The University of Jordan
Electrical Eng. Dept.

Advanced Control Systems EE549 **Second Exam** **Thursday. 3/12/2017**

Student Name and Registration #: 0144235 محمد مفيد أحمد أبوحاشية **Mark out of 30** 3·1

Answers should be written in ink **Exam Duration: 70 min**

Q1) a) Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{bmatrix}$, and that A has **3 identical** eigenvalues.

i) Calculate **two** eigenvectors of A by any method.

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$\boxed{\lambda = 2}$

first eigen vector given By: $\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

second one: $\begin{bmatrix} 0 \\ 1 \\ 2\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

third one: $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$[A - \lambda I]^2 P_2 = 0$

$\Rightarrow \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 8 & -12 & 6-\lambda \end{bmatrix}^2 P_2 = 0$

$\Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{bmatrix}^2 P_2 = 0$

$\Rightarrow \begin{bmatrix} 4 & -4 & 1 \\ 8 & -8 & 2 \\ 16 & -16 & 4 \end{bmatrix} P_2 = 0$

$\therefore P_2 \Rightarrow \operatorname{adj} \begin{bmatrix} 4 & -4 & 1 \\ 8 & -8 & 2 \\ 16 & -16 & 4 \end{bmatrix}$

$P_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?! Not useful 3

By Gauss Elemination

$[A - \lambda I] P_2 = P_1$

$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{bmatrix} P_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Do gauss elimination:

$\Rightarrow \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} P_2 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ so $P_2 = \begin{bmatrix} \frac{r-4}{2} \\ \frac{r-2}{2} \\ r \end{bmatrix}$ take $r = 4$ 1

so $P_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

ii) By inspection, or otherwise what is the characteristic equation ?

$C.E = \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

iii) hence, use Cayley-Hamilton theorem to calculate A^3

$\Delta(A) = 0$, satisfy C.E $\Rightarrow A^3 - 6A^2 + 12A - 8I_n = 0$

Then, : $A^3 = 6A^2 - 12A + 8I_n$

$= 6\begin{bmatrix} 0 & 0 & 1 \\ 8 & -12 & 6 \\ 48 & -64 & 24 \end{bmatrix} - 12\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 8 & -12 & 6 \\ 48 & -64 & 24 \\ 192 & -240 & 80 \end{bmatrix}$ 2

b) Given $\dot{x} = Ax + Bu$; $x \in \mathbb{R}^n$, $u \in \mathbb{R}^1$, where B is an eigenvector of A, use matrix properties to **prove** that the system is always **uncontrollable**.

As we know that $Ax = \lambda x$, so By using rank method:

construct the matrix $M = [B \;\; AB \;\; A^2B \cdots]$; where $AB = \lambda B$ 2

where λ is the eigen value of Matrix A.

$\Rightarrow M = [B \;\; \lambda B \;\; \lambda^2 B \cdots]$ as we see that each column is dependent on the other one, so $\operatorname{rank}(M) < n$. Consequantly, system is Uncontrollable.

Q2 a system is given by $\dot{x} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad ; \quad y = [-3 \quad 1]\, x$

Calculate the exponential matrix as $e^{At} = P\, e^{\Lambda t}\, P^{-1}$ where P contains the eigenvectors as columns.

• find eigenvalues:

$A - \lambda I = \begin{bmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$

$|A - \lambda I| = (\lambda+1)(\lambda-4) - 6 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$\boxed{\lambda = -2, 5}$

for $\lambda = -2$:

$adj \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix}$

$P_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

for $\lambda = 5 \Rightarrow adj \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & -6 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\therefore P = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \Rightarrow P^{-1} = \frac{-1}{7} \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow e^{At} = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{7}$

$= \begin{bmatrix} -2e^{-2t} & e^{5t} \\ e^{-2t} & 3e^{5t} \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{7}$

$= \begin{bmatrix} \frac{6e^{-2t} + e^{5t}}{7} & \frac{-2e^{-2t} + 2e^{5t}}{7} \\ \frac{-3e^{-2t} + 3e^{5t}}{7} & \frac{e^{-2t} + 6e^{5t}}{7} \end{bmatrix}$ ##

3.5

Given that $x(0) = [7 \quad 7]^T$, and that $u(t) = 0$, for $t \geq 0$, determine $y(t)$, for $t \geq 0$.

$x(t) = e^{A(t)} x(0) + \underbrace{e^{At} \int_0^t e^{A(-\tau)} u(\tau) B\, d\tau}_{\text{equal Zero.}}$ $\boxed{t_0 = 0}$

~~$x(t) = 6$~~ $x(t) = \frac{1}{\cancel{7}} \begin{bmatrix} 6e^{-2t} + e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & e^{-2t} + 6e^{5t} \end{bmatrix} \cdot \cancel{7} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4e^{-2t} + 3e^{5t} \\ -2e^{-2t} + 9e^{5t} \end{bmatrix}$

$\therefore y(t) = [-3 \quad 1]\, x(t) = -12e^{-2t} - \cancel{9e^{5t}} - 2e^{-2t} + \cancel{9e^{5t}}$

$\Rightarrow y(t) = -14e^{-2t}$

1.5

What do you notice ? why one of the two exponential terms is missing ? , justify your claim using a system structural test ?

we Notice that e^{5t} missing / physically CC means that e^{5t} cannot be affected by u.

so the system is Uncontrollable due to $\lambda = 5$.

0.5

Further , determine the **transfer function by a matrix method** to confirm your accusation.

$G = C[sI - A]^{-1} B = \frac{1}{s^2 - 3s - 10} [-3 \quad 1] \begin{bmatrix} s-4 & 2 \\ 3 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{[-3s+15 \quad s-5] \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 - 3s - 10} = \frac{s-5}{(s-5)(s+2)} = \frac{1}{s+2}$

Pole-Zero Cancellation "Uncont" unobse

1.5

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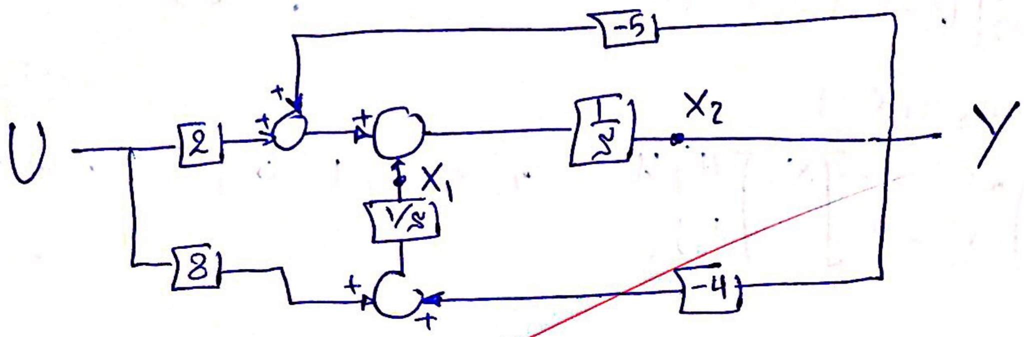
Q3 Given the transfer function $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{2s+8}{s^2+5s+4}$,

Obtain a **block diagram** representation of the system by expressing $Y(s)$ as an encapsulating function of powers of $\frac{1}{s}$.

$$G(s) = \frac{Y}{U} = \frac{2\frac{1}{s} + 8\frac{1}{s^2}}{1 + 5\frac{1}{s} + 4\frac{1}{s^2}} \Rightarrow Y\left[1+\frac{5}{s}+\frac{4}{s^2}\right] = U\left[\frac{2}{s}+\frac{8}{s^2}\right]$$

$$\Rightarrow Y = U\left[\frac{2}{s}+\frac{8}{s^2}\right] - Y\left[\frac{5}{s}+\frac{4}{s^2}\right] = \frac{1}{s}\left[(2U-5Y)+\frac{1}{s}(8U-4Y)\right]$$



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Hence, obtain a two dimensional state space description for the system.

$\boxed{y = x_2}$, $x_1 = \frac{1}{s}(-4x_2 + 8u) \Rightarrow \boxed{\dot{x}_1 = -4x_2 + 8u}$

$x_2 = \frac{1}{s}[x_1 + 2u - 5x_2] \Rightarrow \boxed{\dot{x}_2 = x_1 - 5x_2 + 2u}$

$$\dot{x} = \begin{bmatrix} 0 & -4 \\ 1 & -5 \end{bmatrix} x + \begin{bmatrix} 8 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + 0 \cdot u.$$

2

Is the system controllable? ,is it observable? Verify using $G(s)$, and other tests.

$M = \begin{bmatrix} 8 & -8 \\ 2 & -2 \end{bmatrix} \Rightarrow |M| = 0 \Rightarrow rank(M) = 1$ it is $\overline{CC}$. 3

$N = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix} \Rightarrow |N| = -1 \Rightarrow rank(N) = 2$ it is OO.

$G(s) = \dfrac{2\cancel{(s+4)}}{\cancel{(s+4)}(s+1)} = \dfrac{2}{s+1}$ due to Pole-Zero Cancellation in G(s) it confirms the Uncontrollability in the system.

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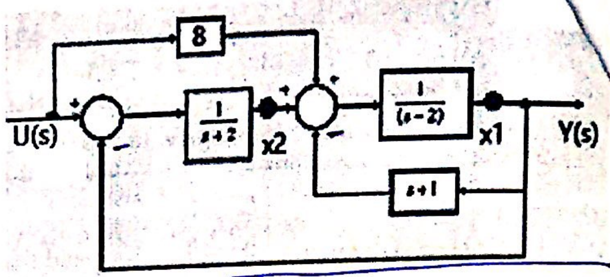
Q4 Given the block diagram shown,



i) Determine **the state space representation,**

$\boxed{y = x_1}$ ---(1)

$X_1 = \frac{1}{s-2}\left[X_2 + 8U - (s+1)X_1\right]$

$\dot{x}_1 - 2x_1 = x_2 + 8u - \dot{x}_1 - x_1 \Rightarrow 2\dot{x}_1 = x_1 + x_2 + 8u \Rightarrow \boxed{\dot{x}_1 = 0.5x_1 + 0.5x_2 + 4u}$ (2)

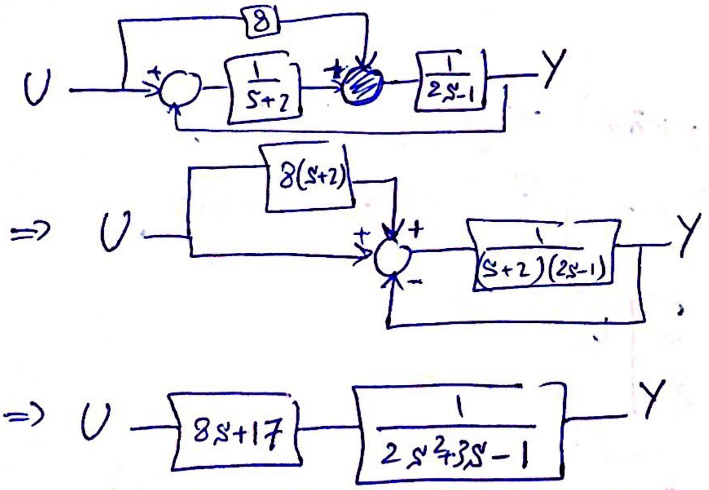
$X_2 = \frac{1}{s+2}\left[U - X_1\right] \Rightarrow \boxed{\dot{x}_2 = -x_1 - 2x_2 + u}$ ---(3)

from (1), (2) & (3):

$\dot{x} = \begin{bmatrix} 0.5 & 0.5 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 4 \\ 1 \end{bmatrix} u$, $y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0 \cdot u$.

3

ii) the **transfer function by block diagram reduction tecniques,**



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3

$\Rightarrow G(s) = \frac{Y(s)}{U(s)} = \frac{8s+17}{2s^2+3s-1}$

iii) Is the system controllable ?. justify by **two** methods .

① $M = [B \; AB] = \begin{bmatrix} 4 & 2.5 \\ 1 & -6 \end{bmatrix} \Rightarrow |M| \neq 0$ so $\text{rank}(M) = 2$

the system is Controllable.

② G(s) Don't suffer from Pole-Zero Cancellation, so that the system is Controllable.

2

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