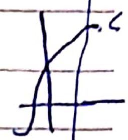
24-9-2019

25/25/30

* chapter ~~9~~ 9: Vector Differential ~~equ~~ calculus (Grad; Div; curl)

↳ revision — scalar — Vector

$f(x) = x^2$ → integral from [1 to 3] $= \int_1^3 f(x)\,dx$ → calc 1 → 3



ch2 and line integral $\int_C f(x)\, d\vec{r}$ → Line integral → comes from 1 Variable

and surface integral $\iint_S f(x)\, d\vec{s}$ — 1st

↳ chapter 2

↳ uses General sol and other sol → know when to use them

ch3 → F.S and F integral and F.T → Learn these stuff to use to solve Partial ~~B~~ equs in ch5 — 2nd

↳ final

chapter 9:

- the 1st several sections consists of review of material from calculus 3, so we will go on quickly

→ two kind of quantities used in engineering

1. a scalar: A quantity represents magnitude (i.e speed)

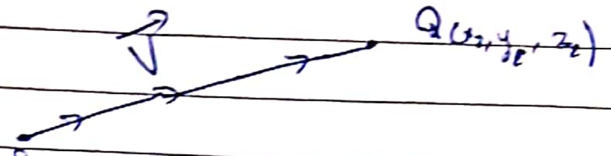
2- a Vector: A quantity represents magnitude and direction (i.e Velocity)

24-9-2019 Partial equation

Scalar func → gives a scalar answer

Vector function → gives a vector answer

• A vector is represented by an arrow: → 2D or 3D → Same work



$P(x_1, y_1, z_1)$ → ( ) for Points

if movement from P to Q → creates a vector

P: "initial Point" Q: "terminal Point"

$$\vec{V} = \vec{PQ} = [x_2 - x_1, y_2 - y_1, z_2 - z_1] = [v_1, v_2, v_3]$$

[v_1, v_2, v_3] → Component form of $\vec{V}$

another form used

$\langle v_1, v_2, v_3 \rangle$

$\langle\rangle$, [ ] → Vector notation

• note that the square or arrow head used for vectors

" " " Brackets are used for scalers.

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1st law of lines

• the Length of $\vec{V}$ "(magnitude or norm) is given by:

$$|\vec{V}| = \sqrt{V_1^2 + V_2^2 + V_3^2}$$

example if $P(4,0,2)$ and $Q(6,-1,2)$ are the initial and terminal Points respectively of the Vector $\vec{V}$, then:

$$\vec{V} = [6-4, -1-0, 2-2] = [2,-1,0] \text{ or } \langle 2,-1,0 \rangle$$

$$|\vec{V}| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}.$$

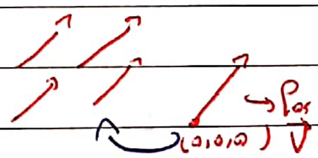
26-9-2019

L2. • If $\vec{n}$ is a vector of Length 1, then $\vec{u}$ is called a unit vector and we denoted by $\hat{n}$

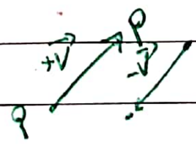
• the direction of a vector $\vec{V}$ is specified by a unit vector

$$\hat{V} = \frac{\vec{V}}{|V|}$$

• If the initial Point of $\vec{V}$ is the origine $(0,0,0)$ then its called Position Vector



• a Vector $-\vec{V}$ has the same length of $\vec{V}$ but opposite direction



this Pos $\vec{V}$ represents them all

26-9-2019

Vector operations

let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ and $\alpha \in R$

① equality $\quad \vec{a} = \vec{b}$

iff $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$

② Scaler multiplication

$\alpha \vec{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$

③ Vector addition:

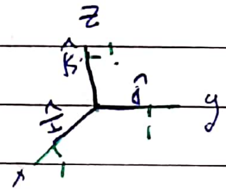
$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

- standard basis vectors

↳ $\hat{I} = \langle 1, 0, 0 \rangle$

$\hat{J} = \langle 0, 1, 0 \rangle$

$\hat{K} = \langle 0, 0, 1 \rangle$



any vector can be represented as:

$\vec{V} \Rightarrow [V_1, V_2, V_3] = [V_1, 0, 0] + [0, V_2, 0] + [0, 0, V_3]$

Component form

$= V_1 \hat{I} + V_2 \hat{J} + V_3 \hat{K}$

eg: ① $\vec{V} = [-2, 3, 1]$

↳ $-2\hat{I} + 3\hat{J} + \hat{K}$

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ⓐ $\vec{w} = 2\hat{i} + 3\hat{k}$ → standard basis form

$\vec{w} = [2, 0, 3]$ → compact form

4- multiplication :

1- Dot product (inner)

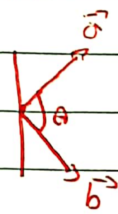
Let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$
and θ is the angle between them
$0 \le \theta \le \pi$,

the dot product of $\vec{a}$ and $\vec{b}$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

note: scaler.

$$= |\vec{a}|\,|\vec{b}| \cos\theta$$



↳ if $\theta = 90, 270$

↳ a and b are orthogonal.

↳ and we denote this by $\vec{a} \perp \vec{b}$

↳ iff $\vec{a} \cdot \vec{b} = 0$

↑ 9.1, 9.2

2- Cross product → 9.3
(Vector Product)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \Rightarrow \text{Det of this}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

$$= (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

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- note that $\vec{a} \times \vec{b}$ is a vector quantity
- $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

eg: let $\vec{a} = [-2, 0, 3]$ and $b = [1, -1, 2]$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & 3 \\ 1 & -1 & 2 \end{vmatrix} = \cancel{a \times} \begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix} \hat{i}$$

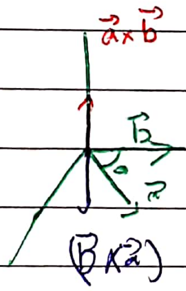
$$- \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} \hat{k}$$

$$= (0 - -3)\,\hat{i} - (-4 - 3)\,\hat{j} + (2 - 0)\,\hat{k}$$
$$= 3\hat{i} + 7\hat{j} + 2\hat{k}$$
$$= [3, 7, 2]$$

* Some algebric properties

① $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

② $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

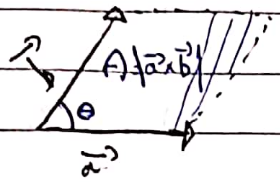


26-9-2019 Partial equ

③ $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta, \ 0 < \theta < \pi$

where,

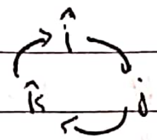
↳ $|\vec{a} \times \vec{b}|$ represents the area of the Parallelogram formed by $\vec{a}$ and $\vec{b}$



- if a vector has no mag represent it by $\vec{V} = \langle 0, 0, 0 \rangle$

④ if $\vec{a} \times \vec{b} = \vec{0}$ then $\vec{a} \parallel \vec{b}$

↳ Parallel



- $\hat{i} \times \hat{j} = \hat{k}$
- $\hat{j} \times \hat{k} = \hat{i}$
- $\hat{k} \times \hat{i} = \hat{j}$

L3, 29-9-2019 Partial.

9.4 - Vector and scaler functions and their fields.

3 Kinds of Domain

2D → plain

3D → space

↳ region in 3D x, y, z

↳ curve in 3D x, y, z

↳ surface in 3D x, y, z

- Let P be any point in a domain of definition.
- typical domains in application are 3 Dimension region ($\subset R^3$), a surface, curve and region in space

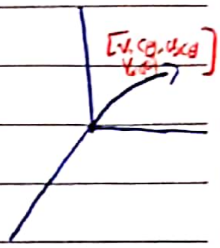
29-9-2019

Def:.

- A vector function gives a vector value for a point P in space

$\vec{V}(P) = [V_1(P), V_2(P), V_3(P)]$

$\vec{V}(x,y,z) = [V_1(x,y,z), V_2(x,y,z), V_3(x,y,z)]$



eg: $P(1,0,3)$

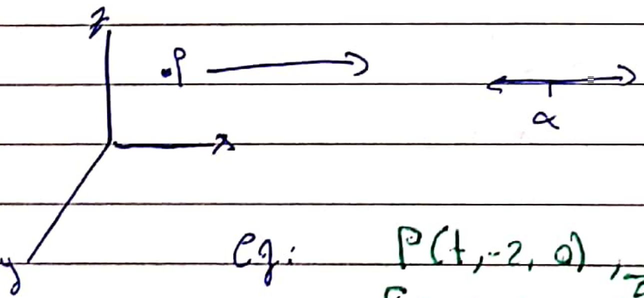
$\vec{V} = [x^2, x-z, z^2 e^y]$

$\vec{V}(P) = [1, 4, 9]$

↳ this vector took a point and made it a vector ↳ (function)

- a scaler function gives a scaler value for a point P in space

$f(P) = f(x,y,z)$



eg: $P(1,-2,0)$, $Q=(3,0,1)$

$f(x,y,z) = e^{zy}$

$f(Q) = 3$ ← $f(P) = 1$ → scaler

↳ this function takes a point and makes it scaler

29-9-2019

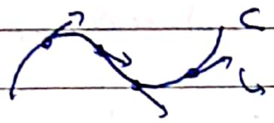
- a Vector range → Vector field

Key word
↳ field → function

- a vector (Scaler) function defines a vector (scaler) field in a domain of defintion

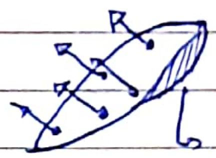
- in engineering: meaning of field = meaning of function
  ↳ thus, (1) a Vector field $\vec{V}$ is a vector-valued function defined on some domain of $\mathbb{R}^3$
  (2) a scaler field f is a real-valued function defined on some domain of $\mathbb{R}^3$

example (Vector field)

① 

Sum of tangent vectors is called Vector field

"field of tangent vectors of a curve"

② 

Sum of norms ↳ is called Vector field

"field of normal vectors of a surface"

27-9

example (scaler field):

- For a given region in $\mathbb{R}^3$, the following function $f: T \rightarrow \mathbb{R}$ defines a scaler field on T:

① if a certain region T is heated in some way, let $f(P)$ denote the temperature at a point P of T

② Let a point P_0 is fixed in T, for any point P in T, let $C(P)$ be the distance of P to P_0

- note that vector functions may also depend on some parameter such that a time t:

(i) $\vec{V}(x,y,z) = [V_1(x,y,z), V_2(x,y,z), V_3(x,y,z)]$

(ii) $\vec{V}(t) = [V_1(t), V_2(t), V_3(t)]$

$\frac{d}{dx}(\vec{V}) = [\frac{dV_1}{dx}, \frac{dV_2}{dx}, \frac{dV_3}{dx}]$

$\frac{d\vec{V}}{dt} = [V_1', V_2', V_3']$

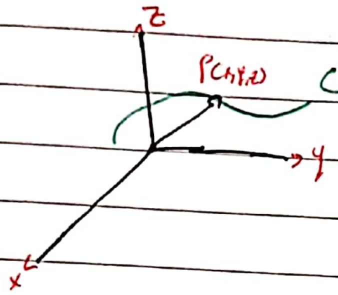
- important: figure a way to turn a 3 variable into one, (Parametrising) $\rightarrow$ such as time

9.5 Curves.

- A curve C can be represented by a Vector Function with a parameter t:

$$\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

"Parametric representation of" "a Curve"



- the direction of the curve C is determined by increasing Value of t take 0, 1, 3

- another representation of a curve C: x, $y = f(x)$, $z = g(x)$

Projection of C onto xy-Plane

Projection of C into xz-plane

- another representation of a curve C is: $f(x,y,z) = 0$, $g(x,y,z) = 0$

"intersection of two" surfaces

- orientation of C is important

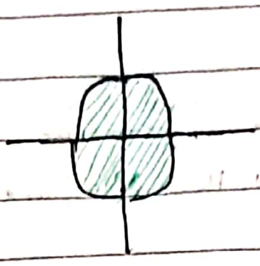
↳ 2 points of t and find out the direction

Five Apple

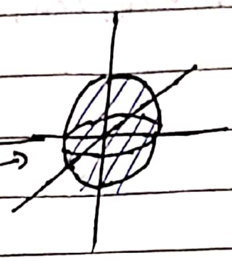
Partial equ

1/10/2019

$f(x,y) =$
$x^2+y^2-1=0$
Plain (curve)



$f(x,y,z) =$



$x^2+y^2+z^2-1=0$
Surface of
Volume.

example: find a parametric representation of the follow curve that happend because 2 surface intersect.

$x^2-y=0$ and $z=3x-1$

Sol 1- $x=t \Rightarrow y=t^2$ and $z=3t-1$

$r(t)=[t,t^2,3t-1]$

OR sol 2 $x=\sqrt{t}$, $y=t$ and $z=3\sqrt{t}-1$
$r(t)=[\sqrt{t},t,3\sqrt{t}-1]$

or sol 3
$x=e^t$, $y=e^{2t}$ and $z=3e^t-1$

This gives that curve has infinite of ways to parametrize.

your job is to find the ~~easier~~ easiest way.

(12)

Five Apple

1/10/2019

- note that there are lots of ways to parametrize a curve

* parametric equation of:

1- straight Line:

↳ the Parametric equation of a straight line in space in the direction of a Vector $\vec{b} = [b_1, b_2, b_3]$ and Passes through a point $A(a_1, a_2, a_3)$ is given by

1st Point A to vector $\vec{a}$

so $\vec{a} = [a_1, a_2, a_3]$

$\vec{r}(t) = \vec{a} + t\vec{b}$ —— ①

$= [a_1, a_2, a_3] + t[b_1, b_2, b_3]$

$= [\underbrace{a_1 + tb_1}_{x(t)}, \underbrace{a_2 + tb_2}_{y(t)}, \underbrace{a_3 + t b_3}_{z(t)}]$

example①

Find the P.E of a Straight line that Passes through $P(2,-1,3)$ in the direction of $\vec{v} = 2\hat{i} + \hat{k}$.

Sol

↳ $\vec{a} = [2,-1,3]$, $\vec{b} = [2,0,-1]$

$\vec{r}(t) = \vec{a} + t\vec{b} = [2+2t, -1, 3-t]$

1/10/2019

example (2).: find the P.E of the straight Line that passes through $P(3,4,-1)$ and $Q(7,2,0)$.

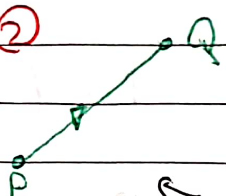
Sol: $\vec{a} = [3,4,-1]$ and $\vec{b} = \vec{PQ} = [4,-2,1]$

$$r(t) = \vec{a} + t\vec{b} = [3,4,-1] + t[4,-2,1]$$
$$= [3+4t,\ 4-2t,\ -1+t]$$

2- Line Segment

$$\vec{r}(t) = (1-t)\vec{P} + t\vec{q} \quad \ldots \text{②}$$

defined → condition $0 \le t \le 1$



example: find the P.E of the Line segment from $P(0,2,-3)$ to $Q(5,7,3)$

if $t=0$ → it gives $\vec{P}$

if $t=1$ it gives $\vec{q}$

soln: $\vec{P} = [0,2,-3]$

$\vec{q} = [5,7,3]$

$$r(t) = (1-t)[0,2,-3] + t[5,7,3]$$
$$= [0,\ 2-2t,\ 3t-3] + [5t,\ 7t,\ 3t]$$
$$= [5t,\ 5t+2,\ 6t-3]$$

→ if you substitute $0=t$ you get Vector $\vec{P}$

→ " " " $1=t$ " " Vector $\vec{q}$

$0 \le t \le 1$ → always write ← exam must see.

(14)

Partial eqns

3/10/2019

L5 3. Circle:

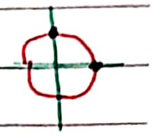
the Parametric equation of the circle:

$x^2+y^2=a^2$, $z=b$ is given by

$\vec{r}(t) = [a\cos t, a\sin t, b]$, $0 \le t \le \pi$

① example: find the Parametric eqn of

① $x^2+y^2=4$, $z=0$



Sol: $\vec{r}(t) = [2\cos t, 2\sin t, 0]$, $0 \le t \le 2\pi$

$\vec{r}(0) = [2,0,0]$, $\vec{r}(\frac{\pi}{2}) = [0,2,0]$

↳ as t increases, $\vec{r}(t)$ moves counter clockwise.

② $x=3$, $y^2+z^2=1$

Sol: $r(t) = [3, \cos t, \sin t]$, $0 \le t \le 2\pi$

(3) $(x-1)^2+z^2=9$, $y=-1$

Sol: $x-1 = 3\cos t$ $\therefore x = 3\cos t + 1$

$\therefore \vec{r}(t) = [1+3\cos t, -1, 3\sin t]$, $0 \le t \le 2\pi$

3/10/2019

4. $y^2 + z^2 + 4z = 5, \quad x = 2$

Sol: $y^2 + z^2 + 4z + 4 = 5 + 4, \quad x = 2$

$\therefore y^2 + (z+2)^2 = 9, \quad x = 2 \implies z + 2 = 3\sin t$

$\therefore z = 3\sin t - 2$

$\Rightarrow \vec{r}(t) = [2, 3\cos t, 3\sin t - 2] \quad 0 \le t \le 2\pi$

4. ellipse:

↳ the P.E of the ellipse:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = c$ is given by

$\vec{r}(t) = [a\cos t, b\sin t, c]$

① example: find the P.E of:

$\frac{y^2}{3} + \frac{z^2}{4} = 1, \quad x = -2$

Sol: $r(t) = [-2, \sqrt{3}\cos t, 2\sin t], \quad 0 \le t \le 2\pi$

② $(x-2)^2 + 16(y+3)^2 = 64, \quad z = 1$

Sol: $\frac{(x-2)^2}{64} + \frac{(y+3)^2}{4} = 1 \implies x - 2 = 8\cos t$

$\therefore x = 2 + 8\cos t$

$\vec{r}(t) = [2 + 8\cos t, -3 + 2\sin t, 1]$

↳ $0 \le t \le 2\pi$

notice that circle is a special case of the ellipse when $a = b$

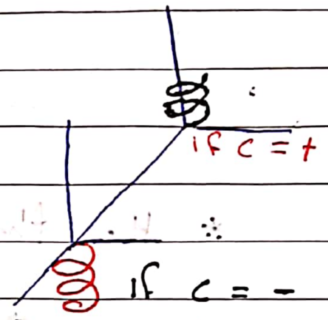
3/10/2019

5- Helix:

the Parametric equation of the helix:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $Z = c\underline{\underline{t}}$ is given by

$\vec{r}(t) = [a\cos t , b\sin t , ct]$ $0 \leq t \leq 2\pi$



9.7 Gradient of a scaler field:

Defn: the gradient of a scaler function $f(x, y, z)$ is given by:

$$grad(f) = \left[\frac{df}{dx} , \frac{df}{dy} , \frac{df}{dz}\right]$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

scaler field $\xrightarrow{grad}$ vector field

3/10/2019

- "del" operator is defined as:

"reads" "nebla" $$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$\Rightarrow$ grad (f) = ∇f

example: $f(x,y,z) = \sin x\, e^{yz}$

$$\nabla f = \left[\cos x\, e^{yz}, z\sin x\, e^{yz}, y\sin x\, e^{yz}\right]$$

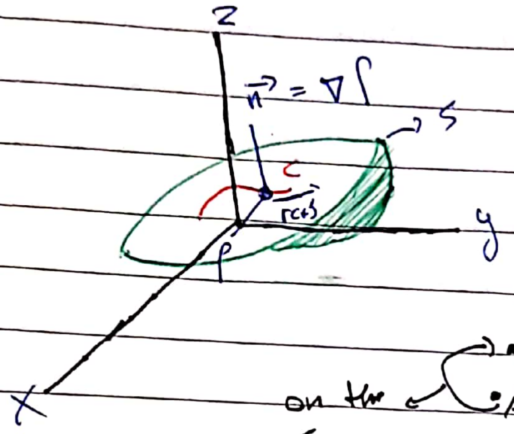
* the physical meaning of gradient:

the ~~gradient~~ gradient describes directions of maximum change.

For example when you take the gradient to the temperature function (scaler field), then the gradient gives the direction in which the temperature is increasing most rapidly

If you plot a point (P) in the gradient of a surface you get a Vector that is the norm of the surface at that point (P)

Gradient as Surface normal Vector :



- A surface S: $f(x,y,z) = c$
- on the S: A ~~curve~~ curve C:

$\vec{r}(t) = [x(t), y(t), z(t)]$

- tangent Vector :

$\vec{r}'(t) = [x(t)', y(t)', z(t)']$

- if C is on S, the Surface eqn becomes :

$f(x(t), y(t), z(t)) = c$

Differential with respect to t :

$$\therefore \frac{\partial f}{\partial x} \cdot \dot{x} + \frac{\partial f}{\partial y} \cdot \dot{y} + \frac{\partial f}{\partial z} \cdot \dot{z} = 0$$

$$\therefore \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \cdot [x', y', z'] = 0$$

$$\therefore \nabla f \cdot r'(t) = 0$$

$\therefore$ the gradient of f at the Point P is a normal Vector to the surface at the point P.

3/10/2019

example : A cone is given by $z^2 = 4(x^2+y^2)$,
find a normal vector at $P(1,0,z)$:

sol:

$$4(x^2+y^2) - z^2 = 0$$

$$\therefore f(x,y,z) = 4(x^2+y^2) - z^2$$

$$\Rightarrow \nabla f = [8x, 8y, -2z]$$

$$\vec{n} = \nabla f(P) = [8, 0, -4]$$

Def: 6-10 Sunday

$$\nabla^2 f = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2$$

is called Laplacian of f.

$$\text{scaler field} \xrightarrow{\nabla^2} \text{scaler field}$$

example let $f(x,y,z) = 3x^2y + e^z$ then

$$\nabla^2 f = 6y + 0 + e^z = 6y + e^z$$

* remark

① $\nabla^2 = \nabla \cdot \nabla$

② iff $\nabla^2 f = 0 \Rightarrow f$ is called harmonic function

* Notation
$\nabla^2 = \Delta$

6-10-2019

* Properties:

① $\nabla(f^n) = n f^{n-1} \nabla f$

② $\nabla(fg) = (\nabla f)(g) + f(\nabla g)$

(3) $\nabla(f/g) = \left((\nabla f)(g) - (f)(\nabla g) \right) / g^2$

(4) $\Delta(fg) = (g)(\Delta f) + 2[\nabla f \cdot \nabla g] + (f)(\Delta g)$

9.8 Divergence of a Vector field.

Defn: the Divergence of the Vector function

$$\vec{V}(x,y,z) = [V_1(x,y,z), V_2(x,y,z), V_3(x,y,z)]$$

is defined as:

$$\operatorname{div}(\vec{V}) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$\text{Vector field} \xrightarrow{\text{div}} \text{scaler field}$$

- using del operator:

$$\operatorname{div}(\vec{V}) = [\partial/\partial x, \partial/\partial y, \partial/\partial z] \cdot [V_1, V_2, V_3]$$
$$= \nabla \cdot \vec{V}$$

example: let $\vec{F} = x e^y i + \sinh y \hat{j} + 3x^2 \cosh(y+z) \hat{k}$

then

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = e^y + \cosh y + 3x^2 \sinh(y+z)$$

Partial eqn

6 - 10 - 2019

theorm: $\operatorname{div}(\operatorname{grad}(f)) = \nabla \cdot \nabla f = \Delta f$
↳ laplacian

* properties:

① $\operatorname{div}(f\vec{v}) = f \operatorname{div}(\vec{v}) + \vec{v} \cdot (\nabla f)$

$\therefore \nabla \cdot (f\vec{v}) = f(\nabla \cdot \vec{v}) + \vec{v} \cdot (\nabla f)$

② $\operatorname{div}((f)(\nabla g)) = \operatorname{div}(\nabla g) f + (\nabla g) \cdot (\nabla f)$

$\therefore = (f)(\nabla^2 g) + (\nabla g) \cdot (\nabla f)$

... us

... vector field.

... of a vector function

..., $V_2(x,y,z)$, $V_3(x,y,z)$]

is defined as:

$$\operatorname{curl}(\vec{v}) = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$\text{Vector field} \xrightarrow{\text{curl}} \text{Vector field}$$

note that: $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$ and $\partial_z = \partial/\partial z$

example let $\vec{V} = yz\hat{i} + 3zx\hat{j} + z\hat{k}$ then

$$\operatorname{curl}(\vec{V}) = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial_x & \partial_y & \partial_z \\ yz & 3zx & z \end{vmatrix} = (0-3x)\hat{i} - (0-y)\hat{j} + (3z-z)\hat{k} = \langle -3x, y, 2z \rangle$$

theorm :

① curl (grad (f)) = $\nabla \times \nabla f = \vec{0}$

② div (curl (f)) = $\nabla \cdot (\nabla \times \vec{f}) = 0$

↳ $\boxed{\nabla^2} \times f$

↳ because ∇^2 is scaler and can't be cross product with a Vector

- more properties

- curl $(\vec{V} + \vec{u})$ = curl $(\vec{V})$ + curl $(\vec{u})$

  $\nabla \times (\vec{V} + \vec{u}) = \nabla \times \vec{V} + \nabla \times \vec{u}$

- curl $(f\vec{V})$ = curl $(\vec{V})(f)$ + $(\vec{V})$ grad (f)

  $\nabla \times (f\vec{V}) = (\nabla \times \vec{V})(f) + (\vec{V})(\nabla f)$

- div $(\vec{u} \times \vec{V})$ = $\vec{V}$. curl $(\vec{u})$ + $\vec{u}$. curl $(\vec{V})$

  $\nabla \cdot (\vec{u} \times \vec{V}) = \vec{V} \cdot (\nabla \times \vec{u}) + \vec{u} \cdot (\nabla \times \vec{V})$

---

chapter 10 Vector integral calculus.

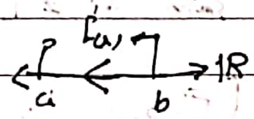
10.1 Line integrals

- a definite integral of $f(x)$ over $[a, b]$ is given by :

$$\int_a^b f(x)\, dx$$

note that

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$



8-10-2019

a Line integral (or curve integral) is an integration along a curve C.

→ in parametric representation

$$C: \vec{r}(t) = [x(t)\hat{i}, y(t)\hat{j}, z(t)\hat{k}] \quad a \le t \le b$$
$$= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

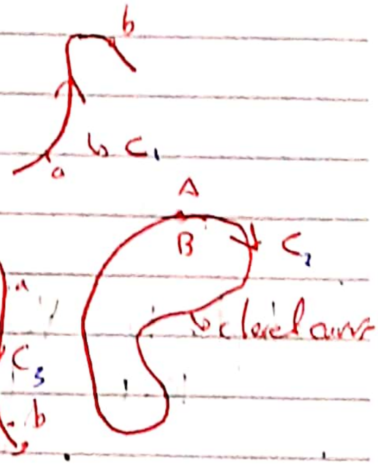
Defin: ① C is a smooth curve if $\vec{r}'(t)$ is continuous.

② A piecewise smooth curve has a finitely many smooth curves.

* Evaluation of Line integrals

1- A Line integral of a Vector function $\vec{F}(x,y,z) = [F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)]$ is given by

$$\underbrace{\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r}}_{\text{eqn. for L.I.}} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\vec{r}'(t)}_{\left(\frac{\partial c}{\partial t}\right)} dt$$



$C = C_1 \cup C_2 \cup C_3$

• Since $d\vec{r} = [dx, dy, dz]$, then:

$$\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz$$

8-10-2019

example: Find the Line integral of the Following Function along the given curve

① $F(x,y) = [-y, -xy]$, $C: x^2+y^2=1$, $y, x \geq 0$ → along path it → 1st quadrant

sol:

$\vec{r}(t) = [\cos t, \sin t]$, $0 \leq t \leq \pi/2$ ...... ① Parametrize

$\vec{F}(\vec{r}(t)) = [-\sin t, -\cos t \sin t]$ ...... ② Plug in

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = [-\sin t, -\cos t \sin t] \cdot [-\sin t, \cos t]$ ③ dot product

$= \sin^2 t - \cos^2 t \sin t$

$\therefore \int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \int_0^{\pi/2} \sin^2 t - \cos^2 t \sin t \, dt$ ...... ④ Solve integral

$= \frac{1}{2} \int_0^{\pi/2} 1 - \cos 2t \, dt + \int_0^{\pi/2} -\cos^2 t \sin t \, dt$

$= \frac{1}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{\pi/2} + \int_1^0 u^2 \, du$

$= \frac{\pi}{4} + \frac{u^3}{3} \Big|_1^0$

$= \frac{\pi}{4} - \frac{1}{3} = \frac{3\pi - 4}{12}$

$u = \cos t$, $du = -\sin t \, dt$

$t \leq \frac{\pi}{2}$ $u \leq ^0_1$

Five Apple

Salhudegn

8-10-2019

② $\vec{F}(x,y,z) = z\hat{i} + x\hat{j} + y\hat{k}$; C: $x^2+y^2=1$, $z=3t$

Soln:

$r(t) = [\cos t, \sin t, 3t]$ ... ① $0 \le t \le 2\pi$

$\vec{F}(r(\vec{t})) = [3t, \cos t, \sin t]$ ... ②

③

$\vec{F}(r(\vec{t})) \cdot \vec{r}'(t) = [3t, \cos t, \sin t] \cdot [-\sin t, \cos t, 3]$

$= -3t\sin t + \cos^2 t + 3\sin t$

$\therefore \int_C \vec{F}(r(\vec{t})) \cdot d\vec{r} = \int_0^{2\pi} (-3t\sin t + \cos^2 t + 3\sin t)\, dt$ ... ④

↳ solve in home
revise calc 2.

answer: 7π

* Some Properties of line integral:

① $\int_C \alpha \vec{F}(r(\vec{t})) \cdot d\vec{r} = \alpha \int_C \vec{F}(r(\vec{t})) \cdot d\vec{r}$ (α is const)

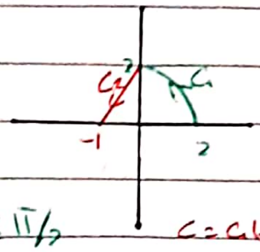
② $\int_C [\vec{F}(r(\vec{t})) \pm \vec{G}(r(\vec{t}))] \cdot d\vec{r} = \int_C \vec{F}(r(\vec{t})) \cdot d\vec{r} \pm \int_C \vec{G}(r(\vec{t})) \cdot d\vec{r}$

③ $\int_{-C} \vec{F}(r(\vec{t})) \cdot d\vec{r} = -\int_C \vec{F}(r(\vec{t})) \cdot d\vec{r}$

④ If $C = C_1 \cup C_2$, then $\int_C \vec{F}(r(\vec{t})) \cdot d\vec{r} = \int_{C_1} \vec{F}(r(\vec{t})) \cdot d\vec{r} + \int_{C_2} \vec{F}(r(\vec{t})) \cdot d\vec{r}$

8 - 10 - 2019

example : evaluate $\int_C \left(x^2 dx + xy\, dy\right)$ along C given below



Soln: $\vec{F}(x,y) = [x^2, xy]$

$C_1 : \vec{r_1}(t) = [2\cos t, 2\sin t] \qquad 0 \le t \le \pi/2$ $\qquad C = C_1 \cup C_2$

$C_2 : r_2(t) = (1-t)[0,2] + t[1,0], \quad 0 \le t \le 1$

$= [-t, 2-2t]$

$\vec{F}(r_1(\vec{t})) = [4\cos^2 t, 4\cos t \sin t]$

$\vec{F}(r_2(\vec{t})) = [t^2, 2t^2 - 2t]$

$\vec{F}(r_1(\vec{t})) \cdot r_1'(\vec{t}) = -8\cos^2 t \sin t + 8\cos^2 t \sin t = 0$

$\vec{F}(r_2(\vec{t})) \cdot r_2'(\vec{t}) = -t^2 + 4t - 4t^2 = 4t - 5t^2$

$$\int_C \vec{F}(r(\vec{t})) = \cancel{\int_{C_1} \vec{F}(r_1(\vec{t}))\, 0} + \int_{C_2} \vec{F}(r_2(\vec{t}))$$

$$= 0 + \int_0^1 4t - 5t^2\, dt$$

$$= 2 - \frac{5}{3} = \frac{1}{3}$$

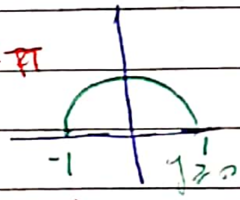
10-10-2019

2 - A line integral for a scalar function $f(x,y,z)$ along a smooth curve $C = \vec{r}(t)$ is given by:

$$\int_C f(\vec{r}(t))\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

example : evaluate the line integral of the given scalar function along the given curve.

① $f(x,y) = 2 + x^2 y$, $C = x^2 + y^2 = 1$, $y \geq 0$



sol $C = \vec{r}(t) = [\cos t, \sin t]$, $0 \leq t \leq \pi$

$f(\vec{r}(t)) = 2 + \cos^2 t \sin t$

$$f(\vec{r}(t))\,|\vec{r}\,'(t)| = (2 + \cos^2 t \sin t) \cdot \cancel{\sqrt{(-\sin^2 t) + (\cos^2 t)}}^{\,1}$$

$$= 2 + \cos^2 t \sin t$$

$$\therefore \int_C f(\vec{r}(t))\,ds = \int_0^{\pi} (2 + \cos^2 t \sin t)\,dt$$

$$= 2\pi + \int_1^{-1} u^2\,du$$

$$= 2\pi + \left(\frac{1}{3} - \frac{-1}{3}\right) = 2\pi + \frac{2}{3}$$

10 - 10 - 2019

ex 2- $f(x,y,z) = x+y+z$, C = line segment from $(2,0,0)$ to $(3,4,5)$

soln:

$r(\vec{t}) = (1-t)[2,0,0] + t[3,4,5]$

$= [2+t, 4t, 5t]$, $0 \le t \le 1$

$f(r(\vec{t})) = 2+t+4t+5t = 2+10t$

$f(r(\vec{t})) \cdot |r(t)| = 2+10t\sqrt{1^2+4^2+5^2}$

$= \sqrt{42}(2+10t)$

$\therefore \int_C f(r(\vec{t}))\, ds = \int_0^1 \sqrt{42}(2+10t)\, dt = \sqrt{42}(2+5) = 7\sqrt{42}$.

ex 3- $f(x,y,z) = y\sin z$, $C = r(\vec{t}) = [\cos t, \sin t, t]$

$0 \le t \le 2\pi$

soln $= \sin t \cdot \sin t = \sin^2 t = f(r(\vec{t}))$

$f(r(\vec{t})) \cdot |r'(t)| = \sin^2 t\sqrt{(-\sin^2 t) + (\cos^2 t) + 1^2} = \sqrt{2}\sin t^2$

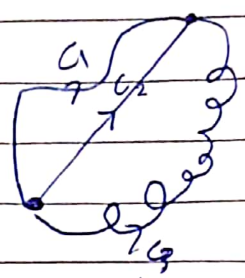
$\therefore \int_C f(r(\vec{t}))\, ds = \sqrt{2}\int_0^{2\pi} \sin^2 t\, dt = \sqrt{2}\int_0^{2\pi} 1 - 2\cos t\, dt$

$= \frac{\sqrt{2}}{2} \cdot (2\pi) = \sqrt{2}\,\pi$

10 - 10 - 2019

If the P.E or curve eqn is complicated use the curl check

$\vec{F}(x,y,z)$



Path dependance :

theorm: the line integral generally depends not only on end points of the path, but also on the path itself,

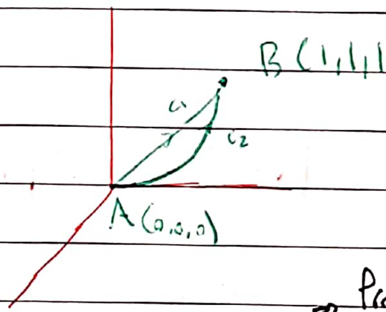
If the curl of $\vec{F}(x,y,z)$ is $\vec{0}$ the integral doesn't depend on the path

$\int_{C_1} = \int_{C_2} = \int_{C_3}$

└→ easiest one

example:

$F(x,y,z) = [5z, xy, x^2 z]$



$C_1 : \vec{r_1}(t) = [t, t, t] \quad 0 \le t \le 1$

line segment

$C_2 : \vec{r_2}(t) = [t, t, t^2] \quad 0 \le t \le 1$

Parabolic segment

Proof?

Try curl

$\int_{C_1} \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \int_0^1 [5t, t^2, t^2] \cdot [1,1,1]\, dt = \int_0^1 5t + t^2 + t^3 dt$

$= \frac{5}{2} + \frac{1}{3} + \frac{1}{4} = \frac{37}{12}$

$\int_{C_2} F(r_2(t)) \cdot dr = \int [5t^2, t^3, t^4] \cdot [1,1,2t]\, dt = \int 5t^2 + t^2 + 2t^5 \, dt$

$= 2 + \frac{1}{3} = \frac{7}{3}$ not equal

∴ Path dependant

13-10

Remark: the line integral is used to find the Work done on an object moving through an electrical or gravitational field.

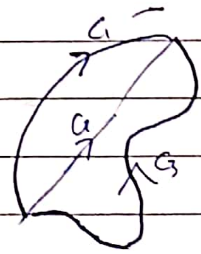
i.e $w = \int_c \vec{F} \cdot d\vec{r}$

10.2 Path dependance of line Integrals.

a line integral $\int_c \vec{F} \cdot d\vec{r}$ is path if it has the Same Value for all curves c with the same end points.

that is, its Value depends only on the end Points of C not C itself

$$\int_{c_1} \vec{F} \cdot d\vec{r} = \int_{c_2} \vec{F} \cdot d\vec{r} = \int_{c_3} \vec{F} \cdot d\vec{r}$$



* therm (Gradient therm):

A Line integral $\int_c \vec{F} \cdot d\vec{r}$ is Path Independent in a domain D iff $\vec{F} = \nabla f$ for some function f defined on D and in this case:

$$\int_c \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Remarks: if $\vec{F} = \nabla f$, then $\vec{F}$ is called conservative and f is called the Potential of $\vec{F}$.

13-10

Theorm: A line integral $\int_C \vec{F}\cdot d\vec{r}$ is path independent iff $\operatorname{curl}(\vec{F}) = 0$

Proof: $\int_C \vec{F}\cdot d\vec{r}$ is path independent $\vec{F} = \nabla f$

now $\operatorname{curl}(\vec{F}) = \operatorname{curl}(\nabla f) = 0$

example: show that $\int_C 2x\,dx + 2y\,dy + 4z\,dz$ is path independent and find its value for endpoints $A(0,0,0)$ and $B(2,2,2)$

Soln:

$\vec{F}(x,y,z) = [2x, 2y, 4z]$.

*method ①

$\nabla f = \vec{F} \Rightarrow \partial f/\partial x = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z)$

$\Rightarrow \partial f/\partial y = 2y \Rightarrow f(x,y,z) = x^2 + y^2 + h(z)$

$\Rightarrow \partial f/\partial z = 4z \Rightarrow \boxed{f(x,y,z) = x^2 + y^2 + 2z^2 + c}$

↳ Potential

$\therefore \int_C \vec{F}\cdot d\vec{r}$ is path independent

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = f(2,2,2) - f(0,0,0) = 16 - 0 = 16$

1-10
4

method (2)

Soln: $\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ 2x & 2y & 4z \end{vmatrix} = [0,0,0]$

$= \int_{C} \vec{F}\cdot d\vec{r}$ is path independent

$C: \vec{r}(t) = (1-t)[0,0,0] + t[2,2,2] = [2t, 2t, 2t]$

$0 \le t \le 1$

$\vec{F}(\vec{r}(t)) = [4t, 4t, 8t]$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 8t + 8t + 16t = 32t$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 32t\,dt = 16t^2\Big|_0^1 = 16$$

example find $\int_C 3x^2dx + 2yz\,dy + y^2dz$ from $A(0,1,2)$ to $B(1,-1,7)$, by showing $\vec{F}$ has a potential

Soln : $\vec{F}(x,y,z) = [3x^2, 2yz, y^2]$

$\nabla f = \vec{F} \Rightarrow \partial f/\partial x = 3x^2 \Rightarrow f(x,y,z) = x^3 + g(y,z)$

$\partial f/\partial y = 2yz \Rightarrow f(x,y,z) = y^2z + x^3 + h(z)$

$\partial f/\partial z = y^2 \Rightarrow f(x,y,z) = x^3 + y^2z + c$

Potential

$\therefore$ Path independent $\Rightarrow \int_C \vec{F}\cdot d\vec{r} = f_B - f_A$

$= 8 - 2 = 6$

15-10

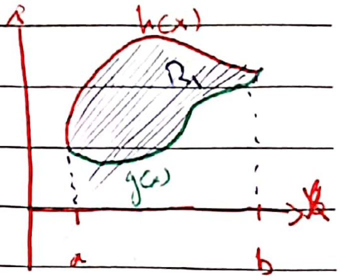
Double integral revision.

10.3 Double integral. (review)

there are two types of regions:

1- Region of type ①:

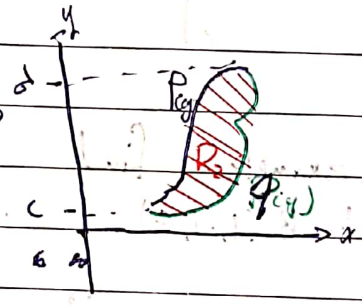
$$R_1 = \{(x,y) : a \le x \le b,\ g(x) \le y \le h(x)\}$$



$$\iint_{R_1} f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx.$$

2- Region of type ②

$$R_2 = \{(x,y) : q(y) \le x \le P(y),\ c \le y \le d\}$$



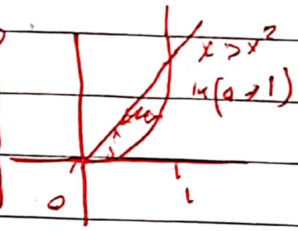
$$\iint_{R_2} f(x,y)\,dA = \int_c^d \int_{P(y)}^{g(y)} f(x,y)\,dx\,dy$$

example evaluate $\iint_R (x+2y)\,dA$ where R is the Region between

$\to y=x,\ y=x^2$

soln: method (one):

$$R_1 = \{(x,y) : 0 \le x \le 1,\ x^2 \le y \le 1\}$$



$$\iint_R (x+2y)\,dA = \int_0^1\int_{x^2}^{x} (x+2y)\,dy\,dx$$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

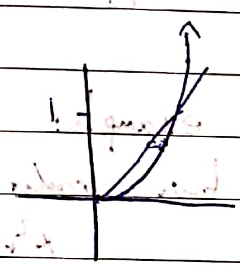
$$= \int_0^1 xy + y^2 \Big|_{x^2}^{x} = \int_0^1 \left(x^2+x^2\right) - \left(x^3+x^4\right) dx$$

$$= \frac{2}{3} - \frac{1}{4} - \frac{1}{5} = \frac{5}{12} - \frac{1}{5} = \frac{13}{60}$$

method (2)

$$R_2 = \{(x,y) : y \le x \le \sqrt{y},\ 0 \le y \le 1\}$$

$$\iint f(x,y)\,dA = \int_0^1\int_y^{\sqrt{y}} (x+2y)\,dx\,dy$$



$$= \int_0^1 \frac{x^2}{2} + 2yx \Big|_y^{\sqrt{y}} dy = \int_0^1 \left(\frac{y}{2} + 2y\sqrt{y}\right) - \left(\frac{y^2}{2} + 2y^2\right) dy$$

$y = \sqrt{y}$

$y^2 - y = 0$

$y = 0, 1$

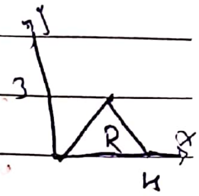
$$= \int_0^1 \frac{y}{2} + 2y^{\frac{3}{2}} - \frac{5}{2}y^2\,dy = \frac{1}{4} + \frac{4}{5} - \frac{5}{6} = \frac{13}{60}$$

Remarks

① $\iint_R dA =$ Area of R

② $\iiint_E dV =$ Volume of E

- evaluate $\iint_R dA$ where R is the region in figure



$\iint_R dA =$ Area
$= \frac{1}{2} \cdot 4 \cdot 3 = 6$

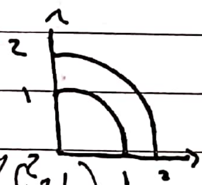
Polar Coordinates:

$R = \{(r,\theta) : \alpha \le \theta \le \beta \,,\ g(\theta) \le r \le h(\theta)$

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\; r\,dr\,d\theta$$

example: evaluate $\iint_R x\,dA$ where R is the Region between two circles
$x^2+y^2=1 \,,\ x^2+y^2=4$ in 1st quadrant

soln $R = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{2} \,,\ 1 \le r \le 2\}$



$$\iint_R x\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 r\cos\theta \cdot r\,dr\,d\theta = \left(\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta\right) \cdot \left(\int_1^2 r^2\,dr\right)$$

15-10 Partial eqn

$= \sin\theta \Big|_0^{\pi} \cdot \dfrac{r^3}{3}\Big|_0^{1} = (1-0)\left(\dfrac{8}{3} - \dfrac{1}{3}\right) = \dfrac{7}{3}$

exercise :- evaluate (hW)

① $\displaystyle\int_0^1\int_x^{2x} (y+x)^2 \, dy\, dx$, ② $\displaystyle\int_0^3\int_{-y}^{y} (x^2+y^2)\, dx\, dy$

③ $\displaystyle\int_0^2\int_0^{y} \sinh(x+y)\, dx\, dy$, ④ $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\cos y} x^2 \sin y \, dx\, dy$

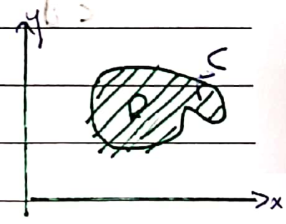
17-10 Partial eqn

10.4 Green theorem in the plain

* green's theorem :-

if R is a closed bounded region in the x-y plane with boundry C with +ve orientation

if $\vec{F}(x,y) = [F_1(x,y), F_2(x,y)]$

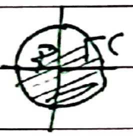


then: $\displaystyle\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$

• in Vectorial form : $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \iint \operatorname{curl}(\vec{F}) \cdot \hat{k}\, dA$

example (Verification) of green's theorem :

let $\vec{F}(x,y) = (y^2 - 7y)\hat{i} + (2xy + 2x)\hat{j}$ and C: $x^2+y^2=1$ with +ve orientation



note: when do we use green's theorem?, 2D or closed curve or CCW

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Particular

17-10-2019

then: $F_1 = y^2 - 7y$ and $F_2 = 2xy + 2x$

if its not CCW assume the opposite and multiply the answer with $-ve$

① $\iint_R (\partial F_2/\partial x - \partial F_1/\partial y)\, dA$

$= \iint_R (2y+2) - (2y-7)\, dA = 9 \iint_R dA$

$= 9\pi(1)^2 = 9\pi$

② $C: \vec{r}(t) = [\cos t, \sin t] \quad 0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = [\sin^2 t - 7\sin t,\ 2\cos t \sin t + 2\cos t]$

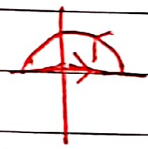
$\vec{F}(\vec{r}(t)) \cdot \vec{r}(t)' = -\sin^3 t + 7\sin t^2,\ 2\cos^2 \sin t + 2\cos^2 t$

$\therefore \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\sin^3 t + 7\sin^2 t + 2\cos^2 t \cdot \sin t + 2\cos^2 t$ using matlab

$= 9\pi$

example $\vec{F}(x,y) = [e^x + 4y,\ \sin 2y + 5x]$ and c is the upper half of the circle $x^2+y^2=4$ with positive orientation

Soln: $F_1 = e^x + 4$, $F_2 = \sin 2y + 5x$



$\oint_C \vec{F} \cdot d\vec{r} = \iint_R 5 - 4\, dA = \iint_R dA = \frac{1}{2}\pi(2)^2 = 2\pi$

17-10

example: evaluate $\oint_C xy\,dx + x^2y^3\,dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,2)$ with +ve orientation

Soln: $F_1 = xy$, $F_2 = x^2y^3$

$$\oint \vec{F}\cdot d\vec{r} = \iint_R \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\,dA = \iint_R 2xy^3 - x\,dA$$

$R = \{0 \le x \le 1,\ 0 \le y \le 2x\}$

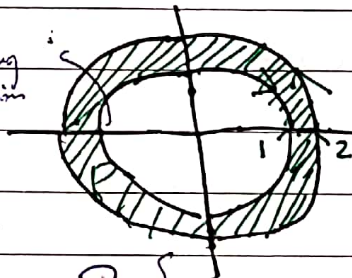
$$\int_0^1\int_0^{2x} 2xy^3 - x\,dy\,dx = \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Big|_0^{2x} dx$$

$$= \int_0^1 8x^5 - 2x^2\,dx = \frac{4}{3}x^6 - \frac{2}{3}x^3\Big]_0^1 = 4/3 - 2/3 = 2/3$$

example: evaluate $\oint_C y^3\,dx - x^3\,dy$ where C is given in the figure below:

soln: $f_1 = y^3$, $f_2 = -x^3$

hole rotating opposite direction



$$\oint \vec{F}\cdot d\vec{r} = \iint_R -3x^2 - 3y^2\,dA$$

$$= -3\iint \underbrace{x^2+y^2}_{r^2}\,dA = -3\int_0^{2\pi}\int_1^2 r^2\,r\,dr\,d\theta$$

$R = \{(r,\theta)\ 0 \le \theta \le 2\pi,\ 1 \le r \le 2\}$

$$= -3(2\pi)\cdot\left(4 - \frac{1}{4}\right) = -\frac{45\pi}{2}$$

10.5 Surfaces for Surface integral.

Recall: The surface in the space is represented by:

$$z = f(x,y) \quad \text{or} \quad g(x,y,z) = 0.$$

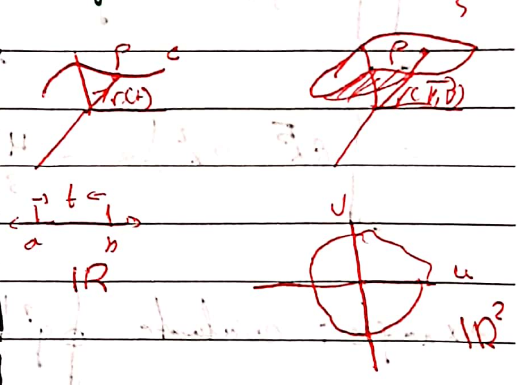
Parametric representation of a surface:

$$S: \vec{r}(u,v) = [x(u,v),\ y(u,v),\ z(u,v)]$$
$$= x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

example: find a P.E of:

① $x^2+y^2=4$, $-1 \le z \le 1$ "cylinder"

$$S: \vec{r}(u,v) = [2\cos u,\ 2\sin u,\ v]$$
$$0 \le u \le 2\pi, \quad -1 \le v \le 1$$



② $x^2+y^2+z^2=9$

Soln:

$$\left.\begin{aligned}\vec{r}(u,v) &= [3\cos v \cos u,\ 3\cos v \sin u,\ 3\sin v] \\ &0 \le u \le 2\pi, \quad -\pi/2 \le v \le \pi/2\end{aligned}\right\}$$ Book method

another representation of sphere.

$$\vec{r}(u,v) = [3\sin v\cos u,\ 3\sin v \sin u,\ 3\cos v]$$
$$0 \le u \le 2\pi, \quad 0 \le v \le \pi$$

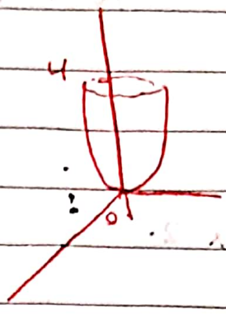
3- Elliptic Paraboloid

Soln:

$$z = x^2 + y^2 \quad , \quad 0 \leq z \leq 4$$

$$0 \leq u \leq 2\pi \quad , \quad 0 \leq v \leq 2$$

$$\hookrightarrow r(\vec{u}, v) = [v\cos u, v\sin u, v^2]$$



4- Cone

$$z = \sqrt{x^2 + y^2} \quad , \quad 0 \leq z \leq 5$$

$x^2 + y^2 = v^2$

$x = v\cos u$

$y = v\sin u$

Soln:

$$r(\vec{u}, v) = [v\cos u, v\sin u, v]$$

$$0 \leq u \leq 2\pi \quad , \quad 0 \leq v \leq 5.$$

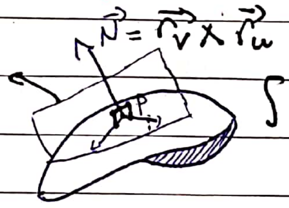
---

* Tangent plane and surface Normal:

Defn: ① the tangent plane of a surface S at a Point P is the plane containing tangent vectors of S at P.

② the Normal Vector of a Surface S at a Point P is the Vector Perpindicular to the tangent plane.

1- a normal Vector $\vec{N} = \vec{r_u} \times \vec{r_v}$



2- a normal unit Vector $\vec{n} = \dfrac{\vec{N}}{|\vec{N}|}$

example: find a normal Vector to:

① $x^2 + y^2 = 4 \quad , \quad 0 \leq z \leq 3$

Soln: $\vec{r}(u, v) = [2\cos u, 2\sin u, v]$

$\vec{r_u} = [-2\sin u, 2\cos u, 0]$

$\vec{r_v} = [0, 0, 1]$

20-10

$$\vec{N} = \vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = [2\cos u, 2\sin u, 0]$$

ex 2- $\frac{x^2}{4} + \frac{z^2}{9} = 1$, $0 \le y \le 4$ (elliptical cylinder)

Soln: $\vec{r}(u,v) = [2\cos u, v, 3\sin u]$

$\vec{r_u} = [-2\sin u, 0, 3\cos u]$

$\vec{r_v} = [0, 1, 0]$

$$\vec{N} = \vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 0 & 3\cos u \\ 0 & 1 & 0 \end{vmatrix} = [-3\cos u, 0, -2\sin u]$$

theorm: if S is given by $g(x,y,z)=0$, then

$$\vec{N} = \nabla g$$

ex Find a $\vec{N}$ of

① $x^2+y^2+z^2=4$

Soln: $x^2+y^2+z^2-4=0 \Rightarrow g(x,y,z)$

$$\vec{N} = \nabla g = [2x, 2y, 2z]$$

② $z = \sqrt{x^2+y^2}$

Soln:

$$g(x,y,z) = \sqrt{x^2+y^2} - z$$

$$\vec{N} = \nabla g = \left[\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right]$$

22-10

① 10.6 Surface integral.

1- Surface integrals for Vector Functions:

a surface integral (flux integral) of a vector function $\vec{F}(x,y,z)$ over a surface S is defined by:

$$\iint_S \vec{F}\cdot\vec{N}\,dA = \iint_R \vec{F}(\vec{r}(u,v))\cdot\vec{N}\,du\,dv$$

where R is the projection of S onto the u-v plane

* notation: $\iint_S \vec{F}\cdot\vec{N}\,dA = \iint_S F_1\,dy\,dz + \iint_S F_2\,dz\,dx + \iint_S F_3\,dx\,dy$

example: evaluate the flux integral

$\iint \vec{F}\cdot\vec{N}\,dA$ where 1- $\vec{F}(x,y,z) = [3z^2, 6, 6xz]$

and S: $y = x^2$ $\quad 0 \le x \le 2 \;,\; 0 \le z \le 3$

$x = u \quad y = x^2 = u^2$

Soln: $\vec{r}(u,v) = [u, u^2, v]$

$\vec{F}(\vec{r}(u,v)) = [3v^2, 6, 6uv]$

der. of $\vec{r}_u = [1, 2u, 0] \Rightarrow \vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$\vec{r}_v = [0, 0, 1]$

$\vec{F}\cdot\vec{N} = 12uv^2 - 6 \quad \therefore \iint_S \vec{F}\cdot\vec{n}\,dA = \int_0^2\int_0^3 6uv^2 - 6\,dv\,du = 72$

note: $\iint_S x\,dz\,dx - 3e^y\,dx\,dy + dy\,dz$

$\vec{F}(x,y,z) = [1, x, -3e^y]$

22-10

[2] Surface integrals for Scaler functions:

$$\iint_S F(x,y,z)\, dA = \iint_R F(\vec{r}(u,v))\, |\vec{N}|\, du\, dv$$

$\hookrightarrow$ normal multiplication

example: evaluate $\iint_S f(x,y,z)\, dA$, where

① $f(x,y,z) = x^2$ and $S = x^2+y^2+z^2 = 1$

Soln: $f(x,y,z) = x^2$

and $S: \vec{r}(u,v) = [\sin v\cos u, \sin v \sin u, \cos v]$

$0 \le u \le 2\pi,\ 0 \le v \le \pi$

$f(\vec{r}(u,v)) = \sin^2 v \cos^2 u$

$g(x,y,z) = x^2+y^2+z^2-1 \Rightarrow \vec{N} = \nabla g = [2x, 2y, 2z]$

$\hookrightarrow = [2\sin v\cos u, 2\sin v \sin u, 2\cos v]$

Solve it alone $|\vec{N}| = \sqrt{4\sin^2 v\cos^2 u + 4\sin^2 v\sin^2 u + 4\cos^2 v} = 2$

$f(|\vec{N}|) = 2\sin^2 v\cos^2 u$

$$\therefore \iint_S x^2\, dA = \int_0^{2\pi}\int_0^{\pi} 2\sin^2 v\cos^2 u\, dv\, du = 2\left(\int_0^{2\pi}\cos^2 u\, du\right)\left(\int_0^{\pi}\sin^2 v\, dv\right)$$

$= \pi^2$

[Tonelli's simplify]

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2- example: $f(x,y,z) = y$, $S: z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$

Soln: $S: \vec{r}(u,v) = [u, v, u + v^2]$ $0 \le u \le 1$, $0 \le v \le 2$

$f(\vec{r}(u,v)) = v$

$g(x,y,z) = x + y^2 - z \Rightarrow \vec{N} = \nabla g = [1, 2y, -1] = [1, 2v, -1]$

$f \cdot |\vec{N}| = v\sqrt{2 + 4v^2}$

$$\therefore \iint_S = \int_0^1 \int_0^2 v\sqrt{2+4v^2}\, dv\, du = \left(\int_0^1 du\right) \cdot \left(\int_0^2 v\sqrt{2+4v^2}\, dv\right)$$

$$= \frac{13\sqrt{2}}{3}$$

<u>note</u>: if Question asks for normal Vector on S use $\nabla g = \vec{N}$

but if it asks for Surface integral on Vars Functions use $\vec{N} = \vec{r_u} \times \vec{r_v}$

where $\vec{r_u} = \dfrac{\partial \vec{r}(u,v)}{\partial u}$ and $\vec{r_v} = \dfrac{\partial \vec{r}(u,v)}{\partial v}$

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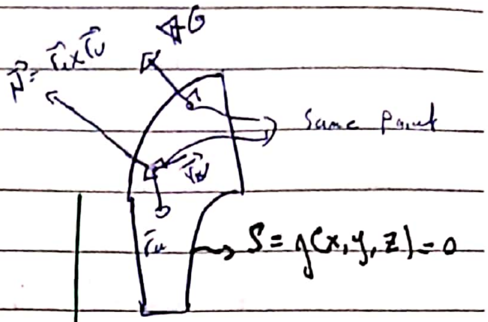
* correction of section 10.6



$\vec n = \alpha \nabla g$

because $\vec n \parallel \nabla g$

example 2.1

Soln: ~~$\vec r(u,v)$~~

$\vec r(u,v) = [\sin v \cos u, \sin v \sin u, \cos v$

$f(\vec r(u,v)) = \sin^2 v \cos^2 u$

$\vec r_u = [-\sin v \sin u, \sin v \cos u, 0]$

$\vec r_v = [\cos v \cos u, \cos v \sin u, -\sin v]$

$$\vec N = \vec r_u \times \vec r_v = \begin{vmatrix} \hat i & \hat j & \hat k \\ -\sin v \sin u & \sin v \cos u & 0 \\ \cos v \cos u & \cos v \sin u & -\sin v \end{vmatrix} = \begin{bmatrix} -\sin^2 v \cos u, -\sin^2 v \sin u, \\ -\sin v \cos v \end{bmatrix}$$

$-\sin v \sin u \cdot \cos v \sin u - \cos^2 u \cos v \sin v$

$= -\sin v \cos v (\sin^2 u + \cos^2 u) = -\sin v \cos v$

$$|\vec N| = \sqrt{(\quad)^2 + (\quad)^2 + (\quad)^2} = \sin v$$

$$f \cdot |\vec N| = \sin v \cdot \sin v^2 \cos v^2 = \sin^3 v \cos v^2$$

$$\therefore \int_0^{2\pi}\int_0^{\pi} \sin^3 v \cos^2 u \, dv \, du = \left(\int_0^{2\pi} \cos^2 u \, du\right)\left(\int_0^{\pi} \sin^3 v \, dv\right)$$

$$= \left(\frac{1}{2}\left(u + \frac{\sin 2u}{2}\right)\Bigg|_0^{2\pi}\right)\left(\int_{-1}^{1} (1 - z^2) \, dz\right)$$

$$= \frac{4\pi}{3}$$

$\sin^3 = \sin \sin^2$

$= \sin (1 - \cos^2)$

$z = \cos x$

$dz = -\sin x \, dx$

27-10

Partial eqn.

10.7 Divergence theorm of Gauss.

Theorm (Divergence Theorm):

Let T be a closed bounded region in space whose boundry is a piecewise smooth orientable surface S with +ve orientation (outward)

Let $\vec{F}(x,y,z)$ be a continuous vector function and has continuous first partial derivatives in T, then

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \iiint_T \operatorname{div}(\vec{F})\,dV$$

OR

$$\iint F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

example (verification of divergence theorm)

→ using theorm

$\vec{F}(x,y,z) = 7x\,\hat{i} - z\hat{k}$; $S: x^2+y^2+z^2=4$; $V = 4/3\,\pi r^3$

① $\iiint_T \operatorname{div}(\vec{F})\,dv = \iiint (7+0-1)\,dv = 6\iiint_T dV = 6\times\frac{4}{3}\pi(2)^3 = 64\pi$

another way

② $\vec{r}(u,v) = [2\sin v\cos u,\ 2\sin v\sin u,\ 2\cos v]$ $\quad 0\le u\le 2\pi$, $0\le v\le\pi$

$\vec{F}(\vec{r}(u,v)) = [14\sin v\cos u,\ 0,\ -2\cos v]$

$$\vec{N} = \vec{r_u}\times\vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin v\sin u & 2\sin v\cos u & 0 \\ 2\cos v\cos u & 2\cos v\sin u & -2\sin v \end{vmatrix} = [-4\sin^2 v\cos u,\ -4\sin^2 v\sin u,\ -4\sin v\cos v]$$

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$$\iint_S \vec{F}\cdot\hat{n}\, dA = \int_0^{2\pi}\int_0^{\pi} -56\sin^3 v\cos^2 u + 8\cos^2 v \sin\, dv\, du$$? a lot of work

note: use divergence theorm the surface must be closed.

Recall: 1- Cylindrical Coordinates

$x = r\cos\theta$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$y = r\sin\theta$ $\quad dv = r\,dr\,d\theta\,dz$

$z = z$ $\quad x^2+y^2 = r^2$

2- spherical Coordinates

$x = \rho\sin\phi\cos\theta$ $\quad dv = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$

$y = \rho\sin\phi\sin\theta$ $\quad x^2+y^2+z^2 = \rho^2$

$z = \rho\cos\phi$

example: Evaluate $\iint_S \vec{F}\cdot\hat{n}\, dA$ where $\vec{F}(x,y,z) = [x^3, y^3, z^3]$

and T: $x^2+y^2 \le 9$, $0 \le z \le 2$

Soln:

$$\iint_S \vec{F}\cdot\hat{n}\, dA = \iiint_T \operatorname{div}(\vec{F})\, dV = \iiint 3x^2 + 3y^2 + 3z^2\, dv$$

$$= 3\int_0^2\int_0^{2\pi}\int_0^3 (r^2+z^2)\, r\, dr\, d\theta\, dz = 315\pi$$

② T: $z \le \sqrt{4-x^2-y^2}$

Soln: $\iint_S \vec{F}\cdot\hat{n}\, dA = \iiint_T \operatorname{div}(\vec{F})\, dV = 3\iiint_T x^2+y^2+z^2\, dv$

$$= 3\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2} \rho^2\cdot\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = 3\left(\int_0^{\frac{\pi}{2}}\sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{2}\rho^4\, d\rho\right)$$

$$3\cdot\left(-\cos\phi\Big|_0^{\frac{\pi}{2}}\right)\cdot(2\pi)\left(\frac{\rho^5}{5}\Big|_0^{2}\right) = \frac{192}{5}\pi$$

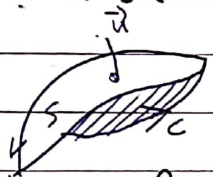
stocks theorm.

theorm:

Let S be a peice wise smooth oriented surface, and let its boundry be a piecewise smooth simple closed curve C:

Let $\vec{F}(x,y,z)$ be a continuous Vector function with continuous first partial derivative, then

$$\iint_S \operatorname{curl}(\vec{F})\cdot\hat{n}\, dA = \oint_C \vec{F}\cdot d\vec{r}$$



Or, $$\iint_R \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_1}{\partial z}\right)N_1 + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)N_2 + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)N_3\, dA$$

$$= \oint f_1 dx + f_2 dy + f_3 dz$$

Example M_1

$\vec{F}(x,y,z) = [y, z, x]$, S: $z = 1 - x^2 - y^2$

$z \geq 0$

c: $x^2 + y^2 = 1$, $z = 0$

$\vec{r}(t) = [\cos t, \sin t, 0]$, $0 \leq t \leq 2\pi$

$\vec{F}(\vec{r}(t)) = [\sin t, 0, \cos t]$

$\vec{F} \cdot d\vec{r} = -\sin^2 t$

$$\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\sin^2 t \, dt = \frac{1}{2}(1 - \cos 2t \Big|_0^{2\pi}) = -\pi$$

M_2 $\text{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = [-1, -1, -1]$

$\vec{r}(u,v) = [u, v, 1 - u^2 - v^2]$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = [2u, 2v, 1]$

$\text{curl}(\vec{F}) \cdot \vec{N} = -2u - 2v - 1$

$$\oint\oint_0 \text{curl}(\vec{F}) \cdot \hat{n} \, dA = \int_0^{2\pi}\int_0^1 -2r^2 (\cos\theta + \sin\theta) - r \, dr \, d\theta$$

$= -\pi$

ex: use Stock theorm

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dA \quad \text{where } \vec{F}(x,y,z) = [z^2, -3xy, x^3 y^3]$$
and $S: z = 5 - x^2 - y^2$

$z \geq 1$

Soln

$c: r(t) = [2\cos t, 2\sin t, 1]$, $0 \leq t \leq 2\pi$

$\vec{F}(\vec{r(t)}) = [1, -12\cos t \sin t, 64\cos^3 t \sin^3 t]$

$\therefore \vec{F} \cdot \vec{r'} = -2\sin t - 24\cos^2 t \sin t$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dA = \oint_C \vec{F} \cdot dr = \int_0^{2\pi} -2\sin t - 24\cos^2 t \sin t \, dt$$
$= 0$

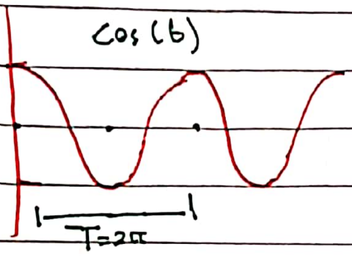
Partial

10/11

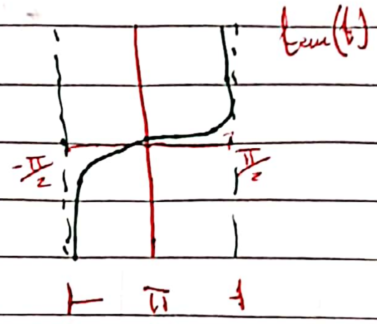
chapter 11: fourier transform

§11.1 fourier series: Defn: A function f is said to be periodic with period $T > 0$ if $f(x) = f(x+T)$

ex:



ex:



Remarks: if a periodic function f with period T, then its also periodic with period $2T, 3T$ ...

the smallest period of f is called the fundamental period

Defn: two functions $f(x)$ and $g(x)$ are called orthogonal from $[a, b]$ iff

$$\int_a^b f(x)\, g(x)\, dx = 0$$

- a set of functions is said to be mutually orthogonal if each pair of functions in the set is orthogonal.

- Orthogonality of Trigonometric functions:

① $$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & , n \neq m \\ L & , m = n \neq 0 \\ 2L & , m = n = 0 \end{cases}$$

Five Apple

② $$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

③ $$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \begin{cases} 0 & , m \neq n \\ L & , m = n \end{cases}$$

* Fourier Series :

If f has Period $2L$ defined on $[-L, L]$, then:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

Periodic

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \quad , n = 0, 1, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \quad , n = 1, 2, \ldots$$

12/11/2019 Fourier Series

Remark: if f is Periodic with Period 2π defined on $[-\pi,\pi]$, then:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$; $n = 0, 1, 2, 3, \dots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$; $n = 1, 2, 3, \dots$

* Fourier Convergence theorem.

assume that f is Periodic with Period $2L$ and Piecewise continuous on $[-L, L]$, then the corresponding Fourier Series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right]$$

converges to the average $\frac{f(x^+) + f(x^-)}{2}$ where

$f(x^-) = \lim_{\substack{n\to\infty \\ n\to 0}} f(x-n)$ Limit from the left

$f(x^+) = \lim_{\substack{n\to\infty \\ n\to 0}} f(x+n)$ Limit from the right

if $f(x)$ cont the Fourier Series value converges to the value of $f(x)$ at a point

but if $f(x)$ is disconnected at a point then Fourier Series value conv to $\frac{f(x^+) + f(x^-)}{2}$

Partial eqn 1

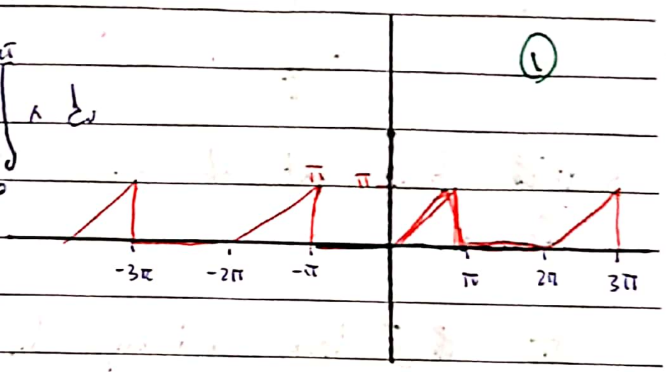
13/11/2019

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] = f(x) \to f(x) \text{ cont on } x$$

$$\to = \frac{f(x^+)+f(x^-)}{2} \text{ if } f(x) \text{ is discont on } x$$

example: for $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x \le \pi \end{cases}$

① sketch the graph of f with three period.
② find the Fourier series of f(x).
③ find the convergence of the Fourier series at $x=0$ and $x=\pi$

① 

② 1. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} x\,dx$

$= \frac{1}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi}{2}$

2. $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\cos(nx)\,dx}_{\text{Partial integral}}$; $n = 1, 2, 3 \ldots$

Diff		Int
x	$+$	$\cos(nx)$
1	$-$	$\frac{\sin(nx)}{n}$
0		$\frac{-\cos(nx)}{n^2}$

$= \frac{1}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi}$; $\sin(nx) = 0$ for every value of n

$= \frac{\cos(nx) - 1}{n^2 x}$

$\cos(n\pi) = -1, 1, -1, 1, \ldots = (-1)^n$

$= \frac{(-1)^n - 1}{\pi n^2}$

13/11/2019

3- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)$, $n = 1, 2, 3 \ldots$

diff Int

x $\sin(nx)$

1 $\oplus$ $\frac{-\cos(nx)}{n}$

0 $\ominus$ $\frac{-\sin(nx)}{n^2}$

$$= \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \cancel{\frac{\sin(nx)}{n^2}}\right]_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\cdot\frac{-\cancel{\pi}\cos(n\pi)}{n} = -\frac{\cos(n\pi)}{n} = \frac{(-1)^1\cdot(-1)^n}{n}$$

$$= \frac{(-1)^{n+1}}{n}$$

$$\therefore f(x) \sim \underbrace{\frac{\pi}{4}}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\boxed{\frac{(-1)^n - 1}{n^2\pi}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)}$$

3 (I) f is cont at $x=0$ the Fourier Series converges to $f(0) = 0$

(II) f has a jump discont at $x=\pi \Rightarrow$ the Fourier series converges to $\frac{f(\pi^+) + f(\pi^-)}{2} = \frac{0+\pi}{2} = \frac{\pi}{2}$

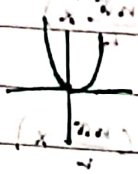
14/11/2019 Partial eqns

* Fourier Series for even and odd functions: *

Defn: A function f is called even function if its graph symmetric about the Y-axis and f is called odd function if its graph is symmetric about the origin.

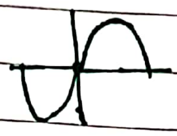
example

① $f(x) = x^2$

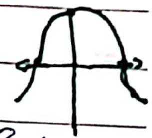


② $f(x) = x^3$

③ $f(x) = \sin x$



④ $f(x) = \cos x$



Remark: Denote the odd function by O and even function by E, then:

(i) O . O = E

(ii) O . E = O

(iii) E . O = O

(iv) E . E = E

Recall ① $\int_{-L}^{L} \text{odd function}\, dx = 0$, ② $\int_{-L}^{L} \text{even function}\, dx = 2\int_{0}^{L} \text{odd func}\, dx$

remember $a_n = \frac{1}{\tau}\int_{-\tau}^{\tau} f(x) \cos(\ )\, dx$, $b_n = \frac{1}{\tau}\int_{-\tau}^{\tau} f(x) \sin(\ )\, dx$

if $f(x)$ even: a_n = value, a_0 = value, $b_n = 0$

if $f(x)$ odd: $a_n = 0$, $a_0 = 0$, b_n = value

* Fourier Sine Series:

Assume that f is an odd Period 2L defined on $[-L, L]$, then the Fourier Sine Series is defined by:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \,, \quad n = 1, 2, 3$$

* Fourier Cosine series:

Assume that f is an even Periodic function with Period 2L defined on $[-L, L]$, then the Fourier cosine series is defined by:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \,, \quad n = 0, 1, 2, 3$$

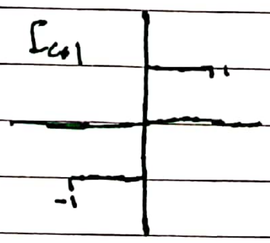
example: for the given function

(a) sketch the graph of f with three Periods.

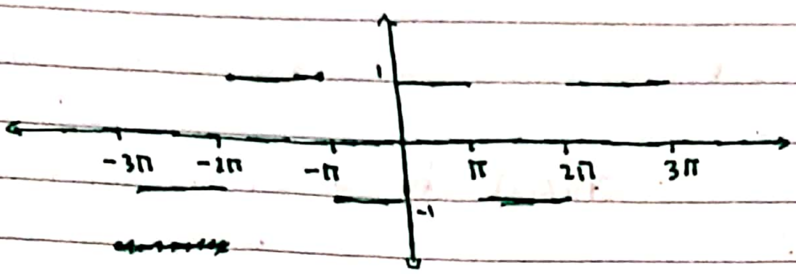
(b) Find the F.S for $f(x)$

(c) discuss the convergence at the Points of discontinuity

$$f(x) = \begin{cases} -1 \,, & -\pi \le x \le 0 \\ 1 \,, & 0 \le x \le \pi \end{cases}$$



a)



b) $a_n = 0$, $n = 0, 1, 2, 3$ Since f is an odd function

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \sin(nx)\, dx = \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{-\cos(n\pi)+1}{n}\right) = \frac{2}{\pi}\left(\frac{(-1)^{n+1}+1}{n}\right)$$

$$= \begin{cases} \frac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

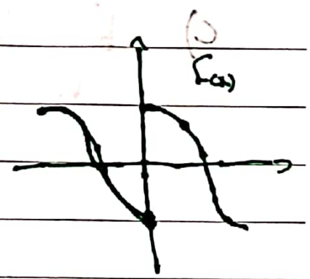
$\begin{cases} \text{even} \Rightarrow 2n \\ \text{odd} \Rightarrow 2n-1 \end{cases}$

$\therefore$ Fourier Sine Series:

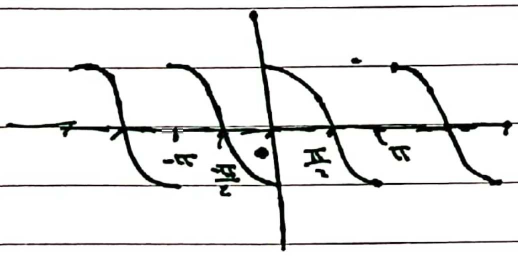
$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin((2n-1)x)$$

c) f has jump discont at $x = 0, \pm\pi, \pm 2\pi$
and F.S.S converges to $\frac{-1+1}{2} = \frac{0}{2} = 0$

Find a, b, c for $f(x) = \begin{cases} -\cos x & -\pi < x \le 0 \\ \cos x & 0 \le x \le \pi \end{cases}$



a)



odd function

b) $a_n = 0,\ n = 0, 1, 2, 3, 4 \ldots$ Since f is an odd function

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \cdot \sin(nx)\, dx = \frac{1}{\pi}\int_0^{\pi} \sin((n-1)x) + \sin[(n+1)x]\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos[(n-1)x]}{n-1} - \frac{\cos[(n+1)x]}{n+1} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos[(n-1)\pi] + 1}{n-1} + \frac{-\cos((n+1)\pi) + 1}{n+1} \right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+2} + 1}{n-1} + \frac{(-1)^{n+2} + 1}{n+1} \right]$$

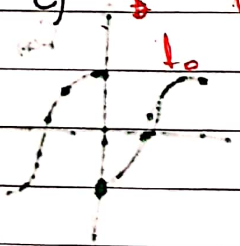
$$= \begin{cases} 0, & n \text{ odd} \\ \frac{1}{\pi}\left[\frac{2}{n-1} + \frac{2}{n+1}\right], & n \text{ even} \end{cases} = \begin{cases} 0, & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$

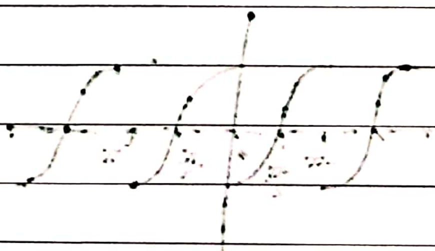
∴ Fourier Sine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)} \sin(2nx)$$

c) f has jump discont at $x = 0, \pm\pi, 2\pi \ldots$

to $\frac{-1+1}{2} = 0$

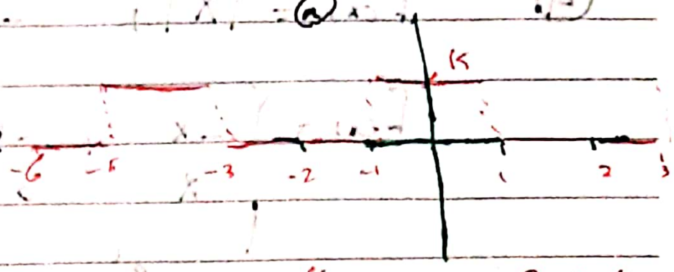




Partial

Fourier Cosine Series

③ $f(x) = \begin{cases} 0, & -2 \le x \le -1 \\ K, & -1 \le x \le 1 \\ 0 & 1 \le x \le 2 \end{cases}$ ⓐ ⓘ



Soln:

ⓐ $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx$ "f(x) even function"

$= \int_0^1 K\,dx = K$ Period 4

$L = 2$

$a_n = \frac{2}{2}\int_0^2 f(x)\,\cancel{\cos}\,\cos\left(\frac{n\pi}{2}x\right)dx$

$= \int_0^1 K\cos\left(\frac{n\pi}{2}x\right)dx = \left. \frac{K\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} \right|_0^1$

$= \frac{2K\sin\left(\frac{n\pi}{2}\right)}{n\pi} = \begin{cases} 0, & n \text{ even} \\ \frac{(-1)^{m+1}2K}{n\pi}, & n \text{ odd} \end{cases}$

$b_n = 0,\ n = 1, 2, 3$ Remarks

Since f is an even function $-, +, -, +, \ldots (-1)^m$

$+, -, +, -, \ldots (-1)^{m+1}$

$\therefore f(x) \sim K + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\,2K}{(2m-1)\pi}\cos\left(\frac{(2m-1)\pi}{2}x\right)$ n even $\Rightarrow n = 2m$

n odd $\Rightarrow n = 2m-1$

$\therefore f(x) \sim K + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,2K}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{2}x\right)$

ⓑ f has a jump discont at $x = \pm 1, \pm 2 \ldots$ and the ~~FS~~ Cosine Series converges to $\frac{0+K}{2} = \frac{K}{2}$

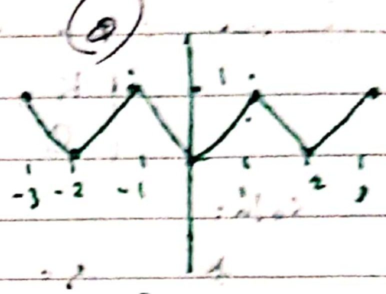
Five Apple

Partial eqn.

Fourier Cosine Series

(4) $f(x) = |x|$; $-1 \le x \le 1$ (a)

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ x & 0 \le x \le 1 \end{cases}$$



"even"

Period 2

$L = 1$

(b) 1- $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx$

$= 2\int_0^1 x\,dx = 2\left(\frac{1}{2}\right) = 1$

2 $a_n = \frac{2}{1}\int_0^1 f(x)\cos\left(\frac{n\pi}{L}x\right)dx = 2\int_0^1 x\cos\left(\frac{n\pi}{1}x\right)dx$

$= \cancel{\frac{2x\sin(n\pi x)}{n\pi}} + \frac{2\cos(n\pi x)}{n^2\pi^2}\Big|_0^1$ 0

diff int

x (+) $\cos(n\pi x)$

1 (−) $\frac{\sin(n\pi x)}{n\pi}$

0 $\frac{-\cos(n\pi x)}{(n\pi)^2}$

$= \frac{2\cos(n\pi)}{n^2\pi^2} - \frac{2}{n^2\pi^2} = \frac{2[(-1)^n - 1]}{n^2\pi^2}$

$= \begin{cases} \frac{-4}{n^2\pi^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

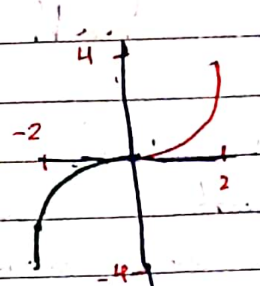
3- $b_n = 0$, $n = 1, 2, 3, 4 \dots$ Since f is an even function

Fourier Cosine Series

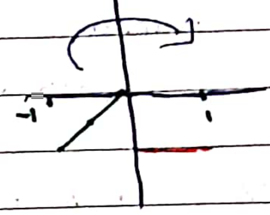
$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2\pi^2}\cos[(2n-1)\pi x]$$

home Work : find F.S

① $f(x) = \begin{cases} x^2 & ; -2 \le x < 0 \\ -x^2 & ; 0 \le x < 2 \end{cases}$



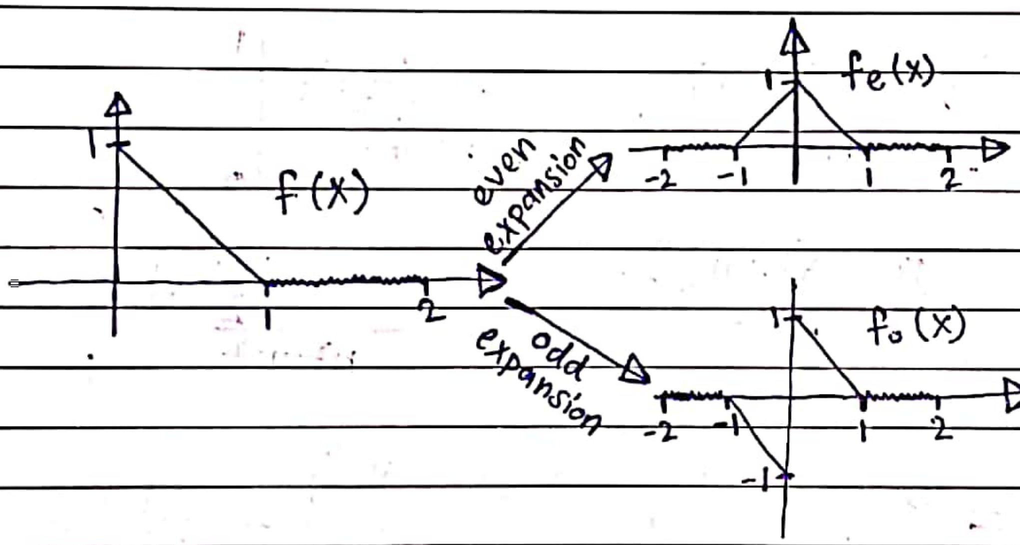
② $f(x) = \begin{cases} -x & , -1 \le x < 0 \\ x & ; 0 \le x \le 1 \end{cases}$



11.4 Half-Range Expansion

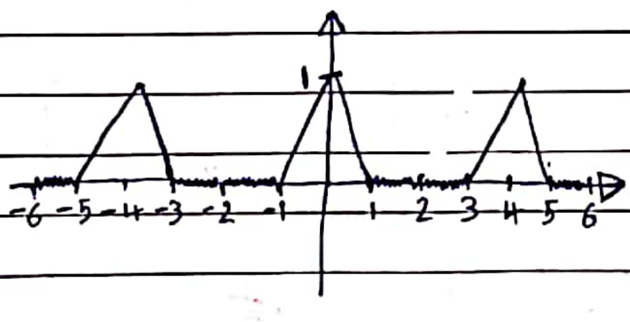
- If only half of the range, i.e $[0, L]$, is of interest, we may extend the function in an odd or even way, and then use the simplified Fourier series expansion for odd or even functions.

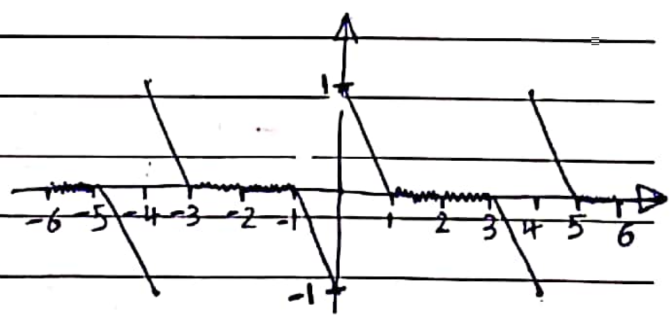
Example: $f(x) = \begin{cases} 1 - x, & 0 \le x < 1 \\ 0, & 1 \le x \le 2 \end{cases}$



$f_e(x)$ is an even periodic function with period $2L$

$f_o(x)$ is an odd periodic function with period $2L$





① Even Expansion

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 3x\,dx + \int_1^2 6-3x\,dx = 3$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad n=1,2,\ldots$$

$$= \int_0^1 3x\cos\left(\frac{n\pi}{2}x\right)dx + \int_1^2 (6-3x)\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \left[\frac{6x\sin\left(\frac{n\pi}{2}x\right)}{n\pi} + \frac{12\cos\left(\frac{n\pi}{2}x\right)}{n^2\pi^2}\right]_0^1$$

$$+ \left[\frac{(12-6x)\sin\left(\frac{n\pi}{2}x\right)}{n\pi} - \frac{12\cos\left(\frac{n\pi}{2}x\right)}{n^2\pi^2}\right]_1^2$$

$$= \cancel{\frac{6\sin\left(\frac{n\pi}{2}\right)}{n\pi}} + \frac{12\cos\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{12}{n^2\pi^2}$$

$$= \begin{cases} 0 & , n \text{ odd} \\ \dfrac{24(-1)^m - 24}{n^2 \pi} & , n \text{ even}, \; m = 1, 2, \ldots \end{cases}$$

$b_n = 0$, $n = 1, 2, \ldots$ since f_e is an even function.

$$\therefore f_e(x) \sim \frac{3}{2} + \sum_{n=1}^{\infty} \frac{24\left((-1)^n - 1\right)}{(2n)^2 \pi^2} \cos(n\pi x)$$

② Odd Expansion

$a_n = 0$, $n = 0, 1, 2, \ldots$ since f_o is an odd function.

$$b_n = \frac{2}{2} \int_0^2 f(x) \sin\left(\frac{n\pi}{2} x\right) dx, \quad n = 1, 2, \ldots$$

is $f(x) \sim \sum_{n=1}^{\infty} \frac{-4}{\pi(2n-1)^2} \cos((2n-1)x) + \frac{2}{2n-1} \sin((2n-1)x)$

Find the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

Soln:

$$f_L(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(\omega_n x) + b_n \sin(\omega_n x)\right]$$

where $\omega_n = \frac{n\pi}{L}$

Question: What happens if we let $L \to \infty$?

Example: $f_L(x) = \begin{cases} 0, & -L < x < -1 \\ 1, & -1 < x < 1 \end{cases}$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\omega x)\, dx$$

is called Fourier Integral representation of $f(x)$

$$\int_0^{\omega} \frac{x\sin x + \cos x - 1}{\pi\, \omega^2}\, dx$$

$$① \; A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\, dx = \frac{1}{\pi}\int_0^1 x\cos(\omega x)\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega}{\omega} + \frac{\cos\omega - 1}{\omega^2}\right] = \frac{\omega\sin\omega + \cos\omega - 1}{\omega^2\pi}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\, dx = \frac{1}{\pi}\int_0^1 x\sin(\omega x)\, dx$$

(3) $$\int_0^\infty \frac{w\sin w+\cos w-1}{w^2\pi}\,\cancel{\cos(0)}^{\,1} + \frac{\sin w - w\cos w}{w^2\pi}\,\cancel{\sin(0)}^{\,0}\, dw$$

$$= f(0) = 0$$

$$\Rightarrow \int_0^\infty \frac{x\sin x+\cos x-1}{x^2\pi}\,dx = 0$$

* Fourier Cosine Integral:

If $f(x)$ is an even function, then:

$$f(x) \sim \int_0^\infty A(w)\cos(wx)\,dw$$

where

$$A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos(wx)\,dx$$

* Fourier Sine Integral:

If $f(x)$ is an odd function, then:

$$f(x) \sim \int_0^\infty B(w)\sin(wx)\,dw$$

where

$$B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin(wx)\,dx$$

Example: Given $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

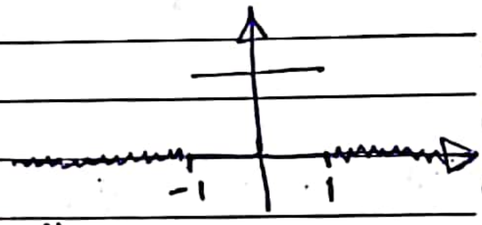
① Find the Fourier integral representation of $f(x)$.

② use the result in part ① to evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx.$$

Soln:

$$f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$



"f is an even function"

① $A(w) = \frac{2}{\pi} \int_0^\infty f(x) \cos(wx)\, dx = \frac{2}{\pi} \int_0^1 \cos(wx)\, dx$

$$= \frac{2 \sin w}{\pi w}$$

$B(w) = 0$, since f is an even function

$$\therefore f(x) \sim \int_0^\infty \frac{2 \sin w}{\pi w} \cos(wx)\, dw$$

② $\cos(wx) = 1 \Longrightarrow x = 0$

since f is cont. at $x = 0 \Longrightarrow \int_0^\infty \frac{2 \sin w}{\pi w}\, dw = f(0) = 1$

$$\Longrightarrow \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Defn: $F_s\{f(x)\} = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(\omega x)\,dx$ is

called Fourier sine transform of $f(x)$, and

$f(x) = F_s^{-1}\{\hat{f}_s(\omega)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega)\sin(\omega x)\,d\omega$ is

called inverse Fourier sine transform of $\hat{f}_s(\omega)$.

② $F_s\{f(x)\} = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(wx)\, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} K \sin(wx)\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{K(1-\cos(wa))}{w}$$

Example: $f(x) = \begin{cases} x^2, & 0<x<1 \\ 0, & x>1 \end{cases}$

① $F_s\{f(x)\} = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{1} x^2 \sin(wx)\, dx$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-x^2 \cos(wx)}{w} + \frac{2x \sin(wx)}{w^2} + \frac{2\cos(wx)}{w^3} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{-w^2 \cos w + 2w \sin w + 2\cos w - 2}{w^3} \right)$$

② $F_c\{f(x)\}$ (H.W.)

* Some important properties:

① $F_c\{\alpha f(x) + \beta g(x)\} = \alpha F_c\{f(x)\} + \beta F_c\{g(x)\}$; α and β are constant

② $F_s\{\alpha f(x) + \beta g(x)\} = \alpha F_s\{f(x)\} + \beta F_s\{g(x)\}$

③ $F_c\{f''(x)\} = -\omega^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0)$

$F_s\{f''(x)\} = -\omega^2 F_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\, \omega f(0)$

Example: find the Fourier cosine transform of

$f(x) = e^{-ax}, \quad a>0$

Soln:

$f(x)\, e^{-ax} \Rightarrow f'(x) = -a e^{-ax} \Rightarrow f''(x) = a^2 e^{-ax} =$

Exercise: find the Fourier sine transform of

$f(x) = \cos(ax)$, $a > 0$

11.10 Fourier Transform

→ Fourier transform is useful in solving partial differential equations.

→ We define the Fourier transform for a piecewise continuous and absolutely integrable function $\left(\text{i.e } \int_{-\infty}^{\infty} |f(x)|\,dx \text{ converges}\right)$ $f(x)$ by:

$$F\{f(x)\} = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

and the inverse Fourier transform by:

$$f(x) = F^{-1}\{\hat{f}(w)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

and

$$\cos x = \frac{e^{ix} + e^{-ix}}{2i}$$

Example: compute the Fourier transform of:

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

* **Fact:** $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

Example: compute the Fourier transform of

$$f(x) = e^{-2x^2}$$

soln:

$$F\{f(x)\} = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(2x^2 + i\omega x)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x^2 + \frac{i\omega}{2}x\right)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x^2 + \frac{i\omega}{2}x + \left(\frac{i\omega}{4}\right)^2 - \left(\frac{i\omega}{4}\right)^2\right)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\omega^2/8} \int_{-\infty}^{\infty} e^{-\left(\sqrt{2}\left(x + \frac{i\omega}{4}\right)\right)^2}\,dx$$

Let $u = \sqrt{2}\left(x + \frac{i\omega}{4}\right)$

$$\Rightarrow du = \sqrt{2}\,dx$$

$$x \to \begin{cases} \infty \\ -\infty \end{cases} \Rightarrow u \to \begin{cases} \infty \\ -\infty \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\omega^2/8} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{2}} = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/8} \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$= \frac{1}{2} e^{-\omega^2/8}$$

ليس للحفظ

* Table of Fourier transform:

$f(x)$	$\hat{f}(\omega)$
① $\begin{cases} 1, & -b<x<b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin b\omega}{\omega}$
② $\begin{cases} 1, & b<x<c \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{-ib\omega}-e^{-ic\omega}}{i\omega\sqrt{2\pi}}$
③ $\frac{1}{x^2+a^2}$, $a>0$	$\sqrt{\frac{2}{\pi}}\,\frac{e^{-a\lvert\omega\rvert}}{a}$
④ $\begin{cases} e^{-ax}; & x>0 \\ 0; & \text{otherwise} \end{cases}$, $a>0$	$\frac{1}{\sqrt{2\pi}\,(a+i\omega)}$
⑤ $\begin{cases} e^{ax}, & b<x<c \\ 0, & \text{otherwise} \end{cases}$, $a>0$	$\frac{e^{(a-i\omega)c}-e^{(a-i\omega)b}}{\sqrt{2\pi}\,(a-i\omega)}$
⑥ $\begin{cases} e^{iax}, & -b<x<b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin b(\omega-a)}{\omega-a}$
⑦ e^{-ax^2}, $a>0$	$\frac{1}{\sqrt{2a}}\,e^{-\omega^2/4a}$

① $F\{\alpha f(x) + \beta g(x)\} = \alpha F\{f(x)\} + \beta F\{g(x)\}$

, α, β constants

② $F\{f'(x)\} = i\omega F\{f(x)\}$

③ $F\{f''(x)\} = (i\omega)^2 F\{f(x)\}$

④ $F\{f^{(n)}(x)\} = (i\omega)^n F\{f(x)\}$

Example: find the Fourier transform of

$f(x) = x e^{-x^2}$ using the fact

$$F\{e^{-ax^2}\} = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$$

soln:

Note that $g(x) = \frac{-1}{2} e^{-x^2}$ then

$g'(x) = x e^{-x^2} = f(x)$

$$\Rightarrow F\{f(x)\} = F\{g'(x)\} = i\omega F\{g(x)\}$$

$$= i\omega \left(\frac{-1}{2}\right) F\{e^{-x^2}\}, \quad \boxed{a=1}$$

$$= i\omega \left(\frac{-1}{2}\right)\left(\frac{1}{\sqrt{2}} e^{-\omega^2/4}\right)$$

$$= \frac{-i\omega}{2\sqrt{2}} e^{-\omega^2/4}$$