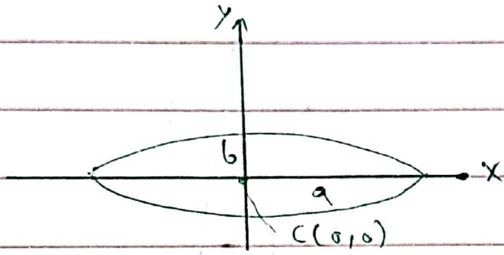


$x = y^2$



2 Ellipse:

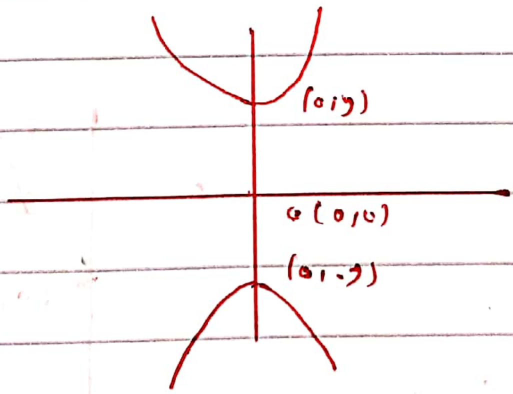
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Ex: $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{5} = 1$

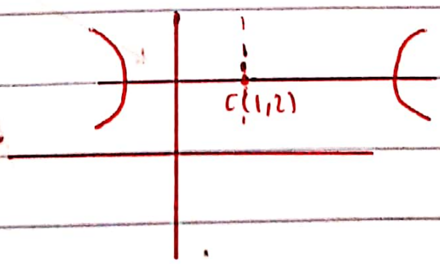
3 Hyperbola:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$



$(x-1)^2 - (y-2)^2 = 9$

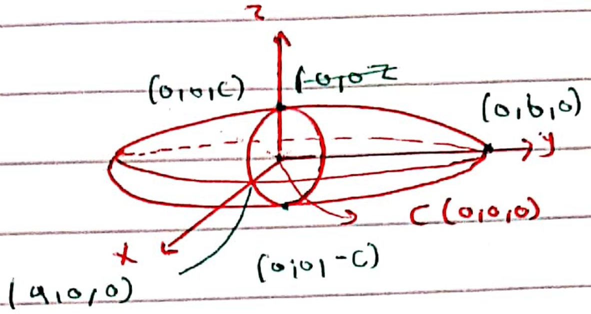
$\frac{(x-1)^2}{9} - \frac{(y-2)^2}{9} = 1$



* Quadric Surfaces:

[1] Ellipsoid:

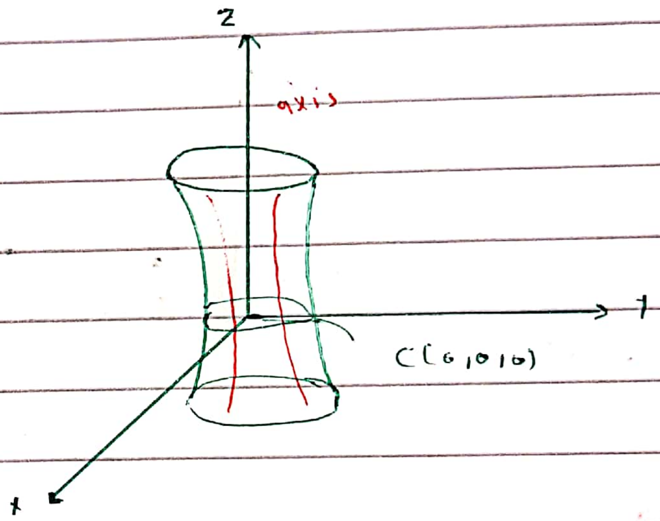
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



[2] Hyperboloid of one sheet:

لما يوجد سالب واحد والمحور اليمين موجب

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

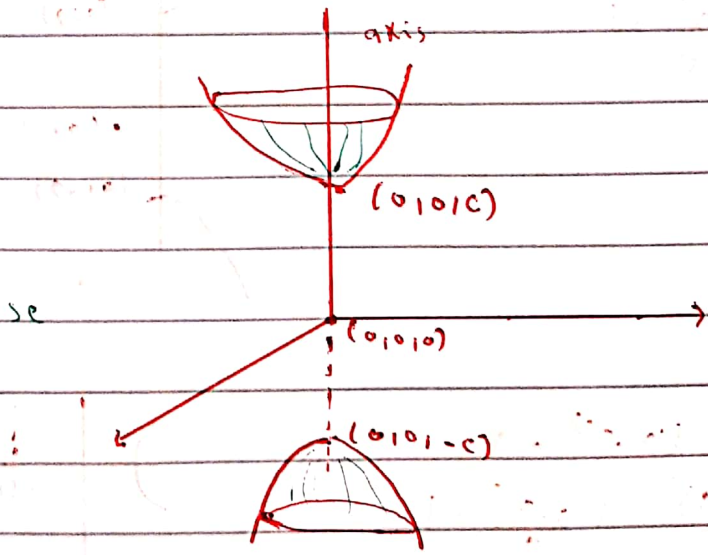


[3] Hyperboloid of 2-sheets:

لما ٢ سالب والمحور اليمين موجب

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

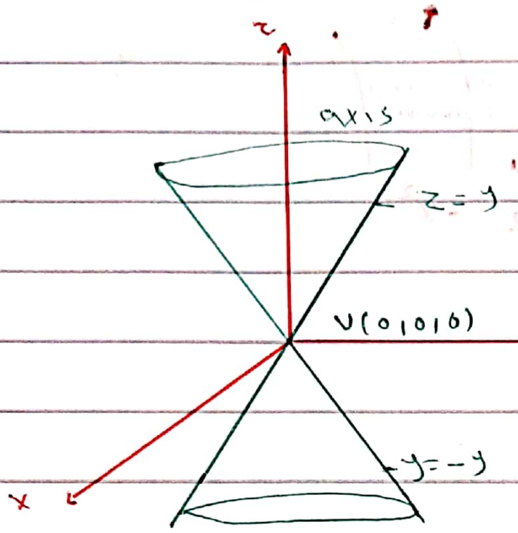
$z > c \Rightarrow$ ellipse



④ Double cone :

المحور حسب اي لحالها .

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \iff \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \iff \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$



$x = 0 \rightarrow$

$$\frac{z^2}{c^2} = \frac{y^2}{b^2}$$

$$z^2 = \frac{c^2}{b^2} y^2$$

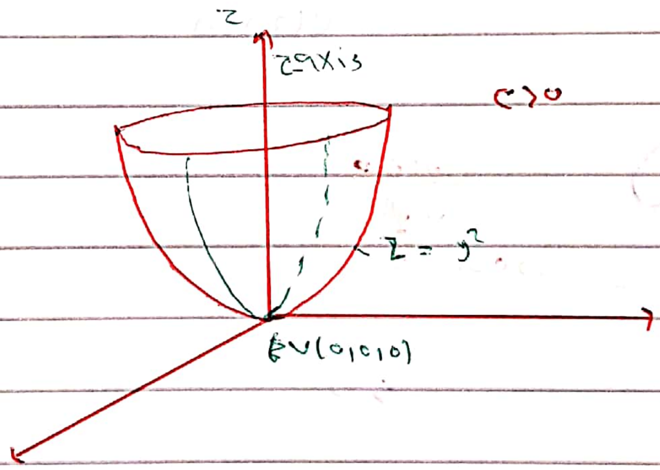
$$z = \pm \frac{c}{b} y$$

⊛ Function

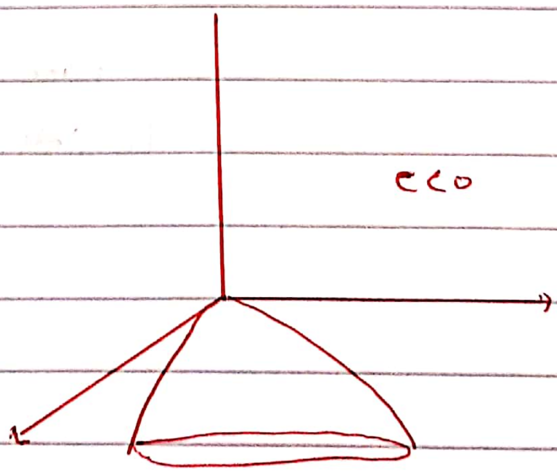
⑤ Paraboloid :

تتغير موجب او سالب

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



$c > 0$



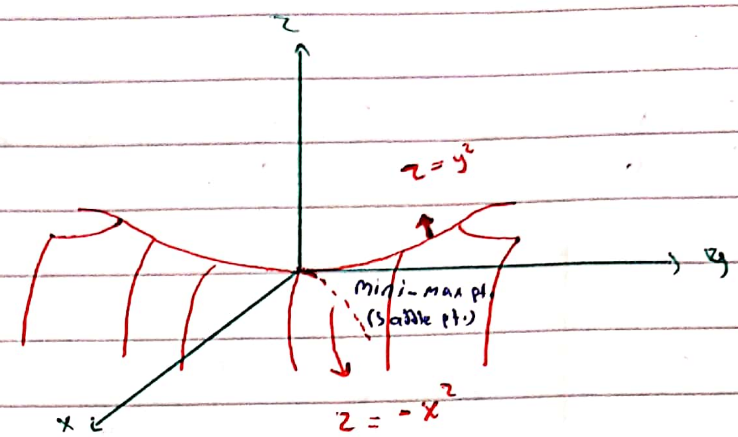
$c < 0$

* Function

$\boxed{6}$ Hyperbolic Paraboloid:

Saddle shaped function:

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



Ex: Identify this surface & graph:

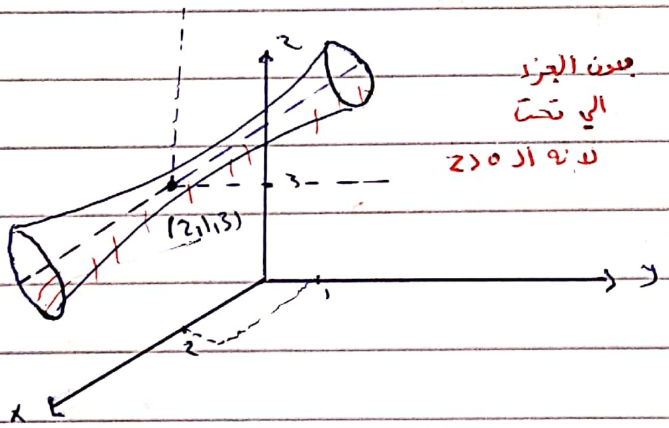
$$z = 3 + \sqrt{1 + (x-2)^2 - (y-1)^2}$$

$$z - 3 = \sqrt{1 + (x-2)^2 - (y-1)^2}$$

$$(z-3)^2 = 1 + (x-2)^2 - (y-1)^2$$

$$(z-3)^2 - (x-2)^2 + (y-1)^2 = 1 \quad \leftarrow \text{Hyperboloid of one sheet}$$

$C(2,1,3)$



بدون الجزء الي تحت لانه ال $z>0$

- اذا كانت المتغيرات كلها خطية Plane.

* اذا كانت 2 خطية (متغيرين) Cylinder

Ex: Identify & graph:

$x^2 - y^2 + z^2 - 4x + 2z = 0$

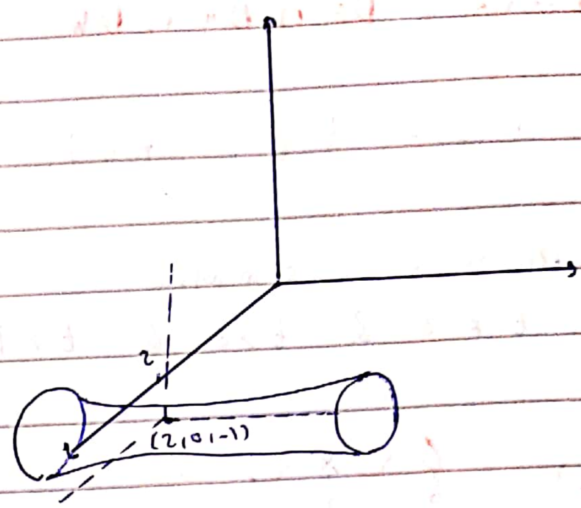
$x^2 - 4x + 2^2 - y^2 + z^2 + 2z + 1^2 = 2^2 + 1^2$

$(x-2)^2 - y^2 + (z+1)^2 = 5$

Hyperboloid of 1-sheet

at axis // y-axis.

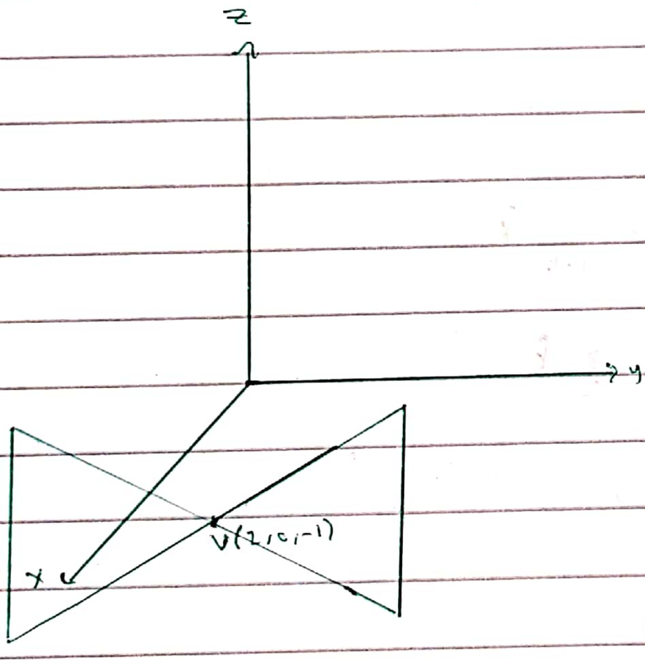
$c(2,0,-1)$



(ii) ------ $+5 = 0$

$= 5 - 5 = 0 \longrightarrow (x-2)^2 - y^2 + (z+1)^2 = 0$

$y^2 = (x-2)^2 + (z+1)^2$ double cone



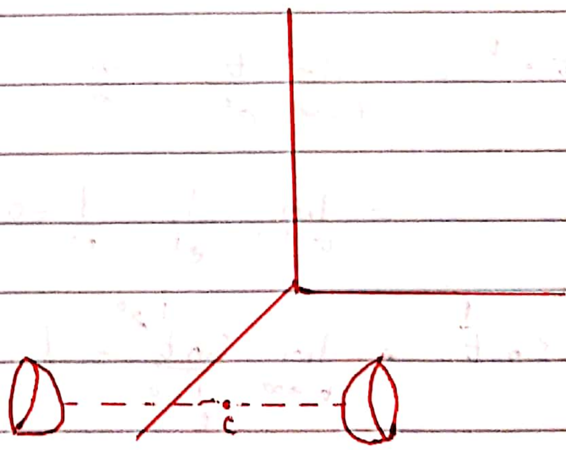
بال One Sheet , two يحدد المحور الاشارة الايجابية

(iii) --- $+6 = 0$

$5 - 6 = -1$

$y^2 - (x-2)^2 - (z+1)^2 = 1$

Hyperboloid of 2-Sheet



Ch#13 Vector Functions.

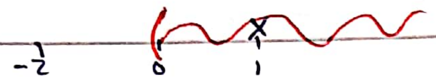
$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ Vector Function

Domain $= D_f \cap D_g \cap D_h$

Ex: $\bar{r}(t) = \frac{t}{t+2}\hat{i} + \ln t\,\hat{j} + \frac{3}{1-t}\hat{k}$

i) Domain = ? ماداخل ال ln أكبر من صفر

$t \neq -2$ & $t > 0$ & $t \neq 1$



~~D = (o,~~ $D\,\bar{r}(t) = (0,1) \cup (1,\infty)$

$= (0,\infty) - \{1\}$

* Range is a set of vectors.

Ex: $\lim\limits_{t\to 3} \bar{r}(t) = \lim\limits_{t\to 3} \left(\frac{t}{t+2}\hat{i} + \ln t\,\hat{j} + \frac{3}{1-t}\hat{k}\right)$

$= \frac{3}{5}\hat{i} + \ln 3\,\hat{j} + \frac{3}{-2}\hat{k}$

② $\lim\limits_{t\to\infty} \langle t e^{-t},\ \tan^{-1} t,\ t \sin\frac{1}{t} \rangle$

$\langle \underset{(i)}{\infty \cdot 0},\ \tan^{-1}\infty,\ \underset{(ii)}{\infty \cdot 0} \rangle$

(i) $\lim\limits_{t\to\infty} t e^{-t} = \lim\limits_{t\to\infty} \frac{t}{e^t} = \frac{\infty}{\infty}$

$= \lim\limits_{t\to\infty} \frac{1}{e^t} = \frac{1}{\infty} = 0$

(ii) $\lim\limits_{t\to\infty} t \sin\frac{1}{t} = \lim\limits_{t\to\infty} \frac{\sin\frac{1}{t}}{\frac{1}{t}} = 1$ ($\frac{1}{t}\to 0$)

$= \langle 0, \frac{\pi}{2}, 1 \rangle$

* graph of the vector function is an oriented space curve with increasing t.

$$C: \bar{r}(t) = \underbrace{f(t)}_{x}\hat{i} + \underbrace{g(t)}_{y}\hat{j} + \underbrace{h(t)}_{z}\hat{k}$$

$\Leftrightarrow$ parametric curve:

$x = f(t)$, $y = g(t)$, $z = h(t)$

Ex: $\bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$.

graph the curve of function

$\Leftrightarrow$ $C:$ $x = \cos t$

$y = \sin t$

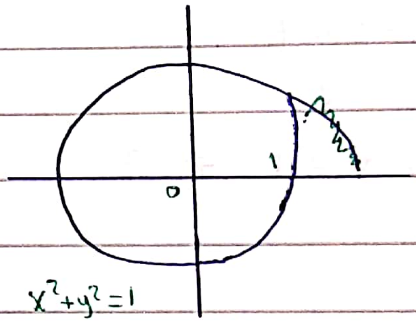
نلغي

eliminate t,

$x^2+y^2 = \cos^2 t + \sin^2 t$

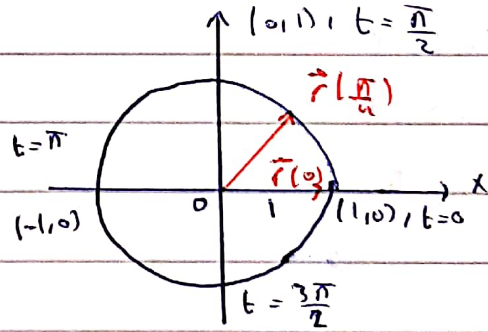
$x^2+y^2 = 1$

Cartesian eqn.



Ex: Sketch $\bar{r}(0) = \hat{i} + 0\hat{j} = \hat{i}$

$\bar{r}(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$



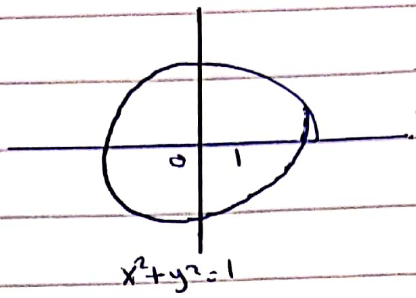
$t=0 \rightarrow \left.\begin{array}{l} x = \cos 0 = 1 \\ y = \sin 0 = 0 \end{array}\right\} (1,0)$

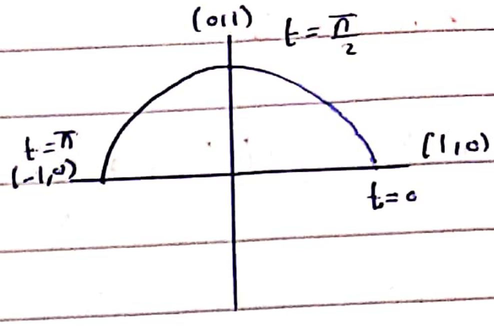
$t=\frac{\pi}{2} \rightarrow \left.\begin{array}{l} x = 0 \\ y = 1 \end{array}\right\} (0,1)$

② $\bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le \pi$

graph:

$x^2+y^2=1$





* Parametric Curve is all or Subset of Cartesian Curve.

③ $\bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 2\,\hat{k}$ $0 \le t \le 2\pi$

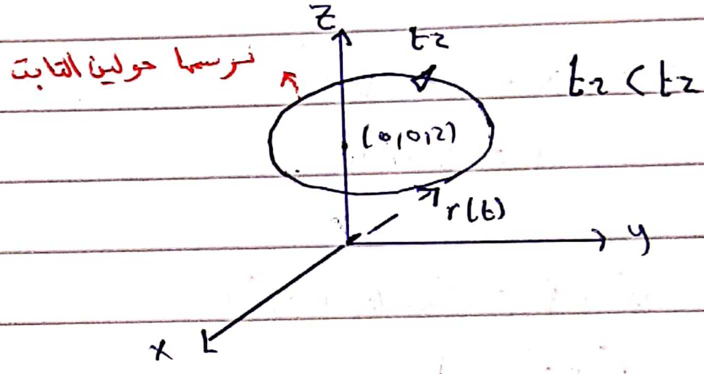
graph:

$x = \cos t$

$y = \sin t$

$z = 2$

$x^2+y^2=1$ $z=2$



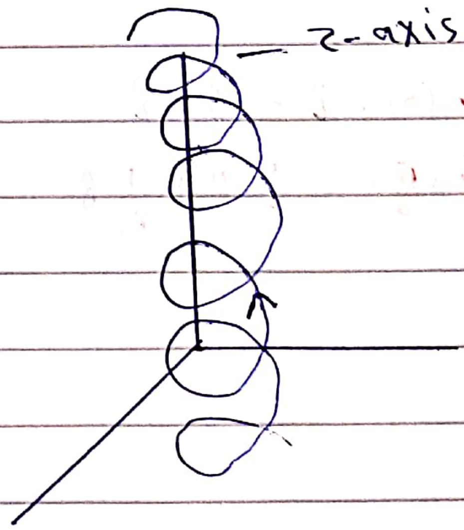
④ Helix:

Ex. $\bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$

$x = \cos t$

$y = \sin t$

$z = t$



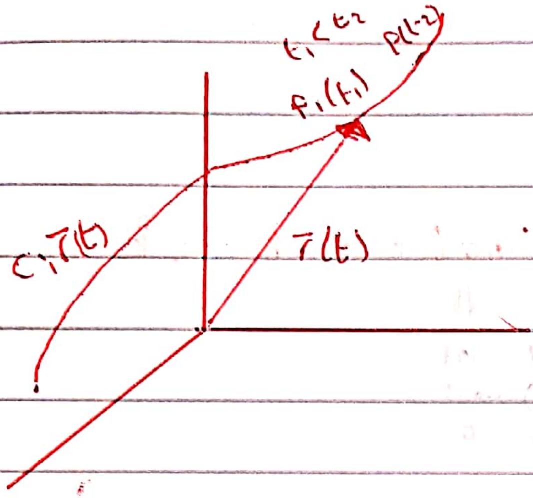
Scanned by CamScanner

C: Curve

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

vector function

graph is an oriented Space curve

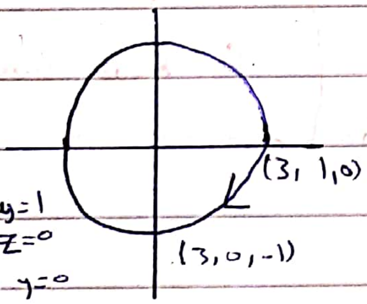


Ex: $\vec{r}(t) = 3\hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k}$ $\quad 0 \le t \le 2\pi$

graph this funct.

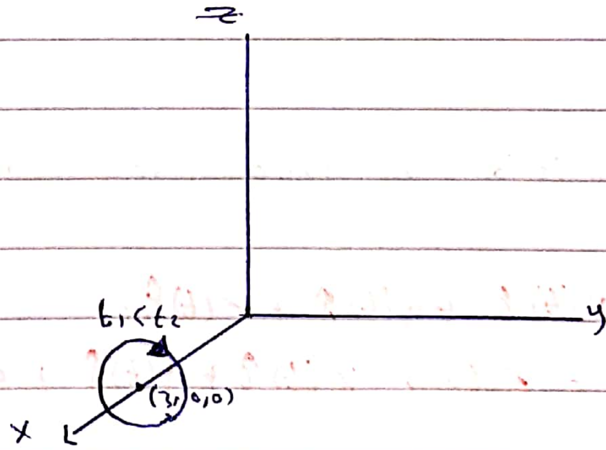
$x = 3, \quad y = \cos t, \quad z = -\sin t$

$y^2 + z^2 = 1, \quad x = 3$



$t = 0 \rightarrow y = 1$
$z = 0$

$t = \frac{\pi}{2} \rightarrow y = 0$
$z = -1$



Ex: Show that the curve $x = t\cos t$, $y = t\sin t$, $z = t$

lies on the cone: $z^2 = x^2 + y^2$

$t^2 \stackrel{?}{=} t^2\cos^2 t + t^2 \sin^2 t$

$= t^2(\cos^2 t + \sin^2 t)$

$t^2 = t^2$ ✓

Scanned by CamScanner

on which surface

* where does this curve: $x = t\cos t$, $y = t\sin t$, $z = t$

$x^2 + y^2 = t^2(\cos^2 t + \sin^2 t)$

$x^2 + y^2 = t^2$

$\hookrightarrow \boxed{t = z}$

$\Rightarrow \boxed{x^2 + y^2 = z^2}$

Cone.

Ex. At what pts. does the helix: $\vec{r}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + t\,\hat{k}$ intersect the sphere:

$\Downarrow$

$x = \sin t$

$y = \cos t$

$z = t$

$x^2 + y^2 + z^2 = 5$

$\underline{\sin^2 t + \cos^2 t} + t^2 = 5$

$1 + t^2 = 5$

$t^2 = 4$

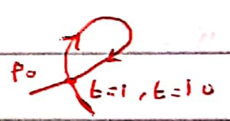
$\boxed{t = \pm 2}$

$\therefore t = -2 \xrightarrow{①} x = \sin(-2)$, $y = \cos(-2)$, $z = -2$

$P_1(\sin(-2), \cos(-2), -2)$

$t = 2 \rightarrow P_2(\sin(2), \cos(2), 2)$

* كل نقطة يمكن ان يكون لها اكثر من (t)



Po have two times

§ 13.2 Differentiation & Integration of vector function

$C: \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$.

① $\frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t) = x'(t)\,\hat{i} + y'(t)\,\hat{j} + z'(t)\,\hat{k}$

② $\int \vec{r}(t)\,dt = \hat{i}\int x(t)\,dt + \hat{j}\int y(t)\,dt + \hat{k}\int z(t)\,dt$

$\boxed{+ C_1\hat{i} \quad + C_2\hat{j} \quad + C_3\hat{k}}$

$+ \vec{C}$ (vector)

$\therefore$ differential & integral of vector function is componentwise.

Ex D: $\bar{r}(t) = \ln(t^2+1)\hat{i} + \cos^2(e^{-t})\hat{j} + 2^t\hat{k}$

Fin) $\bar{r}'(t) = \frac{2t}{1+t^2}\hat{i} + 2\cos(e^{-t})(-\sin e^{-t})(-e^{-t})\hat{j} + 2^t \ln 2\,\hat{k}$

② $\int \left(\frac{1}{1+t^2}\hat{i} + \frac{t}{1+t^2}\hat{j} + 3^t\hat{k}\right) dt$

$= \tan^{-1} t\,\hat{i} + \frac{1}{2}\ln(1+t^2)\hat{j} + \frac{3^t}{\ln 3}\hat{k} + \vec{c}$

③ $\int_0^1 \left(\frac{1}{1+t^2}\hat{i} + \frac{t}{1+t^2}\hat{j} + 3^t\hat{k}\right) dt$

$= \tan^{-1} t\,\hat{i} + \frac{1}{2}\ln(1+t^2)\hat{j} + \frac{3^t}{\ln 3}\hat{k}\Big]_0^1$

$= \overset{\frac{\pi}{4}}{\tan^{-1} 1}\,\hat{i} + \frac{\ln 2}{2}\hat{j} + \frac{3}{\ln 3} - \left(\overset{0}{\tan^{-1} 0}\,\hat{i} + \frac{\overset{0}{\ln 1}}{2}\hat{j} + \frac{1}{\ln 3}\hat{k}\right)$

$\frac{\pi}{4}\hat{i} + \frac{\ln 2}{2}\hat{j} + \frac{2}{\ln 3}\hat{k}$

⊛ Differentiation Rules:

$\bar{r}_1(t), \bar{r}_2(t)$, vector function.

$f(t)$ scalar function.

① $(\bar{r}_1(t) \mp \bar{r}_2(t)) = \bar{r}_1' \mp \bar{r}_2'$

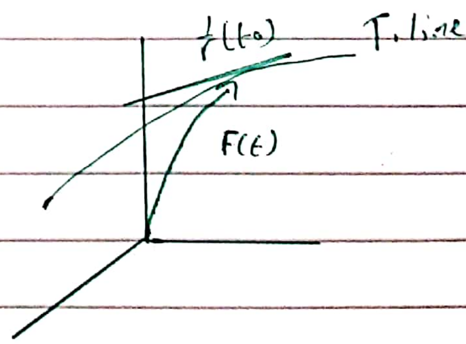
② $(f(t)\,\bar{r}_1(t))' = f(t)\,\bar{r}_1' + f'(t)\,\bar{r}_1(t)$

③ $(\bar{r}_1(t)\cdot\bar{r}_2(t))' = \bar{r}_1\cdot\bar{r}_2' + \bar{r}_1'\cdot\bar{r}_2$

④ $(\bar{r}_1 \times \bar{r}_2)' = \bar{r}_1 \times \bar{r}_2' + \bar{r}_1' \times \bar{r}_2$

* Geometrical meaning of $\bar{r}'(t_0)$

direction of tangent line to the curve C: $\bar{r}(t)$



Scanned by CamScanner

Ex: $C: \bar{r}(t) = (t^2+1)\hat{i} + 4\sqrt{t}\,\hat{j} + e^{t^2-t}\,\hat{k}$

write parametric eqn. of tangent line at the $P_0(2,4,1)$.

$\bar{r}' = 2t\,\hat{i} + \frac{4}{2\sqrt{t}}\,\hat{j} + (2t-1)e^{t^2-t}\,\hat{k}$ | اتجاه خط المماس $t=1$

$C: \quad x = t^2+1$

$y = 4\sqrt{t}$

$z = e^{t^2-t}$

$P_0 \Rightarrow \quad 2 = t^2+1$

$4 = 4\sqrt{t}$ } نبدأ بالمعادلة الأكثر سهولة

$1 = e^{t^2-t}$

$4 = 4\sqrt{t} \rightarrow \boxed{t=1}$

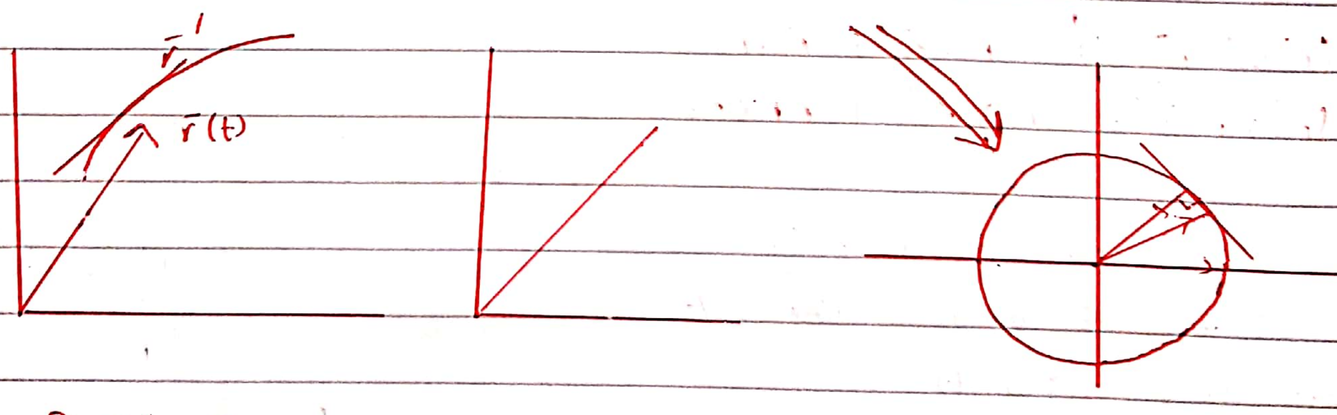
$\bar{r}'(1) = 2\hat{i} + 2\hat{j} + \hat{k}$ | T. line: $x = 2+2t$

$y = 4+2t$

$z = 1+t$

Ex: If the vector function $\bar{r}(t)$ is of a constant norm. show that $\bar{r}'(t)$ is orthogonal to $\bar{r}(t)$ at every t!

$\Leftrightarrow$ if $\|\bar{r}(t)\| = C = \text{Constant} \overset{?}{\Longrightarrow} \bar{r}' \perp \bar{r}$? $\bar{r}\cdot\bar{r}' = 0$



Proof: ? $\bar{r}' \perp \bar{r}$

? $\bar{r}' \cdot \bar{r} = 0$

$\|\bar{r}\|^2 = \bar{r}\cdot\bar{r}$

$c^2 = \bar{r}(t)\cdot\bar{r}(t)$

Int. $\quad 0 = \bar{r}\cdot\bar{r}' + \bar{r}'\cdot\bar{r}$

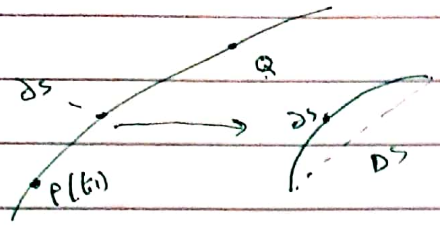
$2\bar{r}\cdot\bar{r}' = 0 \quad = \bar{r}\cdot\bar{r}' = 0 \Rightarrow \bar{r} \perp \bar{r}'$

Scanned by CamScanner

§ 13.3 arc-length & Curvature

$S \equiv$ arc-length parameter

① arc-length $\equiv L$ $C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$



$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

$$\partial s = \sqrt{(\partial x)^2 + (\partial y)^2 + (\partial z)^2}$$

$$\int_P^Q \partial s = \int_P^Q \sqrt{(\partial x)^2 + (\partial y)^2 + (\partial z)^2}$$

$$\int \partial x = x + c$$

$$\int_1^2 \partial x = x\Big]_1^2 = 1$$

$$\int_1^t x \, \partial x = \frac{x^2}{2}\Big|_1^t$$

$$\frac{t^2}{2} - \frac{1}{2} = f(t)$$

$$s(t)\Big|_P^Q = \int_{t_1}^{t_2} \sqrt{(\partial x)^2 + (\partial y)^2 + (\partial z)^2}\ \frac{\partial t}{\partial t}$$

$$L = \int_{t_1}^{t_2} \sqrt{\frac{(\partial x)^2}{(\partial t)^2} + \frac{(\partial y)^2}{(\partial t)^2} + \frac{(\partial z)^2}{(\partial t)^2}}\ \partial t$$

$$\boxed{L = \int_{t_1}^{t_2} \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2}\ \partial t \qquad L = \int_{t_1}^{t_2} \sqrt{(x')^2 + (y')^2 + (z')^2}\ \partial t}$$

$$L = \int_{t_1}^{t_2} \|\vec{r}'(t)\|\ \partial t.$$

$$\int_1^t \partial s = \int_1^t \|\vec{r}'(u)\|\ \partial u$$

$$S(t) = \int_{t_1}^{t} \|\vec{r}'(u)\|\ \partial u \equiv \text{arc. length function, } S(t).$$

ثابت

Ex: ① $C: \vec{r}(t) = t^2\hat{i} + 9t\hat{j} + 4t^{\frac{3}{2}}\hat{k}$ $1 \leq t \leq 4$

Find the arc-length

$$\vec{r}' = 2t\hat{i} + 9\hat{j} + \overset{2}{4}\cdot\frac{3}{\cancel{2}}\, t^{\frac{1}{2}}\hat{k}$$

$$L = \int_1^4 \sqrt{\underbrace{4t^2}_{2\times 2} + 81 + 36t}\ \partial t$$

$$= \int_1^4 \sqrt{(2t+9)^2}\ \partial t$$

Scanned by CamScanner

$$= \int_1^4 (2t+9)\,dt$$

$$= t^2 + 9t \Big]_1^4 = (16+36) - (1+9)$$

② $C: \bar{r}(t) = \sqrt{2}\, t\,\hat{i} + e^t\,\hat{j} + e^{-t}\,\hat{k}$ $\quad t \geq 0$

i) Find the arc-length parameter $S(t)$.

(ii) Reparametrise the curve C_1 using the arc-length parameters.

(i) $\bar{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$

$\bar{r}'(u) = \langle \sqrt{2}, e^u, -e^{-u} \rangle$

$$S(t) = \int_0^t \sqrt{2 + e^{2u} + e^{-2u}}\; du$$

$$= \int_0^t \sqrt{(e^u + e^{-u})^2}$$

$$S(t) = \int_0^t (e^u + e^{-u})\; du$$

$$S(t) = e^u - e^{-u} \Big]_0^t$$

$$S(t) = e^t - e^{-t} - (1-1)$$

$$S(t) = e^t - e^{-t}$$

(ii) $\dfrac{S(t)}{2} = \dfrac{e^t - e^{-t}}{2}$

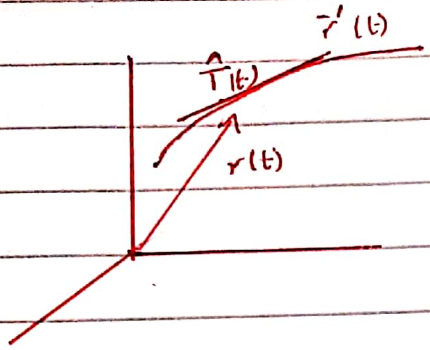
$$\frac{S(t)}{2} = \sinh t$$

$\cancel{\sinh^{-1} \frac{S(t)}{2}} = t$

$$\sinh^{-1} \frac{s}{2} = t$$

C_1 $\bar{r}(s) = \sqrt{2}\, \sinh^{-1}\left(\frac{s}{2}\right) \hat{i} + e^{\sinh^{-1}\left(\frac{s}{2}\right)}\, \hat{j} + e^{-\sinh^{-1}\left(\frac{s}{2}\right)}\, \hat{k}$

* C: $\vec{r}(t)$



$\hat{T}(t)$ = unit tangent vector

$$\hat{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

* Curvature = $K(t)$

Def. Curvature

The magnitude of the rate of change of unit tangent $\hat{T}(t)$ with arc-lengths.

C: $\vec{r}(t)$

$$K(t) = \left\| \frac{\partial T(t)}{\partial s} \right\|$$

$$\hat{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

unit tangent vector

$\boxed{1}$ C: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$K(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$\boxed{2}$ C: $y = f(x)$

$$K(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

Proof $\boxed{2}$:

$$y = f(x)$$

Parametrize:

$x = t \rightarrow y = f(t)$

C: $\vec{r}(t) = \langle t, f(t), 0 \rangle$

Scanned by CamScanner

$\vec{r}'(t) = \langle 1, f'(t), 0 \rangle$

$\vec{r}''(t) = \langle 0, f''(t), 0 \rangle$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix}$$

$= \langle 0, 0, f''(t) \rangle$

$|\vec{r}' \times \vec{r}''| = |f''(t)|$

$|\vec{r}'| = \sqrt{1 + (f'(t))^2}$

① $K(t) = \dfrac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}}$

but $t = x \longrightarrow$

$K(x) = \dfrac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$

Ex. $C: \vec{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle$

Find ① $\hat{T}(t)$ ② $K(t)$

① $\vec{r}' = \langle 2\cos t, 5, -2\sin t \rangle$

$|\vec{r}'| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t}$ $\quad 4 \quad +25$

$= \sqrt{29}$

$\hat{T}(t) = \dfrac{1}{\sqrt{29}} \langle 2\cos t, 5, -2\sin t \rangle$

② $\vec{r}'' = \langle -2\sin t, 0, -2\cos t \rangle$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos t & 5 & -2\sin t \\ -2\sin t & 0 & -2\cos t \end{vmatrix}$$

$= \langle -10\cos t, -(-4\cos^2 t - 4\sin^2 t), 10\sin t \rangle$

$= \langle -10\cos t, 4, 10\sin t \rangle$

$|\vec{r}' \times \vec{r}''| = \sqrt{100\cos^2 t + 16 + 100\sin^2 t}$

$= \sqrt{100 + 16} = \sqrt{116}$

$K(t) = \dfrac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \dfrac{\sqrt{116}}{(\sqrt{29})^3}$

Ex: $C: \bar{r}(t) = \langle t, t^2, t^3 \rangle$

Find the curvature at the pt. $P_0(-1,1,-1)$ نبدأ بحدد الحدود الاقل لايجاد t

$\hookrightarrow t = -1$

$\bar{r}'(t) = \langle 1, 2t, 3t^2 \rangle \big|_{t=-1} = \langle 1, -2, 3 \rangle$

$\bar{r}''(t) = \langle 0, 2, 6t \rangle_{t=-1} = \langle 0, 2, -6 \rangle$

$$\bar{r}'(t) \times \bar{r}''(t) = \bar{r}'(-1) \times \bar{r}''(-1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 2 & -6 \end{vmatrix} = \langle 6, 6, 2 \rangle$$

$|\bar{r}'(t) \times \bar{r}''(t)| = \sqrt{36+36+4} = \sqrt{76}$

$|\bar{r}'(-1)| = \sqrt{1+4+9} = \sqrt{14}$

$$\therefore K(-1) = \frac{\sqrt{76}}{(\sqrt{14})^2}$$

Ex: $C: y = x^2$

Find $K(x)$

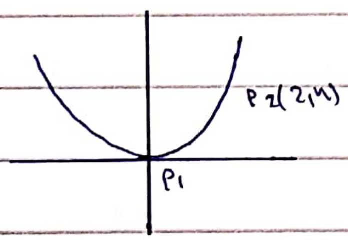
$f' = 2x$

$f'' = 2$

$$K(x) = \frac{|f''(x)|}{(1+(f')^2)^{3/2}}$$

$$K(x) = \frac{2}{(1+4x^2)^{3/2}}$$

② find K at at $P_1(0,0)$ $P_2(2,4)$.



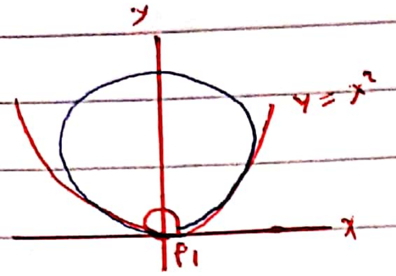
(i) $K(0) = \dfrac{2}{(1+0)^{3/2}} = 2$

(ii) $P_2 \to K(2) = \dfrac{2}{(1+4 \times 4)^{3/2}} = \dfrac{2}{17^{3/2}}$

what happened to K when $x \to \infty$

$$\lim_{x\to\infty} K(x) = \lim \frac{2}{(1+4x^2)^{3/2}} = \frac{2}{\infty} = \boxed{0}$$

* rad of curvature $= \rho = \frac{1}{K}$



P_1; $K=2 \rightarrow \rho_1 = \frac{1}{2}$

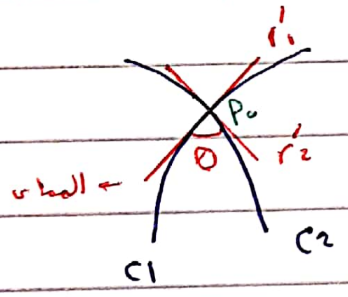
P_2; $K = \frac{2}{17^{3/2}}$ $\quad \rho_2 = \frac{17^{3/2}}{2}$

Ex. C_1: $\vec{r}_1(t) = \langle t, 1-t, 15+t^2 \rangle \Rightarrow t_1 = -1$ بالتعويض

C_2: $\vec{r}_2(t) = \langle 3-t, t-2, t^2 \rangle \rightarrow t_2 = 4$

find the angle of intersection of C_1 & C_2 $\equiv$ acute angle الحادة

at the pt $P_0(-1,2,16)$



θ = angle between two tangent

$\theta = r_1' \& r_2'$

$\vec{r}_1' = \langle 1, -1, 2t \rangle |_{t_1=-1} = \langle 1, -1, -2 \rangle = \vec{r}_1'(-1)$

$\vec{r}_2' = \langle -1, 1, 2t \rangle_{t_2=4} = \langle -1, 1, 8 \rangle = \vec{r}_2'(4)$

$$\cos\theta = \frac{|r_1'(-1) \cdot r_2'(4)|}{|r_1'|\,|r_2'|}$$

$$= \frac{|-1-1-16|}{\sqrt{1+1+4}\,\sqrt{1+1+64}} = \frac{18}{\sqrt{6(66)}}$$

$$\theta = \cos^{-1}(\qquad)$$

Ex: $C: r(t) = \langle t^2, t+1 \rangle \; , \; t \in \mathbb{R}$

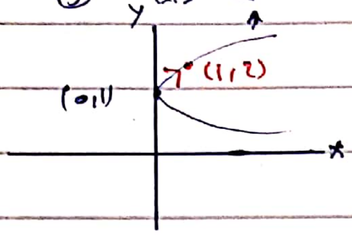
Sketch the curve C.

$\Leftrightarrow C: \quad x = t^2 \; -- ①$

$y = t+1 \qquad t = y-1$

$① \Leftrightarrow \quad x = (y-1)^2$

(y) لانه التربيع على y



$t = 0 \rightarrow x = 0 \quad (0,1)$ بنوخذ زمنين لتحديد الاتجاه

$y = 1$

$t = 1 \rightarrow x = 1 \quad (1,2)$

$y = 2$

Ex: let c be the curve of intersection of cylinder $x^2 = 2y$ ① & the surface $3z = xy$ ② Find the length of this curve c from the origin to the pt. $p(6, 8, 36)$

$$① \rightarrow \cancel{y = \frac{x^2}{2} = x = t \rightarrow y = \frac{t^2}{2}}$$

$$② \cancel{z = \frac{xy}{3} = \frac{1}{3} t \frac{t^2}{2} = \frac{t^3}{6}}$$

$$① \quad y = \frac{x^2}{2} = x = t \rightarrow y = \frac{t^2}{2}$$

$$② \quad z = \frac{xy}{3} = \frac{1}{3} t \frac{t^2}{2} = \frac{t^3}{6}$$

$C: \; x = t \; , \; y = \frac{t^2}{2} \; , \; z = \frac{t^3}{6} \quad \Leftrightarrow \quad C: \; \vec{r}(t) = t\hat{i} + \frac{t^2}{2}\hat{j} + \frac{t^3}{6}\hat{k}$

parametric eqns. of curve C of int

$$L = \int_0^6 \sqrt{(x')^2 + (y')^2 + (z')^2} \, dt$$

$$= \int_0^6 \sqrt{1 + t^2 + \frac{t^4}{4}} \, dt$$

$$= \int_0^6 \sqrt{\left(\frac{t^2}{2} + 1\right)^2} \, dt$$

$$= \int_0^6 \left(\frac{t^2}{2} + 1\right) dt$$

CH. 14 Partial derivatives:

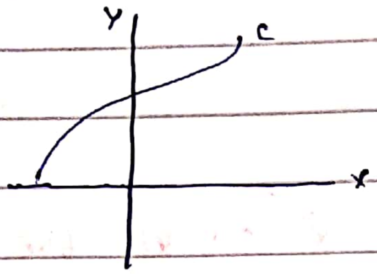
§ 14.1 functions of Several variable:

① Function of one Variable:

$y = f(x)$

$D_f \subseteq \mathbb{R}$, $R_f \subseteq \mathbb{R}$

graph is a curve



② Functions of 2-variables:

$z = f(x, y)$

$D_f \subseteq R^2$, $= R \times R = \{ (x,y) : x, y \in \mathbb{R} \}$

↳ منطقة $\equiv$ region subset of xy-plane

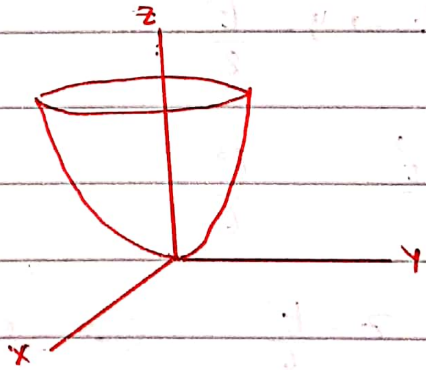
$R_f \subseteq R$

graph is a Surface

Ex: $z = x^2 + y^2$ Paraboloid

$D_f = R^2$

$R_f = [0, \infty)$



③ Functions of 3-variables:

$w = f(x, y, z)$

$D_f \subseteq R^3 = \{ (x,y,z) : x, y, z \in \mathbb{R} \}$

$R_f \subseteq R$

No graph.

Ex: $f(x,y) = \sqrt{4 - x^2 - y^2} + \dfrac{x}{y - x^2}$

Describe & sketch the domain of f.

$4 - x^2 - y^2 \geq 0$ & $y - x^2 \neq 0$

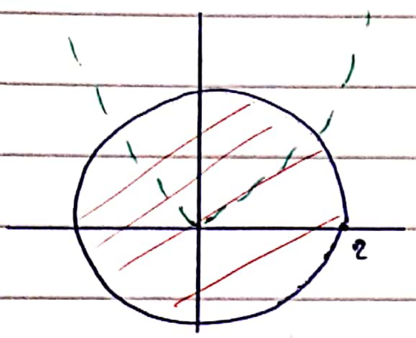
$4 \geq x^2 + y^2$ & $y - x^2 \neq 0$

$x^2 + y^2 = 4$

$\downarrow$

$y - x^2 = 0$

$y = x^2$



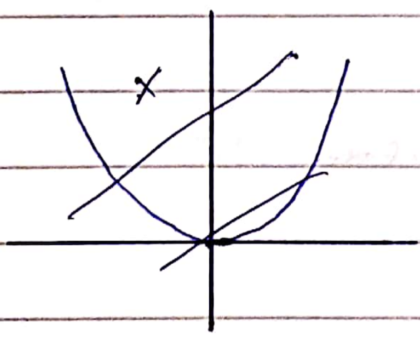
D_f = region inside ~~&~~ of an the circle $x^2 + y^2 = 4$ but not on the parabola $y = x^2$

② $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

D_f : $y - x^2 > 0$

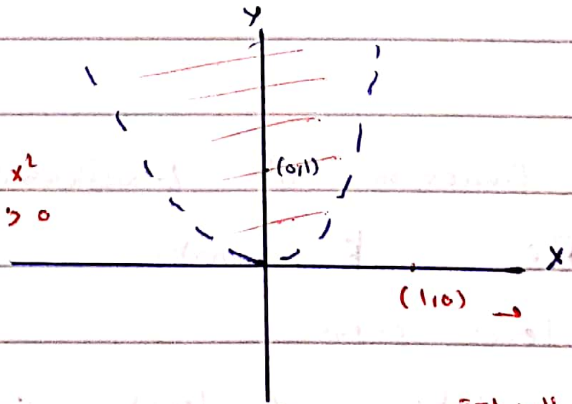
$y - x^2 = 0$

$y = x^2$



$y > x^2$

$1 > 0$



$(1,0) \rightarrow y > x^2$

$0 \leq 1$

خارج المنطقة

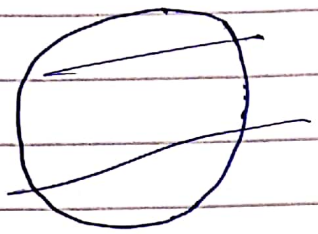
② $F(x,y,z) = \ln(9 - x^2 - y^2 - z^2)$

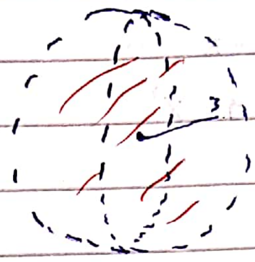
Describe & Sketch the domain of f.

$9 - x^2 - y^2 - z^2 > 0$

$9 > x^2 + y^2 + z^2$

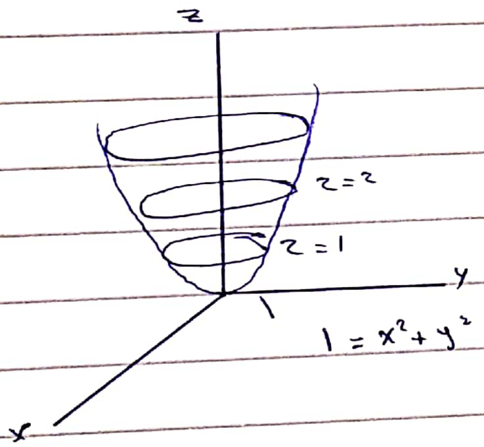
$x^2 + y^2 + z^2 = 0$

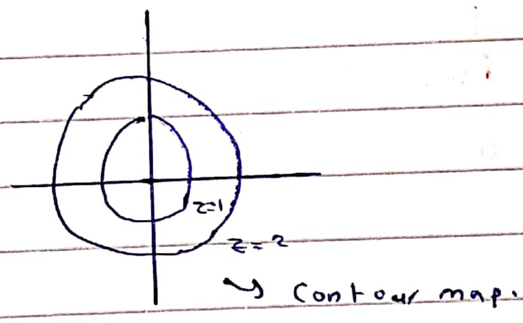




D_f = region inside the sphere $C(0,0,0)$, rad = 3 but not on it

Ex: $z = x^2 + y^2$





* $z = f(x,y)$ ← function of 2-variable → surface

Let $z = k \in \mathbb{R}_f$, $k \equiv$ const.

$k = f(x,y)$ ← k-level curve

* contour map: consists of level curve of several values of k.

Ex $F(x,y) = ye^x$ (k)

Describe & Sketch the contour map.

the level curve.

$k = y e^{x}$ --① $\quad k \in R.$

$\Rightarrow \quad y = k e^{-x}$

① $k = 0 \Rightarrow y = 0$ oR $e^{x} = 0$ never

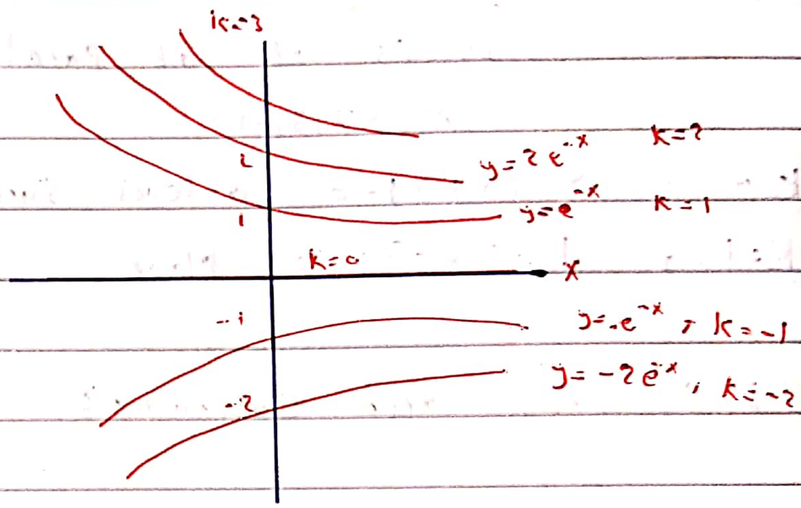
$\downarrow$

x-axis

② $k > 0 \quad = y = k e^{-x}$

$k = 1 \quad = \quad y = e^{-x}$

$k = 2 \quad = \quad y = 2e^{-x}$



$z = F(x,y)$ is graph is Surface:

$\downarrow$

K: level curve

$k = F(x,y)$

2) $w = F(x,y,z)$ No graph

$k = F(x,y,z) \leftarrow$ K-level Surface

Ex: $F(x,y,z) = 2x+3y-z$, Describe the K-level Surface for different values of K

$D_F = \mathbb{R}^3$

$\rightarrow$ $\underbrace{K = 2x+3y-z}$

$\hookrightarrow$ is set of parallel plane

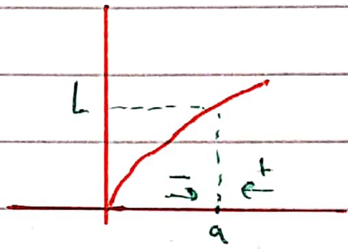
$\vec{n} = \langle 2,3,-1 \rangle$

$K=0 \rightarrow$ $0 = 2x+3y-z$ 0-level surface in a plane

$K=1 \rightarrow$ $1 = 2x+3y-z$ plane

§ 14.2: limit and Cotinuity:

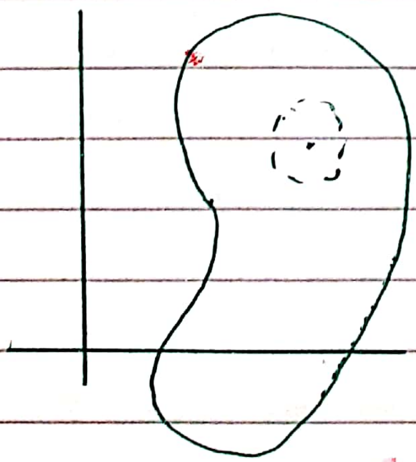
* $\lim_{x \to a} F(x) = L$ exists



2) variables :-

$\lim F(x,y) = L$ exists $\rightarrow$ (For all paths that lead to the P_0 you get the same ~~ans~~ answer)

But limit does not exist take two different paths to get two different answers



Ex ① $\lim \frac{x^4-y^4}{x^2+y^2} = \frac{-15}{5} = -3$ exists

$\hookrightarrow F(x,y) \rightarrow D_f = \mathbb{R}^2 - \{(0,0)\}$

② $\lim\limits_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \frac{0}{0}$!

$\lim \frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{(x^2+y^2)}} = 0$

③ $f(x,y) = \frac{x^4-y^4}{x^2+y^2}$ is f Cont at $(0,0)$

No $(0,0) \notin D_f$

Ex $F(x,y) = \begin{cases} \frac{x^4-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 4, & (x,y) = (0,0) \end{cases}$

① Find $D_f = \mathbb{R}^2$

② is f Cont at $(0,0)$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) \stackrel{?}{=} F(0,0)$

$0 \neq 4$ Not Cot

③ redefine f at $(0,0)$ to be cont

$F(x,y) = \begin{cases} \frac{x^4-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Ex. $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{0}{0}$!!

i) ~~alog~~ along Path $y=0$

$\lim\limits_{x^2\to 0} \frac{0}{x^2} = \lim\limits_{x\to 0} 0 = 0$

ii) along Path $x=0$

$\lim \frac{0}{0+y^2} = \lim\limits_{y\to 0} 0 = 0$

(iii) along $y=x$

$\lim\limits_{x\to 0} \frac{x^2}{x^2+y^2} = \lim\limits_{x\to 0} \frac{1}{2} = \frac{1}{2}$

~~i + ii + iii~~

i and iii or ii and iii

two different paths, two different answer

$\therefore$ limit does not exists

OR along paths $y = mx$

$\lim \frac{x \cdot mx}{x^2+m^2x^2} = \frac{mx^2}{x^2(1+m^2)} = \lim\limits_{x\to 0} \frac{m}{1+m^2}$

depends on an arbitrary

$\therefore$ limit does not exists

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \frac{0}{0}$!

① $y = x^2 = \lim\limits_{x\to 0} \frac{x^2 \cdot x^2}{x^4+x^4} = \frac{1}{2}$

② $y = 2x^2$

$$\lim_{x \to 0} \frac{2x^4}{x^4 + 4x^4} = \frac{2}{5}$$

two different paths $\longrightarrow$ two diff. answer

Ex $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \frac{0}{0} \;!!$$

$$= \lim \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{(\sqrt{x^2+y^2+1}-1)(\sqrt{x^2+y^2+1}+1)}$$

$$= \lim \frac{\cancel{(x^2+y^2)}(\sqrt{x^2+y^2+1}+1)}{\cancel{x^2+y^2}+\cancel{1}-\cancel{1}}$$

$$= 1+1 = 2$$

② $$\lim_{(x,y)\to(0,2)} \frac{y \sin xy}{x} = \frac{0}{0} \;!!$$

$$= \lim_{(x,y)\to(0,2)} \frac{y \sin(xy)}{(xy)} \cdot y$$

$$= 2(1)(2) = \boxed{4}$$

③ $$\lim_{(x,y)\to(0,0)} xy \sin \frac{1}{x^2+y^2} = 0 \cdot \sin\infty = 0 \cdot \text{does not exist}$$

$= 0$, $xy \to 0$

$$-1 \le \sin \frac{1}{x^2+y^2} \le 1$$

& $\sin \frac{1}{x^2+y^2}$ is bounded

$$-xy \le xy \sin \frac{1}{x^2+y^2} \le xy$$

$(x,y)\to(0,0)$ $\quad\downarrow\quad$ sep. $\quad\downarrow\ (x,y)\to(0,0)$

$0 \qquad\qquad 0$

$0 \quad (x,y)\to(0,0)$

Scanned by CamScanner

④ $f(x,y) = \frac{x^2}{x^2+y^2}$

① is bounded

$0 \le x^2 \le x^2+y^2$

$0 \le \frac{x^2}{x^2+y^2} \le 1$

⑤ $\frac{x^4}{x^2+y^2}$

$0 \le \frac{x^2}{x^2+y^2} \le 1$

② $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \frac{0}{0}!$

i) path $y=x$

$\lim\limits_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$

$\neq$

ii) $y=2x$

$\lim\limits_{x\to 0} \frac{x^2}{x^2+4x^2} = \frac{1}{5}$

$\rightarrow$ limit does not exist

③ $\lim\limits_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \frac{0}{0}!$

$\lim\limits_{(x,y)\to(0,0)} = x \cdot \frac{x^2}{x^2+y^2} = 0$, does not exist

$= 0 \cdot bd = 0$

$= 0, \; x \to 0$

& $\frac{x^2}{x^2+y^2}$ is bd

Ex. $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x}\,y}{x+y^2} = \frac{0}{0}!$

① $y=\sqrt{x} \longrightarrow \lim\limits_{x\to 0^+} \frac{\sqrt{x}\,\sqrt{x}}{x+(\sqrt{x})^2}$

$= \lim \frac{x}{x+x} = \frac{1}{2}$

$\neq$

② $y = 2\sqrt{x} \longrightarrow \lim \frac{2x}{x+(2\sqrt{x})^2} =$

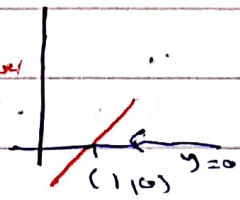
$= \lim \frac{2x}{x+4x} = \frac{2}{5}$

$\therefore$ limit does not exist

Ex. $\lim\limits_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} = \frac{0}{0}$!!

$= \lim\limits_{(x,y)\to(1,0)} \frac{y(x-1)}{(x-1)^2+y^2}$

عشان يصيروا
نفس ال Power
$n = |x|$ Power



(i) $y=0$

$= \lim\limits_{x\to 1} \frac{0}{(x-1)^2+0}$

$\lim\limits_{x\to 1} 0 = 0$

$y - 0 = m(x-1)$

④ $f(x,y) = \dfrac{x^2}{x^2+y^2}$

① is bounded

$0 < x^2 \leq x^2+y^2$

$0 \leq \dfrac{x^2}{x^2+y^2} \leq 1$

② $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2} = \dfrac{0}{0}$!

i) path $y = x$

$\lim\limits_{x\to 0} \dfrac{x^2}{x^2+x^2} = \dfrac{1}{2}$

$\neq$

ii) $y = 2x$

$\lim\limits_{x\to 0} \dfrac{x^2}{x^2+4x^2} = \dfrac{1}{5}$

$\rightarrow$ limit does not exist

③ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2} = \dfrac{0}{0}$!

$\lim\limits_{(x,y)\to(0,0)} = x \cdot \dfrac{x^2}{x^2+y^2} = 0 \cdot$ does not exist

$= 0 \cdot bd = 0$

$= 0, \; x \to 0$

& $\dfrac{x^2}{x^2+y^2}$ is bd

⑤ $\dfrac{x^4}{x^2+y^2}$

$0 \leq \dfrac{x^2}{x^2+y^2} \leq 1$

Ex. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{x}\, y}{x+y^2} = \dfrac{0}{0}$!

① $y = \sqrt{x} \longrightarrow \lim\limits_{x\to 0^+} \dfrac{\sqrt{x}\,\sqrt{x}}{x+(\sqrt{x})^2}$

$= \lim \dfrac{x}{x+x} = \dfrac{1}{2}$

$\neq$

② $y = 2\sqrt{x} \longrightarrow \lim \dfrac{2x}{x+(2\sqrt{x})^2} =$

$= \lim \dfrac{2x}{x+4x} = \dfrac{2}{5}$

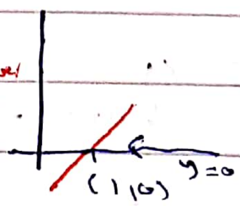
$\therefore$ limit does not exist

Ex. $\lim\limits_{(x,y)\to(1,0)} \dfrac{xy-y}{(x-1)^2+y^2} = \dfrac{0}{0}$!!

$= \lim\limits_{(x,y)\to(1,0)} \dfrac{y(x-1)}{(x-1)^2+y^2}$

عشان يصيروا نفس ال Power

$n = |x|$ Power



$y - 0 = m(x-1)$

(i) $y = 0$

$= \lim\limits_{x\to 1} \dfrac{0}{(x-1)^2+0}$

$\lim\limits_{x\to 1} 0 = 0$

(ii) $y = x - 1$

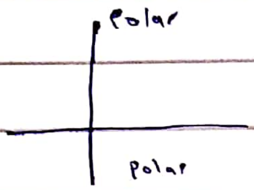
$$\lim \frac{(x-1)^2}{(x-1)^2 + (x-1)^2}$$

$$\lim_{x \to 1} \frac{1}{2} = \frac{1}{2}$$

$\therefore$ does not exist

Ex. $$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = 0 \cdot \ln 0^+$$

$$= 0 \cdot (-\infty) \;!!$$



$(x,y) \to (0,0)$

$(x,0) \to (0,0)$

$$= \lim_{r\to 0} r^2 \ln r^2$$

$$= 2\lim_{r\to 0} r^2 \ln r$$

$$= 2\lim_{r\to 0} \frac{\ln r}{r^{-2}}$$

$$= 2\lim_{r\to 0} \frac{\frac{1}{r}}{-2r^{-3}}$$

$$= \lim_{r\to 0} \frac{1}{r \cdot r^{-3}} = -\frac{1}{r^{-2}} = -\lim_{r\to 0} r^2$$

$$= \boxed{0}$$

Ex $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \frac{0}{0} \;!$$

Polar

$$= \lim_{(r,\theta)\to(0,0)} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2}$$

$$= \lim \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2}$$

$$= \lim_{(r,\theta)\to(0,0)} r(\cos^3\theta + \sin^3\theta) = 0 \cdot (\underbrace{\cos^3\theta + \sin^3\theta}_{bd})$$

$0 \cdot bd$

$= 0$, bd

* Limits :

① $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4} = \dfrac{0}{0}$!!

i) $x = t^2,\ y = t^2,\ z = t$ (عشان نوحدله Power المقام)

$$\lim_{t\to 0} \frac{t^4 + t^2(t^2) + t^2(t^2)}{t^4 + t^4 + t^4}$$

$$\lim_{t\to 0} \frac{3t^4}{3t^4} = \boxed{1}$$

$\neq$

ii) $x = 2t^2,\ y = t^2,\ z = t$

$$\lim_{t\to 0} \frac{2t^4 + t^4 + 2t^4}{4t^4 + t^4 + t^4} = \lim_{t\to 0} \frac{5}{6}\,\frac{t^4}{t^4} = \boxed{\frac{5}{6}}$$

$\therefore$ does not exist.

② $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 \sin^2 y}{x^2 + 2y^2} = \dfrac{0}{0}$!!

اذا تساوت الـ Power does not exist

$$\lim \frac{x^2}{x^2 + 2y^2} \cdot \underset{0}{\sin^2 y}$$

bd.

$= 0$

③ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 + \sin^2 y}{x^2 + 2y^2} = \dfrac{0}{0}$!!

i) $x = 0 \longrightarrow \lim\limits_{y\to 0} \dfrac{\sin^2 y}{2y^2} = \dfrac{1}{2}\left(\dfrac{y}{y}\right)^2$

$= \boxed{\frac{1}{2}}$ $\neq$

ii) $y = 0 \longrightarrow \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = \lim 1 = 1$.

$\therefore$ does not exist

③ $\lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4+4y^2} = \frac{0}{0}$!!

i) $y = x^2 \to \lim_{x\to 0} \frac{x^4 e^{x^2}}{x^4+4x^4} = \boxed{\frac{1}{5}}$

$\neq$

ii) $y = 2x^2 \to \lim_{x\to 0} \frac{2x^4 e^{x^2}}{x^4+16x^4} = \boxed{\frac{2}{17}}$

$\therefore$ does Not exist.

Ex: Find the set of pt. at which the function is cont.

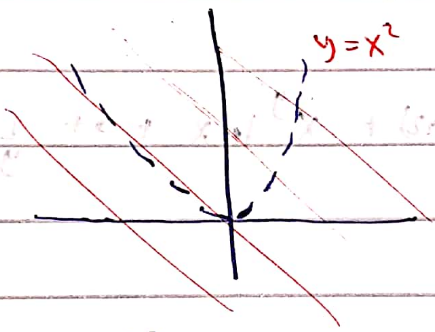
نفس معنى $\equiv$?? Domain

$f(x,y) = \frac{1}{x^2-y}$

$x^2 - y \neq 0$

$x^2 - y = 0$

$y = x^2$



$D_f = \mathbb{R}^2 - \{(x,y) : y = x^2\}$

§14.3 Partial Derivatives:

If f is a function of more more than one variable then differential is partial.

$z = f(x,y)$

$\frac{\partial z}{\partial x} = f_x \equiv$ differentiate f partially with respect to x & treat y as a constant.

$\frac{\partial z}{\partial y} = f_y =$ " " " " " to y
" " x as a constant.

Ex: $f(x,y) = xe^{xy^2} + \cos(x^2 + 2^{-xy}) + x^y + \tan^{-1} xy^2 + x^2 + \ln y$. $\left\{ (e^{2x})' = e^{2x} \cdot 2 \right.$

Find

$f_x = x\, e^{xy^2} \cdot y^2 + e^{xy^2} - \sin(x^2 + 2^{-xy}) \cdot (2x + 2^{-xy}(\ln 2)(-y)) + yx^{y-1} + \frac{y^2}{1+(xy^2)^2} + 2x + 0$

$f_y = x \cdot 2xy\, e^{xy^2} - \sin(x^2 + 2^{-xy})(-x\, 2^{-xy} \ln 2) + x^y \ln x + 0 + \frac{1}{y} + \frac{2xy}{1+(xy^2)^2}$

Ex: $f(x,y) = \int_x^{xy^2} \sin(t^2)\, dt$ ضرب تعويض x مشتقة الحد الاعلى - تعويض x مشتقة الحد الاسفل

Find ① $f_x = \sin(xy^2)^2 \cdot y^2 - (\sin x^2 (1))$

② $f_y = \sin(xy^2)^2 \cdot 2xy - (\sin x^2) \cdot 0$

$= \sin(xy^2)^2 \cdot 2xy$

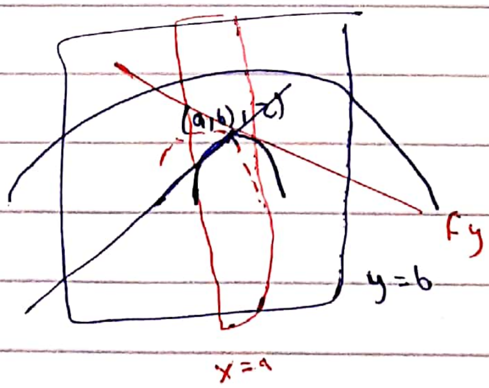
* Geometrical meaning of f_x & f_y at a pt.

$z = f(x,y)$

$f_x(a,b)$

$\boxed{y=b}$

ثابت



$T f_x$

ميل المماس

$f_x(a,b) =$ the slope of tangent line to the curve obtained by the intersection of the surface $z = f(x,y)$ with the plane $y = b$

Scanned by CamScanner

$f_y(a,b) =$ 〃 ―〃―〃 〃 〃 〃 〃 〃 $x = a$

* Higher derivatives:

$z = f(x,y)$

2^{nd} derivative

i) $\dfrac{\partial^2 z}{\partial x^2} = f_{xx}$

ii) $\dfrac{\partial^2 z}{\partial y^2} = f_{yy}$

iii) 2nd mixed partial deri

$\dfrac{\partial^2 z}{\partial y \partial x} = f_{xy}$

$\dfrac{\partial^2 z}{\partial x \partial y} = f_{yx}$

Ex $f(x,y) = x^5 y^6$

Find

① $\dfrac{\partial^2 f}{\partial y \partial x} = f_{xy}$

$f_x = 5x^4 y^6$

$f_{xy} = 5(6)\, x^4 y^5$

② $\dfrac{\partial^5 f}{\partial x \partial x \partial y^2 \partial x} = f_{xyyxx}$

$f_x = 5x^4 y^6$

$f_{xy} = 30\, x^4 y^5$

$f_{xyy} = 30(5)\, x^4 y^4$

$f_{xyyx} = 30(5)(4)\, x^3 y^4$

$f_{xyyxx} = 30(5)(4)(3)\, x^2 y^4$

Scanned by CamScanner

14.4 Tangent plane and linear approximations

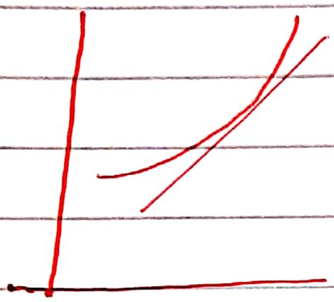
① 1-variable $y = f(x) \to$ Curve $y_F \cong y_T$ (linear app off)

$f(x) \cong f(a) + f'(a)(x-a)$

linear app of f at $x=a$

$f(x) \cong L(x)$

$L(x) \cong f(a) + f'(a)(x-a)$ $L(x)$ = linearisation of f



$T: y - f(a) = f'(a)(x-a) = L(x)$

$L(x) = y = f(a) + f'(a)(x-a)$

* 2-variable $z = f(x,y) \to$ Surface

T plane

$z = f(a,b) + f_x \cdot (x-a) + f_y \cdot (y-b)$

eqn of Tangent plane for f at (a,b)

$L(x,y) = f(a,b) + f_x \cdot (x-a) + f_y (y-b)$

a linearisation of f $z_f \cong z_T$

$f(x,y) \cong f(a,b) + f_x (x-a) + f_y (y-b)$

3) linear app of f at (a,b).

$\Longrightarrow L(x,y) = f(a,b) + f_x(x-a) + f_y(y-b)$

~~3) = T...~~

(6) Ex: $F(x,y) = \frac{x}{x+y}$, $P_0(2,1)$ find:

1) eqn. of Tangent plane:

$Z = f(2,1) + F_x(2,1) \cdot (x-2) + f_y(2,1)(y-1)$

$Z = \frac{2}{3} + \frac{1}{9}(x-2) - \frac{2}{9}(y-1)$ } eqn of ~~tanget~~ tangent plane $F_x(2,1)$

$f(2,1) = \frac{2}{2+1} = \frac{2}{3}$

$F_x = \frac{(x+y)(1) - x(1)}{(x+y)^2} \Big|_{(2,1)}$

$F_x = \frac{(2+1) - 2}{3^2} = \frac{1}{9}$

$F_y = \frac{x(-1)}{(x+y)^2} \Big|_{(2,1)} = \frac{-2}{9}$

2) Normal to this Plane: (x,y,z) معاملات

$\Rightarrow \overline{n} = \langle \frac{1}{9}, \frac{-2}{9}, -1 \rangle$

3) Find the linearization of f at P_0:

$\rightarrow L(x,y) = \frac{2}{3} + \frac{1}{9}(x-2) - \frac{2}{9}(y-1)$

4) approximation $f(x,y)$ linear at P_0:

$f(x,y) \cong L(x,y)$

$\frac{x}{x+2} \cong \frac{2}{3} + \frac{1}{9}(x-2) - \frac{2}{9}(y-1)$

5) Use the approximation to Find the approximated Value of $F(1.99, 1.02)$

$F(1.99, 1.02) \cong \frac{2}{3} + \frac{1}{9}(-0.01) - \frac{2}{9}(0.02)$

$\Delta x = x-2 = 1.99-2 = -0.01$

$\cong \frac{2}{3} - \frac{1}{900} - \frac{4}{900}$

$\Delta y = y-1 = 1.02-1 = 0.02$

$\cong \frac{2}{3} - \frac{1}{180} = \frac{120-1}{180} = \frac{119}{180} = 0.66111$

⑥ Find $f(1.99, 1.02)$

$$= f(1.99, 1.02) = \frac{1.99}{1.99+1.02} = \frac{1.99}{3.01} = 0.66112$$

Ex: Find the approximated Value of $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2}$

let: $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ $P_0(3,2,6)$

1) $f(3,2,6) = \sqrt{3^2+2^2+6^2} = \sqrt{49} = 7$

$\Delta x = 3.02 - 3 = 0.02$

2) $f_x = \frac{2x}{2\sqrt{x^2+y^2+z^2}}\Big|_{(3,2,6)} = \frac{3}{7}$

$\Delta y = 1.97 - 2.5 = -0.03$

3) $f_y = \frac{\cancel{2x}\ 2y}{2\sqrt{x^2+y^2+z^2}}\Big|_{(3,2,6)} = \frac{2}{7}$

$\Delta z \quad 6 - 5.99 = -0.01$

4) $f_z = \frac{2z}{2\sqrt{x^2+y^2+z^2}}\Big|_{\cancel{(3,2,6)}(3,2,6)} = \frac{6}{7}$

$$\sqrt{(3.02)^2+(1.97)^2+(5.99)^2} \cong 7 + \frac{3}{7}(0.02) + \frac{2}{7}(-0.03) + \frac{6}{7}(-0.01)$$

$$\cong 7 + \frac{0.06}{7} - \frac{0.06}{7} - \frac{0.06}{7}$$

$$\cong 7 - \frac{0.06}{7} = 7 - \frac{6}{700}$$

Ex: Verify this approximation at $(0,0)$

$$\sqrt{y^7 + \cos^3 x} \cong 1 + \frac{1}{2}y$$

let $f(x,y) = \sqrt{y + \cos^3 x}$. app of linear at $(0,0)$

$$F(0,0) = \sqrt{0+1} = 1$$

Scanned by CamScanner

$$F_x = \frac{-2\cos x \sin x}{2\sqrt{y+\cos^2 x}} \Big|_{(0,0)} = 0$$

$$F_y = \frac{1}{2\sqrt{y+\cos^2 x}} \Big|_{(0,0)} = \frac{1}{2}$$

$$F(x,y) \cong F(0,0) + F_x \cdot x + F_y \cdot y$$

$$\sqrt{y+\cos^2 x} \cong 1 + 0 \cdot x + \frac{1}{2} y$$

$$\cong 1 + \frac{1}{2} y$$

14.5 The Chain Rule

1- variable

$y = f(x)$, $x = g(t)$, $t = h(w)$

Find ① $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial t}$

② $\frac{\partial y}{\partial w} = \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial t} \cdot \frac{\partial t}{\partial w}$

2- variables ① $z = f(x,y)$, $x = x(t)$, $y = y(t)$

$$z = f \begin{cases} x - t \\ y - t \end{cases}$$

∂t $\quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + f_y \cdot \frac{\partial y}{\partial t}$

③ $z = f(x,y)$ $\quad x = x(r,t)$ $\quad y = y(r,t)$

$$z = f \begin{cases} x \begin{cases} t \\ r \end{cases} \\ y \begin{cases} t \\ r \end{cases} \end{cases}$$

$\frac{\partial f}{\partial t} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t}$

$\frac{\partial f}{\partial r} = f(x) \cdot \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r}$

* Chain Rule :

$z = F(x,y)$, $z = F \begin{cases} x < \begin{smallmatrix} t \\ s \end{smallmatrix} \\ y < \begin{smallmatrix} t \\ s \end{smallmatrix} \end{cases}$

$$\frac{\partial z}{\partial t} = F_x \cdot \frac{\partial x}{\partial t} + F_y \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = F_x \frac{\partial x}{\partial s} + F_y \frac{\partial x}{\partial t}$$

Ex: $f(x,y) = x^2y^3 + xe^{xy}$, $x = t^2 + s$, $y = 2ts$

Find ① $\frac{\partial f}{\partial t} = ??$

$f \begin{cases} x < \begin{smallmatrix} t \\ s \end{smallmatrix} \\ y < \begin{smallmatrix} t \\ s \end{smallmatrix} \end{cases}$ | $f_x \begin{cases} x < \begin{smallmatrix} t \\ s \end{smallmatrix} \\ y < \begin{smallmatrix} t \\ s \end{smallmatrix} \end{cases}$

$$\frac{\partial f}{\partial t} = F_x \cdot \frac{\partial x}{\partial t} + F_y \cdot \frac{\partial y}{\partial t}$$

$$= (2xy^3 + xye^{xy} + e^{xy})(2t) + (3x^2y^2 + x \cdot xe^{xy})(2s)$$

$$= (0 + 0 + e^0)(2) + 0 = \boxed{2}$$

② $\frac{\partial f}{\partial t} \Big|_{\substack{t=1 \\ s=-1}}$

$x = 1 + (-1) = 0$

$y = -2$

Ex: if f is a differentiable function of one variable & $z = f(x^2+y^2)$ Show that $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$

أي عملية حسابية بين متغيرات يعتبر متغير

let $u = x^2+y^2 \rightarrow$

$z = f(u)$, $u = x^2+y^2$

$z = f(u) \rightarrow u < \begin{smallmatrix} x \\ y \end{smallmatrix}$

$$\left(\frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \cdot 2x = f'(x^2+y^2) \cdot 2x \right) y$$

$$\left(\frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y} = \frac{\partial f}{\partial u} \cdot 2y = f'(x^2+y^2) \cdot 2y \right) \cdot x$$

$$f'(x^2+y^2) = \frac{\partial f(x^2+y^2)}{\partial (x^2+y^2)}$$

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 2xy \cancel{f'(x^2+y^2)} - 2xy \cancel{f'(x^2+y^2)}$$

$= 0$

Ex: $g(s,t) = f(s^2-t^2, t^2-s^2)$

اذا طلب نطلع اي بحض متغير بالمرحلة الثانية (chain rule)

Show that

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

let $x = s^2-t^2$, $y = t^2-s^2$

$\therefore g(s,t) = f(x,y)$:

$$g = f \begin{cases} x < \begin{smallmatrix} t \\ s \end{smallmatrix} \\ y < \begin{smallmatrix} t \\ s \end{smallmatrix} \end{cases}$$

$$\left(\frac{\partial g}{\partial s} = f_x \cdot \frac{\partial x}{\partial s} + f_y \cdot \frac{\partial y}{\partial s} = f_x \cdot 2s - 2s\, f_y \right) t$$

$$\left(\frac{\partial g}{\partial t} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} = -2t\, f_x + 2t\, f_y \right) s$$

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = (\cancel{2st\, f_x} - \cancel{2st\, f_y}) - \cancel{2st\, f_x} + \cancel{2st\, f_y}$$

$$= \boxed{0}$$

Ex: $z = f(x,y)$: $x = s+t$, $y = s-t$

Show that $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s} \cdot \frac{\partial z}{\partial t}$

$$z = f \begin{cases} x < \begin{smallmatrix} s \\ t \end{smallmatrix} \\ y < \begin{smallmatrix} s \\ t \end{smallmatrix} \end{cases}$$

$$\frac{\partial z}{\partial s} = z_x \cdot \frac{\partial x}{\partial s} + z_y \cdot \frac{\partial y}{\partial s} = z_x \cdot 1 + z_y \cdot 1$$

$$\frac{\partial z}{\partial t} = z_x \frac{\partial x}{\partial t} + z_y \frac{\partial y}{\partial t} = z_x \cdot 1 - z_y$$

$$\rightarrow = \frac{\partial z}{\partial s} \cdot \frac{\partial z}{\partial t} = (z_x + z_y)(z_x - z_y)$$

$$= (z_x)^2 - (z_y)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

Ex: $w(s,t) = F(u(s,t), v(s,t))$,

where w, F, u, v are differentiable

$u(1,0) = 2$, $v(1,0) = 3$

$u_s(1,0) = -2$, $v_s(1,0) = 5$

$u_t(1,0) = 6$, $v_t(1,0) = 4$

$F_u(2,3) = -1$, $F_v(2,3) = 10$

Find ① $w_s(1,0)$ ② $w_t(1,0)$

$$w = f \begin{cases} u \begin{cases} s \\ t \end{cases} \\ v \begin{cases} s \\ t \end{cases} \end{cases}$$

① $w_s(s,t) = F_u(u(s,t), v(s,t)) \dfrac{\partial u(s,t)}{\partial s} + F_v(u(s,t), v(s,t)) \cdot \dfrac{\partial v}{\partial s}(s,t)$

$w_s(1,0) = F_u(u(1,0), v(1,0)) \cdot u_s(s,t) + F_v(u(1,0), v(1,0)) \cdot V_s(1,0)$

$F_u(2,3) \cdot (-2) + F_v(2,3) \cdot 5$

$(-1)(-2) + (10)(5)$

$= 2 + 50 = \boxed{52}$

② نفس ① بس
بدل s ← $\underline{\underline{t}}$

* 2^{nd} derivative using chain Rule:

Ex: $z = F(x,y)$; $x = r^2 + s^2$, $y = 3rs$

$$z = f \begin{cases} x \begin{cases} r \\ s \end{cases} \\ y \begin{cases} r \\ s \end{cases} \end{cases}$$

Find ① $\dfrac{\partial^2 z}{\partial r \partial s}$

$$f_x \begin{cases} x \begin{cases} r \\ s \end{cases} \\ y \begin{cases} r \\ s \end{cases} \end{cases}$$

$\dfrac{\partial z}{\partial s} = f_x \dfrac{\partial x}{\partial s} + f_y \cdot \dfrac{\partial y}{\partial s}$

$2s\, F_x(x,y) + 3r\, f_y$,

منشتقش المرحلة chain

$\Rightarrow \dfrac{\partial^2 z}{\partial r \partial s} = 2s\, \underline{F_{xr}} + 3r\, f_{yr} + 3 f_y$.

$\dfrac{\partial x}{\partial r} = 2r$

$\dfrac{\partial y}{\partial r} = 3s$

$= 2s\left(f_{xx} \overset{2r}{\dfrac{\partial x}{\partial r}} + f_{xy} \overset{3s}{\dfrac{\partial y}{\partial r}}\right) + 3\left(f_{yx} \overset{2r}{\dfrac{\partial x}{\partial r}} + f_{yy} \overset{3s}{\dfrac{\partial y}{\partial r}}\right) + 3 f_y$.

② $\dfrac{\partial^2 z}{\partial s^2}$ ③ $\dfrac{\partial^2 z}{\partial r^2}$

2^{nd} derivative using the chain rules:

Ex: $z = F(x,y)$ $\quad x = r^2 + s^2$, $y = 3rs$

Find $\dfrac{\partial^2 z}{\partial s^2}$:

$$z = f \begin{cases} x \begin{cases} r \\ s \end{cases} \\ y \begin{cases} r \\ s \end{cases} \end{cases}$$

$$\frac{\partial z}{\partial s} = F_x \frac{\partial x}{\partial s} + F_y \frac{\partial y}{\partial s} \rightarrow$$ (مشتقة اولى)

$$F_x \begin{cases} x \begin{cases} r \\ s \end{cases} \\ y \begin{cases} r \\ s \end{cases} \end{cases} \qquad F_y \begin{cases} x \begin{cases} r \\ s \end{cases} \\ y \begin{cases} r \\ s \end{cases} \end{cases}$$

$$= F_x \cdot (2s) + F_y (3r)$$

2) $$\frac{\partial^2 z}{\partial s^2} = F_x \cdot 2 + 2s\, F_{xs} + 3r\, F_{ys}$$

$$2F_x + 2s\left(F_{xx} \frac{\partial x}{\partial s} + F_{xy} \frac{\partial y}{\partial s}\right) + 3r\left(F_{yx} \frac{\partial x}{\partial s} + F_{yy} \cdot \frac{\partial y}{\partial s}\right)$$

Ex: 1) $F(x,y) = x^2 y^3$ $\qquad x = r^2 + s^2$, $y = 3r^2 s$

Find $\dfrac{\partial^2 F}{\partial s \partial r}$ $\qquad \dfrac{\partial y}{\partial r} = 6rs$, $\dfrac{\partial x}{\partial r} = 2r$

1) $$\frac{\partial F}{\partial r} = F_x \cdot \frac{\partial x}{\partial r} + F_y \cdot \frac{\partial y}{\partial r}$$

$\dfrac{\partial y}{\partial s} = 3r^2$ $\quad \dfrac{\partial x}{\partial r} = 2x$

$$= F_x \cdot 2r + F_y \cdot 6rs$$

2) $\frac{\partial^2 F}{\partial s \partial r} = 2rFx + 6r(Fy \cdot 1 + 5Fy\, s)$

$Fx = 2xy^3 \qquad Fxx = 2y^3$

$\cdot Fxy = 6xy^2 \,,\, (Fy = 3x^2y^2) + Fyy = 6x^2y$

$= 2r(2y^3 \cdot 2s + 6xy^2 \cdot 3r^2) + 6r \cdot 3x^2y^2 + 6rs(6xy^2\, 2s + 6x^2y \cdot 3r^2)$

* Implicit differentiation:

1) $F(x,y) = 0$

Find $\frac{\partial y}{\partial x}$?

$0 = F(x,y) \left\langle \begin{matrix} x \\ y \to x \end{matrix} \right.$

$0 = F(x,y)$ diff bothsind respect to x

$0 = Fx + Fy \cdot \frac{\partial y}{\partial x}$

$0 = Fx + Fy \cdot \frac{\partial y}{\partial x}$

$-Fx = Fy\ \frac{\partial y}{\partial x}$

$\frac{\partial y}{\partial x} = -\frac{Fx}{Fy}$

Ex: $x^2 + 3xy^2 + 2y = 4$ find $\frac{\partial y}{\partial x} = ?$

$\frac{\partial y}{\partial x} = -\frac{Fx}{Fy}$

$\frac{\partial y}{\partial x} = -\frac{2x + 3y^2}{0 + 6xy + 2}$

Proof:

$\frac{\partial z}{\partial y} = -\frac{Fy}{Fz}$

$0 = F(x,y,z) \left\langle \begin{matrix} x \\ y \\ z \left\langle \begin{matrix} x \\ y \end{matrix} \right. \end{matrix} \right.$

$0 = Fy + Fz \cdot \frac{\partial z}{\partial y}$

$\boxed{\frac{\partial z}{\partial y} = -\frac{Fy}{Fz}}$

2) $F(x,y,z) = 0$

$\frac{\partial z}{\partial x} = ?$

$\frac{\partial z}{\partial x} = \frac{-Fx}{Fz}$

$\frac{\partial y}{\partial x} = -\frac{Fx}{Fy}$

$\frac{\partial z}{\partial y} = \frac{-Fy}{Fz}$

Ex: find $\frac{\partial z}{\partial x}$

1) $z = \frac{\ln(x+y^2)}{y^3}$ (اقتران)

$$\frac{\partial z}{\partial x} = \frac{1}{y^3} \cdot \frac{1}{x+y^2}$$

2) $z = \frac{\ln(x+z^2)}{y^3}$ معادلة

$$zy^3 = \ln(x+z^2)$$

$$zy^3 - \ln(x+z^2) = 0 \qquad F(x,y,z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{-1}{x+z^2}}{y^2 - \frac{2x}{x+z^2}}$$

§14.6: Directional Derivative and gradient vector:

1) gradient vector: $\vec{\nabla}$

$\Rightarrow \vec{\nabla}F(x,y) = F_x\hat{i} + F_y\hat{j}$

2) $\vec{\nabla}F(x,y,z) = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

Ex) $F(x,y) = xy$

find

1) $\vec{\nabla}F = y\hat{i} + x\hat{j}$

2) $\vec{\nabla}F(1,2) = 2\hat{i} + \hat{j}$

3) $\vec{\nabla}F(-2,3) = 3\hat{i} - 2\hat{j}$

* Geometrical meaning of $\vec{\nabla}F(x_0,y_0)$

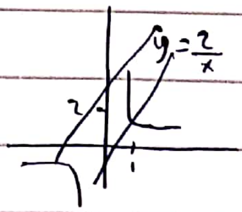
From

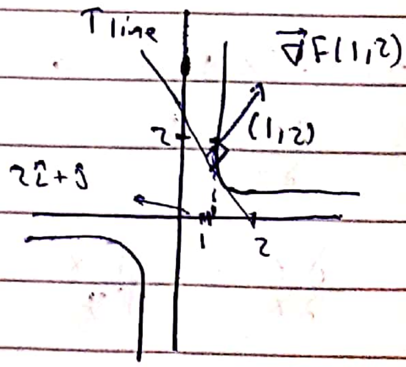
Ex 1) $F(x,y) = F(1,2)$ ← level curve

$xy = 1(2) = 2$

$xy = 2$ → level curve

$y = \frac{2}{x}$





$\vec{\nabla}F(1,2) = 2\hat{i} + \hat{j}$

$\vec{\nabla}F(x_0,y_0)$ = is the direction of normal to the tangent line to the level curve that is obtained by $F(x,y) = F(x_0,y_0)$ ← level curve

$\Rightarrow$ Eqn of normal lines:

N: $\frac{x-1}{2} = \frac{y-2}{1}$

T line: $\frac{x-1}{1} = \frac{y-2}{-2}$

* $F(x,y,z)=0$ $\leftarrow$ surface

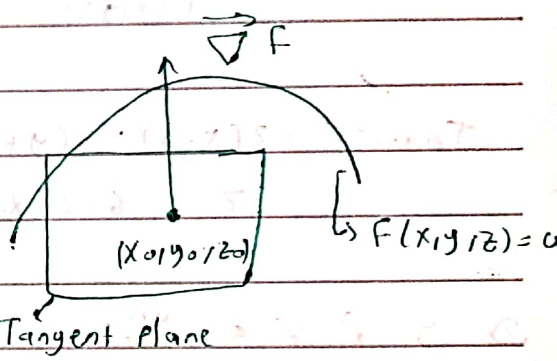
$\Rightarrow$ $\vec{\nabla} F(x_0,y_0,z_0)$ = is normal to tangent plane to this level Surface

$= F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$

T plane: $F_x \cdot (x - x_0) + F_y \, (y - y_0) + F_z \cdot (z - z_0) = 0$

N T plane: $\frac{x - x_0}{F_x} = \frac{y - y_0}{F_y} = \frac{z - z_0}{F_z}$

خط



Tangent plane:

Ex: 1) $z = xy$: write an eqn of the tangent plane at $(3,-2)$

su

$z = xy \rightarrow$ surface $z = F(x,y)$

linear app $\rightarrow$ T plane: $z = F(3,-2) + F_x \cdot (x-3) + F_y \, (y+2)$

$z = -6 - 2(x-3) + 3(y+2)$

$F_x = y \big|_{(3,-2)} = -2$ $\quad F_y = x \big|_{(3,-2)} = 3$

OR eqn $z - xy = 0 \rightarrow \nabla F = -y\hat{i} - x\hat{j} + \hat{k}$

$= -2\hat{i} - 3\hat{j} + \hat{k}$ $\quad \vec{\nabla} f(3,-2,-6)$

* Tangent plane:

function

Surface $z = (x, y)$

Ex: ① $z = xy$

write an eqn. of the tangent plane at $(3, -2)$

* linear app.x

Tplane $z = f(3,-2) + f_x \cdot (x-3) + f_y (y+2)$

$= -6 - 2(x-3) + 3(y+2)$

$f_x = y \big|_{(3,-2)} = -2 \,,\ f_y = x \big|_{(3,-2)} = 3$

OR $\rightarrow$ eqn. $z - xy = 0 \longrightarrow \nabla f = -y\hat{i} - x\hat{j} + \hat{k}$

$F(x, y, z)$ $\quad \nabla F(3, -2, -6)$

$= +2\hat{i} - 3\hat{j} + \hat{k}$

Tplane: $+2(x-3) - 3(y+2) + (z+6) = 0$

$z = -6 - 2(x-3) + 3(y+2)$

② $y = x^2 - z^2 \leftarrow$ surface eqn.

write an eqn of Tangent plane at $(4, 7, 3)$

$$\underbrace{x^2 - z^2 - y}_{F} = 0$$

$$\nabla F = 2x\hat{i} - \hat{j} - 2 \cdot z\hat{k} \Big|_{P_0}$$

$$\nabla F(P_0) = 8\hat{i} - \hat{j} - 6\hat{k}$$

TPL: $8(x-4) - 1(y-7) - 6(z-3) = 0$

Normal to

$$\frac{x-4}{8} = \frac{y-7}{-1} = \frac{z-3}{-6}$$

14.6 Directional Derivative & gradient vector:

② Directional Derivative of F in the direction of a vector $\bar{a}$

$$D_{\bar{a}}F = \bar{\nabla}F \cdot \hat{u}_{\bar{a}} \qquad \hat{u}_{\bar{a}} = \frac{\bar{a}}{\|a\|}$$

Ex $f(x,y,z) = xy + xz + yz$

Find $D_F(1,-1,+3)$ in the direction towards the pt. $Q(2,4,5)$

① $\bar{PQ} = \langle 1, 5, 2\rangle \Rightarrow \hat{u}_{PQ}$

$\|PQ\| = \sqrt{1+25+4} = \sqrt{30}$

$\therefore \hat{u} = \frac{1}{\sqrt{30}} \langle 1,5,2\rangle$

$\bar{\nabla}F = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$

$\bar{\nabla}f(P) = 2\hat{i} + 4\hat{j} + 0\hat{k}$

$D_{\bar{PQ}}f(P) = \bar{\nabla}F \cdot \hat{u}_{PQ}$

$= \frac{1}{\sqrt{30}}(2+20)$

$= \frac{22}{\sqrt{30}}$

② $D_F(P)$ in the direction towards the origin $(0,0,0)$

$\bar{a} = -\hat{i} + \hat{j} - 3\hat{k}$

$\|\bar{a}\| = \sqrt{1+1+9} = \sqrt{11}$

$u_{\bar{a}} = \frac{1}{\sqrt{11}} \langle -1, 1, -3\rangle$

$\therefore D_{\bar{a}}F(P) = \nabla F \cdot \hat{u}_{\bar{a}}$

$= \frac{1}{\sqrt{11}}(-2+4)$

$= \frac{2}{\sqrt{11}}$

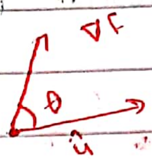
③ $D_{\hat{i}} f = \nabla f \cdot \hat{i}$

$= f_x = 2$

④ $D_{-2\hat{j}} f(P) = \nabla f \cdot (-\hat{j})$

$= -f_y = -4$

* Max & min values of D_f



$D_{\hat{u}} f = \nabla f \cdot \hat{u}$. $\|\hat{u}\| = 1$

$\theta = \pi$ $\quad D_{\hat{u}} f = \|\nabla f\| \, \|\hat{u}\| \cos\theta$

$-\|\nabla f\| \leq D_u f = \|\nabla f\| \cos\theta \leq \|\nabla f\| \cdot 1 = \|\nabla f\|$ $\quad \theta = 0$

min value of D_f & it occurs in the opposite direction of $\vec{\nabla} f$

Max value of D_f & it occurs in the direction of $\vec{\nabla} f$

⑤ Find the max. value of $D_f(P) = \|\nabla f(P)\| = \|2\hat{i} + 4\hat{j}\|$

$= \sqrt{4+16} = \sqrt{20}$

" " min. value " " $= -\|\nabla f(P)\|$

$= -\sqrt{20}$

$\Rightarrow -\sqrt{20} \leq D_f(P) \leq \sqrt{20}$

Ex: if $D_f(1,2) = 5 \rightarrow 5 = \vec{\nabla} f \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle \Rightarrow 5 = \frac{3}{5} f_x - \frac{4}{5} f_y \Rightarrow (25 = 3f_x - 4f_y)^3$ — ①

$\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$

& $D_f(1,2) = 10 \rightarrow 10 = \frac{4}{5} f_x + \frac{3}{5} f_y \Rightarrow (50 = 4f_x + 3f_y)^4$ — ②

$\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$

$275 = 25 f_x$

$f_x = \frac{275}{25} = \boxed{f_x = 11}$

Find $D_f(1,2)$ in the Direction $\vec{a}$ towards the origion

$\vec{a} = -\hat{i} - 2\hat{j} \Rightarrow \hat{u}_a = \frac{1}{\sqrt{5}} \langle -1, -2 \rangle$

$\|\vec{a}\| = \sqrt{5}$

? $\quad D_{f_{\hat{u}_{\vec{a}}}} = \underbrace{\nabla f(1,2)}_{?} \cdot \hat{u}_a$

$\nabla F(1,2) = F_x \hat{i} + F_y \hat{j}$

① $25 - 33 - F_y$

$\cancel{12 = -4 F_y}$

$-8 = -4 F_y$

$\boxed{F_y = 2} \longrightarrow \nabla F = 11\hat{i} + 2\hat{j}$

$\Longrightarrow D_F = \frac{1}{\sqrt{5}} (-11 - 4)$

$= -\frac{15}{\sqrt{5}}$

Ex: if $D_f(3,-2,1) = -5$

$(2\hat{i} - \hat{j} - 2\hat{k})$ ∇F عكس

& $\|\nabla f(3,-2,1)\| = 5$

Find $\vec{\nabla} F(3,-2,1) = ??$

? $\vec{\nabla} F = \|\vec{\nabla} F\| \, u_{\vec{\nabla} F}$

$= 5 \left(-\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})\right) = -\frac{5}{3} \langle 2,-1,-2 \rangle$

$\therefore -(2\hat{i} - \hat{j} - 2\hat{k}) = -\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$ ..... $\vec{\nabla} f$ اتجاه

$\| \quad \| = 3$

$* D_{\bar{a}}f = \bar{\nabla} f \cdot \hat{u}_{\bar{a}}$

① The max. $Df = \|\nabla f\|$

② The direction of this Max

(" " " the Fastest change)

of Df. is the direct of $\bar{\nabla} f$.

$\hat{u} = ?$

Ex. Find the directions in which the Df ; $f(x,y) = y e^{-xy}$ at $(0,2)$ has the value 1

?? $\hat{u}$: $D_{\hat{u}} f = 1$ ---(*)

let $\hat{u} = u_1 \hat{\imath} + u_2 \hat{\jmath}$: $\|\hat{u}\| = 1$

$\sqrt{u_1^2 + u_2^2} = 1$

$u_1^2 + u_2^2 = 1$ ---①

$\bar{\nabla} f = -y^2 e^{-xy} \hat{\imath} + (-xye^{-xy} + e^{-xy}) \hat{\jmath} \Big|_{(0,2)} = -4\hat{\imath} + \hat{\jmath}$

$(*) \rightarrow 1 = \nabla f \cdot \hat{u}$

$1 = \langle -4, 1 \rangle \cdot \langle u_1, u_2 \rangle$

$1 = -4u_1 + u_2$ ---②

$\rightarrow u_2 = 1 + 4u_1$ ---③

3 in 1 $\rightarrow$ $u_1^2 + (1 + 4u_1)^2 = 1$

$u_1^2 + \cancel{1} + 8u_1 + 16u_1^2 = \cancel{1}$

$17u_1^2 + 8u_1 = 0$

$u_1 (17u_1 + 8) = 0$

$u_1 = 0$ (i) oR $u_1 = \frac{-8}{17}$ (ii)

(i) $u_1 = 0 \xrightarrow{③} u_2 = 1 \Longrightarrow \hat{u}_{(1)} = \hat{\jmath}$

(ii) $u_1 = \frac{-8}{17} \xrightarrow{③} u_2 = 1 - \frac{32}{17} = \frac{-15}{17} \Longrightarrow \hat{u}_{(2)} = \frac{-8}{17}\hat{\imath} - \frac{15}{17}\hat{\jmath}$.

Ex: Find all pts. at which the direction ↑ fastest change of the function $f(x,y) = x^2 + y^2 - 2x - 4y$ is $\hat{i} + \hat{j}$

$\vec{\nabla} f$ اتجاه

$\therefore \vec{\nabla} f \parallel \hat{i} + \hat{j}$

$\vec{\nabla} f = (2x-2)\hat{i} + (2y-4)\hat{j}.$

$\vec{\nabla} f \parallel \hat{i} + \hat{j}$

$\nabla f = \alpha(\hat{i} + \hat{j})$

$(2x-2)\hat{i} + (2y-4)\hat{j} = \alpha(\hat{i} + \hat{j})$

$\rightarrow 2x - 2 = \alpha$

$\rightarrow 2y - 4 = \alpha$

$2x - 2 = 2y - 4$

$x - 1 = y - 2$

$\boxed{y = x + 1}$ خط

all pts. on this line is a sol.

Ex: At what pt. on the paraboloid S: $y = x^2 + z^2$ is the tangent plane parallel to $x + 2y + 3z = 1$

$x^2 + z^2 - y = 0$

gradient ال يوازي tangent plane ال

$T_s \parallel \pi$; $\vec{\nabla} f = 2x\hat{i} - \hat{j} + 2z\hat{k}$

$\vec{\nabla} f \parallel \vec{n}_\pi$; $\vec{n}_\pi = \hat{i} + 2\hat{j} + 3\hat{k}$

$\Leftrightarrow \vec{\nabla} f = \alpha\, \vec{n}_\pi$

$2x = \alpha \quad = x = \frac{1}{2}\alpha = \boxed{x = -\frac{1}{4}}$

& $-1 = 2\alpha \rightarrow \alpha = -\frac{1}{2}$

& $2z = 3\alpha \quad z = \frac{3}{2}\alpha \rightarrow \boxed{z = -\frac{3}{4}}$

& $y = x^2 + z^2 \quad \therefore (4)$

(4) $\rightarrow y = \frac{1}{16} + \frac{9}{16} = \frac{10}{16}$

$\therefore pt\left(-\frac{1}{4}, \frac{10}{16}, -\frac{3}{4}\right)$

Ex: Show that the ellipsoid S_1: $3x^2 + 2y^2 + z^2 = 9$ ($F = 0$)

& the sphere S_2: $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ (G)

are tangent to each other at the pt $(1,1,2)$.

? T_{s_1} is the same as T_{s_2}.

T_{s_1} plane: ~~As~~ normal is $\vec{\nabla} F(1,1,2)$

$\nabla F = 6x\hat{i} + 4y\hat{j} + 2z\hat{k} \,\big|_{(1,1,2)}$

$= 6\hat{i} + 4\hat{j} + 4\hat{k}.$

$T_{plane\ S_1}$: $6(x-1)+4(y-1)+4(z-2)=0$

$T_{plane\ S_2}$: $\vec{\nabla} G = \vec{n}_{T_{S_2}}$

$$\vec{\nabla} G = (2x-8)\hat{i} + (2y-6)\hat{j} + (2z-8)\hat{k} \Big|_{(1,1,2)}$$

$$= -6\hat{i} - 4\hat{j} - 4\hat{k} = n_{T_{S_2}}$$

T_{S_2}: $-6(x-1) - 4(y-1) - 4(z-2) = 0$

$T_{S_1} = T_{S_2}$

Ex: Find parametric eqns. of the tangent line to the curve of intersection of the paraboloid S_1: $z = x^2+y^2$ & ellipsoid S_2: $4x^2+y^2+z^2=9$ at $(-1,1,2)$

$F(x,y,z)=0$

S_1: $x^2+y^2-z=0$

$\underbrace{\vec{\nabla} F}_{n_{T_{S_1}}} \times \underbrace{\nabla G}_{n_{T_{S_2}}}$ = direction of line of int of T_{S_1} & T_{S_2}

= direction of the tangent line to the curve of int. of S_1 & S_2

$$\nabla F = \langle 2x, 2y, -1\rangle \Big|_{(-1,1,2)} = \langle -2, 2, -1\rangle$$

$$\nabla G = \langle 8x, 2y, 2z\rangle \Big|_{(-1,1,2)} = \langle -8, 2, 4\rangle$$

$$\vec{\nabla} F \times \nabla G = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -1 \\ -8 & 2 & 4 \end{vmatrix} = 10\hat{i} + 16\hat{j} + 12\hat{k}$$

T_{line}: $x = -1 + 10t$

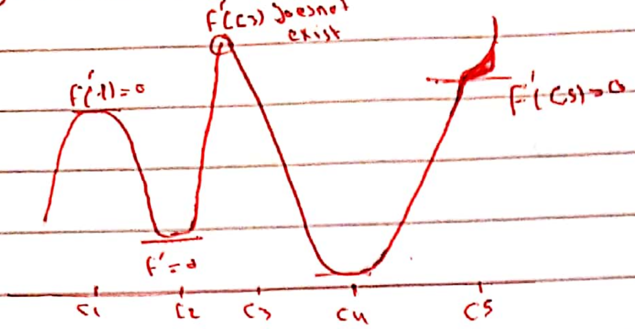
$y = 1 + 16t$

$z = 2 + 12t$

§ 14.7 Maximum & min Values.

$y = f(x) \leftarrow$ curve

اذا كانت المشتقة الاولى صفر
الثانية بتطلع موجبة (min) او سالبة (max)



* $Z = f(x,y) \leftarrow$ Surface

Def. Critical of f. at (x_0, y_0)

if ① $f_x = 0$ & $f_y = 0$

OR ② f_x OR f_y OR both

does not exist

* 2^{nd} partial Derivative test:

to classify the criticals: $f_x = 0$ & $f_y = 0$ for local Max, Min, or Saddle

(x_0, y_0) is this critical & f_{xy} & f_{yx} are cont.

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 \quad \text{at } (x_0, y_0)$$

① if $D > 0$ & (i) $f_{xx} > 0 \xrightarrow{U}$ local min

(ii) $f_{xx} < 0 \xrightarrow{A}$ local max

② $D < 0 \longrightarrow$ Saddle

③ $D = 0 \longrightarrow$ test Fails.

Ex: find criticals & ~~cal~~ classify

Ex: ① $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$

$$f_x = y - \frac{1}{x^2} = y - x^{-2}$$

$$f_y = x - \frac{1}{y^2} = x - y^{-2}$$

$$f_x = 0 \rightarrow y - \frac{1}{x^2} = 0 \rightarrow y = \frac{1}{x^2} \quad \text{①}$$

&

$$f_y = 0 \rightarrow x - \frac{1}{y^2} = 0$$

$$x - \frac{1}{\left(\frac{1}{x^2}\right)^2} = 0$$

$x - x^4 = 0$

$x(1 - x^3) = 0$

$x = 0$ OR $x = 1$

خارج المجال تهمل

عوضناها ما بزبط

$x = 1 \xrightarrow{①} y = 1$

$\therefore (1,1)$ is the critical

$f_{xx} = 2x^{-3} = \left.\frac{2}{x^3}\right|_{(1,1)} = 2$

$f_{yy} = \left.\frac{2}{y^3}\right|_{(1,1)} = 2$

$f_{xy} = 1$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (2)(2) - (1)^2 = 3 > 0$

$\&$ $f_{xx} > 0 \xRightarrow{ST}$

at $(1,1)$ f has local min value $= f(1,1) = 1+1+1 = \boxed{3}$

② $f(x,y) = e^y(y^2 - x^2)$

$f_x = -2xe^y$

$f_y = 2ye^y + e^y(y^2 - x^2)$

$\underline{f_y = e^y(2y + y^2 - x^2)}$

$f_x = 0 \rightarrow -2xe^y = 0 \rightarrow x = 0$

$\&$ $f_y = 0 \rightarrow e^y(2y + y^2 - x^2) = 0$

$\Rightarrow 2y + y^2 - x^2 = 0$

$x = 0 \rightarrow 2y + y^2 = 0$

$y(2 + y) = 0$

$y = 0$ OR $y = -2$

Critical $(0,0)$, $(0,-2)$

(x_0,y_0)	f_{xx}	$f_{yy} =$	$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$	$f_{xy} = -2xe^y$
	$-2e^y$	$e^y(2+4y+y^2-x^2)$		
$(0,0)$	-2	2	$-4<0 \rightarrow$ saddle	0
$(0,-2)$	$-2e^{-2}$	$-2e^{-2}$	$4e^{-4}>0$ & $f_{xx}<0 \Rightarrow$ local Max	0

$e^y(2+2y) + e^y(2y+y^2-x^2)$

$e^y(2+4y+y^2-x^2)$

Ex $f(x,y) = \frac{1}{3}x^3 - 9x + y^3 + 3y^2 + 7$

$f_x = x^2 - 9$

$f_y = 3y^2 + 6y$

$f_x = 0 \rightarrow x^2 - 9 = 0 \rightarrow x^2 = 9 \rightarrow x = 3, x = -3$

$f_y = 0 \rightarrow 3y^2 + 6y = 0 \rightarrow 3y(y+2) = 0 \rightarrow y = 0, y = -2$

Criticals are:

$(3,0), (3,-2), (-3,0), (-3,-2)$

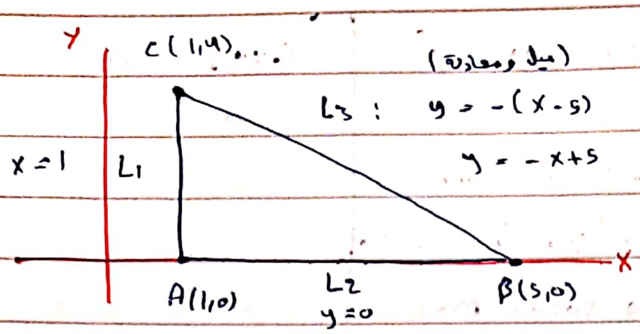
(x_0,y_0)	$f_{xx} = 2x$	$f_{yy} = 6y+6$	$f_{xy} = 0$	$D = f_{xx} f_{yy} - (f_{xy})^2$
$(3,0)$	6	6	0	$36>0$ & $f_{xx}>0 \xrightarrow{+}$ local min.
$(3,-2)$	6	-6	0	$-36<0 \rightarrow$ saddle
$(-3,0)$	-6	6	0	$-36<0 \rightarrow$ saddle
$(-3,-2)$	-6	-6	0	$36>0$ & $f_{xx}<0 \xrightarrow{-}$ local max.

* Thm. If $f(x,y)$ is cont. on a closed region then f has absolute Max & Min

Ex: $f(x,y) = 3 + xy - x - 2y$ ---*

Find the absolute max & min of f over this closed triangular region with vertices $A(1,0)$, $B(5,0)$ & $C(1,4)$

Scanned by CamScanner



(قيل ومعادلة)

$L_3: \; y = -(x-5)$

$y = -x+5$

Ⓘ interior: critical:

$f_x = y-1 = 0 \longrightarrow y=1$

$f_y = x-2 = 0 \longrightarrow x=2$

$\Rightarrow$ critical $(2,1)$

Ⓘⓘ Boundery:

① $L_1: \; x=1 \xrightarrow{*}$

$G_1(y) = 2-y \; , \; 0 \le y \le 4$

$G_1' = -1 \ne 0$

② $L_2: \; y=0 \xrightarrow{*}$

$G_2(x) = 3-x \; , \; 1 \le x \le 5$

$G_2'(x) = -1 \ne 0$

③ $L_3: \; \boxed{y=-x+5} \xrightarrow{*}$

$G_3(x) = 3 + x(-x+5) - x - 2(-x+5)$

$= 3 - x^2 + 5x - x + 2x - 10$

$G_3(x) = -x^2 + 6x - 7 \; , \; 1 \le x \le 5$

$G_3' = -2x + 6$

$= 0 \rightarrow \boxed{x=3}$

$y = -3+5 = \boxed{2}$

pt $(3,2)$

	(x,y)	$f(x,y) = 3+xy-x-2y$
I	$(2,1)$	$5-2-2=1$
B L_1	$(1,0)$	$3-1=\boxed{2} \rightarrow$ max
	$(1,4)$	$3+4-1-8=-2$
L_2	$(5,0)$	-2
L_3	$(3,2)$	$3+6-3-4=\boxed{2}$

2 is the absolute max at $(1,0)$ & $(3,2)$

-2 " " min " $(1,4)$ & $(5,0)$

Scanned by CamScanner

§ Lagrange Multipliers

Lag. Problems:

1. Maximize and/or minimize $F(x,y)$
   Subject to the Constraint: $g(x,y)=0$
2. Max. and/or min. $F(x,y,z)$
   Subject to the constraint: $g(x,y,z)=0$

* Lag. Thm

At the pt. (Critical) P_0 where F has a contrained Max/min.

$\bar{\nabla} F(P_0) \parallel \bar{\nabla} g(P_0)$

i.e there is $\lambda \neq 0$:

$\bar{\nabla} F(P_0) = \lambda \bar{\nabla} \bar{g}(P_0)$

$$\left.\begin{array}{l} \Leftrightarrow F_x = \lambda g_x \\ \& \ F_y = \lambda g_y \\ \& \ F_z = \lambda g_z \end{array}\right\} \text{lag. eqns.}$$

& $g(x,y,z)=0$

Ex. ① find the extreme values of

$F(x,y) = x^2+2y^2$ on the circle

$\underbrace{x^2+y^2=1}_{g=0}$ دائرة

lag eqns: $F_x = \lambda g_x$ → $2x = 2\lambda x$ ...①

$F_y = \lambda g_y$ → $4y = 2\lambda y$ ...②

& $x^2+y^2=1$ → & $x^2+y^2=1$ ...③

$\frac{①}{②} \rightarrow \frac{x}{2y} = \frac{x}{y}$

$xy = 2xy$

$xy = 0$

① $x=0$ ② $y=0$ oR Both $=0$ $X \rightarrow x\,0+0=1$ ③ في ③

i) $x=0 \xrightarrow{③} y^2=1 \rightarrow y=\pm 1$

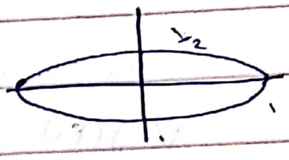
$P_1(0,1)$, $P_2(0,-1)$

ii) $y=0 \xrightarrow{③} x^2=1 \rightarrow x=\pm 1$

$P_3(1,0)$, $P_4(-1,0)$

$P(x,y)$	$F = x^2+2y^2$
$P_1(0,1)$	$\boxed{2}$ = Max at P_1 & P_2
$P_2(0,-1)$	$\boxed{2}$
$P_3(1,0)$	$\boxed{1}$ = Mi at P_3 & P_4.
$P_4(-1,0)$	$\boxed{1}$

Ex: Find the absolute extremes of $f(x,y) = e^{-xy}$ on Region $x^2+4y^2=1$



(I) interior: $x^2+4y^2<1$

(II) Boundary: $x^2+4y^2=1$

(I) interior: $f_x=0$ & $f_y=0$ → criticals

$-ye^{-xy}=0 \Rightarrow y=0$

& $-xe^{-xy}=0 \Rightarrow x=0$

$= P_0(0,0)$

	$P(x,y)$	$f=e^{-xy}$	
I	$P_0(0,0)$	$e^0=1$	
II	$P_1(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}})$	$e^{-\frac{1}{4}}$	
	$P_2(\frac{1}{\sqrt{2}},\frac{-1}{2\sqrt{2}})$	$\boxed{e^{\frac{1}{4}}}$	= M at P_2 & P_3
	$P_3(\frac{-1}{\sqrt{2}},+\frac{1}{2\sqrt{2}})$	$e^{\frac{1}{4}}$	
	$P_4(\frac{-1}{\sqrt{2}},\frac{-1}{2\sqrt{2}})$	$\boxed{e^{-\frac{1}{4}}}$	= m at P_1 & P_4

(II) Boundary: $x^2+4y^2=1$

Lagrange

$f_x = \lambda g_x$ | $-ye^{-xy} = 2\lambda x$ --- ①

& $f_y = \lambda g_y$ | & $-xe^{-xy} = 8\lambda y$ ... ②

& ~~λ~~ $x^2+4y^2=1$ | & $x^2+4y^2=1$ ... ③

*

$\frac{①}{②} \rightarrow \frac{y}{x} = \frac{x}{4y}$

$4y^2 = x^2$ ... ④

④ in ③ → $x^2+x^2=1$

$2x^2=1 \rightarrow x^2=\frac{1}{2}$

$x=\mp\frac{1}{\sqrt{2}}$

$x^2=\frac{1}{2} \xrightarrow{④} 4y^2=\frac{1}{2}$

$y^2=\frac{1}{8} \rightarrow y=\pm\frac{1}{\sqrt{8}}=\pm\frac{1}{2\sqrt{2}}$

$P_1(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}})$, $P_2(\frac{1}{\sqrt{2}},-\frac{1}{2\sqrt{2}})$

$P_3(-\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}})$, $P_4(-\frac{1}{\sqrt{2}},-\frac{1}{2\sqrt{2}})$

* lagrange:

Ex: Find the pt.s on the sphere $x^2+y^2+z^2=36$, that are closest to & farthest from the pt. $A(1,2,2)$

$P(x,y,z)$

∂ = distance function: $\sqrt{(x-1)^2+(y-2)^2+(z-2)^2}$

$f=\partial^2$

let $f(x,y,z) = (x-1)^2+(y-2)^2+(z-2)^2$

Max. & min f s.t

$\underbrace{x^2+y^2+z^2}=36$

$g(x,y,z)=36$

lag. eqn: ~~∇f~~

$2(x-1) = 2\lambda x \Rightarrow x-1 = \lambda x \rightarrow x-\lambda x = 1 \rightarrow x(1-\lambda)=1 \rightarrow x = \dfrac{1}{1-\lambda}$

& $2(y-2) = 2\lambda y \Rightarrow$ " $\qquad y = \dfrac{2}{1-\lambda}$

& $2(z-2) = 2\lambda z \qquad z = \dfrac{2}{1-\lambda}$ $\Bigg\}$ $(*)$

& $x^2+y^2+z^2 = 36$ ..... (4)

$(*)$ in (4) $\rightarrow$

$$\frac{1}{(1-\lambda)^2}+\frac{4}{(1-\lambda)^2}+\frac{4}{(1-\lambda)^2}=36$$

$\dfrac{9}{(1-\lambda)^2}=36$ | $(1-\lambda)^2 = \dfrac{1}{4}$

$\dfrac{1}{(1-\lambda)^2}=4$ | $1-\lambda = \pm\dfrac{1}{2}$

$\lambda = 1\mp\dfrac{1}{2}$

$\lambda_1 = 1-\dfrac{1}{2} = \dfrac{1}{2}$

$\lambda_2 = 1+\dfrac{1}{2} = \dfrac{3}{2}$

(i) $\lambda_1 = \dfrac{1}{2} \xrightarrow{(*)} x = \dfrac{1}{1-\frac{1}{2}} = 2$

$y = \dfrac{2}{\frac{1}{2}} = 4$

$z = \dfrac{2}{\frac{1}{2}} = 4$

$\therefore P_1(2,4,4) \rightarrow f(P_1) = (2-1)^2+(4-2)^2+(4-2)^2 = 9$

(ii) $\lambda_2 = \dfrac{3}{2} \xrightarrow{(*)} x = \dfrac{1}{1-\frac{3}{2}} = -2,\ y=-4,\ z=-4$

$\therefore P_2(-2,-4,-4) \rightarrow f(P_2) = (-2-1)^2+(-4-2)^2+(-4-2)^2$

$= 9+36+36 = 81$

Closest $\rightarrow$ $P_1(2,4,4)$; Shortest $d = \sqrt{9} = \boxed{3}$

Farther $\rightarrow$ pt $\rightarrow$ $P_2(-2,-4,-4)$, largest $= \sqrt{81} = \boxed{9}$

Ex: Find the pt. on the plane $x-y+z=4$; that is closest to the pt. $A(1,2,3)$
min

distance function

$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$

min. f s.t $\underbrace{x-y+z}_{g} = 4$

lag. eqns:

$2(x-1) = \cancel{1}\cdot\lambda \rightarrow x = 1 + \frac{\lambda}{2}$

& $2(y-2) = -\lambda$ $\quad y = 2 - \frac{\lambda}{2}$ $\quad (*)$

& $2(z-3) = \lambda$ $\quad z = 3 + \frac{\lambda}{2}$

& $x-y+z = 4$ --- (4)

(*) in (4) $\rightarrow$

$1 + \frac{\lambda}{2} - 2 + \frac{\lambda}{2} + 3 + \frac{\lambda}{2} = 4$

$2 + \frac{3}{2}\lambda = 4$

$\frac{3}{2}\lambda = 2$

$\lambda = \frac{4}{3} \xrightarrow{(*)}$

$x = 1 + \frac{1}{2}\cdot\frac{4}{3} = 1 + \frac{2}{3}$

$y = ---$

$z = ..$

Ex: Find three positive numbers whose sum is 12.
& the sum of whose squares is as small as possible.

$x \geq 0$, $y \geq 0$ & $z \geq 0$: $\underbrace{x+y+z}_{g} = 12$

$f(x,y,z) = x^2 + y^2 + z^2 =$

min. f. s. to $x+y+z = 12$

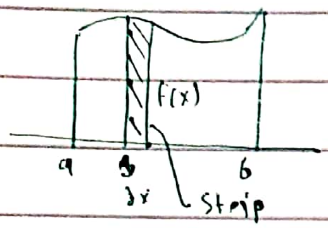
Ch# 15 Multiple integrals:

$\iint$, $\iiint$

* Single integral:

$$\int_a^b f(x)\,\partial x = F(x)\Big|_a^b$$

$$= F(b) - F(a) \in R$$



(1) double integral

$$\iint f(x,y)\,\partial A \overset{①}{=} \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\partial y\,\partial x$$

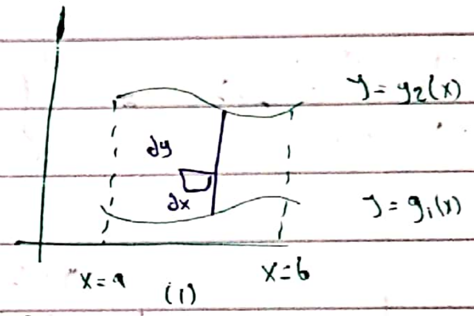
[illegible] $R \subseteq$ xy-plane

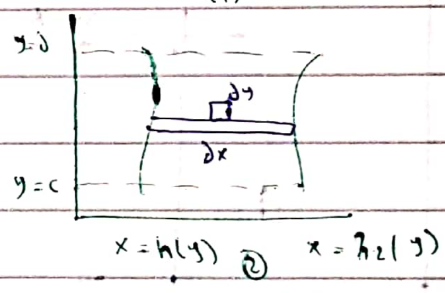
$$\partial A = \partial x\,\partial y$$

or

$$= \partial y\,\partial x$$

$$② \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\partial x\,\partial y$$





Ex $\int_0^3 \left(\int_1^2 (x+y^2)\,\partial x \right) \partial y$

$$= \int_0^3 \left(\frac{x^2}{2} + y^2 x \Big|_1^2 \right) \partial y$$

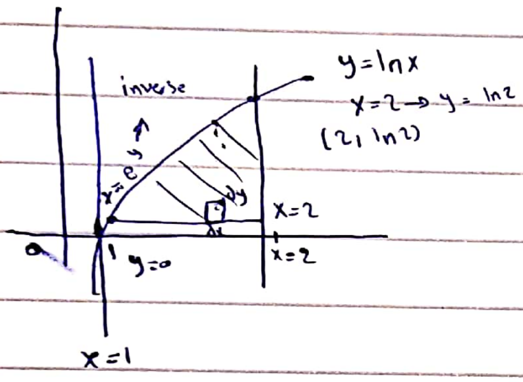
$$= \int_0^3 \left(\frac{2^2}{2} + 2y^2 \right) - \frac{1}{2} - y^2)\,\partial y$$

$$= \int_0^3 \left(\frac{3}{2} + y^2 \right) \partial y$$

$$= \frac{3}{2} y + \frac{y^3}{3} \Big|_0^3 \quad \in R$$

* Reverse the order of integration

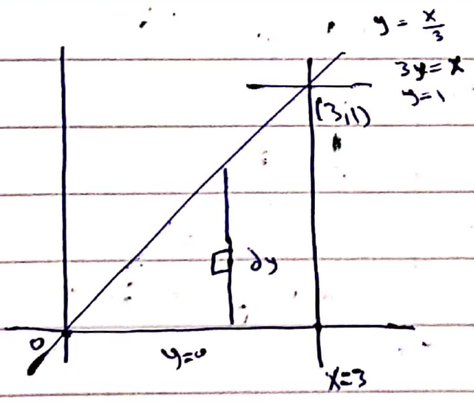
① $\displaystyle\int_{x=1}^{x=2}\int_{0=y}^{\ln x=y} f(x,y)\,dy\,dx = \int_{0}^{\ln 2}\int_{e^y}^{2} f(x,y)\,dx\,dy$



Ex: $\displaystyle\int_{y=0}^{y=1}\int_{3y=x}^{3=x} e^{x^2}\,dx\,dy = \int_{0}^{3}\int_{0}^{\frac{x}{3}} e^{x^2}\,dy\,dx$

$\int e^{x^2}\,dx$
$\int \sin x^2\,dx$
$\int \frac{\sin x}{x}\,dx$

ما بنحلوا بالطرق العادية



$= \displaystyle\int_{0}^{3} e^{x^2}\left(\frac{x}{3} - 0\right)dx$

$= \displaystyle\frac{1}{3}\int_{0}^{3} x e^{x^2}\,dx$

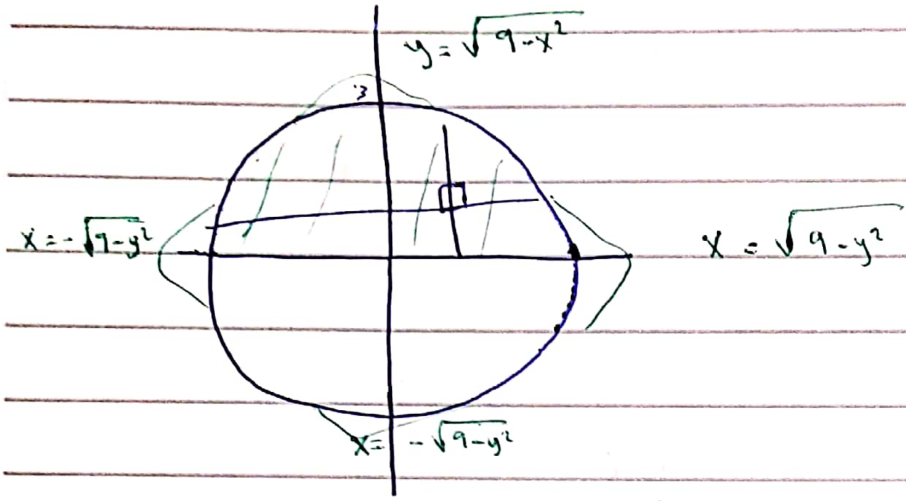
$= \frac{1}{3}\cdot\frac{1}{2}\left(e^{x^2}\Big|_0^3\right)$

$= \frac{1}{6}\left(e^9 - e^0\right)$

$= \frac{1}{6}\left(e^9 - 1\right)$

Ex. Reverse the order of integration

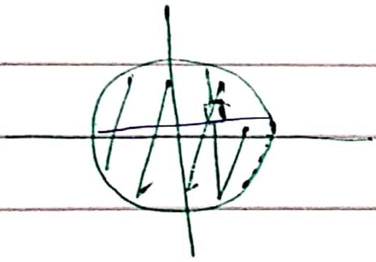
① $\int_{0}^{3}\int_{-\sqrt{9-y^2}=x}^{\sqrt{9-y^2}=x} f(x,y)\,\partial x\,\partial y \;=\; \int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}$



$x^2 = 9-y^2$

$x^2+y^2=9$

②

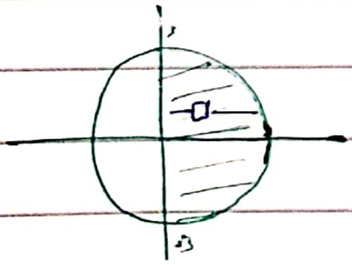


② $\iint_R f(x,y)\,\partial A$.

(i) $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y)\,\partial y\,\partial x$

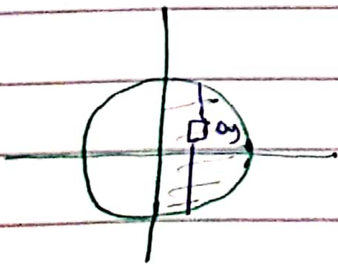
(ii) $\int_{-3}^{3}\int_{x=-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)\,\partial x\,\partial y$

③



$\iint_R f(x,y)\,\partial A$

(i) $\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} f(x,y)\,\partial x\,\partial y$



(ii) $\int_{0}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y)\,\partial y\,\partial x$

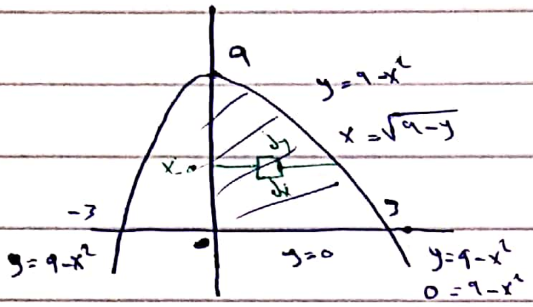
Scanned by CamScanner

(4) $\int_0^9 \int_{0=x}^{\sqrt{9-y}=x} f(x,y)\, \partial x\, \partial y = \int_0^3 \int_0^{9-x^2} f(x,y)\, \partial y\, \partial x$

$x = \sqrt{9-y}$

$x^2 = 9-y$

$y = 9-x^2$



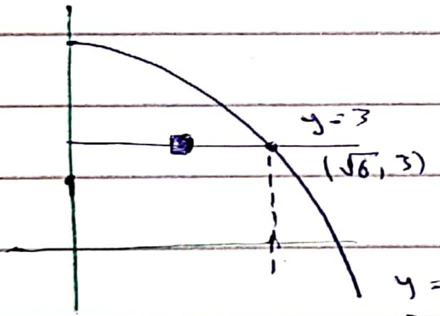
$y = 9-x^2$

$0 = 9-x^2$

$x^2 = 9$

(5) $\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\, \partial x\, \partial y = \int_0^{\sqrt{6}} \int_0^3 f(x,y)\, \partial y\, \partial x + \int_{\sqrt{6}}^3 \int_0^{9-x^2} f(x,y)\, \partial y\, \partial x$

R_1



$y = 9-x^2$

$3 = 9-x^2$

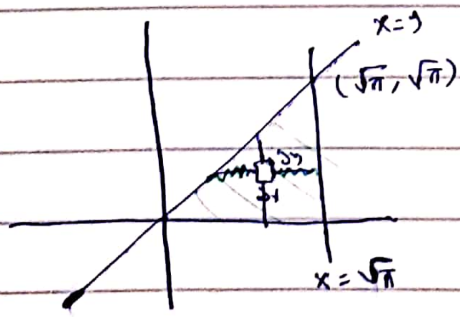
$x^2 = 6$

$x = \sqrt{6}$

Ex. $\int_0^{\sqrt{\pi}} \int_{y=x}^{\sqrt{\pi}} \cos x^2\, \partial x\, \partial y = \int_0^{\sqrt{\pi}} \int_0^{x} \cos x^2\, \partial y\, \partial x$

$= \int_0^{\sqrt{\pi}} x \cos x^2\, \partial x$

$= \frac{1}{2} \sin x^2 \Big|_0^{\sqrt{\pi}}$



$= \frac{1}{2} (\sin \pi - \sin 0)$

$= \frac{1}{2} (0 - 0) = \boxed{0}$

Ex $\int_{-3}^{3}\int_{0=y}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,\partial y\,\partial x = \int_{0}^{\pi}\int_{0}^{3} \sin r^2 \cdot r \cdot \partial r \cdot \partial\theta$

Polar:

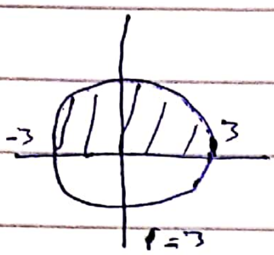
$\partial A = r\,\partial r\,\partial\theta$

$\int_{0}^{\pi} \boxed{-\frac{1}{2}\cos r^2 \Big|_0^3}\,\partial\theta$

ثابت

$-\frac{1}{2}(\cos 9 - \cos 0)\boxed{\int_0^{\pi}\partial\theta}$ π

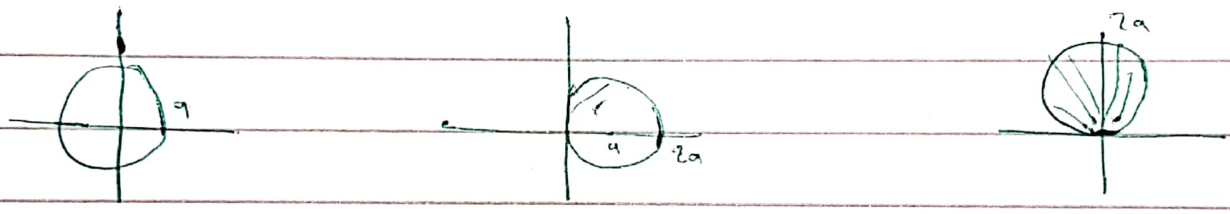
$-\frac{\pi}{2}(\cos 9 - 1)$



$x^2+y^2=9$

$r^2=9$

$r=3$



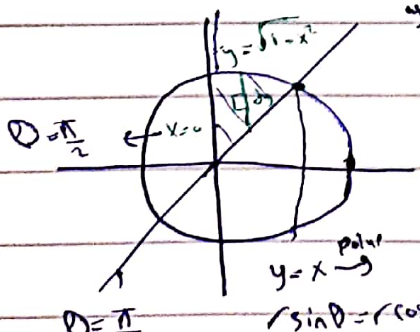
$r=9$ $\quad$ $r=2a\cos\theta$ $\quad$ $r=2a\sin\theta$

$0 \le \theta \le 2\pi$ $\quad$ $0 \le \theta \le \pi$ $\quad$ $0 \le \theta \le \pi$

Ex $\int_{0}^{\frac{1}{\sqrt{2}}}\int_{x=y}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{0}^{1} e^{r^2}\,r\,\partial r\,\partial\theta$

$r=1$

$= \frac{1}{2}e^{r^2}\Big|_0^1 \left(\frac{\pi}{2}-\frac{\pi}{4}\right)$



$y=x$

$x=\sqrt{1-x^2}$ التقاطع

$x^2=1-x^2$

$2x^2=1$

$x^2=\frac{1}{2}$

$x=\pm\frac{1}{\sqrt{2}}$

$y=x \xrightarrow{\text{Polar}}$

$r\sin\theta = r\cos\theta$

$\frac{\sin\theta}{\cos\theta}=1$

$\tan\theta=1$

$\theta=\frac{\pi}{4}$

Scanned by CamScanner

* applications on double integration:

① area: $A = \iint_R \partial A$

$\underline{R}$

$\partial A = \partial y \, \partial x$
$= \partial x \, \partial y$
$= r \, \partial r \, \partial \theta$

② Volume: V =

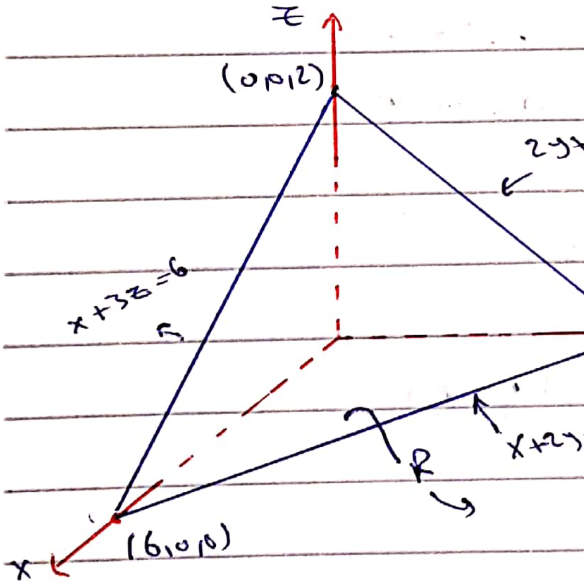
$\partial v = f(x,y) \, \partial A$

$V = \iint_R f(x,y) \, \partial A$

منطقة التكامل
Region of integral (integration)

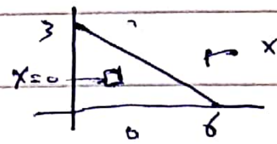
R ⊆ xy-plane
المستوية

Ex: Find the volume of the Solid region bounded by the plane $x+2y+3z=6$ & the Cordinate planes $z=0, x=0, y=0$



$z = \frac{6-x-2y}{3}$

$\Delta z = z - 0$
$= \frac{6-x-2y}{3}$



$x = 6-2y$

$V = \iint_R \frac{6-x-2y}{3} \, \partial A$

$V = \int_0^3 \int_0^{6-2y} \frac{6-x-2y}{3} \, \partial x \, \partial y$

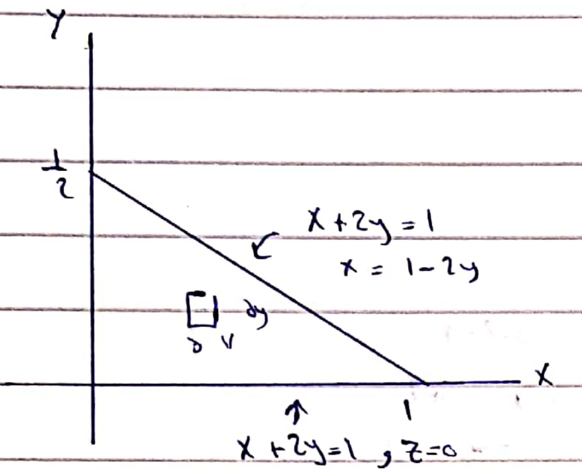
② $V = ?$ bd. by the planes

$x + 2y + 3z = 6$, $x + 2y = 1$ & the cordinate planes

$x = 0$, $y = 0$, $z = 0$

height $= \Delta z = \dfrac{6 - x - 2y}{3}$

$$V = \int_0^{\frac{1}{2}} \int_0^{1-2y} \frac{6 - x - 2y}{3} \, dx \, dy$$



③ $V = ?$ bd. by the cylinder

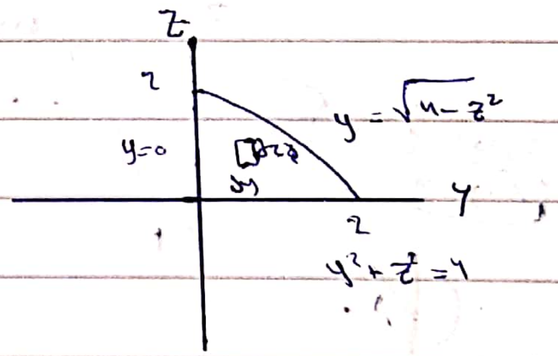
الجوانب $\rightarrow y^2 + z^2 = 4$, & the planes

$\boxed{y = \frac{x}{2} \, , \; x = 0}$, $z = 0$ in the & 1^{st} octant . ($x \geq 0$, $y \geq 0$, $z \geq 0$)

$\Delta x =$ الارتفاع

$x = 2y$, $x = 0$

$\boxed{\Delta x = 2y}$



$$V = \int_0^2 \int_0^{\sqrt{4-z^2}} 2y \, dy \, dz$$

$$y^2 \Big|_0^{\sqrt{4-z^2}}$$

$$= \int_0^2 (4 - z^2) \, dz$$

Scanned by CamScanner

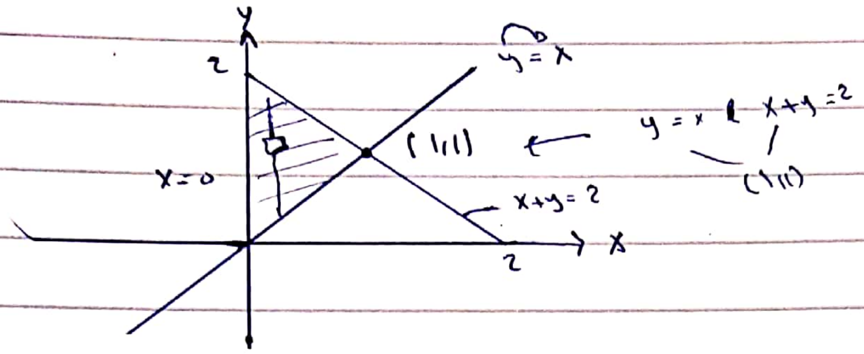
④ $V = ?$ b.d. by the planes : $z = x$, $x + y = 2$, $y = x$

& $z = 0$

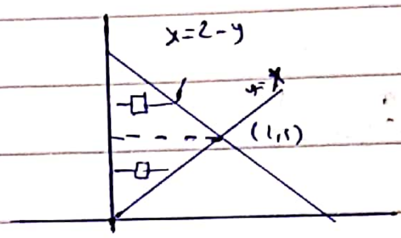
غطاء

قاع $0 = x$

الارتفاع $= \Delta z = x - 0 = x$



$$V = \int_0^1 \int_x^{2-x} x \, dy \, dx$$

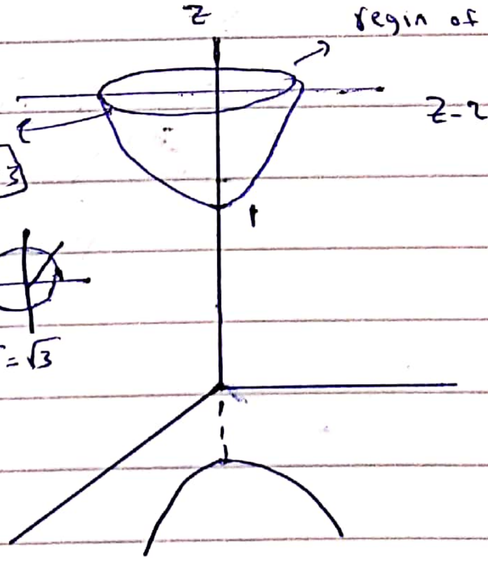


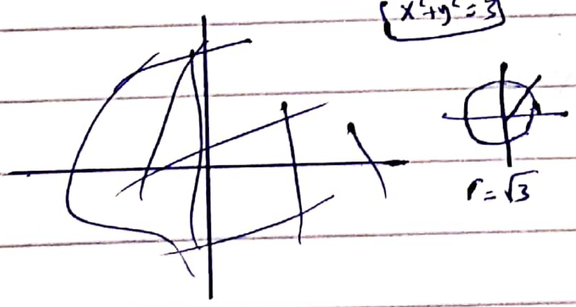
OR $$V = \int_0^1 \int_0^y x \, dx \, dy + \int_1^2 \int_0^{2-y} x \, dx \, dy$$

Ex: $V = ?$ bd. by the hyperboloid.

$-x^2 - y^2 + z^2 = 1$ & the plane:

$z = 2$





$z = \sqrt{1 + x^2 + y^2}$

Δz القاعدة الارتفاع

$\Delta z = 2 - \sqrt{1 + x^2 + y^2}$

$$V = \iint_R 2 - \sqrt{1 + x^2 + y^2} \, dA$$

Polar $$\int_0^{2\pi} \int_0^{\sqrt{3}} (2 - \sqrt{1 + r^2}) \, r \, dr \, d\theta$$

§ 15.6 Triple Integrals:

$$\iiint_{D \subseteq R^3} f(x,y,z)\, \partial v = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, \partial z\, \partial y\, \partial x \quad \in R$$

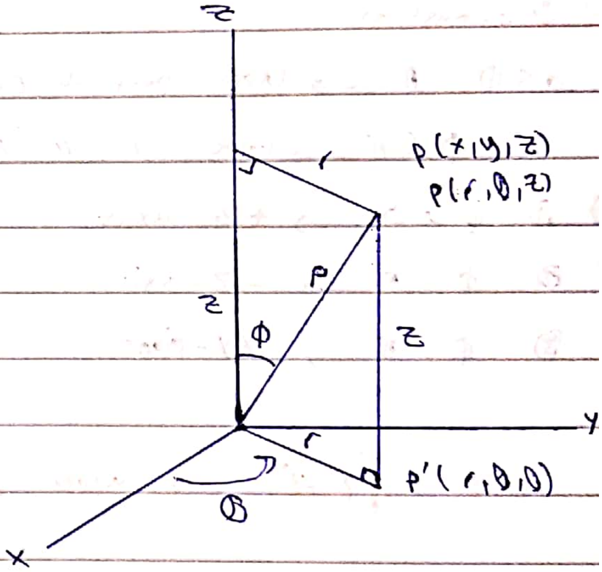
D ↓ Solid region

$$\partial v = \partial z\, \partial A$$
$$= \partial z\, \partial y\, \partial x$$
$$= \partial z\, \partial x\, \partial y$$

$$V = \iiint_{D \subseteq R^3} \partial v$$

§ 15.6 Triple Integrals, § 15.7 & § 15.8

* Coordinate Systems:



$\boxed{1}$ Cartesian Rectangular ; $p(x,y,z)$

$\partial v = \partial z\, \partial A = \partial z\, \partial y\, \partial x = \partial z\, \partial x\, \partial y$

$\boxed{2}$ Cylindrical ; $P(r, \theta, z)$

$\partial v = \partial z\, r\, \partial r\, \partial \theta$

$= r\, \partial z\, \partial r\, \partial \theta$

z = constant $\Rightarrow$ plane // xy-plane

r = constant $\Rightarrow$ Cylinder →

θ = constant $\Rightarrow$ half line

$\boxed{3}$ Spherical : $P(\rho, \theta, \phi)$

$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$

$\partial v = \rho^2 \sin\phi\, \partial \rho\, \partial \phi\, \partial \theta$

① $z = \rho \cos\phi \iff \frac{z}{\rho} = \cos\phi$

② $r = \rho \sin\phi$ $\cancel{\iff}$

Scanned by CamScanner

③ ④ $r^2 + z^2 = \rho^2$

④ $x = r\cos\theta$

$x \overset{②}{=} \rho \sin\phi \cos\theta$

⑤ $y = r\sin\theta$

$y \overset{②}{=} \rho \sin\phi \sin\theta$

⑥ $r^2 = x^2+y^2$

$\overset{②}{\Rightarrow} x^2+y^2+z^2 = \rho^2$

① ρ = constant = sphere

② θ = constant $\Rightarrow$ half plane

③ ϕ = constant

(i) $0 < \phi < \frac{\pi}{2} \longrightarrow$ Upper part of cone.

(ii) $\frac{\pi}{2} < \phi < \pi \longrightarrow$ lower " " "

(iii) ① $\phi = 0 \longrightarrow +z-$axis

② $\phi = \pi \longrightarrow -z-$axis

③ $\phi = \frac{\pi}{2} \longrightarrow$ xy-plane

Ex. $z = \sqrt{\frac{x^2+y^2}{3}}$ $\leftarrow$ upper part cone.

$\Rightarrow$ Spherical

$z = \frac{1}{\sqrt{3}} \sqrt{x^2+y^2}$ أو $\leftarrow$

$z = \frac{1}{\sqrt{3}} r$ $\leftarrow$ cylindrical

$\cancel{\rho} \cos\phi = \frac{1}{\sqrt{3}} \cancel{\rho} \sin\phi$

نقلب الرقم

نحط محل

$\tan\phi \leftarrow z$

$\frac{\sin\phi}{\cos\phi} = \sqrt{3}$

$\tan\phi = \sqrt{3}$

$\phi = \frac{\pi}{3}$

② Convert to Sph.

$z = -\sqrt{3}\sqrt{x^2+y^2}$

$\tan\phi = -\frac{1}{\sqrt{3}}$

$\phi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Ex. Identify Surfaces:

① $\rho = \csc\phi$

$\rho = \frac{1}{\sin\phi}$

$\rho\sin\phi = 1$

$r = 1$, Cylinder z-axis

② $\rho^2 = -\sec 2\phi$

$\rho^2 = -\frac{1}{\cos 2\phi}$

$\rho^2\cos 2\phi = -1$

$\rho^2(\cos^2\phi - \sin^2\phi) = -1$

$\rho^2\cos^2\phi - \rho^2\sin^2\phi = -1$

$z^2 - r^2 = -1$

$z^2 - x^2 - y^2 = -1$

$x^2 + y^2 - z^2 = 1$

hyp. of one sheet

Ex. $P(-1, -\sqrt{3}, 2) \xrightarrow{?}$ cylindrical $P(r, \theta, z)$

(x, y, z)

$z = 2$

$r = \sqrt{x^2+y^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

$\theta_1 = \frac{\pi}{3}$

$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \leftarrow$ لأننا بالربع الثالث

$\therefore P(2, 4\frac{\pi}{3}, 2)$

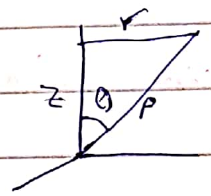
(x, y, z)

② $P(-\sqrt{3}, -3, -2) \longrightarrow$ Sph. $P(\rho, \theta, \phi)$

$\rho^2 = x^2+y^2+z^2$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{3+9+4} = \sqrt{16} = 4$

$\tan\theta = \frac{-3}{-\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$



$\theta_1 = \frac{\pi}{3}$

$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{4}$

$\frac{z}{\rho} = \cos\phi$

$\cos\phi = \frac{-2}{4} = \frac{-1}{2}$

$\phi = 2\frac{\pi}{3}$ second

$\therefore P(4, 4\frac{\pi}{3}, 2\frac{\pi}{3})$

Ex: $x + 2y + 3z = 1 \xrightarrow{?}$ ⓘ Cy ⓘⓘ Sph.

ⓘ Cyl

$r\cos\theta + 2r\sin\theta + 3z = 1$

ⓘⓘ Sph.

عوضنا بدل $r = \rho\sin\phi$ مباشرة

$\rho\sin\phi\cos\phi + 2\rho\sin\phi\sin\phi + 3\rho\cos\phi = 1$

$\rho(\sin\phi\cos\phi + 2\sin\phi\sin\phi + 3\cos\phi) = 1$

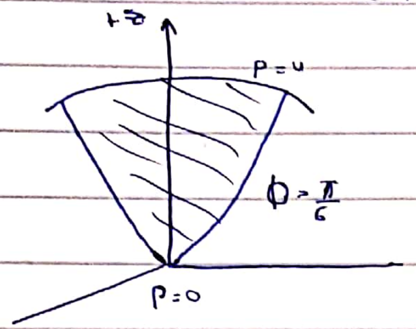
$$\rho = \frac{1}{\sin\phi\cos\phi + 2\sin\phi\sin\phi + 3\cos\phi}$$

Ex: Find the volume of the solid region.

bd. below by cone: $z = \sqrt{3x^2+3y^2}$

& bd. above by sphere: $x^2+y^2+z^2=16$

Sph.



Cone: $z = \sqrt{3}\,\sqrt{x^2+y^2}$

$\tan\phi = \frac{1}{\sqrt{3}}$

$\phi = \frac{\pi}{6}$

Sphere: $\boxed{\rho = 4}$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{4} \rho^2 \sin\phi \, \partial\rho \, \partial\phi \, \partial\theta$$

$$\int\int \left(\frac{\rho^3}{3} \Big|_0^4 \right) \sin\phi \, \partial\phi \, \partial\theta$$

$$= \frac{4^3}{3} \cdot \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \sin\phi \, \partial\phi \, \partial\theta$$

$$= \frac{64}{3} \int_0^{2\pi} \left(-\cos\phi \Big|_0^{\frac{\pi}{6}} \right) \partial\theta$$

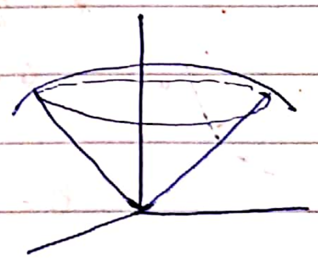
$$= \frac{63}{3} \left(\cos\frac{\pi}{6} - \cos 0 \right) \cdot 2\pi$$

(ii) Cyl. Cone:

$$z = \sqrt{3}\, r$$

Sphere: $r^2 + z^2 = 16 \longrightarrow z = \pm\sqrt{16-r^2}$

لأنه العلوي
بال Sphere
$z = \sqrt{16-r^2}$



$$V = \int_0^{2\pi}\int_0^{2}\int_{\sqrt{3}r}^{\sqrt{16-r^2}} \partial z \, r \, \partial r \, \partial\theta$$

منطقة التقاطع/التكامل

$\partial\theta$

$z = \sqrt{3}\, r \xrightarrow{\text{sph.}}$

$r^2 + 3r^2 = 16$

$4r^2 = 16$

$r^2 = 4$

$\boxed{r = 2}$

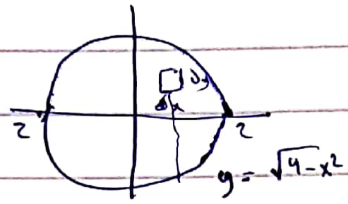
$$= \int_0^{2\pi}\int_0^{2} r\left(\sqrt{16-r^2} - \sqrt{3}\, r\right) \partial r \, \partial\theta$$

(iii) rectangular:

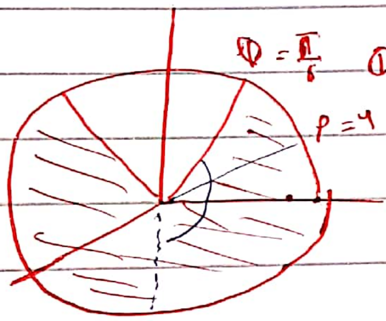
$$V = \iiint_{\sqrt{3}\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} \partial z \, \partial A = \int_{2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{3}\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} \partial z \, \partial y \, \partial x$$

$r = 2 \rightarrow$

$x^2 + y^2 = 4$



Ex:

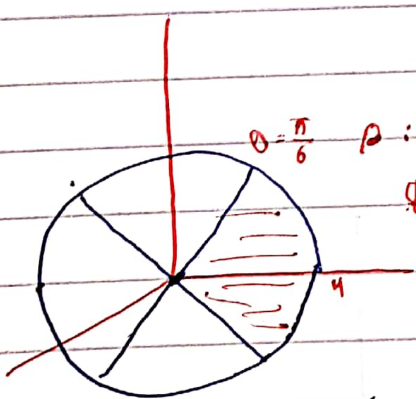


$\phi = \frac{\pi}{6}$ ① outside cone & inside sph.

$$I = \iiint_D x z \, \partial v \; ; \text{ set up in sph}$$

$$I = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\pi} \int_0^4 \rho \sin\phi \cos\theta \cdot \rho \cos\phi \cdot \rho^2 \sin\phi \, \partial\rho \, \partial\phi \, \partial\theta$$

② outside the cone $z^2 = 3x^2 + 3y^2$ & inside Sphere $x^2 + y^2 + z^2 = 16$

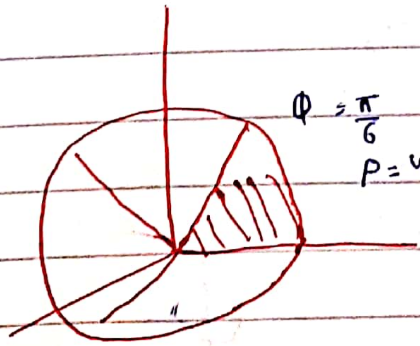


$\theta = \frac{\pi}{6}$ $\rho : 0 \rightarrow 4$

$\phi : \frac{\pi}{6} \rightarrow \frac{5\pi}{6}$

$\theta : 0 \rightarrow 2\pi$

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$



$\phi = \frac{\pi}{6}$

$\rho = 4$

$\rho : 0 \rightarrow 4$

$\phi : \frac{\pi}{6} \rightarrow \frac{\pi}{2}$

$\theta : 0 \rightarrow 2\pi$

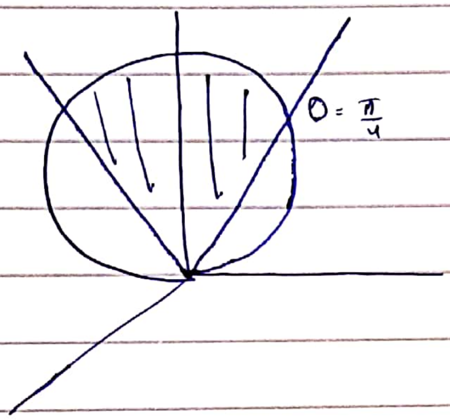
Ex: ?V : bd. below by the cone: $z=\sqrt{x^2+y^2}$

& bd. above by the sphere : $x^2+y^2+z^2=z$

Sph. cone : $\tan\phi = 1 \rightarrow \phi = \frac{\pi}{4}$

Sph. sph: $\rho^2 = \rho\cos\phi$

$\boxed{\rho = \cos\phi}$



$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int\int \frac{\rho^3}{3}\Big|_0^{\cos\phi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \underbrace{\cos^3\phi \sin\phi \, d\phi}\, d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left(\frac{\cos^4\phi}{4} \Big|_0^{\frac{\pi}{4}} \right) d\theta$$

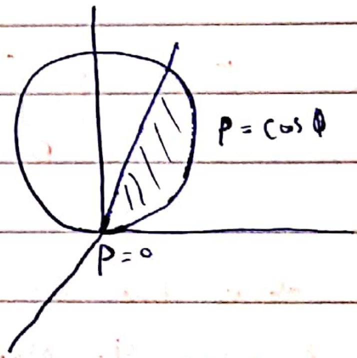
$$= -\frac{1}{3} 2\pi \left(\frac{\cos^4\frac{\pi}{4} - \cos^4 0}{4} \right)$$

(II) below the cone & above the sph.

$\rho : 0 \rightarrow \cos\phi$

$\phi : \frac{\pi}{4} \rightarrow \frac{\pi}{2}$

$\theta : 0 \rightarrow 2\pi$

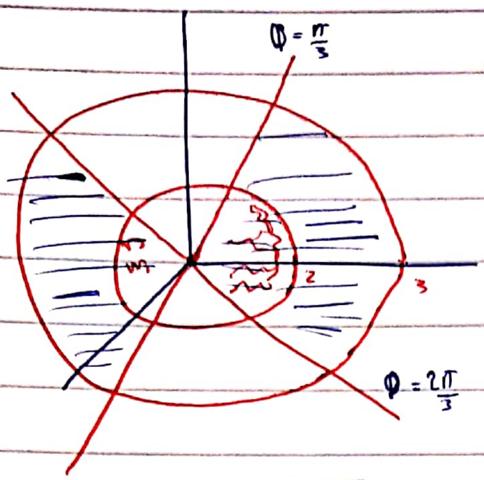


Ex: set up: $I = \iiint_D xy \, dV.$

D: outside the sph. : $x^2+y^2+z^2=4$

& inside " " : $x^2+y^2+z^2=9$

& outside the cone : $z^2 = \frac{x^2+y^2}{3}$



$$= \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_2^3 \rho \sin\phi \cos\theta \cdot \rho \sin\phi \sin\theta \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$z = \pm \frac{1}{\sqrt{3}} \sqrt{x^2+y^2}$

$\tan\phi = \sqrt{3}$

$\phi = \frac{\pi}{3}$ العلوي

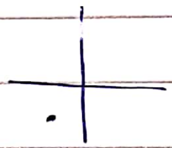
$\phi = \pi - \frac{\pi}{3}$ السفلي

$\phi = \frac{2\pi}{3}$

III ... D نفس : $x \leq 0 , y \leq 0$

نفس حل بس

$\theta : \pi \to \frac{3\pi}{2}$



$\theta : \pi \to \frac{3\pi}{2}$

Ex: $I = \iiint_D z \, dv$

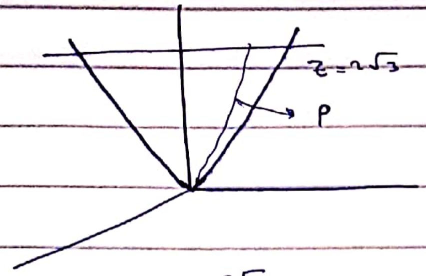
D: above the cone: $z = \sqrt{3x^2+3y^2}$

& below the plane: $z = 2\sqrt{3}$

using Sph

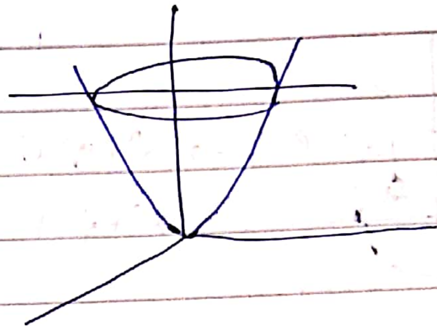
$\rho \cos\phi = 2\sqrt{3}$

$\rho = \frac{2\sqrt{3}}{\cos\phi}$



$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{\frac{2\sqrt{3}}{\cos\phi}} \rho\cos\phi\, \rho^2 \sin\phi\, \partial\rho\, \partial\phi\, \partial\theta$$

② in rectangular:



int. Cone & plane:

$z = z \Rightarrow$

$2\sqrt{3} = \sqrt{3x^2 + 3y^2}$

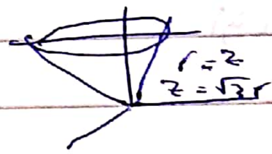
$4(3) = 3x^2 + 3y^2$

$x^2 + y^2 = 4$

$$I = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{3x^2+3y^2}}^{2\sqrt{3}} z\, \partial z\, \partial y\, \partial x$$

③ cyl

$$I = \int_0^{2\pi}\int_0^{2}\int_{\sqrt{3}r}^{2\sqrt{3}} z \cdot r\, \partial z\, \partial r\, \partial\theta$$



Ex:

$$I = \int_{x=0}^{2}\int_{0=y}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\, \partial z\, \partial y\, \partial x$$

Sph.

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{4\cos\phi} (\rho^2)^{3/2}\, \rho^2 \sin\phi\, \partial\rho\, \partial\phi\, \partial\theta$$

$\rho^3 \cdot \rho^2$

ρ^5

$z = 2 - \sqrt{4-x^2-y^2}$

$(z-2) = -\sqrt{4-x^2-y^2}$

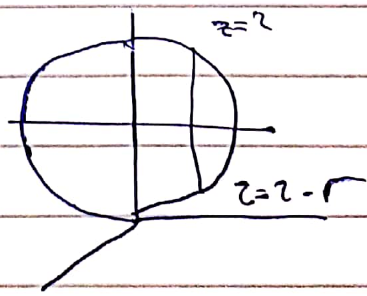
$(z-2)^2 = 4 - x^2 - y^2$

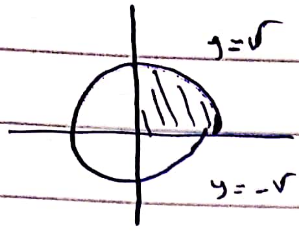
$x^2 + y^2 + (z-2)^2 = 4$

⇓

$\rho = 4\cos\phi$

لـ القطر



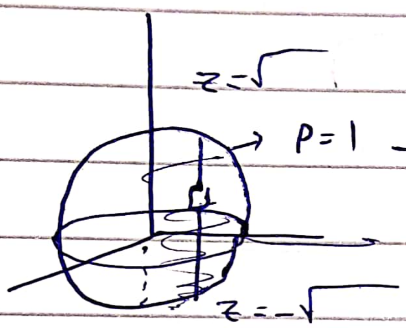


$x^2+y^2=4$

$x \geq 0$ & $y \geq 0$

Ex: $I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{\rho^3} \rho^2 \sin\phi \, \partial\rho \, \partial\phi \, \partial\theta$ =

① rect ② cyl



$\rho=1 \rightarrow x^2+y^2+z^2=1$

$x^2+y^2=1$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{\left(\sqrt{x^2+y^2+z^2}\right)^3} \partial z \, \partial y \, \partial x$$

② cyl: $\int_0^{2\pi}\int_0^{1}\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} e^{\left(\sqrt{r^2+z^2}\right)^3} r \, \partial z \, \partial r \, \partial\theta$

$\rho^2 = x^2+y^2+z^2$

$\rho^2 = r^2+z^2$

$\rho = \sqrt{r^2+z^2}$

$\boxed{r=1}$