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$\frac{26}{30}$

The University of Jordan

Department of Mathematics, Math 202, Second Exam

Name Number: Section: 8 - 9

Q1) (12points) Fill in the blank

$\frac{8}{12}$

1. The eigenvalues of $A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$ are $\lambda_1 = 0, \lambda_2 = 1$

$\begin{vmatrix} -1-\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda)+2 = -2+\lambda-2\lambda+\lambda^2+2$

$\Rightarrow \lambda^2 - \lambda = 0 \quad \lambda(\lambda-1) = 0 \quad \lambda_1 = 0, \lambda_2 = 1$

2. If $r(r-1) = 0$ is the indicial equation of an ODE and the solution corresponds to $r = 1$ is $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$, then the form of the second solution corresponds to $r = 0$ is $y_2 = \sum_{n=0}^{\infty} a y_1 \ln x \cdot x^n$

$r_1 - r_2 = I$

3. Consider $A = \begin{bmatrix} 2 & 2 & 0 \\ a & 1 & 1 \\ b & 0 & 1 \end{bmatrix}$, with $\det A = 12$ and $A^{-1} = \begin{bmatrix} \frac{1}{12} & x & \frac{1}{6} \\ \frac{5}{12} & \frac{1}{6} & \frac{-1}{6} \\ \frac{-1}{4} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Then $x = \frac{-2(24)}{4(72)}$

$\det A^{-1} = \frac{1}{\det A} = \frac{1}{12}$

$\frac{4}{24}x = \frac{-2}{72} \Rightarrow x = \frac{-2(24)}{4(72)}$

$\det A^{-1} = \frac{1}{12}\left(\frac{1}{12}+\frac{1}{12}\right) - x\left(\frac{5}{24}-\frac{1}{24}\right) + \frac{1}{6}\left(\frac{5}{24}+\frac{1}{24}\right)$

$\frac{1}{12} = \frac{1}{12}\left(\frac{2}{12}\right) - x\left(\frac{4}{24}\right) + \frac{1}{6}\left(\frac{6}{24}\right) \Rightarrow \frac{1}{12} = \frac{1}{72} - \frac{4}{24}x + \frac{1}{24} \Rightarrow \frac{4}{24}x = \frac{3\times 1}{3\times 24} + \frac{1}{72} - \frac{1\times 6}{12\times 6}$

$\frac{4}{24}x = \frac{-2}{72}$

4. If the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = 4$, then the corresponding eigenvector of $\lambda_1 = 0$ is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{vmatrix}$: $\lambda_1 = 0 \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x_1 + y_1 = 0$

$y_1 = -2x_1$

$V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

5. If the system $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & a & b \end{array}\right]$ has infinitely many solutions, then

$a =$ 2, and $b =$ 6.

$\xrightarrow{2R_1 - R_2} \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 2-a & 6-b \end{array}\right]$ $2 - a = 0$, $a = 2$ | $6 - b = 0$, $b = 6$

6. Let A be a 3×3 matrix such that $A^2 = A$. Let $B = 3A^{-1}A^TA$. Then

$\det B =$ $3 \det A$

$\det A = \det A^2 = (\det A)^2$ | $\det A^{-1} = \frac{1}{\det A}$, $\det A^T = \det A$

$\det B = 3\left(\frac{1}{(\det A)^2}\right)(\det A)^2 \cdot \det A$

~~$\det B = 3\left(\frac{1}{8\det A}\right)(8\det A)(8\det A)$~~

~~$= 24 \det A$~~

Q2) (6 points) Let A be 2×2 *matrix*. Consider the system:

$y' = Ay + \begin{bmatrix}1\\-2\end{bmatrix}e^t$. If $y_h = c_1\begin{bmatrix}1\\-1\end{bmatrix}e^t + c_2\begin{bmatrix}3\\-2\end{bmatrix}e^{2t}$, use the variation of parameters method to find the particular solution y_P.

$G(t) = \begin{bmatrix}e^t\\-2e^t\end{bmatrix}$

$\vec{y}_p = y\cdot\vec{u}$, $\vec{u} = \int_0^t y^{-1}\cdot G(s)\,ds$, $y = \begin{bmatrix}e^t & 3e^{2t}\\-e^t & -2e^{2t}\end{bmatrix}$

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$y^{-1} = \dfrac{1}{-2e^{3t}+3e^{3t}}\begin{bmatrix}-2e^{2t} & -3e^{2t}\\e^t & e^t\end{bmatrix} = \dfrac{1}{e^{3t}}\begin{bmatrix}-2e^{2t} & -3e^{2t}\\e^t & e^t\end{bmatrix} = \begin{bmatrix}-2e^{-t} & 3e^{-t}\\-e^{-2t} & -2e^{-2t}\end{bmatrix}$

$\vec{u} = \int_0^t \begin{bmatrix}-2e^{-t} & 3e^{-t}\\-e^{-2t} & -2e^{-t}\end{bmatrix}_{2\times2}\cdot\begin{bmatrix}e^t\\-2e^t\end{bmatrix}_{2\times1}ds = \int_0^t\begin{bmatrix}-2-6\\-e^{-t}+4\end{bmatrix}\cdot ds = \int_0^t\begin{bmatrix}-8\\-e^{-t}+4\end{bmatrix}\cdot ds$

$= \begin{pmatrix}-8s\\+e^{-s}+4s\end{pmatrix}\Big|_0^t \Rightarrow \vec{y}_p = \begin{bmatrix}e^t & 3e^{2t}\\-e^t & -2e^{2t}\end{bmatrix}_{2\times2}\cdot\begin{bmatrix}-8t\\e^{-t}+4t-1\end{bmatrix}_{2\times1}$

$= \begin{matrix}-8t-0\\e^{-t}+4t-(1)\end{matrix}$ $\qquad \vec{y}_p = \begin{bmatrix}-8te^t+3e^{2t}(e^{-t}+4t-1)\\8e^{2t}+-2e^{2t}(e^{-t}+4t-1)\end{bmatrix}$

$= \begin{bmatrix}-8t\\e^{-t}+4t-1\end{bmatrix}$

$16 \quad \frac{2}{32}$

Q3) (5 points) Consider the ODE $\underset{P(x)}{x(x^2-4)^2y''} + \underset{q(x)}{3(x-2)y'} + \underset{r(x)}{5y} = 0$. Find the singular points and determine their types.

$x(x^2-4)^2 = 0$

$x=0, x=2, x=-2$

$\boxed{1}\ \lim\limits_{x\to0}\dfrac{3(x-2)}{x(x^2-4)^2}\cdot x = \dfrac{-6}{16} < \infty$

$\lim\limits_{x\to0}\dfrac{5}{x(x^2-4)^2}\cdot x^2 = 0 < \infty$

$\Rightarrow \boxed{x=0 \text{ regular sing. Pt.}}$

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$\boxed{2}\ \lim\limits_{x\to2}\dfrac{3(x-2)}{x(x^2-4)^2}(x-2)$

$= \lim\limits_{x\to2}\dfrac{3(x-2)^2}{x(x^2-4)(x^2-4)}$

$= \lim\limits_{x\to2}\dfrac{3(x-2)^2}{x(x-2)^2(x+2)^2}$

$= \dfrac{3}{32}$

$= \dfrac{1}{8} < \infty$

$\lim\limits_{x\to2}\dfrac{5}{x(x^2-4)^2}(x-2)^2$

$\lim\limits_{x\to2}\dfrac{5(x-2)^2}{x(x-2)^2(x+2)^2}$

$= \dfrac{5}{32} < \infty$

$\Rightarrow \boxed{x=2 \text{ regular sing. Pt}}$

$\boxed{3}\ \lim\limits_{x\to-2}\dfrac{3(x-2)}{x(x^2-4)^2}(x+2)$

$= \lim\limits_{x\to-2}\dfrac{3(x-2)(x+2)}{x(x-2)^2(x+2)^2}$

$= \dfrac{3}{-2(-4)(0)}$

$= \infty$

$\Rightarrow \boxed{x=-2 \text{ is irregular sing. Pt.}}$

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Q4) (3 points) Let A be 2×2 matrix. Given that $y_h = c_1\begin{bmatrix}1\\1\end{bmatrix}e^t + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-t}$ be the homogenous solution of the system $y' = Ay + \begin{bmatrix}t-e^t\\2t\end{bmatrix}$, if the undetermined coefficients method is to be used, find the form of the particular solution of the given system.

$$y' = Ay + \begin{bmatrix}1\\2\end{bmatrix}t + \begin{bmatrix}-1\\0\end{bmatrix}e^t$$

$$y_p = \vec{u}t + \vec{V}e^t.t + \vec{X}e^t$$

$$y_p = \begin{bmatrix}u_1\\u_2\end{bmatrix}t + \begin{bmatrix}V_1\\V_2\end{bmatrix}e^t.t + \begin{bmatrix}X_1\\X_2\end{bmatrix}e^t$$

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Q5) (6 points) Find a power series solution for the ODE: $y'' + (x^2+1)y = 0$, about $x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (x^2+1)\sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} + \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow 2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$\Rightarrow 2a_2 + 6a_3 x + a_0 + a_1 x + \sum_{n=2}^{\infty} x^n\left((n+2)(n+1)a_{n+2} + a_{n-2} + a_n\right)$$

$a_2 = 0$ | $a_0 = 0$ | $(6a_3 + a_1)x = 0$, $6a_3 = -a_1$, $a_3 = \frac{-a_1}{6}$ | $(n+2)(n+1)a_{n+2} + a_{n-2} + a_n = 0$, $a_{n+2} = \frac{-a_n - a_{n-2}}{(n+2)(n+1)}$

α $a_0 \neq 0$ α $a_1 \neq 0$

$$\Rightarrow n=2: \; a_4 = \frac{-a_2 - a_0}{12} = 0$$

$$n=3: \; a_5 = \frac{-a_3 - a_1}{20} = \frac{\frac{a_1}{6} - a_1}{20}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$= a_1 x - \frac{a_1}{6} + \frac{a_2 - a_0}{12} + \frac{\frac{a_1}{6} - a_1}{20} + \dots$$

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