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EE 251: Electromagnetics I
Second Exam (Spring 2017)
April 9th, 2017

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Note that bold letters are vectors

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \qquad \nabla.\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \qquad \varepsilon_o = \frac{10^{-9}}{36\pi}\ \text{F/m}$$

$$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

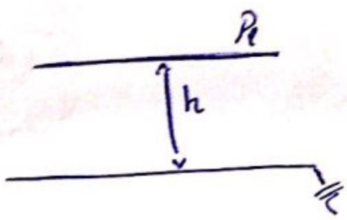
Problem 1 (8 points)

1

Consider a line charge 'ρ_l' placed horizontally at a distance 'h' from a perfect grounded conducting plane of infinite extent. Find the induced charge per unit length on the conductor surface.

Q/l

$$Q = \oint D.ds = \int_s \varepsilon_o \bar{E}\, ds$$



• for infinite line

$$\bar{E}_\rho = \frac{P_l}{2\pi\varepsilon_o R_{\to h}}$$

1

$$Q = \int_s \frac{\cancel{\varepsilon_o}\, P_l}{2\pi \cancel{\varepsilon} R} ds = \frac{P_l S}{2\pi h}$$

$$Q = \frac{P_l S}{2\pi h}\ C$$

$$\frac{Q}{l} = \frac{P_l S}{2\pi h} \div l$$

Problem 2 (8 points)

ε_{r_1} ε_{r_2}

Two homogeneous dielectric regions 1 ($\rho \le 4$ cm) and 2 ($\rho \ge 4$ cm) have dielectric constants 4 and 2, respectively. If $\mathbf{D}_2 = 12\,\mathbf{a}_\rho - 8\,\mathbf{a}_\phi + 10\,\mathbf{a}_z$ nC/m², calculate:

(a) $\mathbf{E}_1$

(b) $\mathbf{P}_2$ and ρ_{pv2}

(c) total surface charge densities above and below the interface ($\rho = 4^+$ and $\rho = 4^-$).

$E_{t1} = E_{t2}$ 0.5

$D_{n1} - D_{n2} = \rho_v$

$D_2 = 12\hat{a}_\rho - 8a_\phi + 10\hat{a}_z$ $\boxed{\varepsilon_{r_2} = 2}$

$\hat{a}_\rho = \hat{a}_n$ 4cm

$\boxed{\varepsilon_{r_1} = 4}$

(a) $D = \varepsilon_r \varepsilon_0 \bar{E}$

$$\bar{E}_2 = \frac{12\rho - 8a_\phi + 10a_z}{2 * \frac{10^{-9}}{36\pi}}$$

$$\bar{E}_2 = \underset{(216\pi)}{(12*18\pi)}a_\rho - \underset{(144\pi)}{(8*18\pi)}a_\phi + (180\pi)a_z \; *10^9$$

$E_2 = E_{n2} + E_{t2}$ 0.5

$E_{2n} = 216\pi\,\hat{a}_\rho$

$E_{t2} = -144\pi a_\phi + 180\pi\hat{a}_z \; *10^9 = E_{t1}$

$D_{2n} = D_{n1} - \rho_v$ ~> بعض النظر ضعر أولا

$D_{n1} = D_{2n} - \rho_v$

$D_{n1} = 12\hat{a}_\rho - \rho_v$

$\varepsilon_r\varepsilon_0 E_{n1} = 12a_\rho - \rho_v$

$$E_{n1} = \frac{12 - \rho_v}{\varepsilon_r\varepsilon_0}$$

$E_1 = E_t + E_n$

$$= \frac{12-\rho_v}{\varepsilon_r\varepsilon_0}\hat{a}_\rho - 144\pi a_\phi + 180\pi a_z$$

unit -> $[10^9\,V/m]$

$$= \frac{12-\rho_v\,\hat{a}_\rho}{\varepsilon_r\varepsilon_0} - 144\pi a_\phi + 180\pi\hat{a}_z$$

(b) $P_2 = \chi_e \varepsilon_r \varepsilon_0 \acute{E}_2$

$$\cancel{(2-1)}\cancel{(\varepsilon_r)}(\varepsilon_0)\frac{12\hat{a}_\rho - 8a_\phi + 10a_z}{\cancel{\varepsilon_r\varepsilon_0}}$$

$P = 12\hat{a}_\rho - 8a_\phi + 10\hat{a}_z$

$\nabla\cdot P = \frac{12}{\rho} +$

$\rho_{pv2} = -\nabla P = \frac{12}{\rho}\;C/m^3$ 0.5

(c) $Q_{total} = Q_{b^+} - Q_{b^-} = \text{Zero}$

$$= \int P_1 \cdot a_n\, dS - \int \nabla\cdot P = \text{Zero}$$

Problem 3: (7 points)

3.5

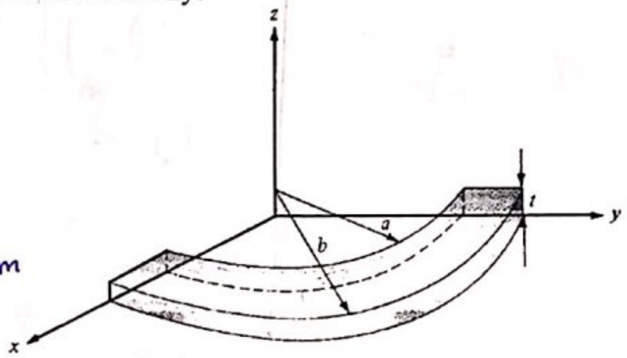
A metal bar of conductivity σ and dielectric constant of ε_r is bent to form a flat 90° sector of inner radius "a", outer radius "b", and thickness "t" as shown below. Find the resistance of the bar between the vertical surfaces at $\phi = 0°$ and $\phi = 90°$.

(a) Using boundary value problem.
(b) Using Gauss's law.
(c) Which method from (a) and (b) is the best choice and why?

$R = \frac{\varepsilon}{\sigma C}$

$R = \frac{V}{I} = \frac{-\int E \cdot dl}{\int J \cdot ds}$

is $\sigma E = J$



(a) using boundary value Problem

using laplace's equ

$\nabla^2 V = 0$ ← conductor

$$\frac{1}{\rho^2} \frac{d^2V}{d\phi^2} = 0$$

$$\frac{d^2V}{d\phi^2} = A$$

$$V = A\phi + B$$

① at $\phi = 0° \to V = V_o$

② at $\phi = \pi/2 \to V = 0$

applying①

$V_o = B$

applying②

$0 = \frac{A\pi}{2} + V_o$

$A = \frac{-2V_o}{\pi}$

$V = \frac{-2V_o}{\pi}\phi + V_o$

$E = -\nabla V$

$E = \frac{+2V_o}{\pi}$ a_ϕ

$$R = \frac{-\int_a^b \frac{2V_o}{\pi} \cdot d\rho}{\int_a^b \int_0^{\pi/2} \sigma \frac{2V_o}{\pi} \cdot \rho \, d\phi \, d\rho}$$

$$= \frac{-\frac{2V_o}{\pi}(b-a)}{\sigma(\pi/2)\frac{2V_o}{\pi} \left.\frac{\rho^2}{2}\right|_a^b}$$

$$R = \frac{4(b-a)}{\sigma\pi(b^2-a^2)}$$

2

(b) Using Gauss's law

$\rho_v = \frac{Q}{v}$

we will get this to this

$$C = \frac{Q}{V}$$

$$R = \frac{\varepsilon}{\sigma C}$$

$$Q = \oint D \cdot ds$$

0.5

$$Q = \int_a^b \int_0^{\pi/2} \varepsilon_0 \varepsilon_r E_z \rho \, d\phi \, d\rho \, \hat{a}_z$$

$$Q = \varepsilon_0 \varepsilon_r E_z (\pi/2) \left. \frac{\rho^2}{2} \right|_a^b$$

$$Q = \frac{\pi}{4} \varepsilon_0 \varepsilon_r E_z \rho^2$$

$$E_z = \frac{4Q}{\pi \varepsilon_0 \varepsilon_r \rho^2}$$

$$-\int \bar{E} \cdot dl \quad -V = -\int_0^t \frac{4Q}{\pi \varepsilon_0 \varepsilon_r \rho^2} dz = \frac{4Q\,t}{\pi \varepsilon_0 \varepsilon_r \rho^2}$$

$$C = \frac{\cancel{Q}\, \pi \varepsilon_0 \varepsilon_r \rho^2}{4 \cancel{Q}\, t} = \frac{\pi \varepsilon_0 \varepsilon_r \overbrace{\rho^2}^{(b^2-a^2)}}{4t} = \frac{\pi \varepsilon_0 \varepsilon_r (b^2 - a^2)}{4t} (F)$$

$$\text{now, } R = \frac{\varepsilon_r \varepsilon_0}{\sigma C} = \frac{4t\, \cancel{\varepsilon_r \varepsilon_0}}{\sigma \pi \cancel{\varepsilon_0 \varepsilon_r} (b^2 - a^2)} = \frac{4t}{\pi \sigma (b^2 - a^2)} \Omega$$

0.5

(c) Which way is better & why??

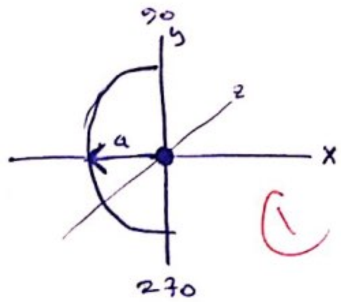
for ~~finan~~ finding Resistence the Boundary Value Problem is more direct. but it's harder than the indirect way "Gauss's" So as can be seen before ~~Part~~ method (a) is the best choice

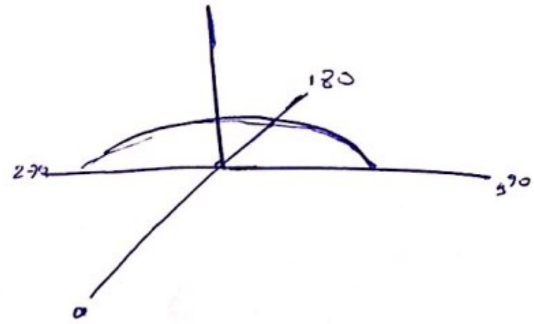
0.5

4.5

Problem 4: (7 points)

A semi-circle of radius "a" and $90^\circ \leq \phi \leq 270^\circ$ placed at z = 0 plane and curry current "I". Find **H** at point O which is the center of the semi-circle.





$$H = \int \frac{I\,dl \times \hat{a}_R}{4\pi R^2} \quad (1) \quad , \; dl = \rho\, d\phi\, \hat{a}_\phi \quad (0.5)$$

$$R = -a\hat{a}_\rho + a_\phi$$

$$|R| = \sqrt{a^2 + \phi^2}$$

$$H = \int \frac{a I\, d\phi\, a_\phi \times (a_\rho + a_\phi)}{4\pi (a^2 + \phi^2)} \quad (1)$$

$$(\rho = a)$$

$$H = \int \frac{a I \; d\phi}{4\pi (a^2 + \phi^2)} (-a_z) \quad (0.5)$$

a_z, $-a_\rho$, a_ϕ

$$H = \frac{a I}{4\pi} \int_{\pi/2}^{270^\circ = 3\pi/2} \frac{d\phi}{(a^2 + \phi^2)} (-a_z) \quad (0.5)$$

$$H = \frac{a I}{4\pi} \frac{1}{a} \tan^{-1}\left(\frac{\phi}{a}\right) \Bigg|_{90^\circ}^{270^\circ}$$

$$= \frac{I}{4\pi} \tan^{-1}\left(\frac{\phi}{a}\right) \Bigg|_{90}^{270}$$

$\frac{270 * \pi}{180}$ $\quad$ $3 * 180$ $\quad$ $3 * 180$ $\quad$ $360 = \frac{440}{2}$