Probability and random Variables

introduction

Probability → uncertainty Possibility → the chance in Randomness

Definition of a set : number of objects

set — $A = \{2, 4, 6, 8\}$ elements

set — $B = \{a, b, z\}$

$2 \in A$ ✓
$9 \notin A$ ✓

2 ways to write a set

→ 1- tabular methods
$A = \{2, 4, 6, 8\}$

→ 2- Rule method
$A = \{$even numbers from 1 to 9$\}$

$C = \{1.5 < \text{Real} \leq 5\}$ → Describes an interval

→ can't be tabular method
$n = \infty$ of Values

→ same

$C = \{$all real numbers between 1.5 and 5, 5 included$\}$

set
- countable
  - $A = \{1, 2, 3, 4, 5\}$
    → 5 elements
  - $D = \{1, 2, 3, \ldots, \infty\}$
    → still countable
    → ∞ elements
- uncountable
  - $F = \{-1 \leq f \leq 7.1\}$
    → interval

set
- finite
  Like A
- infinite
  like D, F

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If an experimal occures in A → means that set A has occoured

Kinds of sets

① empty (null) set : ϕ

② $A \subset B$ → A is contained in B
→ A is a subset of B

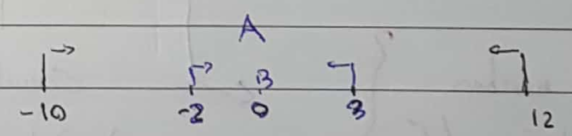
eg: 1- $C = \{2, 4, 6\}$

& $D = \{-1, 0, 0.5, 1, 2, 3, 4, 5, \cancel{6}, 7\}$ → if 6 not in D $C \not\subset D$

$C \subset D$

2- $A = \{-10 \leq a \leq 12\}$ and $B = \{-2 < b < 3\}$

so: $B \subset A$



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③ universal set "S"

the set that contains all other sets
For certain situations

so, $X \subset S$

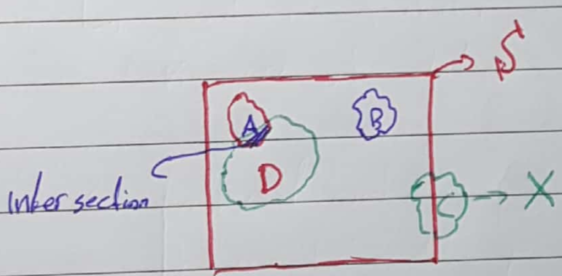
eg: $A = \{a, b, c\} = S$

all subsets $\rightarrow$ $\{a\}, \{b\}, \{c\}$
$\{a, b\}, \{b, c\}, \{a, c\}$ $\rightarrow$ note: $\{a, b\} = \{b, a\}$
$\{\ \}, \{a, b, c\}$

# of subsets = ~~[illegible]~~ 2^n, where n = number of elements

eg: $B = \{2, 4, 6, 8\}$
# of subsets $= 2^4 = 16$.

Venn Diagram $\rightarrow$ very useful.



- A subset of S as well as B and D but not C

A and D share elements

① set equality :- $A = B$ iff $A \subset B$ and $B \subset A$

eg $A = \{2, 3, 5\}$, $B = \{5, 3, 2\}$, $C = \{-1, 2, 3, 5\}$

$A = B$, $A \neq C$

Five Apple

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(3) Set Difference :-

A - B : all elements in A but not in B

eg

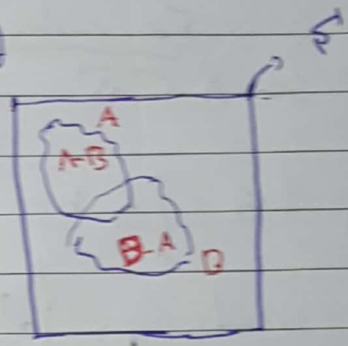
$A = \{-1, 0, 3, 5, 9\}$

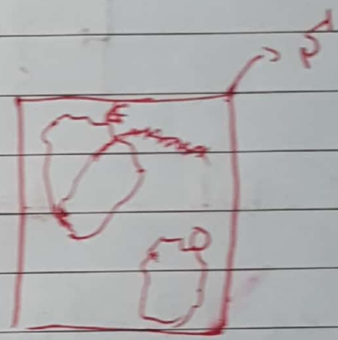
$B = \{-2, -1, 1, 4, 5, 11\}$

$A - B = \{0, 3, 9\}$

$B - A = \{-2, 1, 4, 11\}$

eg





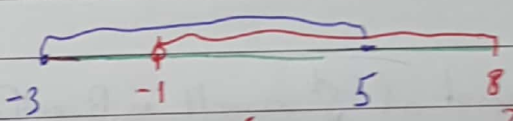
$C - D = C$

$D - C = D$

eg

$A = \{-3 \leq x \leq 5\}$

$B = \{-1 < b \leq 8\}$



$A - B = \{-3 \leq x \leq -1\}$

$B - A = \{+5 < x \leq 8\}$

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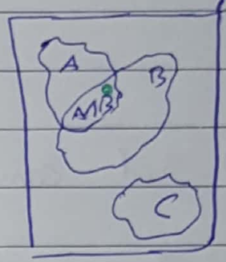
Set operation :.

① intersection $\cap$ " تقاطع "

$A \cap B$: set of all common elements between A and B

$A \cap B = \{-1 < x \leq 5\}$



$\cap$ = " and " " together "
" at the same time " ↳ Key words in exam

. If an event that happens in $A \cap B$
↳ Both A and B happend

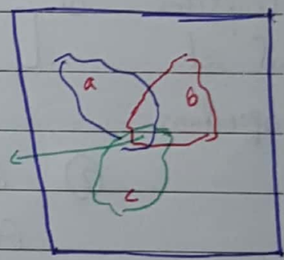
$B \cap C = \emptyset$
↳ B and C are disjoined (mutually exclusive)

$A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n = \bigcap_{i=1}^{N} A_i$

② Union "U" " اتحاد "

$A \cup B$: the set of all elements in A and B

$a \cap b \cap c$

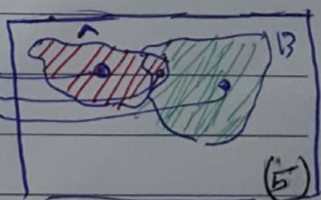


eg : $A = \{1, 5, 6, 7\}$ $B = \{-2, 0, 1, 3, 6, 9\}$

$A \cap B = \{1, 6\}$, $A \cup B = \{1, 5, 6, 7, -2, 0, 3, 9\}$

$\cup$ = " or "
↳ either A or B
↳ Key word in exam

all these represent U



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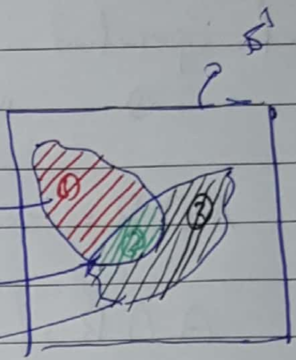
$\bigcup_{i=1}^{n} B_i = B_1 \cup B_2 \cup B_3 \cup \dots \cup B_n$

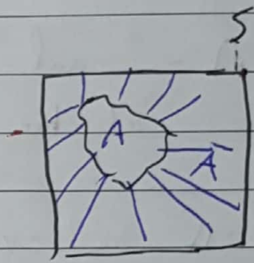


$A \cup B = \underbrace{(A-B)}\cup\underbrace{(B\cap A)}\cup\underbrace{(B-A)}$ { $A-B$, $A\cap B$, $B-A$

are disjointed (mutually exclusive)

③ Set compliment

$\bar{A}$ = all elements not in A



$\bar{\bar{A}} = A$, $A \cap \bar{A} = \emptyset$

$A \cup \bar{A} = S$

$\bar{S} = \emptyset$

$\bar{\emptyset} = S$

Properties of set operations

① $A \cup B = B \cup A$ → Commutative
$A \cap B = B \cap A$

② $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ → Distributive
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

③ $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$ → Associative
same thing for union

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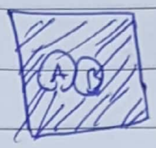
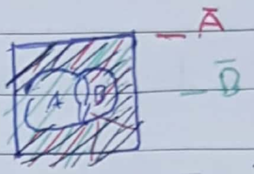
De Morgan's law:

- $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$ → invert everything





- $\bar{A} \cap \bar{B}$

- $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

- Duality Principle

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \checkmark$$

under duality $\downarrow$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

↳ $\varnothing \rightarrow S$
$S \rightarrow \varnothing$
under duality

L2 ch1

- experiment and sample space S

one experiment ↳ trial → an experiment of one out come
↳ one out come

sample space all possible out comes

eg → coin flip → H, T → so $S = \{T, H\}$ ↳ or

eg → roll a dice → 1, 2, 3, 4, 5, 6 so $S = \{1, 2, 3, 4, 5, 6\}$

30 - 9

Mathematical model

/ fair dice

Define events of interest

ex 1 : rolling a dice , $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{\text{even values}\} = \{2, 4, 6\}$, $A \subset S$

$B = \{\text{integer}\} = \{1, 2, 3, 4, 5, 6\}$, $B = S$

$C = \{\text{negative}\} = \{0\} = \phi$

$D = \{4 > x > 3\} = \phi$

trial 1 it gave 2

B , A has occured

$P(S) = 100\% = 1$

↳ Probability of S

$P(X) = [0, 1]$

aspect

ϕ : impossible event

$P(\phi) = 0\% = 0$

$P(A) = 50\% = \frac{3}{6} = 0.5$

fair dice

$P(1) = \frac{1}{6}$

$P(2) = \frac{1}{6}$

.

.

$P(6) = \frac{1}{6}$

$= 1$

→ all equal Probability

if P(element) differs

↳ ~~fair~~ unfair dice

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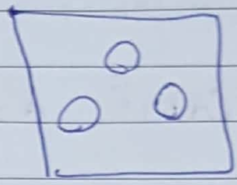
no negative value in Probability

~~Th~~ - Law $P(\bigcup_{n}^{n} A_n) = P(A) + P(B) + P(C)$

↳ iff A and B and C are disjoint

Proof

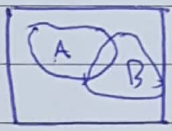


$$P(A \cup B \cup C) = \frac{\text{Area } A + \text{" } B + \text{" } C}{\text{Area } S}$$

$$P(A \cup B \cup C) = P\left(\frac{\text{Area } A}{\text{Area } S}\right) + P\left(\frac{\text{" } B}{\text{" } S}\right) + P\left(\frac{A\,C}{A\,S}\right)$$

so $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

H.W:



$P(A \cup B) = P(A) + P(B)$ iff $(A \cap B) = \phi$

show that
$P(A \cup B) = P(A) + P(B)$
$- P(A \cap B)$

if dice unfair ↳ sum of Possibility of each element

ex $S = \{1, 2, 3, 4, 5, 6\}$

$P(S) = \{0.1, 0.15, 0.05, 0.25, 0.05, 0.4\}$, $A = \{2, 4, 6\}$

$P(A)$

↳ $= 0.15 + 0.25 + 0.4\} = 0.8$ not $3/6$

to calculate $P(A) = P(\{2\} \cup \{4\} \cup \{6\}) = P(2) + P(4) + P(6)$
$= 0.8$

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ex throwing 2 dice

$$S = \begin{Bmatrix} (1,1), & (1,2), & \dots & (1,6) \\ (2,1), & (2,2), & \dots & (2,6) \\ \vdots & & & \\ (6,1), & \dots & & (6,6) \end{Bmatrix} = \quad 6^2 = \text{elements} = 36$$

$P(A)$ if $A = \{sum = 7\} = \frac{6}{36} = \frac{1}{6}$

$P(B)$ if $B = \{8 \leq sum \leq 11\} = \frac{9}{36} = \frac{1}{4}$

$P(C)$ if $C = \{sum > 10\} = \frac{3}{36} = \frac{1}{12}$

$P(A \cap B) = \emptyset$

$P(B \cap C) = \frac{2}{36}$

$P(A \cup B) = \frac{6}{36} + \frac{9}{36} = \frac{15}{36}$

$P(B \cup C) = P(B) + P(C) - P(B \cap C) = \frac{9}{36} + \frac{3}{36} - \frac{2}{36}$
$= \frac{10}{36}$

joint Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

if A and B are disjoint

$\therefore P(A \cup B) = P(A) + P(B)$

We can conclude $P(A \cup B) \leq P(A) + P(B)$

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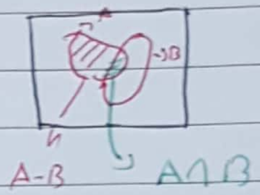
$A^c = \bar{A}$ = A's complement

- $\because S$ $\quad A \cup A^c = S$

$$P(A \cup A^c) = P(S)$$
$$P(A) + P(A^c) = 1$$
$$\therefore P(A^c) = 1 - P(A) \; * \rightarrow \text{complement rule}$$

- $P(\emptyset) = 0$

- $P(A - B) = P(A) - P(A \cap B)$

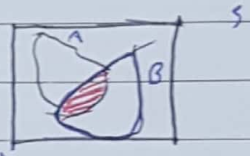


$\therefore P(A) = P(A-B) + P(A \cap B)$

- Conditional Probability: (Lowering the Sample)

$P(A|B) \rightarrow$ Lower the sample space from S to B

given that B has occurred



$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\text{Area}(A \cap B) \,/\, \cancel{\text{Area}(S)}}{\text{Area}(B) \,/\, \cancel{\text{Area}(S)}}$$

and $P(B|A)$

$$= \frac{P(A \cap B)}{P(A)}$$

So $P(A \cap B) = P(B|A) \cdot P(A)$ 'or' $P(A|B) \cdot P(B)$

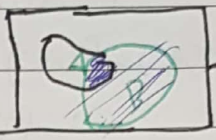
Probability

2/10/2019

L 4. doctor didn't come.

L.5 7-10-2019 Probability

- $P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$



$P(A) = \frac{\text{Area } A}{\text{Area } S}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

given that B has happened $= \frac{\text{Area } (A \cap B)}{\text{Area } (B)}$

- $P(A \cup B | C) = P(A|C) + P(B|C) - P(A \cap B | C)$

- $P(A^c|C) = 1 - P(A|C)$
- $P(\emptyset|C) = 0$
- $P(A|C) \leq 1$
- $P((A-B)|C) = P(A|C) - P(A \cap B|C)$
- $P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$

$\to C$

$P(3rd \cap \underbrace{2nd \cap 1st}_{C}) = P(3rd \mid C) \cdot P(C)$

- if $A \subset B$ then $P(A|C) \leq P(B|C)$

Probability

A = { draw 47Ω }
B = { draw 5% }
C = { draw 100Ω }

R / tolerance

Ω	5%	10%	tot
22	10	24	24
47	$\boxed{28}$	16	$\boxed{44}$
100	$\boxed{24}$	8	$\boxed{32}$
tot	$\boxed{62}$	38	$\boxed{100}$

$P(A) = \frac{44}{100}$

$P(B) = 62/100$

$P(C) = 32/100$

$P(B \cap C) = 24/100$

$P(A \cap B) = 28/100$

$P(A \cap B) = 0$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = 28/62$

$P(A|C) = \frac{P(A \cap C)}{P(C)} = 0$

$P(B|C) = \frac{P(B \cap C)}{P(C)} = 24/32$

example ① E_1 draw 1 Ressitor box = 80

Value	10Ω	15Ω	20Ω	30Ω
Qty	18	12	33	17

$P(10\Omega) = 18/80$

$P(15\Omega) = 12/80$

$P(20\Omega) = 33/80$

$P(30\Omega) = 17/80$

② E_2 - draw 2 Ressitors Without RePlacement. → dependent

$P(2nd\ 30\Omega \cap 1st\ 15\Omega) = P(2nd\ 30 | 1st\ 15) \cdot P(1st\ 15) = 17/79 \cdot 12/80$

↳ OR

$P(2nd\ 30 \cap 1st\ 15) = P(1st\ 15 | 2nd\ 30) \cdot P(2nd\ 30)$

not Logical asking about something in the Past that depend on the Future.

7-10

to Solve this Problem we use

① Bayes rule
② Law of total Probability

* independent events.

for two events with $P(A) > 0$ and $P(B) > 0$ are called statistically independent if

$P(A|B) = P(A)$ (the occurrence of A is not affected by B)

$P(B|A) = P(B)$ (the occurrence of B is not affected by A)

so $P(A \cap B) = P(A)P(B)$

normally

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A).$$

hence: $$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)\cancel{P(A)}}{\cancel{P(A)}} = P(B)$$

note: two events can not be disjoint and independent

$\hookrightarrow A \cap B = 0$ doesn't conquer so must be jointed

7-10

note: two independent events must have $A \cap B \neq \emptyset$

. three events A_1, A_2, A_3 are statistically independent if:

$$P(A_1 \cap A_2) = P(A_1)\,P(A_2) \quad \dots \quad ①$$
$$P(A_1 \cap A_3) = P(A_1)\,P(A_3) \quad \dots \quad ②$$
$$P(A_2 \cap A_3) = P(A_2)\,P(A_3) \quad \dots \quad ③$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3) \quad \dots \quad ④$$

• all 4 must happen to qualify as statistically independent.

. Properties of independent events: N independent events $A_1, A_2, \dots, A_N$ then any A_K is independent of any event formed by unions, intersections and complements of the others.

in general for N events $A_1, A_2, \dots, A_N$ to be called S.I, we require that all conditions

$$P(A_1 \cap A_2) = P(A_1)\,P(A_2)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$$
$$\vdots$$
$$P(A_1 \cap A_2 \cap \dots \cap A_N) = P(A_1)\,P(A_2) \dots P(A_N)$$

be satisfied

number of conditions to check given by:

$$\#C = 2^N - N - 1$$

if $N=3$ $\#C = 4$ to be checked

back to example

E_3 - draw 2 Resistors with replacement

1. $P(2nd\ 30\Omega \cap 1st\ 15\Omega) = P(2nd\ 30 | 1st\ 15)\, P(1st\ 15)$

$= \frac{17}{80} \times \frac{12}{80}$

if A_1, A_2 are S.I events then: A_1 is I of $\bar{A}_2$ A_2 is I of $\bar{A}_1$ $\bar{A}_1$ is I of $\bar{A}_2$	this condition is redundant because they are S.I

. Property: if A_1, A_2 and A_3 are three I events, then any one is I of the joint occurrence of the other two, Also any one is I of the Union of the other two.

Ex given A_1, A_2 are I with $P(A_1) = 0.6$, $P(A_2) = 0.3$
Find:

① $P(A_1 \cap \bar{A}_2)$
② $P(\bar{A}_2 | \bar{A}_1)$

1- using law (because Ind) $= P(A_1) \cdot P(\bar{A}_2) = P(A_1) \cdot [1 - P(A_2)]$
$= 0.6 \cdot 0.7 = 0.42$

2- $P(\bar{A}_2 | \bar{A}_1) = \frac{P(\bar{A}_2 \cap \bar{A}_1)}{P(\bar{A}_1)} = \frac{P(\bar{A}_2) \cdot \cancel{P(\bar{A}_1)}}{\cancel{P(\bar{A}_1)}} = P(\bar{A}_2) = 1 - P(A_2)$
$= 1 - 0.3$
$= 0.7$

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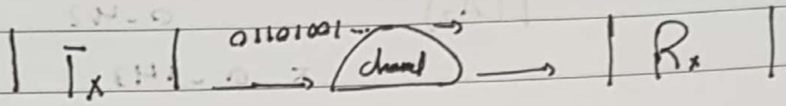
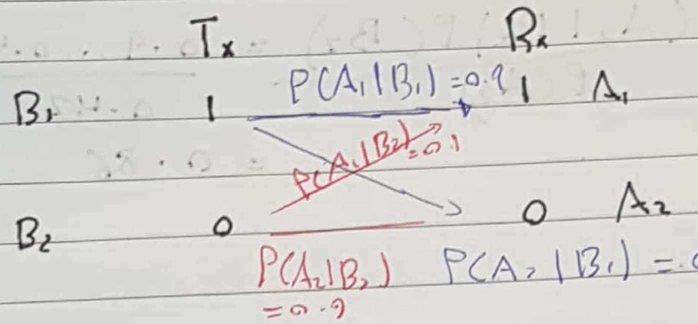
BCC:





$P(B_1) = 0.6$

$P(B_2) = 0.4$

$$\therefore P(A_1) = P(A_1 \cap B_1 \cup A_1 \cap B_2)$$
$$= P(A_1|B_1)P(B_1) + P(A_1|B_2)P(B_2)$$
$$= (0.9)\cdot(0.6) + (0.1)\cdot(0.4)$$
$$= 0.58$$

total Probability Law

$$P(A_2) = P(A_2|B_1)P(B_1) + P(A_2|B_2)P(B_2)$$
$$= (0.1)\cdot(0.6) + (0.9)(0.4)$$
$$= 0.42$$

total Probability law

or $P(A_2) = 1 - P(A_1)$

complement

Bayes Rule

$$P(B_1|A_1) = \frac{P(B_1 \cap A_1)}{P(A_1)} = \frac{P(A_1|B_1)P(B_1)}{P(A_1)} = \frac{0.9 \cdot 0.6}{0.58}$$
$$= 0.931$$

Comp

$$P(B_2|A_1) = 1 - P(B_1|A_1) = 0.069$$

or

Bayes Rule

$$P(B_2|A_1) = \frac{P(B_2 \cap A_1)}{P(A_1)} = \frac{P(A_1 \cap B_2) \cdot P(B_2)}{P(A_1)} = \frac{0.1 \cdot 0.4}{0.58}$$
$$= 0.069$$

$$P(B_1|A_2) = \frac{P(A_2|B_1)P(B_1)}{P(A_2)} = \frac{0.1 \cdot 0.6}{0.42} = 0.143$$

$$P(B_2|A_2) = \frac{P(A_2|B_2)P(B_2)}{P(A_2)} = \frac{0.9 \cdot 0.4}{0.42} = 0.86$$

- combined experiments.

exp:

Flip a coin & roll a dice

$S = \begin{Bmatrix} (H,1), (H,2), \ldots, (H,6) \\ (T,1), (T,2), \ldots, (T,6) \end{Bmatrix} = 12$ element

C = "H and even"

$C = \{(H,2)(H,4)(H,6)\}$

$P(C) = 3/12$

exp → sub exp 1, sub exp 2

$S_1 = \{H, T\}$ $\quad S_2 = \{1, 2, 3, 4, 5, 6\}$

$S = S_1 \times S_2$ ↳ combined

$C = A_{S_1} \times A_{S_2}$

$A_{S_1} = \{H\}$, $A_{S_2} = \{2, 4, 6\}$

S_1	S_2
H	1
T	2
	3
	4
	5
	6

and "∩"

$P(C) = P(A_{S_1} \times A_{S_2})$

$= P(A_{S_1} \cap A_{S_2}) = P(A_{S_1})P(A_{S_2})$ I

$= \frac{1}{2} \cdot \frac{3}{6} = \frac{1}{4}$

9-10

exp Rolling 3 dice

$S' = \{(1,1,1), (1,1,2) \ldots (6,6,6)\}$ $6^3 = 216$

216 element

$S_1 = \{1,2,3,\ldots,6\}$, $S_2 = \{1,\ldots,6\}$, $S_3 = \{1,\ldots,6\}$

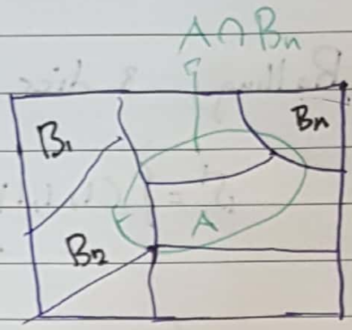
$S = S_1 \times S_2 \times S_3 = \{216\}$

$P(\text{even values}) = P(S_1\text{even} \times S_2\text{even} \times S_3\text{even})$
$= P(S_1\text{even}) \cdot P(S_2\text{even})\, P(S_3\text{even})$
$= \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{3}{6} = \frac{27}{216}$

~~Permutations~~

9-10

· total probability



given N disjoint events B_n
whose $\cup$ equal S

$$\sum_{n=1}^{N} P(B_n) = 1$$

then $P(A) = \sum_{n=1}^{N} P(A|B_n)\,P(B_n)$
is the law of total probability

Proof :. $A \cap S = A$

$\Rightarrow A \cap \left(\sum_{n=1}^{N} B_n\right) = \bigcup_{n=1}^{N} (A \cap B_n)$

$\therefore P(A) = P(A \cap S) = P\left[\bigcup_{n=1}^{N} (A \cap B_n)\right] = \sum_{n=1}^{N} P(A \cap B_n)$

Bayes' theorem:

$$P(B_n|A) = \frac{P(A|B_n)\,P(B_n)}{P(A)}$$

Proof $P(A \cap B_n) = P(A|B_n)P(B_n) = P(B_n|A)P(A)$

Combined experiments

Defn: is an experiment that consists of multiple sub-experiments, like...

1- flip a coin and roll a dice
2- repeating the same experiment

suppose S is 2 sub experiments S_1, S_2

S_1 has M elements
S_2 has N elements

$S = S_1 \times S_2$
has $M \times N$ elements

See ex P15 - Lecture 3

suppose $C = A \times B \longrightarrow$ independent sub experiments
where A is coin flip
B is a dice throw
$C =$ " coin tail and number is odd "
$$P(C) = P(\{T,1,\ T,3,\ T,5\}) = 3/12 = \frac{1}{4}$$

Or $P(C) = P(A \times B) = P(A)P(B) = \frac{1}{2} \cdot \frac{3}{6} = \frac{3}{12} = \frac{1}{4}$

$\hookrightarrow$ this is useful for unfair coin or dice

- Permutations:

Defn: all possible sequence of ordering (order is important) of r elements taken from n elements w/o replacement. $\{a,b\} \neq \{b,a\}$

ex given 4 letters order them in two places:

$$P_r^n = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

ex how many permutation for 4 cards in 52 card deck

$$P_4^{52} = \frac{52!}{(52-4)!} = (52).(51).(50).(49) = 6,497,400$$

Combinations:

Defn: all possible sequence of taking (order is not important) of r elements from n elements w/o replacement.

$$C_r^n = \frac{P_n^n}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r} \rightarrow$$ (read as n choose r)

$\binom{n}{r}$ is called binomial coefficient

$\binom{n}{0} = \binom{n}{n} = 1 \quad \binom{n}{1} = 1 \quad 0! = 1$

note: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$

end of ch1

- bernoulli trial:

Defn: it is an experiment with two outcomes A or $\bar{A}$ such that

$$P(A) = p \quad , \quad P(\bar{A}) = 1 - p$$

eg (T/H , T/F , 1/0 , Hit/miss)

if we repeat a bernoulli trial N time, what is the Probability of getting K occurences of A and N-K occurrences of $\bar{A}$

$$P[A \text{ occurres exactly } k \text{ time}] = \binom{n}{k} p^{k} (1-p)^{N-K}$$

end of chapter 1