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Diff

* Ordinary diff eq :-

1.1 Basic concepts

1) O.D.E eq. which contains derivatives of the unknown

ex:- $2y' + y = 0$

$y' = (x+1)\, e^{-x} y^2$

$y''' - 2x (y'')^3 = 0$

2) classification

$\boxed{1}$ order $\Rightarrow$ the highest order of derivative.

$\boxed{2}$ linear and non linear

ex on non linear :- y^2, $\sqrt{y}$, e^y, $\ln y$, $\sin y$, yy', $y''y'$

3) homo and non homo

$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_0 y = f(x)$

If $f(x) = 0 \rightarrow$ homo

& $f(x) \neq 0 \rightarrow$ non-homo (pure x functions) examples :-

① $y'' - \sin x \, y' - y = \sin x$.

② $y''' - 2x(y'')^3 = 0$.

③ $(x-1)y' + y = 0$

④ $y'' + 3y = 2$

⑤ $y' + e^x y = x^2$

(6) $(x + xy)\,dx + 2y\,dy = 0$

sol.

1) 2^{nd} order, linear, non-homo

2) 3^{rd} order, non-linear, homo

3) 1^{st} order, linear, homo

4) 2^{nd} order, Linear, non-homo

5) 1^{st} order, Linear, non-homo

6) $\Rightarrow (x + xy) + 2y \frac{dy}{dx} = 0$

$\Rightarrow (x + xy) + 2yy' = 0$

↳ 1^{st} order, non-linear, non-homo.

* In O.D.E we are looking for sol $y = f(x)$
so the sol of O.D.E is $y = f(x)$ satisfying
O.D.E.

example: Show that $y(x) = 10 - c e^{-x}$ with constant c, is ased to $y' + y = 10$.

$y + y' = 10$

$y = 10 - c e^{-x}$

$y' = c e^{-x}$

$10 - \cancel{c e^{-x}} + \cancel{c e^{-x}} \stackrel{?}{=} 10$

$y(x)$ is sol for O.D.E

اذا الوجه بنفس السؤال

* Initial Value problem: I.V.P $=$ O.D.E & initial condition

ex:- Show that $y = c e^{2x}$ is a sol of the IVP

$\underbrace{y' = 2y}_{O.D.E}$, $\underbrace{y(0) = 1}_{I.C}$

① $y = c e^{2x}$

$y(x) = 1 = c e^{0}$

$c = 1$

$y = e^{2x}$

$y' = 2y$

$2 e^{2x} \stackrel{?}{=} 2 e^{2x}$ ✓

1.3 separable O.D.E

$f(x)\,dx = g(y)\,dy$

examples:

① $x \sin y \, dx + x^2 \, dy = 0$

② $x\,dx + x^2 y \, dy = 0.$

③ $dx + x^2 y \, dy = 0$

④ $(x + y)\,dx + x^2 \sin y \, dy = 0$

Sol, 1) $x \sin y \, dx = -x^2 \, dy$

$$\frac{x}{x^2} \, dx = \frac{dy}{\sin y}$$

$$\frac{1}{x} \, dx = -\csc y \, dy \qquad \text{(sep)}$$

$$\int \frac{1}{x} \, dx = \int -\csc y \, dy$$

$$\ln x + C = -\ln |\csc + \cot|$$

② $x \, dx = -x^2 y \, dy.$

$$\frac{1}{x} \, dx = -y \, dy$$

$$\int \frac{1}{x} \, dx = -\int y \, dy$$

$$\ln x + C = \frac{-y^2}{2}$$

3) $dx = -x^2 y \, dy$

$\frac{1}{x^2} dx = -y \, dy$ (sep)

$\int \frac{1}{x^2} dx = -\int y \, dy$

$\frac{-1}{x} + c = \frac{-y^2}{2}$

4) non- ~~Sep~~ separable

* example :- Soluc

1) $\frac{dy}{dx} = y^2 \cos x$

$\int \frac{dy}{y^2} = \int \cos x \, dx$ (sep)

$\Rightarrow -\frac{1}{y} = \sin x + c.$

2) $\frac{dy}{dx} = x^2 y^2 + y^2 + x^2 + 1$

$y(0) = 2$

$\frac{dy}{dx} = y^2 (x^2+1) + (x^2+1)$

$\frac{dy}{dx} = (x^2+1)(y^2+1)$

$\frac{dy}{} = (x^2+1) \, dx$ (sep

$y = 2, \; x = 0$

$\tan^{-1} y = \frac{x^3}{3} + x + c$

$\tan^{-1} 2 = \frac{0}{3} + 0 + c$

$c = \tan^{-1} 2$

$\tan^{-1} y = \frac{x^3}{3} + x + \tan^{-1} 2$

* example:-

$$e^{x+y}\, dx = e^{x-2y}\, dy$$

$$e^{x} e^{y}\, dx = \frac{e^{x}}{e^{2y}}\, dy$$

$$e^{x}\, dx = e^{-2y}\, dy.$$

$$dx = \frac{e^{-2y}}{e^{y}}\, dy.$$

$$\int dx = \int e^{-3y}\, dy \qquad \text{(sep)}$$

$$x + C = \frac{e^{-3y}}{3}$$

$$3x + C = e^{-3y}.$$

H.W:-

$$dy - xy\, dx = (4y + 3x + 12)\, dx$$

$$dy = (4y + 3x + 12 + xy)\, dx$$

$$dy = (4y + 12 + 3x + xy)\, dx$$

$$dy = [4(y+3) + x(3+y)]\, dx$$

$$\int \frac{dy}{3+y} = (x+4)\, dx \qquad \text{(sep)}$$

$$\ln(3+y) = \frac{x^2}{2} + 4x + C.$$

* Reduction to separable form

$y' = f\left(\frac{y}{x}\right)$ homo.

Let $u = \frac{y}{x}$.

* Solve

① $2xyy' = y^2 - x^2$

$2y' = \frac{y^2 - x^2}{xy} = \frac{y}{x} - \frac{x}{y}$ homo.

let $u = \frac{y}{x}$, $y = ux$, $y' = u'x + u$.

$2u'x + \cancel{2u} = \cancel{u} - \frac{1}{u}$

$2u'x = -u - \frac{1}{u} = \frac{-u^2 - 1}{u}$

$2\frac{du}{dx}x = \frac{-u^2 - 1}{u}$

$\int \frac{-2u}{u^2+1}\,du = \int \frac{dx}{x}$ sep.

$-\ln u^2+1 = \ln x + C_1$

$-\ln \left(\frac{y}{x}\right)^2 + 1 = \ln x + C$

(2) $xy' = y + 2x^3 \sin^2 \frac{y}{x}$

$y' = \frac{y}{x} + 2x^2 \sin^2 \frac{y}{x}$

let $u = \frac{y}{x}$, $ux = y$

$y' = u'x + u$

$u'x + u = u + 2x^2 \sin^2 u \Longrightarrow \frac{du}{dx} = 2x \sin^2 u$

$\frac{du}{\sin^2 u} = 2x \, dx$

$\int \csc^2 u \, du = \int 2x \, dx$

$-\cot u = x^2 + c$

$-\cot \frac{y}{x} = x^2 + c$

(3) $(x^2 e^{\frac{2y}{x}} + xy)\, dx = x^2 \, dy$

$\frac{dy}{dx} = \frac{(x^2 e^{\frac{2y}{x}} + xy)}{x^2}$

$y' = \frac{(x^2 e^{2y/x} + xy)}{x^2} \Rightarrow y' = e^{2y/x} + \frac{y}{x}$

let $u = \frac{y}{x}$, $y = ux$, $y' = u'x + u$

$\Rightarrow u'x + \cancel{u} = e^{2u} + \cancel{u}$

$u'x = e^{2u} = \int \frac{du}{e^{2u}} = \int \frac{dx}{x}$

$$\Rightarrow \frac{e^{-2u}}{-2} = \ln x + c$$

$$= \frac{e^{-2\frac{y}{x}}}{-2} = \ln x + c.$$

(4) $(x+y)\,dy = (x-y)\,dx$

$$\frac{dy}{dx} = \frac{(x-y)}{(x+y)} \begin{array}{l} /x \\ /x \end{array}$$

$$y' = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

let $u = \frac{y}{x}$, $y = ux$, $y' = u'x + u$

$$u'x + u = \frac{1-u}{1+u}$$

$$u'x = \frac{1-u}{1+u} - u \times \frac{(1+u)}{(1+u)}$$ وحدنا مقامات

$$\frac{du}{dx}x = \frac{1-u-u-u^2}{1+u}$$

$$\frac{du}{dx}x = \frac{1-2u-u^2}{1+u}$$

$$\int \frac{1+u}{1-2u-u^2}\,du = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln(1+2u-u^2) = \ln x + c$$

HW: solve ① $y' = (y + 4x)^2$

مرات يجب انه نفترضها
ما داخل القوس كله
مثلاً اذا كان linear.

let $u = y + 4x$

$y = u - 4x$

$y' = u' - 4$

$u' - 4 = u^2$

$u' = u^2 + 4$

$$\int \frac{du}{u^2+4} = \int dx \quad \text{-----}$$

② $y' = (x + y - 2)^2$

$u = x + y - 2.$

1.4 : Exact O.D.E.

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{---} \circledast$$

if $M_y = N_x$ then $\circledast$ is Exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The sol. : $u(x,y) = c$

اشتق

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$$ show that ?

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \#$$

so $\dfrac{\partial u}{\partial x} = M(x,y)$

$$\frac{\partial u}{\partial y} = N(x,y)$$

so $u(x,y) = \int M(x,y)\,dx + f(y)$

or $= \int N(x,y)\,dy + f(x)$

example :- Solve

(1) $\underbrace{(1-\sin x\tan y)}_{M(x,y)}dx + \underbrace{(\cos x\sec^2 y)}_{N(x,y)}dy = 0$

$$\left.\begin{aligned} M_y &= -\sin x\sec^2 y \\ N_x &= -\sin x\sec^2 y \end{aligned}\right\} M_y = N_x \Rightarrow \text{Exact.}$$

$u(x,y) = c \Rightarrow u(x,y) = c = x + \cos x\tan y$ ([illegible])

$\Rightarrow$

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$$u(x,y) = \int (1-\sin x \tan y)\, dx + f(y)$$

$$u(x,y) = x + \cos x \tan y + f(y)$$

$$\frac{\partial u}{\partial y} = \cos x \sec^2 y + f'(y)$$

$$\cos x \sec^2 y = \cos x \sec^2 y + f'(y)$$

$$f'(y) = 0 \Longrightarrow \text{so } f(y) = C.$$

(2) $$(\underbrace{2x\cos y + 3x^2 y}_{M(x,y)})\,dx + (\underbrace{x^3 - x^2\sin y - y}_{N(x,y)})\,dy = 0$$

$$\left.\begin{aligned} M_y &= -2x\sin y + 3x^2 \\ N_x &= 3x^2 - 2x\sin y \end{aligned}\right] M_y = N_x \quad \therefore \text{Exact}$$

$$u(x,y) = c.$$

$$u(x,y) = \int (x^3 - x^2\sin y - y)\, dy + f(x)$$

$$= x^3 y + x^2\cos y - \frac{y^2}{2} + f(x)$$

$$\frac{du}{dx} = 3x^2 y + 2x\cos y + f'(x)$$

$$2x\cos y + 3x^2 y = 3x^2 y + 2x\cos y + f'(x)$$

$$f'(x) = 0 \longrightarrow f(x) = c.$$

$$\therefore u(x,y) = c = x^3 y + x^2\cos y - \frac{y^2}{2}$$

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* Integrating Factor

F is I.F if :-

$$\left[F\, M(x,y)\,dx + F\, N(x,y)\,dy\right] = 0$$

is exact O.D.E

* How to find F ?

$\boxed{1}$ $F \Rightarrow R(x) = \dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} \Rightarrow \boxed{R(x) = \dfrac{M_y - N_x}{N}}$

$\hookrightarrow F = e^{\int R(x)\,.dx}$

$\boxed{or}$ $\boxed{R(y) = \dfrac{N_x - M_y}{M}} \rightarrow F = e^{\int R(y)\,.dy}$

$\boxed{2}$ Exact equation

example :-

Solve :- $x^{-1} \cosh y\, dx + \sinh y\, dy = 0$

sol.

$$\left.\begin{array}{l} M_y = x^{-1} \sinh y \\ N_x = 0 \end{array}\right] \quad M_y \neq N_x$$

non-exact

So , we will find I.F

$\Rightarrow$

$$R(x) = \frac{M_y - N_x}{N}$$

$$= \frac{x^{-1} \sinh y - 0}{\sinh y} = \frac{1}{x}$$

$$* \quad F = e^{\int \frac{1}{x} \cdot dx} = e^{\ln x} = x$$

now, multiply the eq. by F

$$\cosh y \, dx + x \sinh y \, dy = 0$$

$M_y = \sinh x$

$N_x = \sinh x$] $M_y = N_x$

Exact

The sol. is $u(x,y) = c$

$$u(x,y) = \int \cosh y \, dx + f(y)$$

$$u(x,y) = x \cosh y + f(y)$$

$$\frac{du}{dy} = N(x,y) \Longrightarrow \cancel{x \sinh y} + f'(y) = \cancel{x \sinh y}$$

$$f'(y) = 0 \, , \, f(y) = \text{const}$$

$$u(x,y) = c = x \cosh y .$$

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example :⇒ $\underbrace{(3x^2y+y^2)}_{M}dx+(\underbrace{2x^3+3xy}_{N})dy=0$

sol. $M_y = 3x^2+2y$, $N_x = 6x^2+3y$] $M_y \neq N_x$ non-exact

* find I.F

$$R = \frac{3x^2+2y-6x^2-3y}{-M} = \frac{-3x^2-\cancel{y}}{-y(3x^2+y)} = \frac{1}{y}$$

$$F = e^{\int R \cdot dy} = e^{\ln y} = y$$

Now $(* y)$

↳ $\underbrace{(3x^2y^2+y^3)}_{M}dx+(\underbrace{2x^3y+3xy^2}_{N})dy=0$

$M_y = 6x^2y+3y^2$, $N_y = 6x^2y+3y^2$] $M_y = N_x$ ⇒ exact

The sol. $u(x,y)=c$.

$$u(x,y) = \int 3x^2y^2+y^3 \, dx + f(y)$$
$$= x^3y^2+y^3x+f(y)$$

$$\frac{\partial u}{\partial y} = 2x^3y+3y^2x+f'$$

$2x^3y+3xy^2 = 2x^3y = 3y^2x+f'(y)$

$f'(y)=0$

$f(x) = \text{const}$, $C = x^3y^2+y^3x$

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* 1.5 1^{st} order linear diff. Eq

$$y' + P(x)\,y = f(x) \quad \begin{cases} f(x) = 0 & \text{sep} \\ f \neq 0 & \end{cases}$$

لازم معامل y' = واحد

$$\boxed{e^{\int P(x)\,dx}}\; y = \int e^{\int P(x)\,dx} f(x)\,dx + C$$

↳ integrating factor for 1^{st} order

example : Solve :-

① $xy' + 2y = \sin x$

$$y' + \frac{2y}{x} = \frac{\sin x}{x}$$

$P(x) = \frac{2}{x}$, $f(x) = \frac{\sin x}{x}$

$$e^{\int \frac{2}{x} dx}\, y = \int e^{\int \frac{2}{x} dx}\, \frac{\sin x}{x}\, dx + c$$

$e^{2\ln x}$

$$x^2 y = \int x^2 \frac{\sin x}{x}\, dx + c$$

$$x^2 y = \int x \sin x\, dx + c$$

$$x^2 y = -x\cos x + \sin x + c.$$

(22) $(1+x^2)\,dy + 4xy\,dx = (1+x^2)^{-2}\,dx$

Sol. $(1+x^2)\dfrac{dy}{dx} + 4xy = (1+x^2)^{-2}$

$$y' + \underbrace{\frac{4x}{1+x^2}}_{P(x)}\, y = \underbrace{(1+x^2)^{-3}}_{f(x)}$$

$$e^{\int \frac{4x}{1+x^2}dx}\, y = \int e^{\int \frac{4x}{1+x^2}dx} (1+x^2)^{-3}\, dx + c$$

$$(1+x^2)^2 y = \int (1+x^2)^2 (1+x^2)^{-3}\, dx + c$$

$$(1+x^2)^2 y = \int \frac{1}{1+x^2}\, dx + c.$$

$$(1+x^2)^2 y = \tan^{-1} x + c.$$

* Bernoulli D.E.

$n=0 \to$ 1st order

$n=1 \to$ sep.

$$y' + P(x)\,y = f(x)\,y^n, \quad n \neq 0, 1 \quad \text{----} \; (*)$$

let $u = y^{1-n} \Rightarrow u' = \dfrac{du}{dx} = (1-n)\,y^{-n}\,y'$

then, Multiply (*) by $(1-n)\,y^{-n}$

Find, the value of (n) that makes the eqn Bernoulli

كمرين سؤال $\Rightarrow$

* examples:

① $x^2 y' + 2xy = y^3$

Sol.

$$\frac{x^2 y' + 2xy = y^3}{x^2}$$

$$y' + \frac{2}{x} y = \frac{1}{x^2} y^3 \quad \text{Ber}$$

let $u = y^{1-3} = y^{-2}$

$u' = -2 y^{-3} y'$

Now, $-2y^{-3} \left(y' + \frac{2}{x} y = \frac{1}{x^2} y^3 \right)$

$$-2y^{-3} y' + \frac{-2x^2}{x} y^{-3} y = \frac{-2 y^{-3}}{x^2} y^3.$$

$$-2 y^{-3} y' - \frac{4}{x} y^{-2} = \frac{-2}{x^2}$$

$$u' - \frac{4}{x} u = \frac{-2}{x^2} \quad \text{Lin}$$

$$e^{\int \frac{-4}{x} dx} u = \int e^{\int \frac{-4}{x} dx} \frac{-2}{x^2} dx + c$$

$$x^{-4} u = -2 \int x^{-4} x^{-2} dx + c$$

$$x^{-4} u = -2 \int x^{-6} dx + c$$

$$x^{-4} y^{-2} = \frac{-2}{-5} x^{-5} + c$$

(2) $2xyy' + (x-1)y^2 = x^2 e^x$

$$y' + \frac{x-1}{2x} y = \frac{x e^x}{2} y^{-1}$$ Ber

let $u = y^{1+1} = y^2$

$u' = 2yy'$

$$2yy' + \frac{x-1}{x} y^2 = x e^x$$ lin

$$u' + \frac{x-1}{x} u = x e^x$$

$$e^{\int 1 - \frac{1}{x} dx} u = \int e^{\int 1 - \frac{1}{x} dx} x e^x \, dx + c$$

$$e^{x - \ln x} u = \int e^{x - \ln x} x e^x \, dx + c.$$

$$\frac{e^x}{x} u = \int \frac{e^x}{x} x e^x \, dx + c$$

$$\frac{e^x}{x} y^2 = \frac{e^{2x}}{2} + c.$$

$\boxed{1}$ G. form $y'' + p(x) y' + q(x) y = f(x)$ ⊛
$f = 0 \rightarrow$ homo
$f \neq 0$ non homo

* Chapter 2 :- 2^{nd} order diff. Eq

2.1 Homo linear O.D.E

example :- ⊛ $y'' + y' + 2y = x$
2^{nd} order, linear

$\boxed{2}$ If y is a sol of ⊛ then cy is sol.

$\boxed{3}$ If y_1, y_2 are sol, then $c_1 y_1 + c_2 y_2$ is sol
if ⊛ is linear & homo.

* example : if y_1, y_2 are sol. for
$y y' + x^2 y'' = 0$.
Then $c_1 y_1 + c_2 y_2$ is sol. ✗
↳ $y y' + x^2 y'' = 0$ is non-linear.

$\boxed{4}$ let $y_1\ y_2$ be two sol. of ⊛ if $\frac{y_1}{y_2} = c$
$\Rightarrow y_1 = c y_2$, then y_1, y_2 are linear dep.
other wise $\rightarrow$ linear indep.

examples :- ① $y_1 = e^x$
$y_2 = 2e^x$

sol $\frac{y_1}{y_2} = \frac{1}{2}$ const, so y_1, y_2 are liner dep

③ $y_1 = e^x$
$y_2 = xe^x$

sol.

$$\frac{y_1}{y_2} = \frac{1}{x} \neq c \quad , \; y_1, y_2 \text{ are linear indep.}$$

* Reduction of order

$\boxed{1}$ y – is missing

$$y'' = f(y', x)$$

let $u = y'$
$u' = y''$

اذا كانت x & y are missing
y'' or y بلش بال

$\boxed{\text{example}}$ $y'' + y' = 0$ (y – missing)

let $u = y'$
$u' = y''$

$$u' + u = 0$$

$$\frac{du}{dx} + u = 0 \quad (\underline{\text{sep}})$$

$$\frac{du}{u} = -dx$$

$$\ln u = -x + c$$

$$u = e^{-x+c} = y'$$

$$y' = ce^{-x}$$

$$\frac{dy}{dx} = ce^{-x} \quad (\text{sep})$$

$$\int dy = \int ce^{-x}\, dx$$

$$y = -ce^{-x} + c_1$$

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2 x - is missing

$$y'' = f(y, y')$$

let $u = y'$

$u' = y''$

chain rule. $\left[\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u = y'' \right.$

u

example solve :-

$$y y'' + (y')^2 = 0$$

x - missing

let $u = y'$

$$u' = y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$$

u

$$y \cdot u \frac{du}{dy} + u^2 = 0$$

$$y u \frac{du}{dy} = -u^2$$

$$y \frac{du}{dy} = -u$$

$$\int \frac{du}{u} = -\int \frac{dy}{y}$$

sep.

$\rightarrow$ * $\ln u = -\ln y + c_1$

$$y' = e^{-\ln y + c} = e^{-\ln y} \cdot e^{c_1}$$

$y' = \frac{c}{y}$ sep $\rightarrow \frac{dy}{dx} = \frac{c}{y}$

$$\int y \, dy = \int c_1 \, dx = \frac{y^2}{2} = c_1 x + c_2$$

$\boxed{3}$ Given $\frac{1}{=}$ sol.

consider $y'' + p(x)y' + q(x)y = 0$

Given y_1, find y_2 as follows $\Rightarrow$ $$\boxed{y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx}$$

example:

$$y'' - 4y' + 4y = 0$$

$y_1 = e^{2x}$, find y_2.

Sol.

$$y_2 = e^{2x} \int \frac{e^{-\int -4\,dx}}{(e^{2x})^2}\,dx$$

$$y_2 = e^{2x} \int \frac{e^{4x}}{e^{4x}}\,dx = x\,e^{2x}.$$

$$y = c_1 e^{2x} + c_2 x e^{2x}$$ general sol.

2.2 Homo Linear O.D.E with const coeffe:

$$ay'' + by' + cy = 0 \quad --- \quad (*)$$

The sol e^{rx}

char. eq.

$$ar^2 + br + c = 0 \longrightarrow \text{quadratic eq.}$$

$$\Delta = b^2 - 4ac$$

1. $\Delta > 0 \rightarrow$ has 2 sol r_1, r_2, $r_1 \neq r_2$
2. $\Delta = 0 \rightarrow$ has only one sol or reapeated $r = r_1 = r_2$
3. $\Delta < 0 \rightarrow$ has complex root $r = \lambda \mp i\mu$.

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* Quick Revision:-

$r = \lambda + i\mu$

$i = \sqrt{-1}$

$i^2 = -1$

$-i \overset{?}{=} \frac{1}{i} \Rightarrow -i \overset{?}{=} \frac{1 \times i}{i \times i} \Rightarrow -i = \frac{i}{-1}$ ✓

* Ouler Rules

$e^{(\lambda + i\mu)x} = e^{\lambda x} e^{i\mu x} = e^{\lambda x}(\cos \mu x + i \sin \mu x)$

Δ

① $\Delta > 0 \equiv \boxed{r_1 \neq r_2}$

the sol of ⊛ $y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

② $\Delta = 0$, $\boxed{r = r_1 = r_2}$

the sol of ⊛ : $y_h = c_1 e^{rx} + c_2 x e^{rx}$

③ $\Delta < 0$, $r = \lambda \mp i\mu$

the sol of ⊛ , $y_h = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$

example :- Solve:-

① $y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0$

$(r+1)(r-3) = 0$

$\boxed{r = -1}$ $\boxed{r = 3}$

$y_h = c_1 e^{-x} + c_2 e^{3x}$

$\{e^{-x}, e^{3x}\}$ basis sol.

② $y'' - 3y' = 0$

$r^2 - 3r = 0$

$r(r-3) = 0$

$r = 0, \quad r = 3$

$y_h = c_1 e^{0x} + c_2 e^{3x} = c_1 + c_2 e^{3x}$

③ $2y'' - 12y' + 18y = 0$

$2r^2 - 12r + 18 = 0$

$\cancel{r^2} - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r = 3, 3$

$y_h = c_1 e^{3x} + c_2 x e^{3x}$

example ⇒ find the O.D.E ① which has the sol. $y = c_1 e^{3x} + c_2 e^{-2x}$

sol. $r = 3, \quad r = -2$.

$(r-3)(r+2) = 0$

char $r^2 - r - 6 = 0$

$y'' - y' - 6y = 0$

② which has the sol. $y(x) = e^{2x}(c_1 + x c_2)$

sol. $r = 2, 2$

$(r-2)(r-2) = 0$

$r^2 - 4r + 4 = 0$

$y'' - 4y' + 4y = 0$.

Ex: Solve:- ① $y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r^2 = -1 \quad (r = \pm\sqrt{-1})$

$r = \mp i$

$y_h = c_1 e^{0x} \cos x + c_2 e^{0x} \sin x$

$y_h = \underbrace{c_1 \cos x}_{y_1} + \underbrace{c_2 \sin x}_{y_2}$

② $2y'' + 2y' + 4y = 0$

$2r^2 + 2r + 4 = 0$

$r^2 + r + 2 = 0$

$\Delta = b^2 - 4ac$

$\Delta = 1 - 8 = -7 \quad , \quad r = \frac{-1 \mp \sqrt{1 - 4(1)(2)}}{2}$

$r = \frac{-1 \mp \sqrt{7} i}{2} = \frac{-1}{2} \mp \frac{\sqrt{7}}{2} i$

$y_h = c_1 e^{-\frac{1}{2}x} \cos \frac{\sqrt{7}}{2} x + c_2 e^{-\frac{1}{2}x} \sin \frac{\sqrt{7}}{2} x.$

③ $y'' + 9y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 3$

$r^2 + 9 = 0 \Rightarrow r = \mp 3i$

$y_h = c_1 \cos 3x + c_2 \sin 3x$

$y(0) = 1 \Rightarrow \boxed{c_1 = 1}$

$y'_h = \cos 3x + c_2 \sin 3x$

$y' = 0 + 3c_2 \cos 3x \Rightarrow 3 = 3c_2 \Rightarrow \boxed{c_2 = 1}$

example: Find 2nd order lin. homo. O.D.E which has the sol. $\boxed{y = c_1 \cos 2x + c_2 \sin 2x}$

$r = \mp 2i$

$(r - 2i)(r + 2i) = 0$

$r^2 + 2ri - 2ri + 4 = 0$

$r^2 + 4 = 0$

$y'' + 4y = 0$

2.5 Cauchy Euler Equation C.E.E

① $ax^2y'' + bxy' + cy = 0$ C.E.E.

② The sol for C.E.E. $y = x^r$

③ $ar(r-1) + br + c = 0.$
quadratic eq.

④ Cases:
1) $r_1 \neq r_2 \Rightarrow y = c_1x^{r_1} + c_2x^{r_2}$
2) $r_1 = r_2 = r \Rightarrow y = c_1x^r + c_2x^r \ln x.$
3) $r = \lambda \mp i\mu = c_1 x^\lambda \cos \mu \ln x + c_2 x^\lambda \sin \mu \ln x$

* **example:** Solve

① $2x^2y'' + 3xy' - y = 0$

$2r(r-1) + 3r - 1 = 0$

$2r^2 - 2r + 3r - 1 = 0$

$2r^2 + r - 1 = 0$

$\Rightarrow$

$\Longrightarrow$

$$r = \frac{-1 \mp \sqrt{1-4(2)(-1)}}{4} = \frac{-1 \mp 3}{4}$$

$$r = \left\{ \begin{matrix} -1 \\ \frac{1}{2} \end{matrix} \right\} \rightarrow y = c_1 x^{-1} + c_2 x^{\frac{1}{2}}$$

② $x^2 y'' - 3xy' + 4y = 0$

$r(r-1) - 3r + 4 = 0$

$r^2 - r - 3r + 4 = 0$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r = 2, 2$

$y = c_1 x^2 + c_2 x^2 \ln x$

③ $x^2 y'' + 7xy + 13y = 0$

$r(r-1) + 7r + 13 = 0$

$r^2 - r + 7r + 13 = 0$

$r^2 + 6r + 13 = 0$

$$r = \frac{-6 \mp \sqrt{36 - 4(13)}}{2} = \frac{-6 \mp 4i}{2} = -3 \mp 2i$$

$y = c_1 x^{-3} \cos 2 \ln x + c_2 x^{-3} \sin 2 \ln x$

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(4) $xy'' + 2y' = 0$

Sol, $x^2y'' + 2xy' = 0$

$r(r-1) + 2(r) = 0$

$r(r-1+2) = 0$

$r = 0, -1$

$y = c_1 x^0 + c_2 x^{-1} = c_1 + c_2 x^{-1}$

example :- (1) find the O.D.E which has the sol:

$y = c_1 x^3 + c_2 x^2$

sol. $r = 3$, $r = 2$

$(r-3)(r-2) = 0$

$(r - 2r + 3) + 6 = 0$

$r^2 - 5r + 6 = 0$

$r^2 - r - 4r + 6 = 0$

$r(r-1) - 4r + 6 = 0$

$x^2y'' - 4xy' + 6y = 0$

(2) find the C.E.E which has the characteristic eq. : $r^2 - 5r + 6 = 0$

$r^2 - r - 4r + 6 = 0$

$r(r-1) - 4r + 6 = 0$

$x^2y'' - 4xy' + 6y = 0$

③ find the O.D.E with const. coeff which has the char. eq $r^3 - 5r + 6 = 0$

Sol. $r^2 - 5r + 6 = 0$

$y'' - 5y' + 6y = 0$

---

* let y_1, y_2 be 2 sol. for
$y'' + p(x)y' + q(x)y = 0$

Then $w[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

If $w[y_1, y_2] = 0$ then y_1, y_2 are linear dep.
if $w[y_1, y_2] \neq 0$ then y_1, y_2 are linear indep

example:

$w[x^3, x^2] = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix}$

$= 2x^4 - 3x^4 = -x^4 \neq 0$

y_1, y_2 are linar Indep.

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2.7 * Non homo O.D.E

$$y'' + P(x)\,y' + q(x)\,y = f(x)$$

$f = 0$ homo — Real, Const, C.E-E

$f \neq 0$ non-homo

$$y_g = y_h + y_p$$

non-homo → undeterminal coefficient Method; → variation of Parameter

$f(x)$	y_p
① $K e^{\gamma x}$	$A e^{\gamma x}$
② $K x^n$	$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

ex: $f(x) = x^2$

$y_p = a_2 x^2 + a_1 x + a_0$

③ constant K	A constant
④ $K \cos \omega x$, $K \sin \omega x$ →	$A \cos \omega x + B \sin \omega x$

ex: $f(x) = \cos 2x \Rightarrow y_p = A \cos 2x + B \sin 2x$

⑤ $K e^{\gamma x} \cos \omega x$, $K e^{\gamma x} \sin \omega x$ →	$A e^{\gamma x} \cos \omega x + B e^{\gamma x} \sin \omega x$

* The general sol of the diff eq:

$\ln(y) + \frac{x}{3y} y' = 0$ is given by :-

a) $3x \ln y$

b) $\frac{x^2}{y^3} = c$

c) $\frac{x}{y} = c$

d) $\frac{3x}{y^2}$

e) $x' \ln(y) = c$

Sol.

$\ln y = \frac{-x}{3y} \cdot y'$

$y \ln y = -\frac{x}{3} \frac{dy}{dx}$

$\int \frac{dy}{y \ln y} = \int \frac{-3}{x} dx$

$u = \ln y \Rightarrow du = \frac{1}{y} dy$

$\int \frac{du}{u} = \int \frac{-3}{x} dx$

$\ln u = -3 \ln x + c$

$e^{\ln \ln y} = e^{-3 \ln x + c}$

$\ln y = e^{-3 \ln x} c$

$\ln y = \frac{c}{x^2}$

$x^3 \ln y = c$

* The general sol of diff eq $x^2y'' - 3y' + 4y = 0$, $x > 0$ C.E.E is given by :-

<u>sol</u>

$$r(r-1) - 3r + 4 = 0$$
$$r^2 - r - 3r + 4 = 0$$
$$r^2 - 4r + 4 = 0$$
$$(r-2)(r-2) = 0$$
$$r = 2, 2$$
$$y_h = c_1x^2 + c_2x^2 \ln x.$$

* convert the following Bernoulli eq to linear diff eq.

$\boxed{\text{sol}}$ $x\frac{dy}{dx} + y = x^2y^2$

$$\left(\frac{dy}{dx} + \frac{1}{x}y = xy^2\right)$$

let $u = y^{1-2} = y^{-1}$

$u' = -1y^{-2}y'$

$\hookrightarrow -y^{-2}y' + \frac{-1y^{-2}y}{x} = -y^{-2}xy^2$

$$u' - \frac{1}{x}u = -x.$$

* Find I.F of $(x + ye^y)y' + (\cos x + \ln y)y = 0$ to be exact

<u>sol.</u> $M_y = \cos x + \ln y + 1$

$N_x = 1$

$$R(y) = \frac{\cos x + \ln y + 1 - 1}{-M} = \frac{\cos x + \ln y}{-y(\cos x + \ln y)}$$

$$R(y) = \frac{-1}{y}, \quad F = e^{\int -\frac{1}{y}\,dy} = e^{-\ln y} = \frac{1}{y}$$

sol $r^2+5r+6=0$

$(r+3)(r+2)=0$

$r=-3, -2$

$y=c_1e^{-3x}+c_2e^{-2x}$

② $y(\ln x-\ln y)\,dx-x\,dy=0$

$y\ln\frac{x}{y}\,dx=x\,dy$

$\frac{y}{x}\ln\frac{x}{y}=\frac{dy}{dx}$

$f(\frac{y}{x})=y'$ homo

③ An I.F of diff eq $dx+(\frac{x}{y}-\sin y)\,dy=$ is.

* If the sol. 2^{nd} order linear homo D.E is $y(x)=c_1+c_2\ln x$

Then the D.E can be written as:-

$r=0,0$

$r^2=0$

$(r^2-r)+r=0$

$r(r-1)+r=0$

$x^2y''+xy=0$

* if $(2xy^m + 2y)\,dx + (2x^2y + 2x)\,dy = 0$ is exact, Then $m = ?$

Sol. $M_y = N_x$

$2mxy^{m-1} + 2 = 4xy + 2$

$2m = 4 \longrightarrow m = 2$

(or) $y^{m-1} = y$

$m - 1 = 1 \Rightarrow m = 2$

* if $w(1, g(x)) = 4x^3$, Then $g(x) = ?$

[1] $\begin{vmatrix} 1 & g(x) \\ 0 & g' \end{vmatrix} = 4x^3$

[2] $g' = 4x^3$

[3] $\frac{dg}{dx} = 4x^3$

[4] $\int dg = \int 4x^3\,dx$

$g(x) = x^4 + c$

* The sol. of the problem
$2xyy' = 1+y^2$, $y(2)=3$

Sol. $2x\frac{dy}{dx} = \frac{1+y^2}{y}$

$\int \frac{2y}{1+y^2}\,dy = \int \frac{dx}{x}$

$e^{\ln 1+y^2} = e^{\ln x + c}$ $\Longrightarrow$ $1+y^2 = cx$

$10 = 2c$

$c = 5$

* solve : $y - x + xy' = 0$, $y(0) = -1$, what $y(2)$?

$y - x + xy' = 0$

$y - x = -xy'$

$\frac{y}{x} - 1 = -y'$

$y' = 1 - \frac{y}{x}$

let $u = \frac{y}{x}$

$y' = u'x + u$

$u'x + u = 1 - u$

$\frac{du}{dx}x = 1 - 2u$

$\frac{du}{1-2u} = \frac{dx}{x}$

$-\frac{1}{2}\ln 1-2u = \ln x + 0$

$-\frac{1}{2}\ln(1-2\frac{y}{x}) = \ln x + c$

$y(1) = -1$; $y_{x=1} = -1$

$-\frac{1}{2}\ln(1-2\frac{(-1)}{1}) = \cancel{\ln(1)} + c$

$-\frac{1}{2}\ln 3 = c$

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$\Rightarrow$

$-\frac{1}{2}\ln\left(1-\frac{2y}{x}\right) = \ln x - \frac{1}{2}\ln 3$

$-\frac{1}{2}\ln\left(1-\frac{2y}{2}\right) = \ln(2) - \frac{1}{2}\ln 3$

$-\frac{1}{2}\ln\left(1-\frac{\cancel{2}y}{\cancel{2}}\right) = \ln(2) - \frac{1}{2}\ln 3$

$\cancel{\frac{-1}{2}}\ln(1-y) = -2\ln(2) + \ln 3$

$\ln(1-y) = \ln 3 - \ln 4$

$e^{\ln(1-y)} = e^{\ln(3) - \ln 4}$

$1-y = \frac{3}{4} \longrightarrow 1-\frac{3}{4} = y \Rightarrow y = \frac{1}{4}$

* The fundamental sol. set for

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

$y = c_1 \cos x + c_2 \sin x$

⇒ * undeterment coeff.

examples

① Find y_h

$y'' - y = 0$

$r^2 - 1 = 0$

$r = 1, -1$

$y_h = c_1 e^x + c_2 e^{-x}$

② find y_p

$y_p = a_2 x^2 + a_1 x + a_0$

$y' = 2a_2 x + a_1$

$y'' = 2a_2$

$2a_2 - (a_2 x^2 + a_1 x + a_0) = 2x^2 + 1$

$-a_2 x^2 - a_1 x + 2a_2 - a_0 = 2x^2 + 1$

$-a_2 = 2 \Rightarrow a_2 = -2$

$-a_1 = 0 \longrightarrow a_1 = 0$

$2a_2 - a_0 = 1$

$2(-2) - a_0 = 1$

$-a_0 = 5$

$a_0 = -5$

$y_p = -2x^2 - 5$

✱ last step : $y_g = y_h + y_p = c_1 e^x + c_2 e^{-x} - 2x^2 - 5$

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* Solve :- $y' + y = 2e^{3x}$

① Find y_h

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \mp i$

$y_h = c_1 \cos x + c_2 \sin x$

② Find y_p

$y_p = A e^{3x}$

$y' = 3A e^{3x}$

$y'' = 9A e^{3x}$

$9Ae^{3x} + Ae^{3x} = 2e^{3x}$

$10A = 2$

$A = \frac{1}{5}$

$y_p = \frac{1}{5} e^{3x}$

③ $y_g = y_h + y_p$

$= c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{3x}$

* Solve:- $y^{4} - 4y = 2 \sin x$

① $y_h \Rightarrow r^2 - 4 = 0 \Rightarrow r \mp 2$

$y_h = c_1 e^{2x} + c_2 e^{-2x}$

② $y_p \Rightarrow A \sin x + B \cos x$

$y' = A \cos x - B \sin x$

$y'' = -A \sin x - B \cos x$

$-A\sin x - B\cos x - 4A\sin x + 4B\cos x = 2\sin x$

$-5A\sin x - 5B\cos x = 2\sin x$

$-5A = 2 \Rightarrow \frac{-2}{5} = A$

$-5B = 0 \Rightarrow 0 = B$

$y_p = \frac{-2}{5}\sin x$

(3) $y_g = y_h + y_p$

$= c_1 e^{2x} + c_2 e^{-2x} - \frac{2}{5}\sin x$

example:- Solve:- $y'' - y = 2e^x$

(1) $y_h \Rightarrow r^2 - 1 = 0 \Rightarrow r = \mp 1, \quad y_h = c_1 e^x + c_2 e^{-x}$

(2) $y_p \Rightarrow y = Axe^x$

$y' = Axe^x + Ae^x$

$y'' = Axe^x + Ae^x + Ae^x = Axe^x + 2Ae^x$

$\Rightarrow \cancel{Axe^x} + 2Ae^x - \cancel{Axe^x} = 2e^x$

$2Ae^x = 2e^x$

$2A = 2$

$\boxed{A = 1}, \quad y_p = xe^x$

(3) $y_g = y_h + y_p = c_1 e^x + c_2 e^{-x} + xe^x.$

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* Determine the form of particular sol. y_p :-

① $y'' - 3y' + 2y = 2e^{x}\sin x$

$\boxed{1}$ $y_h \Rightarrow r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r = 1, 2$

$y_h = c_1 e^{x} + c_2 e^{2x}$

$\boxed{2}$ $y_p = A e^{x}\sin x + B e^{x}\cos x$.

② $y'' + 2y' = \boxed{x^4 - 2x^3 + 1} + x^2 e^{-2x} + \cos 2x$.

$\boxed{1}$ $y_h \Rightarrow r^2 + 2r = 0$

$r(r+2) = 0$

$r = 0, -2$

$y_h = c_1 e^{0} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$

ضربناه x لتكرار $\boxed{a_1, a_0}$

$\boxed{2}$ $y_p = (a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0)\, x$

$+ (b_2 x^2 + b_1 x + b_0)\, e^{-2x}\, x$

$+ A\cos 2x + B\sin x$.

③ $y'' + 2y' + 2y = e^{-x} + \cancel{B}2e^{-x}\sin x$

① $y_h \Rightarrow r^2 + 2r + 2 = 0$

$r = -1 \mp i$

$y_h = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$

$\Rightarrow$

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2.10 : Variation of Parameter

$$y'' + P(x)\, y' + q(x)\, y = r(x)$$

① $y_h = c_1 y_1 + y_2$

② $r(x) = r(x)$: the coefficient of y'' is 1

③ y_p

[1] $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \boxed{y_1 y_2' - y_1' y_2}$

[2] $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

[3] $y_g = y_h + y_p$

example $\quad y'' + y = \sec x$

① $y_h = r^2 + 1 = 0 \Rightarrow r = \pm i$, $\quad y_h = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$

② $r(x) = \sec x$

③ $y_p \Rightarrow$ ④ $W[\cos x, \sin x] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 + \sin^2 = 1$

④ $-\cos x \int \frac{\sin x \sec}{1} dx + \sin x \int \frac{\cos x \sec}{1} dx$

$y_p = +\cos x \ln \cos x + x \sin x$

$y_g = y_h + y_p$

Ch: 3 / Higher order Linear O.D.E.

3.1 Homo Linear O.D.E

① $y^{(4)} - y = 0$

② $y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$

③ $y^{(4)} + 2y'' + y = 0$

④ $x^3 y''' - 3x^2 y'' + 6x y' - 6y = 0$.

Sol. ① $y^{(4)} - y = 0$

$r^4 - 1 = 0$

$(r^2-1)(r^2+1) = 0$

$r = \mp 1, \; r = \mp i$

$y_h = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

② $y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$

$r^5 - 3r^4 + 3r^3 - r^2 = 0$

$r^2 (r^3 - 3r^2 + 3r - 1) = 0$

$r^2 (r-1)(r^2 - 2r + 1) = 0$

$r^2 (r-1)(r-1)(r-1) = 0$

$r = 0, 0, 1, 1, 1$

$y_h = c_1 + c_2 x + c_3 e^{x} + c_4 e^{x} x + c_5 x^2 e^{x}$

(3) $y^{(4)} + 2y'' + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)(r^2+1) = 0$

$r = \pm i \quad , \quad r = \pm i$

$y_h = c_1 \cos x + c_2 x \cos x + c_3 \sin x + c_4 x \sin x$

(4) $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$

* eluer cauchy $\longrightarrow a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_2 x^2 y'' + a_1 x y' + a_0 y = 0$

sol,

$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$r(r-1)(r-2) - 3r(r-1) + 6(r-1) = 0$

$(r-1)\left[r(r-2) - 3r + 6\right] = 0$

$(r-1)\left[r(r-2) - 3(r-2)\right] = 0$

$(r-1)(r-2)(r-3) = 0$

$r = 1, 2, 3$

$y_h = c_1 x + c_2 x^2 + c_3 x^3$

* example : $y''' - y = 0$

$r^3 - 1 = 0$

$(r-1)(r^2 + r + 1) = 0$

$r = 1 \quad , \quad r = \dfrac{-1 \mp \sqrt{1-4}}{2}$

$\boxed{r = 1 \quad , \quad r = \dfrac{-1 \mp \sqrt{3}\,i}{2}}$

$y_h = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos \dfrac{\sqrt{3}}{2} x + c_3 e^{-\frac{1}{2}x} \sin \dfrac{\sqrt{3}}{2}$

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H.W $\Rightarrow$ $y_h = c_1 x + c_2 x \ln x + c_3 x (\ln x)^2$

Find O.D.E.

Sol. $x^3 y''' + x y' - y = 0.$

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* non - homo O.D.E.

ex: Solve (undetr. coff)

$$y''' - 3y'' + 3y' - y = x^2 e^x + \sin x$$

find y_p using undetr. coff Method.

① $y_h \Rightarrow y''' - 3y'' + 3y' - y = 0$

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)(r^2 - 2r + 1) = 0$$

$$(r-1)(r-1)(r-1) = 0$$

$$r = 1, 1, 1$$

$$y_h = c_1 e^x + c_2 x e^x + c_3 e^x x^2$$

② $y_p = (a_2 x^2 + a_1 x + a_0) e^x \cdot x \cdot x \cdot x$

$$+ A \sin x + B \cos x$$

* Variation of parameters.

ex: $x^3 y''' - 3x^2 y'' + 6x y' - 6y = 25 x^5$

① $y_h \Rightarrow r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$$= (r-1)(r-2)(r-3) = 0$$

$$r = 1, 2, 3$$

$$y_h = c_1 \underset{\uparrow y_1}{x} + c_2 \underset{\downarrow y_2}{x^2} + c_3 \underset{\downarrow y_3}{x^3}$$

② $R(x)$ after ① نخلي y''' لحالو

$$r(x) = \frac{25 x^5}{x^3} = 25 x^2.$$

④ $y_g = y_h + y_p$

افقي للاول صف

رأسي للاول عمود

$+ \; - \; +$

باخد اسهل عمود بالجدول الاشارات
صف او

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

---

③ y_p : $\boxed{1}$ $W[x, x^2, x^3] = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

$$= x \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - 1 \begin{vmatrix} x^2 & x^3 \\ 2 & 6x \end{vmatrix}$$

$$= x\left(12x^2 - 6x^2\right) - \left(6x^3 - 2x^3\right)$$

$$6x^3 - 4x^3 = 2x^3$$

$\boxed{2}$ $$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix}$$

$$= 1 \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$$

$\boxed{3}$ $$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -1 \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = -\left(+3x^3 - x^3\right)$$

$$= -2x^3$$

$\boxed{4}$ $$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = +1 \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

$\boxed{5}$ $$y_p = \sum_{i=1}^{3} y_i \int \frac{W_i \, r(x)}{W}$$

or $$y_p = y_1 \int \frac{W_1 \, r(x)}{W} + y_2 \int \frac{W_2 \, r(x)}{W} + y_3 \int \frac{W_3 \, r(x)}{W}$$

$$y_p = x \int \frac{x^4 \cdot 25x^2}{2x^3} + x^2 \int \frac{-2x^3 \cdot 25x^2}{2x^3} + x^3 \int \frac{x^2 \cdot 25x^2}{2x^3}.$$

* find the g.s.

$y''' + y' = \sec t$

$\boxed{1}$ $y_h \Rightarrow r^3 + r = 0$

$r(r^2+1) = 0$

$r = 0, \mp i$

$y_h = c_1 + c_2 \cos t + c_3 \sin t$

$\boxed{2}$ $r(t) = \sec t$

$\boxed{3}$ y_p

1) $W[1, \cos t, \sin t] = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin \end{vmatrix}$

$W = \sin^2 + \cos^2 = \boxed{1 = W}$

2) $w_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin & \cos \\ 1 & -\cos & -\sin \end{vmatrix} = \cos^2 + \sin^2 = 1$

$\boxed{w_1 = 1}$

3) $w_2 = \begin{vmatrix} 1 & 0 & \sin \\ 0 & 0 & \cos \\ 0 & 1 & -\sin \end{vmatrix} = -\cos t$

$\boxed{w_2 = -\cos t}$

4) $W_3 = \begin{vmatrix} +1 & \cos & 0 \\ -0 & -\sin & 0 \\ +0 & -\cos & 1 \end{vmatrix} = -\sin t$

$\boxed{W_3 = -\sin t}$

$\boxed{5}$ $y_p = y_1 \int \frac{W_1\, r(t)}{W}\, dt + y_2 \int \frac{W_2\, r(t)}{W}\, dt + y_3 \int \frac{W_3\, r(t)}{W}$

$= \int \frac{1 \quad \sec t}{1}\, dt + \cos t \int \frac{-\cos t \, \sec t}{1} + \sin \int \frac{-\sin t \, \sec t}{1}\, dt$

$= \ln |\sec t + \tan t| - t\cos t + \sin t \ln \cos t$

$\boxed{5}$ $y_g = y_p + y_h$.

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ex: $y''' + y' = 1 + 2x + \cos x$ find y_p ?

(1) $y_h \Rightarrow r^3 + r = 0$

$r = 0, \mp i$

$y_h = c_1 + c_2 \cos x + c_3 \sin x$

(2) $y_p = (a_1 x + a_0) x + (A \cos x + B \sin x) x$

* Chapter 7 Matrix

① $\begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \end{bmatrix}$

num of rows ← $n \times m$ → no. of colum.

② example: let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ ← squar Matrix

find: ① $A+B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$

② $A - C$ = undefined.

③ $-2C = -2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -6 & -8 \end{bmatrix}$

* in multi: no. of colums in first = no. of row in 2nd

④ AB = undefine

⑤ AC = undefine

⑥ $CA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+2 & 3+4 \\ 3+0 & 6+4 & 9+8 \end{bmatrix}$

Inverse: ① squar Matrix $n \times n$

② Identity Matrix I_n

[1] I_n square Matrix

[2] $IA = AI = A$

[3] $I_1 = [1]$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

③ Inverse Matrix

[1] A^{-1}.

[2] $A^{-1}A = AA^{-1} = I$

[3] $\det(A) = |A|$

* example: let A be $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$|A| = 4-6 = -2$, $A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

[4] let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* Linear Algebra:-

ex:- let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

① order of $A = 4 \times 4$
order of $B = 4 \times 2$.

(2) If $C = AB$ find

C_{12}

$\rightarrow$ 1^{st} row of $A \times 2^{nd}$ colum of B

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 2+2+0+4 = 8$$

(3) $$B^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}$$

[*] Inverse :-

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|A| = ad - cb$

(1) $A^{-1}A = AA^{-1} = I$

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$$Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

ex:-

$$2x + 2y = 0$$
$$3x - y = 4$$

Find x, y.

① $Ax = b$

$$\begin{bmatrix} 2 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

② $A^{-1} = \frac{1}{-2-6} \begin{bmatrix} -1 & -2 \\ -3 & 2 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} -1 & -2 \\ -3 & 2 \end{bmatrix}$

③ $X = A^{-1}b$

$$= -\frac{1}{8} \begin{bmatrix} -1 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$= -\frac{1}{8} \begin{bmatrix} -8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$x = 1$

$y = -1$

* Inverse for 3 x 3 Matrix

* adjoint Matrix :

ex :- Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, Find A^{-1}

$$A^{-1} = \frac{1}{|A|} \left[\text{adjoint of } A \right]^T$$

① $|A| = 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \cancel{2} \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -1 + 2 = 1$

③ Adjoint of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$\boxed{\text{if}}$ $A \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

A_{ij} = cofactor of A

$A_{11} = + \begin{vmatrix} e & f \\ h & i \end{vmatrix}$

$A_{12} = - \begin{vmatrix} d & f \\ g & i \end{vmatrix}$

$A_{13} = + \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

sol. for last example:-

$$A_{11} = + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A_{12} = - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$A_{13} = + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{21} = - \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$A_{22} = + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A_{23} = - \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 2$$

$$A_{31} = + \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$$

$$A_{32} = - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{33} = + \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$$

$\Rightarrow$ adj of $A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{1} \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$

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ch #4 :- if $y_1 = 12x_1 + 7x_2$

$y_2 = -8x_1 + 3x_2$

so $\Longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\bar{y}(\alpha) = A\bar{\alpha}$$

* eigen values + eigen vectors.

$$A\vec{\alpha} = \lambda \bar{x}$$

Matrix ↲ ↳ scaler

$$A\bar{x} - \lambda \bar{x} = 0$$

$$(A - \lambda I)\bar{x} = 0 \quad , \quad x \neq 0$$

$$|A - \lambda I| = 0$$

↳ eigen values

$$x^{(\lambda)} \neq 0$$

↳ eigen vector

[*] example : find eigen values & eigen vectors for

$$A = \begin{bmatrix} -4 & +4 \\ 1.6 & 1.2 \end{bmatrix}$$

sol. eigen values.

$$|A - \lambda I| = 0 \Longrightarrow \begin{bmatrix} -4 - \lambda & +4 \\ 1.6 & 1.2 - \lambda \end{bmatrix} = 0$$

$$\Rightarrow (-4-\lambda)(1.2-\lambda) + 1.6(4) = 0$$

$$-4.8 + 4\lambda - 1.2\lambda + \lambda^2 + 6.41 = 0$$

$$\lambda^2 + 2.8\lambda + 1.6 = 0$$

$$(\lambda+2)(\lambda+0.8) = 0$$

$$\lambda = -2, -0.8$$

* eigen vectors

$$X^{(-2)}$$

$$(A - \lambda I)\bar{X} = \bar{0}$$

$$\begin{bmatrix} -4+2 & 4 \\ 1.6 & 1.2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & +4 \\ 1.6 & 3.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 4x_2 = 0$$

$$x_1 = 2x_2$$

$$x_2 = \frac{+x_1}{2}$$

let $x_1 = 1$

$$x_2 = \frac{-1}{2}$$

so $$X^{(-2)} = \begin{bmatrix} 1 \\ \frac{-1}{2} \end{bmatrix}$$

ex:- find eigen values & eigen vectors of $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

[*] eigen values

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda) - 4 = 0$$

$$\lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda+1)(\lambda-3) = 0$$

$$\lambda = -1, 3.$$

[*] eigen vectors

$$x^{(-1)} \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0$$

$$x_2 = -2x_1$$

let $x_1 = 1$, $x_2 = -2$

$$x^{(-1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\rightarrow$

$x^{(3)} \Rightarrow$

$$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + x_2 = 0$

$x_2 = 2x_1$

let $x_1 = 1$

$x_2 = 2$

$$x^{(3)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\boxed{\cancel{X}}$ **Homo. system.**

$y'(x) = A\,y(x)$

$\boxed{ex}$ solve:-

$y_1' = y_1 + y_2$

$y_2' = 3y_1 - y_2$

sol.
$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

*(1) eigen values:-

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 3 = 0$$

$$\lambda^2 - 1 - 3 = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda = -2, 2$$

*(2) eigen vector

$$X^{(-2)} \Rightarrow \begin{bmatrix} 1+2 & 1 \\ 3 & -1+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3x_1 + x_2 = 0$$

$$x_2 = -3x_1$$

$$x_1 = 1, \; x_2 = -3$$

$$X^{(-2)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$X^{(2)} \Rightarrow \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -x_1 + x_2 = 0$$

$$x_1 = x_2 = 1$$

$$X^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

*(3) $\lambda_1 \neq \lambda_2 \Rightarrow y_h = c_1 e^{\lambda_1 x} X^{(-2)} + c_2 e^{\lambda_2 x} X^{(+2)}$

$$= c_1 e^{-2x} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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* ex 1 **Solve**

$$\vec{y}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \vec{y}$$

① Find eigen values

$$\begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(2-\lambda)+1=0$$

$$8-6\lambda+\lambda^2+1=0$$

$$\lambda^2-6\lambda+9=0$$

$$(\lambda-3)(\lambda-3)=0$$

$$\lambda = 3, 3.$$

② Find eigen vectors. $(\lambda=3)$

$$\begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_2 = 0 \\ x_2 = -x_1 \end{matrix}$$

$\boxed{x_1 = 1}$ $\boxed{x_2 = -1}$

$$x^{(3)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda \rightarrow$ so $y = c_1 e^{\lambda t} x^{(\lambda)} + c_2 e^{\lambda t} (x^{(\lambda)} t + \vec{v})$

$$(A - \lambda I)\vec{v} = x^{(\lambda)}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$v_1 + v_2 = 1$$

$$v_2 = 1 - v_1$$

let $v_1 = 0$

$$v_2 = 1$$

$$v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore y_h = c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

ex:- $y' = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}}_{A} y$

① Find eigen values:

$|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$(-1-\lambda)(-1-\lambda) + 1 = 0$

$1 + 2\lambda + \lambda^2 + 1 = 0$

$\lambda^2 + 2\lambda + 2 = 0$

$\frac{-2 \mp \sqrt{4 - 4(2)}}{2} = -1 \mp i$

② Find eigen vectors.

$x^{(-1+i)}$

$$\begin{bmatrix} -1-(-1+i) & 1 \\ -1 & -1-(-1+i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-i x_1 + x_2 = 0$

$x_2 = +i x_1$

let $x_1 = 1$

$x_2 = i$

$\rightarrow x^{(-1+i)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$x^{(-1-i)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\Rightarrow \quad r = \alpha \mp \mu i$

$$y_h = c_1 e^{(\alpha + i\mu)t} X^{(\alpha+i\mu)} + c_2 e^{(\alpha - i\mu)t} X^{(\alpha - i\mu)}$$

$$y_h = c_1 e^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(-1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

ex:

$$y' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} y$$

① find eigen values

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)^2 + 4 = 0$

$\boxed{1-\lambda = \pm 2i}$

$1 - 2i = \lambda$

$1 + 2i = \lambda$

② find eigen vectors

$x^{(1+2i)}$

$$\begin{bmatrix} 1-(1+2i) & 2 \\ -2 & 1-(1+2i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2i x_1 + 2x_2 = 0$

$\boxed{x_2 = i x_1}$

$x_1 = 1, \; x_2 = i$

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$$\Rightarrow y_h = c_1 e^{(1+2i)t}\begin{bmatrix}1\\ i\end{bmatrix} + c_2 e^{(1-2i)t}\begin{bmatrix}1\\ -i\end{bmatrix}$$

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* Non-homo sys. *

$$y' = Ay + g(t)$$

undetermined — Variation of parameters.

ex: find general sol of $y' = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$

$\boxed{1}$ $y_h \Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} (2-\lambda) & -4 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(-3-\lambda) + 4 = 0$

$-6 + \lambda + \lambda^2 + 4 = 0$

$\lambda^2 + \lambda - 2 = 0$

$(\lambda+2)(\lambda-1) = 0$

$\lambda = 1, -2$

$x^{(1)} \Rightarrow$

$$\begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 - 4x_2 = 0$

$x_1 = 4x_2$

let $x_2 = 1$.

$x_1 = 4$

$$x^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$x^{(-2)} \Rightarrow \begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 4x_2 = 0$$

$$x_1 = x_2$$

let $x_1 = 1$

$$x_2 = 1$$

$$x^{(-2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_h = c_1 e^{t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_p = \vec{A} t^2 + \vec{B} t + \vec{C}$$

$$y' = 2\vec{A} t + \vec{B}$$

$$= 2\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$2\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a_1 t^2 + bt + c_1 \\ a_2 t^2 + b_2 t + c_2 \end{bmatrix} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2a_1 t^2 + 2b_1 t + 2c_1 - 4a_2 t^2 - 4b_2 t - 4c_2 \\ a_1 t^2 + b_1 t + c_1 - 3a_2 t^2 - 3b_2 t - 3c_2 \end{bmatrix} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$$

$$\Rightarrow 2\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2a_1 - 4a_2 \\ a_1 - 3a_2 \end{bmatrix} t^2 + \begin{bmatrix} 2b_1 - 4b_2 \\ b_1 - 3b_2 \end{bmatrix} t + \begin{bmatrix} 2c_1 - 4c_2 \\ c_1 - 3c_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -2 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 10 \\ 9 \end{bmatrix} t + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$2a_1 - 4a_2 = -2$

$-2(a_1 - 3a_2 = -1)$

$2a_2 = 0 \Rightarrow a_2 = 0$

$a_1 = -1$

$$A = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$2a_1 = 2b - 4b_2 + 10$

$-2 - 10 = 2b_1 - 4b_2 = -12$

$+\ \cancel{2a_2} = b_1 - 3b_2 + 9$

$-2(\boxed{b_1 - 3b_2 = -9})$

$2b_2 = 6$

$\boxed{b_2 = 3}$

$b_1 = -9 + 9 = 0 \quad \rightarrow \boxed{b_1 = 0}$

$$\vec{B} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$b_1 = 0 = 2c_1 - 4c_2$

$b_2 = \cancel{3} = c_1 - 3c_2 + \cancel{3}$

$-2(0 = c_1 - 3c_2)$

$0 = 2c_2 \Rightarrow c_2 = 0$

$c_1 = 3c_2 = 0$

$$\vec{C} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y_p = \begin{bmatrix} -1 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

$y_g = y_p + y_h$

* find the form of y_p:

$$y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

Sol. $y_h \Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & +1 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$(-3-\lambda)^2 - 1 = 0$

$(-3-\lambda)^2 - 1 = 0$

$(-3-\lambda) = \mp 1$

$-4 = -3-1 = \lambda$

$-2 = -3+1 = \lambda$

$y_h = c_1 e^{-4t} x^{(-4)} + c_2 e^{-2t} x^{(-2)}$

$= c_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y_p = \vec{A} e^{-2t} = (\vec{A} t + \vec{u}) e^{-2t}$.

* example: Solve

$$y' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)^2 = 0$

$\lambda = 1, 1$

$x^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y_h = c_1 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

* variation of parameter :

$$y' = Ay + g(t)$$

ex: solve , $y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

$\boxed{1}$ $y_h' = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = 0$

$$(-3-\lambda)^2 - 1 = 0$$

$$-3-\lambda = \mp 1$$

$$\lambda = -3 \mp 1 = \lambda = -4, -2$$

$$x^{(-4)} \Rightarrow \begin{bmatrix} -3+4 & 1 \\ 1 & -3+4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0$$

$$x^{(-4)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x^{(-2)} \Rightarrow \begin{bmatrix} -3+2 & 1 \\ 1 & -3+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$

$$x^{(-2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_h = c_1 e^{-4t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}1\\1\end{bmatrix}$$

Now to find y_p, Variation of parameter.

$\boxed{1}$ $$Y = [\vec{y}_1\ \vec{y}_2] = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -e^{-4t} & e^{-2t}\end{bmatrix}$$

$\boxed{2}$ $$Y^{-1} = \frac{1}{e^{-6t}+e^{-6t}}\begin{bmatrix} e^{-2t} & -e^{-2t} \\ e^{-4t} & e^{-4t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{4t} & -e^{4t} \\ -e^{-4t} & e^{2t}\end{bmatrix}$$

$\boxed{3}$ $$u' = Y^{-1}g = \frac{1}{2}\begin{bmatrix} e^{4t} & -e^{4t} \\ e^{2t} & e^{2t}\end{bmatrix}\begin{bmatrix} -6 & e^{-2t} \\ 2 & e^{-2t}\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -6e^{2t} - 2e^{2t} \\ -6+2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -8 \cdot e^{2t} \\ -4 \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ -2 \end{bmatrix}$$

$\boxed{4}$ $$u = \int_0^t u'\, d\tau = \begin{bmatrix} \int_0^t -4e^{2\tau}\, d\tau \\ \int_0^t -2\, d\tau \end{bmatrix} = \begin{bmatrix} 2e^{2t}+2 \\ -2t \end{bmatrix}$$

$\boxed{5}$ $$y_p = Yu = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -e^{-4t} & e^{-2t}\end{bmatrix}\begin{bmatrix} -2e^{2t}+2 \\ -2t \end{bmatrix}$$

$$y_g = y_p + y_h$$

Past. Paper

examples

1) The eigen values of $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 + 1 = 0$$

$$(2-\lambda)^2 = -1$$

$$(2-\lambda) = \pm i$$

$$\lambda = 2 \pm i$$

2) The eigen values of $A = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$

are $\lambda_1 = -2$

$\lambda_2 = 3$

the corresponding eign vector of λ_1 is:

sol.

$$\begin{bmatrix} \overset{5}{\cancel{3+2}} & 0 \\ 1 & \underset{0}{\cancel{-2+2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5x_1 + 0\,x_2 = 0$$

$$x_1 = 0$$

$$x_2 \in \mathbb{R} - \{0\}$$

ما بصير x_1
وx_2 يكونوا صفر.

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$\boxed{3}$ $y_1' = 3y_1 + y_2$

$y_2' = y_2 - y_1$

$$y' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} y.$$

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$(3-\lambda)(1-\lambda) + 1 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda = 2, 2$

$$x^{(2)} \Rightarrow \begin{bmatrix} \cancel{3-2}\ 1 & 1 \\ -1 & \cancel{1-2}\ -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$$x^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$v_1 + v_2 = 1$

$$\bar{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y_h = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t \right].$$

* [4] Solve the system.

$$y' = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} y$$

$$\begin{vmatrix} 3-\lambda & 2 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 + 4 = 0$$

$$(3-\lambda) = \mp 2i$$

$$\lambda = 3 \mp 2i$$

$$x^{(3+2i)} \Rightarrow \begin{bmatrix} \cancel{3-(3+2i)}^{-2i} & 2 \\ -2 & \cancel{3-(3+2i)}^{-2i} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2i\, x_1 + 2x_2 = 0$$

$$x_2 = i\, x_1$$

$$x^{(3+2i)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\Rightarrow e^{(3+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$e^{3t} \cdot e^{2it} \begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow e^{3t} \left[\cos 2t + i \sin 2t\right] \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$e^{3t} \begin{bmatrix} \cos 2t & + i \sin 2t \\ i \cos 2t & - \sin 2t \end{bmatrix} \Rightarrow e^{3t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + e^{3t} i \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$$

$$y_h = c_1 \underbrace{e^{3t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix}}_{y_1} + c_2 \underbrace{e^{3t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}}_{y_2}$$

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