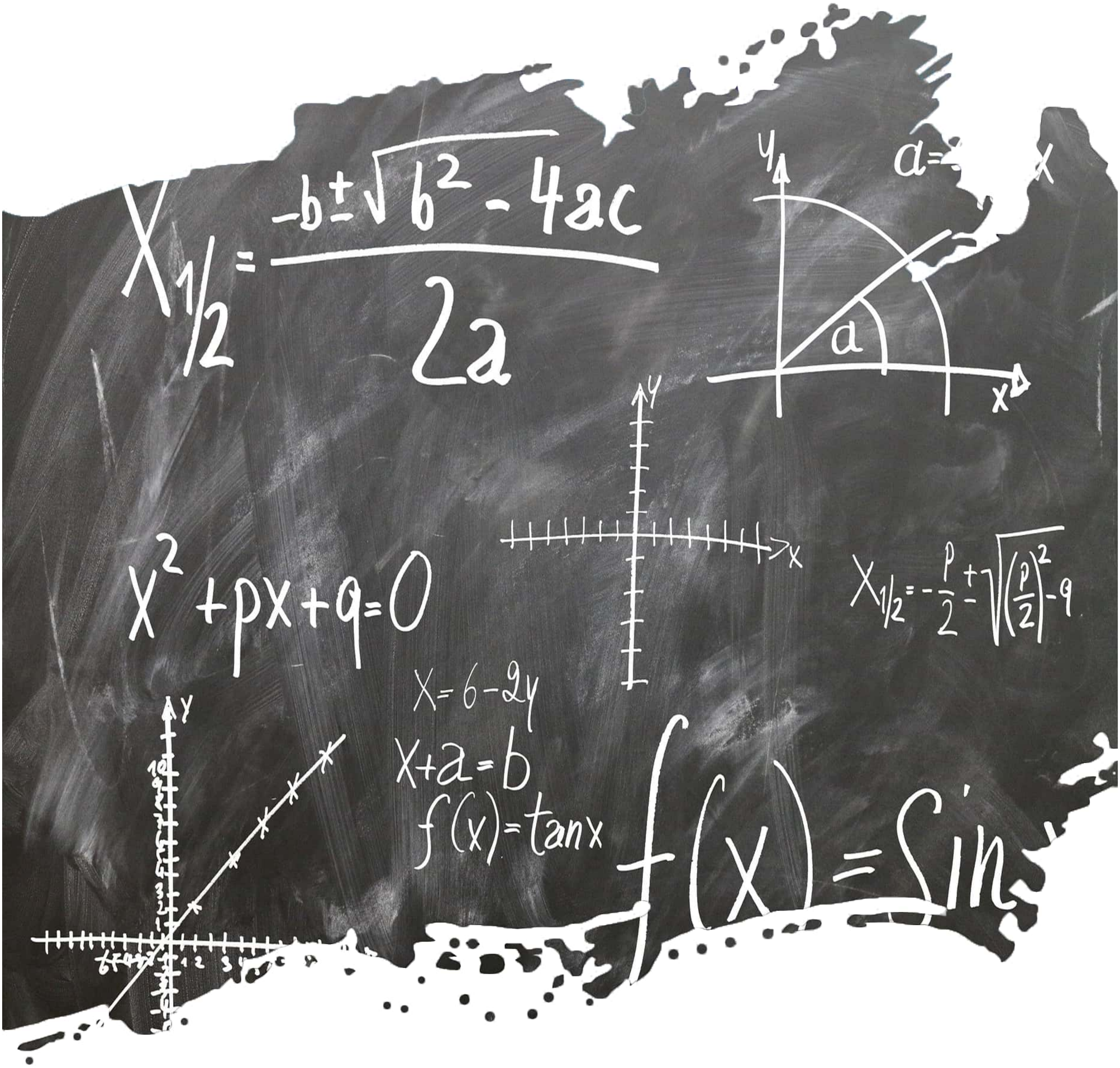
LINEAR ALGEBRA

DR.HASAN NAJJAR

KHALID ALNASER



POWER UNIT

* Matrices:

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 7 & 3 \end{bmatrix} \rightarrow \text{row}$$

column

entry

$A_{2\times3}$

$a_{12} = 4$

Ex: If A is 3×4 Matrix ~~$a_{ij} = 2i$~~

$a_{ij} = 2i - j^2$

Find A

Sol:

$$A = \begin{bmatrix} 1 & -2 & -7 & -14 \\ 3 & 0 & -5 & -12 \\ 5 & 2 & -3 & -10 \end{bmatrix}$$

Ex: If $A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 2 & 5 \end{bmatrix}$

Find:

1) $2A$
2) $A-B$
3) A^t

Sol:

1) $2A = \begin{bmatrix} 4 & 2 & 10 \\ 2 & 6 & 2 \end{bmatrix}$

2) $A - B = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 1 & -4 \end{bmatrix}$

3) $A^t = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 5 & 1 \end{bmatrix}$

Ex: let $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix}$

$\underset{2\times2}{A} \quad \underset{2\times3}{B} = \underset{2\times3}{AB}$

يجوز الضرب

Find (If possible)

1) AB ✓

2) BA ✗

$\underset{2\times3}{B} \quad \underset{2\times2}{A}$

Sol:

$AB = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 & 8 \\ 11 & 13 & 26 \end{bmatrix}$

Ex: Find AC when

$A = \begin{bmatrix} 1 & 2 & 1 \\ 5 & -1 & 2 \end{bmatrix}$

$C = \begin{bmatrix} 3 & 1 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}$

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Sol:

$\underset{2\times 3}{A}\ \underset{3\times 2}{C}$ ✓

$$AC = \begin{bmatrix} 7 & 10 \\ 18 & 3 \end{bmatrix}$$

$\boxed{*}$ Square Matrices :

$A_{n\times n}$ (number of rows = number of columns)

$\boxed{*}$ diagonal matrices :

Ex: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

$\underset{n\times n}{A}$ is diagonal iff : $a_{ij} = 0$ for $i \neq j$

$B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$ diagonal

$\boxed{*}$ upper triangular matrices :

Ex : $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

B is upper triangular matrix iff
$a_{ij} = 0$ For $i > j$

lower triangular matrices:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 7 & 2 & 0 \end{bmatrix}$$

$C_{ij} = 0$ for $i < j$

Symmetric matrices:

Ex: $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 5 & 6 & 7 \end{bmatrix}$, $(A = A^t)$

OR $a_{ij} = a_{ji}$

Skew symmetric matrices

we say A is skew symmetric iff $A = -A^t$

Ex: $A = \begin{bmatrix} 0 & 2 & -5 \\ -2 & 0 & 4 \\ 5 & -4 & 0 \end{bmatrix}$

* System of linear equations:

Ex: ① $2x + y = 5$ → linear eq.

② $3x - \frac{2}{y} + z = 7$ X

③ $\frac{2}{5}x - \frac{1}{3}y + 2z = 1$ ✓

Ex: Solve

$x + 2y - z = 1$

$2x - y + 2z = 2$

$3x + y + z = 3$

Solu-

leading

$$\begin{bmatrix} \boxed{1} & 2 & -1 & 1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$$ → Augmented matrix

* row operations:
1) Multiply any row by nonzero Scalar اضرب بأي ثابت
2) Interchange the position of one row by another بدل الصفوف مع بعض
3) add a multiple of one row to another

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 4 & 0 \\ 0 & -5 & 4 & 0 \end{bmatrix} \begin{matrix} \\ -2r_1 + r_2 \\ -3r_1 + r_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & \boxed{1} & -\frac{4}{5} & 0 \\ 0 & -5 & 4 & 0 \end{bmatrix} \begin{matrix} \\ -\frac{1}{5}r_2 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -4/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ 5r_2 + r_3 \end{matrix}$$

$x + 2y - z = 1$ Free بنخلي واحدمنهم

$y - \frac{4}{5}z = 0$

Free $\leftarrow z = t \quad , \quad t \in \mathbb{R}$

$y = \frac{4}{5}t$

$x = 1 - 2(\frac{4}{5}t) + t$

$x = 1 - \frac{3}{5}t$

Solution $= \left\{ (1 - \frac{3}{5}t, \frac{4}{5}t, t) : t \in \mathbb{R} \right\}$

Ex: Solve:

$x - 3y + z = 2$

$x + y + 5z = 1$

$2x + 2y - z = 4$

$$\left[\begin{array}{ccc|c} \boxed{1} & -3 & 1 & 2 \\ 1 & 1 & 5 & 1 \\ 2 & 2 & -1 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 4 & 4 & -1 \\ 0 & 8 & -3 & 0 \end{array}\right] \quad -r_1 + r_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & \boxed{1} & 1 & -1/4 \\ 0 & 8 & -3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & -1/4 \\ 0 & 0 & -11 & 2 \end{array}\right] \quad -8r_2 + r_3$$

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$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & \boxed{1} & -2/11 \end{array}\right]$$

$z = \frac{-2}{11}$

$y + z = -\frac{1}{4} \Rightarrow y = \frac{-3}{44}$

$x - 3y + z = 2 \qquad x = 3\left(\frac{-3}{44}\right) + \frac{2}{11} + 2 = \ldots$

No Solution اذا راحت المتغيرات و الناتج ظل رقم بكون

Q) If the matrix of the Gaussian elimination has the form

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Find the Solution of the system

S_

$\boxed{w = 2}$

$y + 3w = 5 \rightarrow \boxed{y = -1}$

$x + 2z + w = 2$

$x + 2z = 0$

$z = t \rightarrow x = -2t$

$S\_ = \{(-2t, -1, t, 2) : t \in \mathbb{R}\}$

17/21 For which values of a will the following system has no solutions? Exactly one solution? Infinitely many solutions

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + (a^2 - 14)z = a + 2$$

S —

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

$a^2 - 16 = 0$

$a = 4, -4$

1) IF $a = 4 \Longrightarrow$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$$

$\Longrightarrow$ infinitely & many solutions

2) If $a = -4$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & -8 \end{array}\right]$$

$\Longrightarrow$ No Solution

3) If $a \neq \pm 4$

$$\left[\begin{array}{ccc|c} 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

$$z = \frac{a-4}{a^2-16} = \frac{1}{a+4}$$

$y = ?$

$x = ?$

$\Longrightarrow$ one Solution

22) For which values of λ does the system

$$\left.\begin{aligned} (\lambda-3)x + y &= 0 \\ x + (\lambda-3)y &= 0 \end{aligned}\right\} \longrightarrow \text{homogeneous system}$$

① infenetly
② صفر فقط

have nontrivial solutions?
غير صفري

$$\left[\begin{array}{cc|c} \lambda-3 & 1 & 0 \\ 1 & \lambda-3 & 0 \end{array}\right]$$

If $\lambda=3$

$y=0$

$x=0$

trivial sol

$$\left[\begin{array}{cc|c} 1 & \lambda-3 & 0 \\ \lambda-3 & 1 & 0 \end{array}\right] \quad -(\lambda-3)r_1+r_2$$

$$\left[\begin{array}{cc|c} 1 & \lambda-3 & 0 \\ 0 & -(\lambda-3)^2+1 & 0 \end{array}\right]$$

$$-(\lambda-3)^2+1=0$$

$$\rightarrow (\lambda-3)^2=1$$

$$\lambda-3=\pm 1$$

$$\boxed{\lambda=4,2}$$

The System has non trivial Solutions iff $\lambda=4$ or $\lambda=2$

24| Solve For x, y, z

$$\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1$$

$$\frac{2}{x}+\frac{3}{y}+\frac{8}{z}=0$$

$$\frac{-1}{x}+\frac{9}{y}+\frac{10}{z}=5$$

$\bar{x}=\frac{1}{x}$

$\bar{y}=\frac{1}{y}$

$\bar{z}=\frac{1}{z}$

$\bar{x}+2\bar{y}-4\bar{z}=1$

$2\bar{x}+3\bar{y}+8\bar{z}=0$

$-\bar{x}+9\bar{y}+10\bar{z}=5$

طلع ؛

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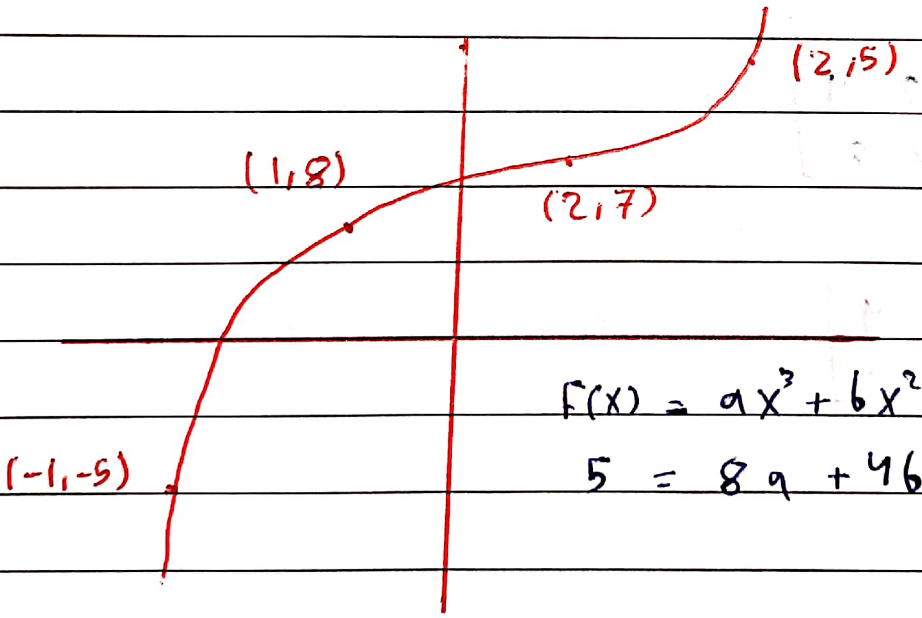
Ex Find a, b, c, d such that

$$\begin{bmatrix} a+2b & c-3d+a \\ b-d & d-2c \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$$

S—

$$\left.\begin{aligned} a+2b &= 4 \\ c-3d+a &= 5 \\ b-d &= -1 \\ d-2c &= 2 \end{aligned}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 1 & 0 & 1 & -3 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -2 & 1 & 2 \end{array}\right]$$

⋮



$$F(x) = ax^3 + bx^2 + cx + d$$

$$5 = 8a + 4b + 2c + d$$

* Properties of Matrices

1) $A + B = B + A$
2) $(KA)(B) = A(KB) = K(AB)$
3) $A(B+C) = AB + AC$

$\boxed{B}$

4) $(B+C)A = BA + CA$
5) $A + 0 = A$
6) $A + (-A) = 0$
7) let $I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A is $n \times n$ matrix

$AI_n = I_nA = A$

Ex:
$$\begin{bmatrix} 2 & 4 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 8 \end{bmatrix}$$

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Q) what conditions of the ~~bs~~ b's such that the following system has no solution.
in

$$x - 2y + z = b_1$$
$$2x + y + 2z = b_2$$
$$x - 7y + z = b_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & b_1 \\ 2 & 1 & 2 & b_2 \\ 1 & -7 & 1 & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \end{array}\right]$$

* Row echelon form of the matrix

we say the matrix have r.e.f if the following hold

1) The first nonzero element in the nonzero row is 1 (leading)

2) All nonzero rows must be in the bottom of the matrix.

3) The leading in the below row must be on the right of the leading in above row

the matrix is in reduced r.e.f

if ①, ②, ③ and ④ hold

①, ②, ③ → Gaussian elimination

①, ②, ③, ④ Gaussian - Jordan elimination

4) all the elements other that the leading in the column that has leading must be zeros.

1/19 which of the following matrices are in r.r.e.f?

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

f) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✗ شرط ④

c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ✓

b) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ ✓

d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✓

4) c) The augmented matrix for a system of linear equations has the form:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(r.r.e.f)

Solve the system:

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* الا ما في leading هو يكون الـ Free

$x_5 = t$, $t, s \in \mathbb{R}$

$x_2 = s$

$x_4 = 8 - 5t$

$x_3 = 7 - 4t$

$x_1 = -2 - 3t + 6s$

Def. let A be a square matrix if there exist a matrix B such that $AB = BA = I$ then we say A is invertible and B is called the inverse of A ($B = A^{-1}$)

* How to find A^{-1} :

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

مشترط $ad - bc$ لا تساوي $\boxed{0}$

Ex find A^{-1} where

$A = \begin{bmatrix} 2 & 5 \\ 7 & -4 \end{bmatrix}$

Sol $A^{-1} = \dfrac{-1}{43} \begin{bmatrix} -4 & -5 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{43} & \frac{5}{43} \\ \frac{7}{43} & \frac{-2}{43} \end{bmatrix}$

$\begin{bmatrix} 2 & 5 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} \frac{4}{43} & \frac{5}{43} \\ \frac{7}{43} & \frac{-2}{43} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ التحقق

Ex: let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 3 & -2 & 0 \end{bmatrix}$

find A^{-1} (If exist)

$[A : I] \xrightarrow{G.J.e} [I : A^{-1}]$

Sol:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 5 & -2 & 1 & 0 \\ 0 & -8 & 3 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -5/3 & 2/3 & -1/3 & 0 \\ 0 & -8 & 3 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 7/3 & -1/3 & \frac{2}{3} & 0 \\ 0 & 1 & -5/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & \frac{-31}{3} & 7/3 & -8/3 & 1 \end{array}\right]$$

إذا آخر صف صار كله $\boxed{0}$ ما في

Singular ← inverse

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 7/3 & -1/3 & 2/3 & 0 \\ 0 & 1 & -5/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & \boxed{1} & \frac{-7}{31} & 8/31 & -3/31 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{18}{93} & \frac{6}{93} & \frac{21}{93} \\ 0 & 1 & 0 & 27/93 & 9/93 & -15/93 \\ 0 & 0 & 1 & -7/31 & 8/31 & -3/31 \end{array}\right]$$

$$A^{-1} = \frac{1}{31}\begin{bmatrix} 6 & 2 & 7 \\ 9 & 3 & -5 \\ -7 & 8 & -3 \end{bmatrix}$$

* Properties of Inverse :

1) $(A^{-1})^{-1} = A$
2) $(AB)^{-1} = B^{-1}A^{-1}$
3) $(\underset{\text{Scalar}}{K}A)^{-1} = \frac{1}{K}A^{-1}$

Lemma:
The Identity of matrices multiplication is unique

Proof :
Assume that I and I' are two identity

$$II' = I$$
$$II' = I'$$
$$\Rightarrow I = I'$$

lemma: If A is invertible, then the inverse of A is unique

Proof :
Assume B and C are two inverses of A
Note that:

$$B = B.I = B.(AC) = (BA)C = IC = C$$

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ B & & C \end{array}$$

$B = B.I = B(AC) = (BA)C = IC = C$

Q) If A is invertible matric and $AB = AC$ Show that $B = C$

Proof:

$$AB = AC$$
$$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$$
$$IB = IC$$
$$B = C$$

Q) Show that $(AB)^{-1} = B^{-1}A^{-1}$

Proof:

$$(AB)(AB)^{-1} = I$$
$$(AB)(B^{-1}A^{-1})$$
$$= A(BB^{-1})A^{-1}$$
$$= A\,I\,A^{-1}$$
$$= AA^{-1}$$
$$= I$$

Since the matrix has unique inverse

$\Rightarrow (AB)^{-1} = (B^{-1}A^{-1})$

Q) Show that $(KA)^{-1} = \frac{1}{K}A^{-1}$

Proof: $(KA)(KA)^{-1} = I$

$$(KA)\left(\frac{1}{K}A^{-1}\right) = K\frac{1}{K}AA^{-1} = I$$

$$\Rightarrow (KA)^{-1} = \frac{1}{K}A^{-1}$$

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Q) Show that : $(A^{-1})^{-1} = A$

Proof:

$$(A^{-1})(A) = I$$
$$(A^{-1})(A^{-1})^{-1} = I$$
$$\Rightarrow A = (A^{-1})^{-1}$$

$$A^{0} = I$$
$$A^{-3} = (A^{-1})^{3} = (A^{3})^{-1}$$
$$A^{-n} = (A^{-1})^{n} = (A^{n})^{-1}$$

* Properties of transpose:

1) $(A+B)^t = A^t + B^t$
2) $(KA)^t = KA^t$
3) $(A^t)^t = A$
4) $(AB)^t = B^t A^t$

Ex) $A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}$

Find

A^t, B^t, $(A+B)^t$, $A^t + B^t$

Sol

$$A^t = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} \qquad (A+B)^t = \begin{bmatrix} 5 & 7 \\ 6 & 6 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} \qquad A^t + B^t = \begin{bmatrix} 5 & 7 \\ 6 & 6 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5 & 6 \\ 7 & 6 \end{bmatrix}$$

* A is Symmetric iff $A = A^t$

* A is skew Symmetric iff $A = -A^t$

Ex: $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 6 \\ 5 & 6 & 1 \end{bmatrix}$

$A^t = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 6 \\ 5 & 6 & 1 \end{bmatrix}$

Ex: $B = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix}$

$B^t = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$

$-B^t = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix}$

Q) True of False

1) For any Square matrix B, $B + B^t$ and BB^t are Symmetric

Proof :

a) $(B + B^t)^t = B^t + (B^t)^t$

$= B^t + B$

$= B + B^t$

$\Rightarrow$ Symmetric

b) $(BB^t)^t = (B^t)^t \; B^t$

$= BB^t$

$\rightarrow$ Symmetric.

② Show that $B-B^t$ is skew. Symmetric

Proof:

$(B-B^t)^t = B^t - (B^t)^t$

$= B^t - B$

$-(B - B^t)$

Q) For any square matrix A, $AA^t - A^tA$ is symmetric

Sol

$(AA^t - A^tA)^t = (AA^t)^t - (A^tA)^t$

$= (A^t)^t A^t - A^t (A^t)^t$

$A A^t - A^t A$

$\Rightarrow$ Symmetric

Q) True or False

if A is invetible, then

$(A^t)^{-1} = (A^{-1})^t$

Proof:

$(A^t) \; (A^t)^{-1} = I$

$(A^t) \; (A^{-1})^t = (A^{-1}A)^t$

$= I^t$

$= I$

$\Rightarrow (A^{-1})^t = (A^t)^{-1}$

Q Solve for a's, b's, c's.

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 5 & 0 & 1 \end{bmatrix}}_{A}^{3\times 3} \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 2 & 1 & 5 & 7 \\ 2 & -1 & 3 & 8 \\ 1 & 7 & 3 & 4 \end{bmatrix}}_{C}$$

Find A^{-1}

$$AB = C$$
$$A^{-1}AB = A^{-1}C$$
$$B = A^{-1}C$$

Q Solve

$$2x - y + z = 5$$
$$7x - y + 2z = 8$$
$$x + y + 3z = 0$$

$$Ax = b$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 7 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} \ \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix}$$

$$A^{-1}Ax = bA^{-1}$$
$$x = \underbrace{\underset{3\times 3}{A^{-1}}\ \underset{3\times 1}{b}}_{3\times 1}$$

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Q True or False
↳ Always ↳ Sometimes

1) $(A+B)^2 = A^2 + 2AB + B^2$ (False)

$\neq$

↳ $(A+B)(A+B) = A^2 + AB + BA + B^2$

2) $(AB^{-1})(BA^{-1}) = I$ (True)

$= AB^{-1}BA^{-1}$

$= AIA^{-1}$

$= AA^{-1} = \boxed{I}$

3) $AB \neq BA$ (False)

4) $(A-B)^2 = (B-A)^2$ (True)

$(B-A)^2 = (-(A-B))^2 = (-1)^2(A-B)^2$

$= (A-B)^2$

§ 1.5 Elementary Matrices:

Def: we say E is an elementary matrix if E can be obtained from I by <u>one</u> row operations.

Q: which of the following is elementary Matrix:

1) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ ✓   نضرب اعلا صف بـ $\boxed{2}$

3) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ ✗

2) $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ ✓

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

نعمل وحدة من العمليات عشان نجيبها

١) اضرب اي الصف بثابت.

٢) بدل.

٣) اجمع اضعاف الصف.

Ex $A = \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$ $E_1 A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ $E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$

(13) let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ Find E_1, E_2, E_3 Such that

$E_3 E_2 E_1 A = I$

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\frac{1}{4}A_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$2A_3 + A_1$

$E_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$-3/4\, A_3 + A_2$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{bmatrix}$

Q) Let $A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$

① Find E_1, E_2, E_3 Show that $E_3 E_2 E_1 = I$.

② Write A^{-1} as product of elementary matrices.

③ Write A as // // // //.

$E_1 = ??$

$* \begin{bmatrix} 1 & 3 \\ 0 & 17 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{17} \end{bmatrix}$ $\qquad$ $E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$(E_3 E_2 E_1) A = I$

② $A^{-1} = E_3 E_2 E_1$

③ $A = E_3^{-1} E_2^{-1} E_1^{-1}$

ال inverse عكس ال E يعني لو كنا نطلع E ضربنا ٤ اضعاف الصف عشان نطلع E^{-1} دايما نطرح ٤ اضعاف الصف.

Q) If $(3A^t - 2I)^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix}$. Find A

$((3A^t - 2I)^{-1})^{-1} = \left(\begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix} \right)^{-1}$

$$3A^t - 2I = \frac{1}{3} \begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1/3 \\ -1 & 2/3 \end{bmatrix}$$

$+2I \qquad +2I$

$$3A^t = \begin{bmatrix} 1 & -1/3 \\ -1 & 2/3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1/3 \\ -1 & 8/3 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & -1/9 \\ -1/3 & 8/9 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1/3 \\ -1/9 & 8/9 \end{bmatrix}$$

Theorem: If A is $n \times n$ matrix, then the following are equivelant :

1) A is invertible
2) The System $Ax = 0$ has only trivial Solution
3) r.r.e.f of A is I_n
4) A is expressible as a product of elementary matrices
5) The System $Ax = b$ has exactly one Solution for every $n \times 1$ matrix b.

Q) If A is $n \times n$ matrix and $A^2 - 3A - I = 0$

Show that $A^{-1} = A - 3I$

↳ A is invetible and

Solution:

$$A^2 - 3A - I = 0$$

$$A^2 - 3A = I$$

$$A\underbrace{(A - 3I)}_{B} = I$$

$$\underbrace{(A - 3I)}_{B} A = I$$

$$B = A - 3I = A^{-1}$$

** Q) If $A^2 + 3A + I = 0$ Show $(A + I)^{-1} = A + 2I$

→ ال $(A+I)$ Invertible لـ يعني ال A invertible

$AB = BA = I$

$$A^2 + 3A + 2I = I$$

$$\underbrace{(A + I)}_{B^{-1}} \underbrace{(A + 2I)}_{B} = I$$

$$A + 2I = (A + I)^{-1}$$

Chapter 28 Determinants

Ex $A = \begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}$

$$|A| = 2 * 5 - -1 * 4 = \boxed{14}$$

$|A| \to \det(A)$

Ex let

$$B = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 2 \\ -1 & 5 & 4 \end{bmatrix} \end{array}$$

Find $|B|$

$$|B| = 2 * \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} - 3 * \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} + 4 * \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix}$$

$$= 4 - 18 + 4 * 8$$

$$= \boxed{18}$$

للصف الثاني

$$|B| = -1 * \begin{vmatrix} 3 & 4 \\ 5 & 4 \end{vmatrix} + 3 \begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix}$$

$$= 8 + 36 - 26$$

$$= \boxed{18}$$

Q) Find $|A|$

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 2 & 2 & 0 & 0 \\ 3 & 1 & 7 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

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Sol:

$$|A| = 3 * \begin{vmatrix} 2 & 1 & 3 \\ 2 & 2 & 0 \\ 3 & 1 & 7 \end{vmatrix}$$

$$= 3 * \left(-2 \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \right)$$

$$= 3(-8 + 10)$$

$$= \boxed{6}$$

* A is invertible iff $\det(A) \neq 0$

Ex: let $A = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -2 & 4 \\ 1 & 5 & 1 \end{vmatrix}$

Find A^{-1} (If exist)

S

$$|A| = -44 + 3 + 21 = -20 \neq 0$$

$\Longrightarrow$ A is invertible

* To find A^{-1}:

$$C(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & & \\ & & \end{bmatrix} \longrightarrow \begin{bmatrix} -22 & 3 & 7 \\ 14 & -1 & -9 \\ 10 & -5 & -5 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 4 \\ 5 & 1 \end{vmatrix} = -22$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -2 \\ 1 & 5 \end{vmatrix} = 7$$

$A \to C(A)$ الصف الاول بالاول

$-44+3+21 = -20 = \det(A)$

$14+2-36 = -20 = \det(A)$

$$adj = (C(A))^t = \begin{bmatrix} -22 & 14 & 10 \\ 3 & -1 & -5 \\ 7 & -9 & -5 \end{bmatrix}$$

$$A\,adj(A) = \det(A)\,I$$

$$\boxed{A^{-1} = \frac{1}{\det(A)}\,adj(A)}$$

$*$ Q) If

$$\underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 5 & 3 & 2 \\ 4 & -1 & 2 \end{bmatrix}}_{A_{3\times 3}} \underbrace{X}_{3\times 4} = \underbrace{\begin{bmatrix} 1 & 2 & 4 & 1 \\ 5 & 1 & 7 & 3 \\ 2 & 0 & 1 & 2 \end{bmatrix}}_{B}$$

find X

Sol

$$AX = B$$

$|A| = 32 \neq 0 \Rightarrow A^{-1}$ exist

$$C(A) = \begin{bmatrix} 11 & -7 & -17 \\ -2 & 10 & 6 \\ 5 & -9 & 1 \end{bmatrix}$$

$$adj(A) = (C(A))^t = \begin{bmatrix} 11 & -2 & 5 \\ -7 & 10 & -9 \\ -17 & 6 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{32}\begin{bmatrix} 11 & -2 & 5 \\ -7 & 10 & -9 \\ -17 & 6 & 1 \end{bmatrix}$$

$$X = \frac{1}{32}\begin{bmatrix} 11 & -2 & 5 \\ -7 & 10 & -9 \\ -17 & 6 & 1 \end{bmatrix}\begin{bmatrix} B \end{bmatrix} =$$

* Cramer's Rule:

* Consider the system

$$AX = b$$

when A is $\underset{3}{n} \times \underset{3}{n}$, b is $n \times 1$

* If A is invertible, then the system has unique solution.

* The solution of this system is given by:

$$x = \frac{|A_1|}{|A|}$$

$$y = \frac{|A_2|}{|A|}$$

$$z = \frac{|A_3|}{|A|}$$

Ex Solve the system

$$2x - y + 3z = 1$$

$$x + 2y - z = 5$$

$$-x + y + 2z = 0$$

Solution:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

A_1 هي A بس بنشيل العامود ~~الصف~~ الاول وبنحط محله b

$|A| = 20$

$$|A_1| = \begin{vmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 30$$

$$|A_2| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 5 & -1 \\ -1 & 0 & 2 \end{vmatrix} = 34$$

$$|A_3| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 5 \\ -1 & 1 & 0 \end{vmatrix} = -2$$

$x = \frac{30}{20} = \frac{3}{2}$ $\quad$ $y = \frac{34}{20} = \frac{17}{10}$ $\quad$ $z = \frac{-2}{20} = \frac{-1}{10}$

* Properties of Det:

1) $\det(A) = \det(A^t)$

2) If the matrix A has a zeros row, then $|A| = 0$

3) If the matrix B is obtained from A by multiply one row in A by non-zero scalar, then $\det(B) = k\det(A)$

Ex $A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$ $|A| = 8$

$B = \begin{bmatrix} 6 & 12 \\ 1 & 6 \end{bmatrix}$, $|B| = 24$

Q) If $B = 2A$, $|A| = 3$, A is 4×4 matrix find $|B|$

$|B| = 48$

$B = KA$

$|B| = K^n |A|$

* 4) If the matrix B is obtained from A by interchanging the position of one row with another , then

$\det(B) = -\det(A)$

Ex $A = \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}$, $|A| = 10$

$B = \begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix}$, $|B| = -10$

* 5) If B is obtained from A by adding a multiple of one row to another, then $\det(B) =$

Ex) $A = \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix}$, $|A| = -34$

$B = \begin{bmatrix} 2 & 4 \\ 13 & 9 \end{bmatrix}$, $|B| = -34$

← ٣ اضعاف الصف الاول

Ex) let $A = \begin{bmatrix} 1 & -2 & 3 & 5 \\ 2 & 7 & -1 & 2 \\ 1 & 0 & 3 & 8 \\ 2 & -4 & 6 & 4 \end{bmatrix}$

Find $|A|$

$B = \begin{bmatrix} 1 & -2 & 3 & 5 \\ 2 & 7 & -1 & 2 \\ 1 & 0 & 3 & 8 \\ 0 & 0 & 0 & -6 \end{bmatrix} -2r_1 + r_4$

$|B| = -6 \begin{vmatrix} 1 & -2 & 3 \\ 2 & 7 & -1 \\ 1 & 0 & 3 \end{vmatrix} = -6(-19 + 33) = -84$

$= |A| = -84$

* 6) If in a matrix A one row is a multiple to another, then $|A| = 0$

* 7) If A is diagonal matrix, then

$|A| = \prod_{i=1}^{n} a_{ii} = a_{11} \cdot a_{22} \cdots a_{nn}$

Ex. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$|A| = 2 * 7 * 5 = 70$

*8) If A is upper or lower triangular matrix, then

$$|A| = \prod_{i=1}^{n} a_{ii}$$

Ex)

$A = \begin{bmatrix} 2 & 7 & 8 & 1 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$|A| = 2*3*7*2 = 84$

Ex)

$A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 2 & 3 & 1 & 2 \\ 1 & 6 & 3 & 7 \\ -2 & 1 & 3 & 4 \end{bmatrix}$

$A_1 = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & -1 & 3 & -8 \\ 0 & 4 & 4 & 2 \\ 0 & 5 & 1 & 14 \end{bmatrix}$, $|A_1| = |A|$

$A_2 = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & \boxed{1} & -3 & 8 \\ 0 & 4 & 4 & 2 \\ 0 & 5 & 1 & 14 \end{bmatrix}$, $|A_2| = -|A|$

$$A_3 = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & -3 & 8 \\ 0 & 0 & 16 & -30 \\ 0 & 0 & 16 & -26 \end{bmatrix} \qquad |A_3| = |A_2|$$

$$A_4 = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & -3 & 8 \\ 0 & 0 & 16 & -30 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad |A_4| = |A_3|$$

$$|A| = -|A_4| = -(1 * 1 * 16 * 4) = -64$$

OR

$$A_3 = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & -3 & 8 \\ 0 & 0 & 1 & -30/16 \\ 0 & 0 & 16 & -26 \end{bmatrix}, \quad |A_3| = \frac{1}{16} |A_2|$$

$$A_4 = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & -3 & 8 \\ 0 & 0 & 1 & -30/16 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad |A_4| = |A_3|$$

$|A_4| = 4$

$|A_3| = 4$

$|A_2| = 16 \cdot 4$

$|A_1| = -64$

$|A| = 64$

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* 9) $\det(AB) = \det(A)\det(B)$

* 10) If A is invertible, then

$$|A^{-1}| = \frac{1}{|A|}$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

A is invertible

Proof:

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det(I)$$

$$\det(A)\det(A^{-1}) = I$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Q) If $\det(A) = 3$, A is 4×4 matrix

find $\det(\operatorname{adj}(A))$

Solution:

$$\operatorname{adj}(A) = \det(A)\,A^{-1} = 3A^{-1}$$

$$\det(\operatorname{adj}(A)) = \det(3A^{-1})$$

عدد الصفوف ←

$$= 3^4 \det(A^{-1})$$

$$= 27$$

عدد الصفوف ←

$$\det(\operatorname{adj}(A)) = (\det(A))^{n-1}$$

$\frac{12}{102}$ Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find:

a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$ d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & c-4f \end{vmatrix}$

a) $= -6$

b) $= 3 * -1 * 4 * -6 = 72$

d) $= -3 * -6 = 18$

19 Solve the eq:

$$\begin{vmatrix} x & 5 & 7 \\ 0 & x+1 & 6 \\ 0 & 0 & 2x-1 \end{vmatrix} = 0$$

Solution:

upper

$$x(x+1)(2x-1) = 0$$

$$x = 0, -1, \frac{1}{2}$$

$\frac{7}{110}$ Show that

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solution:

$= \begin{vmatrix} b+c & c+a & b+a \\ a+b+c & a+b+c & a+b+c \\ 1 & 1 & 1 \end{vmatrix}$ $r_1 + r_2$ If one row is multiple for another row $= 0$

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$= 0$ (because $r_2 = (a+b+c)\, r_3$)

Q) Always true or Sometimes False:

a) $\det(2A) = 2\det(A)$ False

b) $|A^2| = |A|^2$ True $\rightarrow$ $|AA| = |A||A|$

c) $\det(I+A) = 1 + \det(A)$ False $\rightarrow$ $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$0 \overset{?}{=} 1 + 1$ X

d) If $\det(A) = 0$ $\rightarrow$ the system $A\bar{x} = 0$ has infinitely many solutions. True

Inv $\rightarrow$ $Ax = 0$ triv

~~Singul~~

Single $\rightarrow$ $Ax = 0$ Infv.

e) If $\det(A) = 0$ $\rightarrow$ $AX = b$ has ~~A-0~~ Solution

the solution False

Infinite or no

Q) $|A| = 2$, $|B| = -3$, A, B 4×4 matrices

Find:

1) $|2A^{-2}B^t|$

Solution

$$|2A^{-2}B^t| = 2^4\,|A^{-2}B^t|$$
$$= 2^4\,|A^{-1}|\,|A^{-1}|\,|B|^t$$
$$= 16 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot -3 = \boxed{-12}$$

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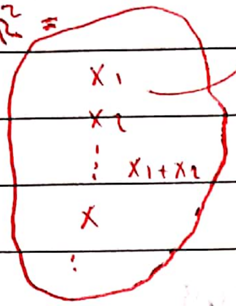
Ch

* Vector Space :

$V = \{(x,1) : x \in \mathbb{R}\}$

$\mathbb{R}^2 =$



addition (+)

Scalar multiplication (.)

$x_1(2,5)$

$x_2(7,2)$

$x_1+x_2=(9,7)$

$x_2+x_1=(9,7)$

$x_1=(5,1)$

$x_2=(9,1)$

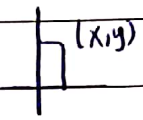
$x_1+x_2=(14,2)$

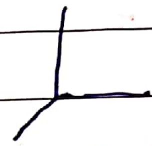
هون صارت خارج الـ V

فما تحققت الشرط

~~$x_2+x_1=$~~ $3x_1=(15,3)$

R_1 

R_2 

R_3 

Def: The vector space V is a nonempty set with two operations (addition, Scalar multiplication) in which the following conditions hold

اول 5 شروط

1) $u+v \in V$ whenever $u,v \in V$
2) $u+v = v+u$ " " " "
3) $(u+v)+w = u+(v+w)$ whenever $u,v,w \in V$
4) There exist $0_v \in V$ (identity) such that $u+0_v = 0_v+u = u$ whenever $u \in V$ (المجموعة V لازم يكون فيها عنصر محايد زي (صفر للجمع، واحد للضرب))
5) For all $u \in V$, there exist $(-u) \in V$ s.t. $u+(-u) = (-u)+u = 0_v$
6) $ku \in V$ whenever $u \in V$, $k \in F \to \mathbb{R}$
7) $k(u+v) = ku+kv$ whenever $k \in \mathbb{R}$, $u,v \in V$
8) $(k_1+k_2)u = k_1u+k_2u$
9) $(k_1k_2)u = k_1(k_2u)$
10) $1u = u$

Ex ① $(\mathbb{R}^n, +, \cdot) \rightarrow$ Vector space

② $(M_{2\times 2}(\mathbb{R}), +, \cdot) \rightarrow$ Vector Space

nm $\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$ زي

* $(\mathbb{R}^n, +, \cdot)$
* $(M_{nm}(\mathbb{R}), +, \cdot)$
* $(P_2(\mathbb{R}), +, \cdot)$ → $3x^2+1$ ✓, $5x^3+2x+1$ X, $x+1$ ✓, $(2x^2+7x+1)$

$P_2 \rightarrow n$

المعاكس → $(-2x^2-7x-1)$

$\frac{15}{227}$ V = the set of all positive real numbers $(\mathbb{R}^+)$

operations:

$x, y \in V$

$x \oplus y = xy$

$k \odot x = x^k$, $k \in \mathbb{R}$

Is $(V, \oplus, \odot)$ Vector space?

Solution:

1) $x \oplus y = xy \in V$ whenever $x, y \in V$

2) $y \oplus x = yx$

$x \oplus y = xy$ ✓

③ $(x \oplus y) \oplus z \overset{?}{=} x \oplus (y \oplus z)$, $z \in V$

$xy \oplus z \overset{?}{=} x \oplus yz$

$xyz = xyz$ ✓

4) $1 \oplus x = x$

1 is identity

5) $x \oplus \frac{1}{x} = x \cdot \frac{1}{x} = 1$

$\frac{1}{x} \in V$

6) $k \odot x = x^k \in V,\ k \in \mathbb{R},\ x \in V$

7) $k \odot (x \oplus y) \overset{?}{=} k \odot x \oplus k \odot y$

$= k \odot (xy) \overset{?}{=} x^k \oplus y^k$

$(x\,y)^k \overset{?}{=} x^k y^k = (xy)^k$ ✓

8) $(k_1 + k_2) \odot x \overset{?}{=} k_1 \odot x \oplus k_2 \odot x$

$x^{k_1+k_2} \overset{?}{=} x^{k_1} x^{k_2} = x^{k_1+k_2}$ ✓

9) $(k_1 k_2) \odot x \overset{?}{=} k_1 \odot (k_2 \odot x)$

$x^{k_1 k_2} \overset{?}{=} (x^{k_2})^{k_1}$ ✓

10) $1 \odot x = x^1 = x$ ✓

$\Longrightarrow$ Vector Space

$\frac{8}{227}$) $V = \mathbb{R}^2$

operations:

$(x_1, y_1), (x_2, y_2) \in V$

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 + 1,\ y_1 + y_2 + 1)$

$k \odot (x_1, y_1) = (k x_1,\ k y_1)$

Is V vector space?

4) $(x, y) \oplus (a, b) \overset{?}{=} (x, y)$

$(x + a + 1,\ y + b + 1) = (x, y)$

$x + a + 1 = x \rightarrow a = -1$

$y + b + 1 = y \rightarrow b = -1$

$(-1, -1)$ Identity

5) $(x, y) \oplus (c, d) = (-1, -1)$

$(x + c + 1,\ y + d + 1) = (-1, -1)$

$x + c + 1 = -1 \rightarrow c = -x - 2$

$y + d + 1 = -1 \rightarrow d = -y - 2$

Inverse of (x, y) is $(-x-2, -y-2)$

$(x_1,y_1)\quad(x_2,y_2)$

7) $k \odot (x \oplus y) \overset{?}{=} \underbrace{k \odot x \oplus k \odot y}$

$= k \odot (x_1 + x_2 + 1,\ y_1 + y_2 + 1)$

$= (kx_1 + kx_2 + k,\ ky_1 + ky_2 + k)$

$=$

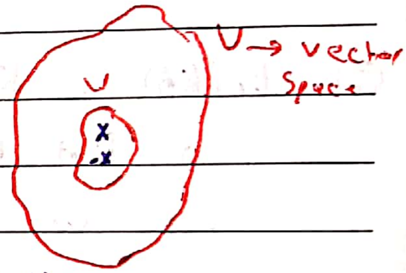
$= (kx_1, ky_1) \oplus (kx_2, ky_2)$

$= (kx_1 + kx_2 + 1,\ ky_1 + ky_2 + 1)$

Not a vector space.

§ 5.2 Subspace:

Let V be a vector space, W subset of V, we say W is a subspace iff W itself is a vector space.



Theorem:

Let W be a nonempty subset of vector space V, then W is a subspace of V iff the following hold:

1) $w_1 + w_2 \in W$ whenever $w_1, w_2 \in W$.
2) $kw \in W$ whenever $k \in \mathbb{R}$, $w \in W$

لو صفرها ما بزبط لانه الزوج الاول X

Q) let $M = \{(x, 0) : x \in \mathbb{R}\}$

Show that M is a vector Space.

Proof:

$M \subseteq \mathbb{R}^2 \rightarrow$ Vector space

حطيناه داخل Vector Space جديدة

1) Let $(x_1, 0), (x_2, 0) \in M$.

$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$

2) $k(x_1, 0) = (kx_1, 0) \in M$

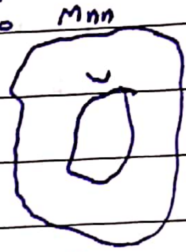
$\Rightarrow$ M is a subspace of $\mathbb{R}^2$

5 / 225) $W = \{A \in M_{nn} : tr(A) = 0\}$

$A \to \begin{bmatrix} a_{11} & \cdots & \\ \vdots & & a_{nn} \end{bmatrix}$

$tr(A) = \sum_{i=1}^{n} a_{ii}$

مجموع ال Diagonal = 0



1) Let $A, B \in W$, want to show $A + B \in W$

Note that:

$tr(A) = 0 \quad , \quad tr(B) = 0$

$tr(A+B) = tr(A) + tr(B)$

$= 0 + 0 = \boxed{0}$

$\Rightarrow A + B \in W$

② $tr(kA) = k\, tr(A) = k \cdot 0 = 0$

$\Rightarrow kA \in W$

$\Rightarrow$ W subspace of M_{nn}

* properties of trace:

1) $tr(A+B) = tr(A) + tr(B)$

2) $tr(kA) = k\, tr(A)$

Q) $H = \{A \in M_{nn} : det(A) = 0\}$

Is H subspace of M_{nn}?

$n = 2$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ |A| = 0$

$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ |B| = 0$

$A+B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \to |A+B| = -1$

$A + B \notin H$

$\to$ H Not Subspace

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$\frac{1}{238}$ c) If $W = \{(a, b, c) : c = a + b,\ a, b \in \mathbb{R}\}$

Is W subspace of $\mathbb{R}^3$?

Proof:

Let ① $u = (a_1, b_1, c_1)$, $v = (a_2, b_2, c_2)$ in W

$\rightarrow$ $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$ ---⊛

Want to show $u + v \in W$

$u + v = (a_1 + a_2, b_1 + b_2, c_1 + c_2) \in W$

From ⊛

$c_1 + c_2 = \underline{a_1 + a_2} + \underline{b_1 + b_2}$.

$\rightarrow u + v \in W$

② $Ku = (Ka_1, Kb_1, Kc$

From ⊛ : $c_1 = a_1 + b_1$

$Kc_1 = Ka_1 + Kb_1$

$\Rightarrow Ku \in W$

From ① and ② W is Subspace

2) b) $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d = 0 \right\}$

Determine whether M is a subspace of $M_{22}(\mathbb{R})$

Proof:

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ in M

①

$\rightarrow$ $a_1 + b_1 + c_1 + d_1 = 0$ and $a_2 + b_2 + c_2 + d_2 = 0$ ⊛

$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

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From ⊛

$$\underbrace{a_1+a_2}+\underbrace{b_1+b_2}+\underbrace{c_1+c_2}+\underbrace{d_1+d_2}=0$$

$\longrightarrow A+B \in M$

② $KA = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}$

From ⊛

$a_1+b_1+c_1+d_1=0$

$ka_1+kb_1+kc_1+kd_1=0$

$\longrightarrow KA \in M$

From ① and ② M is Subspace of M_{22}.

Q) let $H = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

Is H Subspace of $M_{22}(\mathbb{R})$

Solution:

1) $A = \begin{bmatrix} 5 & 8 \\ 9 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 7 \\ 20 & 1 \end{bmatrix}$ in H

$A+B = \begin{bmatrix} 7 & 15 \\ 29 & 2 \end{bmatrix} \notin H$

$\boxed{\text{Not}}$

3/238) a) $W = \{ P \in P_3(\mathbb{R}) : P(0)=0 \}$

Is W Subspace of $P_3(\mathbb{R})$

Solution:

let $P_1(x), P_2(x) \in W$

$\rightarrow P_1(0)=0$, $P_2(0)=0$

$(P_1+P_2)(0) = P_1(0)+P_2(0) = 0+0 = 0$

$P_1+P_2 \in W$

$(KP_1)(0) = KP_1(0)$

$= K \cdot 0 = 0 \rightarrow KP_1 \in W \rightarrow W$ Subspace

Five Ap

* Linear Combination

* Linear Independent

* Span

* Basic

Def: let $S = \{v_1, v_2, \dots, v_n\}$ be subset of vector space V.

let $u \in V$, we say u is linear Combination of S iff there exist $k_1, k_2, k_3, \dots, k_n$ in F such that:

$$u = k_1 v_1 + k_2 v_2 + \dots + k_n v_n.$$

Ex. Let $S = \{(1,0),(0,1)\}$ Let $u = (7,2)$, is u L.C of S?

Solution

$$u \overset{?}{=} k_1(1,0) + k_2(0,1)$$

$$(7,2) = 7(1,0) + 2(0,1)$$

$\rightarrow$ $(7,2)$ L.C of S

Q) Let $S = \{(1,0,1),(1,1,0),(0,1,1)\}$

$u = (2,5,-3)$ Is u L.C of S?

$$(2,5,-3) \overset{?}{=} k_1(1,0,1) + k_2(1,1,0) + k_3(0,1,1)$$

$$2 = k_1 + k_2$$

$$5 = k_2 + k_3$$

$$-3 = k_1 + k_3$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & -1 & 1 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} k_3 = 0 \\ k_2 = 5 \\ k_1 = -3 \end{array}\right] \rightarrow \text{yes L.C}$$

Q) $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

$u = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$ Is u L.C of S?

Solution:

$$\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \stackrel{?}{=} k_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$3 = k_1 + k_2 \rightarrow k_1 = -2$

$1 = k_1 + k_3 \rightarrow k_1 = -1 \rightarrow$ No solution $\Big] \rightarrow$ Not L.C

$2 = k_3$

$5 = k_2$

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* Linear Independent:

Def: Let $S = \{V_1, V_2, \dots, V_n\}$ be subset of vector space V, we say S is L.I iff we cannot write any element in S as L.C of the others.

otherwise S L.d.

Ex $S = \{(1,2,-1), (3,2,5), (-2,4,-14)\}$

Is S L.I?

Theorem: Let $S = \{V_1, V_2, \cdots, V_n\}$ be subset of vector space V. we say S is L.I iff the system $K_1 V_1 + K_2 V_2 + K_n V_n = 0$ has only trivial solution otherwise S L.d.

Solution

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$(0,0,0)$

$$K_1(1,2,-1) + K_2(3,2,5) + K_3(-2,4,-14) = 0$$

$$K_1 + 3K_2 - 2K_3 = 0$$

$$2K_1 + 2K_2 + 4K_3 = 0$$

$$-K + 5K_2 - 14K_3$$

* Span:

Def: let $S = \{v_1, v_2, \dots, v_n\}$ be subset of vectorspace V, we say $\boxed{S \text{ spans } V}$ iff every element in V can be written as L.C of S.

$$\boxed{V = \mathrm{Span}(S)}$$

Ex let $S = \{(1,0), (0,1)\}$ Is S spans $\mathbb{R}^2$?

Solution:

$$(2,3) \overset{?}{=} 2(1,0) + 3(0,1)$$

$$(100, -17) = 100(1,0) + 17(0,1)$$

$$(a,b) = a(1,0) + b(0,1)$$

$$\Rightarrow S \text{ spans } \mathbb{R}^2$$

Ex let $S = \{(1,0,1), (2,1,0), (1,-1,3)\}$ Is S spans $\mathbb{R}^3$?

Solution:

$$(a,b,c) \overset{?}{=} k_1(1,0,1) + k_2(2,1,0) + k_3(1,-1,3)$$

$$a = k_1 + 2k_2 + k_3$$

$$b = k_2 - k_3$$

$$c = k_1 + 3k_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & -1 & b \\ 1 & 0 & 3 & c \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & -1 & b \\ 0 & -2 & 2 & c-a \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & -1 & b \\ 0 & 0 & 0 & c-a+2b \end{array}\right]$$

The system has solution iff

$\boxed{c = a - 2b}$

$\Rightarrow$ there is a condition on the elements of $\mathbb{R}^3$ → Choice of the

$\Rightarrow$ S not Spans $\mathbb{R}^3$.

Ex $S = \{(1,-2), (-2,4)\}$ Is S spans $\mathbb{R}^2$?

Solution

$$(a,b) = k_1(1,-2) + k_2(-2,4)$$

$2*$ $a = k_1 - 2k_2$

$b = -2k_1 + 4k_2$

$2a + b = 0$

$\Rightarrow$ S Not Spans of $\mathbb{R}^2$

* Basic:

Def: Let $S = \{v_1, v_2, \dots, v_n\}$ be subset of vector space V,

We say S is a basic of V iff the following hold

1) S is L.I

2) S Spans V

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Q) Which of the following Can be basic of $\mathbb{R}^3$.

a) $\{(1,2,5), (3,7,-2)\}$ X

b) $\{(1,1,1), (2,-1,5), (3,9,2), (7,0,1)\}$ X

c) $\{(1,1,0), (1,0,1), (0,1,1)\}$ اذا عددهم نفس الي عنا بكفي نفحص شرط واحد بس

$\downarrow$ basic

$\dim(V)$ = number of elements in the basic of V

Q) Find $\dim(P_3(\mathbb{R}))$

Solution

$$P \in P_3(\mathbb{R})$$

$$P = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\dim(P_3(\mathbb{R})) = 4$$

P_2 :

$$a_0 + a_1 x + a_2 x^2$$

$$\dim(P_2) = 3$$

$$\boxed{\dim(P_n) = n+1}$$

$\dim(M_2) = 4$

$\downarrow$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\dim(M_{nm}) = nm$

Ex Find basic For $M_2(\mathbb{R})$

Solution

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{basic} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Q) If $W = \{(a,b,c) : c = a+b,\ a,b \in \mathbb{R}\}$

Find $\dim(W)$

Solution:

$$W = \{(a, b, a+b) : a, b \in \mathbb{R}\}$$

a سحبنا   b سحبنا

$\dim W = 2$

basic of $W = ??$ $= \{(1,0,1), (0,1,1)\}$

Q) If $W = \{A \in M_n : A^t = A\}$

Find $\dim(W)$

Solution

$$A = \begin{bmatrix} a_{11} & * & & \square \\ * & a_{22} & & \\ & & \ddots & \\ \square & & & a_{nn} \end{bmatrix}$$

$$\frac{n^2-n}{2} + n = \frac{n(n+1)}{2}$$

$\boxed{3} = \frac{2(3)}{2} \leftarrow \boxed{n=2}$ لانوخذ مثلاً

Q) Let $S = \{x^2+x+1,\ x-2,\ x+5\}$

Is S basic of $P_2(\mathbb{R})$?

Solution:

$$k_1(x^2+x+1) + k_2(x-2) + k_3(x+5) = \vec{0} = 0 + 0x + 0x^2$$

$$\begin{array}{ll} x^2: & k_1 = 0 \\ x: & k_1 + k_2 + k_3 = 0 \\ 1: & k_1 - 2k_2 + 5k_3 = 0 \end{array} \rightarrow k_2 = k_3 = 0 \rightarrow S.L.I$$

$\rightarrow S$ Spans P_2

$\rightarrow S$ basic

2/ Find the coordinate vector of w relative to the basis $S = \{w_1, w_2\}$ where

b) $w_1 = (2,-4)$

$w_2 = (3,8)$

$w = (1,1)$

$w = k_1 w_1 + k_2 w_2$

$(w)_S = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

↳ coordinate vector of w relative to S

Solution!

$$(1,1) = k_1(2,-4) + k_2(3,8)$$

2/ $1 = 2k_1 + 3k_2$

$1 = -4k_1 + 8k_2$

$3 = 14k_2$

$$k_2 = \frac{3}{14}$$

$$k_1 = \frac{1 - 9/14}{2} = \frac{5}{28}$$

$$(w)_S = \begin{bmatrix} 3/14 \\ 5/28 \end{bmatrix}$$

Q) If $S = \{(2,1,-1), (-1,5,0), (3,2,7)\}$ basis of $\mathbb{R}^3$, $(u)_S = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

find u.

Solution:

$u = 2(2,1,-1) - 1(-1,5,0) + 3(3,2,7)$

$= (14, 3, 19)$

19 / 264 Determine the dimensions of the following subspaces of $\mathbb{R}^4$

W b) all vectors of the form (a,b,c,d) where $d = a+b$, $c = a-b$

M c) all vectors of the form (a,b,c,d) where $a=b=c=d$

Solution:

$W = \{(a, b, a-b, a+b) : a, b \in \mathbb{R}\}$

$\dim(W) = 2$

$(a, b, a-b, a+b) = a(1,0,1,1) + b(0,1,-1,1)$

$\Longrightarrow$ basis of $W = \{(1,0,1,1), (0,1,-1,1)\}$

c) $M = \{(a,a,a,a) : a \in \mathbb{R}\}$

$\dim(M) = 1$

basis $= \{(1,1,1,1)\}$

22) Find standard basis vectors that can be added to the set $\{v_1, v_2\}$ to produce a basis of $\mathbb{R}^4$ where

$v_1 = (1,-4,2,-3)$

$v_2 = (-3,8,-4,6)$.

(a,b,c,d)

u_1 $a(1,0,0,0)$

u_2 $b(0,1,0,0)$ → standard basis

u_3 $c(0,0,1,0)$

u_4 $d(0,0,0,1)$

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§ 5.5

* Row Space:

* Column Space:

* Null Space:

Ex

Let
$$A = \begin{array}{c} \\ \left[\begin{array}{ccccc} 1 & 2 & 4 & 5 & 1 \\ 3 & 2 & 1 & 7 & 2 \\ 1 & 4 & 0 & 2 & 1 \\ 2 & 5 & 1 & 3 & 4 \end{array}\right] \\ \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \end{array} \begin{array}{l} \rightsquigarrow r_1 \\ \rightsquigarrow r_2 \\ \rightsquigarrow r_3 \\ \rightsquigarrow r_4 \\ \\ \end{array}$$

row Space of A = Span $\{r_1, r_2, r_3, r_4\}$, Note that row space is a subspace of $\mathbb{R}^5$.

$r_1 = (1,2,4,5,1)$

Column Space of A = Span $\{c_1, c_2, c_3, c_4, c_5\}$, Note that Column space is a Subspace of $\mathbb{R}^4$

$c_1 = (1,3,1,2)$

Theorem:

① Elementary row operations do not change the row Space of the matrix.

② If A and B are row equivalent, then a given set of Column vectors of A forms a basis of A iff the Corresponding Column vectors of B form a basis of Column Space of B.

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(3) If a matrix R is in r.e.f, then the row vectors with leading 1 form a basic of row space of R, and the column vectors with leading 1 of the row vectors form a basis for the column space of R.

$$A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 1 & 3 & 2 & 5 \\ 3 & 8 & 8 & 9 \end{bmatrix}$$

$$\downarrow$$

r.e.f

$$B = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 1 & - & - \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q) Let $V_1 = (1, -2, 0, 3)$

$V_2 = (2, -5, -3, 6)$

$V_3 = (0, 1, 3, 0)$

$V_4 = (2, -1, 4, -7)$

$V_5 = (5, -8, 1, 2)$

Find a subset of the vectors $\{V_1, V_2, V_3, V_4, V_5\}$ that form a basis of the space spanned by these vectors.

$W = \text{Span}\{V_1, V_2, V_3, V_4, V_5\}$

Solution:

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & +1 & -1 & -3 & -2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 2 & 5 \\ 0 & \boxed{1} & -1 & -3 & -2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow \{v_1, v_2, v_4\}$ basis of W

Def :

let A be a matrix the null Space of A is the solution of the system $AX=0$

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c) let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 7 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

Find a basis for the null Space of A.

Solution:

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 4x_2 + 5x_3 + 2x_4 = 0$

$x_2 + x_3 + \frac{4}{7} x_4 = 0$

$x_4 = t$

$x_3 = s$

$x_2 = -\frac{4}{7} t - s$

$x_1 = -4\left(-\frac{4}{7} t - s\right) - 5s - 2t$

$= -s + \frac{2}{7} t$

Null Space = Solution $= \left\{ \left(-s + \frac{2}{7}t, -\frac{4}{7}t - s, s, t\right) : t, s \in \mathbb{R} \right\}$

$\boxed{s}$ شيلنا $\qquad$ t شيلنا

basis of nullspace $= \left\{ (-1, -1, 1, 0), \left(\frac{2}{7}, -\frac{4}{7}, 0, 1\right) \right\}$

dim (Nullspace) = 2

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§ 5.6

* dim (row space) = dim (column space)

* rank (A) = dim (row space)

* rank (A) = rank (A^t)

* nullity(A) = dim (null space)

* rank (A) + nullity(A) = number of columns

4
288 | In each part find dim (row space) of A, dim (column space) of A, dim (null space) of A, dim (null space) of A^t.

	a	b	c	d	e	f	g
Size of A	3x3	3x3	3x3	5x9	9x5	4x4	6x2
rank (A)	3	2	1	2	2	0	2
dim (row) A	3	2	1	2	2	0	2
dim (column) A	3	2	1	2	2	0	2
dim (null) A	0	1	2	7	3	4	0
dim (null) A^t	0	1	2	3	7	4	4

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5) In each part find: the largest possible value of rank(A) and smallest possible value of nullity(A)

a) A is 4x4 $\rightarrow$ $\boxed{4}$, $\boxed{0}$

b) A is 3x5 $\rightarrow$ $\boxed{3}$, $\boxed{2}$

c) A is 5x3 $\rightarrow$ $\boxed{3}$, $\boxed{0}$

Ch.6 Inner Product Spaces:

Def: An Inner Product on real vector space V is a function that associates a real number $\langle u,v \rangle$ for each pair u,v in V and that function satisfy the following conditions:-

1) $\langle u,v \rangle = \langle v,u \rangle$
2) $\langle ku,v \rangle = k\langle u,v \rangle$, $k \in \mathbb{R}$
3) $\langle u+w, v \rangle = \langle u,v \rangle + \langle w,v \rangle$, $w \in V$
4) $\langle u,v \rangle \geq 0$ $(=0$ iff $v=0)$

Ex: $V = \mathbb{R}^2$ $u = \langle x_1, y_1 \rangle$ $v = \langle x_2, y_2 \rangle$

$\langle u,v \rangle = x_1 x_2 + y_1 y_2$

$I = \langle \cdots, \cdots \rangle$ I.P?

Solution:

1) $\langle u,v \rangle = x_1 \cdot x_2 + y_1 y_2$

$\langle v,u \rangle = x_2 x_1 + y_2 y_1$ ✓

2) $ku = (kx_1, kx_2)$

$\langle ku, v\rangle = kx_1x_2 + ky_1y_2 = k(x_1x_2 + y_1y_2)$

3) $w = (w_1, w_2)$

$u + w = (x_1 + w_1, y_2 + w_2)$

$\langle u+w, v\rangle = (x_1 + w_1)x_2 + (y_1 + w_2)y_2$

$= x_1x_2 + w_1x_2 + y_1y_2 + w_2y_2$

$= \langle u, v\rangle + \langle w, v\rangle$

4) $\langle v, v\rangle = x_2^2 + y_2^2 \geq 0$

$(= 0$ iff $x_0 = y_2 = 0 \to v = 0)$

$\Rightarrow \langle -, - \rangle$ inner product

Q) $V = \mathbb{R}^3$

$u = (u_1, u_2, u_3)$ $\quad$ $v = (v_1, v_2, v_3)$

$\langle u, v\rangle = u_1v_1 + u_3v_3$

Is $\langle -, - \rangle$ I.P ?

1) ✓

2) ✓

3) ✓

4) $v = (0, 2, 0) \neq 0$

$\langle u, v\rangle = 0 \to$ Not Inner Product

Q) $V = P_2(\mathbb{R})$

$p = a_0 + a_1x + a_2x^2$

$q = b_0 + b_1x + b_2x^2$

$\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$ $\quad$ Is $\langle -, - \rangle$ IP?

Sol.: 1) $\langle q, p\rangle = b_0a_0 + b_1a_1 + b_2a_2 = \langle p, q\rangle$

2) $kp = ka_0 + ka_1x + ka_2x^2$

$\langle kp, q\rangle = ka_0b_0 + ka_1b_1 + ka_2b_2$

$= k\langle p, q\rangle$

3) ✓

4) $\langle p, p\rangle = a_0^2 + a_1^2 + a_2^2 \geq 0$

$(= 0$ iff $a_0 = a_1 = a_2 = 0 \rightarrow p = 0)$

$\rightarrow \langle -, -\rangle$ IP

Ex $V = M_{22}(\mathbb{R})$

$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $\quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$\langle A, B\rangle = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$

I: $\langle -, -\rangle$ I.P ?

1) ✓ 2) ✓ 3) ✓ 4) ✓ Yes

Q) ~~$V = P_2(\mathbb{R})$ $p, q \in V$~~

~~$\langle p, q\rangle = \int_0^1 (p+q) f = \int_0^1 pf + \int_0^1 qf$~~

$=$ ~~$\langle p, f\rangle + \langle q, f\rangle$~~

Q) $V = P_2(\mathbb{R})$ $\quad p, q \in V$

$\langle p, q\rangle = \int_0^1 pq\,dx$ Is $\langle -, -\rangle$ IP?

1) ✓ 2) ✓

$\langle p+q, f\rangle = \int_0^1 (p+q)f = \int_0^1 pf + \int_0^1 qf$

$= \langle p, f\rangle + \langle q, f\rangle$

4) $\langle p, p\rangle = \int p^2\,dx$

Since $p^2 \geq 0 \rightarrow \int p^2 \geq 0$

and $\int p^2\,dx = 0$ iff $p = 0$

Def: let V be I.P.S, let $u \in V$, then

$\|u\|^2 = \langle u, u \rangle$

norm

length

Ex Let V be I.P.S $u, v \in V$, $\|u\| = 2$, $\|v\| = 3$
$\langle u, v \rangle = 5$, find $\|u+v\|$

Solution:

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$
$$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$$
$$= 4 + 5 + 5 + 9$$
$$= 23$$

$$\Rightarrow \|u+v\| = \sqrt{23}$$

Def: Let V be I.P.S $u, v \in V$, if $\langle u, v \rangle = 0$
then u and v are orthogonal.

Q) Show that for any $u, v \in V$ if u orthogonal to v,
then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$
$$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$$
$$4 + 0 + 0 + 9$$
$$= 13$$

* Properties of norms

1) $\|u\| \geq 0 \quad \forall u \in V$

2) $\|ku\| = |k| \, \|u\|$

3) $\|u+v\| \leq \|u\| + \|v\|$

§ 6.2

Def: let V be I.P.S, let $S = \{v_1, v_2, \dots, v_n\}$ be subset of V, we say S is an orthogonal set iff
1) $\langle v_i, v_j \rangle = 0 \quad \forall i \neq j$
2) $\langle v_i, v_i \rangle = 1$

orthonormal set

Ex. $V = \mathbb{R}^3$, $\langle u, v \rangle = u \cdot v$

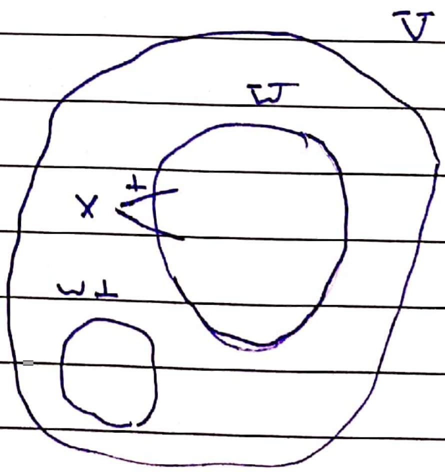
$S = \{(1,0,0), (0,1,0), (0,0,1)\}$ Is S orthogonal set?

yes

• Def: let V be I.P.S W subspace of V

$W^{\perp} = \{v \in V : \langle u, v \rangle = 0 \quad \forall u \in W\}$

orthogonal complement of W



* Properties of $W^{\perp}$:

1) $W^{\perp}$ is a Subspace of V

2) $W \cap W^{\perp} = \{0_V\}$

3) $(W^{\perp})^{\perp} = W$

Proof:

1) let $w_1, w_2 \in W^{\perp}$

$\Rightarrow \langle w_1, u \rangle = 0 \quad \forall u \in W$

$\langle w_2, u \rangle = 0$

Want to Show $w_1 + w_2 \in W^{\perp}$

let $u \in W$

$\langle w_1 + w_2, u \rangle = \langle w_1, u \rangle + \langle w_2, u \rangle$

$= 0 + 0 = 0$

$\Rightarrow w_1 + w_2 \in W^{\perp}$

2) $\langle k w_1, u \rangle = k \langle w_1, u \rangle$

$= k \cdot 0 = 0$

$\Rightarrow k w_1 \in W^{\perp}$

$\Rightarrow W^{\perp}$ Subspace of V

2) let $z \in W \cap W^{\perp}$

$\Rightarrow z \in W$ and $z \in W^{\perp}$

$\Rightarrow \langle z, z \rangle = 0$

$\Rightarrow z = 0$

$\Rightarrow W \cap W^{\perp} = \{0\}$

Q) If $W = \{(x, y, z) : z = 3x - 2y,\ x, y \in \mathbb{R}\}$

1) Show that W is subspace of $\mathbb{R}^3$
2) find basis for W
3) find $W^\perp$
4) find basis for $W^\perp$

Solution

1) let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ in W

$\Rightarrow u_3 = 3u_1 - 2u_2$, $v_3 = 3v_1 - 2v_2$

want to show $u + v \in W$

$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ and Note that:

$u_3 + v_3 = 3(u_1 + v_1) - 2(u_2 + v_2) \Rightarrow u + v \in W$

* $ku = (ku_1, ku_2, ku_3)$ and note that:

$ku_3 = 3(ku_1) - 2(ku_2)$

$\Rightarrow ku \in W$

$\Rightarrow W$ is Subspace of $\mathbb{R}^3$.

2) $W = \{(x, y, 3x - 2y) : x, y \in \mathbb{R}\}$

$\Rightarrow$ basis of $W = \{(1, 0, 3), (0, 1, -2)\}$

Theorem:

3) $\boxed{\text{In the matrix } A, \text{ the row space and null space are orthogonal Complement to each other}}$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

* W span $\{(1,0,3),(0,1,-2)\}$

* row space of A = $\{(1,0,3),(0,1,-2)\}$

$x + 3z = 0$

$y - 2z = 0$

$z = t$

$y = 2t$

$x = -3t$

$W^{\perp}$ = null space = $\{(-3t, 2t, t) : t \in \mathbb{R}\}$

4) basis of $W^{\perp} = \{(-3, 2, 1)\}$

Q) $W = \{(x,y) : y = 3x\}$

Find $W^{\perp}$

basis of $W = \{(1,3)\}$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \end{array}\right]$$

$x + 3y = 0$

$y = t$

$x = -3t$

$L = y = 3x$

$N = y = \frac{1}{3}x$

$-3y = x$

$W^{\perp} = \{(-3t, t) : t \in \mathbb{R}\}$

basis of $W^{\perp} = \{(-3, 1)\}$

$$\boxed{\begin{array}{c} \text{basis of } W^{\perp} = \{(x,y)\} \\ (x_1, y_1) \cdot (1,3) = 0 \\ x_1 + 3y_1 = 0 \end{array}}$$

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Q) If the basis of $W = \{v_1, v_2, \dots, v_n\}$

If $\langle u, v_i \rangle = 0 \;\forall\; i = 1, 2, \dots, n$

$u \in V$, W subspace of V show that $u \in W$

Solution:

Let $x \in W$

Want to show $\langle u, x \rangle = 0$

$x = k_1 v_1 + k_2 v_2 + \dots + k_n v_n$

$\Rightarrow \langle u, x \rangle = \langle u, k_1 v_1 + k_2 v_2 + \dots + k_n v_n \rangle$

$= k_1 \langle u, v_1 \rangle + k_2 \langle u, v_2 \rangle + \dots k_n \langle u, v_n \rangle = 0$

$\Rightarrow u \in W^{\perp}$

* $S = \{v_1, v_2, \dots, v_n\}$ basis of V $u \in V$

u can be written uniquley by S

* Gram - Schmidt process :

(to convert basis to an orthonormal basis)

Ex. Consider the set $\rightarrow$ basis

$S = \{ \underbrace{(1,1,1)}_{u_1}, \underbrace{(0,1,1)}_{u_2}, \underbrace{(0,0,1)}_{u_3} \} \rightarrow$ بدي اشكل V_1, V_2, V_3

Convert S to an orthonormal set

Solution:

$V_1 = u_1 = (1,1,1)$ dot

$V_2 = u_2 - \boxed{\dfrac{\langle u_2, V_1 \rangle}{\| V_1 \|^2} V_1}$ $\text{proj}_{V_1} u_2$

$= (0,1,1) - \dfrac{2}{3}(1,1,1)$

$= \left\langle -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3} \right\rangle$

$V_3 = u_3 - \dfrac{\langle u_3, V_1 \rangle}{\| V_1 \|^2} V_1 - \dfrac{\langle u_3, V_2 \rangle}{\| V_2 \|^2} V_2$

$= (0,0,1) - \dfrac{1}{3}\langle 1,1,1 \rangle - \dfrac{\frac{1}{3}}{\frac{2}{3}} \left\langle -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3} \right\rangle$

$\left\langle -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3} \right\rangle \cdot \left\langle -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3} \right\rangle = \dfrac{6}{9} = \dfrac{2}{3}$

$= \left(0, -\dfrac{1}{2}, \dfrac{1}{2}\right)$

$\Rightarrow \{V_1, V_2, V_3\}$ is an orthogonal set.

$= \left\{ \underbrace{(1,1,1)}_{V_1}, \underbrace{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}_{V_2}, \underbrace{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}_{V_3} \right\}$

Let $q_1 = \dfrac{V_1}{\|V_1\|} = \dfrac{(1,1,1)}{\sqrt{3}} = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

$q_2 = \dfrac{V_2}{\|V_2\|} = \dfrac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\frac{\sqrt{2}}{\sqrt{3}} \; \frac{\sqrt{6}}{3}} = \left(\dfrac{-2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right)$

$$q_3 = \frac{V_3}{\|V_3\|} = \frac{(0, -\frac{1}{2}, \frac{1}{2})}{\frac{\sqrt{2}}{2}} = (0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

$\Rightarrow \{q_1, q_2, q_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

* Theorem:

If $S = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis for I.P.S V, and $u \in V$, then

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \langle u, v_3 \rangle v_3 + \dots + \langle u, v_n \rangle v_n$$

$$u = (7, 8, 3)$$
$$= \langle u, q_1 \rangle q_1 + \langle u, q_2 \rangle q_2 + \langle u, q_3 \rangle q_3$$
$$= \frac{18}{\sqrt{3}} + \dots$$

$$u = k_1 v_1 + k_2 v_2 + \dots + k_n v_n$$
$$\langle u, v_1 \rangle = \langle k_1 v_1 + k_2 v_2 + \dots + k_n v_n, v_1 \rangle$$
$$= \langle k_1 v_1, v_1 \rangle +$$

* arthonormal $\rightarrow$ متعامد مع الي جنبو
+ مع حاله = 1 $\longrightarrow \langle u, u \rangle$

* arthogonal $\rightarrow$ متعامد مع الي جنبو

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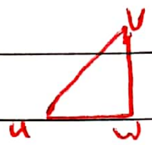
* let V be I.P.S , $u, v \in V$,
$d(u,v) = \| u - v \|$

* properties:

① $d(u,v) \geq$ zero
② $d(v,v) =$ zero
③ $d(u,v) \leq d(u,w) + d(w,v)$ $\quad w \in V$



Q) Let $S = \{x^2, x, 1\}$ be basis of P_2 , for $p, q \in P_2$
define $\langle p,q \rangle = \int_0^1 pq \, dx$ convet $\underline{\underline{S}}$ to an
orthonormal basis .

Solution:

$v_1 = u_1 = x^2$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\| v_1 \|^2} v_1$$

$$= x - \frac{\int_0^1 x^3 \, dx}{\int_0^1 x^4} v_1$$

$$v_2 = x - \frac{\frac{1}{4}}{\frac{1}{5}} x^2$$

$$\boxed{v_2 = x - \frac{5}{4} x^2}$$

$u \in V$
$u = k_1 v_1 + k_2 v_2 + \cdots$
$\langle u, v_1 \rangle$

Second

Five Appl

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Chapter 7 :

Eigen values and Eigen vectors.

$Ax = \lambda x$

$\underbrace{(\lambda I - A)}_{B} x = 0$

متى بكون مش صفري

$|\lambda I - A| = 0$

* A is a square matrix

$$\underset{n\times n}{A}\,\underset{n\times 1}{x} = \lambda x$$

λ: scalar → eigenvalues

Ex $A = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$

Find eigenvalues and eigen vectors of A

$\lambda I - A = \begin{bmatrix} \lambda-2 & -5 \\ 0 & \lambda-3 \end{bmatrix}$

$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$

$|\lambda I - A| = (\lambda-2)(\lambda-3) = 0$

$\lambda = 2 \quad \lambda = 3$ eigen values

If $\lambda = 2$ or 3 → non-trevial

For $\lambda = 2$:

عوضنا ال 2 بال $(\lambda I - A)$

$$\left[\begin{array}{cc|c} 0 & -5 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$-5x_2 = 0 \rightarrow x_2 = 0$

$-1\,x_2 = 0$

$x_1 = t$

Solution $= \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$

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المطلوب $\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$ eigenvector of A for $\lambda = 2$

For $\lambda = 3$:

$$\left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - 5x_2 = 0$

$x_2 = t$

$x_1 = 5t$

Solution $= \left\{ \begin{bmatrix} 5t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ eigen vector of A For $\lambda = 3$

Ex: let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Find eigenvectors and eigenvalues of A.

Solution

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda-2) \begin{vmatrix} \lambda & -2 \\ -1 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-2)(\lambda^2 - 3\lambda + 2) = 0$$

$$= (\lambda-2)^2(\lambda-1) = 0$$

$\lambda = 2, 1$ eigen values

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* For $\lambda=2$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$

$x_1 = t, \; x_3 = -t, \; x_2 = s$

Solution $= \left\{ \begin{bmatrix} t \\ s \\ -t \end{bmatrix} ; \; t, s \in \mathbb{R} \right\}$

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightsquigarrow$ eigenvectors of A for $\lambda=2$

* For $\lambda=1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_3 = 0$

$x_2 - x_3 = 0$

$x_3 = t$

$x_2 = t$

$x_1 = -2t$

Solution: $\left\{ \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ eigenvector

Ex:

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \boxed{?} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = ? \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = ? \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Q) If λ is eigenvalue of A, show that λ^2 is eigenvalue of A^2

Proof:

λ is eigenvalue of A

$\rightarrow$ there exist non zero vector x

such that:

$Ax = \lambda x$

$A(Ax) = A(\lambda x)$

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$$A^2 x = \lambda (A x)$$
$$= \lambda (\lambda x)$$
$$= \lambda^2 x$$

* قاعدة اذا كان عندي A^2 وطلبها
الـ Eigenvalue بطلع λ وبربعها فقط

Ex: If A is invertible matrix and λ is an eigenvalue of A, Show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}:

Proof: λ is eigenvalue of $A \rightarrow$ there exist $X \neq 0$

s.t $AX = \lambda X$

$A^{-1}(AX) = A^{-1}(\lambda X)$

$X = \lambda A^{-1} X \longrightarrow A^{-1}X = \frac{1}{\lambda} X$

قسمة ع λ

* قاعدة: اذا كان A^{-1} وطلب Eigenvalue الها بطلع λ بتقلبها

$\rightarrow \frac{1}{\lambda}$ Eigenvalue of A^{-1}

* Note: If λ is Eigevalue of A, and A is invertible, then $\lambda \neq 0$

* Example: If λ is Eigenvalue of A, Show that $\lambda + c$ is an Eigenvalue of $A + CI$:

Proof:

$AX = \lambda X$

$(A + CI)X = AX + CIX$

$= \lambda X + CX$

$= (\lambda + c) X$

$\rightarrow \lambda + c =$ Eigenvalue of $A + CI$

* قاعدة: اذا كان عندي $A+CI$ وطلب Eigenvalue الها، بطلع λ و بجمعلها c.

* Example: If A is 3×3 matrix and $2, -3, 5$ are Eigenvalues of A. Find the Eigenvalue of $A^{-1} + 3I$:

Sol:

$\frac{1}{2} + 3$, $\frac{-1}{3} + 3$, $\frac{1}{5} + 3$

* $A^{-2} = (A^{-1})^2 = (A^2)^{-1}$

بتقلب وبتربع او بتربع بعدين بتقلب بالحالتين صح ← لما احسب Eigenvalues

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* Ex: find the Eigenvalues of A^9 are:
where

$$A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Sol:

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -3 & -7 & -11 \\ 0 & \lambda-\frac{1}{2} & -3 & -8 \\ 0 & 0 & \lambda & -4 \\ 0 & 0 & 0 & \lambda-2 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda-1)(\lambda-\tfrac{1}{2})(\lambda)(\lambda-2) = 0$$

$\lambda = 1, \frac{1}{2}, 0, 2$ $\rightarrow$ Eigenvalues of A^9 are:
$1, (\frac{1}{2})^9, 0, (2)^9$

* Note: If the matrix is upper, lower, or diagonal, then the entries on diagonal are the Eigenvalues.

* § 7.2 Diagonalizable:

def: We say A is diagonalizable iff there exist an invertible matrix P such that $P^{-1}AP$ is diagonal

$P^{-1}AP$ = Diagonal matrix

Ex: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ find Eigenvalues and Eigenvectors of A :

Solution:

$$\lambda I - A = \begin{bmatrix} \lambda-2 & -1 \\ 0 & \lambda-4 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda-2)(\lambda-4) = 0 \longrightarrow \lambda = 2, 4$$

For $\lambda = 2$:

$A\underset{\downarrow}{x} = \lambda x$
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & -2 & 0 \end{array}\right] \qquad x_1 = t, \quad x_2 = 0$$

$Sol = \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} ; t \in \mathbb{R} \right\}$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Eigenvector of A for $\lambda = 2$

For $\lambda = 4$:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} &2x_1 - x_2 = 0 \\ &x_1 = t, \quad x_2 = 2t \end{aligned}$$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Eigenvector of A for $\lambda = 4$

٢ بكتب ②

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Eigenvector ①
For $\lambda = 2$ لمين؟

* كيف تشكل D ؟

② نفس الاشي للعمود الثاني ولازم ال diagonal الباقي اصفار

Ex: find eigenvalues and eigenvectors.

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda-2)^2(\lambda-1)$$

For $\lambda = 2$:

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ eigenvectors

For $\lambda = 1$:

$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ eigenvector

Geometric multiplicity of λ = number of eigenvector for λ.

Algebraic " " of λ = number of repeated of λ as a root

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Note: If G.m = A.m for all eigenvalues of A, then A is diagonalizable.

Q) Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

a) Is A diagonalizable?

Solution

$\lambda = 2, 1$: G.m = A.m = 2

For $\lambda = 2$: G.m = A.m = 1

$\Longrightarrow$ yes.

١) نجيب Eigenvalues و Eigenvectors

٢) نجيب G.M ، A.m

٣) اذا متساويين yes

b) Find P, D such that

$P^{-1}AP = D$

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex: Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

Is A diagonalizable?

Solution

$$\lambda I - A = \begin{bmatrix} \lambda - 3 & -1 \\ 0 & \lambda - 3 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda - 3)^2 = 0$$

$$\Rightarrow \lambda = 3$$

For $\lambda = 3$:

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-x_2 = 0$

$x_1 = t$

Solution $= \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ eigenvector

G.m for $\lambda = 3 = 1$

A.m for $\lambda = 3 = 2$

$\Rightarrow$ A is not diagonalizable.

$A_{n\times n}$ $\rightarrow$ $|\lambda I - A| = \lambda^n + \cdots$

A have n eigenvalues

Is A diagonalizable?

A.m for any $\lambda = 1$

$\boxed{G.m \leq A.m \quad 1 \leq 1}$

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If $n\times n$ matrix has n distinct eigenvalues, then A is diagonable.

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$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Is A ~~diagonable~~ diagonalizable, if yes find P, D such that $P^{-1}AP = D$

Solution :

$$\lambda I - A = \begin{bmatrix} \lambda+2 & 0 & 0 & 0 \\ 0 & \lambda+2 & -5 & 5 \\ 0 & 0 & \lambda-3 & 0 \\ 0 & 0 & 0 & \lambda-3 \end{bmatrix}$$

$\lambda = -2, -2, 3, 3$

* for $\lambda = -2$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{array}\right]$$

$x_1 = t$

$x_2 = s$

eigenvectors for $\lambda = -2$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$

$x_2 - x_3 + x_4 = 0$

$x_4 = t$

$x_3 = s$

$x_2 = s - t$

eigenvectors:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Chapter 8: (Map)

Linear transformations:

Def: A linear transformation is a function from vector space V to vector space W such that the following conditions hold:

1) $T(v+u) = Tu + Tv \quad , u, v \in V$

2) $T(ku) = k\,Tu \quad , k \in F, u \in V$

Ex Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ s.t

$T(x,y) = (x, y, x+y)$

Is T L.tran?

Solution:

Let $\underset{u}{(x_1, y_1)}, \underset{v}{(x_2, y_2)} \in \mathbb{R}^2$

$u+v = (x_1+x_2, y_1+y_2)$

① $T(u+v) = (x_1+x_2, y_1+y_2, x_1+x_2+y_1+y_2)$

$Tu = (x_1, y_1, x_1+y_1)$

$Tv = (x_2, y_2, x_2+y_2)$

$Tu+Tv = (x_1+x_2, y_1+y_2, x_1+x_2+y_1+y_2)$

② $ku = (kx_1, ky_1)$

$T(ku) = (kx_1, ky_1, k(x_1+x_2))$

$k\,Tu = k(x_1, y_1, x_1+y_1)$

$= (kx_1, ky_1, k(x_1+y_1))$

$\Longrightarrow$ L. trans.

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Q) Let $T: M_{nn}(\mathbb{R}) \to \mathbb{R}$ s.t

$T(A) = tr(A)$

Is T L. trans?

Solution:

1) Let $A, B \in M_{nn}$

$$T(A+B) = tr(A+B)$$
$$= tr(A) + tr(B)$$
$$= T(A) + T(B)$$

2) $$T(KA) = tr(KA)$$
$$= K\,tr(A)$$
$$= K\,T(A)$$

$\Rightarrow T$ L.M.

* Linear map:

Ex:

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(1,0) = (2,1,-1)$

$T(0,1) = (-1,0,2)$

Find

1) $T(3,5)$

2) $T(x,y)$

Solution:

Note that $\{(1,0),(0,1)\}$ basis of $\mathbb{R}^2$

① $(3,5) = K_1(1,0) + K_2(0,1)$

$\Rightarrow (3,5) = 3(1,0) + 5(0,1)$

$\Rightarrow$

$$T(3,5) = T(3(1,0) + 5(0,1))$$
$$= 3T(1,0) + 5T(0,1)$$
$$= 3(2,1,-1) + 5(-1,0,2)$$

$= (1,3,7)$

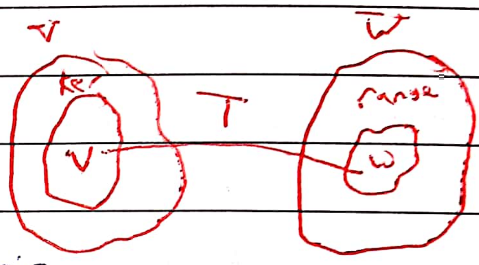
② $(x,y) = \underline{x(1,0) + y(0,1)}$

$T(x,y) = T(x(1,0) + y(0,1))$

$= x\,T(1,0) + y\,T(0,1)$

$= x(2,1,-1) + y(-1,0,2)$

$$\boxed{T(x,y) = (2x-y,\ x,\ 2y-x)}$$



Def: Let $T: V \to W$ be L.M

$\ker T = \{ v \in V : Tv = 0_w \}$ ← $\boxed{W}$ عنصر محايد لل

$\text{range } T = \{ w \in W : \exists\, v \in V,\ Tv = w \}$

Theorem: Let $T: V \to W$ be L.M

Show that $\ker T$ is a subspace of V.

Proof:

Let $u, v \in \ker T$ want to show $u+v \in \ker T$

① $T(u+v) = Tu + Tv = 0 + 0 = 0 \to u+v \in \ker T$

② $T(ku) = k\,Tu = k \cdot 0 = 0$

$ku \in \ker T$

$\Rightarrow \ker T$ subspace of V

Q) Let $T \in \mathcal{L}(V, W)$

Show that $\text{range}T$ is a subspace of W

Proof: Let $w_1, w_2 \in$ Range$T$ want to show $w_1 + w_2 \in$ Range$T$

$w_1 = Tu$, $u \in V$

$w_2 = Tv$, $v \in V$

$T(\underbrace{v+u}_{\in V}) = Tv + Tu = w_1 + w_2$

$\longrightarrow w_1 + w_2 \in$ Range$T$

2) Want to show $kw_1 \in$ range$T$

$T(ku) = kTu = kw_1 \in$ range$T$

$\Rightarrow$ range$T$ subspace of W.

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$T: P_2 \to P_3$

$T(P_{(x)}) = x P(x)$, which of the following in Ker$T$

a) x^2

b) $0 \to T(0) = x \cdot 0 = 0$

c) $1 + x$

"لما تكون صورته تساوي صفر"

$T(P_1 + P_2) = x(P_1 + P_2)$

$= xP_1 + xP_2$

$= T(P_1) + T(P_2)$

$T(kP) = xkP = kxP = kT(P)$ لينير

Q) $T: P_3 \to P_2$

$T(P) = P''$

Find Ker$T$

Solution:

$T(P_1 + P_2) = (P_1 + P_2)''$

$= P_1'' + P_2''$

$= T(P_1) + T(P_2)$.

$T(kP) = (kP)''$

$= kP''$

let $P \in P_3$

$\rightarrow P = a_3 X^3 + a_2 X^2 + a_1 X + a_0$

$T(P) \overset{?}{=} 0$

$T(P) = (a_3 X^3 + a_2 X^2 + a_1 X + a_0)'' = 0$

$\Rightarrow 6a_3 + 2a_2 = 0$

$\rightarrow a_2 = 0 , a_3 = 0$

$\rightarrow P = a_1 X + a_0$

$\rightarrow \ker T = \{a_1 X + a_0 , a_1 , a_0 \in \mathbb{R}\}$

Def: $T : V \rightarrow W$

$\dim(\ker T) = \underline{\text{nullity of } T}$
nullity(T)

$\dim(\text{rang}(T)) = \text{rank}(T)$

$$\boxed{\text{rank}(T) + \text{nullity}(T) = \dim V}$$

* Matrix of L.M:

عدد الأعمدة ← Dim   عدد الصفوف ← Dim

Ex $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ s.t

$T(x,y) = (2x+y , y-3x , x+4y)$

Find Matrix of T.

Solution:

$T(1,0) = (2,-3,1) = 2(1,0,0) - 3(0,1,0) + 1(0,0,1)$

$T(0,1) = (1,1,4) = 1(1,0,0) + 1(0,1,0) + 4(0,0,1)$

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Matrix of $\leftarrow [T] = \begin{bmatrix} 2 & 1 \\ -3 & 1 \\ 1 & 4 \end{bmatrix}$

$T = \mathbb{R}^2 \to \mathbb{R}^3$

$T(x, y) = \begin{bmatrix} 2 & 1 \\ -3 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

ال Dim ← عدد الأعمدة

Q) $T: P_2 \to P_3$ ال Dim حسب عدد الصفوف

$T(p) = xp$ Find $[T]$

Solution:

$T x^2 = x^3 = 1 \cdot x^3 + 0(x^2) + 0(x) + 0(1)$

$T x = x^2 = 0 x^3 + 1 x^2 + 0 x + 0(1)$

$T 1 = x = 0 x^3 + 0 x^2 + 1 x + 0(1)$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Def: let $T: V \to W$ and let $[T]$ be the matrix of L.M.T

nullity T = nullity$([T])$
rank T = rank$([T])$

$$\boxed{\text{rank}([T]) + \text{nullity}([T]) = \dim V}$$

Def: Let $T: V \to W$ be a L.M

T is $\underline{\underline{1-1}}$ iff $\ker T = \{0_V\}$

Ex $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(x,y) = (3x-y, x+y)$

Is T $\underline{\underline{1-1}}$ L.M ?

↳ injective

Solution:

$T(x,y) = (0,0)$

$(3x-y, x+y) = (0,0)$

$3x-y=0$

$x+y=0$

$x=0 \to y=0 \Longrightarrow \ker T = \{(0,0)\} \to T$ is $1-1$

Def: Let $T: V \to W$ a L.M, then T is onto iff

Range ← $R(T) = W$

> If M subspace of W
> and
> $\dim(M) = \dim W \to W = M$

Q) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (3x-y, x+y)$

Is T onto L.M ?

Solution:

$\dim R(T) = \dim V - \dim(\ker T)$

$= 2 - 0$

$= 2$

but $\dim(\text{codomain}) = \dim(\mathbb{R}^2) = 2$

$\Rightarrow \dim(R(T)) = \dim(\text{Codomain})$

$\Rightarrow T$ is onto.

Def : Let $T: V \to V$

if T is $\underbrace{\text{1-1 and onto}}_{\text{bijective}}$, then T is Invertible

Q) if $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (3x-y, x+y)$ find $T^{-1}(a,b)$

Solution :

$$[T^{-1}] = ([T])^{-1}$$

$$[T] = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$[T]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{bmatrix}$$

$$[T^{-1}] = \begin{bmatrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{bmatrix}$$

$$T^{-1}(a,b) = \begin{bmatrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}a + \frac{1}{4}b \\ -\frac{1}{4}a + \frac{3}{4}b \end{bmatrix}$$

Ex: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ s.t $T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 - 4x_2 + 3x_3, 5x_1 + 4x_2 - 2x_3)$

1) Find Matrix of T
2) Find Rank T and nullity T
3) Is T 1-1 ?
4) If ③ yes, Find $T^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$:

Solutions

1) $T(1,0,0) = (3,-2,5) = 3(1,0,0) - 2(0,1,0) + 5(0,0,1)$

$T(0,1,0) = (1,-4,4) = 1(1,0,0) - 4(0,1,0) + 4(0,0,1)$

$T(0,0,1) = (0,3,-2) = 0(1,0,0) + 3(0,1,0) - 2(0,0,1)$

$$\Rightarrow [T] = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

2) Rank $([T])$ = rank T

$$\begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & \frac{1}{3} & 0 \\ 0 & -\frac{10}{3} & 3 \\ 0 & \frac{7}{3} & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & \boxed{1} & \frac{-9}{10} \\ 0 & \frac{7}{3} & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{-9}{10} \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{-9}{10} \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $\text{rank}([T]) = \text{rank}\, T = 3$ عدد ال leading

$\Rightarrow$ $\text{Nullity}(T) + \text{Rank}(T) = 3$ عدد الأعمدة

$\Rightarrow$ $\text{Nullity}(T) = 0 \longrightarrow \text{Ker}\, T = \{0\}$ $\rightarrow$ Free parameter

3) $\Rightarrow$ T is 1-1

4) $[T^{-1}] = ([T])^{-1}$

$$C([T]) = \begin{bmatrix} -4 & 11 & 12 \\ 2 & -6 & -7 \\ 3 & -9 & -10 \end{bmatrix}$$

$-12 + 11 = -1$

$-4 + 24 - 21 = -1$

$15 - 36 + 20 = -1$

$\det([T]) = -1$

$$T^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4a - 2b - 3c \\ -11a + 6b + 9c \\ -12a + 7b + 10c \end{bmatrix}$$

$(T \circ T^{-1}) \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ $\leftarrow$ عالهامش

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* Notes:

اذا بديا اعرف قاعدة ال Matrix

$[T] = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ find $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = [T]\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a+b \\ 3a+5b \end{bmatrix}$$

$$T(a,b) = (2a+b,\ 3a+5b)$$

* Change of Basis :

<u>Ex</u> $S_1 = \{(1,0),(0,1)\}$ basis of $\mathbb{R}^2$

$S_2 = \{(2,1),(-1,1)\}$ " " "

let $u = (5,7)$, find

1) $(u)_{S_1}$ $(u)_{S_2}$

2) Find Matrix P such that

$(u)_{S_1} = P\,(u)_{S_2}$ transition matrix from S_2 to S_1 → $P_{S_2 \to S_1}$

Solution:

$(5,7) = k_1(1,0) + k_2(0,1)$

$= 5(1,0) + 7(0,1)$

$\rightarrow (u)_{S_1} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$(5,7) = k_1'(2,1) + k_2'(-1,1)$

$2k_1' - k_2' = 5$

$k_1' + k_2' = 7$

$3k_1' = 12$

$\rightarrow k_1' = 4$

$k_2' = 3$

$(u)_{S_2} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$(2,1) = 2(1,0) + 1(0,1)$

$(-1,1) = -1(1,0) + 1(0,1)$

$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$\longrightarrow (u)_{S_2} = P^{-1} (u)_{S_1}$

Q) If $u \in \mathbb{R}^3$

$S_1 = \{(1,2,-1),(3,-1,5),(2,1,4)\}$

basis of $\mathbb{R}^3$, S_2 basis of $\mathbb{R}^3$

$$P_{S_2 \to S_1} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad (u)_{S_2} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \text{ find } u.$$

Solution

$(u)_{S_1} = P\,(u)_{S_2}$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 42 \end{bmatrix}$$

$\Longrightarrow u = 7(1,2,-1) + 11(3,-1,5) + 42(2,1,4)$

$u = Q$

* Similar matrices:

Def: we say the matrix A is similar to matrix B iff there exist an invertible matrix P such that:

$$\boxed{B = P^{-1} A P} \longrightarrow \boxed{A \sim B}$$

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Q) True or False

1) If A Similar to B, then $\det(A) = \det(B)$

Solution

True

$$\det(B) = \det(P^{-1} A P)$$

$$= \cancel{\det(P^{-1})} \det(A) \cancel{\det(P)}$$

$$= \det(A)$$

Similar

2) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof:

Such that

$A \sim B \longrightarrow \exists P$ S.t

$$A = P^{-1} B P$$

$B \sim C \Rightarrow \exists q$ S.t

$$B = q^{-1} C q$$

$\Rightarrow$

$$A = P^{-1} q^{-1} C q P$$

$$= (qP)^{-1} C (qP)$$

$M = qP$

$\Rightarrow A = M^{-1} C M$