Partial

Fall017

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* Vector Fields:

ex) let $F: R^2 \to R^3$, be S.T:

* this is vector Field.

$$\underline{F}(x,y) = 2x\hat{i} + xy\hat{j} + 2\hat{k}$$

$(\hat{i}, \hat{j}, \hat{k}) \leftarrow$ يوجد

$$\to \underline{F}(1,1) = 2\hat{i} + \hat{j} + 2\hat{k} = \langle 2, 1, 2 \rangle$$

ex) Consider $F: R^2 \to R$

$$\underline{F}(x,y) = x^2 + y^2 + 5$$

* this is scaler Field

$$\underline{F}(2,1) = 4 + 1 + 5 = 10$$

ex) Consider $\phi(x,y,z) = x^2 + y^2 + yz$, Then:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \to \text{(Vector)}$$

$\nabla \to$ Del operator

$$\nabla\phi = 2x\hat{i} + (2y+z)\hat{j} + y\hat{k}$$

* Consider:

$$\underline{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$$

Then the divergence of $\underline{F}$ is given by:

$$\text{div } \underline{F} = \frac{dF}{dx} + \frac{dF}{dy} + \frac{dF}{dz} \to \text{(scaler)}$$

ex) let $\underline{F}_{(x,y,z)} = x^2 y \hat{i} + yz \hat{j} + x \hat{k}$, Then:

$\text{div } \underline{F} = 2xy + z + \text{zero} = 2xy + z. \rightarrow$ (scaler)

→ Remark:

$$\underline{F}_{(x,y,z)} = f_{(x,y,z)} \hat{i} + g_{(x,y,z)} \hat{j} + h(x,y,z) \hat{k}$$

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

$$\rightarrow \underline{\nabla} \cdot \underline{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \text{div } \underline{F}$$

$$* \quad \underline{\nabla} \cdot \underline{F} = \text{div } \underline{F}$$

let $\underline{F}, \underline{G}$ be a vector field , ϕ be a scaler field.

Then:

1. $\text{div}(k\underline{F}) = k \text{ div } \underline{F}$

2. $\text{div}(\underline{F} + \underline{G}) = \text{div } \underline{F} + \text{div } \underline{G}$

3. $\text{div}(\phi \underline{F}) = \phi \text{ div } \underline{F} + \nabla \phi \cdot \underline{F}$

ex) Derive Rule number 3 → in the next page:

Sol:

let $\underline{F} = f\hat{i} + g\hat{j} + h\hat{k}$, Then:

$$\phi \underline{F} = \phi f\hat{i} + \phi g\hat{j} + \phi h\hat{k}$$

$$\rightarrow \text{div}(\phi \underline{F}) = \frac{\partial}{\partial x}(\phi f) + \frac{\partial}{\partial y}(\phi g) + \frac{\partial}{\partial z}(\phi h)$$

$$= \phi f_x + \phi_x f + \phi g_y + \phi_y g + \phi h_z + \phi_z h$$

$$= \phi(f_x + g_y + h_z) + \nabla\phi \cdot \underline{F}$$

$$= \phi \, \text{div}\underline{F} + \nabla\phi \cdot \underline{F} \quad \#$$

* let $\underline{F}_{(x,y,z)} = f_{(x,y,z)}\hat{i} + g_{(x,y,z)}\hat{j} + h_{(x,y,z)}\hat{k}$, Then:

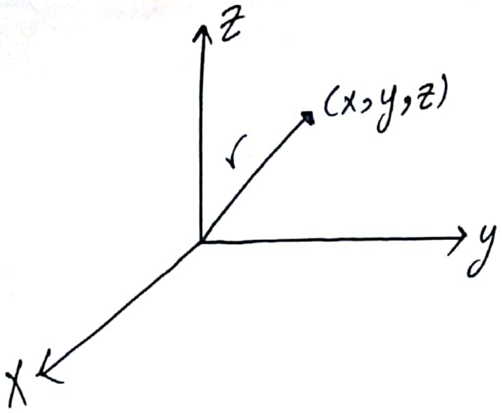
$\text{div}\,\underline{F} = f_x + g_y + h_z$.

& let $\phi_{(x,y,z)}$ be ascaler vector , Then:

$\nabla\phi = \phi_x\hat{i} + \phi_y\hat{j} + \phi_z\hat{k}$.

* div of avector give a scaler Field.

* ∇ of ascaler give a vector Field.



$$\left.\begin{aligned} \underline{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \underline{r} &= \langle x, y, z \rangle \\ \underline{r} &= (x, y, z) \end{aligned}\right] \rightarrow \text{Position vector.}$$

$\rightarrow \|\underline{r}\| = \sqrt{x^2+y^2+z^2}$, (norm) or (magnitude) $\rightarrow$ scaler

$\rightarrow$ if $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Then:

$r = \sqrt{x^2+y^2+z^2} \longrightarrow$ same $\|\underline{r}\|$

$$\rightarrow \frac{dr}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\rightarrow \frac{dr}{dy} = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$

$$\rightarrow \frac{dr}{dz} = \frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

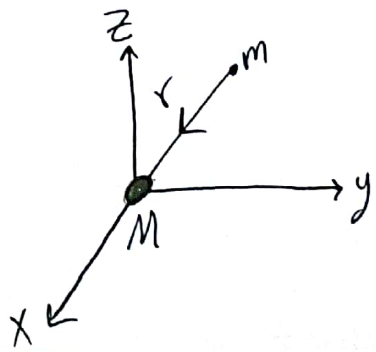
$* \; f_{(x,y,z)} = \left(\sqrt{x^2+y^2+z^2}\right)^3 \longrightarrow f_{(r)} = r^3$

Now: $\dfrac{d}{dx}(r^3) = 3r^2 \cdot \dfrac{x}{r} = 3xr$

$*$ Gravitational Field:

$$\underline{F} = -\frac{GMm}{r^2} \cdot \frac{\overline{r}}{r}$$

$$\underline{F} = \frac{C}{r^3} \cdot \overline{r}$$



ex) General case: Consider $\underline{F} = f(r)\,\bar{r}$,

show that: $\operatorname{div}\underline{F} = 3f(r) + r f'(r)$:

sol:

$$\underline{F} = f(r)\cdot x\hat{i} + y\hat{j} + z\hat{k} = x f(r)\,\hat{i} + y f(r)\,\hat{j} + z f(r)\,\hat{k}$$

$$\rightarrow \operatorname{div}\underline{F} = \frac{\partial}{\partial x}(x f(r)) + \frac{\partial}{\partial y}(y f(r)) + \frac{\partial}{\partial z}(z f(r)).$$

$$* \ \frac{\partial}{\partial x}(x f(r)) = x\cdot f'(r)\cdot\frac{x}{r} + f(r) = \frac{x^2 f'(r)}{r} + f(r)$$

$$* \ \frac{\partial}{\partial y}(y f(r)) = y\cdot f'(r)\cdot\frac{y}{r} + f(r) = \frac{y^2 f'(r)}{r} + f(r)$$

$$* \ \frac{\partial}{\partial z}(z f(r)) = z\cdot f'(r)\cdot\frac{z}{r} + f(r) = \frac{z^2 f'(r)}{r} + f(r)$$

Now $\rightarrow \operatorname{div}\underline{F} = \frac{\partial}{\partial x}(x f(r)) + \frac{\partial}{\partial y}(y f(r)) + \frac{\partial}{\partial z}(z f(r))$

r^2

$$\operatorname{div}\underline{F} = 3f(r) + \frac{x^2 f'(r) + y^2 f'(r) + z^2 f'(r)}{r}$$

$$\operatorname{div}\underline{F} = 3f(r) + \frac{r^2 f'(r)}{r} = 3f(r) + r f'(r) \quad \#$$

ex Consider $\underline{F} = \frac{c}{r^3}\underline{r}$, Find div $\underline{F}$?

<u>sol:</u>

$$\underline{F} = C\,\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} = \frac{Cx}{r^3}\hat{i} + \frac{Cy}{r^3}\hat{j} + \frac{Cz}{r^3}\hat{k}$$

$$\text{div}\,\underline{F} = \frac{\partial}{\partial x}\left(\frac{xC}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{Cy}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{Cz}{r^3}\right)$$

$$* \frac{\partial}{\partial x}\left(\frac{Cx}{r^3}\right) = \frac{\partial}{\partial x}\left(Cxr^{-3}\right) = Cx \cdot -3r^{-4} \cdot \frac{x}{r} + Cr^{-3}$$

$$= -3Cr^{-5}x^2 + Cr^{-3}$$

$$* \frac{\partial}{\partial y} = -3Cr^{-5}y^2 + Cr^{-3}$$

$$* \frac{\partial}{\partial z} = -3Cr^{-5}z^2 + Cr^{-3}$$

$$\text{Now} \rightarrow \text{div}\,\underline{F} = -3Cr^{-5}\left(\overbrace{x^2 + y^2 + z^2}^{r^2}\right) + 3Cr^{-3}$$

$$\text{div}\,\underline{F} = -3Cr^{-3} + 3Cr^{-3} = \text{Zero}.$$

$$* \frac{\partial}{\partial x}\left(f(r)\right) = f'(r) \cdot \frac{x}{r}$$

$$* \frac{\partial}{\partial x}\left(f(r^2)\right) = f'(r^2) \cdot 2r \cdot \frac{x}{r} = f'(r) \cdot 2x$$

$$* \|\underline{r}\| = r \longrightarrow \text{norm}$$

* Curl:

Consider : $\underline{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$

$$\text{div } \underline{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \longrightarrow \text{scaler}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \longrightarrow \text{vector}$$

$$\underline{F} = f\hat{i} + g\hat{j} + h\hat{k}$$

$$\rightarrow \underline{\nabla} \cdot \underline{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \text{div } \underline{F}$$

Now $\rightarrow$ let $\underline{F} = f\hat{i} + g\hat{j} + h\hat{k}$, Then:

$$\text{Curl } \underline{F} = \underline{\nabla} \times \underline{F} \longrightarrow (\text{vector Field})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

ex let $\underline{F} = (x^2 y)\hat{i} + (y^2)\hat{j} + (xz)\hat{k}$, Find Curl $\underline{F}$?

<u>sol:</u>

$$\text{Curl } \underline{F} = \underline{\nabla} \times \underline{F}$$

$$= (0-0)\hat{i} - (z-0)\hat{j} + (0-x^2)\hat{k}$$

$$= -z\hat{j} - x^2\hat{k}$$

ex let $\underline{F}$, $\underline{G}$ be two vector Fields, ϕ be a scaler Field.

Then:

1. $\text{Curl } (k\underline{F}) = k \text{ Curl } \underline{F}$
2. $\text{Curl } (\underline{F} + \underline{G}) = \text{Curl } \underline{F} + \text{Curl } \underline{G}$
3. $\text{Curl } (\phi \underline{F}) = \phi \text{ Curl } \underline{F} + \underline{\nabla} \phi \times \underline{F}$

   (مشتقة الاول * الثاني + الاول * مشتقة الثاني) ← الترتيب مهم

4. $\text{div } (\text{Curl } \underline{F}) = \text{Zero}$
5. $\text{Curl } (\nabla \phi) = \text{Zero}$

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ex) Consider the vector Field:

$\underline{F} = x\hat{i} + y\hat{j} + z\hat{k}$, Does there exist a vector Field $\underline{G}$ such that $\text{Curl } \underline{G} = \underline{F}$?

sol: Suppose there exist vector Field $\underline{G}$ such that

$$\text{Curl } \underline{G} = \underline{F}$$

Then take div to both sides:

$$\text{div}(\text{curl } \underline{G}) \stackrel{?}{=} \text{div} \underline{F}$$

$\text{zero} \stackrel{?}{=} 3 \longrightarrow$ No, Then there is No vector field $\underline{G}$ such that $\text{Curl } \underline{G} = \underline{F}$ #

* We know from calculas II:

$$ds = \sqrt{1 + f'(x)^2}\, dx \quad , L = \int_a^b ds = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

arc length

ex) Consider $y^2 = x^3$, Find the arc length from $(1,1)$ to $(4,8)$

$x_1 \; y_1$ $\quad x_2 \; y_2$

sol: $y = x^{\frac{3}{2}} \longrightarrow \frac{dy}{dx} = 3/2\, x^{\frac{1}{2}}$

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx = \int_1^4 \left(1 + \frac{9}{4}x\right)^{\frac{1}{2}} dx = \frac{\left(1 + \frac{9}{4}x\right)^{3/2}}{\frac{3}{2} \cdot \frac{9}{4}} \Bigg]_1^4$$

* $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

→ if we have a circle equation $x^2 + y^2 = 4$, then the paramatric equation:

$$\left.\begin{aligned} x(t) &= 2\cos t \\ y(t) &= 2\sin t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

Now → $ds = \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\, dx$

$$ds = \sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\, dx = \frac{\sqrt{x'(t)^2 + y'(t)^2}}{\frac{\cancel{dx}}{dt}}\, \cancel{dx}$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

and arc length: $L = \int_{t_1}^{t_2} ds = \int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}\, dt$

ex Consider $x(t) = 2\cos t$, $y(t) = 2\sin t$, $0 \le t \le 2\pi$,

Find the arc length?

sol: $L = \int_{t_1}^{t_2} ds = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt$

$= \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t} = \int_0^{2\pi} \sqrt{4} = 2(2\pi) = 12.56$

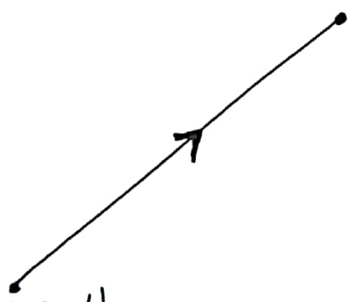
$\rightarrow x^2 + y^2 = 4$

$\rightarrow \frac{x^2}{4} + \frac{y^2}{4} = 1 \longrightarrow$ circle (special case from ellipse)

$$\left.\begin{aligned} x(t) &= 2\cos t \\ y(t) &= 2\sin t \end{aligned}\right\} \; 0 \le t \le 2\pi$$

* $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$
* $dx = x'(t)\, dt$
* $dy = y'(t)\, dt$
* $dz = z'(t)\, dt$

$B(2,3,7)$



$A(1,2,4)$

* parameterization for this line $= \boxed{A + (B-A)t}$

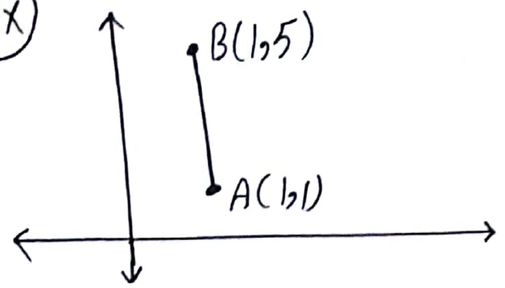
$= \langle 1,2,4 \rangle + \langle 1,1,3 \rangle t$

$= \langle 1+t,\, 2+t,\, 4+3t \rangle$

$$\left.\begin{aligned} x(t) &= 1+t \\ y(t) &= 2+t \\ z(t) &= 4+3t \end{aligned}\right\} \; 0 \le t \le 1$$

*أي parameterization بهذه الطريقة تكون $(0 \le t \le 1)$

ex



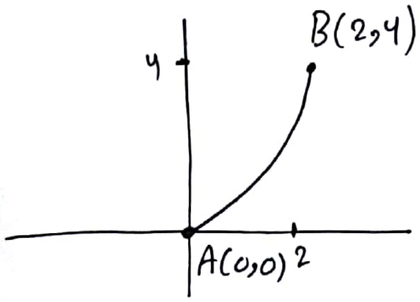
① $\langle 1,1\rangle + \langle 0,4\rangle t$

$= \langle 1, 1+4t\rangle$

$\left.\begin{aligned} x(t) &= 1 \\ y(t) &= 1+4t \end{aligned}\right\} \quad 0 \le t \le 1$

② $\left.\begin{aligned} x(t) &= 1 \\ y(t) &= t \end{aligned}\right\} \quad 1 \le t \le 5$ → (another parameterization)

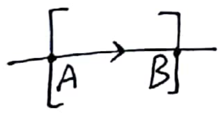
ex $y = x^2$



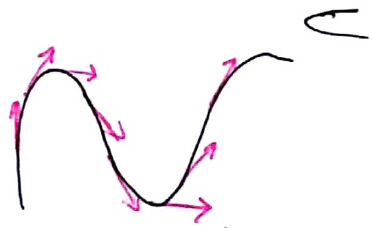
$\left.\begin{aligned} x(t) &= t \\ y(t) &= t^2 \end{aligned}\right\} \quad 0 \le t \le 2$

** line integral:

$\int_a^b f_{(x)}\,dx$ ⟶ one direction (x-axis)



$\int_C f_{(x,y,z)}\,ds$ ⟶ integral on curve



(on space)

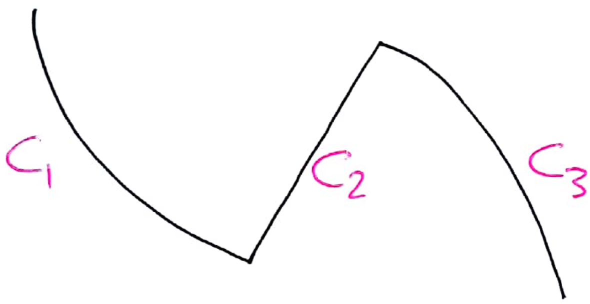
$$\int_C f_{(x,y,z)}\, ds = \int_C \underbrace{x(t)\, y(t)\, z(t)}_{\text{come from parameterization}}\, dx\, dy\, dz$$

C: Smoth Curve or (piece wise smoth)

↳ (Continous & diffrential)

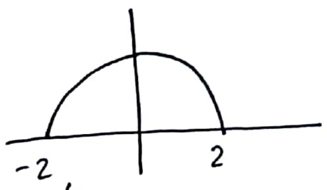
↳ finite Union of smoth curves

C:



$$C = C_1 + C_2 + C_3 = \int_{C_1} ds + \int_{C_2} ds + \int_{C_3} ds$$

ex) evaluate $\int_C [2xy+4]\, ds$ where C: is the upper half of the circle: $x^2+y^2=4$.



→ $x(t) = 2\cos t$, $y(t) = 2\sin t$ $\left\{ \boxed{0 \le t \le \pi} \right\}$ (upper half circle)

$$= \int_{t_1}^{t_2} 2(2\cos t)(2\sin t) + 4 \cdot \underbrace{\sqrt{x'(t)^2 + y'(t)^2}}\, dt$$

↳ $\sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$

→ follow

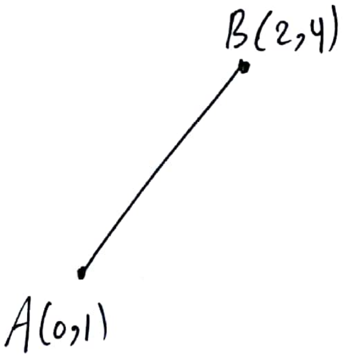
follow:

$$= \int_0^{\pi} \left[2 \cdot (2\cos t)(2\sin t) + 4 \right] \cdot 2\, dt$$

$$= \int_0^{\pi} 16 \cos t \sin t + 8\, dt = \int_0^{\pi} 16 \cos t \sin t\, dt + \int_0^{\pi} 8\, dt$$

$$= \underbrace{\cancel{\frac{16 \sin^2 t}{2}}\Bigg|_0^{\pi}}_{\text{zero}} + 8\pi = \text{Zero} + 8\pi = 8\pi$$

(ex) evaluate: $\int_C 2xy\,dy + y\,dx$ where C: is the line segment from $(0,1)$ to $(2,4)$



$$\Rightarrow (\langle 0,1 \rangle + \langle 2,3 \rangle t)$$

$$= \langle 2t,\ 1+3t \rangle$$

$$\left.\begin{aligned} x(t) &= 2t \\ y(t) &= 1+3t \end{aligned}\right\} \quad 0 \le t \le 1$$

Now $\rightarrow \int_{t_1}^{t_2} \left[2\, x(t)\, y(t)\, y'(t) + y(t)\, x'(t) \right] dt$

$\ast\ dx = x'(t)\,dt$

$\ast\ dy = y'(t)\,dy$

$$= \int_0^1 2(2t)(1+3t) \cdot 3 + (1+3t)(2)\,dt$$

$$= \int_0^1 12t + 36t^2 + 6t + 2\,dt = \frac{18t^2}{2} + \frac{36t^3}{3} + 2t \Bigg]_0^1$$

$$= (9+12+2) - (0+0+0) = 23$$

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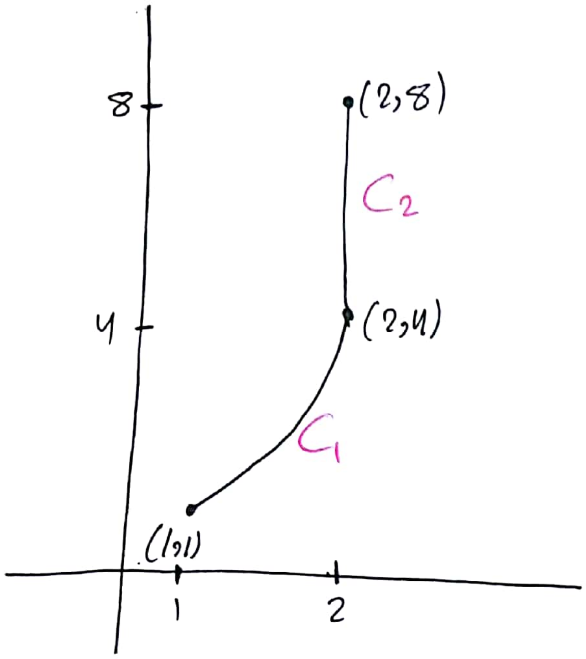
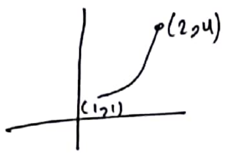
ex) evaluate $\int_C 2xy\,dy$ where C is the part of the parabola $y=x^2$ from $(1,1)$ to $(2,4)$ followed by the segment from $(2,4)$ to $(2,8)$? 15



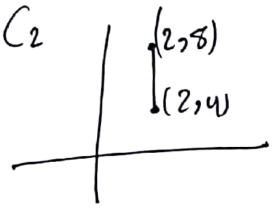
$$\int_C = \int_{C_1} ds + \int_{C_2} ds$$

for C_1:



$x(t) = t$
$y(t) = t^2$ $\Big\}$ $1 \le t \le 2$

for C_2



$= \langle 2,4 \rangle + \langle 0,4 \rangle t = \langle 2, 4+4t \rangle$

$x(t) = 2$
$y(t) = 4+4t$ $\Big\}$ $0 \le t \le 1$

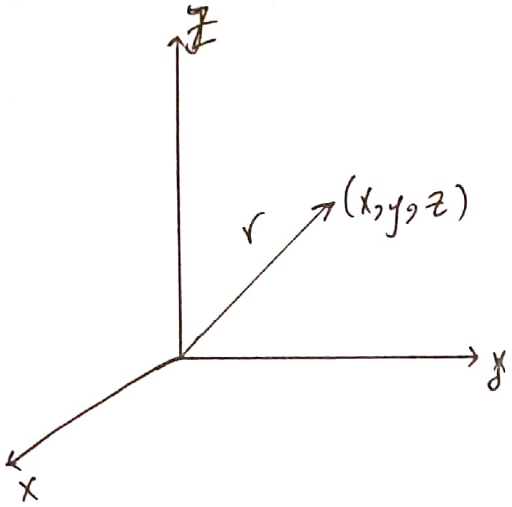
Now $\rightarrow \int_C = \int_{C_1} 2x(t)\,y(t)\,y'(t)\,dt + \int_{C_2} 2x(t)\,y(t)\,y'(t)\,dt$

$= \int_1^2 2(t)(t^2)(2t)\,dt + \int_0^1 2(2)(4+4t)(4)\,dt$

$= \int_1^2 4t^4\,dt + \int_0^1 64+64t\,dt$

$\frac{4t^5}{5}\Big|_1^2 + \left(64t + \frac{64t^2}{2}\Big|_0^1\right) = \left(\frac{128}{5} - \frac{4}{5}\right) + (64+32)$

* line integral :



$\underline{r} = (x, y, z)$

$\underline{r} = x\underline{\hat{i}} + y\underline{\hat{j}} + z\underline{\hat{k}}$

$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$

$\rightarrow \underline{F}(x,y,z) = f(x,y,z)\,\hat{i} + g(x,y,z)\,\hat{j} + h(x,y,z)\,\hat{k}$

$* \; \underline{F}(x,y,z) = \underline{F}(r(t)) = \underline{F}(x(t), y(t), z(t))$

$\Longrightarrow \int_c \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(r(t)) \cdot r'(t)\, dt$

(ex) evaluate $\int_c \underline{F} . d\underline{r}$, where : $\underline{F} = y\hat{i} + xz\hat{j} + y\hat{k}$,

$C : r(t) = \underbrace{2t}_{x(t)}\hat{i} + \underbrace{t}_{y(t)}\hat{j} + \underbrace{3}_{z(t)}\hat{k}$, $0 \le t \le 2$

sol:

$\underline{F}(r(t)) = [y(t), x(t)z(t), y(t)] = [t, 6t, t]$

$r'(t) = [2, 1, 0]$

follow

$\rightarrow F(r(t)) \cdot r'(t) = 2t + 6t + 0 = 8t$

$$\int_0^2 8t\,dt = 4t^2 \Big|_0^2 = 16 - 0 = 16$$

---

ex Evaluate : $\int \underline{F} \cdot d\underline{r}$, where:

$\underline{F} = [y, z, x]$

$C : r(t) = [\underbrace{\cos(t)}_{x(t)}, \underbrace{\sin(t)}_{y(t)}, \underbrace{2t}_{z(t)}]$

from $(1, 0, 0)$ to $(-1, 0, 2\pi)$

sol:

$\left.\begin{matrix} \cos(t) = 1 \\ \sin(t) = 0 \\ 2t = 0 \end{matrix}\right\} \rightarrow \boxed{t = 0}$, $\left.\begin{matrix} \cos(t) = -1 \\ \sin(t) = 0 \\ 2t = 2\pi \end{matrix}\right\} \rightarrow \boxed{t = \pi}$

$\rightarrow \underline{F}(r(t)) = [y(t), z(t), x(t)] = [\sin(t), 2t, \cos(t)]$

$\rightarrow r'(t) = [-\sin(t), \cos(t), 2]$ #

* $\underline{F}(r(t)) \cdot r'(t) = -\sin^2(t) + 2t\cos(t) + 2\cos(t)$

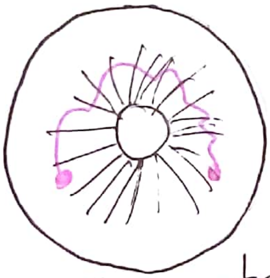
$$\rightarrow \int_C \underline{F} \cdot d\underline{r} = \int_0^{\pi} [-\sin^2(t) + 2t\cos(t) + 2\cos(t)]\,dt$$

Conservation Fields:

→ Independent of path → (Exact equation)

Def: $\underline{F}$ is Conservative: <u>if</u> there exist a scaler field ϕ • $\underline{F} = \nabla \phi$

simply connected (without holes)

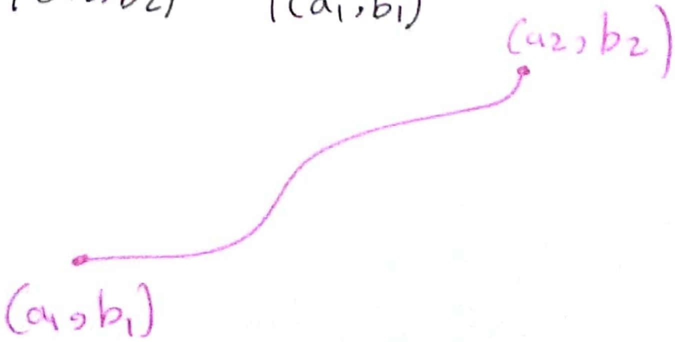


Connected (with holes)

→ **Theorem:** let $\underline{F}(x,y) = f(x,y)\,\underline{i} + g(x,y)\,\underline{j}$ [The region D is simply connected]

<u>If</u> $\dfrac{df}{dy} = \dfrac{dg}{dx}$ → Then $\underline{F}$ is Conservative

and $\int_c \underline{F}.d\underline{r} = \int_c \nabla\phi . dr = \phi_{(a_2,b_2)} - \phi_{(a_1,b_1)}$

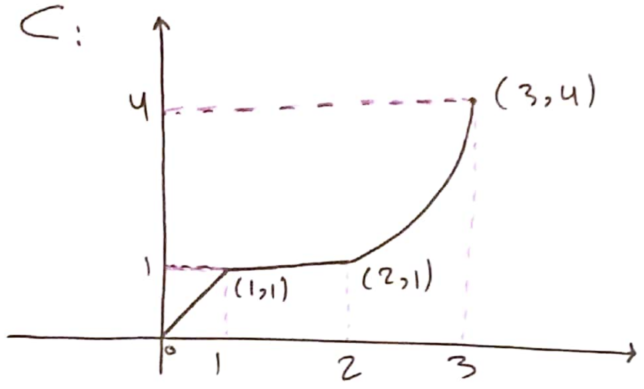


Scanned by CamScanner

ex Evaluate $\int_C \underline{F}\cdot d\underline{r}$

$$\underline{F} = [4xy^2+2]\,\underline{i} + [4x^2y]\,\underline{j}$$

C:



sol:

$$\left.\begin{aligned} \frac{df}{dy} &= 8xy \\ \frac{dg}{dx} &= 8xy \end{aligned}\right\} \underline{F} \text{ is Conservative.}$$

$$\rightarrow \underline{F} = \nabla\phi \rightarrow [4xy^2+2]\,\underline{i} + [4x^2y]\,\underline{j} = \frac{\partial\phi}{\partial x}\,\underline{i} + \frac{\partial\phi}{\partial y}\,\underline{j}$$

$$* \ \frac{\partial\phi}{\partial x} = 4xy^2+2 \quad , \quad * \ \frac{\partial\phi}{\partial y} = 4x^2y$$

$$\phi(x,y) = \int \frac{\partial\phi}{\partial x}\,dx = \int 4xy^2+2\,dx$$

$$\phi(x,y) = 2x^2y^2+2x+k(y)$$

→ 20

#follow

$\boxed{\text{Diff w.r.to } y}$

$$\frac{\partial \phi}{\partial y} = 4x^2 y + k'(y)$$

$$\rightarrow 4x^2 y = 4x^2 y + k'(y)$$

so: $k'(y) = \text{zero} . \longrightarrow \int k'(y) = \int \text{zero} = C = k(y)$

$$\boxed{\phi_{(x,y)} = 2x^2 y^2 + 2x + C}$$

$$\int_c \underline{F} \cdot dr = \int_c \nabla \phi \cdot dr$$

$$= \phi(3,4) - \phi(0,0) = (2)(9)(16) + (6) - \cancel{(c)} + \cancel{(c)}$$

$$= 288 + 6 = 294$$

OR $\underline{F} = [4xy^2 + 2]\,i + [4x^2 y]\,j$

$$\int \quad dx \qquad\qquad \int dy$$

Then: $\phi(x,y) = \int \; dx + \int \; dy \longrightarrow$ (من غير تكرار).

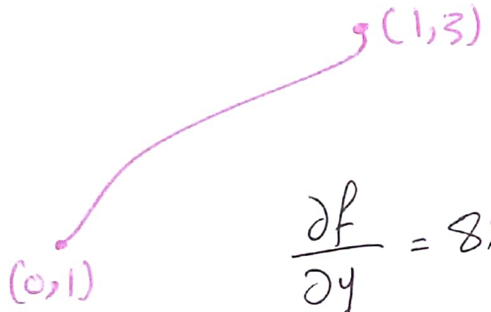
$$\underline{F}(x,y) = f(x,y)\,\underline{\hat{i}} + g(x,y)\,\underline{\hat{j}} \quad , \quad \underline{r} = x\underline{\hat{i}} + y\underline{\hat{j}}$$

$$d\underline{r} = dx\,\underline{\hat{i}} + dy\,\underline{\hat{j}}$$

$$\underline{F} \cdot d\underline{r} = f(x,y)\,dx + g(x,y)\,dy$$

$$\int_c \underline{F} \cdot d\underline{r} = \int f(x,y)\,dx + g(x,y)\,dy$$

ex Evaluate : $\displaystyle\int_{(0,1)}^{(1,3)} (2xy^4)\,dx + (4x^2y^3 - 4)\,dy$



* اذا ما حدد المسار بنتأكد انو conservative .
* اذا طلع conservative ما بعتمد على شكل المسار .

$$\left.\begin{aligned} \frac{\partial f}{\partial y} &= 8xy^3 \\ \frac{\partial g}{\partial x} &= 8xy^3 \end{aligned}\right\} \quad \begin{aligned} &\underline{F} \text{ is conservative.} \\ &\underline{F} = \nabla\phi \end{aligned}$$

$$\int_{(0,1)}^{(1,3)} \underline{F} \cdot d\underline{r} = \int_{(0,1)}^{(1,3)} \nabla\phi \cdot d\underline{r} = \phi(1,3) - \phi(0,1)$$

$$\rightarrow \int 2xy^4\,dx = x^2y^4$$

$$\int 4x^2y^3 - 4\,dy = x^2y^4 - 4y$$

$$\phi(x,y) = x^2y^4 - 4y$$

* let $\underline{F}(x,y,z) = f_{(x,y,z)}\underline{i} + g_{(x,y,z)}\underline{j} + h_{(x,y,z)}\underline{k}$

IF $\boxed{\text{Curl } \underline{F} = \text{zero}}$, then $\underline{F}$ is Conservative.

* $\text{Curl } \underline{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$

* $\underline{F}$ Conservative, then:

① $\int_c \underline{F} \cdot d\underline{r} = \int_c \nabla\phi \cdot d\underline{r} = \phi_B - \phi_A$

② $\oint_c \underline{F} \cdot d\underline{r} = \text{zero}$ (Closed Curve)

ex) Evaluate: $\int_c 3x^2\,dx + 2yz\,dy + y^2\,dz$

from $A:(0,1,2)$ to $B:(1,-1,7)$

sol:

$\text{Curl } \underline{F} = (2y - 2y)\underline{i} - (0-0)\underline{j} + (0-0)\underline{k} = \text{zero}$

$\rightarrow$ Curl $\underline{F}$ = zero $\rightarrow$ So $\underline{F}$ is Conservative.

$\int_A^B \underline{F} \cdot d\underline{r} = \phi_{(1,-1,7)} - \phi_{(0,1,2)}$

$x^3 \rightarrow \int dx$

$y^2 z \rightarrow \int dy$

$y^2 z \rightarrow \int dz$

$\phi_{(x,y,z)} = x^3 + y^2 z$

ex Evaluate:

$$\int \underbrace{\frac{-y}{x^2+y^2}}_{f(x,y)} \, dx + \underbrace{\frac{x}{x^2+y^2}}_{g(x,y)} \, dy$$

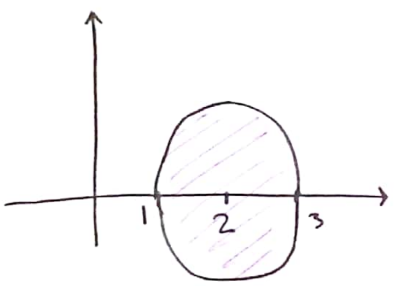
Where:

① C: # $\boxed{(x-2)^2+y^2=1}$

② C: # $\boxed{x^2+y^2=1}$

sol:

①



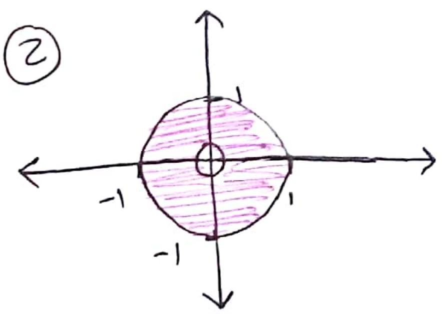
$$* \quad \frac{df}{dy} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$* \quad \frac{dg}{dx} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\oint \underline{F} \cdot d\underline{r} = 0 \implies \frac{df}{dy} = \frac{dg}{dx}$ / holes ما في $\implies$ Conservative

---

②



$x^2+y^2=1 \longrightarrow x(t)=\sin(t), \quad y(t)=\cos(t)$

$0 \le t \le 2\pi$

$$\int \left[\frac{-y(t)}{x^2(t)+y^2(t)} x'(t) + \frac{x(t)}{x^2(t)+y^2(t)} y'(t) \right] dt$$

$$= \int_0^{2\pi} \left[\frac{-\sin(t)}{1} (-\sin(t)) + \frac{\cos(t)}{1} \cos(t) \right] dt = \int_0^{2\pi} 1 \, dt = 2\pi$$

*هاد ما بحله مثل الطريقة ① لأنه في holes #

Green's theorem:

- +ve oriented (C.C.W)

(لما نمشي يكون ال Region على الشمال)

- f, g, and their first partial derivatives are continuous. Then:

$$\oint_C f\,dx + g\,dy = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

dA → $dx\,dy$ / $dy\,dx$

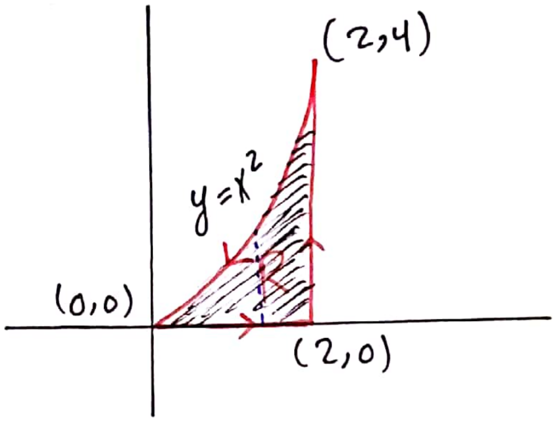
---

ex1 evaluate $\oint_C \big[\underbrace{x^2+y^3}_{f}\big]\,dx + \big[\underbrace{3xy^2}_{g}\big]\,dy$,

where C: consists of the portion of $y=x^2$ from $(2,4)$ to $(0,0)$ followed by the segments from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(2,4)$.

→ sol in page 25

sol:



اذا نبلاقي ال curve مغلق بدون holes ← نطبق (green's theorem)

$$\oint \underline{F} \cdot d\underline{r} = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

$$= \iint_R \left[3y^2 - 3y^2\right] dA = \text{Zero} \;\therefore (\text{conservative})$$

لو ما كان (zero) :

$$\int_0^2 \int_0^{x^2} [\;\cdots\cdots\;]\, dy\, dx$$ (نفضل ان يكون x ثابت و y متغير)

---

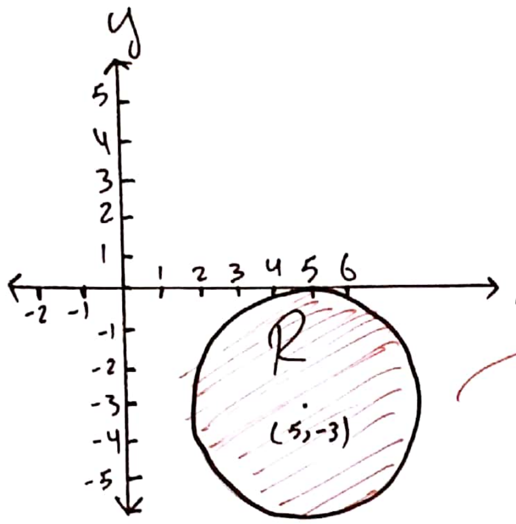
ex2 evaluate $\oint_c \big[\underbrace{7y - e^{\sin x}}_{f}\big]\, dx + \big[\underbrace{15x - \sin(y^3 + 8y)}_{g}\big]\, dy$

where $C: (x-5)^2 + (y+3)^2 = 9$

circle with raduis $= 3$ & center $= (5,-3)$

→ Sol in page 26

Sol:



- closed & no holes:

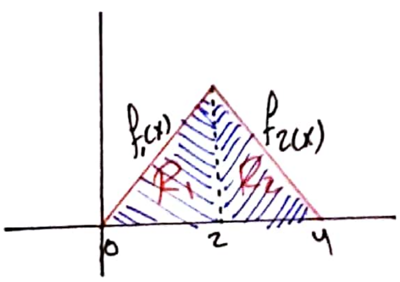
→ Apply green's theorem:

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA$$

$$= \iint_R [15 - 7]\, dA = 8 \iint_R dA$$

area of the region = area of circle with raduis = 3 = 9π

→ $= 8(9\pi) = 72\pi$



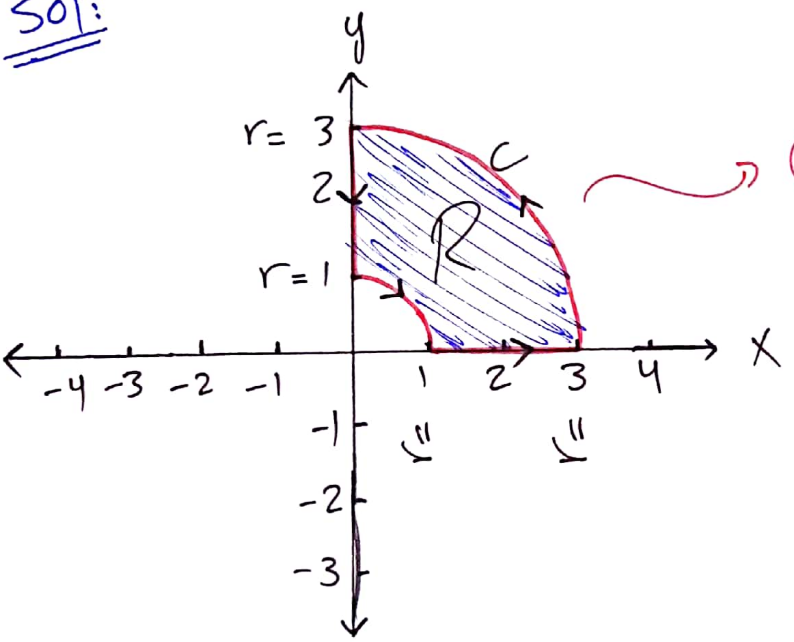
- في مثل هذه الحالة يكون الحل كالتالي :

$$\int_0^2 \int_0^{f_1(x)} [\ldots]\, dA + \int_2^4 \int_0^{f_2(x)} [\ldots]\, dA$$

ex3 Evaluate $\oint [\underbrace{e^x + 6yx}_{f}]\,dx + [\underbrace{8x^2 + \sin y^2}_{g}]\,dy$, (27)

where C: is the boundary of the region enclosed by the Circles of raduises (1, 3) centered at the orgion and lying in the first quadrant.

Sol:



Closed → (green's theorem)

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

$$= \iint_R [16x - 6x]\,dA = \iint_R 10x\,dA$$

Use poler coordenate : $x = r\cos\theta$, $y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_1^3 10\, r\cos\theta \cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} 10\cos\theta \boxed{\int_1^3 r^2\,dr}\, d\theta$$

$\boxed{\int_1^3 r^2\,dr} = \frac{26}{3}$

$$= \frac{260}{3} \underbrace{\int_0^{\frac{\pi}{2}} \cos\theta}_{=1} = \frac{260}{3}$$

note:

[1] $\int_{-C} f(x,y,z)\,ds = \int_C f(x,y,z)\,ds$

[2] $\int_{-C} f(x,y,z)\,dx = -\int_C f(x,y,z)\,dx$

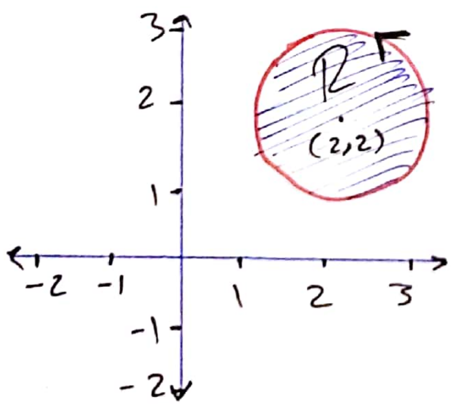
[3] $\int_{-C} \underline{F}\cdot d\underline{r} = -\int_C \underline{F}\cdot d\underline{r}$

ex evaluate:

$$\int_C \underbrace{\frac{-y}{x^2+y^2}}_{f}\,dx + \underbrace{\frac{x}{x^2+y^2}}_{g}\,dy \quad , \text{ where:}$$

(1) $C: (x-2)^2 + (y-2)^2 = 1$

sol:



$$\int_C f\,dx + g\,dy$$

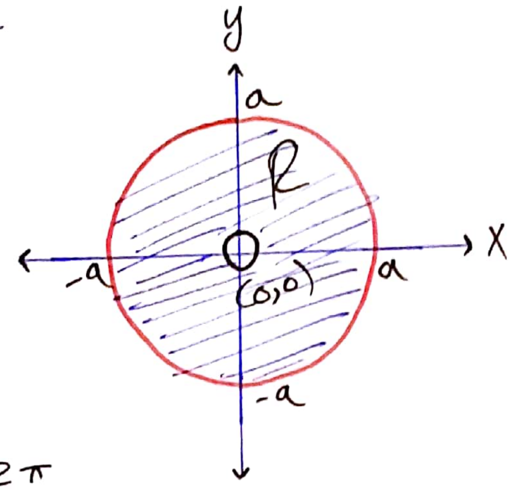
$$= \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

$$= \text{Zero}$$

→ follow

② $C: y^2 + x^2 = a^2$

sol:



$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}\right\} 0 \le t \le 2\pi$$

ما بطبق green's theorem
لأنو في holes

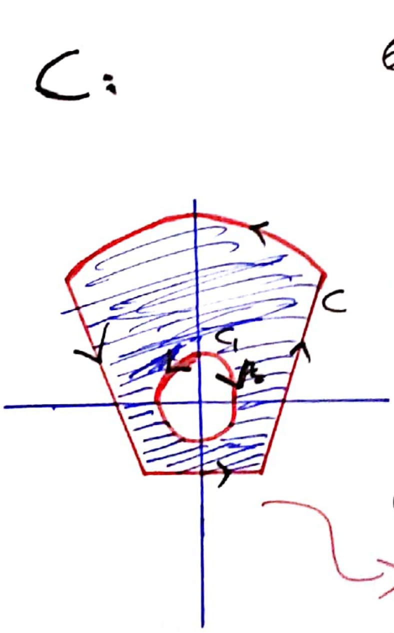
$$= \int_0^{2\pi} \left[\frac{-y(t)}{x^2(t)+y^2(t)} \cdot x'(t) + \frac{x(t)}{x^2(t)+y^2(t)} \cdot y'(t) \right] dt$$

$$= \int_0^{2\pi} \left[\frac{-a\sin t \cdot a\sin t}{a^2} + \frac{a\cos t \cdot a\cos t}{a^2} \right] dt$$

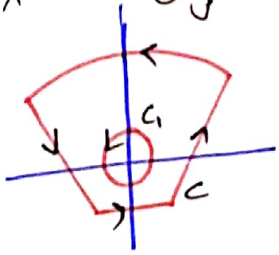
$$= 2\pi$$ → for any circle with any raduis.

③ C:

① $\displaystyle\int_C - \int_{C_1} = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA = 0$



② $\displaystyle\int_C = \int_{C_1} = 2\pi$

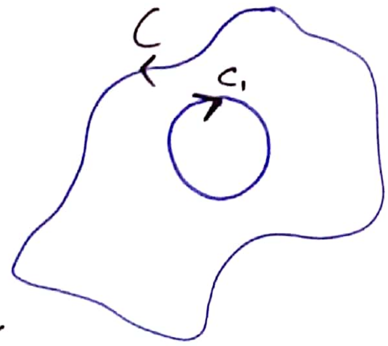


③ التكامل على اي دائرة مركزها $(0,0) = 2\pi$

③ $\displaystyle\int_C + \int_{C_1} = \text{zero} \longrightarrow \int_C = -\int_{C_1}$

- Remark:

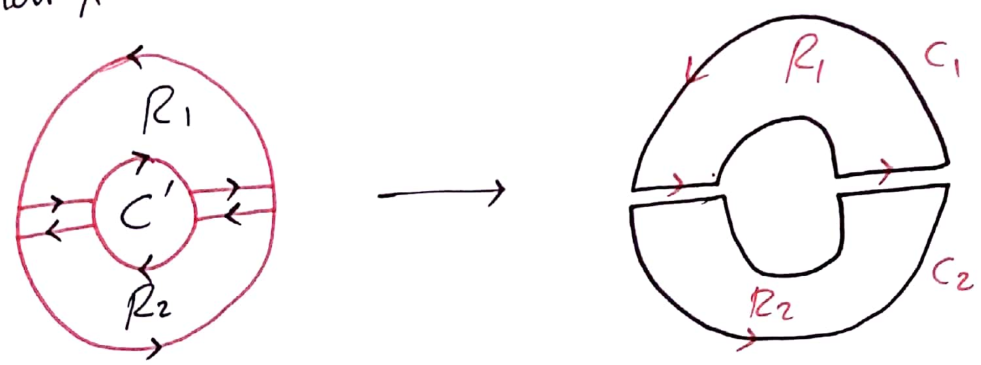
$$\int_C = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \rightarrow \text{for simply connected.}$$



→ not simply connected

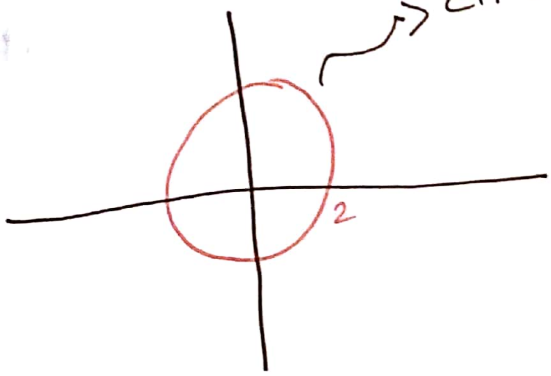
$$\int_C + \int_{C_1} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \rightarrow \text{for not simply connected.}$$

- Remark



$$C_1 = \iint_{R_1} \qquad C_2 = \iint_{R_2}$$

$$\int_{C_1} + \int_{C_2} = \iint_R \rightarrow \int_C + \int_{C'} = \iint_R$$



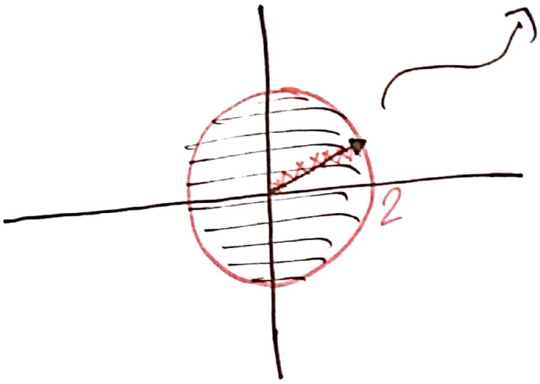
circle

$x^2+y^2=4$

$x(t)=2\cos t$

$y(t)=2\sin t$

$\boxed{0\le t\le 2\pi}$



Disk (surface)

$x^2+y^2=4$

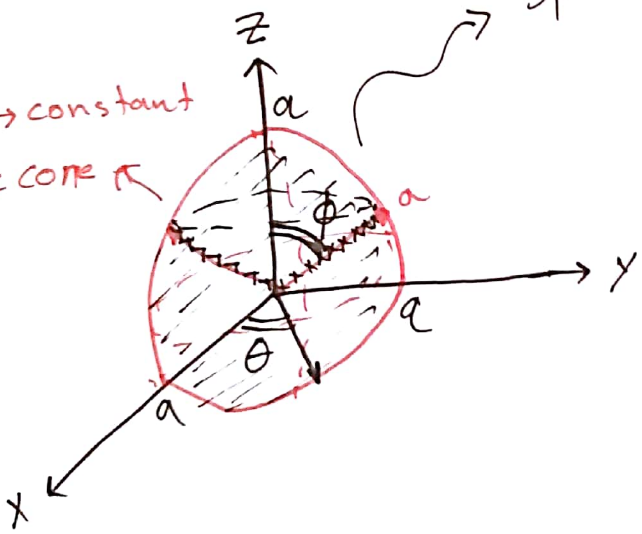
$x=r\cos\theta$

$y=r\sin\theta$

$\left\{\begin{array}{l} 0\le\theta\le 2\pi \\ 0\le r\le 2 \end{array}\right.$

$\phi \rightarrow$ constant

Give cone



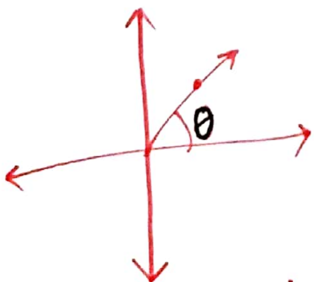
sphere (volume)

$x^2+y^2+z^2=a^2$

$x=\rho\cos\theta\sin\phi$

$y=\rho\sin\theta\sin\phi$

$z=\rho\cos\phi$



$0\le\rho\le a$

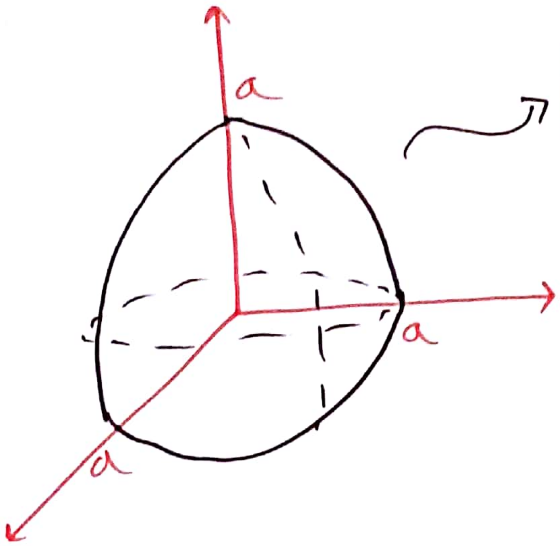
$0\le\theta\le 2\pi$

$0\le\phi\le\pi$

لما نسقط نقطة على

plane xy ← θ = الزاوية

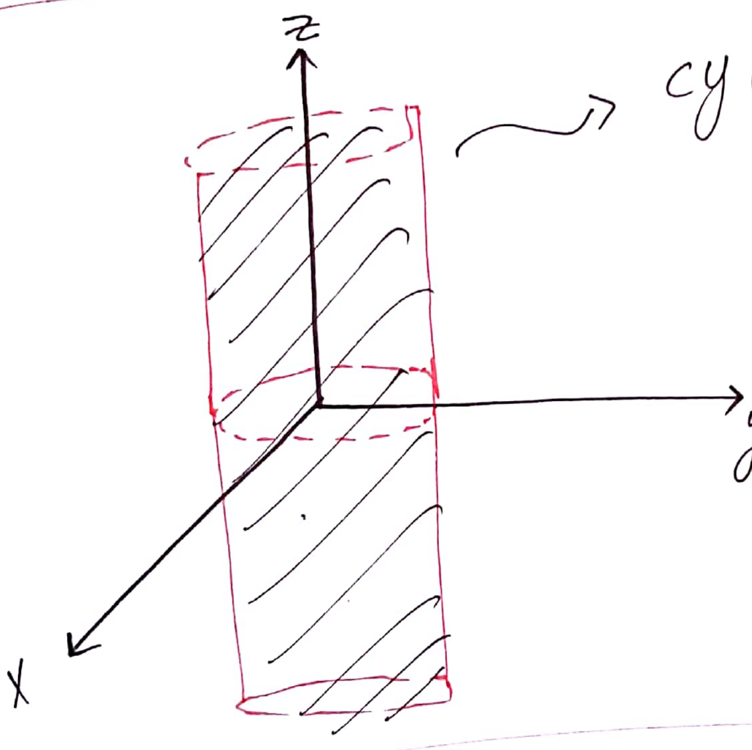
بين النقطة و X



(surface hollow sphere)

$x^2+y^2+z^2=a^2$

$$\boxed{\begin{aligned} x &= a\cos\theta\sin\phi \\ y &= a\sin\theta\sin\phi \\ z &= a\cos\phi \end{aligned}}$$



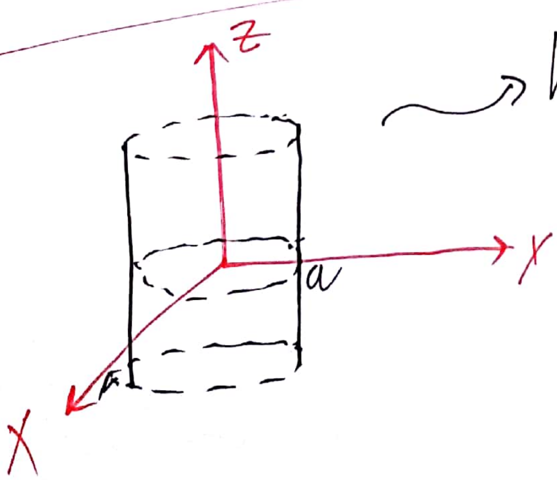
cylinder (solid)

$x^2+y^2=a^2$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$



hollow cylinder (surface)

$x^2+y^2=a^2$

$x = a\cos\theta$

$y = a\sin\theta$

$z = z$

for a hollow sphere (surface)

$x = a\cos\theta\sin\phi$

$y = a\sin\theta\sin\phi$

$z = a\cos\phi$

we know:

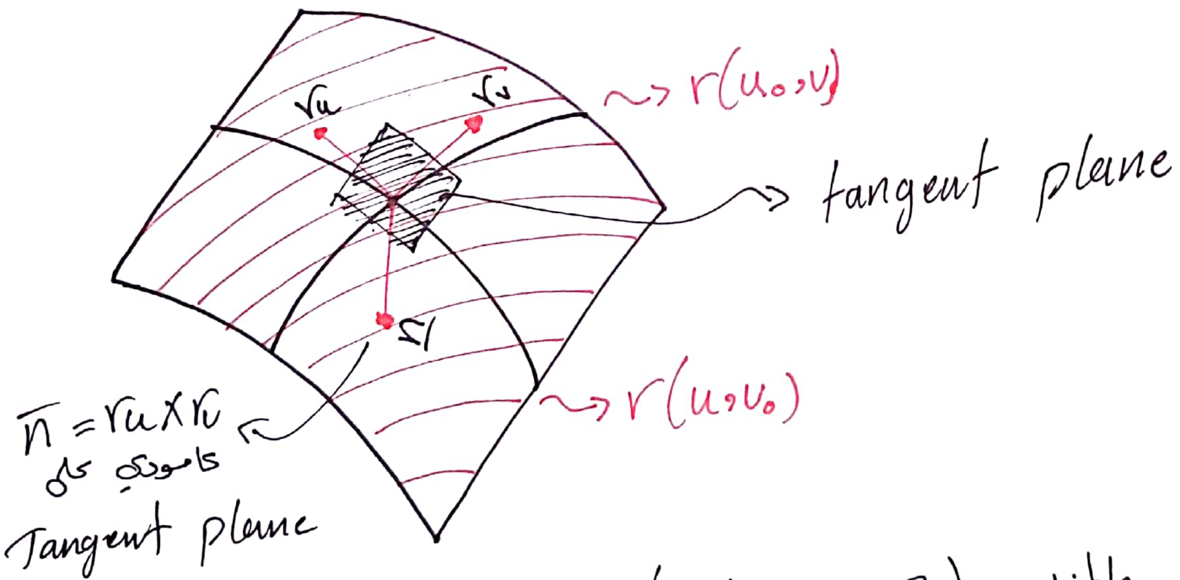
$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$\underline{r} = x(\theta,\phi)\underline{i} + y(\theta,\phi)\underline{j} + z(\theta,\phi)\underline{k}$

→ <u>in general:</u>

we can parameterize a surface as:

$$\underline{r}(u,v) = x(u,v)\underline{i} + y(u,v)\underline{j} + z(u,v)\underline{k}$$



the tangent plane at (x_0,y_0,z_0), with $\underline{n}(a,b,c) =$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Remark → $\underline{n} = r_u \times r_v$

Ex: Find the tangent plane of:

$$\boxed{r(u,v) = u^2 \underline{i} + v^2 \underline{j} + (u+2v)\underline{k}} \text{ at } (1,1,3)$$

sol:

$$a(x-1) + b(y-1) + c(z-3) = 0$$

$\rightarrow$ we know $n = r_u \times r_v \rightarrow$ Find $(r_u), (r_v)$

$$r_u = 2u\hat{i} + 0\hat{j} + \underline{k} = [2u, 0, 1]$$

$$r_v = 0\hat{i} + 2v\hat{j} + 2\underline{k} = [0, 2v, 2]$$

$$n = r_u \times r_v = \begin{vmatrix} \hat{i} & -\hat{j} & \underline{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = (-2v)\underline{i} - (4u)\hat{j} + (4uv)\underline{k}$$

$$= \underbrace{[-2v, -4u, 4uv]}$$

نعوض النقطة الي معطينا ياها بالسؤال

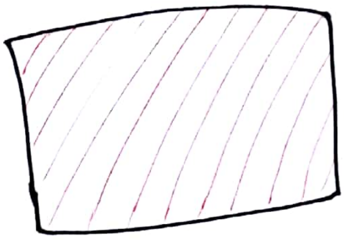
$r(u,v) = u^2 \hat{i} + v^2 \hat{j} + (u+2v)\underline{k}$ $\Longleftarrow$

at $(1,1,3)$

$$\left.\begin{matrix} u^2 = 1 \\ v^2 = 1 \\ u + 2v = 3 \end{matrix}\right\} \rightarrow \begin{matrix} u = 1 \\ v = 1 \end{matrix}$$

Now $\rightarrow N = [-2, -4, 4]$

tangent plane $= -2(x-1) - 4(y-1) + 4(z-3) = 0$



S , σ $\longrightarrow$ (surface المقصود فيهم)

$$\underline{r}(u,v) = x(u,v)\,\underline{i} + y(u,v)\,\underline{j} + z(u,v)\,\underline{\hat{k}}$$

$\rightarrow$ r_u , r_v : normal vector

---

sphere : $x^2 + y^2 + z^2 = a^2$

* $x = a\cos\theta\sin\phi$
* $y = a\sin\theta\sin\phi$
* $z = a\cos\phi$

$$\underline{r}(\phi,\theta) = x(\phi,\theta)\,\underline{\hat{i}} + y(\phi,\theta)\,\underline{\hat{j}} + z(\phi,\theta)\,\underline{\hat{k}}$$

$$= (a\cos\theta\sin\phi)\,\underline{\hat{i}} + (a\sin\theta\sin\phi)\,\underline{\hat{j}} + (a\cos\phi)\,\underline{\hat{k}}$$

$$= \langle a\cos\theta\sin\phi ,\ a\sin\theta\sin\phi ,\ a\cos\phi \rangle$$

$\rightarrow r_\phi = \langle a\cos\theta\cos\phi ,\ a\sin\theta\cos\phi ,\ -a\sin\phi \rangle$

$\rightarrow r_\theta = \langle -a\sin\theta\sin\phi ,\ a\cos\theta\sin\phi ,\ 0 \rangle$

$\underline{n} = r_\phi \times r_\theta =$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \end{vmatrix} \Longrightarrow$$

follow:

$$\underline{n} = (0 + a^2\cos\theta\sin^2\phi)\underline{i} + (0 + a^2\sin\theta\sin^2\phi)\underline{j} +$$

$$(\underline{a^2\cos^2\theta\sin\phi\cos\phi + a^2\sin^2\theta\sin\phi\cos\phi})\underline{k}$$

$$= \underline{a^2\sin\phi\cos\phi}\,(\underbrace{\sin^2\theta + \cos^2\theta}_{1})$$

$\rightarrow$

$$\underline{n} = (a^2\cos\theta\sin^2\phi)\underline{i} + (a^2\sin\theta\sin^2\phi)\underline{j} + (a^2\sin\phi\cos\phi)\underline{k}$$

$$\underline{n} = \underline{r_\phi \times r_\theta} = \langle a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\sin\phi\cos\phi\rangle$$

في حالة sphere

هذه الصيغة ليست حفظ ☹

طول ال (vector) العاسوري

$$\rightarrow \|r_\phi \times r_\theta\| = \sqrt{a^4\cos^2\theta\sin^4\phi + a^4\sin^2\theta\sin^4\phi + a^4\sin^2\phi\cos^2\phi}$$

$$= a^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$$

$$= a^2\sqrt{\sin^2\phi(\underbrace{\sin^2\phi + \cos^2\phi}_{1})}$$

$$= a^2\sqrt{\sin^2\phi}$$

$$= a^2\sin\phi$$

حفظ

- $S: \underline{r}(u,v) = x(u,v)\,\underline{i} + y(u,v)\,\underline{j} + z(u,v)\,\underline{k}$

- $\underline{dS} = \|r_u \times r_v\|\, du\, dv \longrightarrow$ هنا القانون لأي plane وليس للـ x-y plane فقط.

↳ وحدة المساحة

- $S = z = f(x,y) \longrightarrow z - f(x,y) = 0 = G$

- $dS = \|\nabla G\|\, dx\, dy$

→ $\nabla G = \langle -f_x, -f_y, 1 \rangle \Rightarrow \|\nabla G\| = \sqrt{f_x^2 + f_y^2 + 1}$

→ $dS = \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy$ (حفظ)

$$\boxed{* \; S = \text{surface}, \quad \text{Area}(S) = \iint_S dS}$$

$S: y = f(x,z)$

→ $dS = \sqrt{f_x^2 + f_z^2 + 1}\; dx\,dz$

ex: Find the surface area of:

$S = x^2 + y^2 + z^2 = a^2$ (sphere).

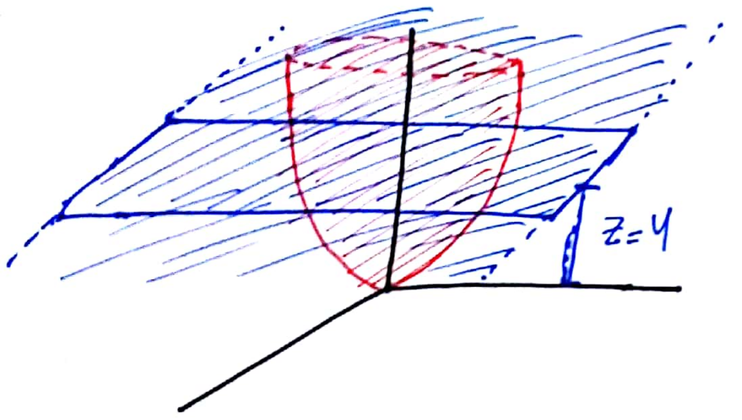
sol:

$$\text{Area} = \iint_S dS = \iint \|r_\phi \times r_\theta\|\, d\phi\, d\theta$$

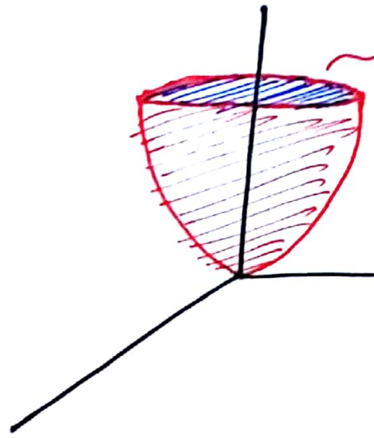
$$= \int_0^{2\pi}\int_0^{\pi} \underbrace{a^2 \sin\phi}_{\|r_\phi \times r_\theta\|}\, d\phi\, d\theta = a^2 \int_0^{2\pi} \cos\phi \Big|_{\pi}^{0}\, d\theta = \int_0^{2\pi} 2a^2\, d\theta$$

$$\boxed{= 4\pi a^2}$$

Ex: evaluate the surface area of the path of $z = x^2 + y^2$ that lies under $z = 4$. (paraboloid)



$\longrightarrow$



Sol:

$x^2 + y^2 = 4 \longrightarrow$ (التقاطع)

$z = x^2 + y^2 \longrightarrow f(x,y)$

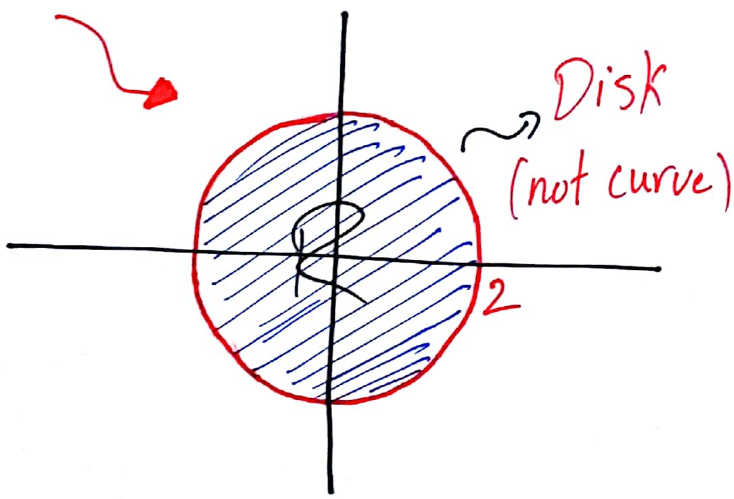
$$dS = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy = \sqrt{4x^2 + 4y^2 + 1}\,dx\,dy$$

$$= \sqrt{4(x^2+y^2)+1}\,dx\,dy$$

$$\rightarrow \text{Area}_{(S)} = \iint_S dS = \iint_R \sqrt{4(x^2+y^2)+1}\,dx\,dy$$

$R \rightarrow$ (projection القطع على plane x-y)

$$= \int_0^{2\pi}\int_0^{2} \sqrt{4r^2+1}\; r\,dr\,d\theta$$

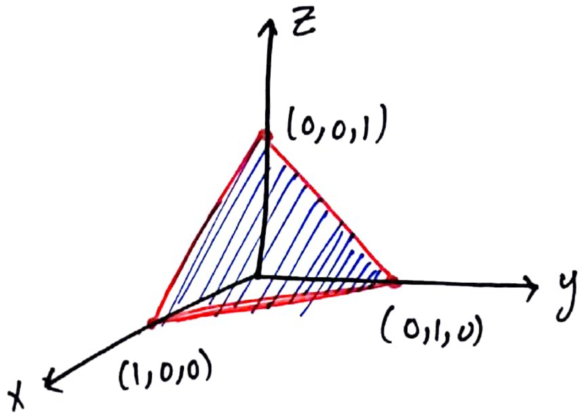


Ex: evaluate $\iint_S xz \, dS$, where

$xz \to$ function

S : the part of $x + y + z = 1$, in the first octant. (الثمن الأول)

sol



$z = 1 - x - y \quad (f(x,y))$

$dS = \sqrt{f_x^2 + f_y^2 + 1} \, dx\,dy$

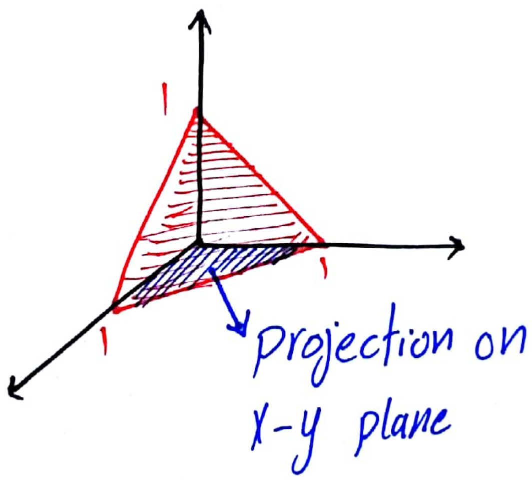
$dS = \sqrt{1 + 1 + 1} \, dx\,dy$

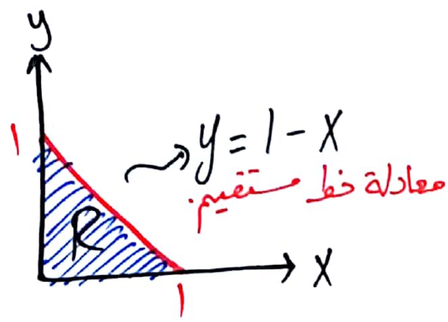
$dS = \sqrt{3} \, dx\,dy$

$$\to \iint_S xz \, dS = \iint x(1 - x - y) \, dS$$

$$= \iint_R x(1 - x - y) \underbrace{\sqrt{3} \, dx\,dy}_{\to dS}$$

$R \to$ projection on x-y plane



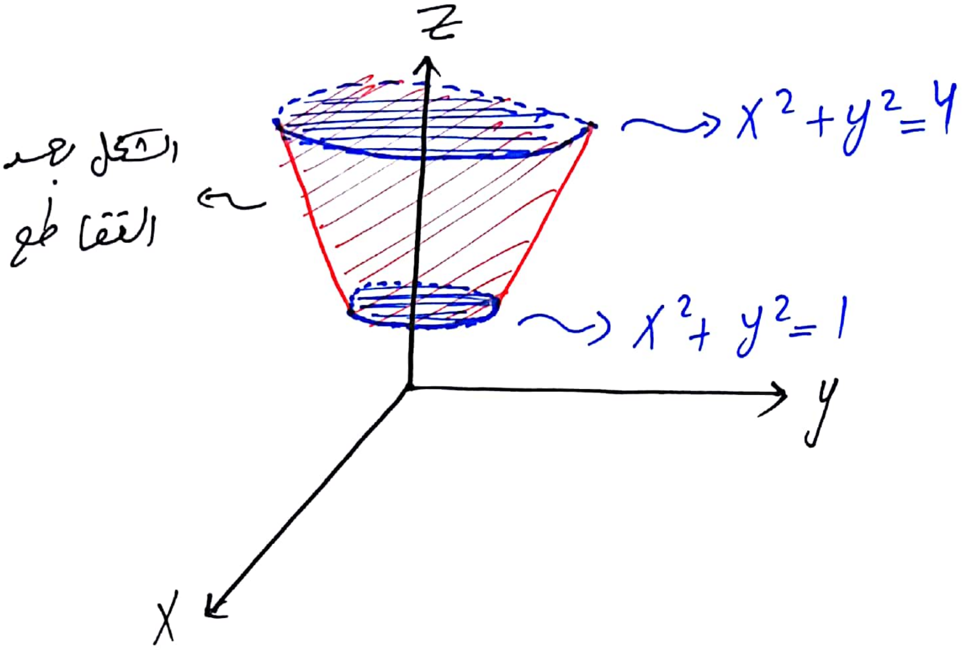


$$= \sqrt{3} \int_0^1 \int_0^{1-x} x(1 - x - y) \, dy \, dx$$

Ex: evaluate $\iint_S y^2 z^2 \, dS$, where:

S: the plane of $z = \sqrt{x^2+y^2}$, that lies between $z=1$, $z=2$.

$\hookrightarrow$ Cone (مخروط)



$z = \sqrt{x^2+y^2}$, $(f(x,y))$

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \longrightarrow f_x^2 = \frac{x^2}{x^2+y^2}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}} \longrightarrow f_y^2 = \frac{y^2}{x^2+y^2}$$

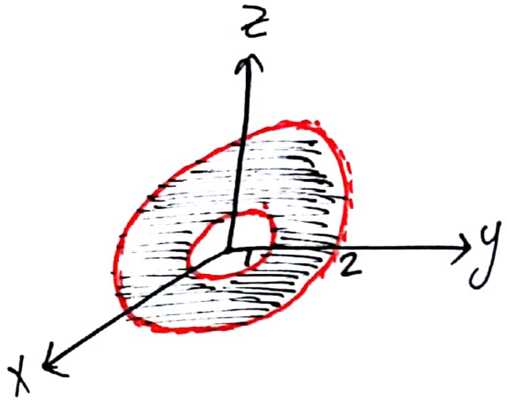
$$\left.\right\} f_x^2 + f_y^2 = 1$$

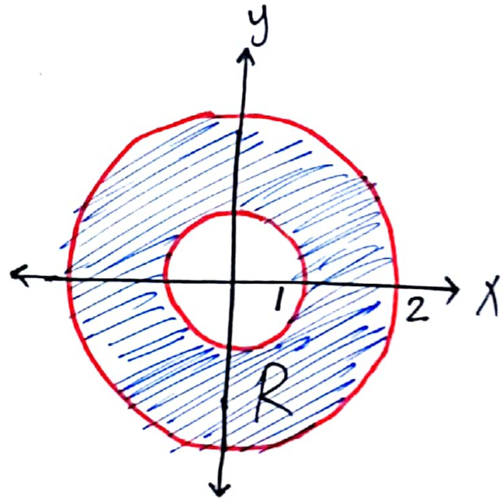
$$dS = \sqrt{\underbrace{f_x^2 + f_y^2}_{=(1)} + 1} \; dx\,dy = \sqrt{2} \; dx\,dy$$

$$\iint_S y^2 z^2 \, ds \Longrightarrow \iint_S y^2 (x^2+y^2) \, dS$$

$$= \iint_{R} y^2 (x^2+y^2)\sqrt{2}\, dx\, dy$$

$R \longrightarrow$ projection on xy plane





$$= \sqrt{2} \int_0^{2\pi}\int_{1}^{2} r^2 \sin^2\theta \cdot r^2 \cdot r\, dr\, d\theta$$

Ex: evaluate $\iint_S x\, ds$, where:

$S: x^2 + y^2 + z^2 = 4$

$* \; S: \underline{r}(u,v) = - - - - -$

$ds = \| r_u \times r_v \|\, du\, dv$

$* \; S: z = f(x,y)$

$ds = \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy$

$\longrightarrow ds = \| r_\phi \times r_\theta \| d\phi\, d\theta$

$= a^2 \sin\phi\, d\phi\, d\theta = 4 \sin\phi\, d\phi\, d\theta$ (بحالة ال sphere)

$= \iint 2\cos\theta \sin\phi * 4\sin\phi\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} 8\cos\theta \sin^2\phi\, d\phi\, d\theta$

* parametrization *

$x = 2\cos\theta \sin\phi$

$y = 2\sin\theta \sin\phi$

$z = 2\cos\phi$

Flux:

→ Let $\underline{F}$ be a vector field, Then the flux of $\underline{F}$ across S is given by:

$$\Phi = \iint_S \underbrace{\underline{F} \cdot \underline{n}}_{\text{Dot product.}} \, dS$$

$\underline{n}$ → unit normal vector. $= \dfrac{n}{\|n\|}$

Ex: find the flux of $\underline{F} = z\hat{\underline{k}}$ across S, where:

$S: x^2 + y^2 + z^2 = a^2$

Sol:

- $r_\phi \times r_\theta = \langle a^2 \cos\theta \sin^2\phi,\ a^2 \sin\theta \sin^2\phi,\ a^2 \sin\phi \cos\theta \rangle$

- $\underline{n} = \dfrac{r_\phi \times r_\theta}{\|r_\phi \times r_\theta\|} = \dfrac{\langle a^2\cos\theta \sin^2\phi,\ a^2 \sin\theta \sin^2\phi, a^2 \sin\phi \cos\phi \rangle}{a^2 \sin\phi}$

always the same.

→ $dS = \|r_\phi \times r_\theta\| \, d\phi \, d\theta = a^2 \sin\phi \, d\phi \, d\theta$

نا تج الضرب = صفر

\# $\underline{F} \cdot \underline{n} \, dS = \langle 0, 0, \underset{a\cos\phi}{z} \rangle \cdot \dfrac{\langle \ldots\ldots, \ldots\ldots, a^2 \sin\phi \cos\phi \rangle}{\cancel{a^2 \sin\phi}} \cdot \underbrace{(\cancel{a^2 \sin\phi} \, d\phi \, d\theta)}_{dS}$

$= a^3 \sin\phi \cos\phi^2 \, d\phi \, d\theta$

$$\Rightarrow \Phi = \int_0^{2\pi}\int_0^{\pi} a^3 \sin\phi \cos^2\phi \, d\phi \, d\theta.$$

Ex: find the flux of $F = x\underline{i} + y\underline{j} + z\underline{k}$, across the surface S where S: the portion of $z = 1 - x^2 - y^2$ which lies above $z = 0$ (x-y plane).

Sol:

$$\Phi = \iint_S F \cdot \underline{n} \, dS$$

- $\underbrace{z + x^2 + y^2}_{G} = 1 \longrightarrow \nabla G = \langle 2x, 2y, 1 \rangle$

- $\underline{n} = \dfrac{\nabla G}{\|\nabla G\|} = \dfrac{\langle 2x, 2y, 1 \rangle}{\sqrt{4x^2 + 4y^2 + 1}}$.

same

- $dS = \sqrt{f_x^2 + f_y^2 + 1} \; dx\,dy = \sqrt{4x^2 + 4y^2 + 1} \; dx\,dy$

$\rightarrow F \cdot \underline{n} \, dS = \langle x, y, z \rangle \cdot \dfrac{\langle 2x, 2y, 1 \rangle}{\cancel{\sqrt{4x^2 + 4y^2 + 1}}} \cdot \cancel{\sqrt{4x^2 + 4y^2 + 1}} \; dx\,dy$

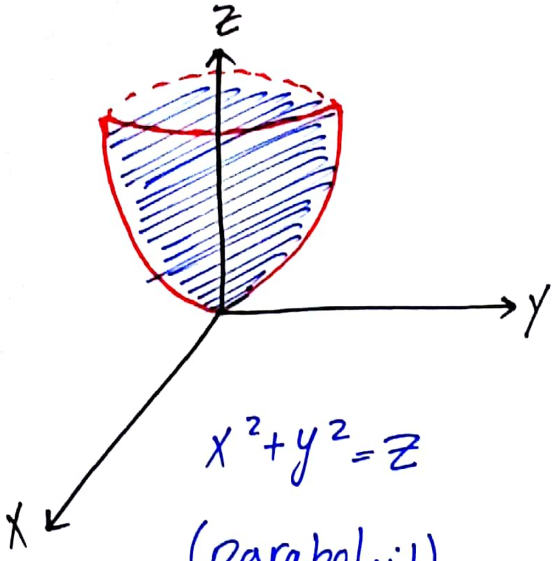
$= 2x^2 + 2y^2 + z \longrightarrow$ z نعوض قيمة

$= 2x^2 + 2y^2 + (1 - x^2 - y^2)$

$= x^2 + y^2 + 1$

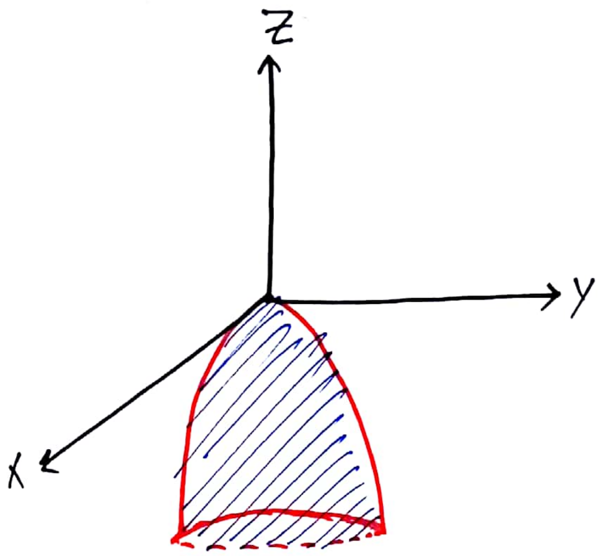
$$\rightarrow \Phi = \iint_R (x^2 + y^2 + 1) \, dx\,dy$$

now: for the previous example:

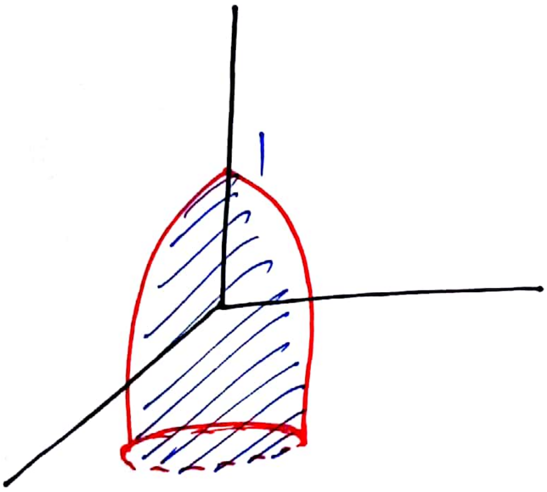


$x^2+y^2=z$

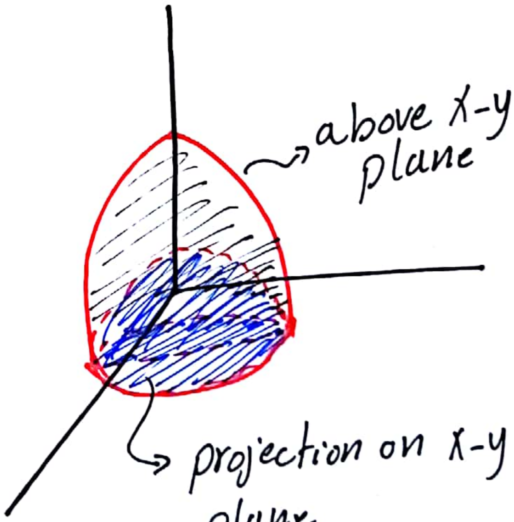
(paraboloid)



$-(x^2+y^2)=z$

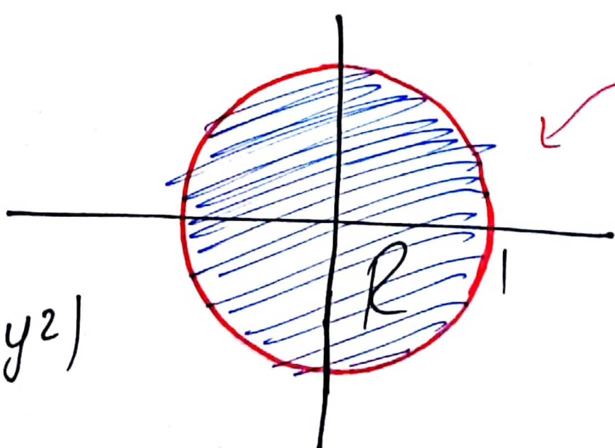


$z=1-(x^2+y^2)$



الشكل المطلوب

للسؤال السابق



$z=1-(x^2+y^2)$

$z=0$ $\quad x^2+y^2=1$ (disk)

follow:

- $\Phi = \iint_R (\underbrace{x^2 + y^2}_{r} + 1)\, dx\, dy$

polar coordinate

- $\Phi = \int_0^{2\pi} \int_0^1 (r^2 + 1)\, r\, dr\, d\theta$