2.1 : 2^{nd} order, linear, homog D.E

1) $a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x)$ (*)

2^{nd} linear

$f(x) = 0 \rightarrow$ homo

$f(x) \neq 0 \rightarrow$ non-homo

2) If y_1, y_2 are soll for (*) then $c_1 y_1, c_2 y_2$ are soll

3) If y_1, y_2 are sol for 2^{nd} linear, homo then

$c_1 y_1 + c_2 y_2$ is sol

4) If y_1, y_2 are sol and $\frac{y_1}{y_2} = c \equiv$ constant

the: $y_1, y_2 \rightarrow$ linearly dep

o w $\rightarrow$ linearly in dep

ex: If $y_1 = x^2$, $y_2 = 2x^2$

$$\frac{y_1}{y_2} = \frac{x^2}{2x^2} = \frac{1}{2} \equiv \text{constant} \qquad \text{L-dep}$$

ex: If $y_1 = xe^x$, $y_2 = e^x$

$$\frac{y_1}{y_2} = \frac{xe^x}{e^x} = x \neq \text{constant} \qquad \text{L-indep}$$

* Reduction of order:-

y-missing

$y'' = f(x, y')$

Let $u = y'$

$u' = y''$

x-missing

$y'' = f(y, y')$

Let $u = y'$

$u' = y''$

$$y'' = \frac{du}{dx} = \frac{du}{dy} = \frac{dy}{dx}$$

$$y'' = \frac{du}{dy}\, u$$

ex: $y'' + y' = 0$

sol: y-missing

let $u = y' \Rightarrow u' = y''$

$u' + u = 0 \Longrightarrow \frac{du}{dx} = -u$ sep

$\frac{du}{u} = -dx \Longrightarrow \ln|u| = -x + c$

$\ln|y'| = -x + c \Longrightarrow y' = e^{-x+c}$

$y_1 = c_1 e^{-x}$ sep

$\frac{dy}{dx} = c_1 e^{-x} \Rightarrow dy = c_1 e^{-x} dx \Rightarrow$

$y = -c_1 e^{-x} + c_2$

ex: $y y'' + (y')^2 = 0$

sol: x-missing

$u = y' \Rightarrow u' = y''$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = y'' \rightarrow y'' = u\frac{du}{dy}$$

$$y\frac{du}{dy} = -u \quad \text{sep} \quad \Rightarrow \frac{du}{u} = \frac{-dy}{y}$$

$$\ln|u| = -\ln|y| + c \Rightarrow \ln|y'| = \ln\frac{1}{y} + c$$

$$y' = e^{\ln|\frac{1}{y}|} e^{c} \Rightarrow y' = \frac{c_1}{y} \Rightarrow \frac{dy}{dx} = \frac{c_1}{y}$$

$$y\,dy = c_1\,dx \rightarrow \frac{y^2}{2} = c_1 x + c_2$$

* consider $y'' + P(x)\, y' + q(x)\, y = C$

Given y_1, we can ~~Exactly~~ final y_2 as follows:

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\, dx$$

ex: $2y'' - 8y' + 8y = 0$ Given $y_1 = e^{2x}$ Find y_2

~~P(x)~~ sol:

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\, dx \qquad , P(x) = \frac{-8}{2} = -4$$

$$y_2 = e^{2x} \int \frac{e^{-\int -4\,dx}}{(e^{2x})^2}\, dx = e^{2x} \int \frac{e^{4x}}{e^{4x}}\, dx \Rightarrow y_2 = e^{2x}\, x$$

ex: $x^2 y'' - 3xy' + 4y = 0$

Given $y_1 = x^2$, Find y_2

$$y_2 = x^2 \int \frac{e^{-\int \frac{-3}{x}\,dx}}{(x^2)^2}\, dx = x^2 \int \frac{e^{\int \frac{3}{x}\,dx}}{x^4}\, dx$$

$$= x^2 \int \frac{x^3}{x^4}\, dx = x^2 \ln|x|$$

2.2: Homo linear D.E with constant coefficient.

$a y'' + b y' + c y = c$; a, b and c are constants

* The characteristic eq:

$ar^2 + br + c = 0$ quadratic eq.

The sol of $y = e^{rx}$, r = the root of chare. eq

* The charac eq has 2 sol:

① $b^2 - 4ac > 0$, the charac eq has 2 different sol : $r_1 \neq r_2$

The g solution (homo sol)

$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

ex: $y'' + 3y' + 2y = 0$

sol:

$r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0 \Rightarrow r = -1, -2$

$y_h = c_1 e^{-x} + c_2 e^{-2x}$

ex: $y'' - 4y = 0$

sol:

$r^2 - 4 = 0$

$(r-2)(r+2) = 0 \Rightarrow r = 2, -2$

$y_h = c_1 e^{2x} + c_2 e^{-2x}$

② $b^2 - 4ac = 0$, the charc. eq has only one sol

$r_1 = r_2 = r$

$y_1 = e^{rx}$, $y_2 = x e^{rx}$

$y_h = c_1 e^{rx} + c_2 x e^{rx}$

ex: $y'' + 4y' + 4 = 0$

sol:

$r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0 \Rightarrow r = -2$

$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$

solve:

① $y'' - 2y' - 3y = 0$

sol:

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0 \Rightarrow r = 3, -1$

$y_h = c_1 e^{3x} + c_2 e^{-x}$

② $y'' - 2y' = 0$

sol:

$r^2 - 2r = 0$

$r(r-2) = 0 \Rightarrow r = 0, 2$

$y_h = c_1 e^{0x} + c_2 e^{2x} \Rightarrow y_h = c_1 + c_2 e^{2x}$

③ $2y'' - 12y' + 18y = 0$

$2r^2 - 12r + 18 = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0 \Rightarrow r = 3$

$y_h = c_1 e^{3x} + c_2 x e^{3x}$

ex: Find O.D.E which has the sol: $y = c_1 e^{3x} + c_2 e^{-2x}$

sol:

$r = 3, -2$

$(r+2)(r-3) = 0$

$r^2 - r - 6 = 0$

$y'' - y' - 6y = 0$

ex: find the 2^{nd} O.D.E which has the sol

$y = e^{2x}(c_1 + y c_2)$

$r = 2, 2 \Rightarrow (r-2)(r-2) = 0 \Rightarrow r^2 - 4r + 4 = 0$

$y'' - 4y' + 4y = 0$

* complex

$z = \lambda \mp \mu i$

λ ↳ Real, μi ↳ imag

$i = \sqrt{-1} \rightarrow i^2 = (\sqrt{-1})^2 = -1$

ex: $z = \underbrace{2}_{\lambda} + \underbrace{3i}_{\mu} \Rightarrow \bar{z} = 2 - 3i$

ex: $z = \underbrace{-3i}_{\mu} \Rightarrow \bar{z} = 3i$

ex: $(1+2i)(1-2i) = (-2i + 2i - 4(-1) \rightarrow 1 + 4 = 5$

ex: $e^{zx} = e^{(\lambda + \mu i)x} = e^{\lambda x} e^{\mu x i}$

$= e^{\lambda x}[\cos \mu x + i \sin x]$

(3) $b^2-4ac<0$, then the charc-eq has complex Roots

$r=\lambda \mp \mu i$

the sol of O.D.E $y = \underbrace{c_1 e^{\lambda x}\cos \mu x}_{y_1} + \underbrace{c_2 e^{\lambda x}\sin \mu x}_{y_2}$

ex: $y''+y=0$

$r^2+1=0 \Rightarrow r^2=-1 \Rightarrow r=\pm i$

$y=c_1 e^{0x}\cos x + c_2 e^{0x}\sin x$

$= c_1\cos x + c_2 \sin x$

ex: $y''+4y=0$

$r^2+4=0 \Rightarrow r^2=-4 \Rightarrow r=\pm 2i$

$y = c_1\cos 2x + c_2 \sin 2x$

ex: $y''+9y=0$

$r^2+9=0 \Rightarrow r^2=-9 \Rightarrow r=\pm 3i$

$y=c_1\cos 3x + c_2\sin 3x$

ex: $2y'' + 2y' + 4y = 0$

$2r^2 + 2r + 4 = 0 \Rightarrow r^2 + r + 2 = 0$

$r = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \frac{-1}{2} \mp \frac{\sqrt{7}}{2} i$

$y = c_1 e^{-x/2} \cos\left(\frac{\sqrt{7}}{2} x\right) + c_2 e^{-x/2} \sin\left(\frac{\sqrt{7}}{x} x\right)$

ex: find the 2^{nd} O.D.E which has the sol
$y = c_1 e^x \cos x + c_2 e^x \sin x$

$r = 1 \mp i$

$(r - (1+i))(r - (1+i))$

$r^2 - r(1-i) - (1+i)r + (1+i)(1-i)$

$r^2 - r + ir - r - ri + \cancel{(1-1)(i} (1 - i + i + 1)$

$r^2 - 2r + 2 = 0$

$y'' - 2y' + 2y = 0$

Find the 2^{nd} O.D.E which has the sol

$y = c_1 e^x \cos x + c_2 e^x \sin x$

$r = 1 \pm i \Rightarrow r = 1+i \,,\; r = 1-i$

$(r - (1 \mp i) - (1+i)r + (1+i)(1-i))$

$r^2 - r + \cancel{ir} - r - \cancel{ri} + (1 - \cancel{i} + \cancel{i} + 1)$

$r^2 - 2r + 2 = 0$

$y'' - 2y' + 2y = 0$

* cauchy - Euler Equations C.E.E

$ax^2 y'' + bx y' + cy = 0$

The sol: $y = x^r$

The charc equ $\Rightarrow ar(r-1) + br + c = 0$

The sol of charc equ has 3 sol:

case 1: $r_1 \neq r_2$

The g.s $\Rightarrow y_h = c_1 x^{r_1} + c_2 x^{r_2}$

Case 2: $r_1 = r_2 = r$

The g.s $\Rightarrow y_h = c_1 x^r + (c_2 x^r (\ln x))$

Case 3: $r = \lambda \mp Mi$

The g.s $\Rightarrow y_h = c_1 x^\lambda \cos(M \ln x) + c_2 x^\lambda \sin(M \ln x)$

ex: solve

1) $2x^2y'' + 3xy' - y = 0$

sol:

$2r(r-1) + 3r - 1 = 0 \Rightarrow 2r^2 - 2r + 3r - 1 = 0$

$2r^2 + r - 1 = 0$

$$r = \frac{-1 \mp \sqrt{1-4(2)(-1)}}{4} = \frac{-1 \pm 3}{4} \begin{cases} -1 \\ \frac{1}{2} \end{cases}$$

$y_h = c_1 x^{-1} + c_2 x^{\frac{1}{2}}$

2) $x^2y'' - 3xy' + 4y = 0$

sol:

$r(r-1) - 3r + 4 = 0 \Rightarrow r^2 - r - 3r + 4 = 0$

$r^2 - 4r + 4 = 0 \Rightarrow (r-2)(r-2) = 0 \Rightarrow \boxed{r = 2, 2}$

$y_h = c_1 x^2 + c_2 x^2 \ln(x)$

3) $x^2y'' + 7xy' + 13y = 0$

sol:

$r(r-1) + 7r + 13 = 0 \Rightarrow r^2 - r + 7r + 13 = 0$

$r^2 + 6r + 13 = 0$

$$r = \frac{-6 \pm \sqrt{36-(4)(13)}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

$y_h = c_1 x^{-3} \cos(2\ln x) + c_2 x^{-3} \sin(2\ln x)$

P.P

4) $x y'' + 2y' = 0$ → y-missing لو اجت بالفيرست / → C.E.E لو اجت بالسكند

sol: نضرب بـ x

$x^2 y'' + 2xy' = 0 \Rightarrow r(r-1) + 2r = 0$

$r^2 - r + 2r = 0 \Rightarrow r^2 + r = 0$

$r(r+1) = 0 \Rightarrow r = 0 , r = -1$

$y_h = c_1 x^0 + c_2 x^{-1} = c_1 + c_2 x^{-1}$

P.P

5) $y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0$

sol:

نضرب بـ x^2

$x^2 y'' + x y' + y = 0 \Rightarrow r(r-1) + r + 1 = 0$

$r^2 \cancel{-r} \cancel{+r} + 1 = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r^2 = \sqrt{-1} \Rightarrow r = \mp i$

$y_h = c_1 \cos \ln x + c_2 \sin \ln x$

ex: Find the D.E which has the sol:

$y = c_1 x^3 + c_2 x^2$

$r = 3 , 2$

$(r-3)(r-2) = 0 \Rightarrow r^2 - 5r + 6 = 0$ charc eq

$r^2 - r - 4r + 6 = 0 \Rightarrow r(r-1) - 4r + 6 = 0$

$x^2 y'' - 4xy' + 6y = 0$

ex: IS the charc eq: $2r^2 + 4r + 3 = 0$ find the 2^{nd} order C.E.E

sol:

$$2r^2 + 4r + 3 = 0 \Rightarrow 2r^2 - 2r + 2r + 4r + 3 = 0$$

$$2r(r-1) + 6r + 3 = 0$$

$$2x^2 y'' + 6xy' + 3y = 0$$

2.6:

$$w[y_1 : y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

ex: $w[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^{-x} e^x = -2$

linear indep

ex: $w[e^x, 2e^x] = \begin{vmatrix} e^x & 2e^x \\ e^x & 2e^x \end{vmatrix} = 2e^{2x} - 2e^{2x} = 0$

linear dep

ex: $w[x^2, x^3] = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^2(3x^2) - x^3(2x)$

$= 3x^4 - 2x^4 = x^4$

linear indep

If $w[y_1, y_2] = 0 \Rightarrow$ linear dep

If $w[y_1, y_2] \neq 0 \Rightarrow$ linear indep

ex: $[3x^2, 2x]$
$$= \begin{vmatrix} 3x^2 & 2x \\ 6x & 2 \end{vmatrix} = 6x^2 - 12x^2 = -6x^2 \neq 0$$
linear indep

2.7: Non homo 2^{nd} O.D.E

$y'' + P(x) y' + q(x) y = f(x) \neq 0$

The g.s $= y_g = y_h + y_p$

y_h: Const. Co, C.E.E (Cachy)

y_p: undetermined Method (constcoeffiant) $= y_h$; Variation of parameters

* Undetermined coeff method

1) $y'' + P(x) y' + q(x) = 0$

$y_h = C_1 y_1 + C_2 y_2$

2) Find y_p

3) $y_g = h_h + y_p$

$f(x)$ y_p

1) $ke^{\gamma x}$ $Ae^{\gamma x}$

2) kx^n $a_n x^n + a_{n-1}, a x^{n-1} + \cdots + a_1 x + a_0$

3) $k\cos(wx)$ → $A\cos(wx) + B\sin(wx)$

4) $k\sin(wx)$

example for No ②

$f(x) = 2x^3$, $y_p = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

ex:

$ke^{\gamma x}\cos wx$ → $Ae^{\gamma x}\cos(wx) + Be^{\gamma x}\sin(wx)$

$ke^{\gamma x}\sin wx$

ex: $y'' - y = 2x^2 + 1$

1) $y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

$y_h = c_1 e^x + c_1 e^{-x}$

2) $y_p = ax^2 + a_1 x + a_0$

$y' = 2a_2 x + a_1 \Rightarrow y'' = 2a_2$

$2a_1^2 - a_2 x^2 - a_1 x - a_0 = 2x^2 + 1$

$2a_2 - a_0 - a_2 x^2 - a_1 x = 2x^2 + 1$

$-a_2 = 2 \Rightarrow a_2 = -2$

$-a_1 = 0 \Rightarrow a_1 = 0$

$2a_2^{-2} - a_0 = 1 \Rightarrow -4 - a_0 = 1 \Rightarrow a_0 = -5$

$y_p = -2x^2 - 5$

3) $y_g = y_h + y_p = c_1 e^x + c_2 e^{-x} - 2x^2 - 5$

ex: $y'' + y = 2e^{3x}$

sol:

1) $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \mp i$

$y_h = c_1 \cos x + c_2 \sin x$

2) $y_p = Ae^{3x} \Rightarrow y' = 3Ae^{3x} \Rightarrow y'' = 9Ae^{3x}$

$9Ae^{3x} + Ae^{3x} = 2e^{3x} \Rightarrow 10Ae^{3x} = 2e^{3x}$

$10A = 2 \Rightarrow A = \frac{1}{5}$

$y_p = \frac{1}{5} e^{3x}$

3) $y_g = y_h + y_p = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{3x}$

ex: $y'' - 4y = 2\sin x$

sol:

1) $r^2 - 4 = 0 \Rightarrow r = \pm 2$

$y_h = c_1 e^{2x} + c_2 e^{-2x}$

2) $y_p = A\cos x + B\sin x \Rightarrow y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

$-A\cos x - B\sin x - 4A\cos x - 4B\sin x = 2\sin x$

$(-A - 4A)\cos x = 0 \Rightarrow -5A = 0 \Rightarrow A = 0$

$(-B - 4B)\sin x = 2\sin x \Rightarrow -5B = 2 \Rightarrow B = -\frac{2}{5}$

$y_p = \frac{-2}{5} \sin x$

$y_g = y_h + y_p$

ex: Determine form of y_h using determined

$y'' - 3y' + 2y = 2e^{x} \sin x$

1) $r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0 \Rightarrow r = 1, 2$

$y_h = c_1 e^{x} + c_2 e^{2x}$

2) $y_p = Ae^{x} \cos x + Be^{x} \sin x$

⋮

ex: $y'' + 2y' = x^4 - 2x^3 + 1 + x^2 e^{-2x} + \cos x$

1) $r^2 + 2r = 0 \Rightarrow r(r+2) = 0 \Rightarrow r = 0, -2$

$y_h = c_1 e^{0x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$

2) $y_p = (a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0)\, x + \cancel{(b_2 x^2 + b_1 x + b_0)}$

$+ ((b_2 x^2 + b_1 x + b_0) e^{-2x} - x) + A\cos x + B \sin x$

⋮

ex: $y'' + 2y' + 2y = e^{-x} + x^2 e^{-x} \sin x$

1) $r^2 + 2r + 2 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \mp i$

$y_h = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

$y_p = Ae^{-x} + ((a_2 x^2 + a_1 x + a_0) e^{-x} \cos(x) * (x))$

$+ (b_2 x^2 + b_1 x + b_0) e^{-x} \sin(x) * (x)$

⋮

ex : $y'' - y = 2e^x$

1) $y_h = r^2 - 1 = 0 \Rightarrow r = 1 , 1$

$y_h = c_1 e^x + c_2 e^{-x}$

نضرب بـ x في حالة التشابه

2) $y_p = Ae^x(x) \Rightarrow Ae^x x \Rightarrow y' = A(e^x x + e^x)$

$y'' = A(e^x x + e^x + e^x) = A(e^x x + 2e^x)$

$Ae^x x + 2Ae^x - Axe^x = 2e^x$

$2Ae^x = 2e^x \Rightarrow 2A = 2 \Rightarrow \boxed{A = 1}$

$y_p = xe^x$

3) $y_g = y_h + y_p = c_1 e^x + c_2 e^{-x} + xe^x$

ex: Let $y'' - 2y' + y = 2e^x$ Determine the form of y_p using underermined

1) $y_h = r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0 \Rightarrow r = 1 , 1$

$y_h = c_1 e^x + c_2 x e^x$

2) $y_p = Ae^x(x)(x) = Ax^2e^x$

نستمر بضرب المعادلة بـ x حتى تختلف عن الـ y_h

⋮