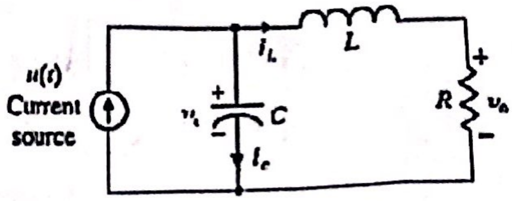
...written in ink. Exam Duration:50 min Registration #: 12

Mark out of 20:

Q1 a) Consider the circuit shown. with **vo** as output , capacitor voltage as state **x1** , and inductor current as state **x2** .



i) Obtain the state space representation .

$i_c = C \frac{\partial v_c}{dt}$, $x_1 = v_c$.

$V_L = L \frac{d i_L}{dt}$, $x_2 = i_L$

$i_t = i_c + i_L$

$u(t) = C\dot{x}_1 + x_2.$

$\dot{x}_1 = \frac{1}{c} x_2 - \frac{1}{c} u(t)$

$\dot{x}_2 = \frac{1}{L} x_1 - \frac{R}{L} x_2.$

$y = R x_2.$

A B

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{c} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{c} \\ 0 \end{bmatrix} u(t)$

$y = [0 \quad R] x + [0] u$

C D

ii) Let $C = 1\ \mu F, L = 1\ H$, and $R = 2\ \Omega$, **calculate the steady state value of x1 due to a unit step input,** ,

$X_{ss} = -A^{-1} B$

$= -\begin{bmatrix} -\frac{R}{L} & -\frac{1}{c} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{c} \\ 0 \end{bmatrix} \frac{1}{\Delta} = \begin{bmatrix} 20 * 10^5 \\ -10 * 10^5 \end{bmatrix}$

0

and the transfer function using <u>two</u> methods.

$T.F = \frac{out}{input}$

~~input~~

~~output~~

$C[SI - A]^{-1} B + D$

$[0 \quad R] \left[\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{c} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \right]^{-1} \begin{bmatrix} -\frac{1}{c} \\ 0 \end{bmatrix}$

$u(t) = i_L + i_c.$

$1 = \frac{V_o}{R} + CsV_c$

$i_t - \frac{V_o}{R} = CsV_c.$

$i_t = CsV_c - \frac{V_o}{R}$

$i_t = Cs \frac{V_o \frac{1}{cs}}{\frac{1}{cs} + Ls + R} - \frac{V_o}{R}$

~~$\frac{V_o}{i_t}$~~ $\frac{1}{1s}$ $Cs + \frac{1}{Ls + R}$

b) Use the **power series expansion** for e^{At} to calculate it when $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$e^{At} = I_n + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \frac{t^3}{3!}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{t^2}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{t^3}{3} \end{bmatrix} + \cdots$

5

Q2. Given matrix A as shown $A=\begin{bmatrix}-3 & 3 & 3\\ -6 & 5 & 4\\ 0 & 1 & 2\end{bmatrix}$.

Show that $\mathbf{0}$ is an eigenvalue of A.

$\prod_i(\lambda_i) = |A| = 0$

$|A| = (-3*6 + 36 - 3*6) = 0$

Hence, use **matrix properties** to determine the remaining eigenvalues. **Do not** determine them using $|A-\lambda I_n|=0$.

$\sum_i(\lambda_i) = \text{Trace} : -3+5+2 : \lambda_1 + \lambda_2 + 0$

$\boxed{4 = \lambda_1 + \lambda_2}$

Determine **the three eigenvectors.** *You can do it in a smart way. Think, but don't waste too much time.*

adjoint method.

$$\begin{array}{ccc} w_1 & & \\ \downarrow & & \end{array}$$
$$\begin{bmatrix}6 & -6 & -3\\ 12 & -6 & -6\\ -6 & 6 & 3\end{bmatrix}$$

$w_1 = \begin{bmatrix}6\\12\\-6\end{bmatrix}$

b) A **2x2** matrix **A** has **4** and **5** as eigenvalues. To calculate A^3, use **Cayley-Hamilton theorem** to obtain a formula involving multiples of A, and I_2 **only**.

$\lambda^2 - 9\lambda + 20$

$\lambda^2 + \lambda - 20 = 0.$

$A * A^2 + A - 20 = 0.$

$A^3 + A^2 - 20AI_n = 0.$

$A^3 = 20A - A^2.$

$= 20\begin{bmatrix}-3 & 3 & 3\\ -6 & 5 & 4\\ 0 & 1 & 2\end{bmatrix} - [$

c). Use the defining relationship of the eigenvalue –eigenvector problem to obtain a relationship relating the eigenvalues of A^2 to those of A.

$\lambda(A^2) = (\lambda(A))^2$

prove

d) **Derive** the **derivative** of the exponential matrix e^{At}. Comment on any restrictions on A.

$d\, e^{At}$ $\quad A . e^{At}$ $\quad \frac{d}{dt} e^{At} = 0 + A + 2A^2t + A^3 t^2 \ldots$

Answers should be w...

Q1) Given that **3** is an eigenvalue of A where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & -27 & 9 \end{bmatrix}$, Calculate the exponential matrix of A **by any method.**

$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0$

$\lambda_1 = 3$

$\lambda_2 = 3$

$\lambda_3 = 3$

$w_1 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ $w_2 \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$ $w_3 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$e^{At} = P^{-1} A P$

Because they are repeted λ.

$$e^{Jt} = \begin{bmatrix} e^{3t} & te^{3t} & t^2 e^{3t} \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}$$

2.5

b)Use matrix properties to **derive** an expression for $e^{A^T t}$ in terms of e^{At}.

$|A^T| = |A|$

$e^{A^T t} = \cancel{e^{A}}$ $e^{A^T t} = \frac{1}{e^{At}}$

0

c)Use a special case of $e^{(A+B)t} = e^{At}e^{Bt}$ together with matrix properties to **derive** an expression for the inverse of e^{At}

$(e^{At})^{-1}$ $e^{A(-t)}$

$A + B = -A$

$\boxed{B = -2A}$

0

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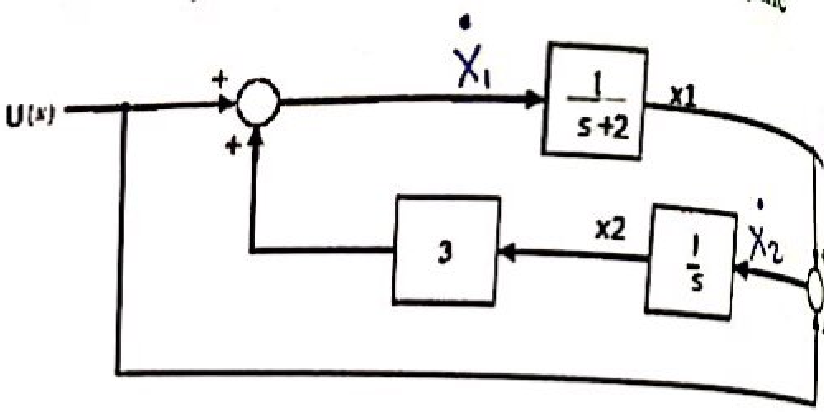
Q4 Given the block diagram shown, Determine **the state space representation**, the **transfer function**, then obtain **a block diagram of diagonal form**. Is the system controllable ?. justify by any method .



$Y(S) = X_1$

$\dot{X}_1 = 3X_2 + u(S)$

$\dot{X}_2 = Y(S) + u(S)$

$\dot{X}_2 = X_1 + u(S)$

$\dot{X} = \begin{bmatrix} \cancel{0}\,2 & 3 \\ 1 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$

$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} X + [0]\, u$

$G(S) = C[SI - A]^{-1} B + D$

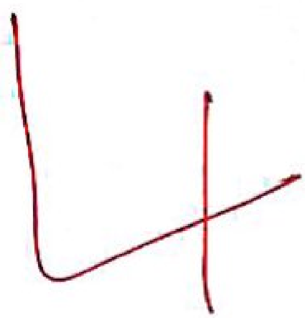
$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} S & -3 \\ -1 & S \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + D$

$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} S & 3 \\ +1 & S \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \frac{1}{S^2 - 3}$

1x2

$= \begin{bmatrix} S & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \frac{1}{S^2 - 3}$

$G(S) = \frac{S+3}{S^2-3}$



$Y(S) = \frac{3}{S}$

$\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$

$\begin{bmatrix} S & -3 \\ -1 & S \end{bmatrix}$

$\begin{bmatrix} 0 & 3 \end{bmatrix}$

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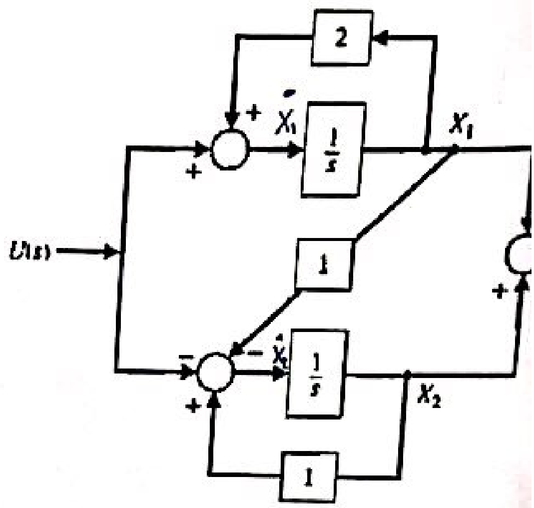
Q2 Given the block diagram shown, Determine the system **transfer function** , and **controllability** by any method you prefer.



$Y(S) = X_1 + X_2$

$\dot{X}_1 = 2X_1 + u(s)$

$\dot{X}_2 = X_2 - u(S) - X_1$

$$\dot{X} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u.$$

$$M_c = \begin{bmatrix} B & AB \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

2

-2

$|M_c| = 0$ un Controllable.

$Y(s) = X_1 + X_2$

$u(S) = \dot{X}_1 - 2X_1$

$u(S) = SX_1 - 2X_1$

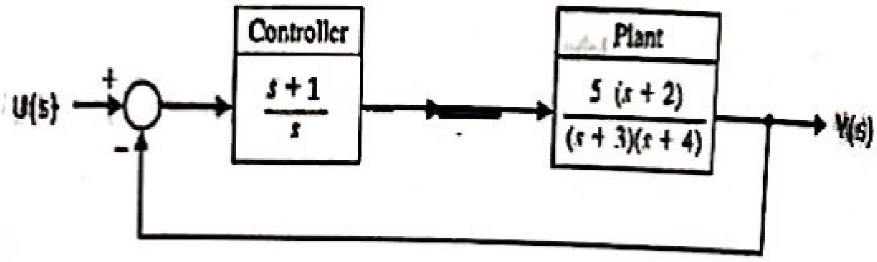
$u(S) = X_2 - X_1 - SX_2$

$T.F = \frac{Y(S)}{u(S)}$

$T.F = \frac{X_1 + X_2}{X_2 - X_1 - SX_2}$

$T.F = \frac{S + 1}{S - 1 - S^2}$

Q3 Given the block diagram shown, Obtain **a <u>state space description</u>** by any method you prefer.



$$\frac{G(s)}{1 + G(s)H(s)}$$

$$T.F = \frac{\frac{S+1}{S} * \frac{5(S+2)}{(S+3)(S+4)}}{1 + \frac{S+1}{S} * \frac{5(S+2)}{(S+3)(S+4)}}$$

$$= \frac{\frac{S+1}{\cancel{S}} * \frac{5(S+2)}{\cancel{(S+3)(S+4)}}}{\frac{S(S+3)(S+4) + (S+1)*5*(S+2)}{\cancel{S(S+3)(S+4)}}}$$

$$(S+1)(5S+10)$$
$$5S^2 + 10S + 5S + 10$$

$$= \frac{S+1\,(5(S+2))}{S(S+3)(S+4) + (S+1)*5*(S+2)}$$

$$(S^2+3S)(S+4)$$
$$S^3 + 4S^2 + 3S^2 + 12S$$

$$= \frac{5S^2 + 10S + 5S + 10}{S^3 + 4S^2 + 3S^2 + 12S + 5S^2 + 10S + 5S + 10}$$

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$$\frac{Y(s)}{u(s)} = \frac{5S^2 + 15S + 10}{S^3 + 12S^2 + 27S + 10}$$

$$u(s) = \dddot{z} + 12\ddot{z} + 27\dot{z} + 10z$$
$$Y(s) = 5\ddot{z} + 15\dot{z} + 10z$$
$$z = x_1 \longrightarrow \dot{z} = \dot{x}_1$$
$$\dot{z} = x_2 \longrightarrow \ddot{z} = \dot{x}_2$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u(s) - 12x_3 - 27x_2 - 10x_1$$

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