PARTIAL

16/6/2019

* see iPhone photos of previous ~~1st~~ lessons first.

9.7 Gradiant :

* equation of the plane that is normal $n = a\hat{i} + b\hat{j} + c\hat{k}$ and passes through $P_0(x_0, y_0, z_0)$ is :

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

Ex: Find eqn of the plane that passes through $P(1, 2, -1)$ and $Q(2, 1, 3)$ and $R(-1, 0, 4)$.

$\overrightarrow{PQ} = \langle 1, -1, 4 \rangle$, $\overrightarrow{PR} = \langle -2, -2, 5 \rangle$

$$\boxed{n = \overrightarrow{PQ} \times \overrightarrow{PR}} = \begin{vmatrix} i & j & k \\ 1 & -1 & 4 \\ -2 & -2 & 5 \end{vmatrix} = 3i - 13j - 4k$$

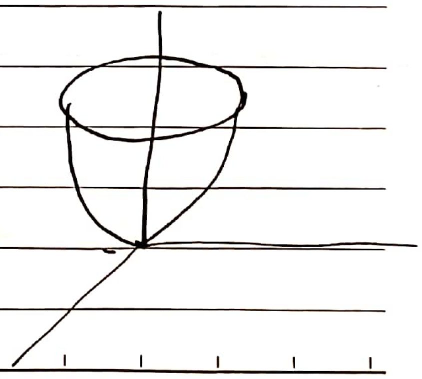
$\therefore$ Eqn of the plane :

$$3(x-1) - 13(y-2) - 4(z-1) = 0$$

* Suppose $F(x, y, z) = k$ is a surface, then $\underbrace{\nabla F\big|_{(x_0, y_0, z_0)}}_{\text{gradiant}}$ is normal to the tangent plane to that surface at (x_0, y_0, z_0).

Ex: Find eqn of the tangent plane to

$z = 4x^2 + 3y^2$ at $(1, 2, 16)$



$z - 4x^2 - 3y^2 = 0$

$\nabla F\big|_{(1,2,16)} = \langle -8x, -6y, 1 \rangle\big|_{(1,2,16)}$

$= \langle -8, -12, 1 \rangle$

$\longrightarrow$

$\therefore$ Eqn of the plane :

$$-8(x-1)-12(y-2)+1(z-16)=0$$

* The Laplacian of u is

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}}$$

Ex. If $u = x^3 y + y^2 z^2$, then find $\nabla^2 u$

$\nabla^2 u = 6xy + 2z^2 + 2y^2$

If $u = 2x^2 i + xy^2 j + z^3 k$, then

$\nabla^2 u = 4i + 2xj + 6zk$

Properties:

1) $\nabla fg = \nabla f g + f \nabla g$

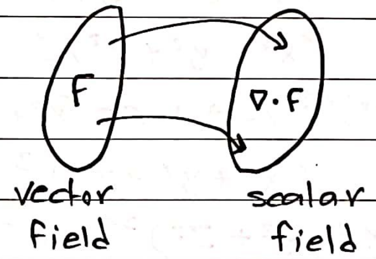
2) $\nabla (f/g) = \dfrac{g\nabla f - f\nabla g}{g^2}$

Def: If $v = [V_1, V_2, V_3]$ is a vector field, then the divergence of V is:

$$\boxed{\operatorname{div}(V) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}} = \underline{\nabla \cdot V}$$

Ex: If $V = \langle 2x + y^2, z^3, y + z \rangle$ then:

$$\nabla \cdot V = 2 + 0 + 1 = 3.$$



$$\boxed{\nabla \equiv \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k}$$

$$\underline{\nabla \cdot V} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (\langle V_1, V_2, V_3 \rangle)$$

$$= \boxed{\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}}$$

Remark: $\operatorname{div}(\operatorname{grad}(f)) = \nabla^2 f$; $\nabla \cdot \nabla f = \nabla^2 f$

$$\nabla \cdot \nabla f = \nabla \cdot (f_x i, f_y j, f_z k) = f_{xx} + f_{yy} + f_{zz} = \nabla^2 f.$$

Ex : show that $\operatorname{div}(\operatorname{grad}(\frac{c}{r})) = 0$ where $r = |\vec{r}|$ and

$\vec{r} = xi + yj + zk$. (c is constant)

$$\longrightarrow c(x^2+y^2+z^2)^{-1/2}$$

$$\operatorname{div}\left(\operatorname{grad}\left(\frac{c}{\sqrt{x^2+y^2+z^2}}\right)\right)$$

$$\operatorname{div}\left(\frac{-cx}{(x^2+y^2+z^2)^{3/2}}\, i + \frac{-cy}{(x^2+y^2+z^2)^{3/2}}\, j + \frac{-cz}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\frac{(x^2+y^2+z^2)^{3/2}(-c) + cx\,\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{3/2}(-c) + y(\tfrac{3}{2})(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$+ \frac{(x^2+y^2+z^2)^{3/2}(-c) + cz(x^2+y^2+z^2)^{1/2}\,2z}{(x^2+y^2+z^2)^3}$$

$$= \frac{-3c(x^2+y^2+z^2)^{3/2} + 3c(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3} = 0$$

Properties :-

1) $\operatorname{div}(kV) = k\operatorname{div}(V)$, k = constant

2) $\operatorname{div}(fV) = f\operatorname{div}(V) + \nabla f \cdot V$

3) $\operatorname{div}(f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$

Ex. $\operatorname{div}((2x+y+z)\langle xz, y^2, 2x\rangle)$

$= \operatorname{div}(\langle (2x+y+z)xz, (2x+y+z)y^2, (2x+y+z)2x\rangle)$

Ex: show that $\operatorname{div}(\operatorname{grad}(\frac{c}{r})) = 0$, $r = (x^2+y^2+z^2)^{1/2}$

$$\operatorname{div}\left(\operatorname{grad}\left(\frac{c}{(x^2+y^2+z^2)^{1/2}}\right)\right)$$

$$= \operatorname{div}\left(\left\langle \frac{-cx}{(x^2+y^2+z^2)^{1/2}}, \frac{-cy}{(x^2+y^2+z^2)^{3/2}}, \frac{-cz}{(x^2+y^2+z^2)^{3/2}} \right\rangle\right)$$

$$\operatorname{div}\left(\underbrace{\frac{-c}{(x^2+y^2+z^2)^{3/2}}}_{f} \underbrace{\langle x, y, z\rangle}_{\vec{v}}\right)$$

$$= \frac{-c}{(x^2+y^2+z^2)^{3/2}}(3) + \left(\frac{3cx}{(x^2+y^2+z^2)^{5/2}} i + \frac{3cy}{(x^2+y^2+z^2)^{5/2}} j + \frac{3cz}{(x^2+y^2+z^2)^{5/2}}\right)$$

$$\cdot \langle x, y, z\rangle = \frac{-3c}{(x^2+y^2+z^2)^{3/2}} + \frac{3c(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} =$$

$$\frac{-3c}{(x^2+y^2+z^2)^{3/2}} + \frac{3c}{(x^2+y^2+z^2)^{3/2}} = 0$$

9.9 Curve of a Vector Field

Let $V = [V_1, V_2, V_3]$, then the curl of V

$$\operatorname{curl}(V) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) i - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) j + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) k$$

* $\nabla \times V$

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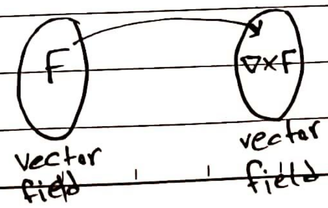
Ex. let $V = [yz, 3xz, z]$, find $\nabla \times V$

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 3xz & z \end{vmatrix} = -3x\, i + y\, j + (3z - z)\, k$$

*


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Properties:

① $\text{curl}(\text{grad}(f)) \equiv 0$

$\nabla \times (\nabla f) \equiv 0$

$\nabla \times (\nabla f) = \nabla \times (f_x i + f_y j + f_z k)$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$$

$= (f_{zy} - f_{yz}) i - (f_{zx} - f_{xz}) j$

$+ (f_{xy} - f_{yx}) k \equiv 0$

② $\text{div}(\text{curl}(V)) \equiv 0$

$$\text{div}\left(\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix} \right)$$

$\text{div}\Big(\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) i - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) j +$

$\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) k \Big)$

$\left(\frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z}\right) - \left(\frac{\partial^2 V_3}{\partial y \partial x} - \frac{\partial^2 V_1}{\partial y \partial z}\right) +$

$\left(\frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}\right) \equiv 0$

③ $\text{curl}(u + v) = \text{curl}(u) + \text{curl}(v)$

④ $\text{curl}(fv) = \text{grad}(f) \times v + f\,\text{curl}(v)$

⑤ $\text{div}(u \times v) = v \cdot \text{curl}(u) - u \cdot \text{curl}(v)$

Ex. If $\vec{r} = xi + yj + zk$ and $r = (|r|)$, then show that:

$\text{div}(\text{curl}(r^5 \vec{r}) + r\vec{r}) = 4r$

$\text{div}(\text{curl}(r^5 \vec{r})) + \text{div}(r\vec{r})$

$|||$

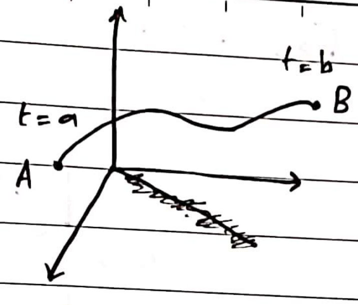
$0 + r\,\text{div}(\vec{r}) + \nabla r \cdot \vec{r}$

$3r + \left(\frac{x}{\sqrt{x^2+y^2+z^2}} i + \frac{y}{\sqrt{x^2+y^2+z^2}} j + \frac{z}{\sqrt{x^2+y^2+z^2}} k \right) \cdot (xi + yj + zk)$

$3r + \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} = 3r + r = \underline{4r}$

Remark: The $\text{curl}(v)$ is a measure of rotation

$\int_C F \cdot dr$



$C: r(t) = x(t)i + y(t)j + z(t)k = r$

* $\int_C F \cdot dr$ (line integral)

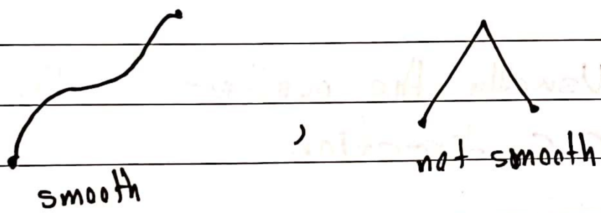
* we integrate along a path

$C: r(t) = \langle x(t), y(t), z(t) \rangle$

which is called the path of integration.

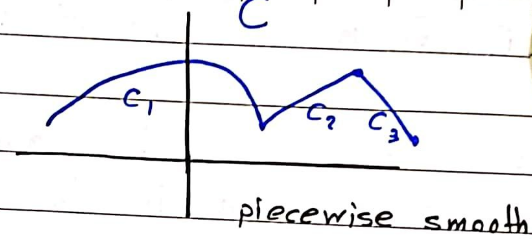
$C: r(t) \quad a \le t \le b$

* $C: r(t)$ is called a smooth curve if $r(t)$ is diff, $r'(t)$ is cont. and $r'(t) \neq 0$ at any $t \in [a, b]$



* A piecewise smooth curve is a curve that consists of finitely many smooth curves.
* We assume that the path of integration is piecewise smooth

Ex



* A line integral of a vector field F over a smooth curve $C: r(t)$

$r(t) = x(t)\,i + y(t)\,j + z(t)\,k$

$a \le t \le b$ is defined by:

$\int_C F \cdot dr$ or

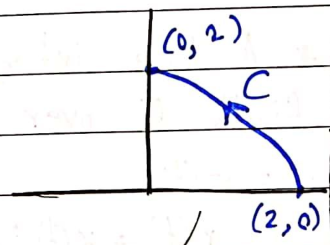
$= \int_C F_1\,dx + F_2\,dy + F_3\,dz$

* $\int_C F \cdot dr = \int_a^b \left(F(r(t)) \cdot \frac{dr}{dt} \right) dt$

note: If C is a closed curve, we may write $\oint F \cdot dr$.

Ex. Evaluate $\int_C F \cdot dr$ when

$F = \langle -y , -xy \rangle$ and C is the circular arc in the figure:



$x^2+y^2=4$

$(\frac{x}{2})^2 + (\frac{y}{2})^2 = 1 \leftarrow \frac{x^2}{4} + \frac{y^2}{4} = 1$

C: $r(t) = \langle 2\cos(t) , 2\sin(t) \rangle$

$0 \le t \le \frac{\pi}{2}$

$F(r(t)) = \langle -2\sin(t), -4\cos(t)\sin(t) \rangle$

$0 \le t \le \frac{\pi}{2}$

$\frac{dr}{dt} = \langle -2\sin(t) , 2\cos(t) \rangle$

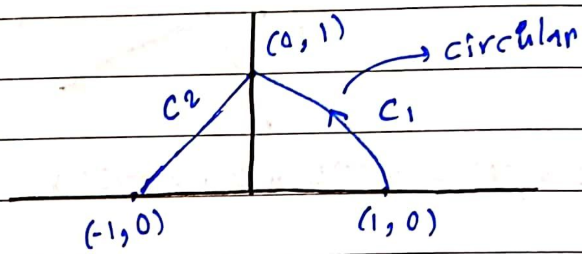
$$\int_C F \cdot dr = \int_0^{\pi/2} F(r(t)) \cdot \frac{dr}{dt} \, dt$$

$$= \int_0^{\pi/2} \left(4\sin^2(t) - 8\cos^2(t)\sin(t)\right) dt$$

$$= 4\left(\frac{1}{2}t - \frac{1}{4}\sin(2t)\right) + 8\,\frac{\cos^3(t)}{3} \Big|_0^{\pi/2}$$

....

Ex. .... C in the figure



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Properties :-

① $\int_{C_1 \cup C_2} F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$

② $\int_{-C} F \cdot dr = -\int_C F \cdot dr$

③ $\int_C kF \cdot dr = k \int_C F \cdot dr$

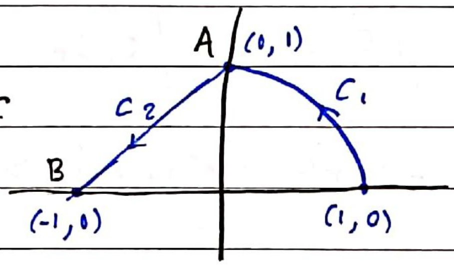
④ $\int_C (F + G) \, dr = \int_C F \, dr + \int_G G \, dr$

⑤ Usually the positive is the c.c. direction.

⑥ If F is a force, then $\int_C F \cdot dr$ represents the work done by this force in the displacement along c.

Ex: Find the work done by $F = \langle -y, -xy \rangle$ moving an object along C in the figure.

$C_1 \cup C_2 = C$



① Along C_1 .... $C_1 : r(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le \frac{\pi}{2}$

② Along C_2 :- $\quad y - 1 = x - 0$

$y = x + 1$

$-C_2 : r(x) = \langle x, x+1 \rangle \quad -1 \le x \le 0$

$(r(t) = \langle t, t+1 \rangle)$

$$\int_{C_1} F . dr = -\int_{-C_2} F . dr = -\int_{-1}^{0} F(r(x)) . \frac{dr}{dx} dx$$

$F(r(x)) = \langle -(x+1), -x(x+1) \rangle$

$\frac{dr}{dx} = \langle 1, 1 \rangle$

$\therefore$ So, $\int_{C_2} F.dr = -\int_{-1}^{0} (-(x+1) + -x(x+1)) dx$

Or, $C_2 : x = 0 + (-1)t \quad 0 \le t \le 1$

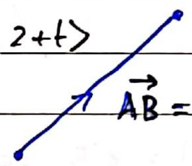
$y = 1 + (-1)t$

$z = 0 + 0t$

$C_2 : r(t) = \langle -t, 1-t \rangle$

$F(r(t)) = \langle -(1-t), +(t)(1-t) \rangle$

Ex. Find the work done by $F = \langle z, x, y \rangle$ along the line segment from $(1, -1, 2)$ to $(2, 1, 3)$



$C : r(t) = \langle 1+t, -1+2t, 2+t \rangle$

$0 \le t \le 1$

$\vec{AB} = \langle 1, 2, 1 \rangle$

$F(r(t)) = \langle 2+t, 1+t, -1+2t \rangle$

$\frac{dr}{dt} = \langle 1, 2, 1 \rangle$

$$\int_C F.dr = \int_0^1 \left(F(r(t)) . \frac{dr}{dt} \right) dt$$

$$= \int_0^1 ((2+t) + 2(1+t) + 1(-1+2t)) dt$$

$\vdots$

② The helix :

$r(t) = \langle \cos(t), \sin(t), 3t \rangle$

from $(1, 0, 0)$ to $(1, 0, 6\pi)$

$\uparrow t = 0 \qquad \uparrow t = 2\pi$

$F(r(t)) = \langle 3t, \cos(t), \sin(t) \rangle$

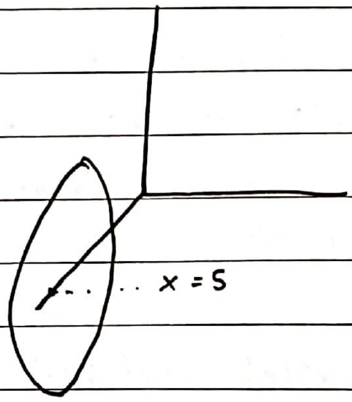
$\frac{dr}{dt} = \langle -\sin(t), \cos(t), 3 \rangle$

$$\int_C F.dr = \int_0^{2\pi} F(r(t)) . \frac{dr}{dt} dt$$

$$= \int_0^{2\pi} -3t \sin(t) + \cos^2(t) + 3\sin(t) \, dt$$

$\vdots$

③ The ellipse : $\frac{y^2}{4}+\frac{z^2}{9}=1$, width $x=5$.



$C: r(t) = \langle 5, 2\cos(t), 3\sin(t)\rangle \quad 0\le t\le 2\pi$

$F(r(t)) = \langle 3\sin(t), 5, 2\cos(t)\rangle$:

$\frac{dr}{dt} = \langle 0, -2\sin(t), 3\cos(t)\rangle$

$\int_C F\cdot dr = \int_0^{2\pi} (F(r(t))\cdot \frac{dr}{dt})\, dt$

$= \int_0^{2\pi} (-10\sin(t) + 6\cos^2(t))\, dt$

$* \; 6(\frac{1}{2}+\frac{1}{2}\cos(2t))$

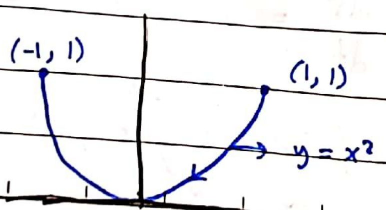
$= 10\cos(t) + 6(\frac{1}{2} + \frac{1}{4}\sin(2t)) \Big|_0^{2\pi}$

$= 6\pi$

Ex. Find $\int_C F\cdot dr$ where

$F = \langle xy, y\rangle$ and C in

① the figure



$y = x^2$, $x = t$

$\to y = t^2 \to C: r(t) = \langle t, t^2\rangle$

$-1\le t\le 1$

$\int_C F\cdot dr = -\int_{-C} F\cdot dr$

$-C: r(t) = \langle t, t^2\rangle \quad -1\le t\le 1$

$F(r(t)) = \langle t^3, t^2\rangle$

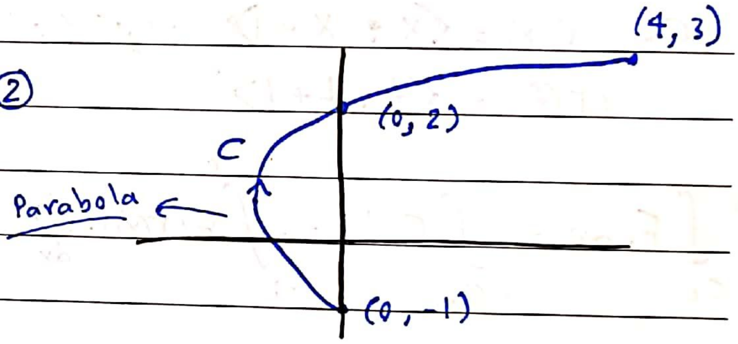
$\frac{dr}{dt} = \langle 1, 2t\rangle$

$\int_C F\cdot dr = -\int_{-1}^{1} F(r(t))\cdot \frac{dr}{dt}\cdot dt$

$= -\int_{-1}^{1} (t^3+2t^3)\, dt$

⋮

②



$(x = a_1 y^2 + a_2 y + a_3)$

$x = k(y-(-1))(y-2)$

$4 = k(3+1)(3-2) \to \underline{k=1}$

$x = (y+1)(y-2)$

$r(y) = \langle (y+1)(y-2), y\rangle$

$-1\le y\le 3$

$F(r(y)) = \langle (y+1)(y-2)y,\ y\rangle$

$$\frac{dr}{dy} = \langle \underbrace{(y+1)+(y-2)}_{2y-1},\ 1\rangle$$

$$\int_C F\cdot dr = \int_{-1}^{3} F(r(y))\cdot \frac{dr}{dy}\,dy$$

$$= \int_{-1}^{3} \big((2y-1)(y+1)(y-2)\,y + y\big)\,dy$$

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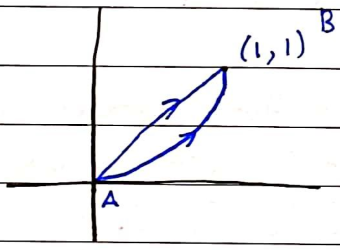
10.1

Remark: The line integral generally depends not only on F and the end points of the path, but also on the path itself.

Ex. $F = \langle 0, xy\rangle$

① Find $\int_{C_1} F.dr$



$\overrightarrow{AB} = \langle 1, 1\rangle$

$C_1 : r(t) = \langle 0 + 1t, 0 + 1t\rangle = \langle t, t\rangle$, $0 \le t \le 1$

$F(r(t)) = \langle 0, t^2\rangle$

$\frac{dr}{dt} = \langle 1, 1\rangle$

$$\int_{C_1} F.dr = \int_0^1 \left(F(r(t))\cdot\frac{dr}{dt}\right).dt$$

$$= \int_0^1 t^2\,dt = \frac{1}{3}$$

② Find $\int_{C_2} F.dr$

$C_2 : r(t) = \langle t, t^2\rangle \quad 0 \le t \le 1$

$F(r(t)) = \langle 0, t^3\rangle$

$\frac{dr}{dt} = \langle 1, 2t\rangle$

$$\int_{C_2} F.dr = \int_0^1 \left(F(r(t))\cdot\frac{dr}{dt}\right)dt$$

$$= \int_0^1 2t^4\,dt = \frac{2}{5}$$

10.2 Path independent of Line Integral.

Theorem: The line integral

$$\int_C \vec{F}\,dr = \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

where F_1, F_2 and F_3 are cont. on a domain D in the space —

- is path independent iff $F = \langle F_1, F_2, F_3 \rangle$ is the gradiant of some scalar field f in D, i.e there is a scalar function f such that

$$\nabla f = \langle F_1, F_2, F_3 \rangle.$$

In this case we say F is conservative and f is called the potential of F.

Also, $\boxed{\int F.dr = f\Big|_{\text{end point}} - f\Big|_{\text{initial point}}}$

Ex (1). show that

$\int_c (2x\,dx + 2y\,dy + 4z\,dz)$ is path Independent.

Is there f: $\nabla f = F = \langle 2x, 2y, 4z \rangle$
$= \langle f_x, f_y, f_z \rangle$

$f_x = 2x$
$f_y = 2y$
$f_z = 4z$

$f_x = 2x \longrightarrow$
$f(x,y,z) = x^2 + w(y,z)$
$f_y = 0 + w_y = 2y$

$w_y = 2y \xrightarrow{\text{int.}} w(y,z) = y^2 + h(z)$

$f(x,y,z) = x^2 + y^2 + h(z)$
$f_z = 0 + 0 + h'(z) = 4z$

$h'(z) = 4z \rightarrow h(z) = 2z^2$

$\therefore$ Thus, $f(x,y,z) = x^2 + y^2 + 2z^2$

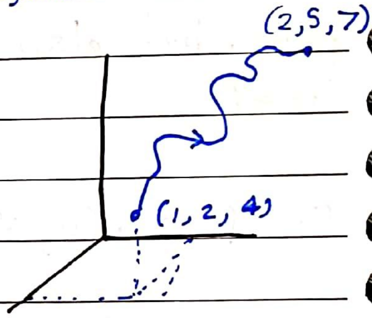
(2) Find $\int_c 2x\,dx + 2y\,dy + 4z\,dz$ where c is the line seg. from $(1,-1,2)$ to $(2,3,1)$.

$\int_c F.dr = f\Big|_{(2,3,1)} - f\Big|_{(1,-1,2)} =$

$\left[(2)^2 + (3)^2 + 2(1)^2\right] - \left[(1)^2 + (-1)^2 + 2(2)^2\right]$

③ Find $\int 2x\,dx + 2y\,dy + 4z\,dz$ where c is in the figure:

$\int \ldots = f\Big|_{(2,5,7)} - f\Big|_{(1,2,4)}$



Ex: Is $F = \langle z, x, y \rangle$ conservative?

is there scalar field f wide
$\nabla f = \langle z, x, y \rangle$
$\langle f_x, f_y, f_z \rangle = \langle z, x, y \rangle$

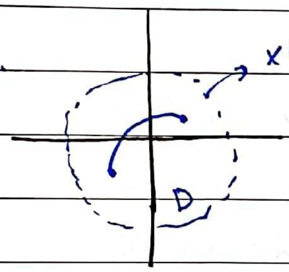
$f_x = z \xrightarrow{\text{int.}} f(x,y,z) = zx + w(y,z)$
$f_y = x$ Diff with respect to y!
$f_z = y$

$\rightarrow f_y = 0 + w_y(y,z) \equiv x$!!!

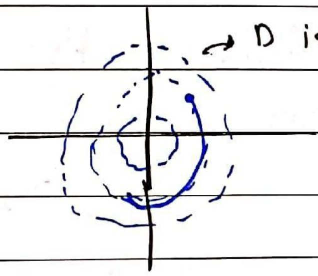
$\therefore$ Thus, F is not cons.

Def: A domain D is connected if any two points in D can be connected by a path that lies entirely in D.

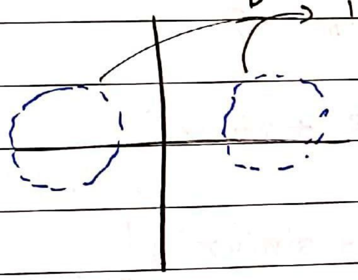
Ex.



D, $x^2+y^2<4$ is connected



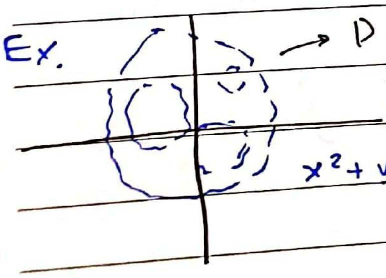
D is connected



D is not connected

Def: A ~~domai~~ connected domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

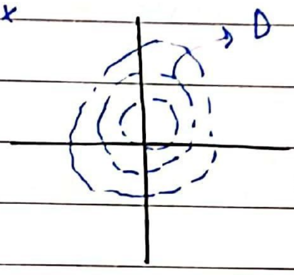
Ex.



D is simply connected

$x^2+y^2<4$

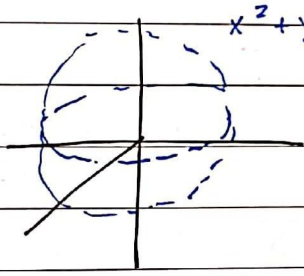
Ex



D

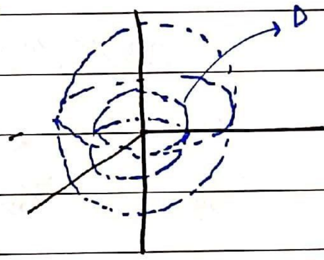
$1<x^2+y^2<4$ is not simply connected.

Ex



$x^2+y^2+z^2<4$ is simply connected

Ex.



D $1<x^2+y^2+z^2<4$ is simply connected

Theorem: Let F_1, F_2 and F_3 be cont. and have cont. partial derivatives in a domain D and $F=\langle F_1, F_2, F_3\rangle$.

(1) If F is conservative in D, then $\text{curl}(F)=0$ in D

(2) If $\text{curl}(F)=0$ in D and D is simply connected, then F is conservative

* If F is conservative on D, and $(0,0,0) \in D$,

then $f(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$

Ex. show that $F = \langle 2xyz^2,\ x^2z^2 + z\cos(yz),\ 2x^2yz + y\cos(yz) \rangle$ is cons.

$\int_C F.dr = f| - f|$ $\rightarrow$

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$\rightarrow$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^2 & x^2z^2 + z\cos(yz) & 2x^2yz + y\cos(yz) \end{vmatrix}$$

$$F = \langle i\ j\ k \rangle = [(2x^2z - yz\sin(yz) + \cos(yz)] - [2x^2z - zy\sin(yz) + \cos(yz)]i - [4xyz - 4xyz]j + (2xz^2 - 2xz^2)k = 0$$

So, F is conservative

(b) Find the potential of F

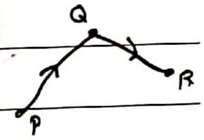
$f(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt =$

$\int_0^x z(t)(0)(0)^2 + \int_0^y x^2(0)^2 + 0(\cos(t)0)\,dt + \int_0^z 2x^2yt + y\cos(yt)\,dt$

$x^2yt^2 + \sin(yt) \Big|_0^z$

$f(x,y,z) = x^2yz^2 + \sin(yz)$

(c) Find $\int_C F.dr$ where C is the line seg. from $P(1,2,-1)$ to $Q(2,1,4)$ followed the line seg. from $Q(2,1,4)$ to $R(-1,3,3)$.



$\int_C F.dr = f\Big|_R - f\Big|_P$

$= x^2yz^2 + \sin(yz)\Big|_{R(-1,3,3)} - (x^2yz^2 + \sin(yz))\Big|_{(1,2,-1)} =$

Theorem: The integral is path independent in a domain D iff its value around every closed path in D is zero.

(F is cons. iff $\oint_C F.dr = 0$)

• Def: $F\cdot dr = F_1\,dx + F_2\,dy + F_3\,dz$ is exact if there's a function f such that:

$$F\cdot dr = df = f_x\,dx + f_y\,dy + f_z\,dz.$$

• Theorem: Suppose F_1, F_2 and F_3 are cont. with $F = \langle F_1, F_2, F_3\rangle$ the integral $\int_C F\cdot dr$ is path-independent where c in a domain D iff $F\cdot dr$ is exact.

Ex. (a) show that $F\cdot dr$ is exact where $F = \langle ye^x + z,\ e^x + zy,\ x\rangle$

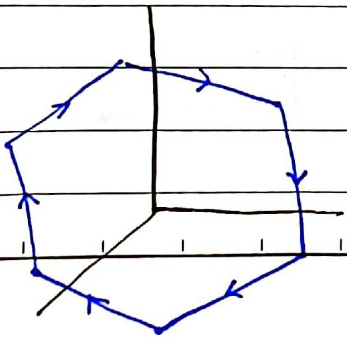
solution: $F\cdot dr$ is exact iff

$\nabla\times F = 0$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^x + z & e^x + zy & x \end{vmatrix}$$

$$= (0-0)i - (1-1)j + (e^x - e^x)k = 0$$

(b) Find $\oint_C F\cdot dr$ C in the figure



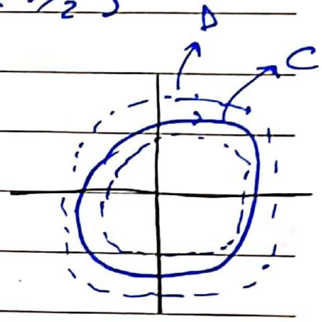
$\oint_C F\cdot dr = 0$

since F is cons.

Ex. let $F = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$

$D = \{ \frac{1}{2} < x^2 + y^2 < 3/2 \}$

(a) Find $\int_C F\cdot dr$



$C: x^2 + y^2 = 1$

$c: r(t) = \langle \cos(t), \sin(t) \rangle$

$F(r(t)) = \langle -\sin(t), \cos(t) \rangle$

$\frac{dr}{dt} = \langle -\sin(t), \cos(t) \rangle$

$$\int_C F\cdot dr = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt = 2\pi$$

(b) is F conservative?

No, in (a) C is closed $\oint_C F\cdot dr = 2\pi$.

(c) Find curl(F)

$$\nabla F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$0i - 0j + \left(\frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} \right)k$$

$$= \left(\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{(y^2 - x^2)}{x^2 + y^2} \right)k = 0$$

Remark: curl $(F)=0$ but F is not cons on D.

This doesn't contradict previous theorem, indeed we can't apply that theorem because D is not simply connected.

• If $f = \tan^{-1}(y/x)$, then $\nabla f = F$.

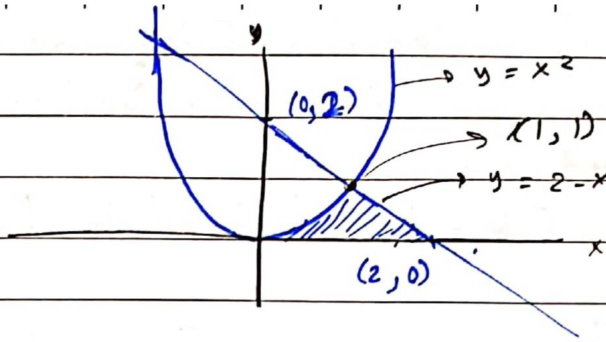
~~Whe~~ we can't apply previous Th. because $f = \tan^{-1}(y/x)$ is not a single-valued function on D.

10.3 Double Integral (Review)

* $\iint_R f(x,y)\, dA$

R is region of int.

Ex. Find $\iint_R (x+y)\, dA$ where R is the region bounded by $y = x^2$, $y + x = 2$ and x-axis



$x^2 = 2 - x \rightarrow x^2 + x - 2$

$\rightarrow (x+2)(x-1) = 0$

$\rightarrow x = -2, \quad x = 1$

$$\int_0^1 \int_0^{x^2} (x+y)\, dy\, dx + \int_1^2 \int_0^{2-x} (x+y)\, dy\, dx$$

or $\int_0^1 \int_{\sqrt{y}}^{2-y} (x+y)\, dx\, dy$ ......

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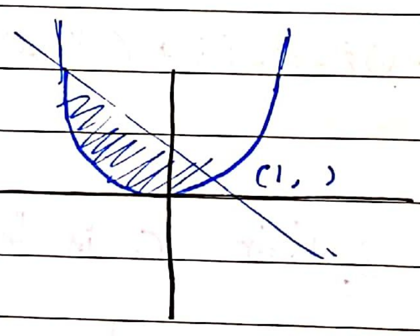
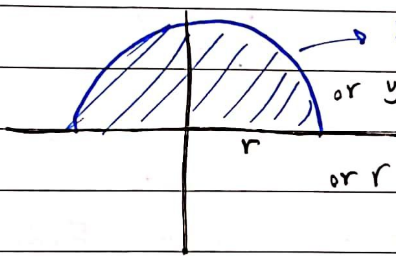
Ex: Find $\iint_R (x+y)\, dA$, R is the region bounded by: $y = x^2$ and $y + x = 2$



$$\int_{-2}^{1} \int_{x^2}^{2-x} (x+y)\, dy\, dx$$

Ex. Find $\iint_R \sqrt{x^2+y^2}\, dA$ where R in the figure:



$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx \quad \text{or}$$

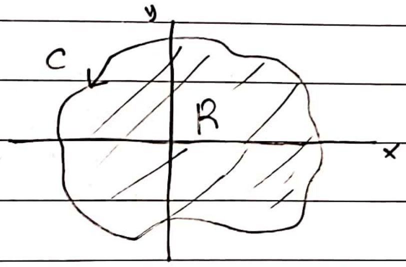
$$\int_{0}^{\pi}\int_{0}^{2} r \underbrace{r\,dr\,d\theta}_{dA}$$

10.4: Green's Theorem

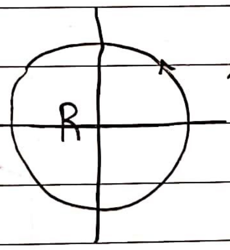
- let R be a closed region in the x-y-plane whose boundary C consists of finitely many ~~small~~ smooth curves.
- let $F_1(x,y)$ and $F_2(x,y)$ be functions that are cont. and have cont. partial derivatives $\frac{\partial F}{\partial y}$ and $\frac{\partial f}{\partial x}$ everywhere in some domain containing R.
- Also, $\vec{F} = \langle F_1, F_2 \rangle$, then

$$\boxed{\oint_C F \cdot dr = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA}$$

motion is counter clockwise



Ex. Find $\int_C F \cdot dr$ where $F = \langle y^2 - 7y ; 2xy + 2x \rangle$ and C is the circle $x^2+y^2=4$ moving counter clock-wise.



$$\int_C F \cdot dr = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

using Green's th.

$$= \iint_R \left((2y+2) - (2y-7) \right) dA$$

$$= \iint_R 9\, dA = \underline{9(\pi 2^2)}$$

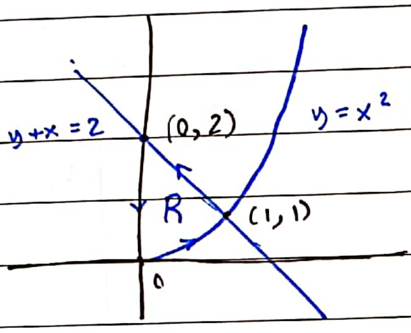
C.C

C

$$\boxed{\oint_C F\cdot dr = \pm \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA}$$

Ex. Find $\int_C F\cdot dr$ where

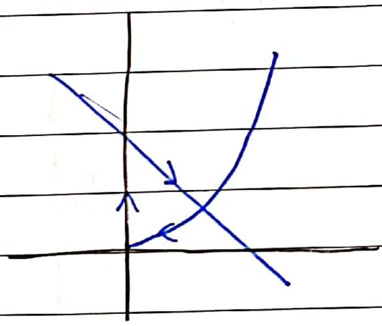
$F = \langle x^2 y, -y \rangle$ and C in the figure (1):



$$\oint_C F\cdot dr = \iint_R 0 - x^2 \, dA$$

$$= \int_0^1 \int_{x^2}^{2-x} -x^2 \, dy\, dx$$

figure (2):

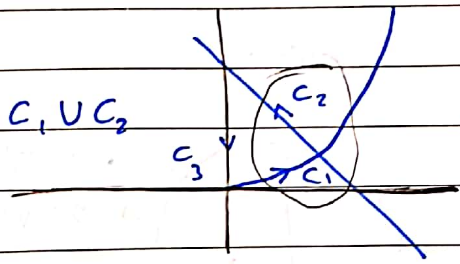


$$\oint_C F\cdot dr = -\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

⋮

figure (3):

$C = C_1 \cup C_2$



$$\oint_C F\cdot dr = \boxed{\int_{C\cup C_3} F\cdot dr} - \int_{C_3} F\cdot dr$$

Green's th.

C_3 : $x = 0 + 0t \qquad 0 \le t \le 1$

$y = 2 + (-2)t$

$z = 0$

$r(t) = \langle 0, 2-2t \rangle$

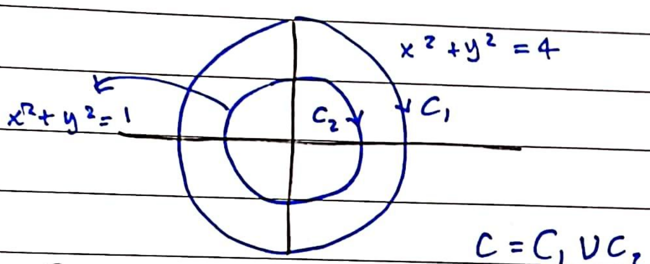
$F(r(t)) = \langle 0, -(2-2t) \rangle$

$\frac{dr}{dt} = \langle 0, -2 \rangle$

$$\int_{C_1} F\cdot dr = \int_0^1 \left(F(r(t))\cdot \frac{dr}{dt}\right) dt$$

$$= \int_0^1 2(2-2t)\, dt \quad \dots$$

figure (4):



$C = C_1 \cup C_2$

on C_1:

$$\oint_{C_1} F.dr \equiv \iint_R -x^2 \, dA$$

$$\int_0^{2\pi}\int_0^2 - r^2 \cos^2(\theta) \, r dr d\theta$$

$\cos^2(\theta) \rightarrow \frac{1}{2}+\frac{1}{2}\cos(2\theta)$

$$= \frac{-r^4}{4}\Big|_0^2 \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Big|_0^{2\pi} = \dots$$

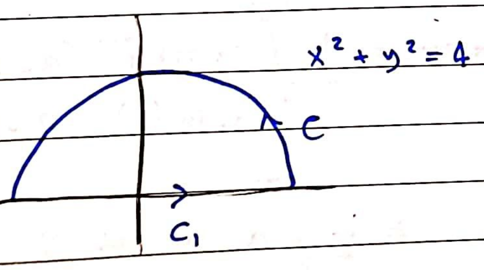
on C_2:

$$\int_{C_2} F.dr \equiv -\iint_R -x^2 \, dA$$

$$= -\int_0^{2\pi}\int_0^1 -r^2\cos^2(\theta) \, r dr d\theta \quad \dots$$

$$\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F_2.dr$$

figure (5): C is the upper half of $x^2+y^2=4$.



$$\int_C F.dr \equiv \boxed{\int_{C\cup C_1} F.dr} - \int_{C_1} F.dr$$

$$\int_{C\cup C_1} F.dr \underset{G.T}{\equiv} \iint_R -x^2 \, dA$$

$$= \int_0^{\pi}\int_0^2 -r^2\cos^2(\theta) \, r dr d\theta$$

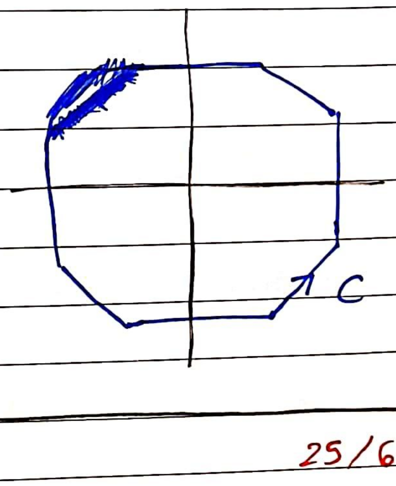
C_1: $x = -2+4t \quad 0 \le t \le 1$

$y = 0+0$

$r(t) = \langle -2+4t, 0 \rangle \quad 0 \le t \le 1 \dots$

Ex. Find $\int_C F.dr$ where

$F = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C is in the figure below:



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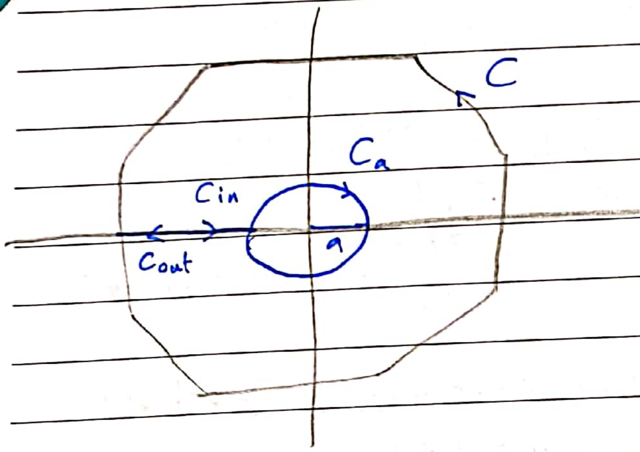
$$\int_C F.dr + \int_{C_a} F.dr + \int_{C_{in}} \cancel{F.dr} + \int_{C_{out}} \cancel{F.dr} \equiv$$

↓

Gauss's th.

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$



$$\int_C F \cdot dr + \int_{C_a} F \cdot dr = \iint_R \left(\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} \right) dA$$

$$= \iint \left(\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2} \right) = 0$$

$$\int_C F \cdot dr = -\int_{C_a} F \cdot dr \quad ✓$$

$-C_a : r(t) = \langle a\cos(t), a\sin(t) \rangle$

$0 < t \leq 2\pi$

$$F(r(t)) = \left\langle \frac{-a\sin(t)}{a^2}, \frac{a\cos(t)}{a^2} \right\rangle$$

$$\frac{dr}{dt} = \langle -a\sin(t), a\cos(t) \rangle$$

$$\int_{-C_a} F \cdot dr = \int_0^{2\pi} \left(F(r(t)) \cdot \frac{dr}{dt} \right) dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi, \quad \boxed{\text{So } \int_C F \cdot dr = 2\pi}$$

$$* \int_C F \cdot dr = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$* \int_C F_1 \, dx + F_2 \, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

① If $F_2 = x$ and $F_1 = 0$, then

Area of R: $\iint_R 1 \, dA = \oint_C x \, dy$

② If $F_1 = -y$ and $F_2 = 0$, then

Area of R: $\iint_R dA = -\oint y \, dx$

③ Area of R $= \frac{1}{2} \int_C x\,dy - y\,dx$

(c.c)

Ex. Find the area of circle $x^2 + y^2 = a^2$.

$x = a\cos(t)$, $y = a\sin(t)$

$C: r(t) = \langle a\cos(t), a\sin(t) \rangle$

$0 \leq t \leq 2\pi$

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} (a\cos(t))(a\cos(t)) - (a\sin(t))(-a\sin(t)) \, dt$$

$$= \frac{1}{2} \int_0^{2\pi} a^2 = a^2 \pi$$

Ex. Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$C: \; r(t) = \langle \underset{x}{\underset{\uparrow}{a\cos(t)}}, \underset{y}{\underset{\uparrow}{a\sin(t)}} \rangle$$

$$0 \le t \le 2\pi$$

$$\text{Area} = \frac{1}{2} \oint_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} \left(a\cos(t)\, b\cos(t) - b\sin(t) \times (-a\sin(t)) \right) dt$$

$$= \frac{1}{2} \int_0^{2\pi} \left(ab\cos^2(t) + ab\sin^2 t \right) dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab\,dt = ab\pi$$

10.5 Surfaces :-

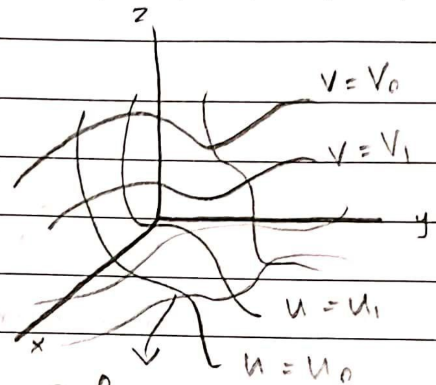
* An equation of a surface is of the form:

$$\boxed{F(x,y,z) = k \equiv \text{constant}}$$

* A parametric equation of a surface is of the form:

$$\boxed{r(u,v) = x(u,v)\,i + y(u,v)\,j + z(u,v)\,k}$$

$(u,v) \in$ Domain



Surface

$$\left.\begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}\right\} \omega(x,y) = V$$

$$\left.\begin{array}{l} y = y(u,v) \\ z = z(u,v) \end{array}\right\} T(y,z) = V$$

$$\boxed{\omega(x,y) - T(y,z) = 0}$$

$$F(x,y,z) = 0$$

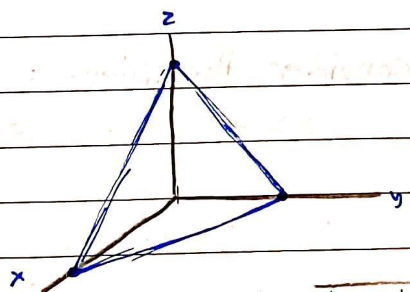
Important Surfaces:-

① Planes:-

eqn of a plane:

$$ax + by + cz = d$$

Ex. $x + 2y + z = 4$



Five Apple

Normal of this plane
$i + 2j + k$

unit normals $\mp \dfrac{i + 2j + k}{\sqrt{6}}$

* every surface (almost) has two normals opposing each other

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- Remark: For the surface $F(x, y, z) = k$ the unit normal at (x_0, y_0, z_0) are:

$$\boxed{\pm \left. \frac{\nabla F}{|\nabla F|} \right|_{(x_0, y_0, z_0)}}$$

- Remark: For the surface $r(u, v) = x(u, v)\, i + y(u, v)\, j + z(u, v)\, k$,

The unit normals are:

$$\boxed{\mp \frac{r_u \times r_v}{|r_u \times r_v|}}$$

Ex. consider the plane:
$$2x + 3y + z = 5$$

$\rightarrow$ $z = 5 - 2x - 3y$ $\rightarrow$

$x = u$, $y = v$,
$z = 5 - 2u - 3v$.

$r(u, v) = ui + vj + (5 - 2u - 3v)k$

Ex. consider the plane:
$$ax + by + cz = d \quad (c \neq 0)$$

$r(u, v) = ui + vj + \left(\dfrac{d - au - bv}{c}\right) k$

Find the unit normals to this surface.

$r_u = i + \dfrac{-a}{c} k$

$r_v = j + \dfrac{-b}{c} k$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -a/c \\ 0 & 1 & -b/c \end{vmatrix}$$

$$= \frac{a}{c} i + \frac{b}{c} j + k$$

The unit normals are:

$$\mp \frac{a/c\, i + b/c\, j + k}{\sqrt{a^2/c^2 + b^2/c^2 + 1}}$$

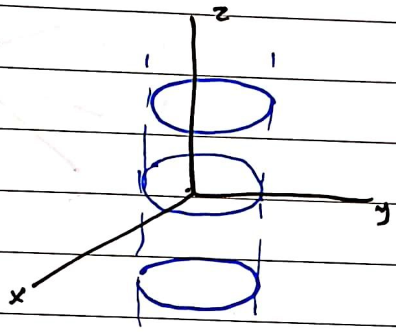
$$= \mp \frac{(a/c\, i + b/c\, j + k)c}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{ai + bj + ck}{\sqrt{a^2 + b^2 + c^2}}$$

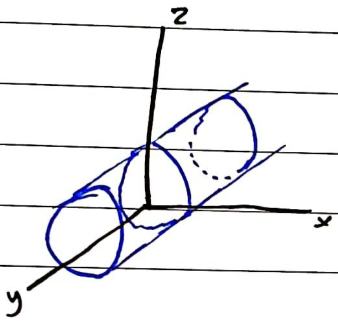
② Cylindrical shapes:

- An equation of two variables

Ex. $x^2+y^2=7$



Ex. $\frac{x^2}{9}+\frac{z^2}{4}=1$



$x = 3\cos(u)$

$y = v$

$z = 2\sin(u)$

$0 \le u \le 2\pi$

$v \in R$

Ex. $\frac{x^2}{9}+\frac{z^2}{4}=1$

A normal vector =

$$\frac{2x}{9} i + 0j + \frac{2z}{4} k = \nabla F$$

$r(u,v) = 3\cos(u)\, i + v j + 2\sin(u)\, k$

~~$* \; r(u,v) = 3\cos(u)\, i + vj +$~~

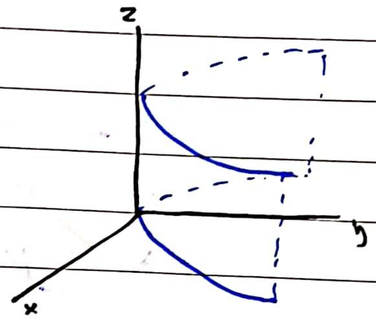
$r_u = -3\sin(u)\, j + 0j + 2\cos(u)\, k$

$r_v = j$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ -3\sin(u) & 0 & 2\cos(u) \\ 0 & 1 & 0 \end{vmatrix} =$$

$-2\cos(u)\, i - 3\sin(u)\, k$

Ex. $y = x^2$



$F = y - x^2 = 0$

$\nabla F = -2x i + j \rightarrow$ inward normal

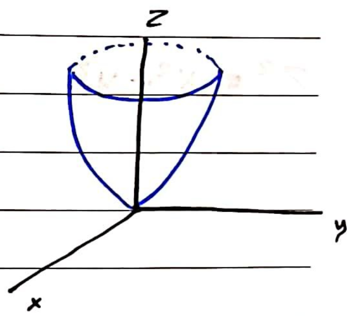
$2x - j \rightarrow$ outward normal

$y = x^2 \rightarrow$ parametric rep:

$x = u$, $y = u^2$, $z = v$ $\rightarrow r(u,v) = u i + u^2 j + v k$

③ Paraboloid:

$z = x^2 + y^2$



$S: x^2+y^2-z$

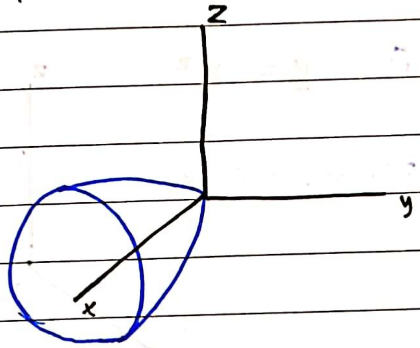
$\nabla S = 2x i + 2y j - k$

$r(u,v) = u i + v j + (u^2+v^2) k$ or

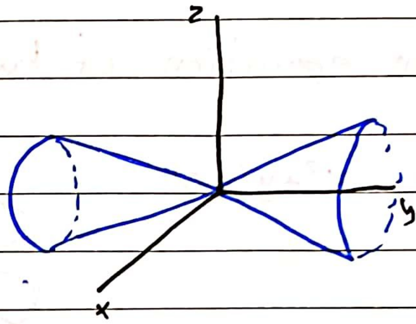
$r(x,y) = x i + y j + (x^2+y^2) k$ or

$r(u,v) = u\cos(v)\, i + u\sin(v) + u^2 k$

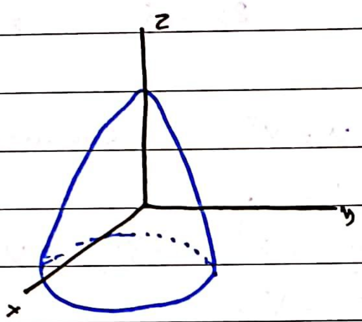
Ex. $x = \frac{y^2}{4} + \frac{z^2}{10}$



Ex. $y^2 = \frac{x^2}{4} + \frac{z^2}{9}$

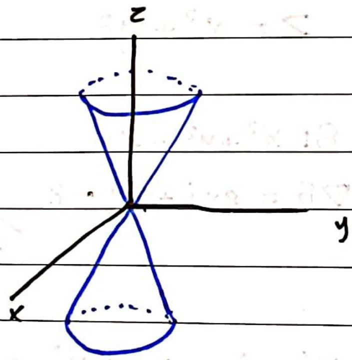


Ex. $z = 9 - x^2 - y^2$



④ Cone :-

$z^2 = x^2 + y^2$



parametric equation of $z^2 = x^2 + y^2$

$r(u,v) = u\cos(u)i + u\sin(u)j + uk$