* Digital signal processing. (DSP)

26.Sep.2019

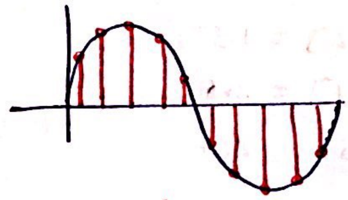
- Discrete Time signals:

$x(t)$: Continuous signal

$x[n]$: Discrete time signal

Descritized signal | Discrete By Nature

* Descritized signal:



when you descritize a sample

$$x(t)\Big|_{t=nT} = x(nT) = x[n]$$

T: The sampling period

$T = \frac{1}{F_s}$ where F_s : The sampling Freq

ex sampling freq = 10 Hz (sample/sec)

$T = \frac{1}{10} = 0.1$ sec

* Discrete by Nature:

ex: Taking records in the lab.

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* Representation of Discrete-Time signals:

[1] Functional Representation

$$x[n] = \begin{cases} 2n, & -2 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

$\equiv -3 < n < 3$ The same Representation

$x[n] = \cos(wn)$ for all n.

$\hookrightarrow$ w: discrete freq (rad/sample) $\longrightarrow$ $x(t) = \cos(\Omega t)$

$\hookrightarrow$ continuous freq. (rad/sec)

[2] Tabular method

n	0	1	2	3	4
$x[n]$	1.5	3.6	2.7	-1.5	3

corresponding value

3. Sequence method :

$x[n] = \{ \ldots, 1.5, 3.6, 2.7, -2.3, 4, \ldots \}$ ← Two-sided sequence

n=-1 ↑ n=0 n=1 n=2

infinite from the left side

infinite from the right side

* There should be an extra information about the zero location

↳ we use vertical arrow

$x[-2] = 1.5$

$x[-1] = 3.6$

$x[0] = 2.7$

ex $x_1[n] = \{1 \quad 2 \quad 3 \quad 4\}$ ← sequence with finite length

If there isn't any info about the zero, the default is the first element → (offline Data)

No Arrow ⟶ means that the first element is index zero

④ length sequence

↳ # of elements inside

$x_1[n] \neq x_2[n]$

order is important

$x_2[n] = \{2 \quad 1 \quad 3 \quad 4\}$ → $x[0] = 2$

$x_3[n] = \{1 \quad 2 \quad 3 \quad 4\}$

$N_1 = -1$ ↳ the first index with non-zero value

$N_2 = 2$ ↳ the last index with non-zero value

N: The length of the sequence ⟶ $\boxed{N = N_2 - N_1 + 1}$

$x_1[n] = \{1.5, 1.6, \underset{\uparrow}{2.7}, -2.3, 4, \dots\}$ → Infinite sequence (Right-sided sequence)

$N_1 = -2$

$x[n] = 0$

$n < -2 \quad (N_1)$ ← بتحقق هاد الشرط

$x_2[n] = \{\dots, 1.5, 1.6, \underset{\uparrow}{2.7}, -2.3, 4\}$ → left-sided Sequence

$x[n] = 0 \qquad N_2 \qquad n > 2$

$x[n] = \{1 \quad 1.5 \quad \underset{\uparrow}{2.6} \quad -3\} \qquad -2 \le n \le 1$

$x[-2] = 1$

$x[0] = 2.6$

$\boxed{4}$ Matrix Representation:

$$\underline{X} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

finite length (N) → column vector

$N - 1 - 0 + 1 = N \qquad N \times 1$

$$\underline{X} = [x[0] \cdot x[1] \quad x[2] \quad \cdots \quad x[N-1]]^{T}$$

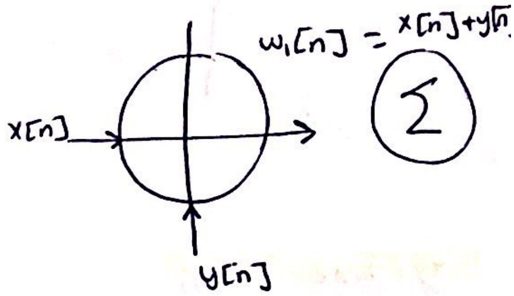
T: Transpose operation

↳ coloumn vector $N \times 1$

* Elementary operations on frequencies :

① Addition.

$w_1[n] = x[n] + y[n]$



Example $x[n] = \{1 \quad 2 \quad \underset{\uparrow}{3} \quad 1\}$

$y[n] = \{1 \quad \underset{\uparrow}{1} \quad 2 \quad 2\}$

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 \\ \hline \end{array}$$

$w_1[n] = \{1 \quad 3 \quad \underset{\uparrow}{4} \quad 3 \quad 2\}$

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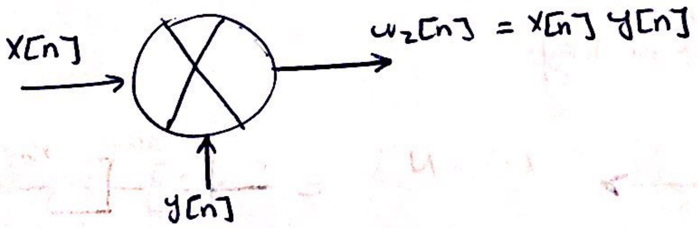
② Multiplication.

* It's very important to fix the indicies.

$w_2[n] = x[n]\, y[n]$

* for the same previous example

$w_2[n] = \{0 \quad 2 \quad \underset{\uparrow}{3} \quad 2 \quad 0\} \equiv \{2 \quad \underset{\uparrow}{3} \quad 2\}$ ← equivalent sequences.

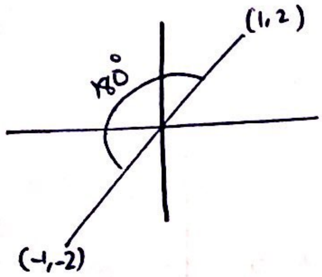


in matlab :
$w_2 = x .* y$
element by element

③ Multiplication by a constant.

$w_3[n] = A\, x[n]$ A: constant (Real number)

$|A|$ → Amplification
→ Attenuation



$A = -1$
$x = \{1 \quad 2\}$
$w_3 = \{-1 \quad -2\}$

→ which means 180° phase-shift.

(4) Time-Delay

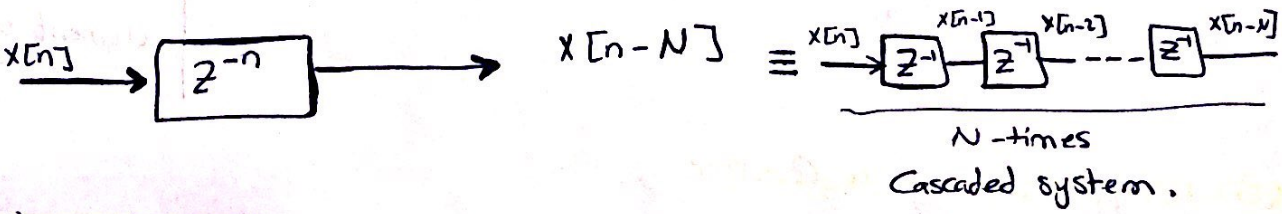
$w_3[n] = x[n-1]$

$x[n] = \{\underset{-2}{1} \quad \underset{-1}{2} \quad \underset{\substack{\uparrow \\ n=0}}{3} \quad \underset{n=1}{1}\}$

$$\left.\begin{aligned}
w_3[-2] &= x[-2-1] = x[-3] = 0 \\
w_3[-1] &= x[-1-1] = x[-2] = 1 \\
w_3[0] &= x[0-1] = x[-1] = 2 \\
w_3[1] &= x[1-1] = x[0] = 3 \\
w_3[2] &= x[2-1] = x[1] = 1
\end{aligned}\right] \rightarrow w_3[n] = \{1 \quad \underset{\uparrow}{2} \quad 3 \quad 1\}$$

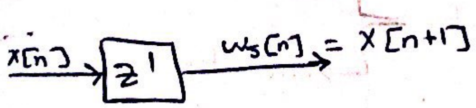
نفس الأرقام بس تغير مكان السهم
(Time-Delay)

يعبر عن
1-single time delay

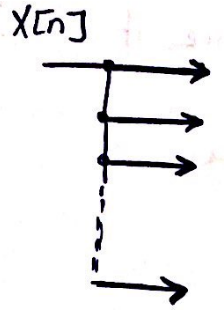


(5) Time-advance

$w_5[n] = x[n+1]$ (non-Causal signal)



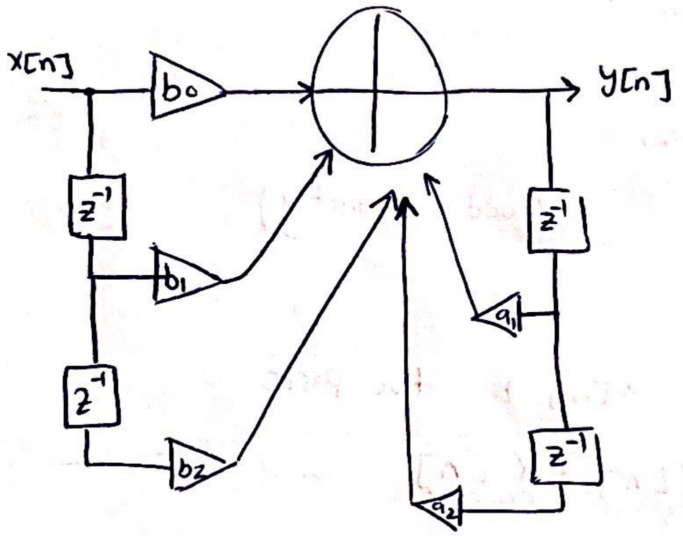
⑥ Pick_off point (نقطة تفرع)



Example:

$$y[n] - a_1 y[n-1] - a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$\hookrightarrow \quad y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$



← schematic of the differance equation

* Classification of Sequences :

[1] Based on symmetry

→ A sequence $x[n]$ is called conjugate symmetric sequence if

$$x_{cs}[n] = x^*_{cs}[-n]$$

cs ← conjugate symmetry

the same value

$(x+jy)^* = x - jy$

if $x[n]$ is real

$$x_{ev}[n] = x_{ev}[-n] \quad \text{(even symmetry)}$$

حالة خاصة من CS

$(re^{j\theta})^* = re^{-j\theta} = r[\cos\theta - j\sin\theta]$

→ A sequence $x[n]$ is called Conjugate anti-symmetric sequence

if $x_{ca}[n] = -x^*_{ca}[n]$

if $x[n]$ is real then,

$$x_{od}[n] = -x_{od}[-n] \quad \text{(odd symmetry)}$$

In general : any sequence $x[n]$ has two parts

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where :

$$x_{cs}[n] = \frac{x[n] + x^*[-n]}{2}$$

$$= \left(\frac{x[n] + x^*[n]}{2}\right)^*$$

$$x_{ca}[n] = \frac{x[n] - x^*[-n]}{2}$$

3.Oct. 2019

$$x_{cs}[n] = \frac{x[n] + x^*[-n]}{2}$$

$$x_{CA}[n] = \frac{x[n] - x^*[-n]}{2}$$

Example

$x[n] = \{1-j \quad 2+j4 \quad \underset{\uparrow}{3-j6} \quad 5\}$

$x^*[-n] = \{5 \quad \underset{\uparrow}{3+j6} \quad 2-j4 \quad 1+j\}$

$x_{cs}[n] \rightarrow$

$1-j$	$2+j4$	$\underset{\uparrow}{3-j6}$	5	0
0	5	$\underset{\uparrow}{3+j6}$	$2-j4$	$1+j$
$1-j$	$7+j4$	$\underset{\uparrow}{6}$	$7-j4$	$1+j$

divide by 2

$x_{cs}[n] = \{0.5-j0.5 \quad 3.5+j2 \quad \underset{\uparrow}{3} \quad 3.5-j2 \quad 0.5+j0.5\}$

$x_{CA}[n] \rightarrow$

طرح

$1-j$	$2+j4$	$\underset{\uparrow}{3-j6}$	5	0
0	5	$\underset{\uparrow}{3+j6}$	$2-j4$	$1+j$
$1-j$	$-3+j4$	$\underset{\uparrow}{-j12}$	$3+j4$	$-1-j$

divide by 2

$x_{CA}[n] = \{0.5-j0.5 \quad -1.5+j2 \quad \underset{\uparrow}{-j6} \quad -1.5+j2 \quad -0.5-j0.5\}$

* Periodic and Aperiodic Sequences :

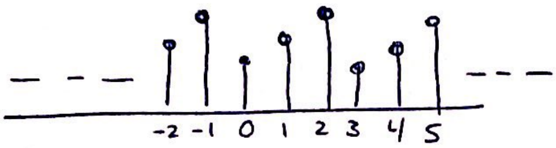
A sequence $x[n]$ is periodic if $x[n] = x[n+KN]$

for all n $(-\infty < n < \infty)$

K: an integer

N: positive integer (the period of the periodical sequence)

↳ N_f



$x[n] = x[n+3]$

* fundemental period أصغر period بتحققها

↳ N_f: The least N that satisfies the periodicity condition.

* The Addition of two or more periodical sequences is also periodic.

$\tilde{x}[n]$ → periodic

$$\tilde{x}_c[N] = \tilde{x}_a[n] + \tilde{x}_b[n]$$

$$N_c = \underbrace{LCM}_{\text{Least common multiplier}}(N_a, N_b)$$

↳ Least common multiplier.

$$LCM(N_a, N_b) = \frac{N_a N_b}{GCD(N_a, N_b)}$$

Example $N_a = 3$ $N_b = 4$

$$N_c = LCM(3,4) = \frac{3 \times 4}{1} = 12$$

$N_a = 6$ $N_b = 4$

$$N_c = \frac{2 \times 3 \times 2 \times 2}{2} = 12$$

* The same Rule applies for multiplication of sequences.

* Energy and Power Sequences :

The total amount of energy in a sequence is defined by :

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$|x+jy|^2 = x^2+y^2 \neq (x+jy)^2$$

$$(x+jy)(x-jy)^*$$

Example

$$x[n] = \begin{cases} \frac{1}{n} & , n>1 \\ 0 & , n<1 \end{cases}$$

$$\mathcal{E}_{x_1} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

finite ⇐ energy sequence

Example

$$x_2[n] = \begin{cases} \frac{1}{\sqrt{n}} & n \geq 1 \\ 0 & n<1 \end{cases}$$

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{(diverges)}$$

$\boxed{\text{finite sequence} \rightarrow \text{Energy}}$

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* The Average power in a sequence x[n] is defined as :

$$P_x = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=-k}^{k} |x[n]|^2$$

Example $x[n] = \begin{cases} 3(-1)^n & , n \geq 0 \\ 0 & , n<0 \end{cases}$ find P_x :

$$P_x = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=0}^{k} (3(-1)^n)^2$$

$$P_x = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=0}^{k} 9$$

$$= \lim_{k\to\infty} \frac{1}{2k+1} 9(k+1) = \lim_{k\to\infty} \frac{9k+9}{2k+1}$$

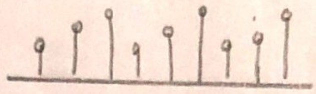
$$= \frac{9}{2} = 4.5 \text{ (watt)}$$

$\frac{\infty^2}{\infty^1}$ → بناخد المعاملات ازا بدنا ال Power

* for Periodic Sequences :

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

$$= \frac{1}{3} \sum_{n=0}^{2} |\tilde{x}[n]|^2$$



$$= \frac{1}{3} \times \left[1^2 + 2^2 + 3^2\right] = \frac{14}{3} \text{ (watt)}$$

* other types of classifications :

[*] Bounded Sequence

$$|x[n]| \leq B_x < \infty$$

[*] Absolutely summable sequence :

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

[*] Sequence summable (Energy sequence)

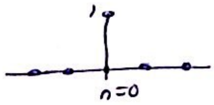
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty = \mathcal{E}_x$$

* Typical frequences sequence Representation :-

(1) Unit Sample Function (Unit Impulse)

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

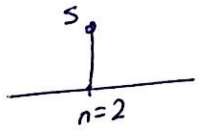
$d(t) = \delta \longrightarrow$



$$\int_{-\infty}^{\infty} d(t) \cdot dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$$

$$\delta[n-k] = \begin{cases} 1 & n=k \\ 0 & n \neq k \end{cases}$$

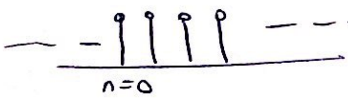


$5\,\delta[n-2]$

(2) Unit step sequence

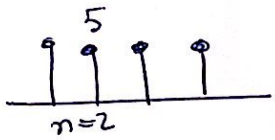
$$u_n[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$x[n] \xrightarrow{A} Ax[n]$



$\longrightarrow \mu_n = \sum_{n=0}^{\infty} \delta[n-m]$

$5\,\mu[n-2] \longrightarrow$



$$\mu[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$$

$$\mu[0] = \delta[0] + \delta[-1] + \delta[-2] + \cdots$$

$$\mu[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Example

$$\mu[-10] = \sum_{k=-\infty}^{-10} \delta(k) = 0$$

$$\delta[-\infty] + \text{---} \ \delta[-1] + \delta[-10] = 0$$

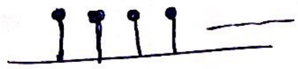
$$\mu[0] = \sum_{k=-\infty}^{0} \delta(k)$$

$$\delta[-\infty] + \text{---} \ \delta[-1] + \delta[0] = 1$$

$$\mu[5] = \sum_{k=-\infty}^{5} \delta[k] = 1$$

sol $\rightarrow$ $$\underbrace{\delta[-\infty] + \text{---} + \delta[-1]}_{\text{zero}} + \underbrace{\delta[0]}_{1} + \underbrace{\delta[1] + \text{---} \ \delta[5]}_{\text{zero}}$$

n=0



n=1

* Sinusoidal Sequences :-

$$x[n] = A\cos(\omega n + \phi)$$

$$x[n] = x[n+N]$$

$$x[n+N] = A\cos(\omega(n+N) + \phi)$$

$$= A\cos(\omega n + \phi + \omega N)$$

$$= A\left[\cos(\omega n+\phi)\cos(\omega N) - \sin(\omega n+\phi)\sin(\omega N)\right]$$

$\rightarrow$ $\cos(\omega N) = 1$

$\sin(\omega N) = 0$

$\omega N = 2\pi r$

$\rightarrow \boxed{\dfrac{2\pi}{\omega} = \dfrac{N}{r}}$ $\longrightarrow$ $\dfrac{\text{int}}{\text{int}} \rightarrow$ rational # $\longrightarrow$ sinusoidal is periodic

$\searrow$ if $\dfrac{N}{r}$ is irrational number $\longrightarrow$ Aperiodic sinusoidal

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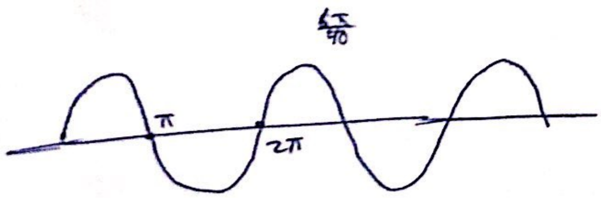
* Ex:- $x[n] = A\cos(0.15\pi n)$

$\omega = 0.15\pi$

$\dfrac{2\pi}{\omega} \rightarrow$ $\dfrac{2\cancel{\pi}}{0.15\cancel{\pi}} = \dfrac{2}{0.15} = \dfrac{200}{15} = \dfrac{40}{3}$ (rational number)

$\rightarrow$ 3 sinusoidals

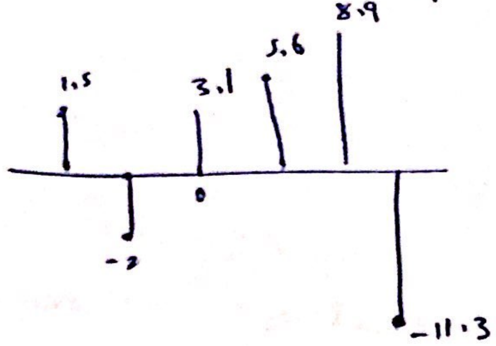


Ex $\rightarrow$ $x[n] = A\cos(0.15n)$

$\dfrac{2\pi}{0.15} = \dfrac{200\pi}{15} = \dfrac{40\pi}{3}$ (irrational number)

Ex

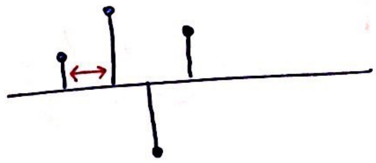
$$x[n] = \{1.5 \quad -2.0 \quad \underset{\uparrow}{3.1} \quad 5.6 \quad 8.9 \quad -11.3\}$$



$$x[n] = 1.5\,\delta[n+2] - 2\,\delta[n+1] + 3.1\,\delta[n] + 5.6\,\delta[n-1]$$
$$+ 8.9\,\delta[n-2] - 11.3\,\delta[n-3]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$$

* The Sampling process:



$$x_a(t)\Big|_{t=nT} = x_a(nT) = x[n]$$

$$x_a(t) = A\cos(\Omega t)\Big|_{t=nT}$$

$$x_a(t)\Big|_{t=nT} = A\cos(\Omega nT) = x[n]$$
$$= A\cos(\omega n)$$

$$\boxed{\omega = \Omega T}$$

discrete — continuous — sampling period

$$x_a(t) = A\cos(\Omega t)$$

① $t = [0 : \overset{T}{0.1} : 1]$ → 11 points

قسمنا (1)sec لـ 10 أجزاء

$T = \frac{1}{10} \rightarrow f = 10$ Hz

(10 samples per sec)

طريقتين للتعبير

$$x[n] = A\cos(0.1\, n\, \Omega)$$
$$= A\cos(\underbrace{0.1\,\Omega}_{\omega}\, n)$$

$n = [0 : 1 : 10]$

② $t = [0 : \underset{T}{0.01} : 1]$ → 101 points

$n = [0 : 1 : 100]$

Example

$x_1(t) = \cos(\overset{\Omega}{6\pi} t)$: $f_1 = 3$ Hz

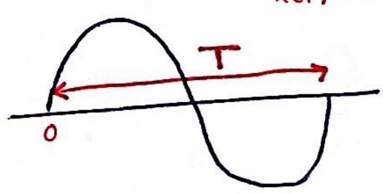
$x_2(t) = \cos(14\pi t)$: $f_2 = 7$ Hz

$x_3(t) = \cos(26\pi t)$: $f_3 = 13$ Hz

$$\boxed{\Omega = 2\pi f}$$

$$\boxed{\Omega = \frac{2\pi}{T}}$$

$x(t) = A\cos(\Omega t)$



Sampled at 10 Hz (10 samples/sec)

$$\boxed{T = 0.1 \text{ sec}} \qquad \boxed{\omega = \Omega T}$$

$n = [0 : 1 : 10]$

$t = [0 : 0.1 : 10]$

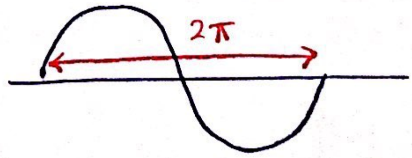
$x_1[n] = \cos(0.6\pi n)$

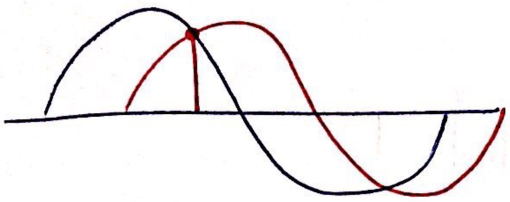
$x_2[n] = \cos(1.4\pi n) \rightarrow \cos((2 - 0.6)\pi n)$

$= \cos(2\pi n - 0.6\pi n)$

$\cos(\underbrace{2\pi n}_{1})\cos(0.6\pi n) + \underbrace{\sin(2\pi n)}_{0}\sin(0.6\pi n)$

$= \cos(0.6\pi n) = x_1[n]$

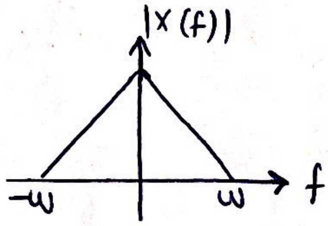




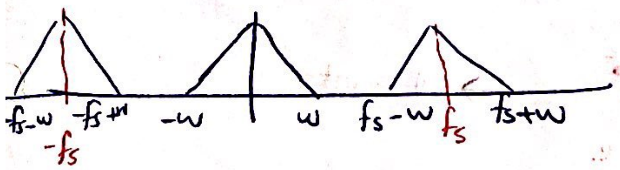
* كأنه ال 2 freq تقاطعوا مع بعض ف نقطة واحدة
عشان هيك طلع الهم نفس القيمة .

$$x_3[n] = \cos(2.6\pi n)$$
$$= \cos(2\pi n + 0.6\pi n)$$
$$= \cos(0.6\pi n) = x_1[n]$$

* يعني ال 3 signals كان الهم freq مختلف بال Time domain بس لما عملت sampling بينوا كأنهم متطابقين .

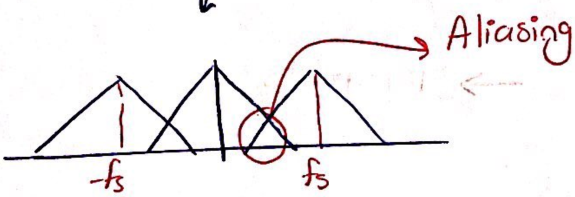


Sampling → Repeat the signal (spectrum) around the sample freq (f_s)

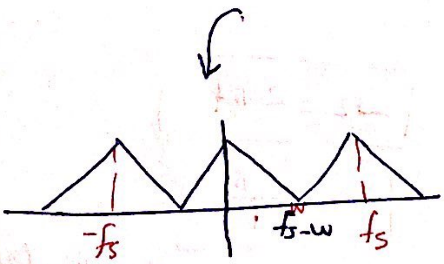


→ Train of deltas .

عشان هيك بنحط LPF عند ال Reciever



→ لما نصغر قيمة f_s بتصير Aliasing



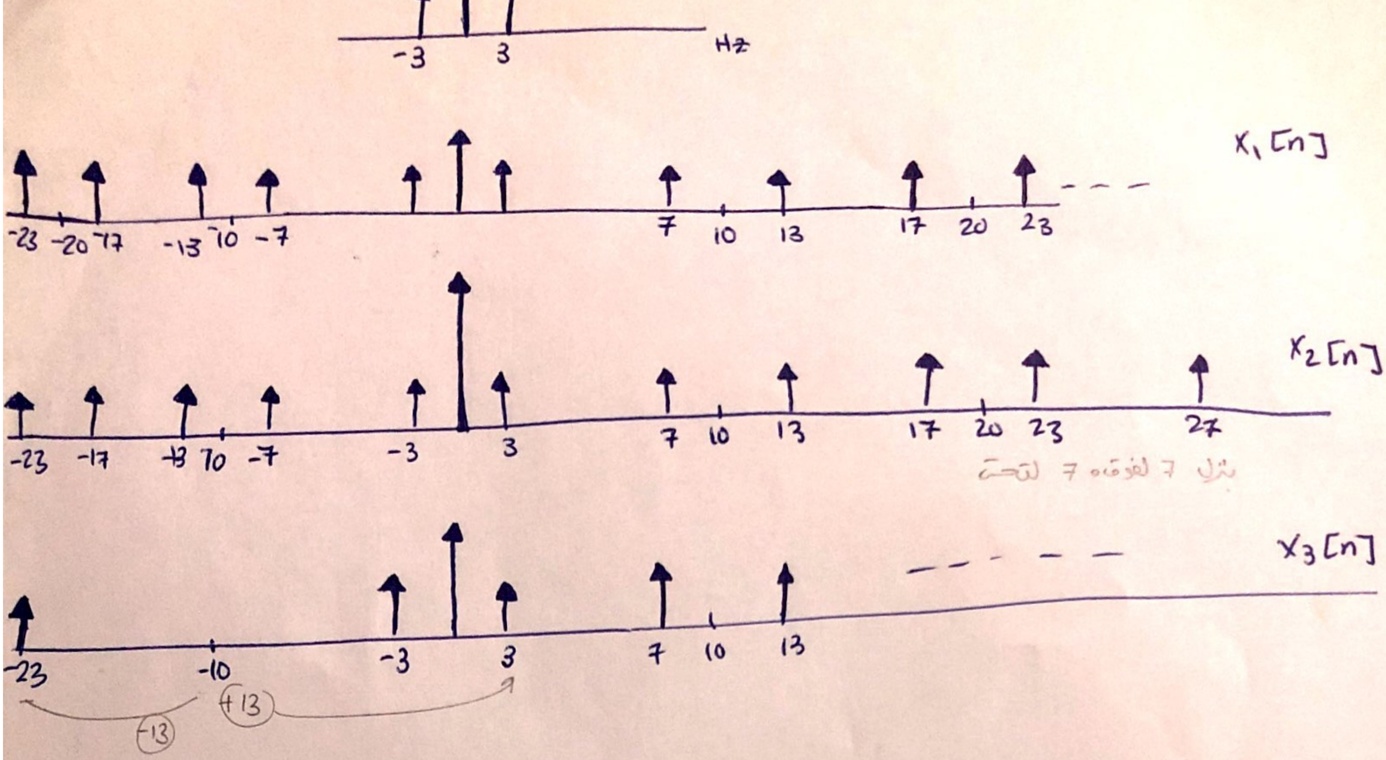
→ $w = f_s - w$

$$\boxed{f_s = 2w}$$

//

6.Oct.2019

$\cos(6\pi t)$



3 Hz
7 Hz
13 Hz
17 Hz
23 Hz

Sampling freq

10 Hz → $k f_s \pm f_m$

10 ± 3 Hz

7 $\overset{-3}{\curvearrowleft}$ 10 $\overset{+3}{\curvearrowright}$ 13

17 $\overset{-3}{\curvearrowleft}$ 20 $\overset{+3}{\curvearrowright}$ 23

بناءً ع
Sampling freq
بتتكرر الإشارة

→ $\cos(46\pi t)$ $\cos(23 \times 2\pi t)$

→ $\cos(4.6\pi n)$

$\cos(\cancel{4\pi n})^{1}\cos(0.6\pi n) - \sin(\cancel{4\pi n})^{0}\sin(0.6\pi n)$

10.10.2019

* Discrete Time Systems:

$$x[n] \longrightarrow \boxed{T[\cdot]} \xrightarrow{T[x[n]] = y[n]}$$

$$x[n] \longrightarrow \boxed{T[\cdot]} \xrightarrow{y[n] = x[n] - 0.5\, x[n-1]}$$

$$x[n] \longrightarrow \boxed{T[\cdot]} \xrightarrow{y[n]} = |x[n]|$$

Classification of Discrete-time systems:

$\boxed{1}$ linear systems:-

$$\alpha x_1[n] + \beta x_2[n] \xrightarrow{T[\cdot]} \alpha y_1[n] + \beta y_2[n]$$

$$T\left[\alpha x_1[n] + \beta x_2[n]\right] = \alpha\, T\left[x_1[n]\right] + \beta\, T\left[x_2[n]\right]$$

$$= \alpha\, y_1[n] + \beta y_2[n]$$

Ex: 1) $y[n] = n\, x[n]$

2) $y[n] = x^2[n]$

3) $y[n] = A\, x[n] + B$

1) $x_1[n] \xrightarrow{T[\cdot]} n\, x_1[n] = y_1[n]$

$x_2[n] \xrightarrow{T[\cdot]} n\, x_2[n] = y_2[n]$

$$x[n] = \alpha x_1[n] + \beta x_2[n]$$

$$x[n] = \alpha x_1[n] + \beta x_2[n] \xrightarrow{T[\cdot]} n\, x[n]$$

$$n\, x[n] = n[\alpha x_1[n] + \beta x_2[n]]$$

$$= \alpha n\, x_1[n] + \beta n\, x_2[n]$$

$$= \alpha y_1[n] + \beta y_2[n] \longrightarrow \text{linear system}$$

2) $x_1[n] \xrightarrow{T[\cdot]} x_1^2[n] = y_1[n]$

$x_2[n] \xrightarrow{T[\cdot]} x_2^2[n] = y_2[n]$

$\alpha x_1[n] + \beta x_2[n] \xrightarrow{T[\cdot]} (\alpha x_1[n] + \beta x_2[n])^2$

$\alpha^2 x_1^2[n] + 2\alpha\beta\, x_1[n]\, x_2[n]$

$+ \beta^2 x_2^2[n] \neq \alpha y_1[n] + \beta y_2[n]$

non-linear system

2) Time Invarient (Shift Invarient) System:

$x[n] \xrightarrow{T[\cdot]} y[n]$

$x[n-K] \xrightarrow{T[\cdot]} y[n-K]$

* Check for Invariance :-

1- Apply $x[n] \xrightarrow{T[\cdot]} y[n]$

2- Apply $x_1[n] = x[n-K] \xrightarrow{T[\cdot]} y_1[n]$

3- Shift $y[n]$ in ① by K if

$y[n-K] = y_1[n]$ ⟶ Time Invariant system.

Example 1) $y[n] = x[n] - x[n-1]$

2) $y[n] = n\,x[n]$

1) ① $y[n] = x[n] - x[n-1]$

② $x_1[n] = x[n-K]$

$y_1[n] = x[n-K] - x[n-K-1]$

③ $y[n-K] = x[n-K] - x[n-K-1] \stackrel{?}{=} y_1[n]$

↳ So, Time Invariant system.

//

10.10.2019

2) ① $y[n] = n\,x[n]$

② $x_1[n] = x[n-k]$

$y_1[n] = n\,x_1[n] = n\,x[n-k]$ $\neq$ → Time Variant System

③ $y[n-k] = (n-k)\,x[n-k]$

$= x[n-k] + n$

$y_1[n] = x[n] + n$

$y[n-k] = x[n-k] + (n-k)$

[3] Causal system :-

$$y[n] = T[x[n],\ x[n-1],\ x[n-2],\ ----]$$

Ex: 1) $y[n] = x[n] + 0.5\,x[n-1]$ → Causal

2) $y[n] = x[-n]$ → Non-Causal → لما نعوض 10 بيطلع $x[-10]$ فهاد اذا — صحيح بس لما نعوض -10 بيطلع $x[10]$ → future value X غلط

[4] Stable System :-

A Discrete time is stable if for every bounded input, the output is also bounded (BIBO)

(BIBO) stable system ⟶ Bounded-input, Bounded-output

Bounded means:

$|x[n]| < B_x < \infty$ for all n, then $|y[n]| < B_y < \infty$

Example ① $y[n] = e^{x[n]}$

② $y[n] = \sum_{k=0}^{n} x[k]$ → $y[10] = x[0] + x[1] + --- x[10]$

sol: $|y[n]| = |e^{x[n]}| \leq e^{x[n]} \leq e^{B_x}$

$|y[n]| \leq e^{B_x}\ B_y$ ⟶ BIBO stable system

Sol: ② $|y[n]| = \left| \sum_{k=0}^{n} x[k] \right|$. ← أول خطوة بأخذ ال absolute value لنفس ال expression بالسؤال

$$|y[n]| = \left| \sum_{k=0}^{n} x[k] \right| \leqslant \sum_{k=0}^{n} |x[k]| \leqslant \sum_{k=0}^{n} B_x$$

$$|y[n]| \leqslant \sum_{k=0}^{n} B_x = (n+1)(B_x)$$

↳ Astable system (Non-stable)

---

5) Static and Dynamic systems

(Memoryless)

$y[n] = \frac{1}{2} x[n]$ (static)

$y[n] = x[n] - \frac{1}{2} x[n-1]$ (Dynamic)

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- Passive and lossless systems :-

A discrete-time system is said to be passive if for every finite energy input sequence $x[n]$, the output $y[n]$ will have at most the same energy.

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\mathcal{E}_x} \geq \underbrace{\sum_{n=-\infty}^{\infty} |y[n]|^2}_{\mathcal{E}_y}$$

~~lossy~~ lossless system بكون = كان !!

Ex: $y[n] = 5x[n]$

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |5x[n]|^2$$

$$\mathcal{E}_y = 25 \sum_{n=-\infty}^{\infty} |x[n]|^2 = 25\, \mathcal{E}_x \quad \text{(Active system)}$$

Ex: $y[n] = \frac{1}{2} x[n]$ → passive lossy

$$\mathcal{E}_y = \frac{1}{4} \mathcal{E}_x$$

Ex: $Y[n] = x[n-1]$

$\mathcal{E}_y = \mathcal{E}_x$ → passive and lossless.

Impulse and Step Responses :-

$x[n] = \delta[n]$ $\qquad$ $y[n] = h[n]$

$x[n] = \mu[n]$ $\qquad$ $y[n] = \overset{\text{step}}{S[n]}$

$$h[n] = T[\delta[n]]$$

$$S[n] = T[\mu[n]]$$

Ex: $y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + a_3 x[n-3]$

Find $h[n]$:- $\rightarrow$ Impulse Response.

Sol: $h[n] = a_0 \delta[n] + a_1 \delta[n-1] + a_2 \delta[n-2] + a_3 \delta[n-3]$

$h[0] = \underline{a_0 \delta[0]} + a_1 \delta[-1] + a_2 \delta[-2] + a_3 \delta[-3]$

$h[0] = a_0$

$h[1] = a_0 \delta[1] \underline{+ a_1 \delta[0]} + a_2 \delta[-1] + a_3 \delta[-2]$

$h[1] = a[1]$

$h[2] = a_2$

$h[3] = a_3$

$h[4] = 0$

$\vdots$

So, $h[n] = \{\underset{\uparrow}{a_0} \quad a_1 \quad a_2 \quad a_3\} \rightarrow$ finite sequence

FIR system $\rightarrow$ Finite Impulse Response System.

Ex: $y[n] = x[n] + \frac{1}{2}y[n-1]$

Find $h[n]$. (causal)

$$h[n] = \delta[n] + \frac{1}{2}h[n-1]$$

$$h[0] = \delta[0] + \cancel{\frac{1}{2}h[-1]}^{\,0}$$

بحتاج initial conditions وعشانه نعرف كل

$$h[0] = 1$$

$$h[1] = \cancel{\delta[1]}^{\,0} + \frac{1}{2}h[0]$$

$$h[1] = \frac{1}{2} * 1 = \frac{1}{2}$$

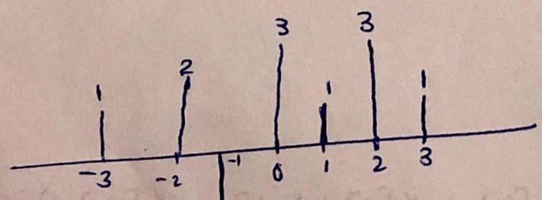
$$h[2] = \delta[2] + \frac{1}{2}h[1]$$

$$h[2] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$h[3] = \frac{1}{8}$$

$$\text{So, } h[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases} \equiv h[n] = \left(\frac{1}{2}\right)^n \mu[n]$$

IIR → Infinite Impulse Response System



$$x[n] = \underbrace{1\,\delta[n+3]}_{x[-3]} + \underbrace{2\,\delta[n+2]}_{x[-2]} + \underbrace{(-1)\,\delta[n+1]}_{x[-1]} + \underbrace{3\,\delta[n]}_{x[0]} + \underbrace{\delta[n-1]}_{x[1]} + \underbrace{3\,\delta[n-2]}_{x[2]} + \underbrace{\delta[n-3]}_{x[3]}$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$x[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n]$$

$$y[n] = T\left[\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right]$$

If the system is linear $\longrightarrow$ $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,T[\delta[n-k]]$

* دخلناها جوا لأنه linear

ينتج عنه تعطي $h[n,k]$

For a time-Invariant (shift-Invariant) system.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \rightarrow$$ The convolution Sum

$n-k=m$ ↳ shift → multiply → add →

system.

LTI system: linear time-Invariant system

اذا ما في overlap بطلع ال output = zero

$$y[n] = x[n] \circledast h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$y[n] = h[n] \circledast x[n]$$: Commulative

$x[0]$ (by default أول عنصر)

Ex: $x[n] = \{1\ \ 2\ \ 0\ \ 1\}$

$h[n] = \{1\ \ 1\ \ 2\ \ 2\}$

Find $y[n] = x[n] \circledast h[n]$

Sol: $$y[n] = \sum_{k=0}^{3} x[k]\, h[n-k]$$

$$y[0] = \sum_{k=0}^{3} x[k]\, h[-k]$$

$$= x[0]h[0] + x[1]\,\cancel{h[-1]} + x[2]\,\cancel{h[-2]} + x[3]\,\cancel{h[-3]}$$

$$y[0] = (1)(1) = 1$$

$$y[1] = \sum_{k=0}^{3} x[k]\, h[1-k]$$

$$= x[0]h[1] + x[1]h[0] + \cancel{x[2]h[-1]} + \cancel{x[3]h[-2]}$$

$$= (1)(1) + (2)(1) + 0 + 0$$

$$= 3$$

بتوصل ل $y(6)$ $y[2]$

عدد نقاط x + عدد نقاط h

-1

$= 7$

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LTI system: linear time - Invarient system

$$y[n] = x[n] \circledast h[n]$$

$$y[n] = \sum_{K=-\infty}^{\infty} h[K]\, x[n-K]$$

$$y[n] = h[n] \circledast x[n]$$: commulative

Ex:

$x[n] = \{\underset{0}{1} \quad \underset{1}{2} \quad \underset{2}{0} \quad \underset{3}{1}\}$, find $y[n] = x[n] \circledast h[n]$

$h[n] = \{\underset{0}{1} \quad \underset{1}{1} \quad \underset{2}{2} \quad \underset{3}{2}\}$

$$y[n] = \sum_{K=0}^{3} x[K]\, h[n-K]$$

$$y[0] = \sum_{K=0}^{3} x[K]\, h[-K]$$

$$y[n] = x[0]\,h[n] + x[1]\,h[-1] + x[2]\,h[-2] + x[3]\,h[-3]$$

$$y[0] = (1)(1) = 1$$

$$y[1] = x[0]\,h[1] + x[1]\,h[0] + x[2]\,h[-1] + x[3]\,h[-2]$$

$$= (1)(1) + (2)(1) = 3$$

⋮

$$y[n] = \{1 \quad 3 \quad 4 \quad 7 \quad 5 \quad 2 \quad 2\}$$

Ex:- $x[n] = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}$, find $y[n]$ Graphically.

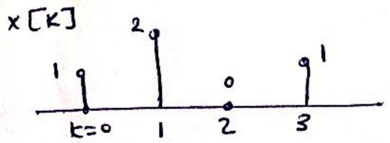
$h[n] = \{\underset{0}{1} \quad \underset{1}{1} \quad \underset{2}{2} \quad \underset{3}{2}\}$

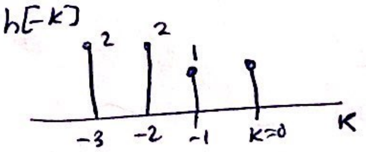
$$y[n] = \sum_{K=0}^{3} x[K]\, h[n-K]$$

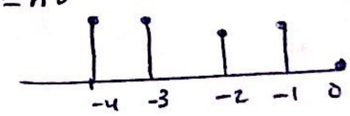
$z[K] = h[-1-K]$

$z[-4] = h[-1+4] = h[3]$

$z[-3] = h[-1+3] = h[2]$

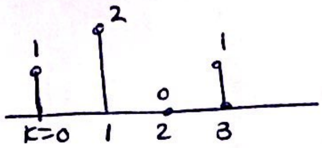






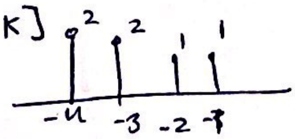
$y[-1] =$

$x[K]$



so, $y[-1]=0$

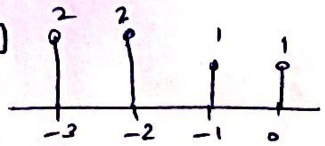
$h[-1-K]$



$y[0]=1$

$y[0] =$

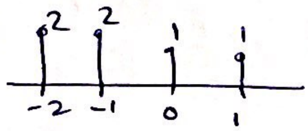
$h[0-K]$



$y[0]=1$

$y[1] =$

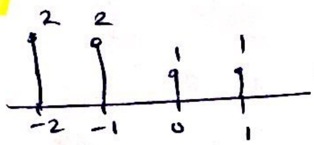
$h[1-K]$



$y[1] = 1+2 = 3$

$y[2]$

$h[-K+2]$



$y[2] = 4 \rightarrow 2+2$

* another way for solution :-

Tabular method :

n	0	1	2	3	4	5	6
x[n]	1	2	0	1			
h[n]	1	1	2	2			
	1	2	0	1			
+	–	1	2	0	1		
	–	–	2	4	0	2	
	–	–	–	2	4	0	2

$y[n] = \{1\ 3\ 4\ \underset{\uparrow}{7}\ 5\ 2\ 2\}$

15. oct. 2019

Ex: $x[n] = \{1 \quad \underset{\uparrow}{2} \quad 0 \quad 1\}$

$h[n] = \{1 \quad 1 \quad \underset{\uparrow}{2} \quad 2\}$

$N(\text{length}) = N_1 + N_2 - 1$

starting index 1 + starting index 2 = starting index for the output

Start of midterm material

Discrete-Time Fourier Transform (DTFT)

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

Ex: $x[n] = \{1 \quad \underset{\uparrow}{2} \quad -3 \quad 1\}$

$$\boxed{e^{jw} = 1\angle w}$$

$$= \cos w + j\sin w$$

$$= \sqrt{\cos^2 w + \sin^2 w}\ \angle \tan^{-1}\left(\frac{\sin w}{\cos w}\right)$$

$$= 1 \angle w$$

$$X(e^{jw}) = 1e^{jw} + 2e^{jw(0)} - 3e^{-jw} + 1e^{-j2w}$$

$$X(e^{jw}) = e^{jw} + 2 - 3e^{-jw} + e^{-j2w}$$

$$X(e^{j\frac{\pi}{3}}) = e^{j\frac{\pi}{3}} + 2 - 3e^{-j(\frac{\pi}{3})} + e^{-j2(\frac{\pi}{3})}$$

Ex: $x[n] = \delta[n]$

find $\Delta(e^{jw}) \rightarrow = \underbrace{\sum_{n=\infty}^{\infty} \delta[n]\, e^{-jwn}}_{\text{sifting property}} = e^{-jw(0)} = 1$

sifting property $\rightarrow \delta[0]$, $n=0$

Geometric Series :-

$$S_N = \sum_{n=0}^{N} \alpha^n = 1 + \alpha + \alpha^2 + \alpha^3 + \text{------} \alpha^{N-1} + \alpha^N$$

$$S_{N+1} = \sum_{n=0}^{N} \alpha^n = 1 + \alpha + \alpha^2 + \alpha^3 + \text{------} \alpha^N + \alpha^{N+1}$$

$$\text{so,} \quad S_{N+1} = S_N + \alpha^{N+1}$$

$$S_{N+1} = 1 + \alpha S_N$$

$$S_N + \alpha^{N+1} = 1 + \alpha S_N$$

$$S_N(1-\alpha) = 1 - \alpha^{N+1}$$

$$S_N = \frac{1-\alpha^{N+1}}{1-\alpha}$$

$$\sum_{n=0}^{5} (2)^n = \frac{1-2^6}{1-2} = \frac{-63}{-1} = 63$$

$$1 + 2 + 4 + 8 + 16 + 32 = 63$$

$$\sum_{n=1}^{5} (2)^n = \sum_{n=0}^{5} (2)^n - 1$$

$$\sum_{n=0}^{5} \left(\tfrac{1}{2}\right)^n = \frac{1-\left(\frac{1}{2}\right)^6}{1-\frac{1}{2}} = ?$$

For $|\alpha| < 1$ and $N \longrightarrow \infty$

$$S_\infty = \frac{1}{1-\alpha}$$

Ex:- $x[n] = \left(\tfrac{1}{2}\right)^n \mu[n]$, find $X(e^{jw})$

Sol: $X(e^{jw}) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n e^{-jwn}$ $\quad \left|\tfrac{1}{2}\right| \left(e^{-jw}\right) < 1$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} e^{-jw}\right)^n$$

$$X(e^{jw}) = \frac{1}{1 - \frac{1}{2} e^{-jw}}$$

[illegible] (n)

index = −1

$x(-1)$ [illegible]

Ex: $X(e^{jw}) = e^{jw} + 2 - 3e^{-jw} + e^{-j2w}$ while $x[n] = \{1, \underset{\uparrow}{2}, -3, 1\}$

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{jw})$$

//

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

* Two properties for the DTFT

1- continuous function in ω → while $x[n]$ is discrete function.

2- $X(e^{j\omega})$ is periodic with a period of 2π

$$\cos(\cancel{2\pi n}) + j\sin(\cancel{2\pi n})$$

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

always = 1

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, e^{-j2\pi n} = \sum x[n]\, e^{-j\omega n}$$

$$= X(e^{j\omega})$$

$$\boxed{e^{-j\pi n} = (-1)^n}$$

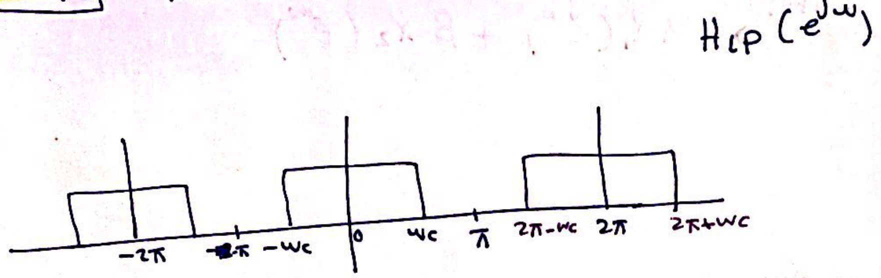
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} . d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$\boxed{\text{Example}}$ The Ideal lowpass Filter



$H_{LP}(e^{j\omega})$; The transfer function

$$H_{LP}(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & 0 \leq |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

** Find $H_{LP}[n]$.

$$H_{LP}[n] = \frac{1}{2\pi}\int_{-w_c}^{w_c} (1)\, e^{j\omega n} . d\omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j\omega n}}{jn} \right]_{-w_c}^{w_c}$$

$$H_{LP}[n] = \frac{e^{j w_c n} - e^{-j w_c n}}{2\pi j n} = \frac{\sin(w_c n)}{\pi n} \quad \text{for all } n$$

for $n=0$

$$h_{LP}[0] = \frac{1}{2\pi}\int_{-w_c}^{w_c} 1 .\, e^{j0}\, d\omega = \left. \frac{\omega}{2\pi} \right]_{-w_c}^{w_c} = \frac{2w_c}{2\pi} = \frac{w_c}{\pi}$$

$$h_{LP}[n] = \begin{cases} \frac{w_c}{\pi} & n=0 \\ \frac{\sin w_c n}{\pi n} & n \neq 0 \end{cases} \quad \text{for all } n .$$

22 Oct. 2019

* Theorem of the DTFT :-

$$x_1[n] \xleftrightarrow{DTFT} X_1(e^{jw})$$

$$x_2[n] \xleftrightarrow{DTFT} X_2(e^{jw})$$

[1] linearity :-

$$\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{DTFT} \alpha X_1(e^{jw}) + \beta X_2(e^{jw})$$

[2] Time Reversal :

$$x_1[-n] \xleftrightarrow{DTFT} X_1(e^{-jw})$$

$$x[n] = \{1 \quad \underset{\uparrow}{2} \quad -3 \quad 1\}$$

$$X(e^{jw}) = e^{jw} + 2 - 3e^{-jw} + e^{-j2w} \rightarrow X(\cancel{e^{jw}}) = e^{-jw} + 2 - 3e^{jw} + e^{j2w}$$

$$= X(e^{-jw})$$

$$g[n] = x[-n] = \{1 \quad -3 \quad \underset{\uparrow}{2} \quad 1\}$$

$$G(e^{jw}) = e^{j2w} - 3e^{jw} + 2 + e^{-jw}$$

[3] Time-Shifting Theorem :-

$$x_1[n-n_0] \xleftrightarrow{DTFT} e^{-jwn_0} X_1(e^{jw})$$

$$\sum_{n=-\infty}^{\infty} x_1[n-n_0]\, e^{-jwn}$$

let $m = n - n_0$

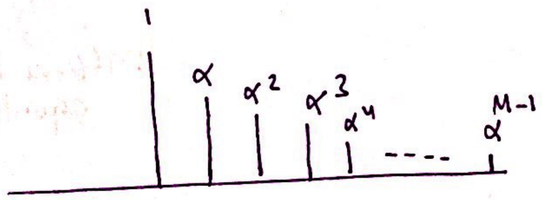
$n = m + n_0$

$$\sum_{m=-\infty}^{\infty} x_1[m]\, e^{-jw(m+n_0)}$$

$$e^{-jwn_0} \sum_{m=-\infty}^{\infty} x_1[m]\, e^{-jwm}$$

22.oct.2019

Ex:- $y[n] = \begin{cases} \alpha^n & 0 \le n \le M-1 \\ 0 & \text{Otherwise} \end{cases} \qquad |\alpha| < 1$



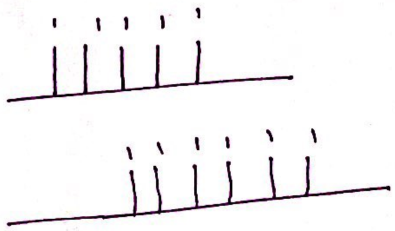
بيلزم باكل

$x[n] = \alpha^n \mu[n] \qquad |\alpha| < 1$

$$X(e^{jw}) = \sum_{n=0}^{\infty} \alpha^n e^{-jwn} = \sum_{n=0}^{\infty} (\alpha e^{-jw})^n$$

$$= \frac{1}{1-\alpha e^{-jw}}$$

$$y[n] = \alpha^n \left[\mu[n] - \mu[n-M]\right]$$



$n=0$

$n=M$

$$y[n] = \alpha^n \mu[n] - \alpha^n \mu[n-M]$$

$$= \alpha^n \mu[n] - \alpha^n \cdot \alpha^{n-M} \mu[n-M]$$

$$Y(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} - \frac{\alpha^M \cdot e^{-jwM}}{1-\alpha e^{-jw}}$$

$$Y(e^{jw}) = \frac{1-(\alpha e^{-jw})^M}{1-\alpha e^{-jw}}$$

$$Y(e^{jw}) = \sum_{n=0}^{M-1} \alpha^n e^{-jwn}$$

$$= \sum_{n=0}^{M-1} (\alpha e^{-jw})^n = \frac{1-(\alpha e^{-jw})}{1-\alpha e^{-jw}}$$

Note :

$$\boxed{\sum_{n=0}^{N} r^n = \frac{1-r^{N-1}}{1-r}}$$

Example : $d_0 V[n] + d_1 V[n-1] + d_2 V[n-2] = P_0 \delta[n] + P_1 \delta[n-1]$

$+ P_2 \delta[n-2]$

Differential equation.

Find $V(e^{jw})$:-

$$d_0 V(e^{jw}) + d_1 e^{-jw} V(e^{jw}) + d_2 e^{-j2w} V(e^{jw}) =$$

$$P_0(1) + P_1 (e^{-jw})(1) + P_2 e^{-j2w} (1)$$

So,
$$V(e^{jw}) = \frac{P_0 + P_1 e^{-jw} + P_2 e^{-j2w}}{d_0 + d_1 e^{-jw} + d_2 e^{-j2w}}$$

Example : $a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] =$

$b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$

Find $H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$; The transfer function (the ratio of the output over the input in the frequency domain)

$x[n] = \delta[n]$

$X(e^{jw}) = \Delta(e^{jw}) = 1$

$$a_0 Y(e^{jw}) + a_1 e^{-jw} Y(e^{jw}) + a_2 e^{-j2w} Y(e^{jw}) =$$

$$b_0 X(e^{jw}) + b_1 e^{-jw} X(e^{jw}) + b_2 e^{-j2w} X(e^{jw})$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{b_0 + b_1 e^{-jw} + b_2 e^{-j2w}}{a_0 + a_1 e^{-jw} + a_2 e^{-j2w}}$$

24.oct. 2019

Frequency shifting :-

$$e^{jw_0 n} x[n] \xleftrightarrow{DTFT} X(e^{j(w-w_0)})$$

Ex: $y[n] = (-1)^n \alpha^n \mu[n]$, $|\alpha| < 1$, find $Y(e^{jw})$

$$y[n] = e^{j\pi n} \alpha^n \mu[n] \quad , \quad |\alpha| < 1$$

$$x[n] = \alpha^n \mu[n] ,$$

$|\alpha| < 1 \rightarrow X(e^{jw}) = \dfrac{1}{1 - \alpha e^{-jw}}$

$$X(e^{jw}) = \frac{1}{1 - \alpha e^{-j(-\pi + w)}} = \frac{1}{1 - \alpha e^{-jw} e^{j\pi}}$$

$e^{j\pi} \rightarrow -1$

$$= \frac{1}{1 + \alpha e^{-jw}}$$

* Alternative Solution :

$$y[n] = (-1)^n \alpha^n \mu[n]$$

$$= (-\alpha)^n \mu[n] \quad , \quad |\alpha| < 1$$

$$Y(e^{jw}) = \frac{1}{1 - -\alpha e^{-jw}} = \frac{1}{1 + \alpha e^{-jw}}$$

* Differentiation in frequency :

$$n\, x[n] \xleftrightarrow{DTFT} j \frac{dX(e^{jw})}{dw}$$

$$\frac{d}{dw} \left[X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} \right]$$

$$\left[\frac{dX(e^{jw})}{dw} = \sum_{n=-\infty}^{\infty} x[n] (-jn) e^{-jnw} \right] \times j$$

$$\frac{j\, dX(e^{jw})}{dw} = \sum_{-\infty}^{\infty} n\, x[n] e^{-jwn}$$

Ex:- $y[n] = (n+1)\alpha^n \mu[n]$, $|\alpha| < 1$

Find $Y(e^{jw})$

$$y[n] = n\alpha^n \mu[n] + \alpha^n \mu[n]$$

$$= n\,x[n] + x[n]$$

where $= \dfrac{1}{1-\alpha e^{-jw}}$

$$\left(\frac{dX(e^{jw})}{dw} = \frac{-1\,(\alpha j e^{-jw})}{(1-\alpha e^{-jw})^2} \right) \times j$$

$$= \frac{\alpha e^{-jw}}{(1-\alpha e^{-jw})^2}$$

$$Y(e^{jw}) = \frac{\alpha e^{-jw}}{(1-\alpha e^{-jw})^2} + \frac{1}{1-\alpha e^{-jw}}$$

$$Y(e^{jw}) = \frac{\alpha e^{-jw} + 1 - \alpha e^{-jw}}{(1-\alpha e^{-jw})^2}$$

$$= \frac{1}{(1-\alpha e^{-jw})^2}$$

Note:

$$e^{-\lambda t}\Big| = e^{-\lambda T n}$$

$t = nT$

$$\boxed{\alpha = e^{-\lambda T}}$$

24.oct.2019

* Convolution Theorem:-

$$x[n] \circledast h[n] \xleftrightarrow{DTFT} X(e^{jw}) \cdot H(e^{jw})$$

Ex:- $x[n] = \{1 \quad 2 \quad 0 \quad 1\}$

$h[n] = \{1 \quad 1 \quad 2 \quad 2\}$, find:

$y[n] = x[n] \circledast h[n]$ using the F.T

$$X(e^{jw}) = 1 + 2e^{-jw} + 0\,e^{-j2w} + e^{-j3w}$$

$$H(e^{jw}) = 1 + e^{-jw} + 2e^{-j2w} + 2e^{-jw}$$

$$\begin{array}{l} = 1 + 2e^{-jw} + 0\,e^{-2jw} + e^{-j3w} \\ \quad - + e^{-jw} + 2e^{-j2w} + 0\,e^{-j3w} + e^{-j4w} \\ \quad - + - + 2e^{-j2w} + 4e^{-j3w} + 0\,e^{-j4w} + 2e^{-j5w} \\ \quad - + - + - + 2e^{-jw3} + 4e^{-j4w} + 0\,e^{-j5w} + 2e^{-j6w} \end{array}$$

$$\rightarrow \sum y[n]\, e^{-jwn}$$

$$Y(e^{jw}) = X(e^{jw}) \cdot H(w)$$

$$= 1 + 3e^{-jw} + 4e^{-j2w} + 7e^{-j3w} + 5e^{-j4w} + 2e^{-j5w} + 2e^{-j6w}$$

$$y[n] = \{\underset{\uparrow}{1} \quad 3 \quad 4 \quad 7 \quad 5 \quad 2 \quad 2\}$$

Example

$x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$

$h[n] = \beta^n \mu[n]$, $|\beta| < 1$

Find $Y[n] = x[n] \circledast h[n]$ using F.T :-

$$X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} , \quad H(e^{jw}) = \frac{1}{1-\beta e^{-jw}}$$

$$Y(e^{jw}) = \frac{1}{(1-\alpha e^{-jw})(1-\beta e^{-jw})}$$

partial Fraction

$$= \frac{A_1}{1-\alpha e^{-jw}} + \frac{A_2}{1-\beta e^{-jw}}$$

$$= \frac{A_1 - A_1\beta e^{-jw} + A_2 - A_2\alpha e^{-jw}}{(1-\alpha e^{-jw})(1-\beta e^{-jw})}$$

Scanned with CamScanner

24.oct.2019

$$A_1 + A_2 = 1 \quad --- \ ①$$

$$A_1\beta + A_2\alpha = 0 \quad --- \ ②$$

$$\beta A_2 - A_2\alpha = \beta$$

$$A_2 = \frac{\beta}{\beta-\alpha}$$

$$\alpha A_1 - \beta A_1 = \alpha$$

$$A_1 = \frac{\alpha}{\alpha-\beta}$$

$$X(e^{j\omega}) = \frac{\frac{\alpha}{\alpha-\beta}}{1-\alpha e^{-j\omega}} + \frac{\frac{\beta}{\beta-\alpha}}{1-\beta e^{-j\omega}}$$

$$Y[n] = \frac{\alpha}{\alpha-\beta}(\alpha)^n \mu[n] + \frac{\beta}{\beta-\alpha}(\beta)^n \mu[n]$$

*** Parsevals Relation :-**

$$\sum_{-\infty}^{\infty} x_1[n] \cdot x_2^*[n] =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega}) \cdot X^*(e^{j\omega})\, d\omega$$

$\rightarrow$ if $x_1[n] = x_2[n]$

$$\underbrace{\sum_{-\infty}^{\infty} |x_1[n]|^2}_{E_x} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_1(e^{j\omega})|^2\, d\omega$$

24.oct.2019

Example :-

$x[n] = \{1\ 2\ 0\ 1\}$

$\sum x = 1^2 + 2^2 + 1^2 = 6$

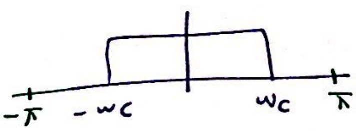
$X(e^{jw}) = 1 + 2e^{-jw} + e^{-j3w}$

$X(e^{jw}) = 1 + 2e^{-jw} + e^{-j3w}$

$|X(e^{jw})| = (1 + 2e^{-jw} + e^{-j3w})(1 + 2e^{jw} + e^{j3w})$

$= 1 + e^{j2w} + e^{j3w} + 2e^{-jw} + 4 + 2e^{jw2} + e^{-j3w} + 2e^{-j2w} + 1$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 + 4 + 1 \, dw = \frac{6(2\pi)}{2\pi} = 6$$

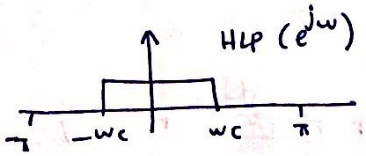


27.10.2019

* Parsevals Relation :-

$$\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{jw})|^2 \, dw$$

Ex :

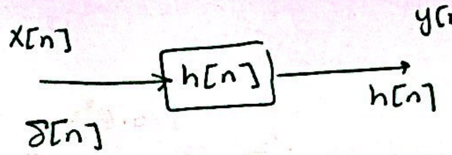


$$h_{LP}[n] = \frac{\sin w_c n}{\pi n} \qquad -\infty < n < \infty$$

$$\sum_{-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-w_c}^{w_c} 1 \cdot dw = \frac{2w_c}{2\pi} = \frac{w_c}{\pi}$$

* Frequency Domain Analysis of LTI system :

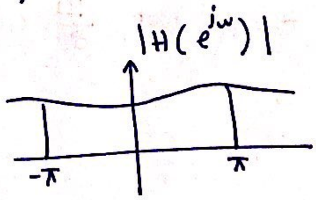
$$y[n] = x[n] * h[n]$$

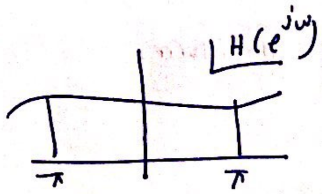


$$Y(e^{jw}) = \frac{X(e^{jw})}{X(e^{jw})} \longrightarrow \text{The Transfer function.}$$

$$1 = \frac{Y(e^{jw_0})}{X(e^{jw_0})} \longrightarrow Y(e^{jw_0}) = X(e^{jw_0})$$

$$0 = \frac{Y(e^{jw_1})}{X(e^{jw_1})} \longrightarrow Y(e^{jw_1}) = 0$$





$$x[n] = A_0 \cos(w_0 n + \phi_0)$$

$$Y[n] = A_0 \, |H(e^{jw_0})| \cos\left(w_0 n + \phi + \angle H(e^{jw_0})\right)$$

*** For general Input :-**

$$x[n] = \sum_{i=1}^{L} A_i \cos(w_i n + \phi_i)$$

→ The output of the system.

$$y[n] = \sum_{i=1}^{L} A_i \, |H(e^{jw_i})| \cos(w_i n) + H\angle e^{(jw_i)} + \phi_i)$$

Find the output of the system with the input Response $h[n] = (\frac{1}{2})^n u[n]$

and an input $x[n] = 10 - 5\sin(\frac{\pi}{2}n) + 20\cos(\pi n)$

$$H(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

$w_1 = 0$

$$H(e^{j0}) = \frac{1}{1-\frac{1}{2}} = 2\angle 0^\circ$$

$w = \pi/2$

$$H(e^{j\frac{\pi}{2}}) = \frac{1}{1 - \frac{1}{2}e^{-j\pi/2}} = \frac{1}{1+\frac{1}{2}j} = \frac{2}{\sqrt{5}} e^{-j26.6^\circ} = \frac{2}{\sqrt{5}}\angle -26.6^\circ$$

$w = \pi$

$$H(e^{j\pi}) = \frac{1}{1 - \frac{1}{2}e^{-j\pi}} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}\angle 0^\circ$$

$$y[n] = 20 - 5 \cdot \frac{2}{\sqrt{5}} \sin(\frac{\pi}{2}n - 26.6) + 20 \cdot \frac{2}{3}\cos(\pi n)$$

$$y[n] = 20 - \frac{10}{\sqrt{5}} \sin(\frac{\pi}{2}n - 26.6^\circ) + \frac{40}{3}\cos(\pi n) ,$$

for $(-\infty < n < \infty)$

DFT : Discrete fourier Transform

31. oct. 2019

* Finite length discrete Transforms :-

- all sequences are of finite length (N)
- $x[n]$ starts from $n=0$ up to $n=N-1$

* Orthogonal Transforms :-

forward direction.

The analysis eqn.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, \psi^*[k,n] \quad , \quad 0 \leq k \leq N-1$$

$$X[0] = \sum_{n=0}^{N-1} x[n]\, \psi^*[0,n]$$

$$X[1] = \sum_{n=0}^{N-1} x[n]\, \psi^*[1,n]$$

$$\vdots$$

$$X[N-1] = \sum_{n=0}^{N-1} x[n]\, \psi^*[N-1,n]$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, \psi[k,n] \longrightarrow \text{Reverse direction}$$

The synthesis equation

$\psi[k,n]$ are called the basis sequences

$$\frac{1}{N} \sum_{n=0}^{N-1} \psi[k,n]\, \psi^*[l,n] = \begin{cases} 1 & , \ k = l \\ 0 & , \ k \neq l \end{cases} \rightarrow \text{orthogonal}$$

* The Discrete Fourier Transform (DFT) :-

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi kn}{N}}$$

where $\underset{\downarrow \text{ epsi}}{\psi}[k,n] = e^{\frac{j2\pi kn}{N}}$

31.oct.2019

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{\frac{j2\pi kn}{N}} \qquad 0 \le n \le N-1$$

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{\frac{j2\pi kn}{N}} \cdot e^{\frac{-j2\pi Ln}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi(L-K)n}{N}}$$

$\hookrightarrow$ for $k=L$ $\quad \frac{1}{N}\sum_{n=0}^{N-1} 1 = \frac{N}{N} = 1$

$\hookrightarrow$ for $K \neq L$ $\quad \frac{1}{N}\sum_{n=0}^{N-1}\left(e^{\frac{j2\pi(K-L)}{N}}\right)^n$

$$= \frac{1}{N} \cdot \frac{1 - e^{\frac{j2\pi(K-L)}{\cancel{N}}\cdot\cancel{N}}}{1 - e^{\frac{j2\pi(K-L)}{N}}}$$

$$\boxed{\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}}$$

$\Psi[K,N] = \cos\left(\frac{2\pi Kn}{N}\right)$

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi Kn}{N}} \qquad 0 \le K \le N-1$$

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[K]\, e^{\frac{j2\pi Kn}{N}} \qquad 0 \le n \le N-1$$

let $W_N = e^{-j\frac{2\pi}{N}}$

$W_2 = e^{-j\frac{2\pi}{2}}$

$W_4 = e^{-j\frac{2\pi}{4}}$

$\vdots$

$W_{100} = e^{-j\frac{2\pi}{100}}$

$\times$ 1 $\quad X[K] = \sum_{n=0}^{N-1} x[n]\, W_N^{Kn} \qquad 0 \le K \le N-1$

$x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X[K]\, W_N^{-Kn} \qquad 0 \le n \le N-1$

$\circledast$ $\quad X[0] = \sum_{n=0}^{N-1} x[n]\, W_N^{(0)(n)} = x[0] + x[1] + \cdots + x[N-1]$

$X[1] = \sum_{n=0}^{N-1} x[n]\, W_N^{(1)(n)} = x[0] + x[1]W_N^1 + x[2]W_N^2 + \cdots + x[N-1]W_N^{N-1}$

$X[2] = \sum_{n=0}^{N-1} x[n]\, W_N^{2n} = x[0] + x[1]W_N^2 + x[2]W_N^4 + \cdots X[N-1]W_N^{2(N-1)}$

$\vdots$

$X[N-1] = \sum_{n=0}^{N-1} x[n]\, W_N^{(N-1)(n)} = x[0] + x[1]W_N^{(N-1)} + x[2]W_N^{2(N-1)} + \cdots + x[N-1]W_N^{(N-1)(N-1)}$

$$\underset{N\times 1}{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ \vdots \\ X[N-1] \end{bmatrix}} = \underset{N\times \cancel{N}}{\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_N^1 & w_N^2 & \cdots & w_N^{N-1} \\ 1 & w_N^2 & w_N^4 & \cdots & w_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & w_N^{(N-1)} & w_N^{2(N-1)} & \cdots & w_N^{(N-1)^2} \end{bmatrix}}} \underset{\cancel{N}\times 1}{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}$$

$\underline{X}$ $\qquad w_N = e^{-j\frac{2\pi}{N}}$ $\qquad D_N : D_{Matrix}$

$$\underline{X} = D_N x \qquad D_N = \text{dft mtx}(N)$$

D_2, $N=2$ $\qquad w_2 = e^{\frac{-j2\pi}{2}} = e^{-\pi j}$

$$D_2 = \begin{bmatrix} 1 & 1 \\ 1 & w_N \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & e^{-\pi j} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w_4^1 & w_4^2 & w_4^3 \\ 1 & w_4^2 & w_4^4 & w_4^6 \\ 1 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix} \qquad w_4 = e^{\frac{-j2\pi}{4}} = e^{-j\frac{\pi}{2}}$$

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$e^{\frac{-j3\pi}{3}} = \cos\frac{3\pi}{2} * -j \quad \sin\frac{3\pi}{2}$$

7/11/2019

* Circular shift of a sequence :-

$x[n] = \{1 \ \underset{\uparrow}{2} \ 0 \ 1\}$

→ two circular steps $\{0 \ 1 \ 1 \ 2\}$

$y[n] = x[n-1]$

$y[0] = x[-1] = 1$

$y[1] = x[0] = 2$

$y[2] = x[1] = 0$

$y[3] = x[2] = 1$

$y[n] = x[n-1] = \{\underset{\uparrow}{1} \ 2 \ 0 \ 1\}$

⟶ linear shift :

$1201 \ | \ 1201 \ | \ \underbrace{1 \ 2 \ 0 \ 1}_{0 \ 1 \ 2 \ 3} \ | \ 1201 \ |$

* The modulo operation :

$\text{mod}(m, N) \rightarrow$ modulo

$\text{rem}(m, N) \rightarrow$

$r = \langle m \rangle_N$

$r = m + lN$

where l: choosen such that the $r = m + lN$ is a number between 0 and $N-1$

$\langle 50 \rangle_7 = 50$ modulo 7

$= 50 + (\underset{-7}{l})(7) = 1$

$\langle -50 \rangle_7 = -50 + \underset{8}{l}(7) = 6$

$\text{rem}(-50, 7) = -1$

$\text{mod}(-50, 7) = 6$

$\text{rem}(50, 7) = 1$

$\text{mod}(50, 7) = 1$

$\boxed{\text{Example}}$ $x[n] = \{ \underset{\uparrow}{1} \quad 2 \quad 0 \quad 1 \}$

$y[n] = x[\langle n-2 \rangle_4]$

$y[0] = x[\langle -2 \rangle_4] = x[2] = 0$

$y[1] = x[\langle -1 \rangle_4] = x[3] = 1$

$y[2] = x[\langle 0 \rangle_4] = x[0] = 1$

$y[3] = x[\langle 1 \rangle_4] = x[1] = 2$

$y[n] = \{ 0 \quad 1 \quad 1 \quad 2 \}$

(2N-1) points

$0 \le n \le 2N-2$

* The circular convolution :-

$y[n] = x[n] \circledast h[n]$

$y_L[n] = \sum_{k=0}^{N-1} x[k]\, h[n-k]$

	0	1	2	3	4	5	6
x[n]	1	2	0	1			
y(n)	1	1	2	2			

$y[n] = \{ \underset{\uparrow}{1} \quad 3 \quad 4 \quad 7 \quad 5 \quad 2 \quad 2 \}$

linear conv.

older conv.

↓

new circular conv.

$y_c[n] = \sum_{k=0}^{N-1} x[k]\, h[\langle n-k \rangle_N]$

$y_c[n] = x[n] \; Ⓝ \; h[n]$

$= h[n] \; Ⓝ \; x[n]$

(2N-2) بعطي عدد نقاط circular conv. ≠

x,h ← مش متساوين

بنخلي للأكبر

circular conv.
N points

$0 \le n \le N-1$

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Example : $x[n] = \{1 \ 2 \ 0 \ 1\}$

$h[n] = \{1 \ 1 \ 2 \ 2\}$

find $y_c[n]$

$$y_c[0] = \sum_{k=0}^{3} x[k]\, h[\langle -k \rangle_4]$$

$$= x[0]\, h[\langle 0 \rangle_4] + x[1]\, h[\langle -1 \rangle_4] + x[2]\, h[\langle -2 \rangle_4] + x[3]\, h[\langle -3 \rangle_4]$$

$$= x[0]\, h[0] + x[1]\, h[3] + x[2]\, h[2] + x[3]\, h[1]$$

$$= (1)(1) + (2)(2) + (0)(2) + (1)(1) = 6$$

$$y_c[1] = \sum_{k=0}^{3} x[k]\, h[\langle 1-k \rangle_4]$$

$$= x[0]\, h[\langle 1 \rangle_4] + x[1]\, h[\langle 0 \rangle_4] + x[2]\, h[\langle -1 \rangle_4] + x[3]\, h[\langle -2 \rangle_4]$$

$$= x[0]\, h[1] + x[1]\, h[0] + x[2]\, h[3] + x[3]\, h[2]$$

$$= (1)(1) + (2)(1) + (0)(2) + (1)(2) = 5$$

Tabular method for circular conv.

	0	1	2	3	$\langle 4 \rangle_4$	$\langle 5 \rangle_4$	$\langle 6 \rangle_4$
x[n]	1	2	0	1			
y[n]	1	1	2	2			
	1	2	0	1			
	1	1	2	0	1		
	0	2	2	4	0	2	
	4	0	2	2	4	0	2

$y_c[n] = \{6 \ 5 \ 6 \ 7\}$

$y_c[2] =$

$y_c[3] =$

$x[n] \xleftrightarrow{DFT} X[k]$

٤ نقاط

٤ نقاط

$\{1 \ 2 \ 0 \ 1\} \rightarrow \{4 \ \ 1-j \ \ -2 \ \ 1+j\}$

DTFT $\longrightarrow$ 7 points

↳ linear conv.

* Discrete Fourier transform Theorems :-

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$x[n] \xleftrightarrow{DFT} X[k]$

$g[n] \xleftrightarrow{DFT} G[k]$

① linearity Theorem :

$\alpha\, x[n] + \beta\, g[n] \xleftrightarrow{DFT} \alpha X[k] + \beta G[k]$

② Circular Time shifting Theorem :

$x[\langle n-n_0\rangle_N] \xleftrightarrow{DFT} W_N^{kn_0} X[k]$

③ circular frequency shifting Theorem :

$W_N^{-k_0 n}\, x[n] \xleftrightarrow{DFT} X[\langle k-k_0\rangle_N]$

④ Duality Theorem :

$G[n] \xleftrightarrow{DFT} N g[\langle -k\rangle_N]$

Example $g[n] = \{1 \;\; 2 \;\; 0 \;\; 1\}$

$G[k] = \{4 \;\; 1-j \;\; -2 \;\; 1+j\}$

$x[n] = G[n] = \{4 \;\; 1-j \;\; -2 \;\; 1+j\}$

Find $X[k]$:-

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} * \begin{bmatrix} 4 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \begin{bmatrix} 4 + 1\cancel{-j} -2 + 1 \cancel{+j} \\ 4 \cancel{-j} -1 +2 \cancel{+j} -1 \\ 4 -1 \cancel{+j} -2 -1 \cancel{-j} \\ 4 \cancel{+j} +1 -2 \cancel{-j} +1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 8 \end{bmatrix}$$

* Alternative Solution:-

$4\,[\, g[\langle 0\rangle_4] \;\; g[\langle -1\rangle_4] \;\; g[\langle -2\rangle_4] \;\; g[\langle -3\rangle_4] \,]$

$= 4\,[\, g[0] \;\; g[3] \;\; g[2] \;\; g[1] \,]$

$= 4[1 \quad 1 \quad 0 \quad 2] \qquad = [4 \quad 4 \quad 0 \quad 8]$

---

(5) Circular Convolution Theorem :-

$$x[n] \; (N) \; g[n] \xleftrightarrow{DFT} X[K]\,G[K]$$

Example:

$x[n] = \{1 \quad 2 \quad 0 \quad 1\}$

$g[n] = \{ \qquad \qquad \}$

$$X[K] = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix} \qquad G[K] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+1+2+2 \\ 1-j-2+2j \\ 1-1+2-2 \\ 1+j-2-2j \end{bmatrix} = \begin{bmatrix} 6 \\ -1+j \\ 0 \\ -1-j \end{bmatrix}$$

point by point multiplication.

$$X[K] \;.*\; G[K]$$

$$= \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix} \begin{bmatrix} 6 \\ -1+j \\ 0 \\ -1-j \end{bmatrix} = \begin{bmatrix} 24 \\ \cancel{-1}+j+j\cancel{+1} \\ 0 \\ \cancel{-1}-j-j\cancel{+1} \end{bmatrix} = \begin{bmatrix} 24 \\ 2j \\ 0 \\ -2j \end{bmatrix}$$

عشان نرجع Time domain N

$$y[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ 2j \\ 0 \\ -2j \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 26 \\ 24-2-2 \\ 24-\cancel{2j}+0\cancel{+2j} \\ 24+2+0+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$

سؤال بالمحاضرة السابقة

$y[n] = \text{ifft}(\text{fft}(x) \;.*\; \text{fft}(g));$ (using Matlab)

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(6) Modulation Theorem :-

$$x[n]\,g[n] \xleftrightarrow{DFT} \boxed{\frac{1}{N}\sum_{l=0}^{N-1} X[l]\,G[\langle k-l\rangle_N]}$$

$\hookrightarrow Y[k]$

$\hookrightarrow$ $Y[0]$
$Y[1]$
$Y[2]$
$Y[3]$

multiplication in time $\longleftrightarrow$ Circular convolution in freq

(7) Parseval's Relation :-

$$E_x = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

$$\sum_{n=0}^{N-1} x[n]\,g^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\,G^*[k]$$

Example

$x[n] = \{1 \quad 2 \quad 0 \quad 1\}$

$X[k] = \{4 \quad 1-j \quad -2 \quad 1+j\}$

$$\Sigma_x = \sum_{n=0}^{3} |x[n]|^2 = 1^2 + 2^2 + 1^2 = 6$$

OR

$$\Sigma_x = \frac{1}{N}\sum_{k=0}^{3} X[k]^2 = \frac{1}{4}\left(4^2 + (1^2 + (-1)^2) + (-2)^2 + (1^2+1)^2\right) = \frac{24}{4} = 6$$

magnitude

$(1-j)(1+j) = 1-j+j+1 = 2$

complex conjigate بضرب المعرفة بـ

real part بعطي الـ

$x[n] = \{1 \;\; 2 \;\; 0 \;\; 1\}$

$g[n] = \{1 \;\; 1 \;\; 2 \;\; 2\}$

Find $y_L[n] = x[n] \circledast g[n]$

	0	1	2	3	4	5	6
	1	2	0	1			
	1	1	2	2			
	1	2	0	1			
	_	1	2	0	1		
	_	_	2	4	0	2	
	_	_	_	2	4	0	2
$y_L[n] =$	{1	3	4	7	5	2	2}

$x_e[n] = \{1\,2\,0\,1\,0\,0\,0\}$ → extended by 3 zero's (مشان تتطابق مع الناتج)

extended

$g_e[n] = \{1 \;\; 1 \;\; 2\;2 \;\; 0\,0\,0\}$

$y_c[n] = x_e[n] \circledast_7 g_e[n] \xleftrightarrow{DFT} X_e[k]\, G_e[k]$

0	1	2	3	4	5	6	$\langle 7\rangle_7$	$\langle 8\rangle_7$	$\langle 9\rangle_7$
1	2	0	1	0	0	0			
1	1	2	2	0	0	0			
1	2	0	1	0	0	0	0		
_	1	2	0	1	0	0	0	0	
_	_	2	4	0	2	0	0	0	0
_	_	_	2	4	0	2	0	0	0
_	_	_	_	0	0	0	0	0	0
_	_	_	_	_	0	0	0	0	0
_	_	_	_	_	_	0	0	0	0

الجواب ⟹ $\{1\; 2\; 4\; 7\; 5\; 2\; 2\}$

نفس جواب ال linear convolution.

in matlab :-

$y[n] = \text{ifft}(\text{fft}(x, \overbrace{7}^{M}) \;.*\; \text{fft}(g, 7))$

$M = 2*N - 1$

End of midterm material

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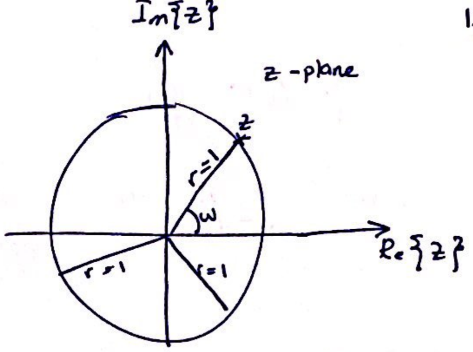
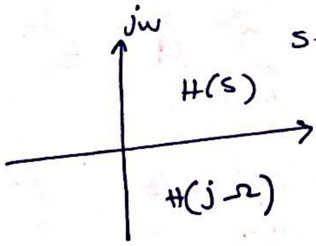
The Z- Transform :-

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \left(r e^{j\omega}\right)^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty}$$





$\boxed{s = j\Omega}$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Example :- $x[n] = 2^n \mu[n]$

* إذا كانت magnitude $1 <$

ما بينفع نحل Fourier transform.

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} 2^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (2 e^{-j\omega}) \quad \text{diverge}$$

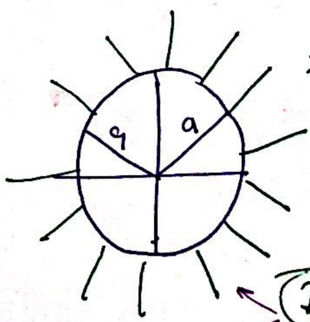
$X(e^{j\omega})$ does not exist.

Example :- $x[n] = a^n \mu[n]$

find $X[z]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n \xrightarrow{\text{sketch}}$$



Z

هي أي نقطة بتحقق الشرط

$|a| < |z|$

ROC

Region of convergence

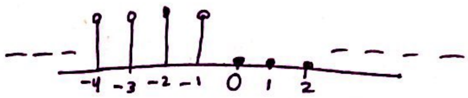
$$= \frac{1}{1 - a z^{-1}} \qquad |a z^{-1}| < 1$$

لازم أكتب الشرط مع إجابتي $\boxed{|a| < |z|}$

مش ممكن نحل $X(z)$ بدون ROC

Example :

$\mu[-n-1]$



$$x[n] = -a^n \mu[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

let $m = -n$

$$X(z) = -\sum_{n=1}^{\infty} a^{-m} z^{m}$$

$$= -\sum_{m=1}^{\infty} (a^{-1}z)^m$$

$$\boxed{\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}}$$

$$= r + r^2 + r^3 + \cdots$$

$$= r(1 + r + r^2 + r^3 + \cdots)$$

$$= r\left(\frac{1}{1-r}\right)$$

$$= \frac{-a^{-1}z}{1-a^{-1}z} \quad |a^{-1}z| < 1$$

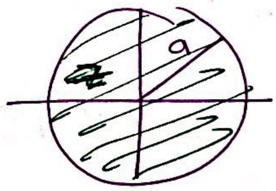
↑ تصح فقط اذا كان $1 > |r|$

نضرب البسط والمقام $\frac{az^{-1}}{az^{-1}}$

$$= \frac{-1}{az^{-1} - 1} \rightarrow \frac{1}{1-az^{-1}}$$

→ نفس z-transform لل example 1 السابق بس الاختلاف بال ROC

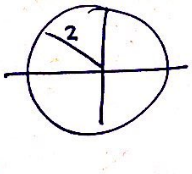
ROC → $|z| < |a|$



left-sided seq → from (a) and inside

Right-sided seq → (a) and outside.

(ex) $\sum_{n=0}^{\infty} (2z^{-1})^n$



$\left(\frac{2}{3}\right)^n$ → داخل (converge)

$\left(\frac{2}{1}\right)^n$ → خارج (diverge)

Example : $x[n] = \{1 \quad \underset{\uparrow}{2} \quad 0 \quad 1\}$

$$X(z) = 1 \cdot z^{1} + 2 \cdot z^{0} + 0 \cdot z^{-1} + 1 \cdot z^{-2} \qquad \boxed{\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$$

Definition

$$= \underbrace{z}_{z=\infty\ \to\ \infty} + 2 + \underbrace{z^{-2}}_{z=0\ \to\ \infty}$$

$z = \infty$ ∞ بطلع .

$\frac{1}{0}$ ∞ بطلع . $z = 0$

→ compinations of +ve and -ve powers .

So, the ROC : The entire z-plane except $z = 0$ and $z = \infty$

Example :- $x[n] = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}$

$$x(z) = 1 + 2z^{-1} + z^{-3}$$

→ ~~compination of +ve and -ve~~ only -ve Powers

So, ROC : the entire z-plane except $z = 0$

Example :-

$x[n] = \{1 \quad 2 \quad 0 \quad \underset{\uparrow}{1}\}$

$$z(x) = z^{3} + 2z^{2} + 1$$

→ only +ve powers .

So, ROC :- the entire z-plane except $z = \infty$

Example:- $x[n] = \delta[n]$

$$x(z) = \sum_{n=-\infty}^{\infty} \delta[n]\, z^{-n} = z^{-(0)} = 1$$

ROC : the entire z-plane .

Example :- $x[n] = \delta[n-1]$

$$x(z) = \sum_{n=-\infty}^{\infty} \delta[n-1]\, z^{-n}$$

$$= z^{-1}$$

ROC : the entire z-plane except at $z = 0$

///

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Example :- $x[n] = \delta[n+1]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1]\, z^{-n}$$

$$= z^{1}$$

ROC : the entire z-plane except ∞

$\dfrac{1}{1-\frac{1}{3}z^{-1}} \rightarrow |z| > \frac{1}{3}$

$\cancel{|z| < \frac{1}{3}}$

$\left(\frac{1}{3}\right)^n \mu[n]$

if $|z| > \frac{1}{3}$

$\left(-\frac{1}{3}\right)^n \mu[-n-1]$

Example :-

$$x[n] = \underbrace{\left(\tfrac{1}{2}\right)^n \mu[n]}_{x_1(n)} + \underbrace{\left(\tfrac{-1}{3}\right)^n \mu[n]}_{x_2(n)}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\left(\tfrac{1}{2}\right)^n + \left(\tfrac{-1}{3}\right)^n\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\tfrac{-1}{3}\right)^n z^{-n}$$

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n}_{} + \underbrace{\sum_{n=0}^{\infty} \left(\tfrac{-1}{3} z^{-1}\right)^n}_{}$$

Both must be converge

(the intersection between the 2 ROC) is the final ROC .

$1 + \frac{1}{3}z^{-1} + 1 - \frac{1}{3}z^{-1}$

$= 2 - \left(\frac{1}{6} z^{-1}\right)$

$2\left(1 - \frac{1}{12} z^{-1}\right)$

$\frac{1}{12} \rightarrow$ zero point

O circle بتمثلها ب

$R_1 \cap R_2$

توحيد مقامات

$$\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\left(\frac{-1}{3}z^{-1}\right)}$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$$

$R_1 : \left|\frac{1}{2} z^{-1}\right| < 1$

$|z| > \left|\frac{1}{2}\right|$

$R_2 : \left|\frac{-1}{3} z^{-1}\right| < 1$

$\left|\frac{-1}{3}\right| < |z|$

so,

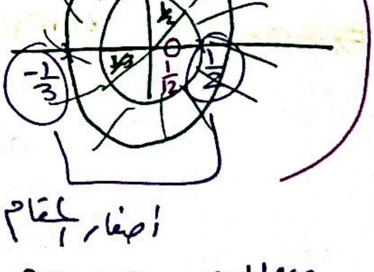
ROC : $|z| > \frac{1}{2}$

تمثل نقاط divergence ال X(Z)

وممنوع أي قيمة Pole تكون موجودة جوا

ال Region of convergence

لأنه X(Z) ال = ∞



أصفار ، أقطاب

Pole - zero pattern

or pole - zero constellation - plane

$$x(z) = \frac{(z+2)}{(z+1)\,(z+3)}$$