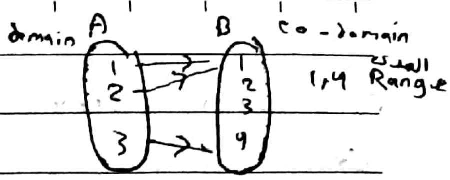
Chapter 1: Functions and models.



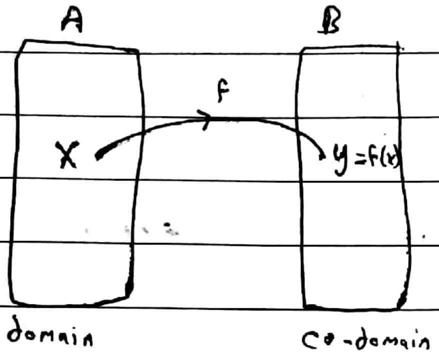
§ 1.1 : Four ways to represent a Function.

اقتران

Defn: A Function f is a rule that assigns to each element X in a set A (domain) exactly one element, called y = f(x), in a set B (Co-domain).

المجال

المجال المقابل



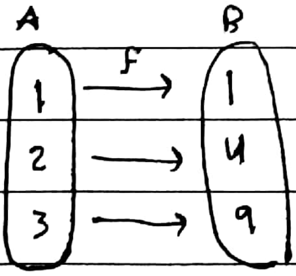
Representations of Functions:

There are four ways to represent a function:

1 Verbal Method: (word description).

Ex. The circumference of a square is four times one of its sides.

2 Numerical Method: (table of values).

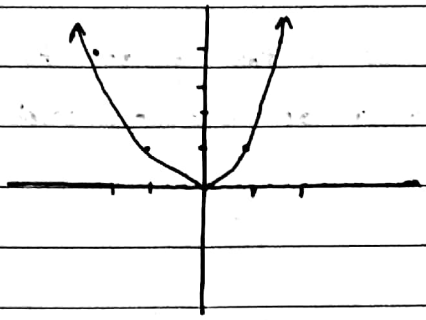


x	y = f(x)
1	1
2	4
3	9

Range

③ Graphical Method: (graph)

Ex.



④ Algebraic Method: (explicit formula)

Ex. $F(x) = x^2$

* Remark: f(x) is read as "f of x"

Ex. find the domain of the following function:

① الجذر التربيعي (-)
② الصفر في المقام
ينفعوا

① $F(x) = 2x^7 - 3x^2$

Soln: مجموعة Real Numbers

Dom(F) = IR

Dom من محور x
Rang من محور y

② $g(x) = 2 - \sqrt[3]{7x-1}$

Soln

Dom(g) = IR = (-∞, ∞)

③ $h(x) = x^2 + 5\sqrt{x^4+1} - \frac{3x+2}{x^2+3}$

Soln

Dom(h) = IR = (-∞, ∞)

④ $F(x) = 3x^5 - 7\sqrt{3x-6}$

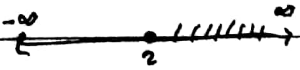
Soln: ~~[illegible]~~ "inequality" متباينة

تتغير اشارة المتباينة
" إذا ضربنا ب سالب او عملنا مثل ٣ ← $\frac{1}{3}$ "

مساواة

$\boxed{[\ 6\ .}$

$3x - 6 \geq 0$



$3x \geq 6$

$x \geq 2$

فترة
“interval”

$Dom(F) = [2, \infty)$

⑤ $g(x) = \frac{x-1}{\sqrt{x^2-x-6}}$

دائما لما يكون الجذر بالمقام نستثني الصفر

Soln

$x^2 - x - 6 \geq 0$

0 لانه ما في اشارة مساواة

$(x-3)(x+2) > 0$

اذا المتباينة اكبر من صفر نأخذ ال+
〃 〃 اصغر من صفر نأخذ الـ-

$-3 \quad \ominus \times \ominus$
$0 \quad \ominus \times \oplus$
$4 \quad \oplus \times \oplus$

اتحاد

$Dom(F) = (-\infty, -2) \cup (3, \infty)$

في حالة المقام بدون جذر بنطلع اللي بخلي صفر وبنستثني

⑥ $F(x) = \frac{3x^2 - x}{2x^2 - 8}$

Soln:

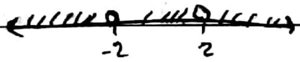
$2x^2 - 8 = 0$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

كل الاعداد ماعدا

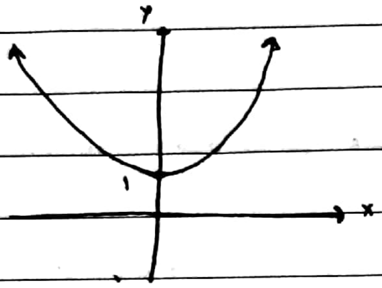


$Dom(F) = \mathbb{R} \setminus \{-2, 2\}$

$= (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Ex. Find the $\overset{x}{\text{domain}}$ and $\overset{y}{\text{Range}}$ of the following function:

①



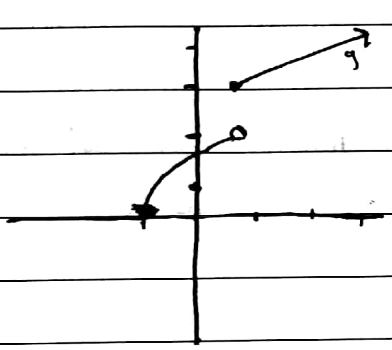
$F: \mathbb{R} \rightarrow \mathbb{R}$

Soln:

$Dom(F) = \mathbb{R} = (-\infty, \infty)$

$Rang(F) = [1, \infty)$

②



$g: \overset{\text{domain}}{[-1, \infty)} \rightarrow \mathbb{R}$ co-domain IR دايما

$Dom(g) = [-1, \infty)$

$Rang(g) = [0, 2) \cup [3, \infty)$

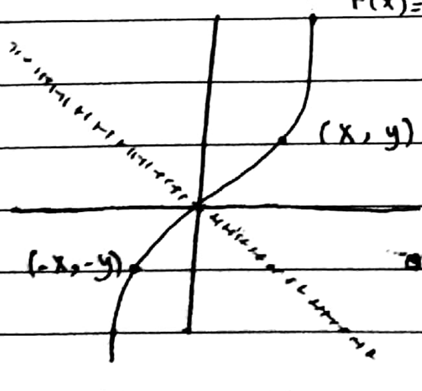
Defn: ① A function f is said to be $\underline{\overset{\text{فردي}}{\text{odd}}}$ function $\overset{\text{if and only if}}{\text{iff}}$ $F(-x) = -F(x)$

② A function f is said to be $\underline{\overset{\text{زوجي}}{\text{even}}}$ function iff $F(-x) = F(x)$

Remark:

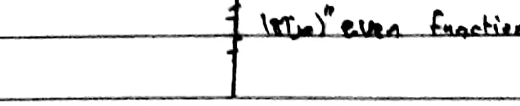
The graph of the function is symmetric about the origin, and the graph of the even function is symmetric about the y-axis.

① $F(x) = x$



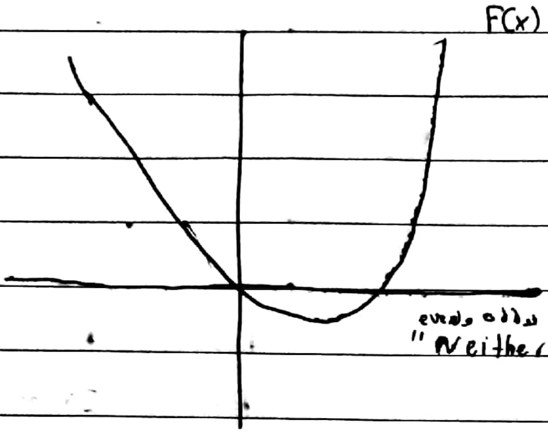
odd function

② $F(x) = x^2$



"even function"

③ $F(x) = x^2 - x$



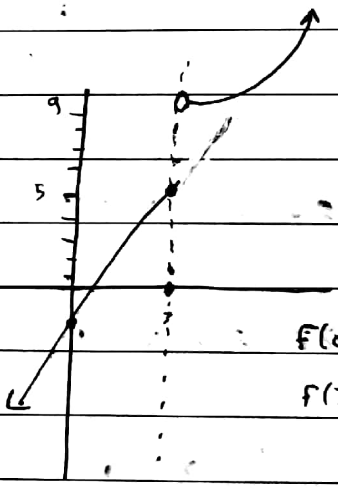
even و odd لا
"Neither"

المتشعب
* Piecewise Function:

Defn: A piecewise function is a function defined by different rules for different parts of the domain.

Ex.

$$f(x) = \begin{cases} 2x-1 & x \leq 3 \\ x^2 & \text{if } x > 3 \end{cases}$$



$f(0) = -1$
$f(3) = 5$

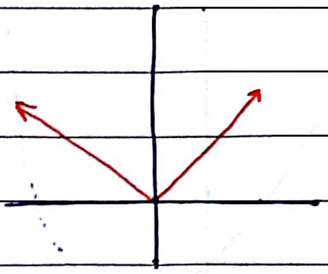
$Dom(f) = \mathbb{R} = (-\infty, \infty)$

مش مطلوب
$Range(f) = (-\infty, 5] \cup (9, \infty)$

اقتران القيمة المطلقة

* Absolute Value Function

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$



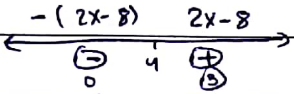
$Dom(f) = \mathbb{R} = (-\infty, \infty)$

$Rang(f) = [0, \infty)$

Ex. Rewrite the given function without the absolute value sign.

① $f(x) = |2x - 8|$

Soln:



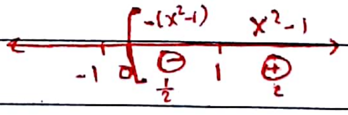
$2x - 8 = 0$

$\boxed{x = 4}$

$$f(x) = \begin{cases} 8 - 2x, & x < 4 \\ 2x - 8, & x \geq 4 \end{cases}$$

② $f(x) = |x^2 - 1|, \ x \geq 0$

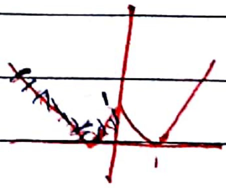
Soln: $f(1) = 0$
$f(3) = 8$
$f(0) = 0$
$f(-1) =$ undefined خارج الفترة



$x^2 - 1 = 0$

$\boxed{x = \pm 1}$

$$f(x) = \begin{cases} 1 - x^2, & 0 \leq x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$$

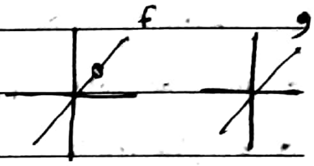


$Dom(f) = [0, \infty) = Rang(f)$

#1 19 if $f(x) = x + \sqrt{2-x}$ and $g(u) = u + \sqrt{2-u}$

is it true that $f=g$? yes

#2 19 if $f(x) = \frac{x^2-x}{x-1}$ and $g(x) = x$ is it true $f=g$? $\boxed{\text{No}}$



§ 1.2 Mathematical Models.

ظواهر حياتية

Many real-world phenomena can be modeled using basic functions:

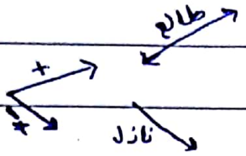
الاقتران الخطي

1 Linear Models :

y الموقع الذي يقطع

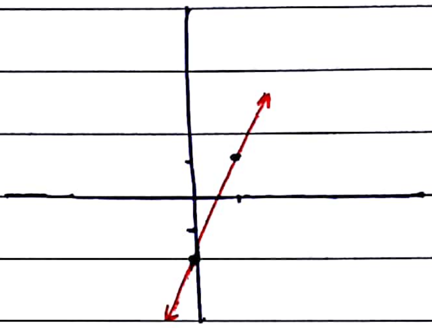
- In the form $\boxed{y = f(x) = mx + b}$, the graph is a line ~~???~~ with slope m (ميل) and y-intercept b.

* if m



Ex. $y = f(x) = 3x - 2$

x	0	1
y	-2	1



معامل x الميل

Slope = 3

y-int = -2

$Dom(f) = \mathbb{R} = (-\infty, \infty)$

$Rang(f) = \mathbb{R} = (-\infty, \infty)$

كثير الحدود

2 Polynomial functions: موجب ولا يوجد كسور

دائما Dom ← $\mathbb{R}$

* A function of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

n صح للترميز (نفس الاس)

صحيح مافي كسور

where is a non-negative integer and $\underbrace{a_n, a_{n-1}, \dots, a_1, a_0}$ عدد مضروب بـ x قوة صفر او عدد صحيح موجب are constants, is called a ~~P~~ polynomial المعاملات "Coefficients"

اعلى اس هو معامل الترميز

if $a_n \neq 0$, then n is called the degree of the ~~P~~ Polynomial.

Ex:

	degree.	name	example
	0	constant	$f(x)=2$
	1	خطي linear	$f(x)=3x+1$
	2	تربيعي quadratic	$F(x)=x^2-x-6$
	3	cubic	$F(x)=x^3+5x$
	4	quartic	$F(x)=5-x^4$

مع $a_4=-1$ $a_0=5$ ...

دائما

$Dom(f) = \mathbb{R} = (-\infty, \infty)$

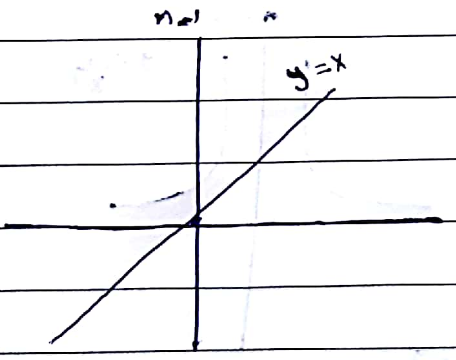
اقترانات القوة

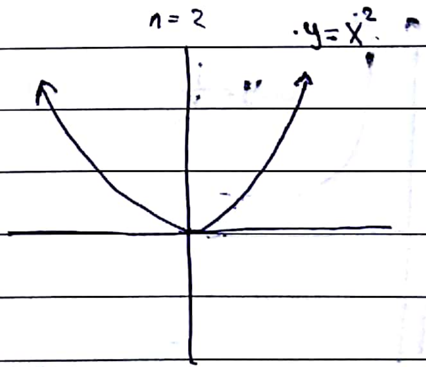
3 Power Functions :

A function of the form $\boxed{y = F(x) = x^a}$ ← عدد حقيقي كل الاعداد ما عدا الصفر

where a is a non-zero constant is called a power function.

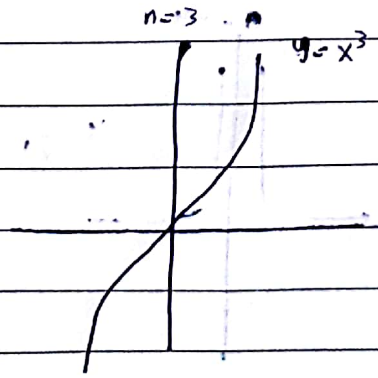
$$\text{I } \overset{\text{موجب}}{a = n} \begin{cases} \xrightarrow{\text{even}} & f \text{ is even function زوجي} \\ \xrightarrow{\text{odd}} & f \text{ is odd ~~fund~~ Function فردي} \end{cases}$$

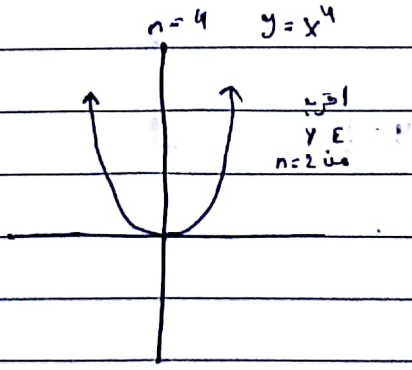


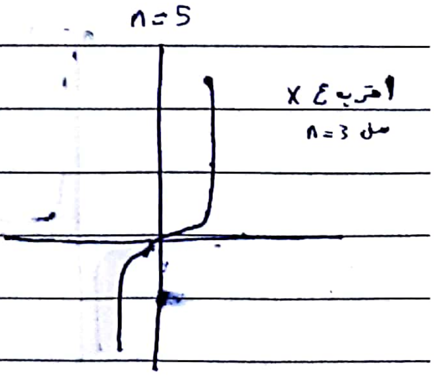


$Dom(x)=\mathbb{R}$

$$Rang(x) = \begin{cases} \xrightarrow{\text{odd}} \mathbb{R} \\ \xrightarrow{\text{even}} [0, \infty) \end{cases}$$

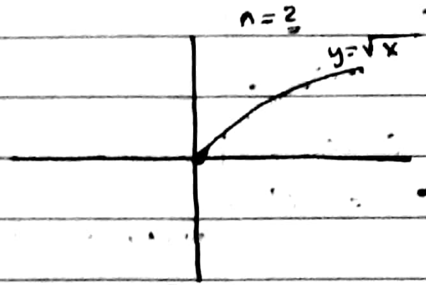


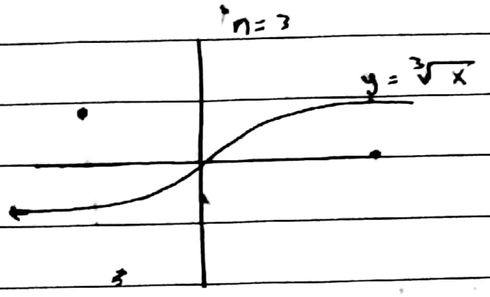




الاس كسر

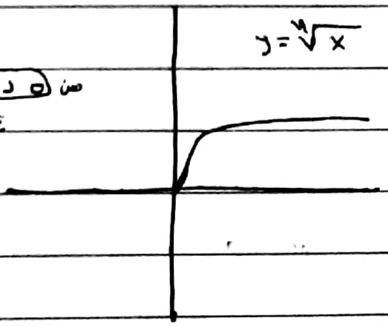
If $a = \frac{1}{n}$ (n positive integer $\neq 1$) then the power function is called root (radical) function.

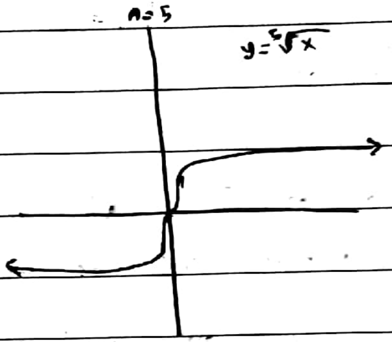




$\text{Dom}(f) \begin{cases} \text{odd} & \mathbb{R} \\ \text{even} & [0,\infty) \end{cases}$ $\qquad$ $\text{Rang}(f) \begin{cases} \text{odd} & \mathbb{R} \\ \text{even} & [0,\infty) \end{cases}$

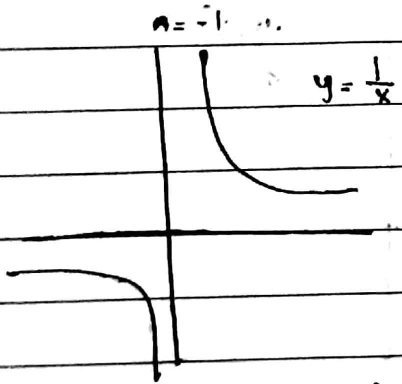
من $\boxed{٥ \text{ ل } ١}$ رح تقرب ع y

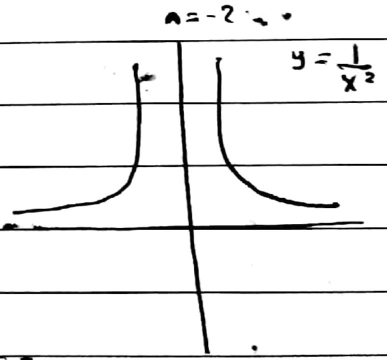




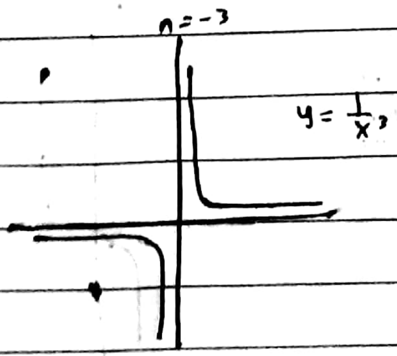
من ١- ل ١ رح تقرب ع x

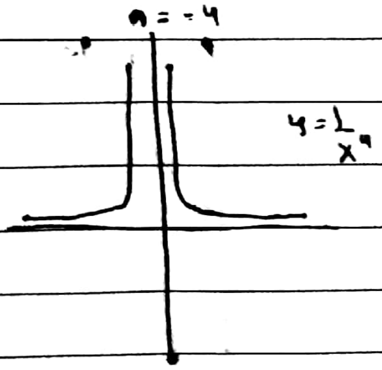
If $a = n$ (n negative integer)





$\text{Dom}(f) = \mathbb{R} \setminus \{0\}$ $\qquad$ $\text{Rang}(f) \begin{cases} \text{odd} & \mathbb{R} \setminus \{0\} \\ \text{even} & (0,\infty) \end{cases}$





[4] Rational Functions: تعيين Domain $\frac{\text{Poly}}{\text{Poly}}$

A function of the form $y = f(x) = \frac{P(x)}{q(x)}$ where P and q are polynomials is called a ~~P~~ rational function:

Ex: $F(x) = \frac{4x^2-25}{x^2-2x-24}$

$x^2 - 2x - 24 = 0$

$(x+4)(x-6) = 0$

$x = -4, 6$

$\text{Dom}(F) = \mathbb{R} \setminus \{-4, 6\}$

⑤ Trigonometric functions:

Trigonometric functions can be defined as functions of an angle in terms of the ratio of two of sides of aright triangle containg tha angle :
نسبة

$\sin\theta = \frac{a}{c}$

$\cos\theta = \frac{b}{c}$

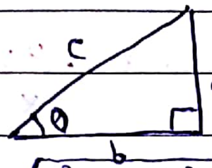
$\tan\theta = \frac{a}{b}$

قتا

$\csc\theta = \frac{c}{a}$

$\sec\theta = \frac{c}{b}$

$\cot\theta = \frac{b}{a}$



$\boxed{c^2 = a^2+b^2}$

Remark:

① $\tan\theta = \frac{\sin\theta}{\cos\theta}$

② $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

③ $\csc\theta = \frac{1}{\sin\theta}$

④ $\sec\theta = \frac{1}{\cos\theta}$

* Special Angles :

		90°	180°	270°	
θ	0 2π	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	
$\sin\theta$	0	1	0	-1	
$\cos\theta$	1	0	-1	0	

	30°	45°	60°
θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$

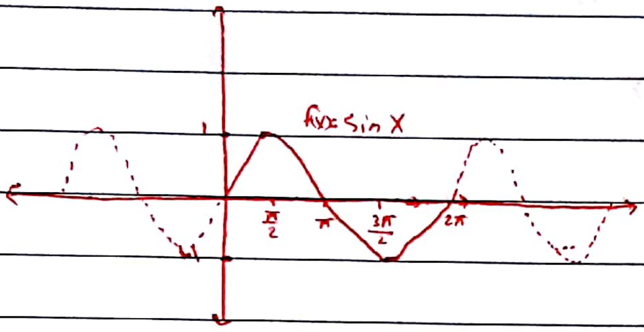
Remarks: Trigonametric ~~Fnc~~ Functions are $\overset{\text{دورية}}{\text{periodic}}$ Functions

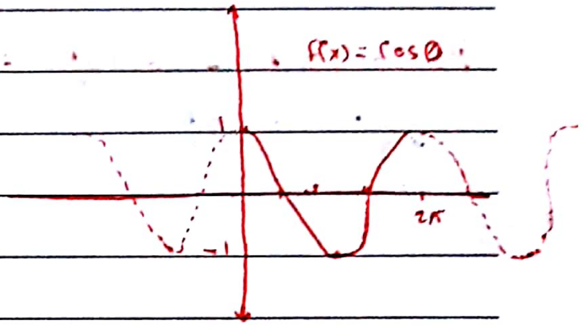
⊙ Period sin/cos $\boxed{2\pi}$ يتكرروا كل دورتين

① Period Tan $\boxed{\pi}$

المطلوب بحساب ال tan ,sin cos

* Graphs



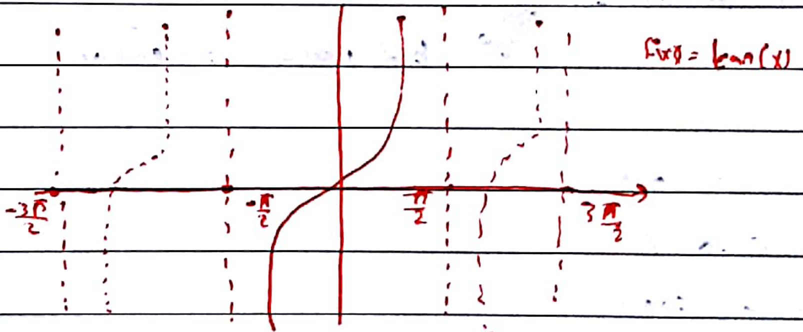


Dom(sin x) = IR

Rang (sin x) = [-1,1]

Dom(cos x) = IR

Rang (cos x) = [-1,1]



$\text{Dom}(\tan x) = \mathbb{R} \setminus \left\{ \pm \frac{\overset{\text{فردي}}{(2n-1)}\pi}{2},\ n = 1,2,3,\ldots \right\}$

$\boxed{\text{or}}$

$\mathbb{R} \setminus \left\{ \frac{\pi}{2} \pm n\pi,\ n = 0,1,2,3,\ldots \right\}$

عشان ال tan = $\frac{\sin}{\cos}$

وال cos بتكون

صفر في ال $\frac{\pi}{2}$ وال $\frac{3\pi}{2}$

فاي اشي عشه بروح ال Dom

Sin, Csc ⊕ $\boxed{\frac{(b-1)\pi}{b}}$ $\frac{5\pi}{6}, \frac{3\pi}{4}, \frac{2\pi}{3}$	All ⊕ $\boxed{\frac{\pi}{b}}$ → عدد $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$
tan, cot ⊕ $\boxed{\frac{(b+1)\pi}{b}}$ $\frac{7\pi}{6}, \frac{5\pi}{4}, \frac{4\pi}{3}$	cos, sec ⊕ $\boxed{\frac{(2b-1)\pi}{b}}$ $\frac{11\pi}{6}, \frac{7\pi}{4}, \frac{5\pi}{3}$

نتعلمها ولانه 6-5 = 1

Ex: خ بالربع الثاني

① $\sin(\frac{\textcircled{5}\pi}{6}) = \sin(\frac{\pi}{6}) = +\frac{1}{2}$

② $\csc(\frac{5\pi}{4}) = -\csc(\frac{\pi}{4}) = -\frac{1}{\sin(\frac{\pi}{4})} = -\sqrt{2}$

③ $\tan(\frac{3\pi}{4}) = \tan(\frac{\pi}{4}) = -\frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} = -\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1$

④ $\cos(\frac{11\pi}{3}) = \cos(\frac{5\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$

$6\ \frac{11\pi}{3} - 2\pi = \frac{5\pi}{3}$

لانه مش موجود،
بالجدول بنقص 2π

مطابقات

* Some important identities :

① $\sin^2(ax) + \cos^2(ax) = 1$

$\xrightarrow{\div \sin^2(ax)}$ $1 + \cot^2(ax) = \csc^2(ax)$

$\xrightarrow{\div \cos^2(ax)}$ $\tan^2(ax) + 1 = \sec^2(ax)$

② $\sin(-x) = -\sin x$

$\cos(-y) = \cos x$

$\tan(-x) = -\tan x$

③ $\sin(2x) = 2\sin x \cos x$

$\cos(2x) = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

④ $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

و

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

Ex. If $\sin\theta = \frac{1}{3}$, $\underbrace{\frac{\pi}{2} < \theta < \pi}_{\text{الربع الثاني}}$

Find $\tan\theta$ and $\sin 2\theta$

Solv:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{1}{9} + \cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{8}{9}$$

$$\Rightarrow \cos\theta = \pm \frac{\sqrt{8}}{3}$$

$$\Rightarrow \boxed{\cos\theta = -\frac{\sqrt{8}}{3}}$$

Now ① $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{8}}{\frac{\sqrt{8}}{3}} = \frac{-1}{\sqrt{8}}$

② $\sin 2\theta = 2\sin\theta\cos\theta$

$$2\left(\frac{1}{3}\right)\left(\frac{-\sqrt{8}}{3}\right) = \frac{-2\sqrt{8}}{9}$$

#5 / 33 Find the domain of $F(X) = \frac{\cos x}{1 - \sin x}$

Soln:

$$1 - \sin x = 0$$

$$\sin x = 1$$

$$X = \frac{\pi}{2} \pm 2n\pi , n = 0, 1, 2, \dots$$

$$\text{Dom}(F) = \mathbb{R} \setminus \left\{\frac{\pi}{2} \pm 2n\pi , n = 0, 1, 2, \dots\right\}$$

§ 1.3 New functions from old functions.

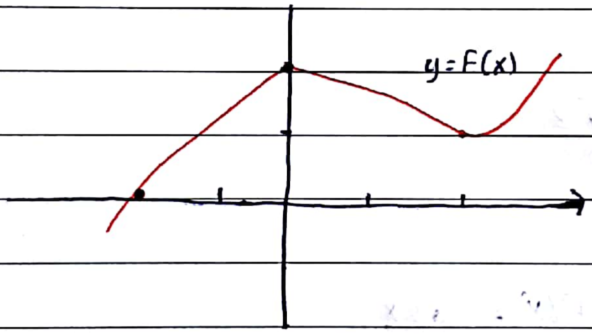
Let f be a function whose graph is know, and
Let C be a ~~function~~ positive constant. Then the following basic transformations of graphs can be done to $\boxed{y = f(x)}$:

① $y = f(x) \pm c$ shift graph up / down

② $y = f(x \overset{\text{Right}}{\mp} c)$ shift graph right / left

عند ضربها بـ سالب تنقلب
③ $y = -f(x)$ reflection in the x-axis

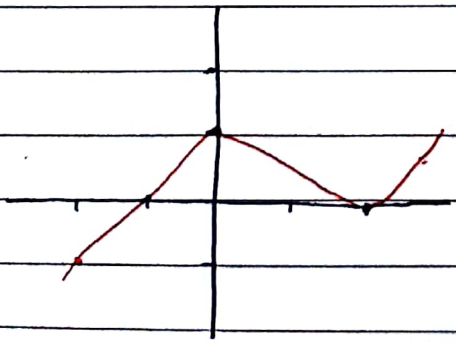
Ex. The graph of f(x) is given below:



Sketch the graph of :

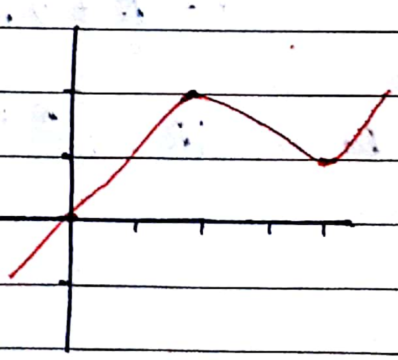
① $y = f(x) - 1$

Sol n :



② $y = f(x-2)$

Sol n :



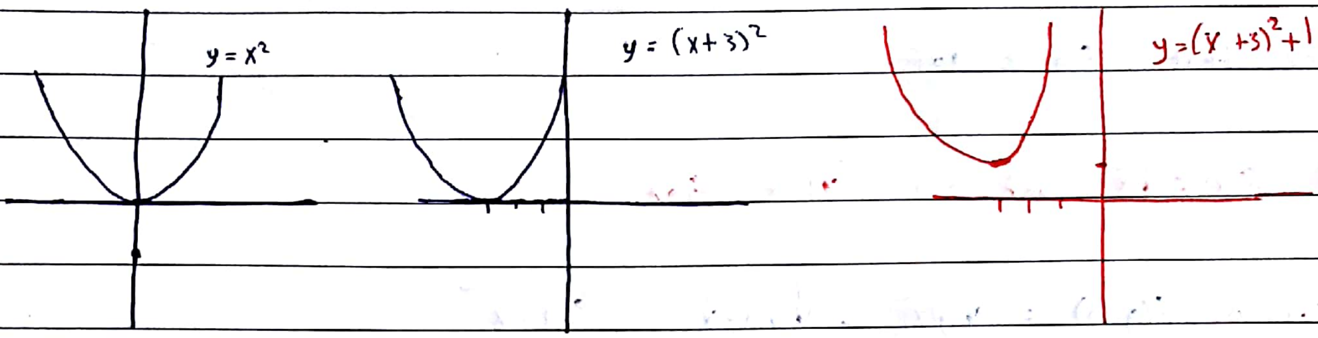
Ex. Sketch the graph of $f(x) = x^2 + 6x + 10$

Soln:

$f(x) = x^2 + 6x + 10$ نقسم معامل X على 2 نربع ونضيفه ونطرحه

اكمال مربع $= (x^2 + 6x + 9) - 9 + 10$

$= (x+3)^2 + 1$



* Combinations of functions:

① Sum: $(f+g)(x) = f(x) + g(x)$

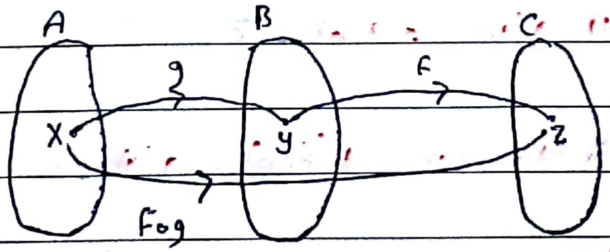
② Difference: $(f-g)(x) = f(x) - g(x)$

③ Product: $(f \cdot g)(x) = f(x) \cdot g(x)$

④ Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

(دمج) تركيب الاقترانات

5 Composition of two functions:



"f circle g"

$\boxed{(f \circ g)(x) = f(g(x))}$

Ex. given $f(x) = 2x - 3$ and $g(x) = \cos x$

Find:

① $(f \circ g)(x) = f(g(x)) = 2g(x) - 3 = 2(\cos x) - 3$

② $(g \circ f)(x) = g(f(x)) = \cos f(x) = \cos(2x - 3)$

* Note that $f \circ g \neq g \circ f$

#38/44 $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{1-x}$ Find

① $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{\sqrt[3]{1-x}} = \sqrt[6]{1-x}$

$\boxed{\frac{1}{6}} = \frac{1}{2} \cdot \frac{1}{3}(1-x)$

② $(g \circ f)(x) = g(f(x)) = \sqrt[3]{1 - f(x)} = \sqrt[3]{1 - \sqrt{x}}$

③ $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = \sqrt[4]{x}$

④ $(g \circ g)(x) = g(g(x)) = \sqrt[3]{1 - g(x)} = \sqrt[3]{1 - \sqrt[3]{1-x}}$

إضافي

⑤ $(f \circ g \circ f)(x) = f(g(f(x))) = f(\sqrt[3]{1 - f(x)}) = \sqrt{\sqrt[3]{1 - f(x)}} = \sqrt[6]{1 - \sqrt{x}}$

مهم للامت

#52/44

x	1	2	3	4	5	6
f(x)	3	1	4	2	2	5
g(x)	6	3	2	1	2	3

find:

① $f(g(1)) = f(6) = 5$

② $g(f(1)) = g(3) = 2$

③ $f(f(1)) = f(3) = 4$

④ $g(g(1)) = g(6) = 3$

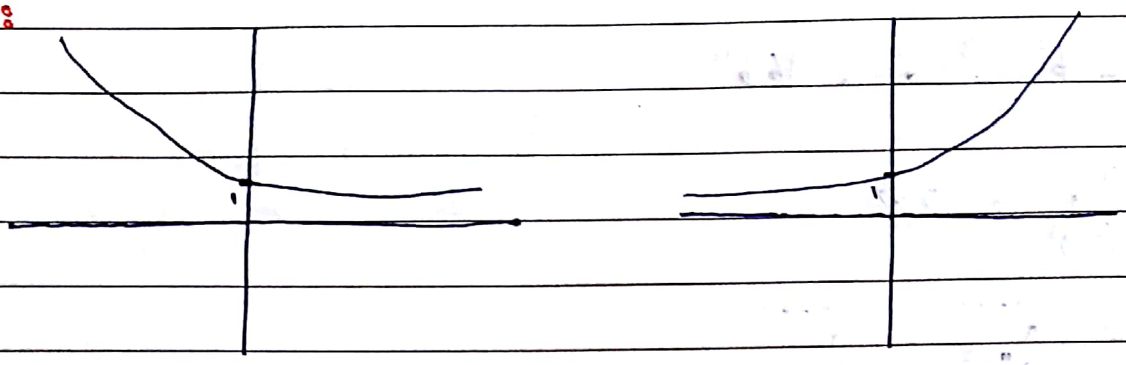
⑤ $(g \circ f)(3) = g(f(3)) = g(4) = 1$

⑥ $(f \circ g)(6) = f(g(6)) = f(3) = 4$

§ 1.4 Exponential Functions:

Defn: A function of the form $F(x) = b^x$, where $b > 0$ and $b \neq 1$, is called exponential function

*** Graph:**



كسر $\boxed{0 < b < 1}$ $\boxed{b > 1}$

⊙ $Dom(b^x) = \mathbb{R} = (-\infty, \infty)$

$Rang(b^x) = (0, \infty)$

(ما رح تقطع الـ x)

⊙ The y-intercept 1 and $y = 0$ is ~~called~~ the horizantal asymptote.

⊙ If $b = e \approx 2.7$, then the function is called natural exponentian function.

*** Laws of exponents:**

① $a^n \cdot a^m = a^{n+m}$

② $a^{-n} = \frac{1}{a^n}$

③ $\frac{a^n}{a^n} = a^{n-m}$

④ $(a^n)^m = a^{nm}$

⑤ $(a \cdot b)^n = a^n \cdot b^n$

⑥ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

⑦ $a^{\frac{n}{m}} = \sqrt[m]{{}^{n}a} = \left(\sqrt[m]{a}\right)^n$

⑧ $a^0 = 1$ $(a \neq 0)$

Ex. Use the laws of exponents to rewrite and simplify :

① $\frac{4^{-3}}{2^{-8}} = \frac{(2^2)^{-3}}{(2)^{-8}} = \frac{2^{-6}}{2^{-8}} = 2^{-6-(-8)} = 2^2 = 4$

② $\frac{1}{\sqrt[3]{x^4}} = \frac{1}{x^{\frac{4}{3}}} = x^{-\frac{4}{3}}$

③ $b^8\,(2b)^4 = b^8 \cdot 2^4 \cdot b^4 = 16\,b^{12}$

④ $\frac{(2y^4)^3}{4y^5} = \frac{2^3 y^{12}}{2^2 y^5} = 2y^7$

⑤ $\frac{a^n \cdot a^{2n+1}}{a^{n-2}} = \frac{a^{3n+1}}{a^{n-2}} = a^{2n+3}$.

Ex. Find the domain :

① $f(x) = \frac{1-e^{x^2}}{1-e^{1-x^2}}$

Soln :

$1 - e^{1-x^2} = 0$

$e^{1-x^2} = 1 = e^0$

$\rightarrow 1 - x^2 = 0 \longrightarrow x = \pm 1$

$\text{Dom}(F) = \mathbb{R} \setminus \{\pm 1\}$

or $= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

② $f(x) = \frac{1+x}{e^{\cos x}}$

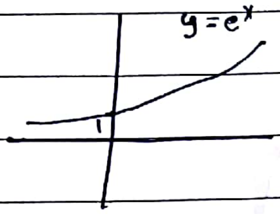
Soln :

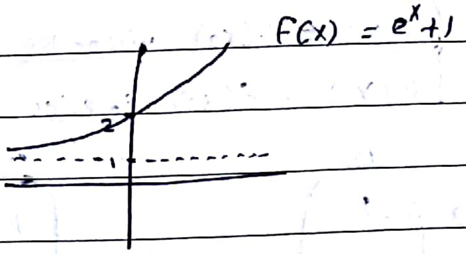
$\text{Dom}(F) = \mathbb{R} = (-\infty, \infty)$ لا يمكن ان يساوي المقام صفر

Ex. Sketch the graph of f and find the range:

① $f(x) = e^x + 1$

Soln:

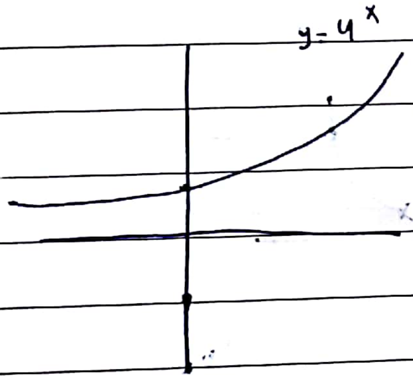


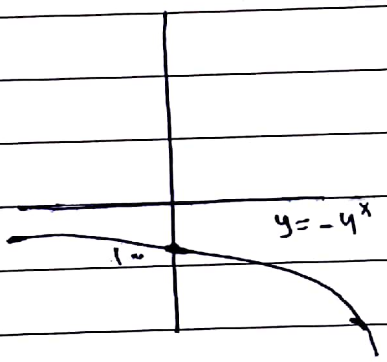


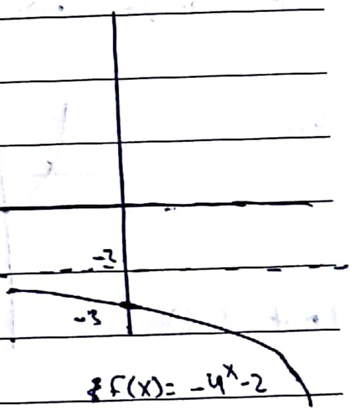
Rang(f) = $(1, \infty)$

② $f(x) = -4^x - 2$

Soln:





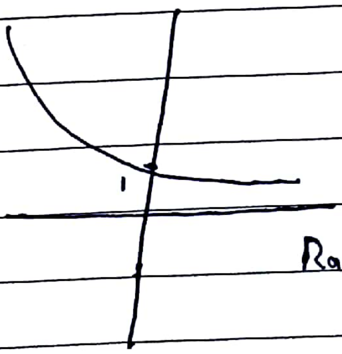


Rang(F) = $(-\infty, -2)$

③ $f(x) = e^{-x}$

Soln:

$f(x) = e^{-x} = (e^{-1})^x = \left(\frac{1}{e}\right)^x$



Rang(F) = $(0, \infty)$

§ 1.5 Inverse functions and logarithms

Defn: A function f is called a one-to-one function if it never takes on the same value twice, that is:

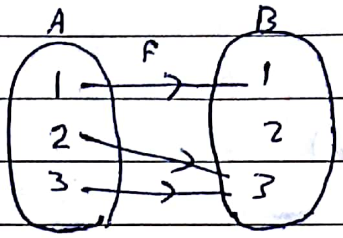
$$\boxed{f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2}$$

or,

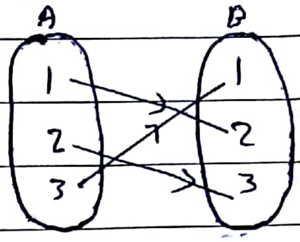
$$\boxed{f(x_1) = f(x_2) \text{ implies } x_1 = x_2}$$

عنصرين مختلفين

ارتبطوا مع عنصر واحد



(not 1-1)

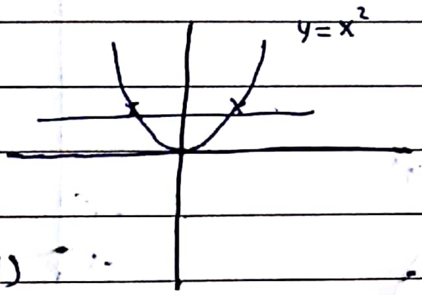


(1-1)

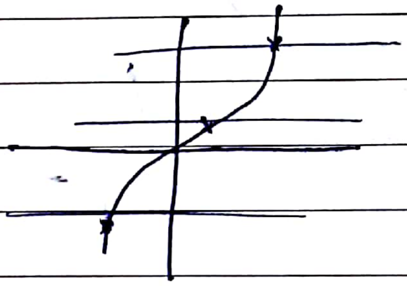
- **Horizontal line test:** A function is 1-1 iff no horizontal line intersects its graph more than once.

يقطع

Ex.



(not 1-1)



f is (1-1)

حدد فيما اذا كان

Ex. Determine whether the function is 1-1 or not:

① $f(x) = 3x + 2$

Soln:

$$f(x_1) = f(x_2) \longrightarrow 3x_1 + 2 = 3x_2 + 2$$

$$3x_1 = 3x_2$$

$$\boxed{x_1 = x_2}$$

② $f(x) = \sqrt{x}$

Soln:

$$f(x_1) = f(x_2) \rightarrow \sqrt{x_1} = \sqrt{x_2}$$

$$\rightarrow x_1 = x_2$$

f is 1-1

note $\sqrt{x^2} = |x|$

$(\sqrt{x})^2 = x$

③ $f(x) = 5 - x^3$

Sol: $f(x_1) = f(x_2) \longrightarrow 5 - x_1^3 = 5 - x_2^3$

$\longrightarrow -x_1^3 = -x_2^3$

$x_1^3 = x_2^3$

$x_1 = x^2$

f is (1-1)

④ $f(x) = 5x^2 + 3$

Sol:

$f(x_1) = f(x_2) \longrightarrow 5x_1^2 + 3 = 5x_2^2 + 3$

$5x_1^2 + = 5x_2^2$

$x_1^2 = x_2^2$

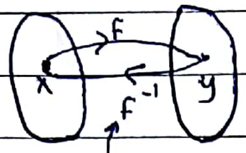
$\longrightarrow |x_1| = |x_2|$

f is not (1-1)

* اذا كان الاقتران (1 to 1) يكون له اقتران عكسي (F^{-1})

Defn: Let F be a 1-1 function with domain A and range B. Then the inverse of f, denoted f^{-1}, has domain B and range A and is defined by:

$$f^{-1}(y) = x \quad \text{iff} \quad f(x) = y$$



"read f inverse"

Note: Do Not confuse f^{-1} with $\frac{1}{f}$.

Ex. If $f(x) = e^x + 1$ find $f^{-1}(2)$?

$f(0) = e^0 + 1 = 1 + 1 = \boxed{2}$

$f^{-1}(2) = 0$

* Important:

① $Dom(f^{-1}) = Rang(f)$

$Rang(f^{-1}) = Dom(f)$

② $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for all $x \in Dom(f)$

$(f \circ f^{-1})(x) = \boxed{f(f^{-1}(x)) = x}$ for all $x \in Rang(f)$

تستخدم لايجاد ال inverse

Ex. Find the inverse function of

$f(x) = (3x-2)^5 + 7$

Soln:

$f(f^{-1}(x)) = (3f^{-1}(x) - 2)^5 + 7 = x$

$(3f^{-1}(x) - 2)^5 = x - 7$

$3f^{-1}(x) - 2 = \sqrt[5]{x-7}$

$\therefore F(x)^{-1} = \frac{\sqrt[5]{x-7} + 2}{3}$

Ex. Given $F(x) = \frac{3x-5}{4-2x}$

① Find the domain and Range of $F(X)$.

② evaluate (احسب) $F^{-1}(-2)$

Soln:

① $4-2x = 0$

$2x = 4$

$\boxed{x = 2}$

$\therefore Dom(F) = \mathbb{R} \setminus \{2\} = (-\infty, 2) \cup (2, \infty)$

Now, $F(F^{-1}(x)) = \frac{3F^{-1}(x) - 5}{4 - 2F(x)^{-1}} = x$

$\longrightarrow 3F^{-1}(x) - 5 = 4x - 2xF^{-1}(x)$

$2x F(x)^{-1} + 3F^{-1}(x) = 4x + 5$

$F(x)^{-1}(2x+3) = 4x+5$

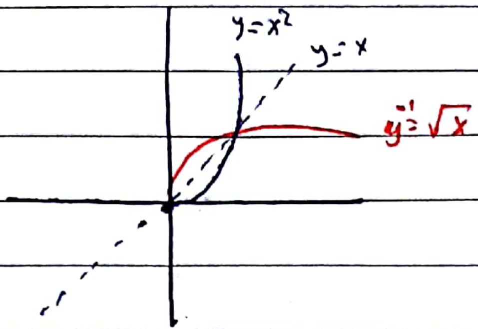
$\therefore F(x)^{-1} = \frac{4x+5}{2x+3}$

$Rang(F) = Dom(F^{-1}) = \mathbb{R} \setminus \{\frac{-3}{2}\} = (-\infty, \frac{-3}{2}) \cup (\frac{-3}{2}, \infty)$

② $F(-2)^{-1} = \frac{4(-2)+5}{2(-2)+3} = \frac{-3}{-1} = \boxed{3}$ or عن طريقة التجريب في $F(x)$

* Graphically , an inverse is areflection a cross the line y = x.

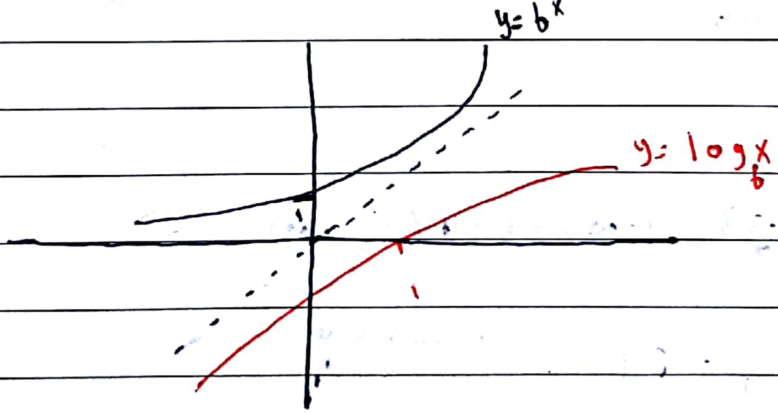
Ex :



* *Logarithmic Functions* :

If $b>0$ and $b\neq1$, the exponental Function $f(x)=b^x$ is either increasing or decreasing and so it is 1-1 by the horizontal line test. It therefore, has an inverse function which is called the logarithmic Function with base b and is denoted by $f(x)=\log_b x$.

$\boxed{\log_b x = y}$ iff $\boxed{b^y = x}$



$Dom(\log_b x) = (0,\infty) = Rang(b^x)$

$Rang(\log_b x) = \mathbb{R} = Dom(b^x)$

⊙ $\log_b b^x = x$ for all $x \in \mathbb{R}$

~~log~~ $b^{\log_b x} = x$ for all $x > 0$

* Remarks:

① $\log_b 1 = 0$

② $\log_b b = 1$

* Laws of Logarithms:

① $\log_b(xy) = \log_b x + \log_b y$

② $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

③ $\log_b(x^r) = r\log_b x$

$\log \longrightarrow \log_{10}$

$\ln \longrightarrow$ ~~log~~ $\log_e$

Ex. Use the laws of logarithms to evaluate:

① $\log_3(270) - \log_3(10) = \log_3\left(\frac{270}{10}\right) = \log_3 27 = \log_3(3^3) = 3\underbrace{\log_3 3}_{1} = \boxed{3}$

② $\log_2(12) + \log_2(3)$ ~~+~~ $- \log_2(9) = \log_2\left(\frac{12 \times 3}{9}\right) = \log_2(4) = \log_2(2^2) = 2\cancel{\log_2 2} = \boxed{2}$

⊙ If the base of the logarithm is:

① $b = 10$ then $f(x) = \log_{10} x = \log x$

② $b = e$ then $f(x) = \log_e x = \ln x$ is called "natural logarithm"

* Remark:

① $\ln(e^{x}) = x$

② $e^{\ln x} = x$

③ $\ln 1 = 0$

④ $\ln e = 1$

* change of base formula:

$$\log_b x = \frac{\log_a x}{\log_a b} = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

Ex. Rewrite $\log_8 5$ interms of natural logarithm ?

$\log_8 5 =$ ~~log 5~~ $\frac{\ln 5}{\ln 8}$

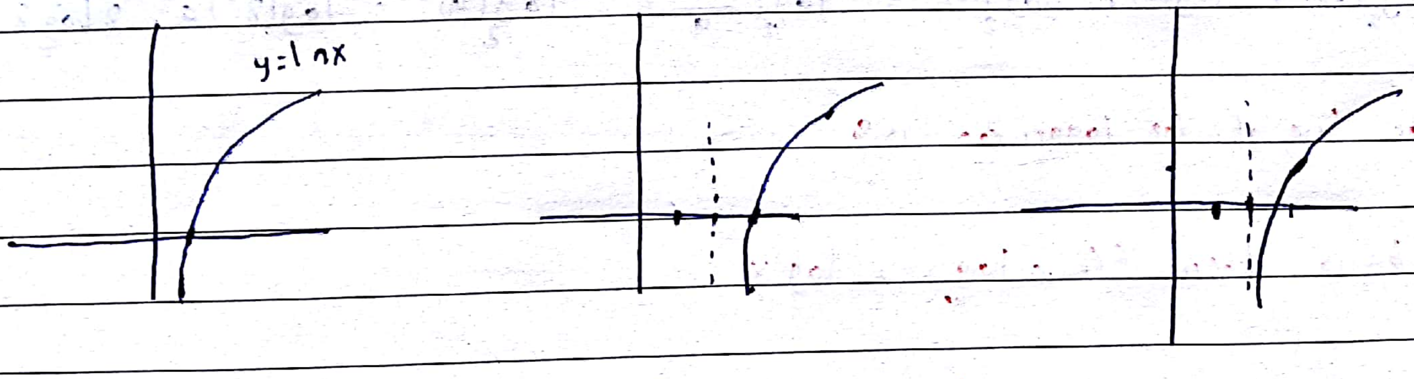
Ex. Find the domain of $y = \ln(x-2) + 1$ then sketch the graph?

Soln:

$x - 2 > 0$ * اللي داخلة اكبر من صفر

$, x > 2$

$\therefore$ Dom (f) = $(2, \infty)$



Dom(f) = $(2, \infty)$

Rang(f) = $\mathbb{R}$

مهم جدا

Ex. Solve for x:

① $2^{x-5} = 3$

Soln:

$\log_2 2^{x-5} = \log_2 3$

$x-5 = \log_2 3$

$\therefore \boxed{x = 5 + \log_2 3}$

② $e^{5-3x} = 10$

Soln:

$\ln e^{5-3x} = \ln 10$

$5-3x = \ln 10$

$3x = \ln 10 + 5$

$\therefore \boxed{x = \frac{\ln 10 + 5}{3}}$

③ $2^x x - 3\,2^x = 0$

Soln

$2^x (x-3) = 0$

$2^x > 0 \longrightarrow x - 3 = 0$

$\therefore \boxed{x = 3}$

④ $e^{2x} - 3e^x + 2 = 0$

Soln:

$(e^x)^2 - 3(e^x) + 2 = 0$

تحليل +
تبسيط

$(e^x - 1)(e^x - 2) = 0$

$e^x - 1 = 0$	$e^x - 2 = 0$
$e^x = 1$	$e^x = 2$
$x = \ln 1 = \boxed{0}$	$x = \ln 2$

⑤ $\ln x = 5$

Soln:

$\therefore e^{\ln x} = e^5$

$\longrightarrow \boxed{x = e^5}$

⑥ $\ln x + \ln(x-1) = 1$

Soln:

$\ln[x(x-1)] = 1$

$\longrightarrow x^2 - x = e$

$\longrightarrow x^2 - x - e = 0$

$x = \frac{1 \pm \sqrt{1+4e}}{2}$

القانون العام

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

ال وما يعامل معاملة الجذر
التربيعي بدون يساوي

#57
67) find the domain and Range of $f(x) = \ln(e^x - 3)$.

Soln:

$e^x - 3 > 0$

$e^x > 3$

~~$\ln x >$~~

$x > \ln 3$

$\therefore \text{Dom}(f) = (\ln 3, \infty)$

$f(f^{-1}(x)) = \ln(e^{f^{-1}(x)} - 3) = x$

$e^{f^{-1}(x)} - 3 = \cancel{x}\ e^x$

$\Longrightarrow e^{f^{-1}(x)} = e^x + 3$

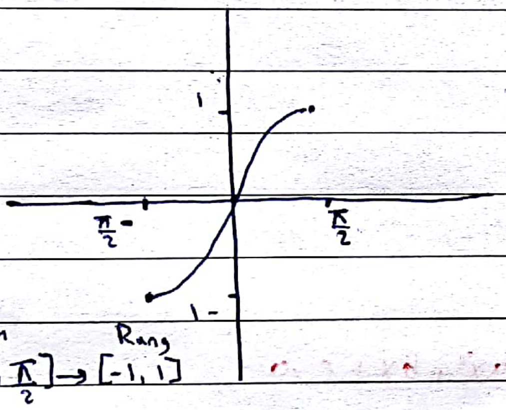
$\therefore f^{-1}(x) = \ln(e^x + 3)$ → بالنظر دائما موجب

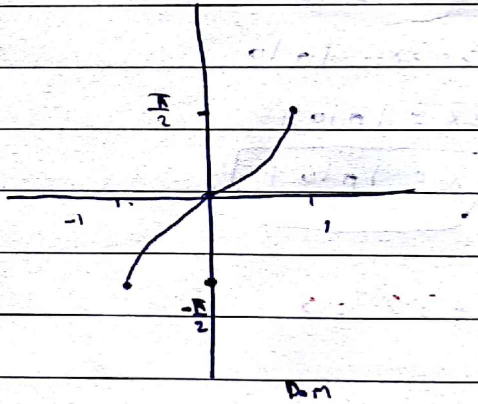
$\text{Rang}(f) = \mathbb{R} = (-\infty, \infty) = \text{Dom}(f^{-1})$

---

* Inverse Trigonometric functions:

1 Inverse Sine function:





Dom Rang
$\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

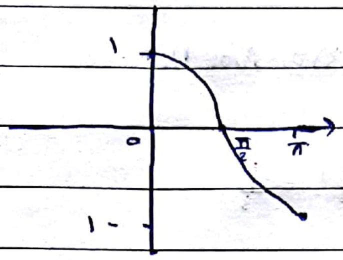
Dom Rang
$\sin^{-1} x : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

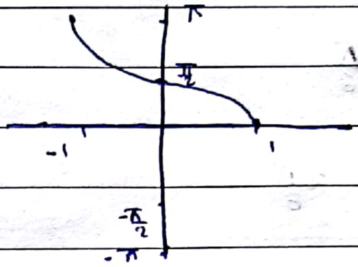
⊙ $\sin \sin^{-1} x = x$ for $x \in [-1, 1]$

$\sin^{-1} \sin x = x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

لم اذا ما كانت ال X ضمن ال Dom [illegible]

② Inverse Cosine Function:





$\cos x : \underset{\text{Dom}}{[0,\pi]} \longrightarrow \underset{\text{Rang}}{[-1,1]}$

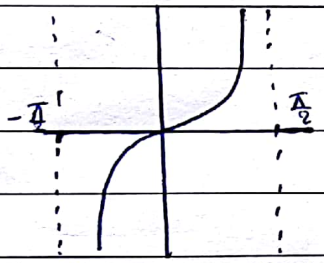
$\cos^{-1} x : \underset{\text{Dom}}{[-1,1]} \longrightarrow \underset{\text{Rang}}{[0,\pi]}$

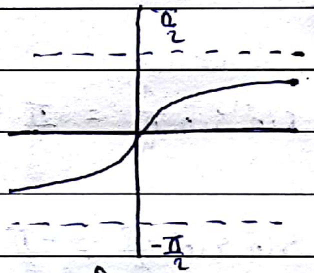
⊙ $\cos \cos^{-1} x = x$ For $x \in [-1,1]$

$\cos^{-1} \cos x = x$ For $x \in [0,\pi]$

لاذا

③ Inverse tangent Function:





$\tan x : \underset{\text{Dom}}{(-\frac{\pi}{2}, \frac{\pi}{2})} \longrightarrow \underset{\text{Rang}}{\mathbb{R}}$

$\tan^{-1} : \underset{\text{Dom}}{\mathbb{R}} \longrightarrow \underset{\text{Rang}}{(-\frac{\pi}{2}, \frac{\pi}{2})}$

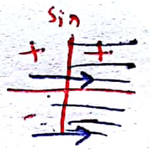
① $\tan \tan^{-1} x = x$ For $x \in \mathbb{R}$

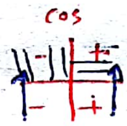
$\tan^{-1} \tan x = x$ For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

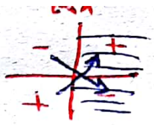
* Remarks: $\sin^{-1} x = \arcsin x$

$\cos^{-1} x = \text{arc} \cos x$

$\tan^{-1} = \text{arc} \tan x$







Ex: Evaluate

① $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

② $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

③ $\tan^{-1}(-1) = -\frac{\pi}{4}$

④ $\sin\sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}$

⑤ $\sin\sin^{-1}(3) =$ Not Defined

⑥ $\cos\cos^{-1}\left(-\frac{1}{3}\right) = -\frac{1}{3}$

⑦ $\tan\tan^{-1}(13) = 13$

Ex. Evaluate

① $\sin^{-1}\sin\frac{\pi}{3} = \frac{\pi}{3}$

مكافئتها

② $\sin^{-1}\sin\frac{3\pi}{4} = \sin^{-1}\sin\overset{\uparrow}{\frac{\pi}{4}} = \frac{\pi}{4}$

③ $\sin^{-1}\sin\frac{11\pi}{6} = \sin^{-1}\sin\frac{-\pi}{6} = \frac{-\pi}{6}$

④ $\sin^{-1}\sin\frac{5\pi}{4} = \sin^{-1}\sin -\frac{\pi}{4} = -\frac{\pi}{4}$

⑤ $\cos^{-1}\cos\frac{2\pi}{3} = \frac{2\pi}{3}$

⑥ $\cos^{-1}\cos\frac{7\pi}{4} = \cos^{-1}\cos\frac{\pi}{4} = \frac{\pi}{4}$

⑦ $\tan^{-1}\tan\frac{5\pi}{6} = \tan^{-1}\tan -\frac{\pi}{6} = \frac{-\pi}{6}$

مهم للامتحان

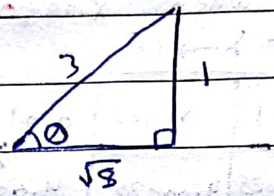
Ex. Evaluate:

① $\tan(\sin^{-1}(\frac{1}{3})) = \tan\theta = \frac{1}{\sqrt{8}}$

Soln

~~tel~~ let $\theta = \sin^{-1}(\frac{1}{3})$

$\sin\theta = \frac{1}{3}$

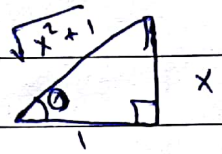


② $\cos(\tan^{-1}x) = \cos\theta = \frac{1}{\sqrt{x^2+1}}$

Soln:

Let $\theta = \tan^{-1}x$

$\tan\theta = \frac{x}{1}$



③ $\cos(2\sin^{-1}(\frac{5}{13})) = \cos(2\theta) = 1 - 2\sin^2\theta$

$= 1 - 2(\sin\sin^{-1}\frac{5}{13})^2$

Soln:

Let $\theta = \sin^{-1}(\frac{5}{13})$

$= 1 - 2(\frac{5}{13})^2$

$= 1 - \frac{50}{169}$

④ $\sin(2\sin^{-1}x) = \sin 2\theta = 2\sin\theta\cos\theta$

$= 2\sin\sin^{-1}(x)\cos\theta$

let $\theta = \sin^{-1}(x)$

النهايات المشتقات

Chapter 2: Limits and Derivatives:

§ 2.2 The limit of a function.

Ex. Consider the function $F(x) = \dfrac{x}{\sqrt{x+4}-2}$

Examine the values of the function when x is near 0.

تقترب من الصفر من اليسار ← $x \to 0^-$ تقترب من اليمين $x \to 0^+$

Soln:

x	-0.01	0.01	-0.001	0	0.0001	0.001	0.01
$F(x)$	3.9975	3.9998	3.99998 ~~3.999~~	Undefined	4.000025	4.00025	4.0025

تقترب من الـ٤ $F(x) \to 4$ $F(x) \to 4$

$$\left.\begin{array}{l}\lim\limits_{x\to 0^-} f(x) = 4 \\ \lim\limits_{x\to 0^+} f(x) = 4\end{array}\right\} \text{one sided limit}$$

$$\lim_{x\to 0} f(x) = 4 \; \} \text{ two sided limit.}$$

remark (1) $\lim\limits_{x\to c} f(x) = L$ read as "the limit of $f(x)$ as x approaches c is L" تقترب

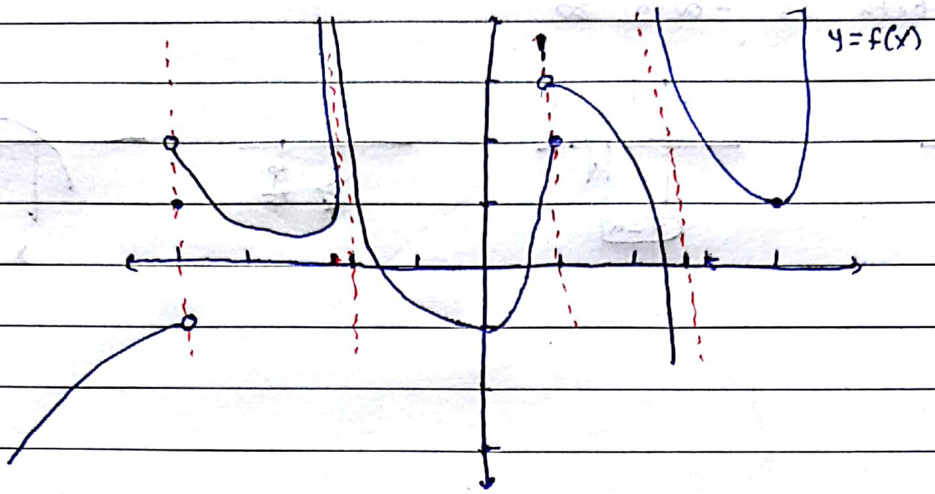
remark (2) Note that $f(0) = \frac{0}{0}$ (undefine) in the previous example.

when dealing with limits, we are examining values as x approaches c, but not equal to c.

Remark 3: ① If $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = L \Rightarrow \lim\limits_{x\to c} f(x) = L$ "exist"

② If $\lim\limits_{x\to c^-} f(x) \neq \lim\limits_{x\to c^+} f(x) \Rightarrow \lim\limits_{x\to c} f(x)$ does not exist (DNE)

Ex. Consider the Graph of f below, find the indicated limits :



① $\lim_{x\to -4^-} f(x) = -1$

② $\lim_{x\to -4^+} f(x) = 2$

③ $\lim_{x\to -4} f(x) = \text{DNE}$

④ $\lim_{x\to -2^-} f(x) = \infty$

⑤ $\lim_{x\to -2^+} f(x) = \infty$

⑥ $\lim_{x\to -2} f(x) = \infty$ (DNE)

⑦ $\lim_{x\to 1} f(x) = \text{DNE}$

Soln:

$\lim_{x\to 1^-} f(x) = 2$

$\lim_{x\to 1^+} f(x) = 3$

⑧ $\lim_{x\to 3} f(x) = \text{DNE}$

$\lim_{x\to 3^-} f(x) = -\infty$

$\lim_{x\to 3^+} f(x) = \infty$

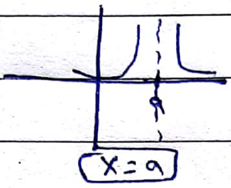
⑨ $\lim_{x\to 4} f(x) = 1$

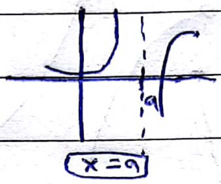
soln:

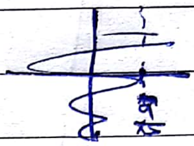
$\lim_{x\to 4^-} f(x) = 1$

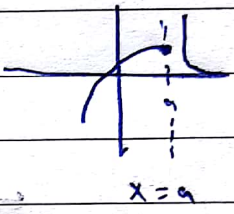
$\lim_{x\to 4^+} f(x) = 1$

Defn: The line x=a is a vertical محاذي عامودي asyptote if the limit from the left, right, or both is $-\infty$ <u>or</u> ∞





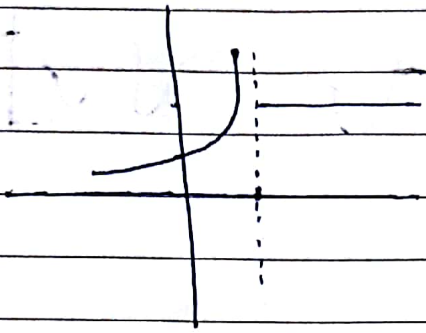


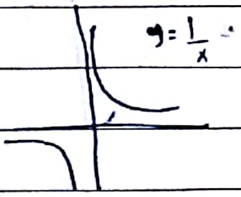


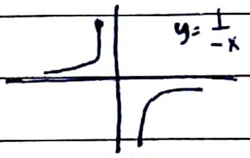
vertical asymptote

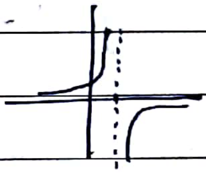
Ex. Find the VA of $f(x) = \begin{cases} 2 , & x \geq 1 \\ \frac{1}{1-x} , & x < 1 \end{cases}$

Soln:









$y = \frac{1}{-(x-1)} = \frac{1}{1-x}$

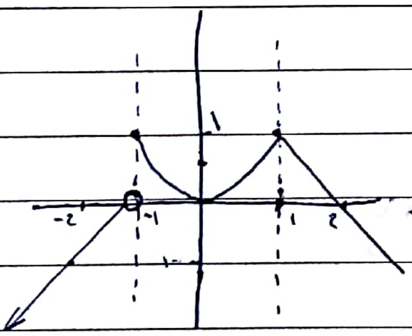
$$\left.\begin{array}{l} \lim_{x \to 1^-} f(x) = \infty \\ \lim_{x \to 1^+} f(x) = 2 \end{array}\right\} \Rightarrow x = 1 \text{ V.A}$$

Ex. Sketch the graph of the funkion and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists.

تحدد

$\frac{11}{93}$ $f(x) = \begin{cases} 1 + x , & x < -1 \\ x^2 , & -1 \leq x \leq 1 \\ 2 - x , & x \geq 1 \end{cases}$

Soln:



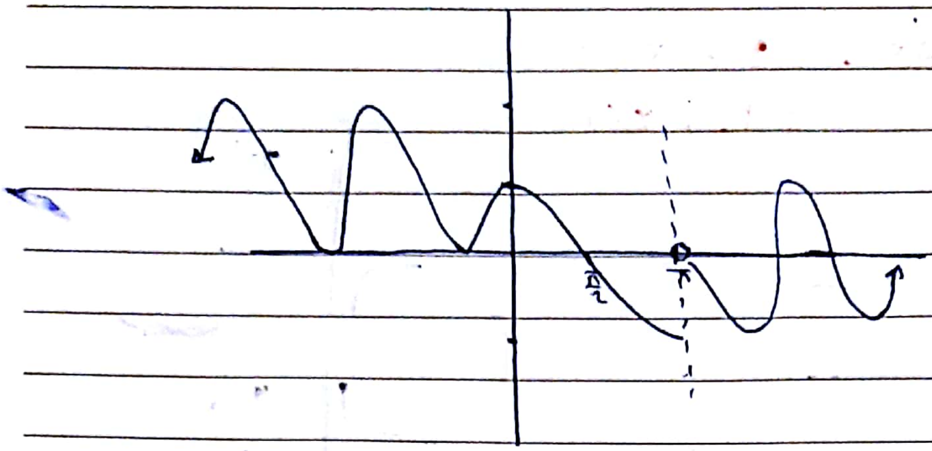
$$\left.\begin{array}{l} \lim_{x \to -1^-} f(x) = 0 \\ \lim_{x \to -1^+} f(x) = 1 \end{array}\right\} \rightarrow \lim_{x \to -1} f(x) = \text{DNE}$$

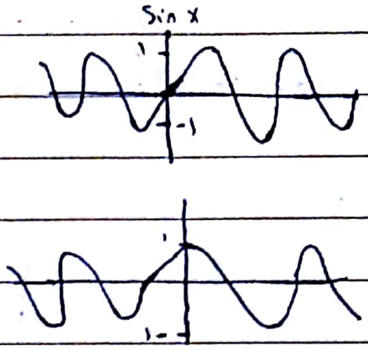
∴ limit exists for all $x \neq -1$

$[\mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, \infty)]$

$\frac{12}{93}$ $f(x) = \begin{cases} 1 + \sin x , & x < 0 \\ \cos x , & 0 \leq x \leq \pi \\ \sin x , & x > \pi \end{cases}$

Soln:





$\lim_{x\to\pi^-} f(x) = -1$

$\lim_{x\to\pi^+} f(x) = 0$

$\Big\}$ $\lim_{x\to\pi} f(x) = DNE$

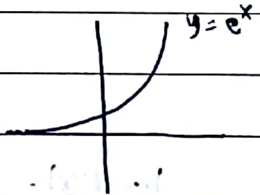
∴ limit exist for all $a \neq \pi$

$[\mathbb{R}\setminus\{\pi\} = (-\infty, \pi) \cup (\pi, \infty)]$

* Some important limits:

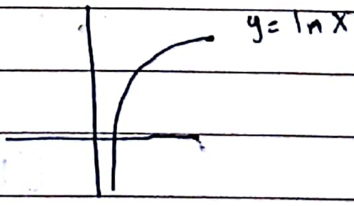
① $\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to-\infty} e^x = 0$



② $\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty$



③ $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$

$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$

§2.3 Calculating limits using the laws.

* limit laws:

① $\lim_{x\to a} \alpha = \alpha$ (α ~~const~~ constant)

② $\lim_{x\to a} x = a$

③ $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

$\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

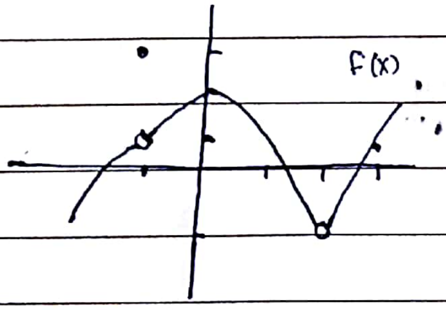
If $\lim_{x\to a} g(x) \neq 0 \rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

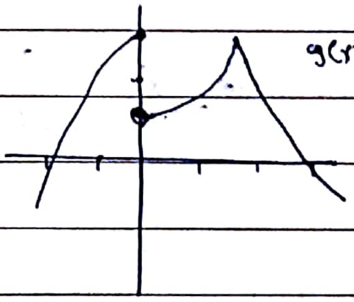
⑤ $\lim_{x\to a} \alpha f(x) = \alpha \lim_{x\to a} f(x)$

④ $\lim_{x\to a} [f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n$

$\rightarrow \lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$

Ex. #2 102 The graph of f and g is given below. Use the graph to evaluate the limit or state that the limit DNE.





① $\lim_{x\to 2} [f(x) + g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x) = -1 + 3 = 2$

② $\lim_{x\to 0} [f(x) - g(x)]$ = DNE as $\lim_{x\to 6} g(x)$ DNE

③ $\lim_{x\to 3} \frac{f(x)}{g(x)}$ DNE as $\lim_{x\to 3} g(x) = 0$

④ $\lim_{x\to 2} [x^2 f(x)] = \lim_{x\to 2} x^2 \cdot \lim_{x\to 2} f(x) = 4 \cdot (-1) = -4$

⑤ $f(-1) + \lim_{x\to -1} g(x) = 3+2 = 5$

- Ex. Given that $\lim_{x\to 3} f(x) = 16$ calculate:

$$\lim_{x\to 3} \frac{\sqrt{(x^2-4)\, f(x)}}{f(x)+3x+2} = \frac{\sqrt{[(\lim_{x\to 3} x^2 - \lim_{x\to 3} 4]\ \lim_{x\to 3} f(x)}}{\lim_{x\to 3} f(x) + 3\lim_{x\to 3} x + \lim_{x\to 3} 2}$$

or

$$= \frac{\sqrt{\lim_{x\to 3}(x^2-4)\ \lim_{x\to 3} f(x)}}{\lim_{x\to 3} f(x) + \lim_{x\to 3}(3x+2)}$$

$$= \frac{\sqrt{(9-4)(16)}}{16+3(3)+2}$$

$$= \frac{\sqrt{80}}{27}$$

$$= \frac{\sqrt{5\times 16}}{16+11} = \frac{\sqrt{80}}{27}$$

Theorem: If f is a polynomial or rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$

Ex. Evaluate the following limits

① $\lim_{x\to -2} \sqrt{3x^2+5x+1} = \sqrt{\lim_{x\to -2}(3x^2+5x+1)} = \sqrt{12-10+1} = \sqrt{3}$

② $\lim_{x\to 2} \left(\frac{x^2-2}{x^2-3x+5}\right)^2 = \left(\lim_{x\to 2} \frac{x^2-2}{x^2-3x+5}\right)^2 = \left(\frac{2}{7}\right)^2 = \boxed{\frac{4}{49}}$

$$\boxed{⊛ \ \lim_{x \to a} \frac{f(x)}{g(x)} :}$$

① If $\lim_{x \to a} f(x) \neq 0$ and $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

Ex. $\lim_{x \to \pi} \frac{1 - \cos x}{2 - x} = \frac{1 - (-1)}{2 - \pi} = \frac{2}{2 - \pi}$

② If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

Ex: $\lim_{x \to 0} \frac{e^x - 1}{x^2 + 3x + 5} = \frac{0}{5} = 0$

③ If $\lim_{x \to a} f(x) \neq 0$ and $\lim_{x \to a} g(x) = 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \pm \infty$ (DNE)

Ex. Determine the infinite limits

~~#~~ $\frac{32}{94}$ $\lim_{x \to 5^-} \frac{x + 1}{x - 5} = -\infty$

لتحديد ∞ موجب او سالب نعوض ٤.٩ مثلاً اقل (من اليسار)

\# $\frac{33}{94}$ $\lim_{x \to 1} \frac{2 - x}{(x - 1)^2} = +\infty$

Soln:

$\lim_{x \to 1^-} \frac{2 - x}{(x - 1)^2} = +\infty$ $\lim_{x \to 1^+} \frac{2 - x}{(x - 1)^2} = +\infty$

\# $\frac{42}{94}$ $\lim_{x \to 0^+} \left(\frac{1}{x} - \ln x \right)$

$= +\infty - (-\infty)$

$= \infty + \infty$

$= \boxed{\infty}$

Ex. find the V.A of:

① $F(x) = \frac{2x+6}{x^2-9}$

Soln:

$x^2 - 9 = 0 \longrightarrow x^2 = 9 \longrightarrow x = \pm 3$

شطبنا لانه تقترب لانها مش $\frac{0}{0}$

اذا $\frac{0}{0}$ ممنوع نشطب

$\lim\limits_{x\to -3} \frac{2\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \lim\limits_{x\to -3} \frac{2}{x-3} = \frac{2}{-6} = \boxed{\frac{-1}{3}}$

$\lim\limits_{x\to 3} \frac{2\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \lim\limits_{x\to 3^{\pm}} \frac{2}{x-3} = \pm\infty$

$\boxed{\therefore x = 3 \text{ is a V.A}}$

مابنشطب لانه function

② $F(x) = \frac{x^2-2x}{x^3-4x^2+4x} = \frac{x(x-2)}{x(x-2)^2}$

Soln:

$x(x-2)^2 = 0 \Rightarrow x = 0, 2$

$\lim\limits_{x\to 0} \frac{\cancel{x}\cancel{(x-2)}}{\cancel{x}(x-2)^{\cancel{2}}} = \lim\limits_{x\to 0} \frac{1}{x-2} = -\frac{1}{2}$

$\lim\limits_{x\to 2} \frac{\cancel{x}\cancel{(x-2)}}{\cancel{x}(x-2)^{\cancel{2}}} = \lim\limits_{x\to 2^{\pm}} \frac{1}{x-2} = \pm\infty$

$\boxed{\therefore x = 2 \text{ is a V.A}}$

غير معرفة

* **Undetermined Quantities:**

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 1^{\infty}, \infty^{0}, 0^{0}$

Ex. Find the Following limit:

① التحليل

② التخلص من الجذر بالمرافق

③ تجميع الكسور

① $\lim_{x\to 4} \overset{\frac{0}{0}}{\frac{x^2-x-12}{x^2-16}} = \lim_{x\to 4} \frac{(\cancel{x-4})(x+3)}{(\cancel{x-4})(x+4)} = \lim_{x\to 4} \frac{(x+3)}{(x+4)} = \boxed{\frac{7}{8}}$

جذر الاول - جذر الثاني

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② $\lim_{x\to -3} \overset{\frac{0}{0}}{\frac{x^2-9}{2x^2+7x+3}} = \lim_{x\to -3} \frac{(x-3)(\cancel{x+3})}{(\cancel{x+3})(2x+1)} = \lim_{x\to -3} \frac{x-3}{2x+1} = \frac{-6}{-5} = \boxed{\frac{6}{5}}$

③ $\lim_{h\to 0} \overset{\frac{0}{0}}{\frac{(h-4)^2-16}{h}} = \lim_{h\to 0} \frac{[(h-4)-4][(h-4)+4]}{h} = \lim_{h\to 0} \frac{\cancel{h}(h-8)}{\cancel{h}} = \lim_{h\to 0} h-8 = \boxed{-8}$

④ $\lim_{t\to 2} \overset{\frac{0}{0}}{\frac{\cancel{t-2}}{(t-2)^{\cancel{3}2}}} = \lim_{t\to 2} \frac{1}{(t-2)^2} = +\infty$

Recall: $x^2-a^2 = (x-a)\underbrace{(x+a)}_{\text{conjugate (المرافق)}}$

Ex. Evaluate the Following limit:

① $\lim_{x\to 1} \overset{\frac{0}{0}}{\frac{\sqrt{x+3}-2}{x-1}} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$

$= \lim_{x\to 1} \frac{(x+3)-4}{(x-1)(\sqrt{x+3}+2)}$

$= \lim_{x\to 1} \frac{(\cancel{x-1})}{(\cancel{x-1})(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \boxed{\frac{1}{4}}$

② $\overset{\frac{0}{0}}{\lim_{x \to 4}} \dfrac{\sqrt{x} - \sqrt{x^2-12}}{4-x} \cdot \dfrac{\sqrt{x} + \sqrt{x^2-12}}{\sqrt{x} + \sqrt{x^2-12}}$

$= \lim_{x \to 4} \dfrac{x - (x^2-12)}{(4-x)(\sqrt{x} + \sqrt{x^2-12})}$ نضرب (-)

في الضرب السالب فقط لواحد

$= \lim_{x \to 4} \dfrac{x^2 - x - 12}{(x-4)(\sqrt{x} + \sqrt{x^2} - 12)}$

$= \lim_{x \to 4} \dfrac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}(\sqrt{x} + \sqrt{x^2} - 12)} = \boxed{\dfrac{7}{4}}$

Recall: $* \ \dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad - bc}{bd}$

$* \ \dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \cdot \dfrac{d}{c}$

$* \ \dfrac{x-a}{a-x} = -1$

Ex. Calculate:

(#23 / 103) ① $\overset{\frac{0}{0}}{\lim_{x \to 3}} \dfrac{\frac{1}{x} - \frac{1}{3}}{x-3} = \lim_{x \to 3} \dfrac{\frac{3-x}{3x}}{\frac{x-3}{1}} = \lim_{x \to 3} \dfrac{\overset{-1}{\cancel{3-x}}}{3x} \cdot \dfrac{1}{\cancel{x-3}} = \boxed{\dfrac{-1}{9}}$

② (#24 / 103) : $\overset{\infty - \infty}{\lim_{x \to 0}} \left(\dfrac{1}{x\sqrt{1+x}} - \dfrac{1}{x} \right)$

$= \lim_{x \to 0} \dfrac{x - x\sqrt{1+x}}{x^2\sqrt{1+x}}$

$\boxed{\frac{0}{0}}$

$= \lim_{x \to 0} \dfrac{\cancel{x}(1 - \sqrt{1+x})}{x^{\cancel{2}}\sqrt{1+x}} \cdot \dfrac{1 + \sqrt{1+x}}{1 + \sqrt{1+x}}$

$= \lim_{x \to 0} \dfrac{1 - (1+x)}{x\sqrt{1+x}\,(1 + \sqrt{1+x})}$

$= \lim_{x \to 0} \dfrac{-\cancel{x}}{\cancel{x}\sqrt{1+x}\,(1 + \sqrt{1+x})} = \boxed{-\dfrac{1}{2}}$

Ex. Given $f(x) = \begin{cases} x & , x \le 0 \\ x^2 & , 0 < x \le 2 \\ 8 - x & , 2 < x < 5 \\ -2 & , x = 5 \\ x - 2 & , x > 5 \end{cases}$

Find

① $\lim f(x) = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} x = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

② $\lim_{x \to 2} f(x) = DNE$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (8 - x) = 6$

③ $\lim_{x \to 5} f(x) = 3$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (8 - x) = 3$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x - 2) = 3$

نفس القاعدة من اليمين واليسار

④ $\lim_{x \to 3} f(x) = \lim_{x \to 3} (8 - x) = 5$

+ نقاط التشعب فقط اللي ممكن يكون ال lim هنا الهم DNE

Ex. Evaluate :

① $\lim_{x \to 0} \frac{x^2 + x}{|x|} = DNE$

Sol :

نعيد تعريف

$|x| = \begin{cases} -x & , x < 0 \\ x & , x \ge 0 \end{cases}$

$f(x) = \frac{x^2 + x}{|x|} = \begin{cases} \frac{x^2 + x}{-x} & , x < 0 \\ \frac{x^2 + x}{x} & , x \ge 0 \end{cases}$

$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} \frac{x(x-1)}{-x} = \boxed{-1}$

$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \frac{x(x-1)}{x} = \boxed{1}$

② $\lim_{x \to -6^+} \frac{2x+12}{|x+6|} = \boxed{2}$

Solu:

$$|x+6| = \begin{cases} -(x+6) , & x < -6 \\ (x+6) , & x \geq -6 \end{cases}$$

$$F(x) = \frac{2x+12}{|x+6|} = \begin{cases} \frac{2x+12}{-(x+6)} , & x < -6 \\ \frac{2x+12}{(x+6)} , & x \geq -6 \end{cases}$$

$\lim_{x \to -6^+} \frac{2(x+6)}{x+6} = \boxed{2}$

③ $\lim_{x \to 3} \frac{|x-3|}{6-2x}$ (H.w)

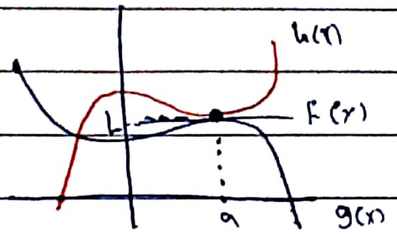
نظرية الحصر (الشطيرة)

* Squeeze Theorem:

If $g(x) \le f(x) \le h(x)$ for all x in an interval (فترة) that contains a, and

$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$

Then $\lim_{x \to a} f(x) = L$



Ex. If

$4x - 2 \le f(x) \le x^2 + 2$

For $0 \le x \le 3$ Find $\lim_{x \to 2} f(x)$

Soln:

$\lim_{x \to 2} (4x - 2) = 6$

$\lim_{x \to 2} (x^2 + 2) = 6$

$\xrightarrow{\text{Sq. Thm}} \lim_{x \to 2} f(x) = 6$

Ex. Find $\lim_{x \to 0} \overset{0 \cdot \infty}{x^2 \sin\left(\frac{1}{x}\right)}$

[Sin , cos , tan⁻]

Soln:

دائما (cos / sin)

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$-\frac{\pi}{2} \xleftarrow{\tan^{-}} \frac{\pi}{2}$

اذا كانت الـ lim ∞ موجودة

اما اذا يمين اشي (∞) واليسار اشي (-∞) DNE

$-x^2 \le \underbrace{x^2 \sin\left(\frac{1}{x}\right)}_{f(x)} \le x^2$

$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$

$\xrightarrow{\text{Sq Thm}} \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

§ 2.5 Continuity

Defn: A function f is Continuous at a number a if ① $\lim_{x \to a} f(x) = f(a)$

① if f is not Continuous at a, we say that f is discontinuous at a

* Notes: To determine continuity, we must check:

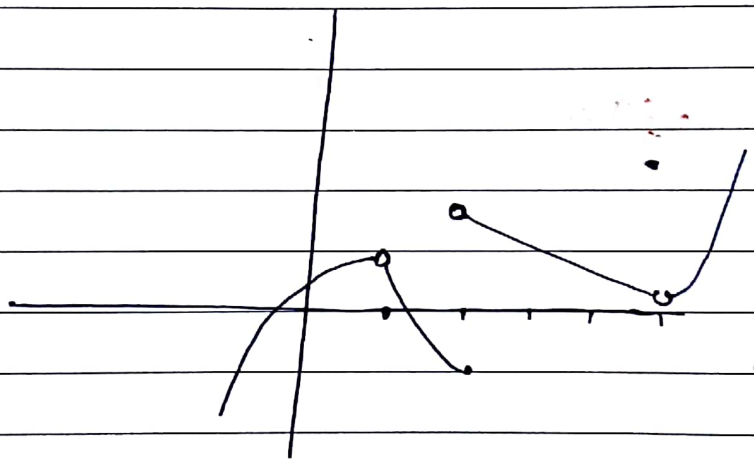
① f(a) is defined ($a \in Dom(f)$)

② $\lim_{x \to a} f(x)$ exists ($\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$)

② = ①

③ $\lim_{x \to a} f(x) = f(a)$

Ex.



f is discontinuous at:

① x = 1 [f(1) undefined]

② x = 2 [$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$]

③ x = 5 [$\lim_{x \to 5} f(x) \neq f(5)$]

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Explane why

$$f(x) = \begin{cases} \cos x , & x < 0 \\ 0 , & x = 0 \\ 1 - x^2 , & x > 0 \end{cases}$$

is discont. at $a=0$

Soln:

① $f(0) = 0$

② $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \cos x = 1$
$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (1 - x^2) = 1$
$\rightarrow \lim\limits_{x \to 0} f(x) = 1$

③ $\lim\limits_{x \to 0} f(x) = 1 \neq 0 = f(x)$

$\therefore$ f is discont. at $a=0$

* الRational دائما cont ما عدا اصفار المقام

Ex: Determine where the function is disont.

① $f(x) = \dfrac{x^2 - 7x + 10}{x - 5}$

Soln:

$x - 5 = 0 \rightarrow \boxed{x = 5}$

$\therefore$ f is discont. at $a=5$ [f(5) Undefined]

* المتشعب دائما متصل بس لازم افحص نقاط التشعب

② $f(x) = \begin{cases} 2x - 1 , & x < 4 \\ 6 , & x = 4 \\ x^2 - 1 , & x > 4 \end{cases}$

Soln:

(i) $f(4) = 6$

(ii) $\lim\limits_{x \to 4^-} f(x) = 7 \neq 15 = \lim\limits_{x \to 4^+} f(x)$

$\Rightarrow$ f is discont at $a = 4$

③ $f(x) = \begin{cases} \frac{x^2-7x+10}{x-5} & , x \neq 5 \\ 7 & , x = 5 \end{cases}$

Soln:

(i) $F(5) = \boxed{7}$

(ii) $\lim_{x \to 5} f(x) = \lim_{x \to 5} \overset{\frac{0}{0}}{\frac{x^2-7x+10}{x-5}}$

$= \lim_{x \to 5} \frac{\cancel{(x-5)}(x-2)}{\cancel{x-5}} = ③$

(iii) $\lim_{x \to 5} f(x) = 3 \neq 7 = f(5)$

∴ f is discont at $a=5$.

#46 125 Find the values of a and B that make f continuous everywhere.

$f(x) = \begin{cases} \frac{x^2-4}{x-2} & , x < 2 \\ ax^2-bx+3 , & 2 \leq x < 3 \\ 2x-a+b & , x \geq 3 \end{cases}$

Soln:

F is continuous at $a=2$ and $a=3$:

① $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \Rightarrow \lim_{x \to 2^-} \overset{\frac{0}{0}}{\frac{x^2-4}{x-2}} = \lim_{x \to 2^+} (ax^2-bx+3)$

[illegible]

$\Rightarrow 4a-2b+3 = 4$

$\Rightarrow \boxed{4a-2b=1}$ — (1)

② $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) \Rightarrow 9a-3b+3 = 6-a+b$

$\Rightarrow \boxed{10a-4b=3}$ — (2)

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(2) + ~~(-~~1) -2(1), to get

$10a - 4b = 3$

$\underline{-8a + 4b = -2}$

$2a = 1$

$\boxed{a = \frac{1}{2}}$ ---- (3)

(3) in (1), to obtain:

$4(\frac{1}{2}) - 2b = 1$

$\boxed{b = \frac{1}{2}}$

Defn8 ① f has a jump discontinuity at a if

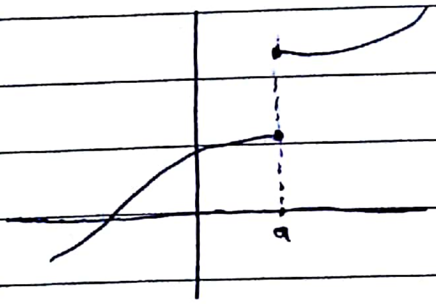
اختل الشرط ③ $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

② f has a removable discontinuity at a if

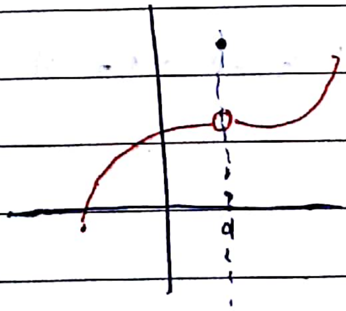
اختل الشرط ①او③ $\lim_{x \to a} f(x)$ exists but either f(a) undefined or $\lim_{x \to a} f(x) \neq f(x)$.

③ f has an infinite discontinuity at a if

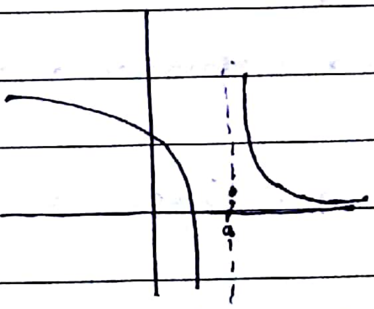
X = a is a V.A.



Jump discont.



Removable discont.



Infinite discont.

Ex Find the discontinuities and classify them as jump, removable, or V.A :

① $F(x) = \frac{x-2}{x^2+x-6}$

Solu:

$x^2 + x - 6 = 0$

$(x-2)(x+3) = 0$

$x = 2 , -3$

∴ f is disconti at $\overbrace{a = -3 \text{ and } a = 2}^{\text{Dom خارج ال}}$ ← الشرط الأول اختل

اذا ال lim موجودة removable , اذا ال lim $= \pm\infty$ infinite

Now, $\lim_{x\to 2} = \lim_{x\to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(x+3)} = \boxed{\frac{1}{5}}$

∴ Removable disconti. at $a = 2$

$\lim_{x\to -3} = \lim_{x\to -3} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(x+3)} = \pm\infty$

∴ ~~t~~ $x = -3$ is A V.A (infinite disconti.)

② $g(x) = \frac{2+x}{2-|x|} = \begin{cases} \frac{2+x}{2+x} , & x < 0 \\ \frac{2+x}{2-x} , & x \geq 0 \end{cases}$

Soln:

g is cont. at $a = 0$ (why?) ' ال lim = الصورة

∴ g is disco. at $x = \pm 2$

ما في يمين يسار $\lim_{x\to -2} g(x) = 1 \Rightarrow$ Removable at $a = -2$

لانه مش نقطة تشعب $\lim_{x\to 2} g(x) = \infty$ V.A at $a = 2$

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③ $h(x) = \begin{cases} x+1, & x \geq 2 \\ 2x, & x < 2 \end{cases}$

Soln:

$\lim_{x \to 2^+} h(x) = 3 \neq 4 = \lim_{x \to 2^-} h(x)$

$\therefore$ Jump discont at $a = 2$

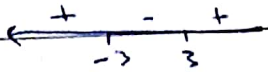
Defn: A function f is said to be continuous function if it is continuous at every point in the domain.

Ex. Determine the intervals(s) on which the function is continuous:

① $f(x) = 1 + \sqrt{x^2-9}$

Soln:

$x^2 - 9 \geq 0$



$(x-3)(x+3) \geq 0$

f is cont. on $(-\infty, -3] \cup [3, \infty)$

② $f(x) = 1 - \sqrt{1+x^2}$

Soln:

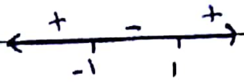
f is cont. on $\mathbb{R} = (-\infty, \infty)$

③ $f(x) = 1 - \sqrt{1-x^2}$

Soln:

$1 - x^2 \geq 0$

$\boxed{x^2 - 1 \leq 0}$



$(x-1)(x+1) \leq 0$

f is cont. on $[-1, 1]$

(H.W) ④ $f(x) = \frac{x^2}{x^2+3x} + \sqrt{x+2} \rightarrow$ dom.

نستثني اصفار المقام

جوابه → f is cont. on $[-2, 0) \cup (0, \infty)$

(H.W) find the values a and b that make g continuous everywhere:

$$g(x) = \begin{cases} ax+b+3 & , x \leqslant -2 \\ bx^3+ax^2 & , -2 < x \leqslant 2 \\ \frac{x^2-2x}{x-2} & , x > 2 \end{cases}$$

$$\lim_{x \to -2^-} ax+b+3 = \lim_{x \to -2^+} bx^3+ax^2$$

$$-2a+b+3 = 8b+4a$$

$$\boxed{-2} \times \boxed{6a+7b=3} - ①$$

$$\lim_{x \to 2^-} bx^3+ax^2 = \lim_{x \to 2^+} \frac{x^2-2x}{x-2} \qquad \lim_{x \to 2} \frac{x(x-2)}{x-2} = \boxed{2}$$

$$③ \times \boxed{8b+4a=2} - ②$$

$$-12a-14b=-3$$

$$12a+24b=6$$

$$10b=3$$

$$b=\boxed{\frac{10}{3}}$$

$$6a+7 \times \frac{10}{3} = 3$$

$$6a+\frac{70}{3}=3$$

$$6a=-\frac{62}{3}$$

$$\boxed{a=-\frac{62}{18}}$$

Theorem: If f and g are con't. at a and c is a constant, then the following functions are also continuous:

① cf ② $f \pm g$ ③ $f \cdot g$ ④ $\frac{f}{g}$ if $g(a) \neq 0$

Ex. Suppose f and g are cont. functions such that $g(-2)=3$ and $\lim_{x \to -2} [11f(x) + g(x)f(x)] = 30$ find $f(-2)$

RP

الصورة تساوي [illegible]
عند الاتصال

Sol:

$$\lim_{x \to 2} [11f(x) + g(x)f(x)] = 11 \lim_{x \to -2} f(x) + \lim_{x \to -2} g(x) \cdot \lim_{x \to -3} f(x)$$

$$= 11 f(-2) + g(-2) f(-2) = 30$$

$$= f(-2)(11 + \overset{3}{\cancel{g(-2)}}) = 30$$

$$\therefore f(-2) = \frac{30}{14} = \boxed{\frac{15}{7}}$$

Theorem: ① Any polynomial is continuous everywhere.

② Any rational function is cont. on its domain.

③ exponential and logarithmic functions are continuous on their domain.

④ Trigonometric functions and their inverses are cont. on their domain.

Theorem: If f is cont. at b and $\lim_{x \to a} g(x) = b$ then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

Ex. If f is cont. and $g(1) = 3$, $\lim_{x \to 1} g(x) = 5$, $f(3) = 4$ and $f(5) = 6$ find $\lim_{x \to 1} f(g(x))$

Soln:

$$\lim_{x \to 1} f(g(x)) = f(\lim_{x \to 1} g(x)) = f(5) = 6$$

Ex Evaluate $\lim_{x \to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) =$

$$= \sin^{-1}\left(\lim \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}\right)$$

$$= \sin^{-1}\left(\lim_{x \to 1} \frac{\cancel{(1-x)}}{\cancel{(1-x)}(1+\sqrt{x})}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right)$$

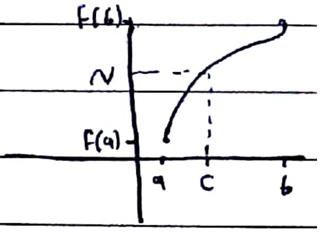
$$= \frac{\pi}{6}$$

Theorem: The Intermediate Value Theorem (IVT):

Suppose that f is

① Continuous on $[a, b]$

② N any number between $f(a)$ and $f(b)$.



IVT $\Longrightarrow$ there exist at least a number c in (a, b) such that $f(c) = N$.

Ex. Show that $f(x) = \frac{x}{x+1}$ takes on the value 0.499 for some x in $[0, 1]$

Soln:

① f is cont $[0, 1]$

② $f(0) = 0 < 0.499 < f(1) = 0.5$

IVT $\Longrightarrow$ there exists at least x in $(0,1)$ such that $f(x) = 0.499$

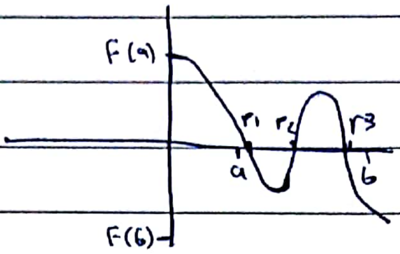
واحد موجب وواحد سالب

Ex. If f is Cont. on $[a,b]$ and $f(a)\,f(b) < 0$

then how many roots are there for f in (a,b)?

[✓ ① at least one] ② exactly one . ③ at most one.

توضيح



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Ex. Use the IVT to show that there is a root of $x^4+x-3=0$ on $(1,2)$.

Soln.

① $f(x)=x^4+x-3$ is cont. on $[1,2]$ because its a polynomial

② $f(1)=-1<0<f(2)=15$

$\xrightarrow{IVT}$ there exists at least x in $(1,2)$ such that $f(x)=0$

First

~~[illegible]~~ ~~HA.~~ ~~(6) Tan x~~

~~[illegible]~~

§ 2.6 limit at infinity: Horizontal asymptote:

Theorm: $\lim_{x\to\infty} \frac{1}{x^n} = 0$ and $\lim_{x\to-\infty} \frac{1}{x^n} = 0$.

Ex. Evaluate the following limit:

نقسم كل حد على اعلى اس

① $\overset{\frac{\infty}{\infty}}{\lim_{x\to\infty}} \frac{7x^2+4x-3}{2x^2-12x+16} = \lim_{x\to\infty} \frac{\frac{7x^2}{x^2} + \frac{4x}{x^2} - \frac{3}{x^2}}{\frac{2x^2}{x^2} - \frac{12x}{x^2} + \frac{16}{x^2}}$

حسب النظرية $\rightarrow 0$ 0

$$= \lim_{x\to\infty} \frac{7 + \frac{4}{\cancel{x}} - \frac{\cancel{7}}{x^2}}{2 - \frac{12}{\cancel{x}} + \frac{16}{\cancel{x^2}}} = \boxed{\frac{7}{2}}$$

Anothe method:

بنخلي اعلى x فوق على اعلى x تحت.

$\lim_{x\to\infty} \frac{7x^2+4x-3}{2x^2-12x+16} = \lim \frac{7x^2}{2x^2} = \boxed{\frac{7}{2}}$

Horizontal

حسب النظرية وال H.A: $\boxed{y=9}$

② $\lim_{x\to\infty} \frac{9x-4}{7x^2+4x-3} = \lim_{x\to\infty} \frac{9x}{7x^2} = \lim_{x\to\infty} \frac{9}{7x} = 0$

③ $\lim_{x\to-\infty} \frac{2x^4-12x^2+16}{x^2-12x+1} = \lim_{x\to-\infty} \frac{2x^4}{x^2}$

$= \lim_{x\to-\infty} 2x^2$

$= \infty$ ال H.A ماله لانه يجب ان يطلع رقم.

Def: If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ then $\boxed{y=L}$ is a horizontal asymptote

في حالة ال rational فقط نطلع ال lim من هنا فقط من ال [illegible] لانهم متساويين

Ex. Find the ~~[illegible]~~ H.A. of:

① $f(x) = \frac{7x^2+4x-3}{2x^2-12x+16}$ ② $g(x) = \frac{9x-4}{7x^2+4x-3}$ ③ $h(x) = \frac{2x^4-12x^3+16}{x^2-12x+1}$

Soln:

From the previous example.

① $y = \frac{7}{2}$ ② $y = 0$ ③ No H.A

Theorm:

$$\lim_{x \to \pm\infty} \frac{ax^p + \cdots}{bx^q + \cdots} = \begin{cases} \xrightarrow{p=q} \frac{a}{b} \Rightarrow y = \frac{a}{b} \text{ H.A.} \\ \xrightarrow{p<q} 0 \Rightarrow y = 0 \text{ H.A.} \\ \xrightarrow{p>q} \pm\infty \Rightarrow \text{No H.A.} \end{cases}$$

Ex. Find the V.A and H.A :

① $f(x) = \frac{3x^2+2x-3}{2(x-1)(x+2)}$

Soln:

$2(x-1)(x+2) = 0$

$\Rightarrow x = 1 \ , \ x = -2$

$\therefore x = 1$ and $x = -2$ V.A.

$\lim \frac{3x^2+2x-3}{2(x-1)(x+2)} = \frac{3}{2}$ بعد ضرب الأقواس

$\therefore y = \frac{3}{2}$ H.A.

② $f(x) = \frac{2x}{\sqrt{x^2+5}}$

Soln: V.A [illegible]

No V.A.

[illegible]

$$\lim_{x\to\infty} = \frac{2x}{\sqrt{x^2+5}} = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2}} = \lim_{x\to\infty} \frac{2x}{|x|} = \lim_{x\to\infty} \frac{2x}{x} = \boxed{2}$$

$$\lim_{x\to-\infty} = \frac{2x}{\sqrt{x^2+5}} = \lim_{x\to-\infty} \frac{2x}{\sqrt{x^2}} = \lim_{x\to-\infty} \frac{2x}{|x|} = \lim_{x\to-\infty} \frac{2x}{-x} = \boxed{-2}$$

$\therefore y = -2$ and $y = 2$ H.A.

Ex: Evaluate:

$\infty - \infty$

① $$\lim_{x\to\infty} \sqrt{x^2+3x} - x \cdot \frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}+x}$$

$$= \lim_{x\to\infty} \frac{3x}{\sqrt{x^2}+x} = \lim_{x\to\infty} \frac{3x}{2x} = \boxed{\frac{3}{2}}$$

لانه هاي تباعد كمان لا نعتمد التنتين

② $$\lim_{x\to\infty} \sqrt{x^2+4x+5}+x$$

$= \infty + \infty$

$= \boxed{\infty}$

$\infty - \infty$

③ $$\lim_{x\to-\infty} \sqrt{x^2+4y+5}+x \cdot \frac{\sqrt{x^2+4x+5}-x}{\sqrt{x^2+4x+5}-x}$$

$$= \lim_{x\to-\infty} \frac{\cancel{x^2}+4x+5-\cancel{x^2}}{\sqrt{x^2+4x+5}-x}$$

$$= \lim_{x\to-\infty} \frac{4x}{\underset{\boxed{-x}}{|x|}-x}$$

$$= \lim_{x\to-\infty} \frac{4x}{-2x} = \boxed{-2}$$

④ $\lim_{x\to\infty} \frac{\sqrt{3x^4+7x^2+1}}{x(x-1)}$

$= \lim_{x\to\infty} \frac{\sqrt{\cancel{x}3}\,x^2}{x^2} = \boxed{\sqrt{3}}$

⑤ ~~$\lim_{x\to\infty}$~~ Ex. Find the V.A and H.A

$f(x) = \frac{1+2e^x}{1-e^x}$

Solve:

$1-e^x=0 \Longrightarrow e^x=1 \Rightarrow \boxed{x=0}$ V.A

$\overset{\frac{\infty}{\infty}}{\lim_{x\to\infty}} \frac{1+2e^x}{1-e^x} = \lim_{x\to\infty} \frac{2e^x}{-e^x} = \boxed{-2}$

$\overset{\frac{1}{1}}{\lim_{x\to-\infty}} \frac{1+2e^x}{1-e^x} = \boxed{1}$

$\therefore y=-2$ and $y=1$ H.A.

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$\lim_{x\to\infty} \frac{\sqrt{x}+x^2}{2x-x^2} = \lim \frac{x^2}{-x^2} = \boxed{-1}$

#37
138

$\overset{\infty-\infty}{\lim_{x\to-\infty}} x^2+2x^7 = \lim_{x\to-\infty} x^2(1+2x^5)$

$= \infty \cdot -\infty$

$= -\infty$

#27
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$\lim_{x\to\infty} \sqrt{9x^2+x}-3x = \frac{1}{6}$ $\boxed{\text{H.w}}$ ضرب مرافق

قواعد الاشتقاق

Chapter 3 : differentiation rules:

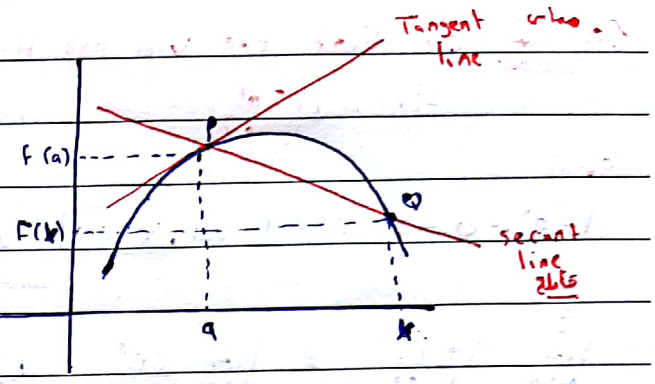
مشتقة

§ 3.1 + § Derivative. (§ 3.1 – § 3.3)

Consider a curve with equation $y = f(x)$ and points $P(a, f(a))$ and $Q(x, f(x))$ on it.

لما النقطة Q تقترب من ال P يصبح Tangent

ال Tangent يمس الخط بنقطة



The slope of the secant line is :

$$\text{(الميل)}\ m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

⊙ as $x \to a \implies F(x) \to f(a)$ and the Secent line become a tangent line, and the slope of the tangent line

$$m = \lim_{x \to a} \frac{F(x) - f(a)}{x - a}$$

Defn: The derivative of $y = f(x)$ at $x = a$ is given by:

(شرطة) Prime

$$F'(a) = \lim_{x \to a} \frac{F(x) - f(a)}{x - a}$$

⊠

الفرق

⊙ Let $h = x - a \implies x = a + h$

now, as $x \to a \implies h \to 0^{\pm}$

Thus, $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

⊙ If we replace a with x, then:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex. Find the derivative of $f(x) = 3x^2 - x + 1$

Soln:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{[3(x+h)^2 - (x+h)+1] - (3x^2 - x + 1)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{x} - h + \cancel{1} - \cancel{3x^2} + \cancel{x} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 1)}{\cancel{h}}$$

$$= 6x - 1$$

* other notations:

Let $y = f(x)$, then we denote to the derivative by:

$$y' = f(x)' = \frac{\partial y}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial (x)} f(x)$$

Ex. $\frac{\partial}{\partial x}(3x^2 - x + 1) = 6x - 1$

⊙ If we want to find the derivative at a specific number a, we use the following notation:

$$y'(a) = f'(a) = \left.\frac{\partial y}{\partial x}\right|_{x=a} = \left.\frac{\partial f}{\partial x}\right|_{x=a}, \left.\frac{\partial}{\partial x} f(x)\right|_{x=a}$$

Ex. let $f(x) = 3x^2 - x + 1$ find $f'(1)$

Soln:

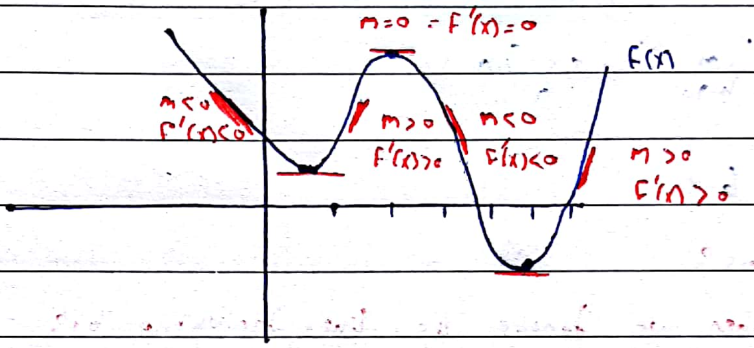
$$f'(x) = 6x - 1$$

$$f'(1) = 6(1) - 1 = 5$$

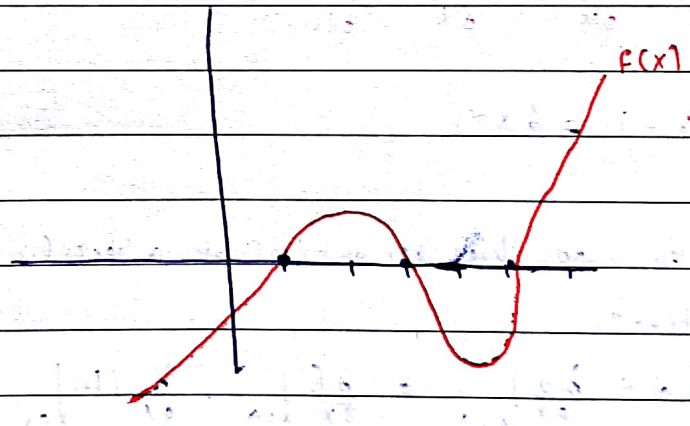
② ~~Geo~~ Geometrical meaning of the derivative of f(x) at x = a is the slope of the tangent line at x = a

⊙ Physical meaning of the derivative of the distance function s(t) at t = a is the velocity of an object at t = a

Ex. Sketch the graph of the derivative f', given the graph of f(x)



Soln:



① The function $f(x)$ is قابل للاشتقاق differentiable ($f'(x)$ exists) at $x = a$ iff

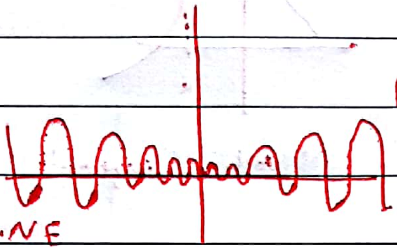
① f is continuous at $x = a$

② $f'(a^-) = f'(a^+)$

③ $\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$

Remark: f differentiable $\Longrightarrow$ f continuous

$\nLeftarrow$



$f(x) = \sin\left(\frac{1}{x}\right)$

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ D.N.E

#59+60 / 151 Determine wether $f'(0)$ exists:

① $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Soln:

① $\lim_{x\to 0} f(x) = \lim_{x\to 0} x$ ~~x~~ $\sin\left(\frac{1}{x}\right) = 0 = f(0)$

$\Longrightarrow$ f is cont. at $x = 0$

(ii) $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h\sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{x\to 0} \sin\left(\frac{1}{h}\right)$ D.N.E

لأنه ممكن من اليمين تكون ١,١- ومن اليسار ١-,١

$\therefore$ f is not differentiable at $x = 0$

② $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

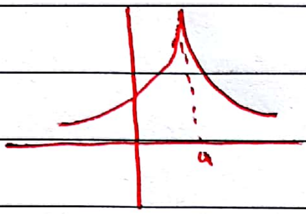
Soln:

(i) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \underset{\text{Sq.Thm}}{\uparrow}= 0 = f(0) \Longrightarrow$ f is cont at $x = 0$

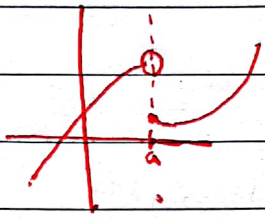
(ii) $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = \lim_{h \to 0} h \sin(\frac{1}{h}) = 0$

$\therefore$ f is diff. at $x = 0$

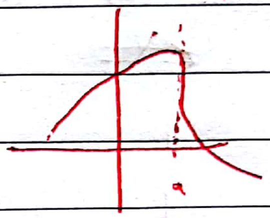
* How can a function fail to be differentiable:



"corner"



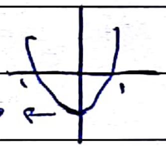
"discontinuity"



"vertical tangent"

Ex: find the numbers at which $f(x) = |x^2 - 1|$ is differentiable.

Soln:



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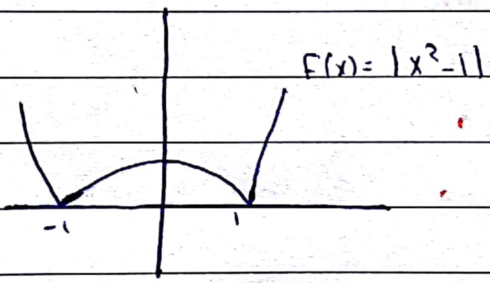
ينعكس اللي تحت



f is not diff. at $x = -1, 1$ "corner"

$\therefore$ f is differentinble for $\mathbb{R} \setminus \{-1, 1\}$

#41+42
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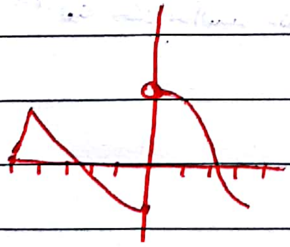
The graph of f is given.

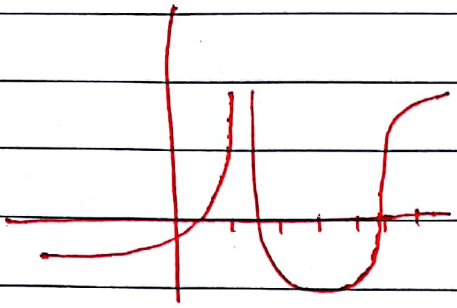
state, with reasons, the numbers at which f is Not diff.



$x = -4$ (corner)

$x = 0$ (discont.)

②



$x = 1$ (undefined) 'discont'')

$x = 5$ (vertical tangent))

* Differentiation Rules:

Function	derivative	
$F(x) = \alpha$ (α ~ constant)	$F'(x) = 0$	
$F(x) = x$	$F'(x) = 1$	
$F(x) = x^n$	$F'(x) = n x^{n-1}$	
$f(x) = \alpha\, g(x)$	$F'(x) = \alpha\, g'(x)$	
$F(x) = g(x) \pm h(x)$	$F'(x) = g'(x) \pm h'(x)$	في مشتقة الاول)
$F(x) = g(x) \cdot h(x)$	$F'(x) = g'(x)\, h(x) + g(x) \cdot h'(x)$	↑ (الاول في مشتقة الثاني+الثاني :
$F(x) = \dfrac{g(x)}{h(x)}$	$F'(x) = \dfrac{g'(x)\ h(x) - g(x)\, h'(x)}{[h(x)]^2}$	
$F(x) = e^x$	$f'(x) = e^x$	اللي بيبتدوا بحرف (c) سالب
$F(x) = \sin x$	$F'(x) = \cos x$	
$F(x) = \cos x$	$f'(x) = -\sin x$	
$F(x) = \tan x$	$f'(x) = \sec^2 x$	
$F(x) = \cot x$	$F'(x) = -\csc^2$	
$f(x) = \sec x$	$F'(x) = \sec x \tan x$	
$F(x) = \csc x$	$f'(x) = -\csc x \cot x$	

Ex: find the derivative:

① $f(x) = 3x^5 - x^2 + 3x - \overset{\text{const.}}{\overset{\uparrow}{\sqrt{3\pi^7}}}$

$\Rightarrow f'(x) = 15x^4 - 2x + 3$

② $f(x) = ex + e^x - e$

$\Rightarrow f'(x) = e + e^x$

Recall: (i) $\frac{1}{x^n} = x^{-n}$

(ii) $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

③ $f(x) = \sqrt[5]{x^4} - \frac{3}{x^4} + x^{2.47} - \sin(e^3\pi)$

$= x^{\frac{4}{5}} - 3x^{-4} + x^{2.47} - \sin(\overset{\text{const.}}{\overset{\uparrow}{e^3\pi}})$

$\Rightarrow F'(x) = \frac{4}{5} x^{\frac{-1}{5}} + 12x^{-5} + 2.47 \cdot x^{1.47}$

④ $h(x) = (\sqrt{x} + 2x^{\frac{1}{3}})(\frac{1}{x^2} + 4x^{1.2})$

$= (x^{\frac{1}{2}} + 2x^{\frac{1}{3}})(x^{-2} + 4x^{1.2})$

$\Rightarrow h'(x) = (\frac{1}{2}x^{-\frac{1}{2}} + \frac{2}{3}x^{\frac{-2}{3}}) \cdot (x^{-2} + 4x^{1.2}) + (x^{\frac{1}{2}} + 2x^{\frac{1}{3}}) \cdot (-2x^{-3} + 4.8x^{0.2})$

⑤ $f(x) = x^2 \sin x$

$\Rightarrow f'(x) = 2x \cdot \sin x + x^2 \cdot \cos x$

⑥ $g(z) = \overset{\text{ضرب}}{\overset{\uparrow}{-6z^3e^2}} + \overset{\text{const.}}{\overset{\uparrow}{e^2}} z$

$\Rightarrow g'(z) = [-18z^2e^2 + (-6z^3) \cdot e'^x] + e^2$

⑦ $g(x) = \underset{\sim}{x}\,\underset{\sim}{e^x}\ \underset{\sim}{\cot x}$

$g'(x) = [1 \cdot e^x + x \cdot e^x] \cot x + (x e^x) \cdot (-\csc^2 x)$

⑧ $f(t) = \dfrac{4\sqrt{t} + 2t^2}{t^2 - 9} \rightarrow \dfrac{4t^{\frac{1}{2}} + 2t^2}{t^2 - 9}$

$\Rightarrow f'(t) = \dfrac{(t^2-9)(2t^{-\frac{1}{2}} + 4t) - (4t^{\frac{1}{2}} + 2t^2) \cdot (2t)}{(t^2-9)^2}$

⑨ $g(w) = \dfrac{1 + \sqrt{w}}{3we^w} = \dfrac{1 + w^{\frac{1}{2}}}{3we^w}$

$\Rightarrow g(w)' = \dfrac{(3we^w) \cdot (\frac{1}{2} w^{-\frac{1}{2}}) - (1 + w^{\frac{1}{2}})(3 \cdot e^w + 3w \cdot e^w)}{(3we^w)^2}$

⑩ $f(x) = \dfrac{e^x}{2 - \tan x}$

$f'(x) = \dfrac{(2 - \tan x)(e^x) - e^x(-\sec^2 x)}{(2 - \tan x)^2}$

⑪ $f(x) = \dfrac{x + 3\cos x}{\csc x}$

Soln:

Method ①: $f'(x) = \dfrac{\csc x\,(1 - 3\sin x) - (x + 3\cos x)(-\csc x \cdot \cot x)}{\csc^2 x}$

method ②:

$f(x) = \sin x\,(x + 3\cos x)$

$f'(x) = \cos x\,(x + 3\cos x) + \sin x\,(1 - 3\sin x)$

⑫ $f(t) = t^2 + \underset{\smile}{2t}\,e^t\ \underset{\smile}{\sec t}$. (H.w)

$f'(t) = 2t + ((2 * e^t + e^t\,2t) * \sec t) + (2te^t * \sec t \tan t)$

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Ex. Evaluate $\lim_{x\to 0} \frac{e^x - 1}{x}$

مش مهم Sol.:

$f(x) = e^x$

$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$

$\lim_{x\to 0} \frac{e^x - 1}{x}$

مشتقة التعريف لـ e^x

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \frac{d}{dx}(e^x)\Big|_{x=0}$$

$$= e^x \Big|_{x=0}$$

$$= e^0 = \boxed{1}$$

Ex. Find the values of x where $f(x) = 6x^5 - 5x^3$ has a horizontal tangent line.

$\hookleftarrow$ يعني الميل = 0 = $f'(x)$

Soln.:

$$f'(x) = 30x^4 - 15x^2 = 0$$

$$\Rightarrow 15x^2(2x^2 - 1) = 0$$

$$\Rightarrow x = 0 \quad , \quad x = \pm \frac{1}{\sqrt{2}}$$

Ex. Find $h'(0)$ for the functions below, where $f(x)$ and $g(x)$ are unspecified differentiable functions with $f(0) = 3$, $f'(0) = 4$, $g(0) = -2$, and $g'(0) = 5$

قابل للاشتقاق

① $h(x) = (x^2 + g(x)) \cdot f(x)$

$$\Rightarrow h'(x) = (2x + g'(x)) \cdot f(x) + (x^2 + g(x)) \cdot f'(x)$$

$$h'(0) = (2(0) + \underbrace{g'(0)}_{5}) \cdot \underbrace{f(0)}_{3} + ((0)^2 + \underbrace{g(0)}_{-2}) \underbrace{f'(0)}_{4}$$

$$= 15 - 8 = \boxed{7}$$

② $h(x) = \frac{2e^x - g(x)}{f(x)}$ H.w

$$h'(x) = \frac{f(x).(e^x - g'(x)) - f'(x)(2e^x - g(x))}{(f(x))^2}$$

$$= \frac{f(0)(1 - g'(0)) - f'(0).(2 - g(0))}{(f(0))^2}$$

$$= \frac{3(1-5) - 4(2+2)}{(3)^2} = \frac{-12 - 16}{9} = \boxed{\frac{-28}{9}}$$

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Equation of tangent line and normal line:

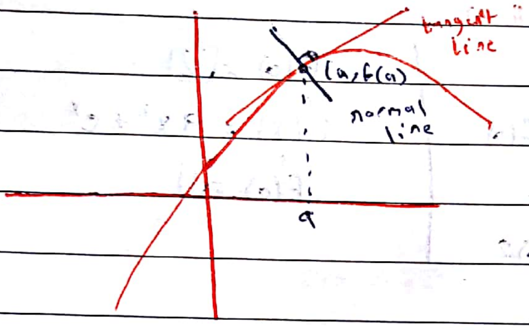
* **Recall:** If the two lines L_1 and L_2 are

يوازي

(i) Parallel $\Rightarrow$ Slope L_1 = Slope L_2

(ii) perpendicular $\Rightarrow$ Slope $L_1 *$ Slope $L_2 = -1$

$\Rightarrow$ Slople $L_2 = \frac{-1}{\text{Slope } L_1}$



① Equation of tangent line to the curve $y=f(x)$ at $x=a$ is given by:

$$\boxed{y = f'(a)(x-a) + f(a)}$$

② Equation of normal line to the curve $y=f(x)$ at $x=a$ is given by:

$$\boxed{y = \frac{-1}{f'(a)}(x-a) + f(a)}$$

Ex. Find the eqn of the tangent and normal lines to the graph of:

① $f(x) = e^x + x$ at $x = 2$

Soln: $f(x) = e^x + x \rightarrow f(2) = e^2 + 2$

$f'(x) = e^x + 1 \longrightarrow f'(2) = e^2 + 1$

∴ eqn of tangent line :

$y = f'(2)\ (x-2) + f(2)$

$= (e^2+1)\ (x-2) + (e^2+2)$

∴ eqn of Normal line :

$y = \frac{-1}{f'(2)}\ (x-2) + f(2)$

$= \frac{-1}{e^2+1}\ (x-2) + (e^2+2)$

② $f(x) = 3x^3 + 2e^x$ at $x = 0$ (H.W)

tangent $y = f'(a)\ (x-a) + f(a)$ } Normal

$y = f'(0)\ (x-0) + f(0)$

$y = 1\ (x-0) + 2$

$y = \frac{-1}{f'(0)}\ (x-0) + f(0)$

$y = \frac{-1}{1}\ (x-0) + 2$

$f(0) = \boxed{2}$

$f'(x) = 9x^2 + e^x$

$f'(0) = \boxed{1}$

③ $y = e^x \cos x$ at $(\underset{a}{0}, \underset{f(a)}{1})$

$f(0) = \boxed{1}$

$f'(x) = (e^x \cos x) + (-\sin x\, e^x)$

$f'(0) = \boxed{1}$

tangent

$y = f'(0)(x-0)f(0)$

$= 1(x-0)1$

normal

$y = \frac{-1}{f'(0)}\ (x-0)\ f(0)$

$= -1\ (x-0)1$

Ex. Find the eqn of the tangent and normal lines to the curve $y = x^{\frac{3}{2}}$ if the tangent line is parallel to $y = 1+3x$

$f(x)' = \frac{3}{2}x^{\frac{1}{2}} = 3$ $\boxed{x = 4}$ ← a

* Higher derivatives:

⊙ The first derivative of $y=F(x)$ is denoted by:

$$y' = f'(x) = \frac{\partial y}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x)$$

⊙ The derivative of the first derivative is called the second derivative denoted by: $y'' = f''(x) = \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f(x)$

⊙ The derivative // // second // // // third // //
// // : $y''' = F'''(x) = \frac{\partial^3 y}{\partial x^3} = \frac{\partial^3 f}{\partial x^3} = \frac{\partial^3}{\partial x^3} f(x)$

⊙ // // // // third // // fourth // //
// // : $y^{(4)} = f^{(4)}(x) = \frac{\partial^4 y}{\partial x^4} = \frac{\partial^4 f}{\partial x^4} = \frac{\partial^4}{\partial x^4} f(x)$

Ex. Find the third derivative of:

① $f(x) = 3x^5 - x^2 + x$

$f'(x) = 15x^4 - 2x + 1$

$f''(x) = 60x^3 - 2$

$f'''(x) = 180x^2$

② $f(x) = x^2 e^x$

$f'(x) = 2xe^x + x^2 e^x$

$= e^x(2x + x^2)$

$f''(x) = (2+2x)e^x + (2x+x^2)e^x$

~~F(x)~~ $= (2+4x+x^2)e^x$

$f'''(x) = (4+2x)e^x + (2+4x+x^2)e^x$

§ 3.4 The chain rule.

Theorm: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Remark: ① $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$

② $\frac{d}{dx}\sqrt{g(x)} = \frac{g'(x)}{2\sqrt{g(x)}}$

Ex. differentiate the following:

① $f(x) = 3(5x^2+7x)^4$

$f'(x) = 12(5x^2+7x)^3 \cdot (10x+7)$

② $f(x) = \frac{5}{(x^2+4x-10)^2}$

$f(x) = 5(x^2+4x-10)^{-2}$

$f'(x) = -10(x^2+4x-10)^{-3} \cdot (2x+4)$

③ $f(x) = \sqrt{2x^3-x}$

$f'(x) = \frac{6x^2-1}{2\sqrt{2x^3-x}}$

④ $f(x) = 3\sin(2x) + e^{\frac{x}{10}}$ $\quad \frac{1}{10}x$

$f'(x) = 3\cos(2x) \cdot 2 + e^{\frac{x}{10}} \cdot \frac{1}{10}$

⑤ $f(x) = \sec^5(e^{3x^2-1})$

$f'(x) = 5\sec^4(e^{3x^2-1}) \cdot \sec(e^{3x^2-1}) \cdot \tan(e^{3x+1}) \cdot e^{3x^2-1} \cdot 6x$

x^5
$\sec^x$
e^x
$3x^2-1$

⑥ $f(x) = \sin(x^3 + e^3) - \cos^3(\pi x)$

$f'(x) = \cos(x^3 + e^3) \cdot 3x^2 - 3\cos^2(\pi x) \cdot (-\sin(\pi x)) \cdot \pi$

e^{x^3}
$e^{\cos x}$
$e^{\pi x}$

#46
204

$y = [x + (x + \sin^2 x)^3]^4$

$\frac{\partial y}{\partial x} = 4[x + (x + \sin^2 x)^3]^3 \cdot [1 + 3(x + \sin^2 x)(1 + 2\sin x \cdot \cos x)]$

Ex. find the eqn of tangent and normal line to $f(x) = \sqrt{3x^2 + 4}$ at $x = 2$

Soln:

$f(x) = \sqrt{3x^2 + 4} \Rightarrow f(2) = 4$

$f'(x) = \frac{36x}{2\sqrt{3x^2 + 4}} \Rightarrow f'(2) = \frac{6}{4} = \frac{3}{2}$

① Eqn. of tangent line

$y = \frac{3}{2}(x - 2) + 4 \Rightarrow \boxed{y = \frac{3}{2}x + 1}$

② Eqn. of normal line

$y = -\frac{2}{3}(x - 2) + 4$

Ex. Let $f(x) = \sec x$, find $f'''(x)$

Soln:

$f'(x) = \sec x \tan x$

$f''(x) = \sec x \tan x \cdot \tan x + \sec x \cdot \sec^2 x$

$= \underline{\sec x \tan^2 x} + \sec^3 x$

$f'''(x) = \underline{\sec x \tan x \tan^2 x + \sec x \, 2\tan x \sec^2 x} + 3\sec^2 x \cdot \sec x \tan x$

Theorm: $\frac{\partial}{\partial x} b^{g(x)} = \ln b \cdot g'(x) \cdot b^{g(x)}$

Remark: $\frac{\partial}{\partial x} e^{g(x)} = g'(x)\, e^{g(x)}$

Ex. Differentiate

① $f(x) = 2^x \implies f'(x) = (\ln 2)\, 2^x$

② $f(x) = 3^{\sin x} \implies f'(x) = (\ln 3) \cos x \cdot 3^{\sin x}$

③ $f(x) = 7^{(x^2+3)^5} \implies f'(x) = (\ln 7) . 5 (x^2+3)^4 . 2x\, 7^{(x^2+3)^5}$

§ 3.5 Implicit differentiation

ضمني

Ex. Find $\frac{\partial y}{\partial x}$:

① $x^2 - 4\underline{x}\,\underline{y} + y^2 = -4$

Soln:

$$2x - (4 \cdot y + 4xy') + 2y \cdot y' = 0$$

$$\implies 2y \cdot y' - 4xy' = 4y - 2x$$

~~2~~ $\implies y'(2y - 4x) = 4y - 2x$

$$\therefore\ y' = \frac{4y - 2x}{2y - 4x}$$

② $e^y \sin x = x + xy$

Soln:

$$e^y \cdot y' \sin x + e^y . \cos x = 1 + 1.y + x.y'$$

$$e^y \sin x\, y' - xy' = 1 + y - e^y \cos x$$

$$\therefore\ y' = \frac{1 + y - e^y \cos x}{e^y \sin x - x}$$

Ex. Find an eqn of the tangent line to the curve $x^2 + xy + y^2 = 3$ at $(\underset{a}{1}, \underset{f(a)}{2})$

Sol.:

$$2x + 1y + x \cdot y' + 2y \cdot y' = 0$$

$$xy' + 2yy' = -(2x+y)$$

$$y' = \frac{-(2x+y)}{x+2y}$$

$$F'(1) = \frac{dy}{dx}\Big|_{\substack{x=1 \\ y=2}} = \frac{-4}{5}$$

Eqn. of tangent line

$$y = \frac{-4}{5}(x-1) + 2$$

⊛ Derivative of inverse function:

$$F(F^{-1}(x)) = x$$

$$F'(F^{-1}(x)) \cdot (F^{-1})(x) = 1$$

$$\Rightarrow \boxed{(F^{-1})'(x) = \frac{1}{\underset{②}{\underline{F'}}(\underbrace{F^{-1}(x)}_{①})}}$$

Ex. let $f(x) = x + e^x$ find $(F^{-1})'(1)$

Sol.:

(i) $F(0) = 0 + e^0 = \boxed{1}$

$\Rightarrow F^{-1}(1) = 0$

(ii) $F'(x) = 1 + e^x$

(iii) $(F^{-1})'(x) = \dfrac{1}{F'(F^{-1}(1))}$

$$= \frac{1}{F'(0)}$$

$$= \boxed{\frac{1}{2}}$$

* Derivative of inverse trigonometric functions:

① $\frac{\partial}{\partial x}[\sin^{-1} g(x)] = \frac{g'(x)}{\sqrt{1-[g(x)]^2}}$

② $\frac{\partial}{\partial x}[\tan^{-1} g(x)] = \frac{g'(x)}{1+[g(x)]^2}$

③ $\frac{\partial}{\partial x}[\sec^{-1} g(x)] = \frac{g'(x)}{|g(x)|\sqrt{(g(x))^2-1}}$

Remark: (i) $\frac{\partial}{\partial x}[\cos^{-1} g(x)] = -\frac{\partial}{\partial x}[\sin^{-1} g(x)]$

(ii) $\frac{\partial}{\partial x}[\cot^{-1} g(x)] = -\frac{\partial}{\partial x}[\tan^{-1} g(x)]$

(iii) $\frac{\partial}{\partial x}[\csc^{-1} g(x)] = -\frac{\partial}{\partial x}[\sec^{-1} g(x)]$

Ex. find the derivative of

① $y = \tan^{-1}\sqrt{x}$

$y' = \frac{\frac{1}{2\sqrt{x}}}{1+(\sqrt{x})^2} = \frac{1}{2(1+x)\sqrt{x}}$

#51
216

② $y = \sin^{-1}(2x+1)$

$\Rightarrow y' = \frac{2}{\sqrt{1-(2x+1)^2}}$

③ $y = x\sec^{-1}(x^3)$.

$y' = 1\cdot\sec^{-1}(x^3) + x \cdot \frac{3x^2}{|x^3|\sqrt{(x^3)^2-1}}$

④ $y = \sqrt{\cos^{-1} x}$

$y' = \frac{\frac{-1}{\sqrt{1-x^2}}}{2\sqrt{\cos^{-1}x}} = \frac{-1}{2\sqrt{\cos^{-1}x(1-x^2)}}$

5.8
216
$\boxed{5}$ $y = \cos^{-1}(\sin^{-1} x)$

$$\Rightarrow y' = -\frac{\frac{1}{\sqrt{1-x^2}}}{\sqrt{1-(\sin^{-1}x)^2}} = \frac{-1}{\sqrt{(1-x^2)(1-(\sin^{-1}x)^2)}}$$

#30
216
$\boxed{6}$ $y = \sqrt{\tan^{-1} x}$ (H.w)

$$y' = \frac{\frac{1}{1+(x)^2}}{2\sqrt{\tan^{-1}x}} = \frac{1}{2(1+x^2)\sqrt{\tan^{-1}x}}$$

مش مهم

Ex. Evaluate

$$\lim_{h\to 0} \frac{\tan^{-1}(1+h)-\frac{\pi}{4}}{h}$$

$$= \frac{d}{dx}(\tan^{-1}x)\Big|_{x=1}$$

$$= \left(\frac{1}{1+x^2}\right)\Big|_{x=1}$$

$$= \boxed{\frac{1}{2}}$$

§ 3.6 Derivatives of logarithmic functions.

Chain [illegible] rule

Theorem: $\frac{d}{dx}\left[\log_b f(x)\right] = \frac{f'(x)}{(\ln b)\, f(x)}$

Remark: $\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)}$

Remark: (i) $\log_b(x \cdot y) = \log_b x + \log_b y$

(ii) $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

(iii) $\log_b x^r = r \log_b x$

Ex. differentiate the following:

① $f(x) = e^{3x+1} - 2\ln x + \pi$

$\Rightarrow f'(x) = 3e^{3x+1} - 2 \cdot \frac{1}{x}$

② $f(x) = \log_2 x + \log_3 \sin x$

$\Rightarrow f'(x) = \frac{1}{(\ln 2)\, x} + \frac{\cos x}{(\ln 3) \sin x}$

$= \frac{1}{(\ln 2)\, x} + \frac{1}{\ln 3} \cot x$

③ #19 223

$y = \ln(e^{-x} + xe^{-x})$

$= \ln(e^{-x}(1+x))$

$= \ln e^{-x} + \ln(1+x)$

$= -x \overset{1}{\cancel{\ln e}} + \ln(1+x)$

$\Rightarrow y' = -1 + \frac{1}{1+x}$

④ $F(x) = \ln\left(\frac{x^2 - x}{\sqrt{x-2}}\right)$

$= \ln(x^2 - x) - \frac{1}{2}\ln(x-2)$

$\Rightarrow F'(x) = \frac{2x-1}{x^2-x} - \frac{1}{2}\cdot\frac{1}{x-2}$

#26
Ex. 223

$y = \ln(1 + \ln x)$

Find y''

Soln:

$y' = \frac{\frac{1}{x}}{1 + \ln x} = \frac{1}{x(1+\ln x)} = [x(1+\ln x)]^{-1}$

$y'' = -[x(1+\ln x)]^{-2}\cdot[1\cdot(1+\ln x) + x\cdot\frac{1}{x}]$

$= \frac{-(2+\ln x)}{[x(1+\ln x)]^2}$

#33
Ex. 223

Find an eqn. of the tangent line to $y = \ln(x^2 - 3x + 1)$ at $(\underset{a}{3}, \underset{F(a)}{0})$

Soln:

$F'(x) = \frac{2x-3}{x^2-3x+1}$

$F'(3) = \frac{6-3}{9-9+1} = \boxed{3}$

$\therefore$ eqn of tangent line:

$y = 3(x-3) + 0$

$= 3x - 9$

نستخدم الاشتقاق الـ log ① x^x ② اذا كان معقد

* Logarithmic differentiation:

Ex.: differentiate:

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نضرب بـ ln ونبسط
نشتق نضرب بـ y

① #45 223 $y = x^{\sin x}$

(i) $\ln y = \ln x^{\sin x} = \sin x \ln x$

(ii) $\frac{y'}{y} = \cos x \ln x + \sin x \frac{1}{x}$

(iii) $y' = \left[\cos x \ln x + \frac{\sin x}{x}\right]$ ⓨ

$= \left[\cos x \ln x + \frac{\sin x}{x}\right] x^{\sin x}$

#42 223 ② $y = \sqrt{x}\, e^{x^2 - x} (x+1)^{2/3}$

(i) $\ln y = \frac{1}{2} \ln x + (x^2 - x)\overset{1}{\cancel{\ln e}} + \frac{2}{3} \ln(x+1)$

(ii) $\frac{y'}{y} = \frac{1}{2} \cdot \frac{1}{x} + (2x - 1) + \frac{2}{3} \cdot \frac{1}{x+1}$

(iii) $y' = \left[\frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\right] \cdot \sqrt{x}\, e^{x^2 - x} (x+1)^{2/3}$

③ $y = \frac{x^{\frac{3}{4}} \sqrt{x^2 + 1}}{(3x+2)^5\, e^{x^3}}$ (H.W)

$\ln\left(\frac{a \cdot b}{c \cdot d}\right) = \ln a + \ln b - \ln c - \ln d$

$\ln y = \ln\left(\frac{x^{\frac{3}{4}} \sqrt{x^2+1}}{(3x+2)^5\, e^{x^3}}\right)$

$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2+1) - 5 \ln(3x+2) - x^3 \cancel{\ln e}$

$\frac{y'}{y} = \frac{3}{4} \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2+1} - 5 \frac{3}{3x+2} - 3x^2$

$\cancel{y} \cdot \frac{y'}{\cancel{y}} = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} - 3x^2 * \left(\frac{x^{\frac{3}{4}} \sqrt{x^2+1}}{(3x+2)^5\, e^{x^3}}\right)$

§ 3.10 Linear Approximation:

$$f(x) \approx L(x) = f'(a)(x-a) + f(a)$$

For all x near to a

Ex: Find the linearization $L(x)$ of $f(x) = \sqrt{x}$ at $a = 4$

Soln:

$f(x) = \sqrt{x} \Rightarrow f(4) = 2$

$f'(x) = \frac{1}{2\sqrt{x}} \rightarrow f'(4) = \frac{1}{4}$

$\therefore L(x) = f'(4)(x-4) + f(4)$

$= \frac{1}{4}(x-4) + 2$

$$\boxed{L(x) = \frac{1}{4}x + 1}$$

Ex: Estimate قدّر $\sqrt{3}$

Soln:

$\sqrt{x} \approx \frac{1}{4}x + 1$ near $a = 4$

$\Rightarrow \sqrt{3} = \frac{1}{4} \times 3 + 1 = \boxed{1.75}$

Ex. Use linear approximation to estimate:

① $e^{0.1}$

Soln:

$f(x) = e^{x}$, $a = 0$

$f(0) = 1$

$f'(x) = e^{x} \rightarrow f'(0) = 1$

$L(x) = 1(x-0) + 1 = x + 1$

$\rightarrow e^{x} \approx x + 1$ near $a = 0$ $\therefore e^{0.1} \approx 0.1 + 1 = 1.1$

② $\ln(1.02)$

Soln:

$f(x) = \ln x \quad , a=1$

$f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \rightarrow f'(1) = 1$

$\therefore L(x) = 1(x-1) + 0 = x-1$

$\Rightarrow \ln x \approx x - 1$ for x close to $a=1$

$\therefore \ln(1.02) \approx 1.02 - 1 = 0.02$

#28
256 $\cos(29^\circ)$

Soln:

$f(x) = \cos x \quad , \quad a = \cancel{30}^\circ \ \frac{\pi}{6}$

$f(\cancel{30}\,\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$f'(x) = -\sin x \rightarrow f'(\cancel{30}\,\frac{\pi}{6}) = -\frac{1}{2}$

$\therefore L(x) = -\frac{1}{2}(x - \cancel{30}\,\frac{\pi}{6}) + \frac{\sqrt{3}}{2}$

$\Rightarrow \cos x \approx -\frac{1}{2}(x - \cancel{30}\,\frac{\pi}{6}) + \frac{\sqrt{3}}{2}$

~~$\therefore \cos(29^\circ) = -\frac{1}{2}(29-30) + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2}$~~

$\therefore \cancel{\cos x}\ 29^\circ = 29 \times \frac{\pi}{180} = \frac{29\pi}{180}$

$\cos\left(\frac{29\pi}{180}\right) = -\frac{1}{2}\left(\frac{29\pi}{180} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}$

$= -\frac{1}{2}\left(\frac{-\pi}{180}\right) + \frac{\sqrt{3}}{2} = \frac{\pi}{360} + \frac{\sqrt{3}}{2}$

Ex. Verify $e^x \cos x \approx x + 1$ at $a = 0$

Soln:

$f(x) = e^x \cos x \longrightarrow f(0) = 1$

$f'(x) = e^x \cos x - e^x \sin x \Rightarrow f'(0) = 1$

$\therefore L(x) = f'(0)(x - 0) + f(0)$

$\therefore f(x) = \underline{e^x \cos x} \approx L(x) = \underline{x+1}.$

§ 3.11 Hyperbolic Functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

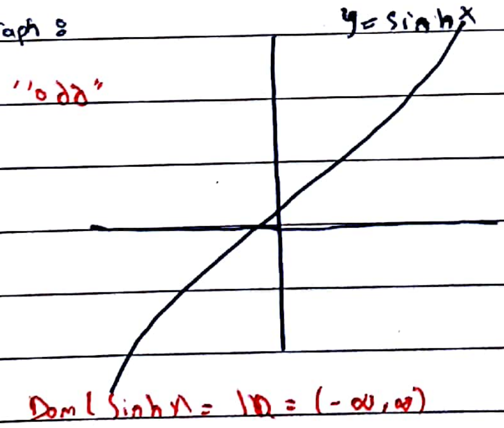
Ex. Evaluate:

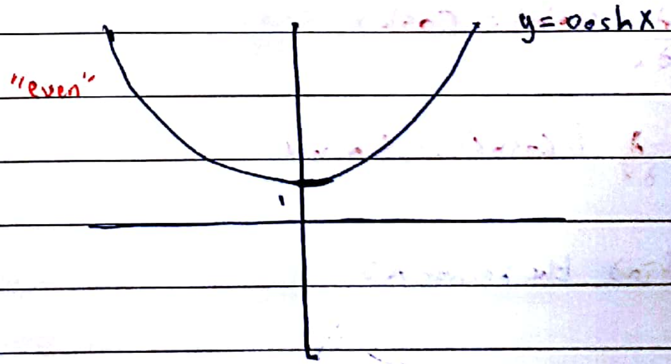
#1 204 $\sinh(0) = \frac{e^0 - e^{-(0)}}{2} = \frac{1-1}{2} = \frac{0}{2} = \boxed{0}$

$\cosh(5) = \frac{e^{\ln 5} + e^{-\ln 5}}{2} = \frac{e^{\ln 5} + e^{\ln 5^{-1}}}{2}$

$= \frac{5 + 5^{-1}}{2} = \frac{5 + \frac{1}{5}}{2} = \frac{\frac{26}{5}}{2} = \frac{26}{5} \times \frac{1}{2} = \frac{26}{20} = \frac{13}{5}$

*Graph:





Dom(sinhx) = ℝ = $(-\infty, \infty)$

Rang(sinhx) = ℝ = $(-\infty, \infty)$

Dom(coshx) = ℝ = $(-\infty, \infty)$

Rang(coshx) = $[1, \infty)$

* Hyperbolic identities:

(1) $\sinh(-x) = -\sinh(x)$

$\cosh(-x) = \cosh(x)$

(2) ~~$\sinh^2 x - \cosh^2 x = 1$~~ $\cosh^2 x - \sinh^2 x = 1$

(3) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh xy$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Ex: If $\cosh x = \frac{5}{3}$, $x>0$

Find $\sinh x$.

Soln:

~~$\sinh^2 x - \cosh^2 x = 1$~~

~~$\Rightarrow \sinh^2 x = 1 + \frac{25}{9} = \frac{34}{9}$~~

~~$\rightarrow \sinh x = \pm \frac{\sqrt{34}}{3}$~~

~~$\rightarrow \sin x = \pm \frac{\sqrt{34}}{3}$~~

$\sinh^2 x = \cosh^2 x - 1$

$= \frac{25}{9} - 1 = \frac{16}{9}$

$\Longrightarrow \sinh x = \pm \frac{4}{3}$

لانو x>0 بكون موجب $\Longrightarrow \boxed{\sinh x = \frac{4}{3}}$

ودايما ال Coh بكون موجب

+ Derivative of hyperbolic functions:

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

Ex. Find the derivative:

① $y = \ln(\cosh x) \rightarrow y' = \frac{\sinh}{\cosh x} = \tanh x$

② $y = e^x \sinh x \rightarrow y' = e^x \sinh x + e^x \cdot \cosh x$

③ $y = e^{\cosh(3x)} \longrightarrow y' = e^{\cosh(3x)} \cdot \sinh(3x) \cdot 3$

④ $y = \sinh(\cosh x)$

$\rightarrow y' = \cosh(\cosh x) \cdot \sinh x$

Ex. $\overset{\frac{\infty}{\infty}}{\lim_{x \to \infty}} \dfrac{\sinh x}{e^x} = \lim_{x \to \infty} \dfrac{\frac{e^x - e^{-x}}{2}}{\frac{e^x}{1}}$

$= \lim_{x \to \infty} \dfrac{e^x - e^{-x}}{2e^x}$

$= \lim_{x \to \infty} \dfrac{1 - e^{-2x}}{2}$

$= \dfrac{1 - 0}{2} = \boxed{\dfrac{1}{2}}$

Chapter 4: Applications of differetation

§ 4.1 $\overset{\text{maxima}}{\text{Maximum}}$ and $\overset{\text{minima}}{\text{minimum}}$ values.

defn: let c be anumber in the domain D of a function f(x), then f(c) is:

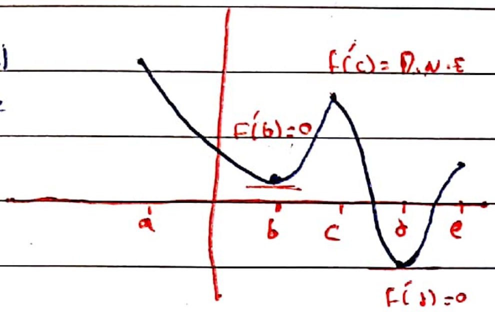
(1) local maximum value of f if $f(c) \geq f(x)$ when x is near c

(2) local minimum value of f if $f(c) \leq f(x)$ when x is near c.

(3) absolute maximum value of ~~F~~ f on D if $f(c) \geq f(x)$ for all x in D

(4) absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D

أطراف الفترة ممنوع يكونوا local



f(b) local minimum value

f(c) local maximum value

f(d) ~~local~~ abs. minimum value

f(a) abs. maximum value

* Fermat's ~~Feo~~ Theorm:

If f has a local max or min at c, and if $f'(x)$ exists, then $f'(c) = 0$

Defn: A $\overset{\text{النقاط الحرجة}}{\text{critical number}}$ of a function f is anumber c in the domain of f such that $f'(c) = 0$ or DNE.

Ex. Find the critical number(s):

① $f(x) = \sqrt{x}$

Soln:

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$f' \rightarrow$ DNE at $x = 0$

$\therefore x = 0$ is a critical number

② $f(x) = \frac{1}{x}$

Soln:

$f'(x) = \frac{-1}{x^2}$

$f'(x)$ D.N.E at $x=0$

No critical numbers (خارج ال Dom)

③ $f(x) = x^3 - 12x \quad , [0,3]$

Soln:

$f'(x) = 3x^2 - 12 = 0$

$\rightarrow \quad x^2 = 4$

$x = \pm 2$

$\therefore \; x = 2$ is a critical number.

④ $y = x^2 \ln x.$

Soln.

$f'(x) = 2x \ln x + x^2 \frac{1}{x}$

$= 2x \ln x + x = 0$

$\Rightarrow \; x(2\ln x + 1) = 0$

$\Rightarrow \; \boxed{x=0}$ or $\ln x = \frac{-1}{2}$

$\boxed{x = e^{-\frac{1}{2}}}$

$\therefore \; x = e^{-\frac{1}{2}}$ is a critical numbers

Ex. find the absolute max and min values of:

① $f(x) = 3x^2 - 12x + 1$, $[-3, 5]$

Soln:

$f'(x) = 6x - 12 = 0$

$\Rightarrow \boxed{x = 2}$ Critical number [c.n]

x	-3	2	5
F(x)	64	-11	16

Abs. max = 64

ABs. min = -11

② $f(x) = (x^2 - 5)^3$, $[-1, 2]$

Soln:

$f'(x) = 3(x^2 - 5)^2 \cdot 2x = 0$

$\Rightarrow x = 0$, $x = \pm\sqrt{5} \rightarrow$ خارج ال Dom

$\Rightarrow x = 0$ C.N.

x	-1	0	2
f(x)	-64	-125	-1

Abs. max = -1

Abs. min = -125

③ $f(x) = x^3 - 3x^2 + 1$, $[0.5, 4]$ (H.w)

$\frac{1}{2}$

$f'(x) = 3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x \overset{\times}{=} 0$ $\boxed{x = 2}$

X	0.5	2	4
F(x)	0;375	~~0~~ -3	17

abs max = 17

abs min = -3

④ $f(x) = x \cdot \sqrt{9-x^2}$; $[-3,3]$ (H.W)

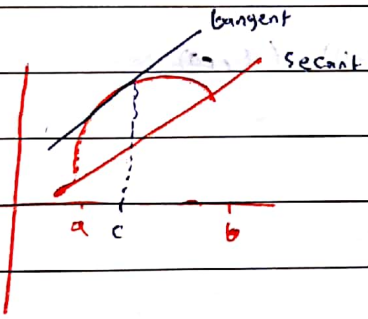
§4.2 The mean value Theorem (MVT)

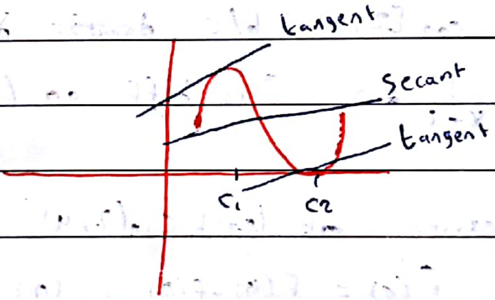
* Theorem: (MVT): Assume (i) f is con. on $[a,b]$ → cont.ال مغلقة لأنه

(ii) f is diff on (a,b)

then there is at least one number c in (a,b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$





Rolles Theorem:

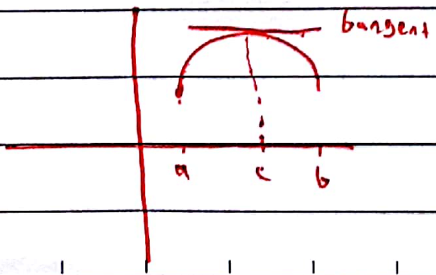
Assume (i) f is con. on $[a,B]$

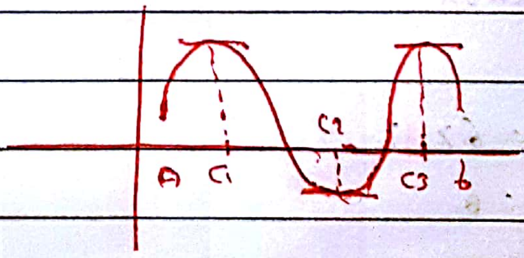
(ii) f is diff on (a,B)

(iii) $f(a) = f(b)$

then there is at least one number c in (a,b) such that:

$$f'(c) = 0$$





Ex. ① Verify that the given function satisfies the hypotheses of the MVT on the given interval then find all numbers c that satisfy the conclusion of the MVT:

(a) $F(x) = x^3 - 3x + 2$; $[-2, 2]$

Soln:

(i) $f(x)$ is con. $[-2,2]$ because its a polynomial.

(ii) $F'(x) = 3x^2 - 3 \longrightarrow$ F is diff. on $(-2,2)$

$\overset{MVT}{\Longrightarrow}$ there exists at least c in $(-2,2)$ such that:

$$f'(c) = \frac{F(2) - f(-2)}{2-(-2)}$$

$$\rightarrow 3c^2 - 3 = \frac{4-0}{4} = 1$$

$$\Rightarrow \boxed{c = \pm \frac{2}{\sqrt{3}}}$$

(b) $F(x) = -\ln(x-1)$, $[2,4]$

(i) $F(x)$ is con $[2,4]$, b/c ~~dom(f)~~ = dom(ln(x-1)) = $(1, \infty)$

(ii) $F'(x) = \frac{1}{x-1} \Rightarrow$ f is diff on $(2,4)$

$\overset{MVT}{\Longrightarrow}$ there exists at least c in $(2,4)$

S.t $$F'(c) = \frac{F(4) - f(2)}{4-2} = \frac{\ln 3 - \overset{0}{\cancel{\ln 1}}}{2}$$

$$\Rightarrow \cancel{\ln \overset{3}{3}/(c-1) = 2}$$

$$\Rightarrow \ln 3 (c-1) = 2$$

$$\Rightarrow \boxed{C = \frac{2}{\ln 3} + 1}$$

② Let $F(x) = 3x^2 - 12x + 5$ find a number c in $(1,3)$ that satisfies Rolle's theorem.

Soln:

$F'(x) = 6x - 12$

$F'(c) = 0$

$\Rightarrow 6c - 12 = 0$

$\Rightarrow \quad c = \frac{12}{6} = \boxed{2}$

③ Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all x how large can $f(2)$ possibly be ??

متصلة ومشتقة في MVT

Soln: By MVT:

$$\frac{f(2) - f(0)}{2-0} = f'(c) \leq 5$$

$$\Rightarrow \frac{f(2) - (-3)}{2} \leq 5$$

$$f(2) \leq 10 - 3 = 7$$

$\therefore$ Max $f(2) = \boxed{7}$

④ Is there a diff. function such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x ?

Soln: If this function exists. By MVT

$$\frac{f(2) - f(0)}{2-0} = f'(c) \leq 2$$

$$\rightarrow \frac{4-(-1)}{2} \leq 2$$

$\frac{5}{2} \leq 2$! → تناقض

$\therefore$ Ans $\boxed{No}$

§ 4.3 How Derivatives affect the Shape of a Graph.

* What does f' say about the graph of f ??

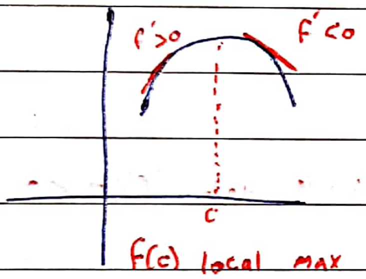
→ If f'>0 on I, then f is increasing on I.

→ If f'<0 on I, then f is decreasing on I.

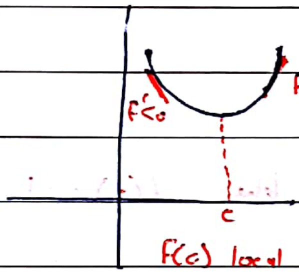
⊙ If c is a critical number and f is continuous then:

(i) If f' changes from ⊕ to (-) at c, then f has a local max at c.

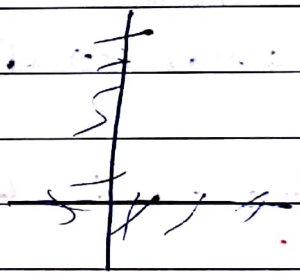
(ii) If f' changes from ⊖ to ⊕ at c, then f has a local min at c.

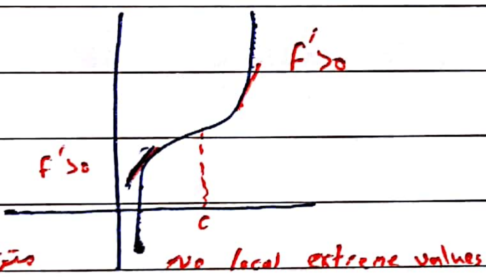


f(c) local max



f(c) local min





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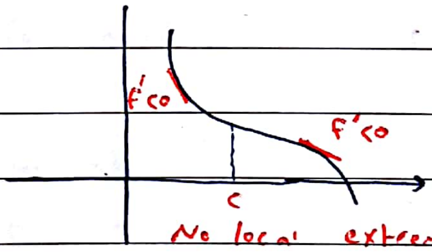
No local extreme values



No local extreme values

Ex. Use the 1^{st} derivative test to find the intervals of increasing and decreasing and identify the local max/min.

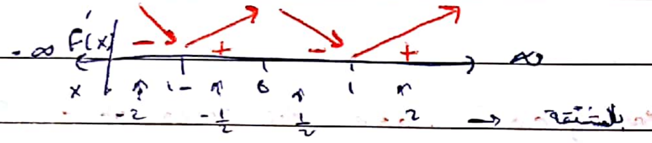
① $F(x) = x^4 - 2x^2 + 1$

Sol:

$f'(x) = 4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$x = 0 \ , \ x = \pm 1$ C.N.



f decreasing $(-\infty, -1) \cup (0, 1)$

f increasing $(-1, 0) \cup (1, \infty)$

$f(-1) = 0$ local min

$f(0) = 1$ local max

$f(1) = 0$ local min

② $F(x) = x + 2\sin x$, $[0, 2\pi]$

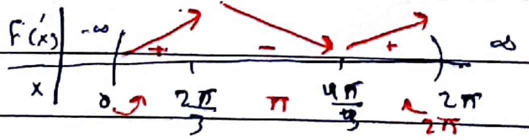
Sol:

$f'(x) = 1 + 2\cos x = 0$

$\cos x = -\frac{1}{2}$

~~$\cos x =$~~

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$ C.N.



f increasing $(0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi)$

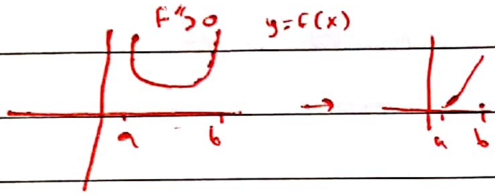
f decreasing $(\frac{2\pi}{3}, \frac{4\pi}{3})$

$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sqrt{3}$ ⇒ local max.

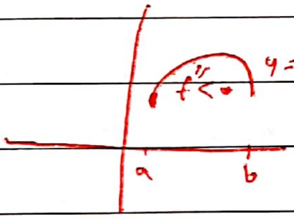
$f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$ local min

* what does f'' say about the graph of f ?

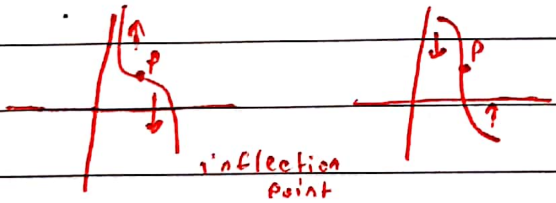
→ If $f'' > 0$ on I, then f' is increasing on I, and the graph of f in $\overset{\text{مقعر}}{\text{concave}}$ upward on I.



→ If $f'' < 0$ on I, then f' is decreasing on I, and the graph of f is concave downward on I.



@ A point P on a curve is an inflection point if the concavity changes sign at P.

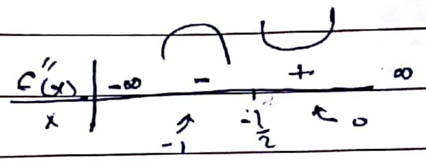


Ex. Determine the intervals on which $f(x) = 2x^3 + 3x^2 - 72x + 17$ is concave up/down and then find any inflection point

$f'(x) = 6x^2 + 6x - 72$

$f''(x) = 12x + 6 = 0$

$x = \frac{-1}{2}$



f is concave down $(-\infty, -\frac{1}{2})$

f is concave up $(-\frac{1}{2}, \infty)$

$(-\frac{1}{2}, f(-\frac{1}{2}))$ inflection point

Ex. determine the intervals on which $f(x) = x\sqrt{4-2x}$. the given function is increasing / decreasing, concave up/down, and. then find any local extrema and inflection points.

Sketch the graph of f.

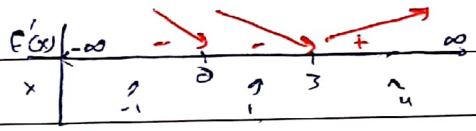
① $f(x) = x^4 - 4x^3$

Soln:

(A) $f'(x) = 4x^3 - 12x^2$

$\rightarrow 4x^2(x-3) = 0$

$\rightarrow$ $x = 0$, $x = 3$ C.N.



f increasing $(3, \infty)$

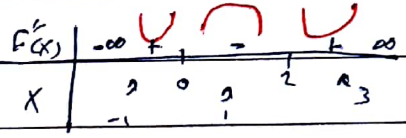
f decreasing $(-\infty, 0) \cup (0, 3)$

$f(3) = -27$ local min.

(B) $f''(x) = 12x^2 - 24x = 0$

$\rightarrow 12x(x-2) = 0$

$\rightarrow x = 0, 2$



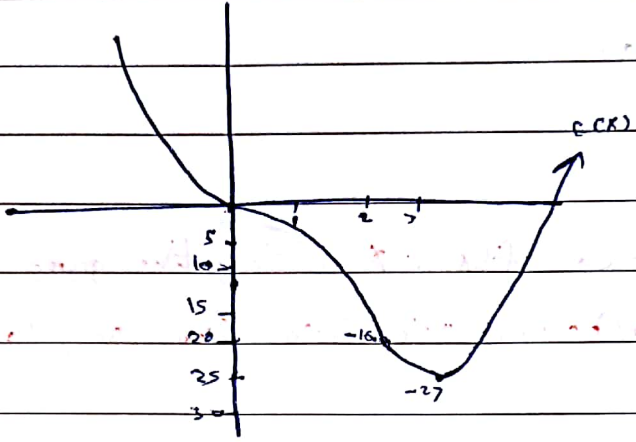
f concave up $(-\infty, 0) \cup (2, \infty)$

f concave down $(0, 2)$

(0 , f(0) = (0,0) inf . pt.

(2 , f(2) = (2, −16) , inf. pt

1c)



⑨ $f(x) = x\sqrt{4-2x}$

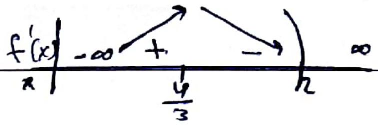
Soln:

$4 - 2x \geq 0$

$x \leq 2$

Dom(f) = $(-\infty, 2]$

$f'(x) = \dfrac{4-3x}{\sqrt{4-2x}}$



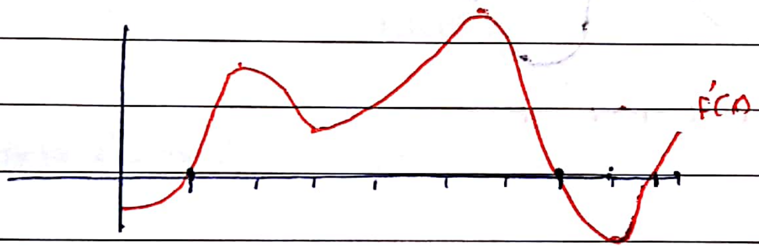
$f''(x) = \dfrac{3x - 8}{(4-2x)\sqrt{4-2x}}$

② $f(x) = x^{2/3}(6-x)^{1/3}$ (H.W) (A+B) بدون رسم

④ $f(x) = x^4 - 8x^2$ (Hw. A+B+C) (H.W)

#36
302

The graph of f' is shown below



فوق محور X .
متزايد
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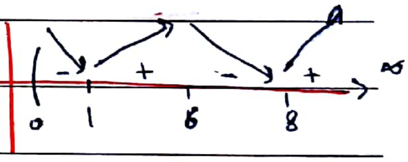


(a) on what intervals is f increasing / decreasing?

f decreasing $(0,1) \cup (6,8)$

f increasing $(1,6) \cup (8,\infty)$

(b) At what values of x does f have a local max / min

Soln:

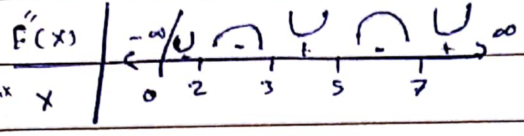
f have a local min at x = 1 , x = 8

f have a local max at x = 6

(c) on what intervals is f concave up/down.

Soln:

اللي عندهم
Local min/max



f concave down (2,3) ∪ (5,7)

f concave up (0,2) ∪ (3,5) ∪ (7,∞)

(d) State the x coordinate(s) of the point(s) of inflection.
اذكر

Soln:

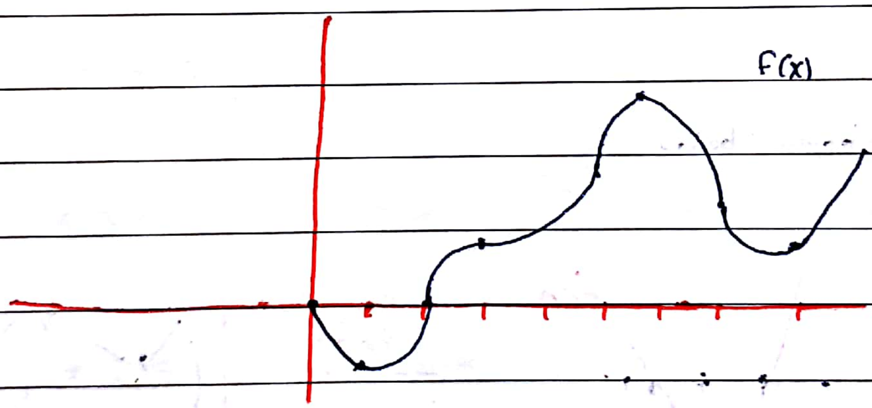
نقاط الانعطاف (بتحول من conc

f have an inflection point at x = 2,3,5,7

(e) assuming that f(0) = 0, sketch agraph of f.

Soln:

حسب ال $f'(x), f''(x)$



* The second derivative test (SDT):

If $f'(c) = 0$, then if

$f''(c) > 0 \longrightarrow f(c)$ local min

$f''(c) < 0 \longrightarrow f(c)$ local max

$f''(c) = 0 \longrightarrow$ Test inconclusive.

Ex: use the SDT to find the local max/min extrema:

① $f(x) = x^5 - 5x^3$

Soln:

$f'(x) = 5x^4 - 15x^2$

$\rightarrow 5x^2(x^2 - 3) = 0$

$x = 0 \quad x = \pm\sqrt{3}$ C.N.

$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3)$

~~$f(a)$~~ $f'(-\sqrt{3}) = <0 \rightarrow f(-\sqrt{3})$ local max

$f'(0) = \quad = 0 \rightarrow$ Test inconc.

$f'(\sqrt{3}) = >0 \rightarrow$ local min

② $f(x) = x^4 - 24x^2$ (H.W).

$f'(x) = 4x^3 - 48x$

$4x(x^2 - 12)$

$x = 0 \quad x = \pm\sqrt{12}$

~~$f(-\sqrt{12})$~~ $f''(x) = 12x^2 - 48$

$f''(-\sqrt{12}) = <0$ local max

$f''(0) = <0$ local max

$f''(\sqrt{12}) = >0$ local min

§ 4.4 Indeterminate Forms and L'Hospitals' rule

* Recall: Indeterminate forms are:

$\underbrace{\frac{0}{0}, \frac{\infty}{\infty}}_{①}, \underbrace{\infty - \infty}_{②}, \underbrace{\infty \cdot 0}_{③}, \underbrace{1^{\infty}, 0^{0}, \infty^{0}}_{④}$

مباشرة ← ①

L'Hospital's rule:

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to x} \frac{f'(x)}{g'(x)}$

Ex. Evaluate:

① $\overset{\frac{0}{0}}{\lim_{x\to 0}} \frac{1-e^x}{\sin x} = \lim_{x\to 0} \frac{-e^x}{\cos x} = \frac{-1}{1} = \boxed{-1}$

② $\overset{\frac{0}{0}}{\lim_{x\to 0}} \frac{x-\sin x}{x^2} = \lim_{x\to 0} \overset{\frac{0}{0}}{\frac{1-\cos x}{2x}} = \lim_{x\to 0} \frac{\sin x}{2} = \frac{0}{2} = \boxed{0}$

③ $\overset{\frac{\infty}{\infty}}{\lim_{x\to\infty}} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

④ $\overset{\frac{\infty}{\infty}}{\lim_{x\to\infty}} \frac{e^x}{x^2} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \boxed{\infty}$

* H.w:

#8 311) $\lim_{x\to 3} \overset{\frac{0}{0}}{\frac{x-3}{x^2-9}} = \lim_{x\to 3} \frac{1}{2x} = \boxed{\frac{1}{6}}$

#9 311) $\lim_{x\to 4} \overset{\frac{0}{0}}{\frac{x^2-2x-8}{x-4}} = \lim_{x\to 4} \frac{2x-2}{1} = \boxed{6}$

#11 311) $\lim_{x\to 1} \overset{\frac{0}{0}}{\frac{x^3-2x^2+1}{x^3-1}} = \lim_{x\to 1} \frac{3x^2-4x}{3x^2} = \boxed{\frac{-1}{3}}$

#19 311) $\lim_{x\to\infty} \overset{\frac{\infty}{\infty}}{\frac{\ln x}{\sqrt{x}}} = \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \overset{\lim_{x\to\infty}}{=} 2\frac{1}{x}\sqrt{x} = \lim_{x\to\infty} x \cdot 2\frac{1}{\cancel{x}} x^{-\frac{1}{2}} \mp 2\sqrt{x} \quad x$

#32 312] $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{x} = \lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \frac{1}{2x^2} = \frac{1}{\infty} = \boxed{0}$

تجميع الكرين

Ex: Evaluate:

$-\infty - \infty$

$\infty - \infty$

① $\lim_{x\to 0^+} [\ln x - \ln(x^2+x)]$

$\Rightarrow \lim_{x\to 0^+} \ln\left(\frac{x}{x^2+x}\right)$

$= \ln\left(\lim_{x\to 0^+} \overset{\frac{0}{0}}{} \frac{x}{x^2+x}\right)$

$= \ln\left(\lim_{x\to 0} \frac{1}{2x+1}\right)$

$= \ln(1) = \boxed{0}$

$\infty - \infty$

② $\lim_{x\to 0^+} \left[\frac{1}{x} - \frac{1}{\tan^{-1} x}\right]$

$= \lim_{x\to 0^+} \overset{\frac{0}{0}}{} \frac{\tan^{-1}x - x}{x \tan^{-1}x}$

$= \lim_{x\to 0^+} \left(\frac{\frac{1}{x^2+1} - 1}{\tan^{-1}x + x \frac{1}{x^2+1}} \right) \cdot \frac{(x^2+1)}{(x^2+1)}$

$= \lim_{x\to 0^+} \overset{\frac{0}{0}}{} \frac{1 - (x^2+1)}{(x^2+1)\tan^{-1}x + x}$

$= \lim_{x\to 0^+} \frac{-2x}{2x\tan^{-1}x + (x^2+1)\frac{1}{x^2+1} + 1}$

$= \frac{0}{0+1+1} = \frac{0}{2} = \boxed{0}$

$\infty - \infty$

③ $\lim_{x\to\infty} (e^x - x)$

$\lim_{x\to\infty} x\left(\frac{e^x}{x} - 1\right) = \infty \cdot \infty = \boxed{\infty}$

∞ by L.R

Recall: $a \cdot b = \frac{a}{\frac{1}{b}} = \frac{b}{\frac{1}{a}}$

$\Rightarrow 0 \cdot \infty = \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$

$= \frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}$

Ex. Calculate:

① $\overset{\infty \cdot 0}{\lim_{x \to \infty}} x \ln\left(1 - \frac{1}{x}\right)$

$$= \lim_{x \to \infty} \overset{\frac{0}{0}}{} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\frac{\frac{\cancel{1}}{\cancel{x^2}}}{1 - \frac{1}{x}}}{-\frac{1}{\cancel{x^2}}} = \lim_{x \to \infty} \frac{-1}{1 - \frac{1}{x}} = \frac{-1}{1 - 0} = \boxed{-1}$$

② #44
312

$\overset{0 \cdot \infty}{\lim_{x \to 0^+}} \sin x \ln x$

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x \to 0^+} \overset{\frac{-\infty}{\infty}}{} \frac{\ln x}{\csc x}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}$$

$$= \lim_{x \to 0^+} \frac{1}{\underbrace{-x}_{0} \underbrace{\csc \cot x}_{\infty}}$$

$$= \lim_{x \to 0^+} \overset{\frac{0}{0}}{} \frac{\sin x \tan x}{-x}$$

$$= \lim_{x \to 0^+} \frac{\cos x \tan x + \sin x \sec^2 x}{-1} = \frac{0 + 0}{-1} = \frac{0}{-1} = \boxed{0}$$

Recall: (1) $f(x) = e^{\ln f(x)}$

(2) $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)}$

Ex. find

① $\overset{\infty^0}{\lim_{x\to\infty}} x^{\frac{1}{x}} = e^{\lim_{x\to\infty} \ln(x^{\frac{1}{x}})}$

$= \cancel{\lim_{x\to\infty}} \; \overset{\frac{\infty}{\infty}}{e^{\lim_{x\to\infty} \frac{\ln x}{x}}}$

$= e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}}$

$= e^0 = \boxed{1}$

② 57] 312 0^0 $\lim_{x\to 0^+} x^{\sqrt{x}} =$

$= e^{\lim_{x\to 0^+} \ln(x^{\sqrt{x}})}$

$0 \cdot \infty$

$= e^{\lim_{x\to 0^+} \sqrt{x} \ln x}$

$\boxed{\frac{-\infty}{\infty}}$

$= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}}$

$= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^{3/2}}}}$

$= e^{\lim_{x\to 0^+} -2x^{\frac{1}{2}}} = e^0 = \boxed{1}$

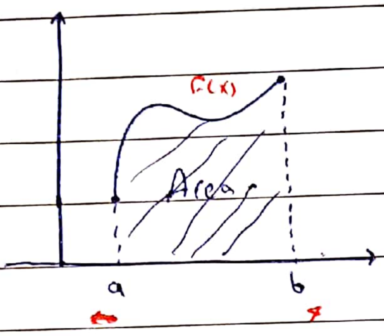
H.w $\frac{59}{312}$] $\overset{-1^\infty}{\lim_{x\to 0}} (1-2x)^{\frac{1}{x}}$

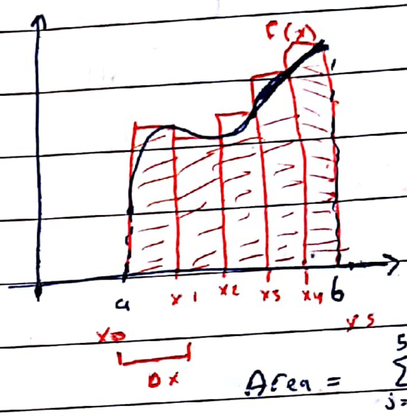
$\frac{68}{312}$] $\overset{1^\infty}{\lim_{x\to\infty}} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$

Chapter 5: Integrals:

§ 5.2 The definite integral.

• Our goal is to compute the area under the graph (the area between the graph and the x-axis) for positive and continuous function f(x)





Area = $\sum_{j=1}^{5} f(x_j)\Delta x$

• If f(x) is positive and continuous on [a,b], then the definite integral of f(x) over [a,b] is:

$\int_a^b f(x)dx$ = Area between the graph of f and x-axis from x=a to x=b

* Theorem: If f is continuous on [a,b], then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_j = a + j\Delta x$

* Remarks:

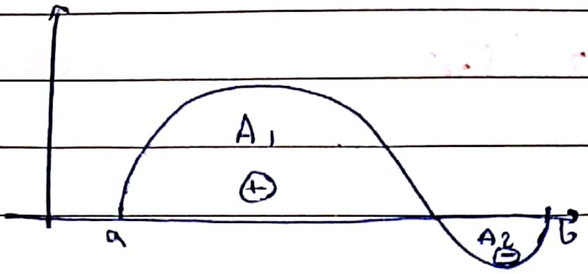
If f(x) is

(i) positive and cont. on (a,b), then:

$\int_a^b f(x)\,dx$ = positive number..

(ii) negative and cont. on [a,b], then:

$\int_a^b f(x)\,dx$ = negative number



$${}_a\int^b f(x)\,dx = A_1 - A_2$$

* Some properties of definite intergal :

① $\int_a^b \alpha \, dx = \alpha(b-a)$, (α ~ constant)

② $\int_a^b \alpha f(x)\,d(x) = \alpha \int_a^b f(x)\,dx$

③ $\int_a^b [f(x) \pm g(x)]\,d(x) = \int_a f(x)\,dx \pm \int_a^b g(x)\,dx$

④ $\int_a^a f(x)\,dx = 0$

⑤ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

⑥ $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

⑦ If $f(x) \geq g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

⑧ If $m \leq f(x) \leq M$ on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Ex. If $\int_0^3 f(x)\,dx = 5$ and $\int_0^3 g(x)\,dx = 12$, find:

① $\int_0^3 [f(x) + 4g(x)]\,dx$

$= \int_0^3 f(x)\,dx + 4\int_0^3 g(x)\,dx$

$5 + 4(12) = 53$

② $\int_0^3 [2f(x) - 5g(x)]\,dx = \underline{\underline{-50}}$ (H.w)

$2\int_0^3 f(x)\,dx - 5\int_0^3 g(x)\,dx$

$= 10 - 60 = \boxed{-50}$

Ex. If $\int_0^1 f(x)\,dx = 1$, $\int_0^2 f(x)\,dx = 4$, and $\int_1^4 f(x)\,dx = 7$. Evaluate:

① $\int_0^4 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^4 f(x)\,dx = 1 + 7 = 8$

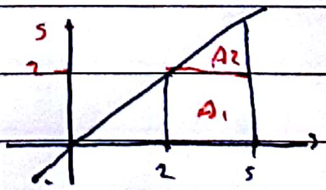
② $\int_1^2 f(x)\,dx = \int_1^0 f(x)\,dx + \int_0^2 f(x)\,dx = -1 + 4 = 3$

③ $\int_2^4 f(x)\,dx = -4 + 1 + 7 = 4$ (H.w)

$\int_2^0 f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^4 f(x)\,dx = -4 + 1 + 7 = \boxed{4}$

Ex. Let $f(x) = \begin{cases} 3, & x < 2 \\ 2x, & x \geq 2 \end{cases}$ find

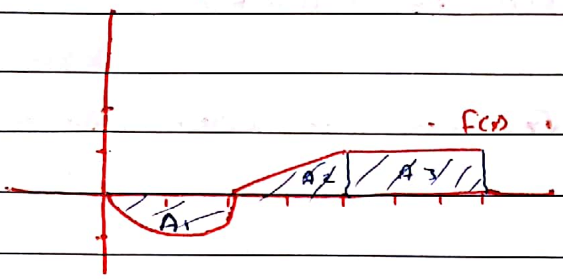
$\int_0^5 f(x)\,dx = \int_0^2 3\,dx + \int_2^5 2x\,dx$



$= 3(2-0) + 2\int_2^5 x\,dx$

$= 6 + 2[\overset{\text{المستطيل}}{(3 \times 2)} + \overset{\text{المثلث}}{(\frac{1}{2}(3)(3))}] = 6 + 12 + 9 = \boxed{27}$

Ex. given the graph of $f(x)$ below:



Evaluate:

$$\int_0^7 f(x)\,dx = \left[-\frac{1}{2}\pi(1)^2\right] + \left[\frac{1}{2}(2)(2)\right] + [3\times 2)$$

$$= -\frac{\pi}{2} + 2 + 6$$

$$= \frac{16-\pi}{2}$$

Ex. compute $\int_{2}^{2} \sin^{-1}x\,\sqrt{1+x^2}\,dx = 0$

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56 + 57
390

verify the following inequality without evaluating the integrals:

① $\int_0^1 \sqrt{1+x^2}\,dx \leq \int_0^1 \sqrt{1+x}\,dx.$

Sol:

$x^2 \leq x$ on $[0,1]$

$1+x^2 \leq 1+x$ on $[0,1]$

$\sqrt{1+x^2} \leq \sqrt{1+x}$ on $[0,1]$

$$\Rightarrow \int_0^1 \sqrt{1+x^2}\,dx \leq \int_0^1 \sqrt{1+x}\,dx$$

② $2 \leq \int_{-1}^{1} \sqrt{1+x^2} \leq 2\sqrt{2}$

Soln:

$-1 \leq x \leq 1 \rightarrow 0 \leq x^2 \leq 1$

$\rightarrow 1 \leq 1+x^2 \leq 2$

$\rightarrow 1 \leq \sqrt{1+x^2} \leq \sqrt{2}$

$\therefore 2 \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$

§ 5.4 Indefinite integral (Antiderivative)

Defn: $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$

Ex.

$F_1(x) = x^2$, $F_2(x) = x^2-1$, $F_3(x) = x^2+\sqrt{\pi 3}$, ... are called the antiderivative of $f(x) = 2x$

Defn: If $F(x)$ is an antiderivative of $f(x)$, then $F(x)+c$ is the general antiderivative of $f(x)$.

Defn: The indefinite integral of a function $f(x)$ is its general antiderivative and denoted by:

$$\int f(x)\,dx = F(x) + c$$

For example, $\int 2x\,dx = x^2 + c$

* Table of indefinite integrals:

① $\int \alpha\,dx = \alpha x + c$, ($\alpha$ is constant).

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $(n \neq -1)$

② $\int \frac{1}{x}\,dx = \ln|x| + c$

③ $\int b^x\,dx = \frac{b^x}{\ln b} + c$

$\int e^x\,dx = e^x + c$

④ $\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int \tan x\,dx = \ln|\sec x| + c$

$\int \cot x \, dx = \ln|\sin x| + c$

$\int \sec x \, dx = \ln|\sec x + \tan x| + c$

$\int \sec^2 x \, dx = \tan + c$

$\int \csc^2 x \, dx = -\cot x + c$

$\int \sec x \tan x \, dx = \sec x + c$

$\int \csc x \cot x \, dx = -\csc x + c$

⑤ $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c$

$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + c$

⑥ $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

⑦ $\int \sinh x \, dx = \cosh x + c$

$\int \cosh x \, dx = \sinh x + c$

Some properties of indefinite integral:

① $\int \alpha f(x) \, dx = \alpha \int f(x) \, dx$

② $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

③ $\int f(x) \cdot g(x)\, dx \neq \int f(x)\, dx \cdot \int g(x)\, dx$

④ $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$

⑤ $\int f(ax+b)\, dx = \frac{F(ax+b)}{a} + c$

Ex. Evaluate the integrals

① $\int [\sqrt{x} + \frac{1}{x^2} - 1]\, dx$

Sol: $= \int [x^{\frac{1}{2}} + x^{-2} - 1]\, dx$

$= \frac{x^{\frac{3}{2}}}{3/2} + \frac{x^{-1}}{-1} - x + c$

$= \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{x} - x + c$

② $\int \frac{x^2+1}{\sqrt{x}}\, dx = \int \frac{x^2}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}}\, dx$

$= \int x^{\frac{3}{2}} + x^{-\frac{1}{2}}\, dx$

$= \frac{x^{\frac{5}{2}}}{5/2} + \frac{x^{\frac{1}{2}}}{1/2} + c$

$= \frac{2}{5} x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + c$

③ $\int \sqrt{x}\, (x^2 + 2x + 1)\, dx$

$\int x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + x^{\frac{1}{2}}\, dx$

$= \frac{2}{7} x^{\frac{7}{2}} + \frac{4}{5} x^{\frac{5}{2}} + \frac{2}{3} x^{3/2} + c$

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H.W

④ $\int (x^2 + 2x + 1)^7 \, dx$

⑤ $\int e^x + x^e + e \, dx$

$e^x + \frac{x^{e+1}}{e+1} + ex + c$

⑥ $\int 4^x \ln 4 \, dx = \ln 4 \, \frac{4^x}{\ln 4} + c$

$= 4^x + c$

⑦ $\int \sqrt{e^x} + 3^{-x} \, dx = \int e^{\frac{1}{2}x} + 3^{-x} \, dx$

$= \frac{e^{\frac{1}{2}x}}{1/2} + \frac{\frac{3^{-x}}{\ln 3}}{-1} + c$

$= 2e^{\frac{1}{2}x} \cancel{x} - \frac{3^{-x}}{\ln 3} + c$

H.W

⑧ $\int \frac{e^x + 1}{e^x} \, dx$

⑨ $\int e^{3\ln x} + \ln e^{-x} \, dx$

$= \int x^3 - x \, dx$

$= \frac{x^4}{4} - \frac{x^2}{2} + c$

H.w

⑩ $\int \frac{e^x}{e^x+1} \, \partial x$

H.w

⑪ $\int \frac{x}{x+1} \, \partial x$

② $\int \frac{3}{2x-5} + e^{14-3x} \, \partial x$

$= \frac{3 \ln|2x-5|}{2} + \frac{e^{14-3x}}{-3} + c$

~~= e~~

Ex. Evaluate the ingral.

① $\int \cos^2 x \cdot \, \partial x = \frac{1}{2} \int \cancel{B} 1 + \cos 2x \, \partial x$

$= \frac{1}{2} [x + \frac{\sin 2x}{2}] + c$

H.w

② $\int \tan^2 x \, \partial 1$

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③ $\int \frac{1+\sin x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} dx$

$= \int \sec^2 x + \sec x \tan x \, dx$

$= \tan x + \sec x + c$

H.w

④ $\int \sec x \, [\sec x + \tan x + \cos x] \, dx$

⑤ $\int \frac{1}{1-\cos x} dx = \int \frac{1+\cos x}{1-\cos^2 x} dx$

$= \int \frac{1+\cos x}{\sin^2 x} dx = \int \csc^2 x + \csc x \cot x \, dx$

$= -\cot x - \csc x + c$

H.w

⑥ $\int \frac{dx}{4+4\sin x}$

Ex. Evaluate the integral.

① $\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + c$

② $\int \frac{1}{\sqrt{1-4x^2}} dx = \int \frac{1}{\sqrt{1-(2x)^2}} dx = \frac{\sin^{-1}(2x)}{2} + c$

③ $\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$

H.w
④ $\int \frac{1}{1+4x^2}\,dx$

H.w
⑤ $\int \frac{1+x}{1+x^2}\,dx$

⑥ $\int \cosh(3x-1)\,dx$

$= \frac{\sinh(3x-1)}{3} + c$

§ 5.3 The Fundamental theorm of calculus (FTC)

* The FTC, part ①:

If f is cont. on [a,b], then $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

Ex.

#9 399) $g(x) = \int_1^x \ln(1+t^2)\,dt \Rightarrow g'(x) = \ln(1+x^2)$

#11 399) $F(x) = \int_x^0 \sqrt{1+\sec t}\,dt = -\int_0^x \sqrt{1+\sec t}\,dt$

$\Longrightarrow F'(x) = -\sqrt{1+\sec x}$

* Remark:

① $\frac{d}{dx}\left[\int_a^{h(x)} f(t)\,dt\right] = f(h(x))\cdot h'(x)$

② $\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$

Ex.

#13 399) $h(x) = \int_1^{e^x} \ln(t)\,dt \Rightarrow h'(x) = \ln(e^x)\cdot e^x = xe^x$

#63 400) $y = \int_{\cos x}^{\sin x} \ln(1+2v)\,dv \Rightarrow y' = \ln(1+2\sin x)\cos x - \ln(1+2\cos x)\cdot -\sin x$

Ex. #67 401) : let $f(x) = \int_2^x e^{t^2}\,dt$ find an equation of the tangent line to the curve $y = f(x)$ at $x = 2$

* The FTC, part ② :

If f is cont. on $[a,b]$, then $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

where $F(x)$ is any anti derivative of $f(x)$

Ex. Evaluate the integrals:

29 / 400 $\int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx = \int_1^4 2x^{-\frac{1}{2}} + x^{3/2}\,dx$

$= 4\sqrt{x} + \frac{2}{5}\sqrt{x^5}\Big|_1^4$

$= \left(4(2) + \frac{2}{5}(2)^5\right) - \left(4(1) + \frac{2}{5}(1)\right)$

$= 8 + \frac{64}{5} - 4 - \frac{2}{5} = 4 + \frac{62}{5}$

\# 47 / 400 $\int_0^{\pi} f(x)\,dx$, where $f(x) = \begin{cases} \sin x , & 0 \leq x < \frac{\pi}{2} \\ \cos x , & \frac{\pi}{2} \leq x \leq \pi \end{cases}$

Solⁿ

$\int_0^{\pi} f(x)\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx + \int_{\frac{\pi}{2}}^{\pi} \cos x\,dx$

$= -\cos x\Big|_0^{\frac{\pi}{2}} + \sin x\Big|_{\frac{\pi}{2}}^{\pi}$

$= (0 - -1) + (0 - 1)$

$= 1 - 1$

$= \boxed{0}$

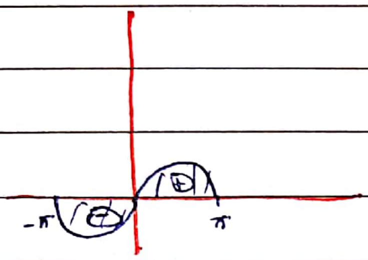
\# 45 / 409 $\int_{-1}^{2} (x - 2|x|)\,dx$ H.w

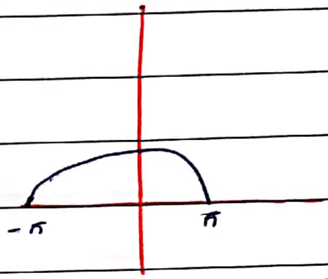
(من ‎-1‎ لـ 0 نعوض ‎-x‎)

ومن 0 لـ 2 نعوض x

Trick **Remark:** (Symmetry Property).

If f is cont. on ℝ $[-a,a]$ and $\begin{cases} \text{odd} \rightarrow \int_{-a}^{a} f(x)\,dx = 0 \\ \text{even} \quad \text{then} \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \end{cases}$





$$\int_{-\pi}^{\pi} \sin x \, dx = 0 \qquad \int_{-\pi}^{\pi} \cos x \, dx = 2\int_{0}^{\pi} \cos x \, dx$$

Ex. Evaluate $\int_{-1}^{1} \frac{\overset{\text{odd}}{\sin x}}{\underset{\text{even}}{\cosh x}} \, dx$ odd $= 0$

§ 5.5 The substitution Rule:

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

Ex. Evaluate the integral

① $\int e^x \cos(e^x)\,dx. = \int \cos u \, du$

$u = e^x$ $\qquad = \sin u + c$

$du = e^x\,dx$ $\qquad = \sin(e^x) + c$

② $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

Solve:

$u = \tan^{-1}x$

$du = \frac{1}{1+x^2}\,dx$

$\cancel{\int e^{u}}$

$\therefore \int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx = \int e^{u}\,du$

$= e^{u} + C$

$= e^{\tan^{-1}x} + C$

H.w

③ $\int \frac{\sin(\ln x)}{x}\,dx$

④ $\int \frac{x}{\sqrt{1-x^4}}\,dx$

⑤ $\int \frac{e^{x}}{1+e^{2x}}\,dx$

⑥ $\int \frac{e^{2x}}{e^{4x}+1}\,dx$

⑦ $\int \sin^2 x \cos x \, \partial x$

⑧ $\int \frac{\partial x}{\sqrt{4 - 12x^2}}$

⑨ $\int \frac{(\ln x)^2 \, \partial x}{x} = \int u^2 \, \partial x.$

$= \frac{u^3}{3} + c$

$u = \ln x$

$\partial u = \frac{1}{x} \partial x$

$= \frac{(\ln x)^3}{3} + c$

⑩ $\int e^x \sqrt{1 + e^x} \, \partial x = \int \sqrt{u} \, \partial u$

$= \frac{2}{3} u^{3/2} + c$

$u = 1 + e^x$

$\partial u = e^x \, \partial x$

$= \frac{2}{3} (1 + e^x)^{3/2} + c$

H.w

Ex. Evaluate $\int x \sqrt{x+1} \, \partial x$

Ex. Evaluate.

① $\int_1^2 (x+1)(x^2+2x)^3 \, dx$

$= \frac{1}{2}\int_1^2 (2x+2)(x^2+2x)^3 \, dx$

$u = x^2+2x$

$du = (2x+2)\, dx$

$x \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$

$u \begin{smallmatrix} 8 \\ 3 \end{smallmatrix}$

$\therefore \int_1^2 (x+1)(x^2+2x)^3 \, dx = \frac{1}{2}\int_3^8 u^3 \, du$

= [illegible]

$= \frac{1}{2} \frac{u^4}{4} \Big|_3^8$

$= \frac{8^4}{8} - \frac{3^4}{8}$

H.W

② $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} \, dx$

H.W

③ $\int_0^{\frac{\pi}{4}} \tan^5 x \sec^2 x \, dx$

H.W

④ $\frac{e^{1/x}}{x^2} \partial x$