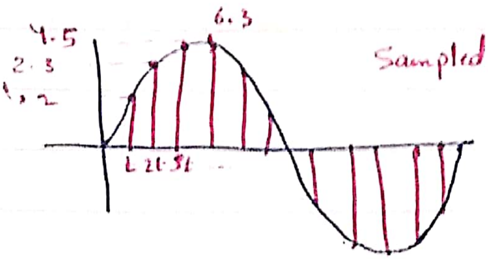
Dsp

Discrete time Signals : (Sequence)



Sampled at nT

$-\infty < n < \infty$

$x[2.5] \Rightarrow$ Undefined not Zero

$X_a(t)\Big|_{t=nT} = x[nT] = x[n]$

lower Case.

$x[n] = \{2 \ \underset{\uparrow}{3} \ 1 \ 5\}$ $\Rightarrow$ $\boxed{1}$ Sequence Representation.

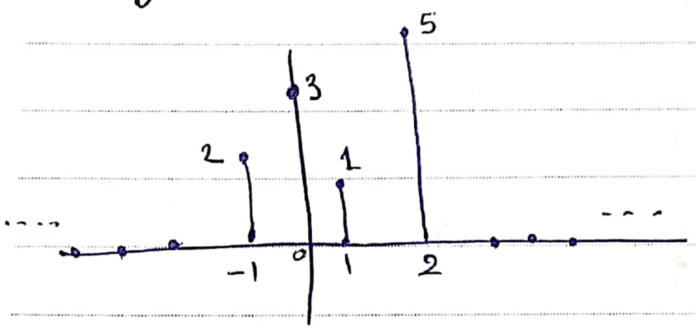
this is $x[0]$

الترتيب مهم .

$x[-1] = 2$ $x[0] = 3$ $x[1] = 1$ $x[2] = 5$

otherwise we assume the default that the first element. is $x[0]$

$\boxed{2}$ graphical Representation.



all the Other Values are Zeros.

$\boxed{3}$ Functional Representation.

$$x[n] = \begin{cases} 2 & -2 < n < 1 \\ 3 & 1 \leq n < 3 \\ 0 & \text{otherwise} \end{cases} \Rightarrow$$

$x[n] = \{2 \ \underset{\uparrow}{2} \ 3 \ 3\}$

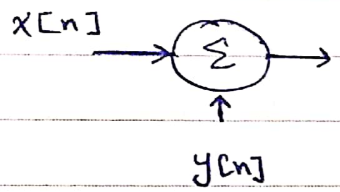
$x[n] = A\cos(wn)$.

④ Tabular method   non-integer values are non-define.

n	-1	0	1	2
$x[n]$	2	3	1	5

Elementary operations on sequences :-

① Addition



$\omega_1[n] = x[n] + y[n]$

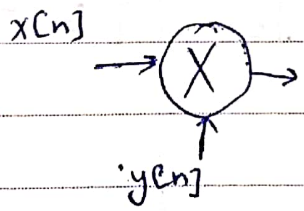
$x[n] = \{2 \ \underset{\uparrow}{3} \ 1 \ 5\}$

$y[n] = \{1 \ \underset{\uparrow}{1} \ 2 \ 2\}$

$$\begin{array}{ccccc} 2 & \underset{\uparrow}{3} & 1 & 5 & \underline{\underline{0}} \\ \underline{\underline{0}} & \underset{\uparrow}{1} & 1 & 2 & 2 \\ \hline 2 & \underset{\uparrow}{4} & 2 & 7 & 2 \end{array}$$

$\omega_1[n] = \{2 \ \underset{\uparrow}{4} \ 2 \ 7 \ 2\}$

② Multiplication.



$\omega_2[n] = x[n] .* y[n]$

$\omega_2[n] = \{0 \ \underset{\uparrow}{3} \ 1 \ 10 \ 0\}$

$W_2[n] = \{3 \ \underset{\uparrow}{1} \ 10\}$

3) Multiplication by scalar value.

$x[n] \longrightarrow \boxed{A} \longrightarrow w_3[n] = A\,x[n]$

$A = 5$ $\quad w_3[n] = \{ 10 \quad \underset{\uparrow}{15} \quad 5 \quad 25 \}$

4) Time - delay :

$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow w_4[n] = x[n-1]$ D flip flop.

$w_4[0] = x[-1] = 2$
$w_4[1] = x[0] = 3$ $\quad w_4[n] = \{ \underset{\uparrow}{2} \quad 3 \quad 1 \quad 5 \}$
$w_4[2] = x[1] = 1$
$w_4[3] = x[2] = 5$

$x[n] \longrightarrow \boxed{z^{-N}} \longrightarrow w_4[n] = x[n-N]$

5) Time advance

$x[n] \longrightarrow \boxed{z} \longrightarrow w_5[n] = x[n+1]$

$x[n] = \{ 2 \quad \underset{\uparrow}{3} \quad 1 \quad \cancel{3}\ 5 \}$

- $w_5[n] = \{ 2 \quad 3 \quad \underset{\uparrow}{1} \quad 5 \}$

6) pick-off point

$x[n]$ —— $x[n]$
$\quad$ —— $x[n]$

A length of sequence

$N_1 \le n \le N_2$

N_1 and N_2 are finite

$\hookrightarrow N = N_2 - N_1 + 1$
number of elements

N_1 is finite and N_2 extends to ∞ ⇒ Right-Sided sequence.
N_2 is finite and N_1 extends to $-\infty$ ⇒ left-sided sequence
N_1 & N_2 extend to ∞ and $-\infty$ ⇒ two-sided Sequence.

* Classification of Discrete time Signals
(1) Based on symmetry :-
- A sequence is conjugate symmetric sequence if

$$X_{cs}[n] = X^*_{cs}[-n]$$

= if the sequence is Real :
$X_{cs}[n] = X_{cs}[-n]$ ⇒ $X_{ev}[n] = X_{ev}[-n]$ Even symmetry

- A sequence is conjugate anti-symmetric sequence

$$X_{ca}[n] = -X^*_{ca}[-n]$$

= if the sequence is Real

$$X_{odd}[n] = -X_{odd}[-n]$$

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Any sequence $x[n] = x_{cs}[n] + x_{ca}[n]$

$$x_{cs}[n] = (x[n] + x^*[-n])/2$$

$$x_{ca}[n] = (x[n] - x^*[-n])/2$$

Example:- $x[n] = \{0 \quad 1+j4 \quad -2+j3 \quad \underset{\uparrow}{4-j2} \quad -5-j6 \quad -j2 \quad 3\}$

$x^*[-n] = \{3 \quad j2 \quad -5+j6 \quad \underset{\uparrow}{4+j2} \quad -2-j3 \quad 1-j4 \quad 0\}$

$x[n]\Big\{$ $x_{cs} = \{1.5 \quad 0.5+j3 \quad -3.5+j4.5 \quad \underset{\uparrow}{4} \quad -3.5-j4.5 \quad 0.5-j3 \quad 1.5\}$

$+$

$x_{ca} = \{-1.5 \quad 0.5+j \quad 1.5-j1.5 \quad \underset{\uparrow}{-j2} \quad -1.5-j1.5 \quad -0.5+j1 \quad 1.5\}$

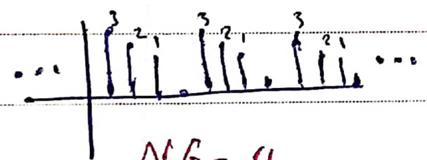
② periodic or Aperiodic sequencies

if the sequence is periodic

$\tilde{x}[n] = \tilde{x}[n+N]$ ↳ where N is a positive integer.

$-\infty < n < \infty$ ↳ N is called the period.

the least N satisfies the above equation.



N_f : fundamental period.

$N_f = 4$

The addition of two periodic sequences

$\tilde{x}_a[n]$ and $\tilde{x}_b[n]$

(N_a) (N_b)

$N = LCM(N_a, N_b)$

↳ Least common multiple.

$$= \frac{N_a * N_b}{GCD(N_a, N_b)}$$

Greatest common divisor.

For the Multiplication of two sequences

$X[n] = \tilde{X}_a[n] \cdot \tilde{X}_b[n]$

$N = LCM(N_a, N_b)$.

③ Energy and power Signals.

1- Energy sequence.

The amout of Energy is A sequence is defined by:

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

infinte sequence:

Example $X[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$ $\{0 \ 1 \ \frac{1}{2} \ \frac{1}{4} \ ... \}$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) = \frac{\pi^2}{6}$$

$\Downarrow$ Energy sequence.

Example $x[n] = \begin{cases} \frac{1}{\sqrt{n}} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \text{divarges.}$$

Finite sequences:

Ex $X[n] = \{1 \ \underset{\uparrow}{2} \ 3 \ 5\}$

$\mathcal{E}_x = (1)^2 + (2)^2 + (3)^2 + (5)^2$ $\Rightarrow$ All finite sequences are Energy sequences

$\mathcal{E}_x = 39$

2 - Power sequence :-

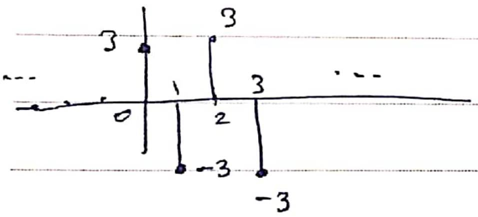
The amount of average power is sequence is defined by :-

$$P_x = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{n=-k}^{k} |x[n]|^2$$

⇒ any energy sequence has a Zero average power.

⇒ for periodic sequences $px = \frac{1}{\underset{\text{periodic}}{N}} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$

Ex :- $x[n] = \begin{cases} 3(-1)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$



$$P_x = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{n=0}^{k} (3(-1)^n)^2$$

$$P_x = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{n=0}^{k} 9$$

$$P_x = \lim_{k \to \infty} \frac{9(k+1)}{2k+1} = \frac{9}{2} = 4.5 \text{ W}$$

x[n] is a power Sequence

Energy $= \infty$

Other types of classifications :-

(1) Bounded sequence : if the mag. $|x[n]| < B_x < \infty$ for all n

(2) Absolutely Summable sequence : $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

No.

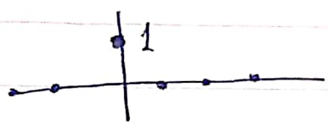
③ Square Summable Sequence

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$ (Energy Sequence).

Some Basic sequences.

1) Unit Sample Sequence (Unit impulse)

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases}$$



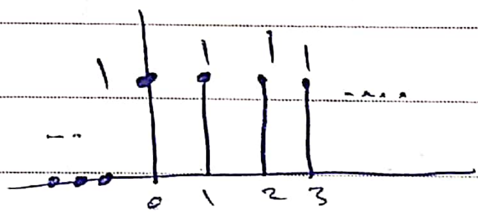
$$\delta[n-k] = \begin{cases} 1, & n=k \\ 0, & \text{otherwise.} \end{cases}$$

$$x[n] = \{1 \quad \underset{\uparrow}{2} \quad 3 \quad 5\}$$

$$x[n] = \delta[n+1] + 2\,\delta[n] + 3\,\delta[n-1] + 5\,\delta[n-2]$$

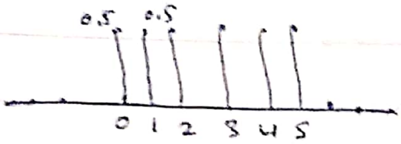
2) Unit step sequence.

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$u[n-k] \begin{cases} 1, & n \geq k \\ 0, & n < k \end{cases}$$

Example



$x[n] = \{0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5\}$

$x[n] = 0.5\,\mu[n] - 0.5\,\mu[n-6]$

$$\mu[n] = \sum_{N=0}^{\infty} \delta[n-N]$$

$$\mu[n] = \sum_{m=-\infty}^{n} \delta[m]$$

---

3) Sinusoidal and exponential sequences.

$x[n] = A\cos(\underset{\uparrow}{w}n + \phi) \qquad -\infty < n < \infty$

discrete time - frequency. rad/sample.

$x(t) = A\cos(\underline{\Omega}t + \phi)$.

continuous time frequency rad/s

$x[n] = A\alpha^n \qquad -\infty < n < \infty$

A: complex or Real $\qquad \alpha$: complex or Real.

$x[n] = A\,e^{(\sigma + jw)n} \Rightarrow \alpha = e^{(\sigma + jw)}$

$$\boxed{e^{jx} = \cos x + j\sin x}$$ Euler formula.

Example $x[n] = A\,e^{(\sigma + jw)n}$ $\qquad$ A is Real

$x[n] = A\,e^{\sigma n} * A e^{jwn} = A\,e^{\sigma n}\,[\cos(wn) + j\sin(wn)]$

$= \underbrace{A e^{\sigma n}\cos(wn)}_{x_{re}[n]} + \underbrace{A j e^{\sigma n}\cos(wn)}_{x_{imag}[n]}$.

– sinusoidal sequence :-

$x[n] = A\cos(wn+\phi)$ $\quad -\infty < n < \infty$

for periodicity $\quad x[n] = x[n+N]$

$x[n+N] = A\cos[w(n+N)+\phi]$

$= A\cos[(wn+\phi)+wN]$

$= A\cos(wn+\phi)\,\cancel{\cos(WN)} - A\sin(wn+\phi)\,\underbrace{\cancel{\sin(WN)}}_{\text{zero}}$

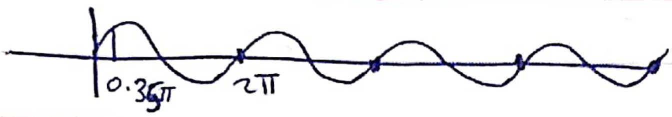
WN should be $2\pi r$, 0 is trivial solution

integers # of samples

integer # of cycles

relatively #

$$\boxed{\frac{N}{r} = \frac{2\pi}{w}} \rightarrow \text{Rational number.}$$



Example :- ① $x[n] = (\underset{w}{0.35\pi n} + 30^\circ)$

$$\frac{2\pi}{0.3\pi} = \frac{200}{35} = \frac{40}{7} \qquad \boxed{N = 40} \quad \boxed{r = 7}$$

② $x[n] = e^{j(0.2\pi)n}$ $\quad (w = 0.2\pi)$

$$\frac{2\cancel{\pi}}{0.2\cancel{\pi}} = \frac{10}{1} \rightarrow \text{10 samples each cycle.}$$

– The Sampling process

$x_a(t) = A\cos(\Omega t)$

$t = nT$

Sampling frequency f_T [sample/sec] or [HZ]

Sampling period $T = \frac{1}{f_T}$

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No

$A\cos(\Omega T n)$ $\quad\quad \Omega T = \omega$

$x[n] = A\cos[\omega n]$

Example :-

$x_1(t) = \cos(6\pi t) \longrightarrow f_1 = 3\,Hz$

$x_2(t) = \cos(14\pi t) \longrightarrow f_2 = 7\,Hz$

$x_3(t) = \cos(26\pi t) \longrightarrow f_3 = 13\,Hz$

$f_T = 10\,Hz \quad\quad T = 0.1\,sec$

$\therefore - \; x_1[n] = \cos(0.6\pi n)$

$x_2[n] = \cos(1.4\pi n)$

$x_3[n] = \cos(2.6\pi n)$

$x_2[n] = \cos((2-0.6)\pi n)$

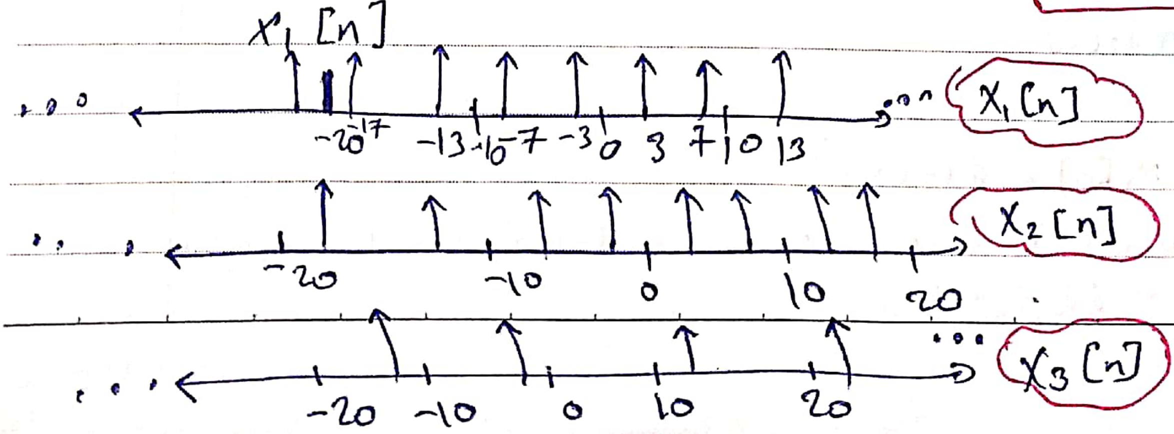
$= \cos(\cancel{2\pi n})\cos(0.6\pi n) + \sin(\cancel{\pi n})^{0}\sin(0.6\pi n)$

$= \cos(0.6\pi n) = x_1[n]$

$x_3[n] = \cos(2.6\pi n) = \cos((2+0.6)\pi n)$

$= \cos(0.6\pi n) = x_1[n]$

Aliasing happen when the f_T is less than $\boxed{(2W \text{ [nyquest rate]}}$



Discrete-Time systems :-

$x[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n]$

$y[n] = T[x[n]]$

Classifications of DT systems :-

$\boxed{1}$ Linearity (linear OR non linear system) :-

$x[n] = \alpha x_1[n] + \beta x_2[n]$
$y[n] = T[x[n]] = T[\alpha x_1[n] + \beta x_2[n]]$
$y[n] = \alpha T[x_1[n]] + \beta T[x_2[n]]$

$y[n] = \alpha y_1[n] + \beta y_2[n] \Rightarrow$ linearity where

$y_1[n] = T(x_1[n])$
$y_2[n]\ T[x_2[n]]$

$\boxed{\text{Examples}}$ (a) $y[n] = n\,x[n]$
(b) $y[n] = A x[n] + B$

(a) $x[n] = \alpha x_1[n] + \beta x_2[n]$
$y_1[n] = n\,x_1[n]$
$y_2[n] = n\,x_2[n]$

$y[n] = n[\alpha x_1[n] + \beta x_2[n]]$
$y[n] = \alpha n x_1[n] + \beta n x_2[n]$
$y[n] = \alpha y_1[n] + \beta y_2[n]$ → the system is linear ✓

(b) $y[n] = a x[n] + B$

$x[n] = \alpha x_1[n] + \beta x_2[n]$

$y_1[n] = a x_1[n] + B$

$y_2[n] = a x_2[n] + B$

$y[n] = a[\alpha x_1[n] + \beta x_2[n]] + B$

$y[n] = a\alpha x_1[n] + a\beta x_2[n] + B$

$y[n] = \alpha y_1[n] + \beta x_2[n] + B \neq \alpha y_1[n] + \beta y_2[n]$

it's a non linear system.

2- Shift - Invariant system (Time Invariant system).

If $y_1[n]$ is the response of $x_1[n]$ then the response of $x[n] = x_1[n - n_o]$ is simply $y_1[n - n_o]$.

How to check ?

$x[n] \xrightarrow{T} y[n]$

$x[n - n_o] \xrightarrow{T} y[n - n_o]$

1) we apply $x[n]$ to the system and we get $y[n]$

2) apply $x_1[n] = \alpha[n-K]$ to the system to get $y_1[n]$

3) Shift $y[n]$ by $K \Rightarrow y[n-k]$

if $y[n-k] = y_1[n] \rightarrow$ Shift invariant system

Ex:- 1) $y[n] = x[n] - x[n-1]$

2) $y[n] = n x[n]$

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No.

① $y[n] = x[n] - x[n-1]$

$x_1[n] = x[n-k]$

$\boxed{\cancel{n \to n-k}}$

$y_1[n] = x[n-k] - x[n-k-1]$

$y[n-k] = x[n-k] - x[n-k-1]$ $=$ So it is time invariant system. / shift - Invariant system.

② $y[n] = n\cancel{x}[n]$

$x_1[n] = x[n-k]$

$y_1[n] = n\,x[n-k]$

$y[n-k] = (n-k)x[n-k]$ $\neq$ Not shift invariant system = shift variant system

3- Causal OR non Causal systems :-

$y[n] = T\,[x[n], x[n-1], x[n-2], \ldots]$

$y[n_0]$ depends on $x[n_0]$ or previous instants ($n < n_0$)

Ex :- ① $y[n] = x[n] - x[n-1]$ ⇒ Causal system

② $y[n] = x[n+1]$ ⇒ Non - Causal system

③ $y[n] = x[-n]$ ⇒ Non - Causal system

4) Stable OR Unstable (Astable) Systems :-

Bounded input Bounded Output Stability Criteria (BIBO)

$x[n] = A\cos(wn)$

Ex $y[n] = c\,x[n]$ c: constant y is stable

$|x[n]| \leq B_x < \infty$ $|y[n]| \leq B_x = A \cdot c$

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No.

Ex : ① $y[n] = e^{x[n]}$

② $y[n] = \sum_{k=0}^{n} x[k]$

1) $|y[n]| = |e^{x[n]}| \leq e^{|x[n]|}$ $\quad x[n] \leq e^{\beta x}$

$$|y[n]| = e^{x[n]} \leq e^{x[n]} \leq e^{\beta x} \leq \cancel{e^{\beta x}}$$

βy

the system is bounded $\Rightarrow$ BIBO Stable

2) $y[n] = \sum_{k=0}^{n} x[k]$ (accumulater)

$$|y[n]| = \left|\sum_{k=0}^{n} x[k]\right| \leq \sum_{k=0}^{n} |x[k]| \leq \sum_{k=0}^{n} \beta x$$

upper Limit

Not BIBO Stable system

Unstable system.

$\Leftarrow (n+1)\,\beta x \neq \beta y$

time varying.

5-) Static and dynamic systems (Memoryless, Systems with Memory)

ⓐ $y[n] = a\,x[n]$ → Memoryless – Static system

ⓑ $y[n] = n^2 x[n]$ → Memoryless – Static system

ⓒ $y[n] = \frac{1}{2} x[n] + \frac{1}{3} x[n-1]$ → dynamic System

previous time

ⓓ $y[n] = \frac{1}{2} x[n] + \frac{1}{3} x[n-1] + \frac{1}{10} y[n-1]$ → dynamic system

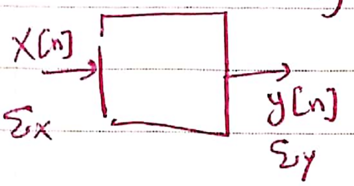
6- Passive and lossless systems:-

the amount of

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad \text{(passive System)}$$

$\downarrow$ $\mathcal{E}_y$ $\quad$ $\downarrow$ $\mathcal{E}_x$

if $\mathcal{E}_y = \mathcal{E}_x$ (passive and lossless system)



**Impulse and step response:-

$x[n] \longrightarrow \boxed{T[.]} \longrightarrow y[n] = T[x[n]]$

$x[n] = \delta[n]$ $\qquad$ $y[n] = h[n]$ (impulse response).

$\Downarrow$

((the response of the system when the input is an impulse))

$h[n] = T[\delta[n]]$

$x[n] = u[n] \Longrightarrow y[n] = S[n]$ (Step response)

$S[n] = T[u[n]]$

Example $\qquad$ finite impulse Response system (FIR)

$y[n] = 0.3\,x[n+1] + 0.2\,x[n] + 0.6\,x[n-1]$

find the impulse response of the system $h[n]$

$y[n] = h[n] = 0.3\,\delta[n+1] + 0.2\,\delta[n] + 0.6\,\delta[n-1]$

$$h[n] = \begin{cases} 0.3, & n=-1 \\ 0.2, & n=0 \\ 0.6, & n=+1 \\ 0, & \text{otherwise} \end{cases}$$

$h[n] = \{0.3 \quad \underset{\uparrow}{0.2} \quad 0.6\}$

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Ex: $y[n] = \frac{1}{2} y[n-1] + x[n]$ $\Rightarrow$ infinite ~~Response~~ impulse Response.
(IIR)

$\boxed{\text{Find the } h[n] \text{ for the system.}}$

$$h[n] = \frac{1}{2} h[n-1] + \delta[n]$$

$$h[0] = \frac{1}{2} \underbrace{h[-1]}_{\text{zero}} + \delta[0]$$

$$h[0] = \delta[0] = 1$$

$$h(1) = \frac{1}{2} h[0] + \delta(1) = \frac{1}{2}$$

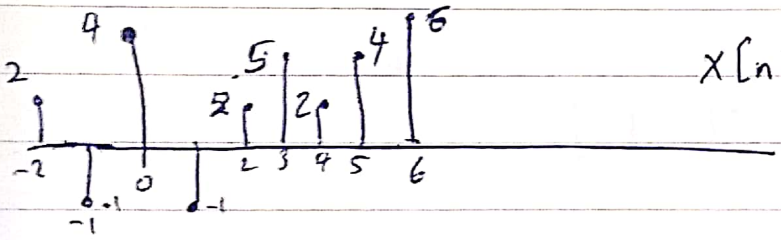
$$h(2) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$h(3) = \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$$

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

** input - output Charectarization of linear time invarient (LTI DT systems)

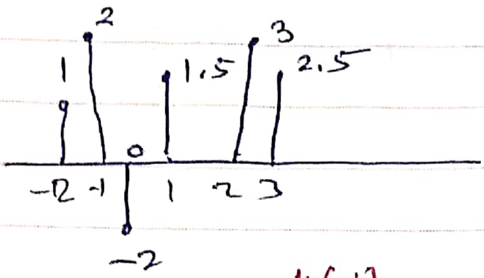


$$x[n] = \{2 \quad -1 \quad \underset{\uparrow}{4} \quad -1 \quad 2 \quad 5 \quad 2 \quad 4 \quad 6\}$$

$$x[n] = 2\delta[n+2] - 1\delta[n+1] + 4\delta[n] + -1\delta[n-1]$$
$$+ 2\delta[n-2] + 5\delta[n-3] + 2\delta[n-4] + 4\delta[n-5] + 6\delta[n-6].$$

$$x[n] = \sum_{k=\infty}^{\infty} x[k]\,\delta[n-k]$$

Example



$x[-1]$ $x[0]$ $x[1]$

$$x[n] = \delta[n+2] + \underline{2}\,\delta[n+1] \underline{-2}\,\delta[n] + \underline{1.5}\,\delta[n-1] + 3\delta[n-2] + 2.5\,\delta[n-3].$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].$$

$$x[n] \longrightarrow \boxed{T[\cdot]} \xrightarrow{y[n]}$$

$$\therefore\ y[n] = T\left[\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right]$$

if the system is linear

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot T[\delta[n-k]]$$

if the system is also time invariant

$$\boxed{\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \\ y[n] &= x[n] \circledast h[n] \end{aligned}}$$

Convolution Sum

for linear time invariant (LTI) systems

$$\|$$

$$y[n] = h[n] \circledast x[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k].$$

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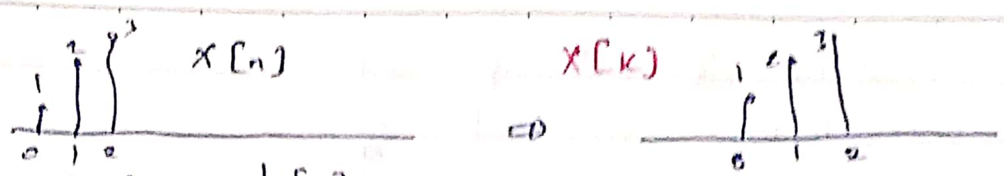
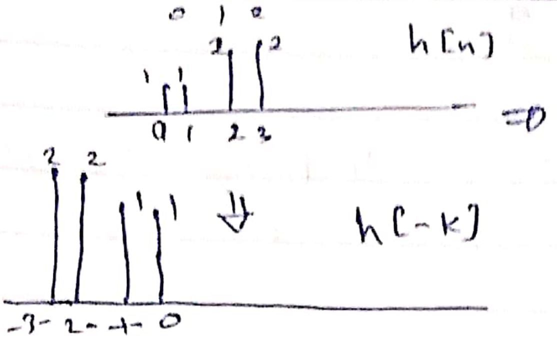
Example





$y[n] = 0 \quad n < 0$

$y[0] = 1 \times 1 = 1 \qquad y[0] = 1$

$y[1] = \sum h[1-k] \cdot x[k]$

$y[1] = 1 \times 1 + 2 \times 1 = 3$

$y[2] = \sum h[2-k]\, x[k]$

$y[2] = 3 + 2 + 2 = 7$

$y[3] = 3 + 4 + 2 = 9$

$y[4] = 6 + 4 = 10$

$y[5] = 2 \times 3 = 6$

$y[6] = 0$ (No overlap)...

$y[n] = \{ \underset{\uparrow}{1} \;\; 3 \;\; 7 \;\; 9 \;\; 10 \;\; 6 \}$. y(n) samples $= \underbrace{N_1}_{\text{samples in x}} + \underbrace{N_2}_{\text{samples in y}} - 1$

Tabular method :- (for Convolution Sum).

n	0	1	2	3	4	5	6	---
x[n]	1	2	3					
h[n]	1	1	2	2				
	1	2	3					
	–	1	2	3				
	–	–	2	4	6			
	–	–	–	2	4	6		

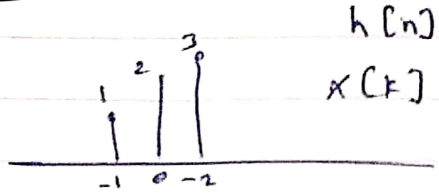
$y[n] = \{1 \;\; 3 \;\; 7 \;\; 9 \;\; 10 \;\; 6\}.$

No.

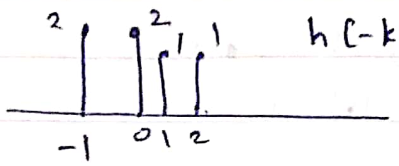
Example $x[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\}$

$h[n] = \{1 \quad 1 \quad \underset{\uparrow}{2} \quad 2 \quad 3$

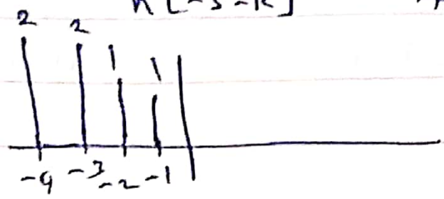
$x[k]$



$h[-k]$



$h[-3-k]$



the first y will be at $n=-3$

n	0	1	2	3	4	5	6
x[n]	1	2.	3				
h(n)	1	1	2.	2			

how many digits after the decimal point.
2 elements.

```
 1  2  3
 -  1  2  3
 -  -  2  4  6
 -  -  -  2  4  6
```

assume that there is a decimal point here.

$y[n] = \{1 \quad 3 \quad 7 \quad \underset{\uparrow}{9} \quad 10 \quad 6\}$

2 1

① the first element index = index of the starting for x[n] + index of the starting of x[k].

② the length of the result = $N_1 + N_2 - 1$

Stability condition in Terms of impulse Response :- (LTI system)

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

$$\le \sum_{k=-\infty}^{\infty} |h[k]|\, \underline{B_x}^{\text{upper limit of } x[n-k]}$$

$$\boxed{S_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty \quad \text{Absoultly summable.}}$$

$$B_x \cdot S_h = B_y$$

**So the impulse response should be absolutely summable.

Example :- $h[n] = \alpha^n \mu[n]$ $\qquad |\alpha| < 1$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |\alpha^n| = \sum_{n=0}^{\infty} |\alpha|^n$$

(Stable system)

$$S_h = \frac{1}{1-|\alpha|} \quad ✓ \qquad B_y = B_x\, S_h$$

Causality in Terms of the impulse Response :-

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$y[n_0] = \sum_{k=-\infty}^{\infty} h[k]\, x[n_0-k]$$

$$= \sum_{k=0}^{\infty} h[k]\, x[n_0-k] + \sum_{k=-\infty}^{-1} h[k]\, x[n_0-k]$$

First sum

$$y[n_0] = [h[0]\, x[n_0] + h[1]\, x[n_0-1] + h[2]\, x[n_0-2] + \cdots]$$
$$+ [h[-1]\, x[n_0+1] + h[-2]\, x[n_0+2] + \cdots]$$

second sum

* $h[k] = 0$ for $k < 0$ $\Rightarrow$ for causal System.

Discrete time Fourier Transform (DTFT)

$$x[n] \xrightarrow{\mathcal{F}} X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

Examples. $x[n] = \delta[n]$

① $\Delta(e^{jw}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-jwn} = e^{-j(w)(0)} = 1$

② $x[n] = \alpha^n \mu[n]$ $\quad |\alpha| < 1$

$$X(e^{jw}) = \sum_{n=0}^{\infty} \alpha^n \cdot e^{-jwn} = \sum_{n=0}^{\infty} (\alpha e^{-jw})^n = \frac{1}{1-\alpha e^{-jw}}$$

③ $x[n] = \{1 \;\; 2 \;\; \underset{\uparrow}{3} \;\; 5\}$

$$X(e^{jw}) = 1 \times e^{-jw(-2)} + 2e^{-jw(-1)} + 3e^{-jw(0)} + 5e^{-jw(1)}$$
$$= e^{2jw} + 2e^{jw} + 3 + 5e^{-jw}.$$

* $X(e^{jw})$ is continuous in w
* $X(e^{jw})$ is periodic in w with period of 2π

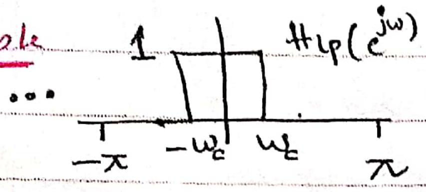
$$X(e^{jw}) = X(e^{(jw+2\pi)})$$

$$\left(X e^{(jw+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(w+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} \cdot \underbrace{e^{-j2\pi}}_{=1}$$

$$= X(e^{jw}) .\ \#$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw$$

Example



$$H_{lp}(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-w_c}^{w_c} 1\, e^{jwn}\, dw .$$

Find $H_{lp}[n]$

$$h_{lp} = \frac{1}{2\pi}\, \frac{e^{jwn}}{jn}\Big]_{-w_c}^{w_c}$$

$$h_{lp}[n] = \frac{e^{jw_cn} - e^{-jw_cn}}{2\pi jn} = \boxed{\frac{\sin(w_cn)}{\pi n}}$$ for all n and $n \neq 0$.

for $n=0$ $$h_{lp}[0] = \int_{-w_c}^{w_c} \frac{1}{2\pi} \cdot 1 \cdot 1 \, dw$$

$$= \frac{w_c(\cancel{2})}{\cancel{2}\pi} = \frac{w_c}{\pi}$$

$$\boxed{h_{lp}[n] = \begin{cases} \dfrac{w_c}{\pi} & n=0 \\ \dfrac{\sin(w_cn)}{\pi n} & n \neq 0 . \end{cases}}$$

No.

Discrete time Fourier transform (~~DTFT~~ DTFT)

Theorems:-

$x[n] \xleftrightarrow{DTFT} X(e^{jw})$

$h[n] \xleftrightarrow{DTFT} H(e^{jw})$

* ① Linearity:-

$\alpha x[n] + \beta h[n] \xleftrightarrow{DTFT} \alpha X(e^{jw}) + \beta H(e^{jw})$

② Time Reversal theorem.

$g[-n] \xleftrightarrow{DTFT} G(e^{-jw})$

③ Time-shifting theorem.

$x[n-n_0] \xleftrightarrow{DTFT} e^{-jwn_0} X(e^{jw})$

Example

$y[n] = \begin{cases} \alpha^n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$ $\quad |\alpha| < 1$

Find $Y(e^{jw})$

$y[n] = \{1 \quad \alpha \quad \alpha^2 \dots \alpha^{M-1}\}$

$Y(e^{jw}) = 1 + \alpha e^{-jw} + \alpha^2 e^{-jw2} + \dots + \alpha^{M-1} e^{-j(M-1)w}$

$= \sum_{n=0}^{M-1} r^n = \frac{1-r^M}{1-r} \Rightarrow \frac{1-(\alpha e^{-jw})^M}{1-(\alpha e^{-jw})}$

Alternative Solution

Shift in time

$y[n] = \alpha^n [M[n] - M[n-M]]$

$= \alpha^n M[n] - \alpha^n M[n-M]$ $\quad \alpha^n - \alpha^M$

$y[n] = \alpha^n M[n] - \alpha^M \cdot \alpha^{n-M} M[n-M]$

$$Y(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} - \frac{\alpha^{M} e^{-jwM}}{1-\alpha e^{-jw}} = \frac{1-\alpha^{M} \cdot e^{-jwM}}{1-\alpha e^{-jw}}$$

Exampl $d_0 V[n] + d_1 V[n-1] = P_0 x[n] + P_1 x[n-1]$

- find the transfer function $H(e^{jw}) = \frac{V(e^{jw})}{X(e^{jw})}$
- Apply $x[n] = \delta[n]$ & find $V(e^{jw})$ that corresponds to $x[n] = \delta[n]$

$$d_0 V(e^{jw}) + d_1 e^{-jw} V(e^{jw}) = P_0 X(e^{jw}) + P_1 e^{-jw} X(e^{jw})$$

$$\Rightarrow V(e^{jw}) \left[d_0 + d_1 e^{-jw} \right] = X(e^{jw}) \left[P_0 + P_1 e^{-jw} \right]$$

$$H(e^{jw}) = \frac{V(e^{jw})}{X(e^{jw})} = \frac{P_0 + P_1 e^{-jw}}{d_0 + d_1 e^{-jw}}$$

② Either we go to the diffrence eq.

$$d_0 V[n] + d_1 [V[n-1]] = P_0 \delta[n] + P_1 \delta[n-1]$$

$$d_0 V(e^{jw}) + d_1 e^{-jw} V(e^{jw}) = P_0 \cdot 1 + P_1 e^{-jw} \cdot 1$$

$$H(e^{jw}) = V(e^{jw}) \Big|_{x[n] = \delta[n]} = \frac{P_0 + P_1 e^{-jw}}{d_0 + d_1 e^{-jw}}.$$

④ Frequency shifting theorem :-

$$e^{j\omega_0 n}\, g[n] \xleftrightarrow{DTFT} G(e^{j(\omega-\omega_0)})$$

Ex :-

$y[n] = (-1)^n \alpha^n \mu[n]$ $|\alpha|<1$

Find $Y(e^{j\omega})$?

$$\alpha^n \mu[n] \longleftrightarrow \frac{1}{1-\alpha e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{1-\alpha e^{-j(\omega-\pi)}} = \frac{1}{1-\alpha e^{j\pi} \cdot e^{-j\omega}} = \frac{1}{1+\alpha e^{-j\omega}}$$

⑤ Differentation in frequency.

$$n g[n] \xleftrightarrow{DTFT} j \frac{dG(e^{j\omega})}{d\omega}$$

$$\frac{d}{d\omega}\left[G(e^{j\omega})\right) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-j\omega n}\Big]$$

$$j \frac{dG(e^{j\omega})}{d\omega} = \cancel{j} \sum -\cancel{j} n\, g[n] e^{-j\omega n} \Rightarrow j \frac{dG(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} n\, g[n] e^{-j\omega n}$$

Ex $y[n] = (n+1)\alpha^n \mu[n]$ $|\alpha|<1$

Let $x[n] = \alpha^n \mu[n]$

$y[n] = n\, x[n] + x[n]$

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$j \frac{d X(e^{-j\omega})}{d\omega} = j \frac{-\alpha j e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha(e^{-j\omega}))^2} + \frac{1}{(1-\alpha e^{-j\omega})} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

smile for life

⑥ Convolution Theorem:-

$$g[n] \circledast h[n] \xrightarrow{DTFT} G(e^{jw})\, H(e^{jw})$$

Ex:-

$g[n] = \{1\ 2\ 3\ 5\}$

$h[n] = \{1\ 1\ 2\ 2\}$

$G(e^{jw}) = 1 + 2e^{-jw} + 3e^{-j2w} + 5e^{-jw3}$

$H(e^{jw}) = 1 + e^{-jw} + 2e^{-jw2} + 2e^{-jw3}$

The modulation theorem:-

$$h[n]g[n] \xrightarrow{DTFT} \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta})\, G(e^{j(w-\theta)})\, d\theta$$

* Gibbs phenomenon.

- parseval Relation

$$\sum_{n=-\infty}^{\infty} g[n]\, h^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{jw}) H^*(e^{jw})\, dw.$$

If $g[n] = h[n]$

$$E_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{jw})|^2 dw$$

Example $g[n] = \{1\ \underset{\uparrow}{2}\ 3\}$

$E_g = 1^2 + 2^2 + 3^2 = 14$

$G(e^{jw}) = 1.e^{jw} + 2 + 3e^{-jw}$

$G(e^{jw}).G^*(e^{jw}) = (e^{jw} + 2 + 3e^{-jw}).(e^{-jw} + 2 + 3e^{jw})$

$= 1 + 2e^{jw} + 3e^{2jw} + 2e^{-jw} + 4 + 6e^{jw} + 3e^{-2jw} + 6e^{-jw} + 9$

$= 14 + (8e^{-jw} + 8e^{jw}) + (3e^{-j2w} + 3e^{2jw})$

smile for life

No.

$$= \int_{-\pi}^{\pi} 14 + 8e^{-jw} + 8e^{jw} + 3e^{-j2w} + 3e^{j2w} = 14$$

$$x[n] = \sum_{i=1}^{L} A_i \cos(w_i n + \theta_i) \qquad -\infty < n < \infty$$

$$H(e^{jw}) = |H(e^{jw})| \angle H(e^{jw})$$

$$y[n] = \sum_{i=1}^{L} A_i |H e^{jw_i}| \cos[w_i n + \phi_i + \angle H(e^{jw_i})]$$

$$Y(e^{jw}) = X(e^{jw}) \cdot H(e^{jw})$$

$$y = X \cdot H = A_1 \angle \theta_1 \cdot H_1 \angle \theta_1 = A_1 \cdot H_1 \angle \theta_1 + \theta_1$$

Example

$$x[n] = 10 - 5\sin\left(\frac{\pi n}{2}\right) + 20\cos \pi n \qquad -\infty < n < \infty$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] \quad ; \text{ Find } y[n]$$

$$H(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

$$w = 0 \Rightarrow H(e^{0}) = 2\angle 0^\circ$$

$$w = \frac{\pi}{2} \Rightarrow H\left(e^{j\frac{\pi}{2}}\right) = \frac{1}{1 + \frac{1}{2}j} = \frac{2}{\sqrt{2}} \angle -26.6^\circ$$

$$w = \pi \Rightarrow H(e^{j\pi}) = \frac{1}{1 - \frac{1}{2}e^{-j\pi}} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} \angle 0^\circ$$

$$y[n] = 10 * 2 - \frac{5 * 2}{\sqrt{5}} \sin\left(\frac{\pi n}{2} - 26.6^\circ\right)$$

$$+ \frac{20 * 2}{3} \cos(\pi n)$$

$\Rightarrow$ $\boxed{\text{First EXAM}}$

No.

Tuesday 5/3 Finite_length Discrete Transforms

N- length sequence

Transpose.

$$X = [\, x[0] \quad x[1] \quad x[2] \; \ldots \; x[N-1] \,]^{T}$$

* orthogonal Transforms

We always start with $x[0]$ up to $x[n-1]$ and we have N- length.

$$X[k] = \sum_{n=0}^{N-1} x[n] \, \Psi^{*}[k,n], \quad 0 \leq k \leq N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, \Psi[k,n], \quad 0 \leq n \leq N-1$$

(synthesis equation)

Inverse of orthogonal ~~transformers~~ transforms

$\Psi[k,n]$ is called the basis sequence

* According $\Psi[k,n]$:

Discrete Fourier Transform (DFT).

Discrete Cosine Transform (DCT)

Haar Transform

- The Discrete Fourier Transform (DFT).

$$\begin{cases} X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j2\pi kn/N} & 0 \leq k \leq N-1 \\ \Psi[k,n] = e^{j2\pi kn/N} \end{cases}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j2\pi kn/N} \qquad 0 \leq n \leq N-1$$

Let $W_N = e^{-j2\pi/N}$ $\Rightarrow$ ex $\begin{cases} W_4 = e^{-j2\pi/4} \\ W_8 = e^{-j2\pi/8} \end{cases}$

smile

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No.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn} \qquad 0 \le n \le N-1$$

$$X[0] = \sum_{n=0}^{N-1} x[n] = x[0] + x[1] + \dots + x[n-1]$$

$$X[1] = \sum_{n=0}^{N-1} x[n]\, W_N^{n} = x[0] + x[1]\, W_N^{1} + \dots x[N-1]\, W_N^{N-1}$$

$\cancel{X[2]}$

$$X[2] = \sum_{n=0}^{N-1} x[n]\, W_N^{2n} = x[0] + x[1]\, W_N^{2N} + \dots + x[N-1]\, W_N^{2(N-1)}$$

$$X[N-1] = \sum_{n=0}^{N-1} x[n]\, W_N^{(N-1)n} = x[0] + x[1]\, W_N^{(N-1)} + \dots + x[N-1]\, W_N^{(N-1)(N-1)}$$

$\cancel{X[0] = \sum_{n=0}}$

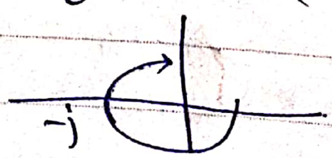
$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & & & & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{D_N} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\underline{x}}$$

Matrix vector by $N \times N$

$$\underline{X} = D_N \cdot \underline{x}$$

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$W_4 = e^{-j2\pi/4} = (e^{-j\pi/2})^1$$



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$D_2 = \begin{bmatrix} 1 & 1 \\ 1 & w_2^1 = -1 \end{bmatrix}$ D_N = dft mtx(N)

$X = D_4 \, x$ (DFT)

Compare $\updownarrow$

$X =$ fft x (DFT)

Inverse :-

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{-nk}$$

$$x[0] = \frac{1}{N} \sum_{k=0}^{N-1} \left[X[0] + X[1] + \dots + X[N-1] \right]$$

$$x[1] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{-k} = \frac{1}{N} \left[X[0] + X[1] W_N^{-1} + \dots + X[N-1] W_N^{-(N-1)} \right]$$

$$x[2] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{-2k} = \frac{1}{N} \left[X[0] + X[1] W_N^{-2} + \dots + X[N-1] W_N^{-2(N-1)} \right]$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^{-1} & W^{-2} & \dots & W^{-(N-1)} \\ \vdots & W^{-2} & & & W^{-2(N-1)} \\ 1 & W_N^{-(N-1)} & & \dots & W_N^{-(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[N-1] \end{bmatrix}$$

$$\underline{x} = \frac{1}{N} D_N^{*} \underline{X}$$

$$D_N^{-1} = \frac{1}{N} D_N^{*}$$

$$D_N^{-1} = \frac{1}{N} D_N^{*}$$

$$\omega_4^1 = e^{-j\frac{2\pi}{4}\cdot 1}$$

$$= e^{-j\frac{\pi}{2}}$$

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$D_4 \cdot D_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$$(D^4)^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

DN = dft mtx (N); — N: length.

inverse

$$ID_N = \frac{\text{conj}(DN)}{N}$$

Circular shift of a sequence.

$r = \langle m \rangle_N$ m modulo N

$r = m + LN$

where L is an integer (+ve / -ve) chosen to make $r = m + Ln$ a number between 0 and N-1

$\langle 25 \rangle_7 = 25 + (L)(7)$ $\boxed{L = -3}$

$25 - 21 = 4$ r $\boxed{r = 4}$

$\langle -16 \rangle_7 = -16 + L\cdot 7$ $\boxed{L = 3}$

$\boxed{r = 5}$

$\langle 50 \rangle_8 = 50 + L \cdot 8$ $\boxed{L = -6}$

$\boxed{r = 2}$

$0 \le r \le 7$.

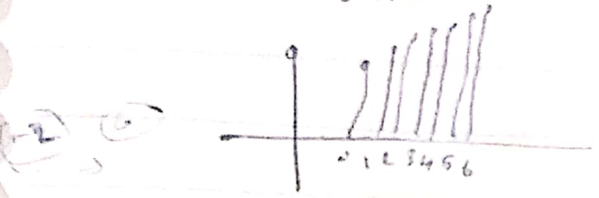
smile

$\langle -50 \rangle_8 = -50 + 7(8) = \boxed{r = 6}$

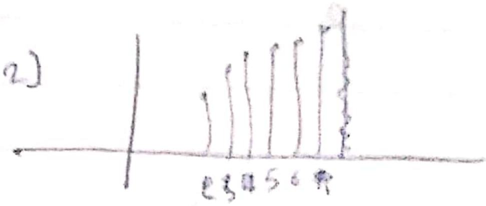
Circular shift of a sequence

$y[n] = x[\langle n - n_0 \rangle_N]$

linear shift



$y[n] = x[n-2]$



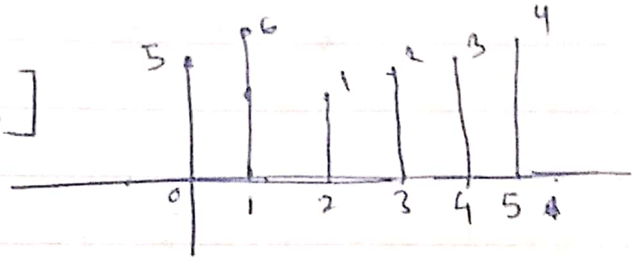
$y[n] = x[\langle N-2 \rangle_6]$

$-2+(1)(6)$

4

$y[0] = x[\langle 0-2 \rangle_6]$

$y[0] = x[4]$



$y[1] = x[\langle -1 \rangle_6] = x[5]$

$y[2] = x[\langle 0 \rangle_6] = x[0]$

$y[3] = x[\langle 1 \rangle_6] = x[1]$

$y[4] = x[2]$

$y[5] = x[3]$

$x[\langle n-2 \rangle_6] = x[\langle n+4 \rangle_6]$

$x[n]$: N length
$h[n]$: N length

circular conv

$X[k]$ N length
$H[k]$ N length

point by point mul in frequency.

Circular Convolution

$$y_L[n] = \sum_{m=0}^{N-1} g[m]\, h[n-m]$$

Same length (N) length.

$0 \le n \le 2N-2$

length $2N-1$

$y_c[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$ $\quad 0 \le n \le N-1$

N length Sequence.

$y_c[n]$ is N sequence

$y_L[n]$ is $\underset{\text{length}}{\underline{2N-1}}$ sequence.

(Thursday) 14-3.

$$y_L[n] = x[n] * h[n] = \sum_{m=0}^{N-1} x[m]\, h[n-m] \qquad 0 \le n \le 2N-2$$

$$y_c[n] = x[n] \circledast h[n] = \sum_{m=0}^{N-1} x[m]\, h[\langle n-m\rangle_N]$$

$$y_c[0] = \sum_{m=0}^{N-1} x[m]\, h[\langle 0-m\rangle_N] = x[0]h[0] + x[1]h[N-1] + x[2]h[N-2] + \cdots$$

$$y_c[1] = \sum_{m=0}^{N-1} x[m]\, h[\langle 1-m\rangle_N] = x[0]h[1] + \cdots + x[N-1]h[2]$$

$$\begin{bmatrix} y_c[0] \\ y_c[1] \\ \vdots \\ y_c[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & & & \vdots \\ \vdots & & & \\ h[N-1] & h[N-2] & & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$\boxed{\text{circulant matrix}}$

Example: $x[n] = \{1 \; 2 \; 3 \; 5\}$ $\quad h[n] = \{1 \; 1 \; 2 \; 2\}$

$$y_c[n] = x[n] \circledast h[n] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 16 \\ 19 \\ 17 \\ 14 \end{bmatrix}$$

Tabular method

n	0	1	2	3	$\langle 4 \rangle_4$	$\langle 5 \rangle_4$	$\langle 6 \rangle_4$
$x[n]$	1	2	3	5			
$h[n]$	1	1	2	2			

1	2	3	5			
5	1	1	2	3	5	
6	10	2	4	6	10	
4	6	10	2	4	6	10

$y[n] = \{16 \quad 19 \quad 17 \quad 41\}$

$$x[n] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 2+j3 \\ -3 \\ -2-j3 \end{bmatrix}$$

$$H[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1+j \\ 0 \\ -1-j \end{bmatrix}$$

$$Y_c[k] = X[k] \cdot H[k] = \begin{bmatrix} 11 \\ -2+j3 \\ -3 \\ -2-j3 \end{bmatrix} \cdot * \begin{bmatrix} 6 \\ -1+j \\ 0 \\ -1-j \end{bmatrix} = \begin{bmatrix} 66 \\ -1-5j \\ 0 \\ -1+j5 \end{bmatrix}$$

No.

$$y_c[n] = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 66 \\ -1-j5 \\ 0 \\ -1+j5 \end{bmatrix} = \begin{bmatrix} 16 \\ 19 \\ 17 \\ 14 \end{bmatrix}$$

Linear convolution for circular $y[n] = x_c[n] \circledast h_c[n]$

n	0	1	2	3	4	5	6	<7> 7		
$x_c[n]$	1	2	3	5	0	0	0			
$h_c[n]$	1	1	2	2	0	0	0			
	1	2	3	5	0	0	0			
	—	1	2	3	5	0	0	0		
	—	—	2	4	6	10	0	0	0	
	—	—	—	2	4	6	10	0	0	0
	1	3	7	14	15	16	10			

- MATLab

fft(x)

fft(X,m)

~~y[n] = ifft~~

y[n] = ifft(fft(x,m) .* fft(h,m));

Choose $M = 2N-1$

"Sunday"

DFT theorems :- $W_N = e^{-j2\pi/N}$

① Linearity ~~Linearity~~ :-

$\alpha g[n] + \beta h[n] \xleftrightarrow{DTFT} \alpha G[K] + \beta H[K]$

② Circular time shifting

$g[\langle n - n_r \rangle_N] \xleftrightarrow{DFT} W_N^{kn_r} G[K]$

③ Circular frequency shifting

$W_N^{-k_1 n} g[n] \xleftrightarrow{DFT} G[\langle K - K_1 \rangle_N]$

No.

④ duality theorem :-

$$G[n] \xrightarrow{DFT} N g[\langle -k \rangle_N]$$

$g[n] = \{1 \quad 2 \quad 3 \quad 5\}$ $\qquad G[k] = \{11 \quad -2+j3 \quad -3 \quad -2-j3$

$h[n] \; \{11 \quad -2+j3 \quad -3 \quad -2-j3\}$ $\qquad H\{k\} = \{1 \quad 5 \quad 3 \quad 2\} = \{4 \quad 20 \quad 1$

$\langle -1 \rangle_4 = 3 \qquad \langle -2 \rangle_4 = -2+4 = 2 \qquad \langle -3 \rangle_4 = -3+4 = 1 \qquad \langle 0 \rangle_4 = 0$

$$H[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 11 \\ -2+j3 \\ -3 \\ -2-j3 \end{bmatrix} = \begin{bmatrix} 4 \\ 20 \\ 12 \\ 8 \end{bmatrix}$$

⑤ Convolution theorem:-

$$g[n] \; Ⓝ \; h[n] \xrightarrow{DFT} G[k] \; H[k]$$

$$\sum_{m=0}^{N-1} g[m] \; h[\langle n-m \rangle_N]$$

⑥ Modulation theorem

$$g[n] \, h[n] \xleftrightarrow{DFT} \frac{1}{N} \sum_{l=0}^{N-1} G[l] \; H[\langle k-l \rangle_N]$$

⑦ parseval's relation.

$$\sum_{n=0}^{N-1} g[n] \, h^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} G[k] \; h^*[k]$$

$$E_g = \sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |G[k]|^2 \qquad o/p \quad g[n] = h[n]$$

$$E_g = 1^2 + 2^2 + 3^2 + 5^2 = 39$$

$$E_g = \frac{1}{4}\left[(11)^2 + (-2)^2 + (3)^2 + (-3)^2 + (-2)^2 + (-3)^2\right] = \frac{156}{4} = 3$$

* Relation between DTFT and DFT :

$x[n] = \{1 \quad 2 \quad 3 \quad 5\}$

$$X(e^{jw}) = \sum_{n=0}^{N-1} x[n] e^{-jwn} \qquad X(e^{jw}) = 1 + 2e^{-jw} + 3e^{-j2w} + 5e^{-j3w}$$

$$X(e^{jw})\Big|_{w_k = \frac{2\pi k}{M}} = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi k}{M}\right)n}$$

; M very large number

$\frac{2\pi(1)}{1000}, \frac{2\pi(2)}{1000} - - - -$

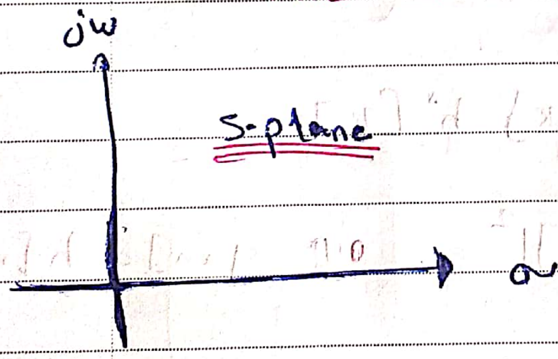
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi n/M}$$

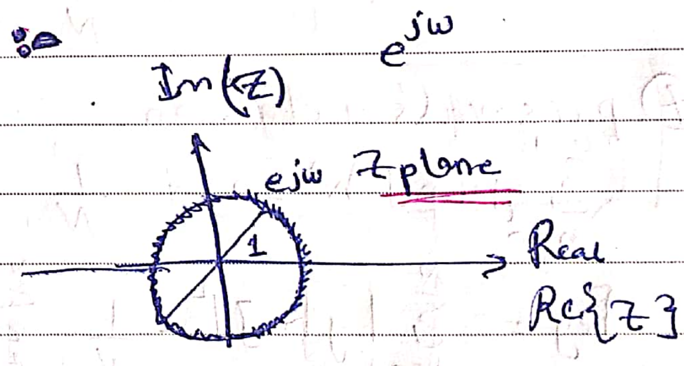
$X_e[n]$ is M-length sequence.

$$X_e[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ 0 & N \le n \le M-1 \end{cases}$$

$$X_e[k] = \sum_{n=0}^{N-1} X_e[n] e^{-j\frac{2\pi k n}{M}} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{M}} + 0$$

The Z-transform :





Definition ;

$$X(Z) = \sum_{n=-\infty}^{\infty} x[n] Z^{-n} \qquad X(Z) = Z\{x[n]\}$$

$$x[n] = Z^{-1}\{X[Z]\} \qquad x[n] \xleftrightarrow{Z} X(Z).$$

No.

$x_1[n] = 2^n \mu[n]$

$x_2[n] = \left(\frac{1}{2}\right)^n \mu[n]$

greater than 1

$X_1(e^{j\omega}) = \sum_{n=0}^{\infty} 2^n e^{-j\omega n} = \sum_{n=0}^{\infty} (2e^{-j\omega})^n$ div.

$X_2(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\omega}\right)^n = \frac{1}{1-\left(\frac{1}{2}e^{-j\omega}\right)}$.

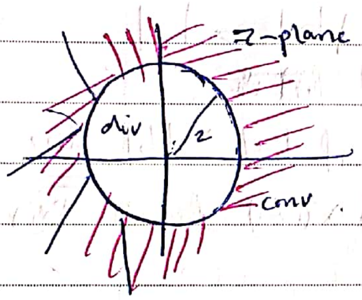
$x[n] \rightarrow \boxed{h[n]}_{LTI} \rightarrow y[n] = x[n] \circledast h[n]$

$X_1(Z) = \sum_{n=0}^{\infty} 2^n Z^{-n} = \sum_{n=0}^{\infty} (2Z^{-1})^n = \frac{1}{1-2Z^{-1}}$

$|2Z^{-1}| < 1$

$|2| < |Z|$

ROC | Region of Convergence.

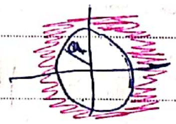


$Y(Z) = X(Z) \cdot H(Z)$

Example ① $x[n] = a^n \mu[n]$

$X(Z) = \sum_{n=0}^{\infty} (aZ^{-1})^n \Rightarrow X(Z) = \frac{1}{1-aZ^{-1}}$

$|aZ^{-1}| < 1$ $\boxed{|a| < |Z|}$



ROC

② $x[n] = -a^n \mu[-n-1]$ (left sided sequence).

$X(Z) = \sum_{-\infty}^{-1} (-aZ^{-1})^n$ if we let $m = -n$

$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \dots$

$= r[1 + r + r^2 + r^3 \dots]$

$X(Z) = \sum_{m=1}^{\infty} (-a^{-1}Z)^m \Rightarrow X(Z) = \frac{-a^{-1}Z}{1-a^{-1}Z}$

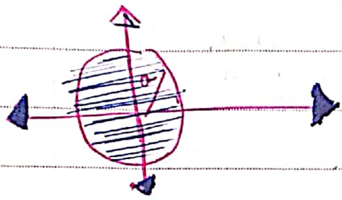
$|r| < 1$ $\leftarrow \frac{r}{1-r}$

$\boxed{|a^{-1}Z| < 1 \quad |Z| < |a|}$

ROC

smile for life

$$\frac{-a^{-1}z}{1-a^{-1}z} \Rightarrow \frac{-1}{\frac{1}{a^{-1}z}-1} \Rightarrow \frac{1}{1-az^{-1}}$$



right sided sequence → (ring) a Outside circle

left sided sequence → inside a circle. (disk)

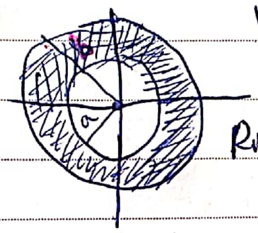
Example: $x[n] = a^n u[n] + b^n u[-n-1]$

Find $X(Z)$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$|b| > |a|$

[if $|a| > |b|$ div. $X(Z)$ doesn't exist]



$|b| > |a|$

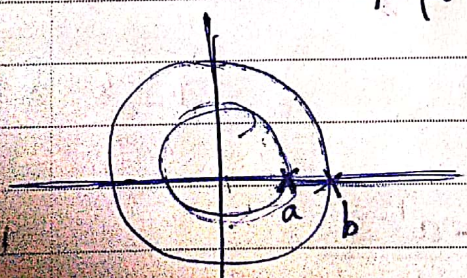
Ring

$$X(z) = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}}$$

$R_{x_1} \cap R_{x_2}$

$\boxed{|a| < |z| < |b|}$ ROC

$$X(z) = \frac{\cancel{1}-bz^{-1}-\cancel{1}+az^{-1}}{1-(a+b)z^{-1}+abz^{-2}} = \frac{(a-b)z^{-1}}{1-(a+b)z^{-1}+abz^{-2}}$$



poles

$1-az^{-1}=0$

$az^{-1}=1$ $\boxed{z=a}$

$\boxed{z=b}$

No. 24/3/2019

The inverse Z-Transform:-

* The inspection Methods:-

$$X(z) = \frac{1}{1-az^{-1}} \qquad |a| < |z|$$

$$x[n] = (a)^n \mu[n]$$

* Partial function Expansions:-

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \quad \rightarrow M < N$$

$$X(z) = \frac{b_0 \prod_{k=1}^{M} (1-c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1-d_k z^{-1})}$$

if $M \geq N \Rightarrow$ Long division

$$A_k = (1-d_k z^{-1})\, X(z) \Big|_{z=d_k}$$

$$= \left(\frac{A_1}{1-d_1 z^{-1}} + \frac{A_2}{1-d_2 z^{-1}} + \cdots + \frac{A_N}{1-d_N z^{-1}} \right) (1-d_1 z^{-1})$$

$$A_1 = A_1 + A_2 \frac{\cancel{(1-d_1 z^{-1})}^{0}}{1-d_1 z^{-1}} + \text{——} + \frac{A_N \cancel{(1-d_N z^{-1})}^{0}}{\cancel{(1-d_N z^{-1})}} \Bigg|_{z=d_1}$$

smile for life

Ex:- $x(z) = \dfrac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$ $\quad |z| > \frac{1}{2}$

$$x(z) = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

$$A_1 = \frac{1}{1-\frac{1}{2}(\frac{1}{4})^{-1}} = \frac{1}{1-\frac{4}{2}} = -1$$

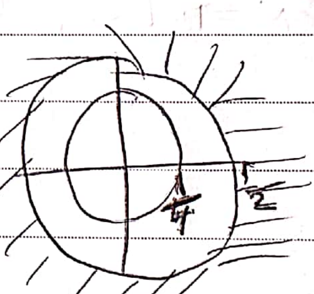
$$A_2 = \frac{1}{1-\frac{1}{4}(\frac{1}{2})^{-1}} = \frac{1}{1-\frac{1}{2}} = 2$$

$$x(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

$x[n] =$

$$\frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

$$\frac{A_1 - \frac{1}{2}A_1 z^{-1} + A_2 - \frac{1}{4}A_2 z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$$



$A_1 + A_2 = 1$ — (1)

$2\times(\frac{1}{2}A_1 + \frac{1}{4}A_2) = 0$ — (2)

$\cancel{A_1 + A_2 = 1}$

$A_2 - \frac{1}{2}A_2 = 1$

$\frac{1}{2}A_2 = 1$

$\boxed{A_2 = 2}$

No.

$$X(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$x[n] = \left(\frac{1}{4}\right)^n \mu[n] + 2\left(\frac{1}{2}\right)^n \mu[n]$$

$$\boxed{\frac{1}{4} < |z| < \frac{1}{2}}$$

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$X(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$x[n] = -\left(\frac{1}{4}\right)^n \mu[n] - 2\left(\frac{1}{2}\right)^n \mu[-n-1]$$

Ex : $X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$ $|z| < \frac{1}{4}$

$$X(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$x[n] = \left(\frac{1}{4}\right)^n \mu[-n-1] + -2\left(\frac{1}{2}\right)^n \mu[-n-1]$$

Ex $X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$ $|z|>1$ right sided.

$$\begin{array}{r|l} & 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & -2z^{-2}+3z^{-1}+2 \\ & 0+5z^{-1}-1 \end{array}$$

$$\Rightarrow X(z) = 2 + \frac{5z^{-1}-1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$X(z) = 2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

No.

$$A_1 = 2 + \frac{(-1+5z^{-1})}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}(1-\frac{1}{2}z^{-1})\Big|_{z=\frac{1}{2}}$$

$$= \frac{-1+5(2)}{1-2} = -9$$

$$\boxed{A_1 = -9}$$

$$A_2 = \frac{-1+5(1)}{1-\frac{1}{2}} = \frac{4}{\frac{1}{2}} = 8$$

$$\boxed{A_2 = 8}$$

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}} \Rightarrow x[n] = 2\delta[n] - 9\left(\tfrac{1}{2}\right)^n \mu[n] + 8(1)^n \mu[n]$$

property of z transform

① Linearity

$$x[n] = \alpha x_1[n] + \beta x_2[n]$$

$$\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{z} \alpha X_1[n] + \beta X_2[n]$$

$$R_x = R_{x_1} \cap R_{x_2}$$

Ex a) $x[n] = \cos(w_0 n)\mu[n]$

b) $x[n] = \sin(w_0 n)\mu[n]$

@ $x[n] = \frac{e^{jw_0 n} + e^{-jw_0 n}}{2}\mu[n]$

$$x[n] = \frac{1}{2}e^{jw_0 n}\mu[n] + \frac{1}{2}e^{-jw_0 n}\mu[n]$$

$$e^{jw_0 n}\mu[n] \xleftrightarrow{z} \frac{1}{1-e^{jw_0}z^{-1}} \qquad |z|>1$$

$$e^{-jw_0 n}\mu[n] \xleftrightarrow{z} \frac{1}{1-e^{-jw_0}z^{-1}} \qquad |z|>1$$

smile.

No.

$$X(z) = \frac{1}{2}\left(\frac{1}{1-e^{jw_0}z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-e^{-jw_0}z^{-1}}\right) \quad |z|>1$$

$$= \frac{1}{2}\left(\frac{1-e^{-jw}z^{-1} + 1-e^{jw}z^{-1}}{1-\cancel{2}\left(\frac{e^{jw_0}+e^{-jw_0}}{2}\right)z^{-1}+z^{-2}}\right) = \frac{1-\cos(w_0)z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-2}}$$

$$x[n] = \cos(w_0 n)\,\mu[n] \xleftrightarrow{z} \frac{1-\cos(w_0)z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-2}}$$

(b) $$x[n] = \frac{e^{jw_0 n} - e^{-jw_0 n}}{2j}\,\mu[n]$$

$$x(n) = \frac{1}{2j}e^{jw_0 n}\mu[n] - \frac{1}{2j}e^{-jw_0 n}\mu[n]$$

$$e^{jw_0 n}\mu[n] \xleftrightarrow{z} \frac{1}{1-e^{jw_0}z^{-1}} \quad |z|>1$$

$$e^{-jw_0 n}\mu[n] \xleftrightarrow{z} \frac{1}{1-e^{-jw_0}z^{-1}} \quad |z|>1$$

$$X(z) = \frac{1}{2j}\left[\frac{1}{1-e^{jw_0}z^{-1}}\right] - \frac{1}{2j}\left[\frac{1}{1-e^{-jw_0}z^{-1}}\right]$$

$$= \frac{\sin(w_0)z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-1}}$$

No.

② Scaling the Z-domain

$x[n] \xleftrightarrow{z} X(z)$ ROC $r_1 < |z| < r_2$

$a^n x[n] \xleftrightarrow{z} X(a^{-1} z)$ ROC $|a| r_1 < |z| < |a| r_2$

Ex :-

ⓐ $x[n] = a^n \cos(w_0 n) \mu[n]$

$$\cos(w_0 n)\mu[n] \xleftrightarrow{z} \frac{1 - \cos(w_0) z^{-1}}{1 - 2\cos(w_0) z^{-1} + z^{-2}}$$

$$a^n \cos(w_0 n)\mu[n] \xrightarrow{z} \frac{1 - \cos(w_0)\left(\frac{z}{a}\right)^{-1}}{1 - 2\cos(w_0)\left(\frac{z}{a}\right)^{-1} + \left(\frac{z}{a}\right)^{-2}}$$

$$= \frac{1 - a\cos(w_0) z^{-1}}{1 - 2a\cos(w_0) z^{-1} + a^2 z^{-2}}$$

$$x[n] = a^n \sin(w_0 n)\mu[n] \xleftrightarrow{z} \frac{a \sin w_0 z^{-1}}{1 - 2a\cos(w_0) z^{-1} + a^2 z^{-2}}$$

No.

* الثاني 28/3/2019

3- Differentiation in the Z-transform.

for both sides $* -z$

$x[n] \xleftrightarrow{Z} X(z)$ $\sum_{n=-\infty}^{\infty} x[n] z^{-n} = \left(\sum -n x[n] z^{-n-1}\right)$

then

$n x[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$ $\boxed{-z \frac{dX(z)}{dz} \longleftrightarrow \sum n x[n] z^{-n}}$

Example:- $x[n] = n a^n \mu[n]$

let $x_1[n] = a^n \mu[n] \xleftrightarrow{Z} X_1(z) = \frac{1}{1-az^{-1}}$ $|z| > |a|$

ROC doesn't change

$x[n] = n x_1[n] \xleftrightarrow{Z} -z * \left(\frac{-az^{-2}}{(1-az^{-1})^2}\right)$

$$\boxed{X(z) = \frac{az^{-1}}{(1-az^{-1})^2}}$$

Special case

$x[n] = \mu[n]$ ramp function. $n\mu[n] \longleftrightarrow \frac{z^{-1}}{(1-z^{-1})^2}$ $|z| > 1$

4- Convolution :-

$x_1[n] \xleftrightarrow{Z} X_1(z)$.

$x_2[n] \xleftrightarrow{Z} X_2(z)$

$x_1[n] \circledast x_2[n] \xleftrightarrow{Z} X_1(z) . X_2(z)$ ~~Max~~ ~~Intersect~~

ROC = $ROC_1 \cap ROC_2$

Example $x_1 = \{1 \quad 2 \quad 3 \quad 5\}$

$x_2[n] = \{1 \quad 1 \quad 2 \quad 2\}$

No.

$$X_1(Z) = 1 + 2Z^{-1} + 3Z^{-2} + 5Z^{-3}$$
$$X_2(Z) = 1 + Z^{-1} + 2Z^{-2} + 2Z^{-3}$$

$$1 + 2Z^{-1} + 3Z^{-2} + 5Z^{-3}$$
$$+ 1Z^{-1} + 2Z^{-2} + 3Z^{-3} + 5Z^{-4}$$
$$+ 2Z^{-2} + 4Z^{-3} + 6Z^{-4} + 10Z^{-5}$$
$$+ 2Z^{-3} + 4Z^{-4} + 6Z^{-5} + 10Z^{-6}$$

$$= 1 + 3Z^{-1} + 7Z^{-2} + 14Z^{-3} + 15Z^{-4} + 16Z^{-5} + 10Z^{-6}$$

$$x[n] = \{\underset{\uparrow}{1}\ 3\ 7\ 14\ 15\ 16\ 10\}$$

Ex :- AN LTI (causal) with an impulse Response
Right sided sequence

poles
natural response

$h[n] = \left(\frac{1}{2}\right)^n u[n]$ and find the Output of the System $y[n]$ if the Input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$

output
force response

$$H(Z) = \frac{1}{1-\frac{1}{2}Z^{-1}} \quad \text{ROC: } |Z| >$$

$$y[n] = x[n] * h[n]$$
$$Y(Z) = X(Z)\, H(Z).$$

$$X(Z) = \frac{1}{1-\frac{1}{4}Z^{-1}} \quad \text{ROC: } |Z|$$

$$Y(Z) = \frac{1}{1-\frac{1}{2}Z^{-1}} * \frac{1}{1-\frac{1}{4}Z^{-1}} \qquad \text{ROC: } |Z| > \frac{1}{2}$$

$$Y(Z) = \frac{A_1}{1-\frac{1}{2}Z^{-1}} + \frac{A_2}{1-\frac{1}{4}Z^{-1}}$$

$$A_1 = \frac{1 \cdot 1}{1-\frac{1}{4}(2)} = \frac{1}{\frac{1}{2}} = 2$$

$$A_2 = \frac{1 \cdot 1}{1-\frac{1}{2}(4)} = -1$$

ROC: $|Z| > \frac{1}{2}$

$$Y(Z) = \frac{2}{1-\frac{1}{2}Z^{-1}} - \frac{1}{1-\frac{1}{4}Z^{-1}}$$

$$y[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

$$Y_n + Y_F$$

$$\frac{Y(z)}{X(z)} \rightarrow \cancel{H(z)} \Rightarrow \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$Y(z) \cdot (1-\tfrac{1}{2}z^{-1}) = X(z)$$

$$Y(z) - \tfrac{1}{2} z^{-1} Y(z) = X(z)$$

$y[n] - \frac{1}{2} y[n-1] = x[n]$ → difference equation.

$$y[n] = \tfrac{1}{2} y[n-1] + x[n]$$

الأحد ٤/٧

Causality and stability :-
Region of Convergence :-

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

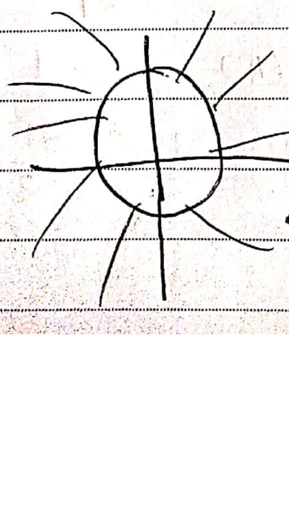
for Causal system $h[n] = 0$ for $n<0$

Stable system $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$|H(z)| = \sum_{n=-\infty}^{\infty} |h[n]\, z^{-n}| \le \sum_{n=-\infty}^{\infty} h[n]\, |z^{-n}|$$

$|z^{-n}| = 1$

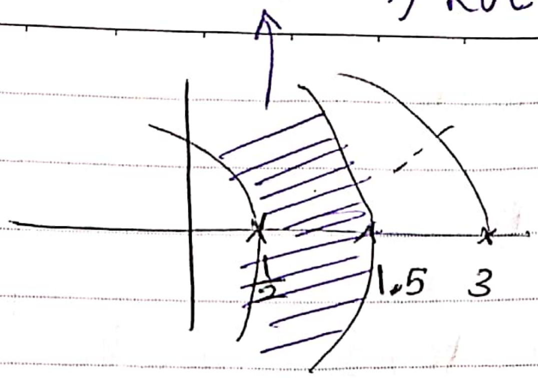
if ROC Contains the Unit circle ⇒ Stable system



Causal system ⇒ right sided sequence

Stability ROC

$Z =$



$$\frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}} + \frac{A_3}{1-3z^{-1}}$$

ROC for Causal system $= |Z| > 3$

∃ right sided sequences.

$$A_1 \left(\tfrac{1}{2}\right)^n \mu[n] + A_2\, \mu[n] \left(\tfrac{1.5}{1}\right)^n + A_3\, \mu[n]\, (3)^n \Rightarrow \text{Causal system}$$

↳ R.O.C for Stability

$$h[n] = A_1 \left(\tfrac{1}{2}\right)^n \mu[n] + -A_2 (1.5)^n \mu[-n-1] + -A_3 (3)^n \mu[-n-1]$$

$$\sum_{n=-\infty}^{-1} (1.5)^n \Rightarrow \sum_{n=1}^{\infty} (1.5)^{-n} = \sum_{n\geq 1}^{\infty} \left(\frac{1}{1.5}\right)^n = \frac{\frac{1}{1.5}}{1-\frac{1}{1.5}}$$

↓

absolutely summable.

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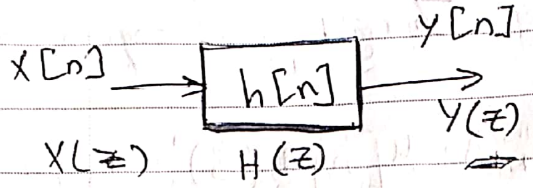
No.

Digital Filter Design :-

Discrete-time



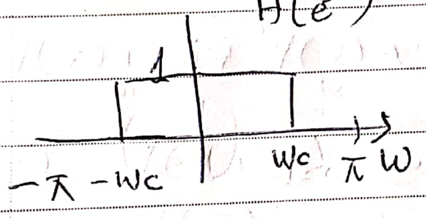
4 types of filters:-

1- Low pass filter $H_{LP}(z)$

2- High pass filter $H_{HP}(z)$

3- band Pass filter $H_{BP}(z)$

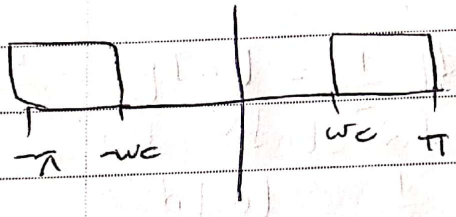
4- band Stop Filter / band reject filter $H_{BS}(z)$



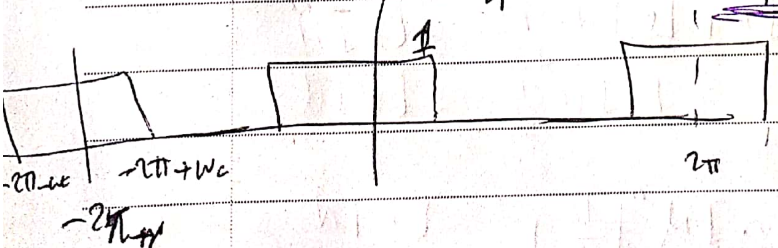
Low pass

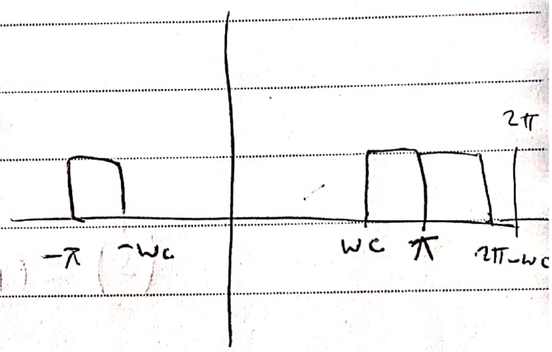
~~LP BPF~~

Highpass



$H_{LP}(e^{jw})$ (periodic)





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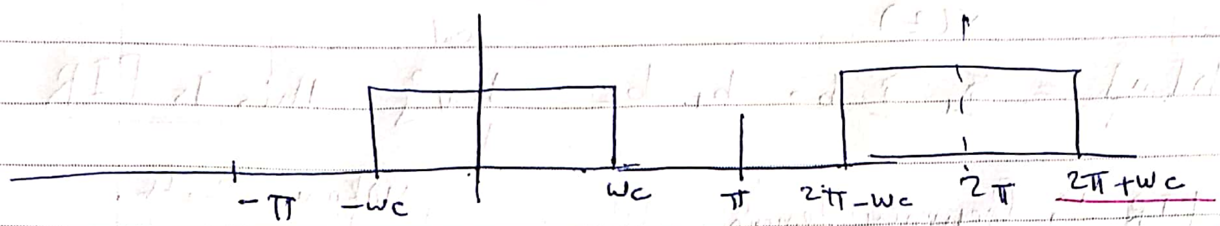
Tuesday 8/4

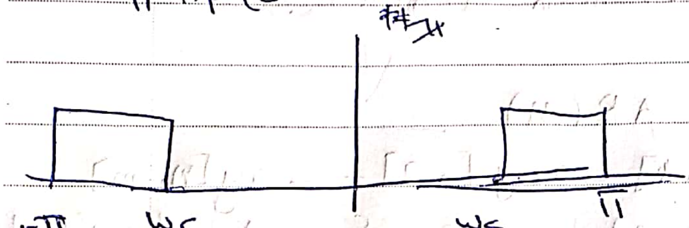
$H_{LP}(e^{jw})$



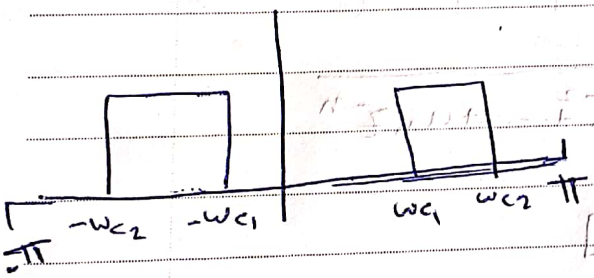
$j(2\pi + w_c)$

$= e^{j2\pi} \cdot e^{jw_c}$

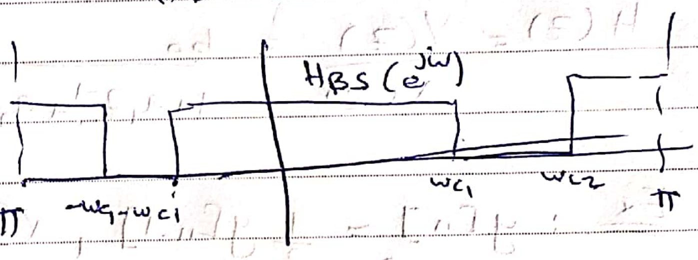
$H_{HP}(e^{jw})$



$H_{BP}(e^{jw})$



$H_{BS}(e^{jw})$



* Base on Impulse Response
1- finite Impulse Response fillters (FIR)
2- infinite Impulse Response filter (IIR)

① FIR:

$$y[n] = b_0 x[n] + b_1 x[n-1] + \dots + b_N x[n-N]$$
$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \dots + b_N z^{-N} X(z)$$

smi:)e for life

No.

$$Y(z) = X(z)\left(b_0 + b_1 z^{-1} + \dots + b_N z^{-N}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + \dots b_N z^{-N}$$

$$h[n] = \{b_0 \; b_1 \; b_2 \dots b_N\}$$ this Is FIR

unknown Coeff.

FIR in frequency domain

Same with

moving Average filter in time domain $MA(N)$.

(2) IIF :- Auto Regressive $AR(M)$

$$y[n] = b_0 x[n] - a_1 y[n-1] - a_2 y[n-2] - \dots a_M y[n-m]$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_M y[n-m] = b_0 x[n]$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \dots + a_M z^{-m} Y(z) = b_0 X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_M z^{-m}}$$

Ex : $y[n] = \frac{1}{2} y[n-1] + x[n]$

let $x[n] = \delta[n]$

$$h[0] = \frac{1}{2} \cancel{y[-1]} + \delta[0]$$

$h[0] = 1$

$$h[1] = \frac{1}{2} h[0] + x[1] = \frac{1}{2}(1) = \left(\frac{1}{2}\right)$$

$$h[2] = \frac{1}{2} h[1] = \frac{1}{4}$$

$$\vdots$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

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No.

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + X(z)$$

$$Y(z) = \left[1 - \frac{1}{2} z^{-1}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$h[n] = \left(\frac{1}{2}\right)^n \mu[n]$$

$X(z) = 1$

$\hookrightarrow x[n] = \delta[n]$

$x[n] \neq \delta[n]$ طب لو

$$x[n] = \left(\frac{1}{4}\right)^n \mu[n]$$ find $y[n]$

$$X(z) = \frac{1}{1 - \frac{1}{4} z^{-1}} \qquad |z| > \frac{1}{4}$$

$$Y(z) = H(z) \cdot X(z)$$

$$\frac{1}{1 - \frac{1}{2} z^{-1}} \cdot \frac{1}{1 - \frac{1}{4} z^{-1}} \qquad |z| > \frac{1}{2} = \frac{A_1}{1 - \frac{1}{2} z^{-1}} + \frac{A_2}{1 - \frac{1}{4} z^{-1}}$$

right sided

$y_h = y_n \rightarrow$ natural

homogeneous

$y_p = y_f$ forced

particular

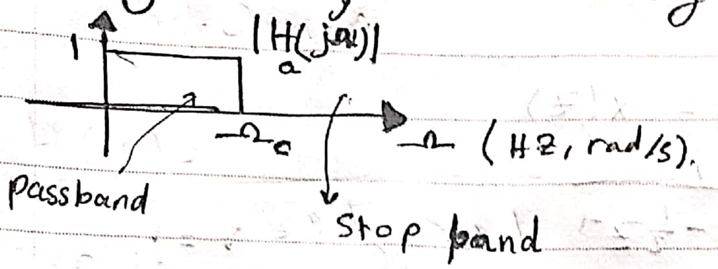
$$= A_1 \left(\frac{1}{2}\right)^n \mu[n] + A_2 \left(\frac{1}{4}\right)^n \mu[n]$$

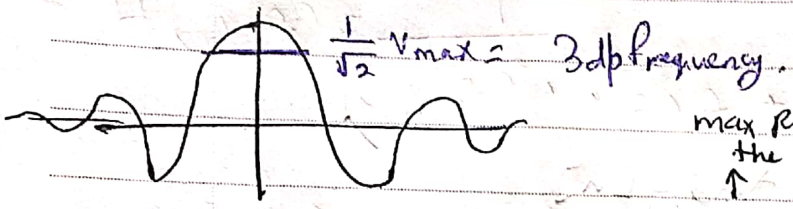
$$a_0 y[n] + a_1 y[n-1] + \dots + a_M y[n-M] =$$
$$b_0 x[n] + b_1 x[n-1] + \dots + b_N x[n-N]$$

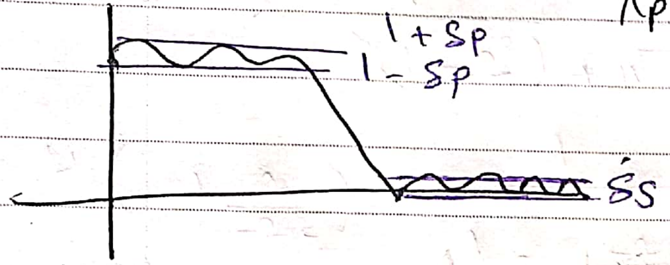
$\downarrow$

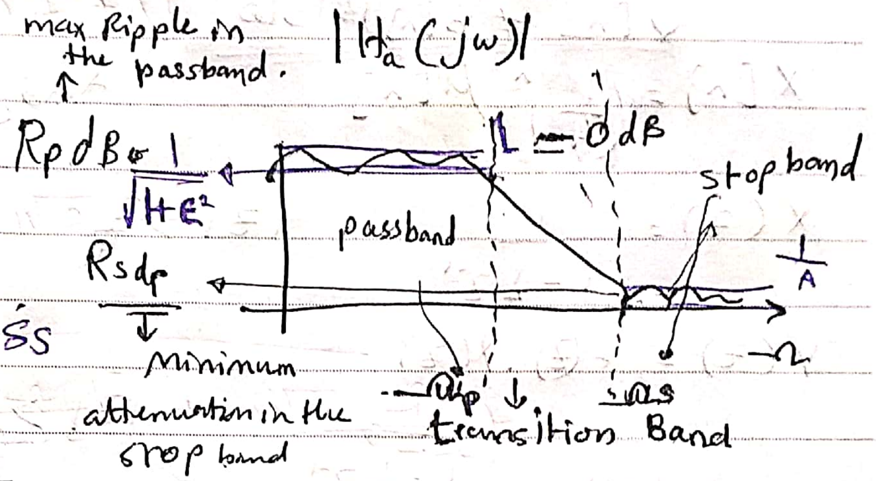
ARMA(M, N).

⊛ Analoga Low pass filter design :-









always negative ← $R_p = 20\log_{10} \frac{1}{\sqrt{1+\epsilon^2}}$

$R_p = 10\log_{10} \frac{1}{1+\epsilon^2}$

$R_s = 20\log_{10} \frac{1}{A} = 10\log\left(\frac{1}{A^2}\right)$

always negative

⇒ transition Ratio or selectivity parameter (k)

$K = \frac{\Omega p}{\Omega s}$ $K < 1$ for low pass filter.

⇒ The discrimination parameter (k_1)

$k_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}}$ $k_1 \ll 1$

smile for life

Butterworth Approximation:-

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$ → order of the filter

First order Butterworth

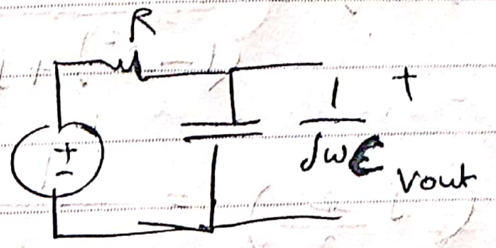
$$H(s) = \frac{1}{s+1}$$

# of poles decided the Order.

2nd Order $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$

$$H(jw) = \frac{1}{1+j\Omega}$$

$$H(jw) = \frac{1}{1+j\left(\frac{\Omega}{\Omega_c}\right)}$$



$\cancel{H(j\omega)} =$

$$\frac{V_{out}}{V_{in}} = \frac{\frac{1}{jwC}}{R+\frac{1}{jwC}} \times jwC$$

$$\frac{V_o}{V_{in}} = \frac{1}{jRwC+1} = \frac{\frac{1}{RC}}{\frac{1}{RC}+jw}$$

Ω_c

$$|H(j\Omega)|^2 = \frac{1}{1+j\frac{\Omega}{\Omega_c}} \cdot \frac{1}{1-j\frac{\Omega}{j\Omega_c}}$$

$$= \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^2}$$

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2nd Order

$$H(s) = \frac{1}{s^2 + \sqrt{2}\,s + 1}$$

$$H(j\Omega) = \frac{1}{((j\Omega)^2 + \sqrt{2}\,j\Omega + 1)}$$

$$H(j\Omega) = \frac{1}{(1-\Omega^2) + j\sqrt{2}\,\Omega}$$

$$H\left(j\frac{\Omega}{\Omega_c}\right) = \frac{1}{\left(1 - \left(\frac{\Omega}{\Omega_c}\right)^2\right) + j\sqrt{2}\left(\frac{\Omega}{\Omega_c}\right)}$$

$$|H(j\Omega)^2| = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^4}$$

$$|H_a(j\Omega)^8| = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = \frac{1}{1+\varepsilon^2} \Rightarrow \boxed{\varepsilon^2 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \quad 1$$

No.

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \frac{1}{A^2} \Rightarrow \boxed{\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = A^2-1}$$ (2)

$$② \div ① \Rightarrow \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{A^2-1}{\varepsilon^2} \Rightarrow \left(\frac{\Omega_s}{\Omega_p}\right)^{N} = \frac{\sqrt{A^2-1}}{\varepsilon}$$

$$N \log_{10}\left(\frac{\Omega_s}{\Omega_p}\right) = \log_{10} \frac{\sqrt{A^2-1}}{\varepsilon} \Rightarrow N = \frac{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}{\log_{10} \frac{\sqrt{A^2-1}}{\varepsilon}}$$

$$\boxed{N = \frac{\log_{10}\left(\frac{1}{k_1}\right)}{\log_{10}\left(\frac{1}{k}\right)}}$$

Ex $R_p = -1$ dB with $\Omega_p = 1$ KHz
$R_s = -40$ dB with $\Omega_s = 5$ KHz

Find the minimum order of N for a maximally flat filter

$$-1 = 10 \log \frac{1}{1+\varepsilon^2}$$

$\varepsilon = 0.5$

$$-40 = 10 \log \left(\frac{1}{A^2}\right)$$

$A = 100$

$$N = \frac{\log_{10}(196.5)}{\log_{10}(5)} = 3$$

the minimum order $N = 4$

No.

Chebyshev Approximation

Type 1 Chebyshev
Type 2 Chebyshev

Chebyshev polynomial

$$T_N(\Omega) = \begin{cases} \cos\left(N\cos^{-1}(\Omega)\right) & |\Omega| \le 1 \\ \cosh\left(N\cosh^{-1}(\Omega)\right) & |\Omega| > 1 \end{cases}$$

Recursive formula

$$T_r(\Omega) = 2\Omega\, T_{r-1}(\Omega) - T_{r-2}(\Omega) \qquad r \geq 2$$

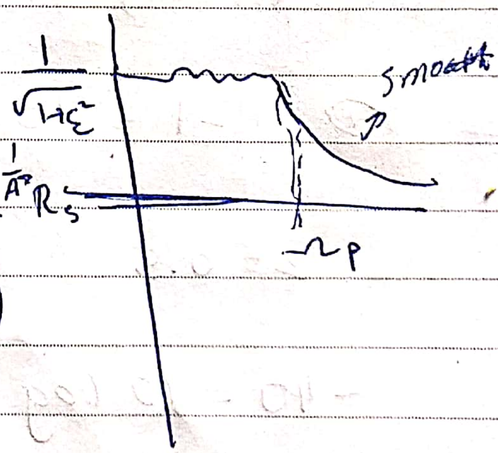
where $T_0(\Omega) = 1$ and $T_1(\Omega) = \Omega$

$$T_2(\Omega) = 2\Omega(\Omega) - 1 = 2\Omega^2 - 1$$

$$T_3(\Omega) = 2\Omega(2\Omega^2 - 1) - \Omega = 4\Omega^3 - 3\Omega$$

Type 1

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$



$$|H_a(j\Omega_p)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

$$|H_a(j\Omega_p)|^2 = \frac{1}{1+\varepsilon^2}$$

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{1}{A^2}$$

$$T_N\left(\frac{\Omega_s}{\Omega_p}\right) = \sqrt{\frac{A^2-1}{\varepsilon^2}} = \frac{\sqrt{A^2-1}}{\varepsilon}$$

$$= \cosh\left(N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right) \quad --- \quad \left(\frac{\Omega_s}{\Omega_p}\right) > 1$$

$$\therefore N = \frac{\cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\left(\frac{1}{k_1}\right)}{\cosh^{-1}\left(\frac{1}{k}\right)}$$

$\boxed{\text{Example}}$

$R_p = -1\ dB$ $\qquad \Omega_p = 1\ kHz$

$R_s = 40\ dB$ $\qquad \Omega_s = 5\ kHz$

find the minimum N order

$$\frac{1}{k_1} = 196.5 \qquad \frac{1}{k} = 5$$

$$N = \frac{\cosh^{-1}(196.5)}{\cosh^{-1}(5)} = 2.6 \rightarrow \boxed{N=3}$$