Topics in Control

2/2/2017

* Simulations Using simulink:

Ex 1:

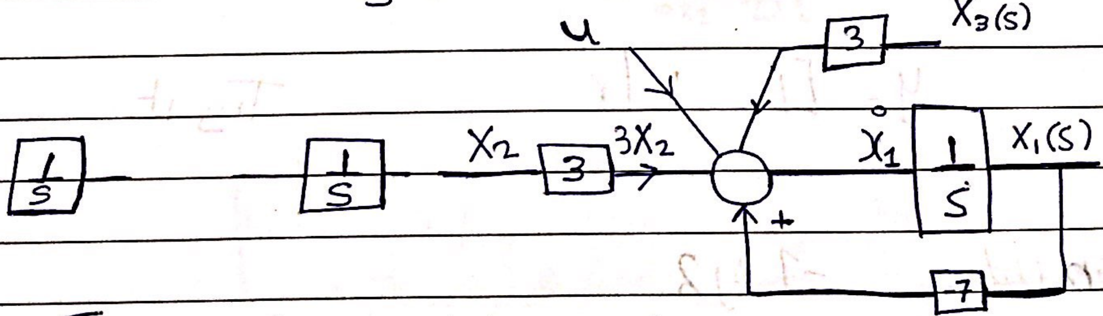
Given $\dot{x} = \begin{bmatrix} -7 & 3 & 3 \\ -6 & 1 & 4 \\ 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x$

$\dot{x}_1(t) = -7x_1 + 3x_2 + 3x_3 + u$

$sX_1(s) = -7X_1(s) + 3X_2(s) + 3X_3(s) + U(s)$

$X_1(s) = \underbrace{\frac{1}{s}}_{\text{integrator}} \left[-7X_1(s) + 3X_2(s) + 3X_3(s) + U(s) \right]$

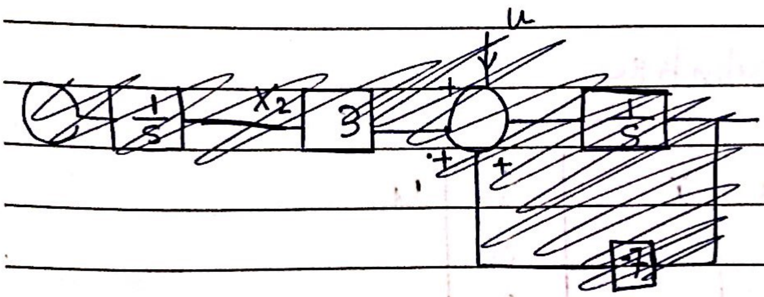


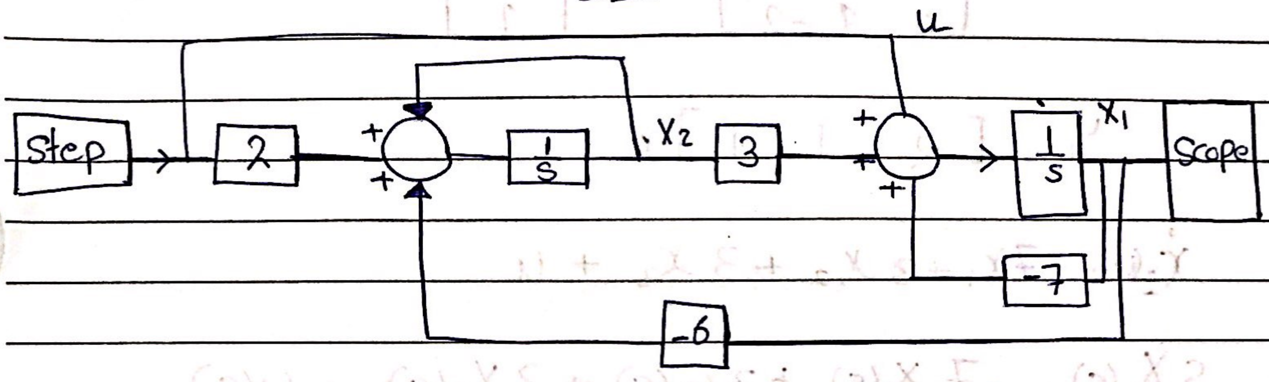
Finish the Simulation at home.

Ex2:

$\dot{x} = \begin{bmatrix} -7 & 3 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$

2/2/2017





Ex:3 Given $\dot{x} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$ Try it

eigen value $-1 \pm j2$

5/2/2017

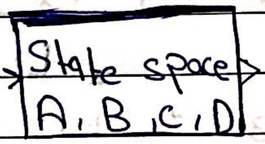
Example: Simulate using Simulink.

$$\dot{x} = \begin{bmatrix} -7 & 3 & 3 \\ -6 & 1 & 4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

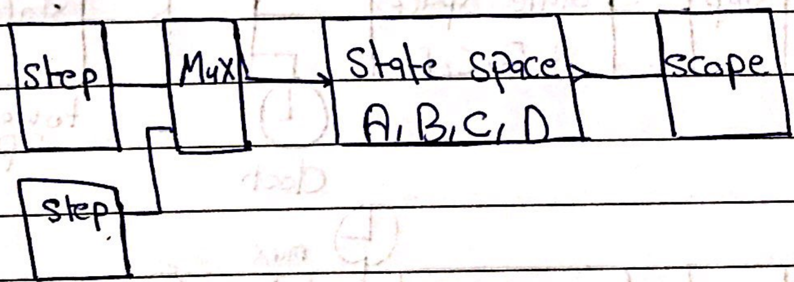
$$y = I_3 x \Rightarrow \begin{matrix} y_1 = x_1 \\ y_2 = x_2 \\ y_3 = x_3 \end{matrix} \Rightarrow C = eye(3)$$

eye(size(A))

>> A=[-7 3 3 ; -6 1 4 ; 0 1 -2] , B=[1 0;0 1;1 1] ; C=eye(size(A)); D=[0 0;0 0;0 0]



$$y_{3\times1} = I_3 x + \underset{3\times2}{D}\, \underset{2\times1}{u}$$



$$\dot{x} = \left[\begin{array}{c|cc} -7 & 3 & 3 \\ \hline 0 & 1 & -4 \\ 0 & 1 & -2 \end{array}\right] \Rightarrow \text{egen values} \quad -7 \ \& \ \lambda \begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix}$$

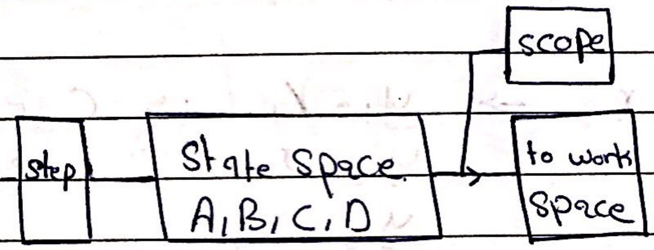
← لو كانت فيها اصفار اكثر تكون اسهل

5/2/2017

Check X_{ss} ,

$X = -A^{-1}B$ ∴ → staedy State.

7/2/2017



$$\overset{\circ}{X} = AX + Bu$$

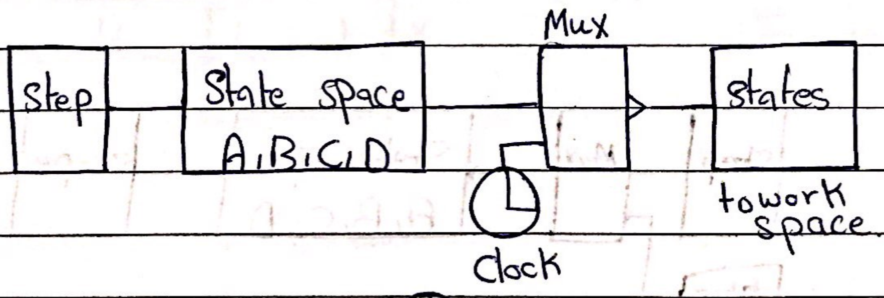
$$\overset{\circ}{X} = \begin{bmatrix} -5 & 3 & 3 \\ 0 & -2 & 1 \\ 0 & -1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

D

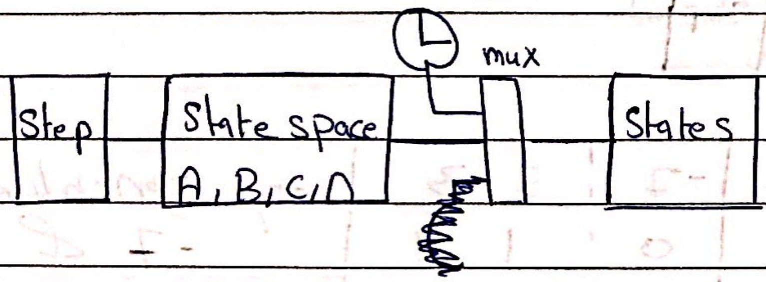
$$y = I_3 X + O_{3\times1} u$$

[0;0;0]

بنقدر نعمله هيك

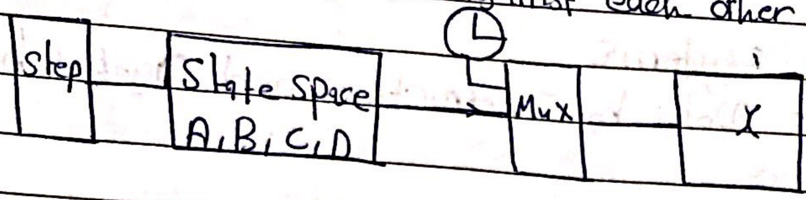


غيرنا ترتيب لل Mux input



7/2/2017

to draw two signal against each other



plot(X(:,2) , X(:,3))

X_2 X_3

plot3(X(:,3) , X(:,2) , X(:,1)) Three dimentional

12/2/2017

⇒ Needed Review from control I (441)

« Routh's stability criterian »

Having obtained the CE $1 + G(s)\,H(s) = 0$

⇒ Best illustrated by an Example :

if the CE = $S^3 + 6S^2 + 11S + 6 + k = 0$

$$
\begin{array}{c|cc}
S^3 & 1 & 11 \\
S^2 & 6 & 6+k \\
S^1 & \frac{60-k}{6} & 0 \\
S^0 & 6+k &
\end{array}
$$

$*6$ إذا ضربنا بـ ٦
بنتخلص من الكسور

هذا العمود اللي يهمني
عشان الـ stability
يهمني ما يكونا في تغيير
بالإشاره .

For stability, because $1 > 0$ we require all number's in the first Column to be positive.

$60 - k > 0$

$6 + k > 0$ $-6 < k < 60$

Check using matlab.

```
>> k=10 ; nc= k ; dc= [ 1 6 11 6+k ] ; sys = tf(nc,dc);
Cpoles = roots (dc) , step(sys)
```

12/2/2017

Example: let CE = $S^5+2S^4+2S^3+4S^2+3S+6=0$

$$\begin{array}{llll} S^5 & 1 & 2 & 3 \\ S^4 & 2 & 4 & 6 \\ S^3 & 8 & 8 & 0 \\ S^2 & 2 & 6 & \\ S^1 & -16 & 0 & \\ S^0 & 6 & & \end{array}$$

unstable

$A(s) = 2S^4 + 4S^2 + 6 = 0$

$\frac{d(A(s))}{ds} = 8S^3 + 8S$

⇒ system is unstable with two poles of positive real parts

14/2/2017

N.B : i) If the C.E has a difference in sign then the system is unstable $S^5+4S^4+3S^3+2S^2+S-1=0$

ii) If the C.E has a missing power then the system is ustable $S^5+3S^3+2S^2+S+1=0$

iii) A second order system is stable whenever the signs of the C.E are the same.

$$\left.\begin{array}{l} 10^{100}S^2+143S+5117=0 \\ -10^{100}S^2-143S-5117=0 \end{array}\right\} \text{stable}$$

S^2	a	c
S^1	b	0
S^0	c	

Modeling of systems

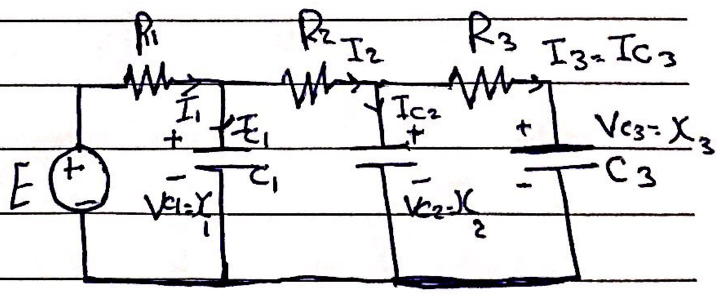
H.W₁

Conside the following circuit :

① ⇒ obtain a detailed block diagram representation of the Circuit using itegrators only, with $E(s)$ as the set value and V_{C3} as output

② ⇒ obtain a state space model of the circuit with X_3 output, $E(t)$ input

③ ⇒ obtain the transfer function of the circuit using three different methods
in concept



16/2/2017

⇒ The Transfer function matrix

Given $\dot{X} = AX + Bu \quad X \in R^n, u \in R^m \quad m \leq n$

$y = CX + Du \quad y \in R^2 \quad 1 \leq n$

It can be shown

$$G(s) = C\,[S I_n - A]^{-1} B + D$$

Using matlab:

```
>> A = [0 1; -2 -3] , B = [0;1] ; C = [5 1] ; D = 0
>> syms S ; G = C* inv (S* eye (size (A)) - A)*B + A ,
   Pretty (G) ⇒ بيرتبها
```

$$G = \frac{S+5}{S^2+3S+2}$$

```
>> a = rand (3) ; size (a)
>> G = ss2tf (A, B, C, D)
```

16/2/2017

N.B given $S(A,B,C,D)$

It can be shown that the steady state value of X due to a unit step is

$$X_{ss} = -A^{-1}B = \lim_{t\to\infty} X(t)$$

provided the system is stable and $|A| \neq 0$

Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

either use

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} B \, \cancel{u}(\tau) \, d\tau$$

then find $\lim_{t\to\infty} X(\tau)$

or use $X_{ss} = -A^{-1}B$

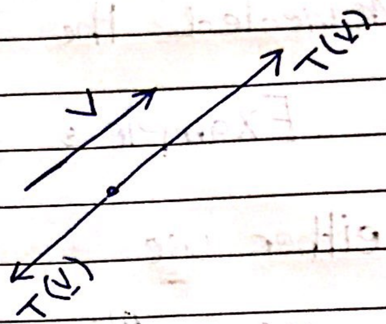
19/2/2017

Review of certain linear Algebra Concepts

* Eign value and Eign vector &c
They arise out of the following question Given a transformation $T(\cdot)$ does there exists a non-Zero vector $\underline{V}$ such that $T(\underline{V})$ remains in the same direction i.e

$$T(\underline{V}) = \lambda \underline{V}$$

$\uparrow$ magnifying or



* Given the fact that linear transform can be represented by matrices (say A), then

$$T(\underline{V}) = A\underline{V} = \lambda \underline{V}$$

or $(A - \lambda I)\underline{V} = 0 \rightarrow (A - \lambda I_n)\underline{V} = 0$

$\uparrow$ nxn matrix

For $\underline{V}$ not to be zero $A - \lambda I_n$ shouldn't have an inverse i.e $A - \lambda I_n$ should be singular i.e

$$|A - \lambda I_n| = 0$$

Five Apple

$$T(\underline{V}) = A\underline{V} = \lambda\,\underline{V}$$

$|A - \lambda I_n| = 0$ results is an nth order polynomial is solved to determine λ

To determine the eignvectors for each λ_i $i = 1, \dots, n$ solve

$$[A - \lambda_i I_n]\, v_i = 0 \qquad v_i \neq 0$$

Example 80

$$A = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

$$|A - \lambda I_n| = \begin{vmatrix} -5-\lambda & 3 & 3 \\ -6 & 3-\lambda & 4 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$(-1)[4(-5-\lambda) + 18] - \lambda[(-5-\lambda)(3-\lambda) + 18] = 0$$

→ Check it at Home

→ matlab

>> poly (A) => $S^3 + 2S^2 - S - 2 = 0$

$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$

>> roots([1 2 -1 -2])

given $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -2$

21/12/2017

Example :

» let $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$

» poly (A)

» roots ([1   -3   -10])

ans = 5 , -2

$$A = \begin{bmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

$$(-1-\lambda)(4-\lambda) - 5 = 0$$

$$-4 + \lambda - 4\lambda + \lambda^2 - 5 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$\lambda = 5 \quad , \quad \lambda = -2$$

$$|A - (2I)| = 0$$

لما نعوض قيمة egin value

~~Example~~

$*\frac{1}{3} - r_1$ $\quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$

↓

determinant = 0

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

V_2 arbitrary let $V_2 = 1$, $V_1 = -2$

» [r e] = eig (A) → r = vector
e = value

$\gg$ format rat → rational

- 5 المعادلة

$|A-\lambda I| \Rightarrow A = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} * -\frac{1}{2} - r_2$

$6-6=0$

$\begin{bmatrix} 0 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$

V_1 arbitrary $V_1 = 1$, $V_2 = 3$

Example :

$A = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix}$

[r e] = eig (A) eign value = 1 , -2 , -1

(-5 +3 +0) لما نجمع ال diagonal ← as a check

diagonal = eign values)) مجموع
Sum
((Trace))

$\gg$ det (A):

|A| = 2 → حاصل ضرب eign value

zero ⇐ eign value)) يعني واحد من ال 0 = |A| لما يكون

23/2/2017

Example 8: let $A = \overset{A}{\begin{bmatrix} -3 & 3 & 3 \\ -6 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}} \overset{V}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \overset{\lambda}{3} \overset{V}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

it can be shown that $\lambda_1 = 1$; $\lambda_2 = 3$, $\lambda_3 = 0$

Check: $\underbrace{\sum A_{ii}}_{\text{Trace}} = \sum \lambda_i = 4$

$$|A| = \prod_{i=1}^{n} \lambda_i = 0$$

To determine eignvector associated with $\lambda_2 = 3$

$$[A - (3) I_3] \underline{V} = 0 \Rightarrow \begin{bmatrix} -6 & 3 & 3 \\ -6 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix} \underline{V} = 0$$

must be singular or $|\ | = 0$ or has no inverse

eigen vector $\neq 0$

$\boxed{r_2 - r_1}$ $\begin{bmatrix} -6 & 3 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \underline{V} = 0$

$\boxed{r_2 - r_3}$ $\begin{bmatrix} -6 & 3 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{V} = 0 \Rightarrow \begin{bmatrix} -6 & 3 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 0\,V_1 + 0\,V_2 + 0 \cdot V_3 = 0$

let V_3 be arbitrary Say 1

$\Rightarrow -V_2 + V_3 = 0$

$V_2 = V_3 = 1$

$\Rightarrow -6V_1 + 3V_2 + 3V_3 = 0$

$-6V_1 + 3 \times 1 + 3 \times 1 = 0$

$V_1 = 1$

$\Rightarrow$ eigen vector associated with $\lambda_2 = 3$ $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Ex: Determine the two eigen vectors associated with $\lambda_1 = 1$, $\lambda_3 = 0$

Check the answer using

$[r\ e] = eig(A)$ To find eigen vector

⇒ Determination of eigen vector using the Adjoint of a matrix

* Best illustrated by an example

Given $A = \begin{bmatrix} -3 & 3 & 3 \\ -6 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ determine the eigen vector associated with $\lambda = 1$

$$V_1 = \text{adj}\,(A - (1) I_n)$$

$$= \text{adj}\left(\begin{array}{ccc} + & - & + \\ \end{array} \right.$$

$$= \text{adj}\left(\begin{bmatrix} -4 & 3 & 3 \\ -6 & 4 & 4 \\ 0 & 1 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 6 & -4 & -2 \\ -6 & 4 & +2 \end{bmatrix}$$

eign vector

هذا اللي يهمنا

بقدر اختار

اي column

لأنهم نفس النسبة

ع ان هيك بنحسب

اول column بس

⇒ properties of matrices:

let $\lambda[\cdot]$ defines the eigenvalue of a matrix, $\nu[\cdot]$ denote the eigenvectors of a matrix.

① $\sum_{i=1}^{n} \lambda_i[M] = trace(M) = \sum_{i=1}^{n} M_{ii}$

② $\prod_{i=1}^{n} \lambda[M] = |A|$ $\qquad A = M$

③ $|A^T| = |A|$, $|A_1A_2| = |A_1||A_2|$

④ $\lambda[A^T] = \lambda[A]$; $\nu[A]$ & $\nu[A^T]$ are not the Same

⑤ $\lambda[A_1 A_2] = \lambda[A_2 A_1]$ generally $A_1A_2 \neq A_2A_1$

⑥ $Trace(A_1A_2) = Trace(A_2A_1)$

⑦ $\lambda[A + KI_n] = \lambda[A] + K$ i.e eigen value are Shifted by K

⑧ $\nu[A + KI_n]$ are the Same as $\nu[A]$

⑨ if $A = A^T$ i.e A is symmetric then $\lambda[A]$ are real (cannot be complex)

$\nu[A]$ = prependicular i.e $(x^T \cdot x = 0)$

26/12/2017

(10) $\lambda[A^{-1}] = \dfrac{1}{\lambda[A]}$

إذا كان ال A ← inverse

يعني انه $\lambda \neq 0$

(11) $\lambda[A_1 A_2]^{-1} = \lambda[A_1]^{-1} \lambda[A_2]^{-1}$

(12) $\lambda[A_1 + A_2] \neq \lambda[A_1] + \lambda[A_2]$ → eigen value

مش مرتبين

matlab Ex: given A_1, A_2, Validate the previous properties.

at Home :P

$$>> A_1 = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} ; \quad A_2 = \begin{bmatrix} -3 & 3 & 3 \\ -6 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

28/2/2017

$$\lambda\left[\begin{array}{c:c} A_{nn} & 0 \\ C & B_{m\times m}\end{array}\right] = \lambda\left[\begin{array}{c|c} A_{n\times n} & D \\ 0 & B_{m\times m}\end{array}\right] = \lambda[A] \,\underline{\cup}\, \lambda[B] ;$$

$\underline{\cup}$ indicates union Counting multiplicity

$\lambda\begin{bmatrix}\text{diagonal} \\ \text{matrix}\end{bmatrix}$ or $\lambda\begin{bmatrix}\text{upper triangular} \\ \text{matrix}\end{bmatrix}$ or $\lambda\begin{bmatrix}\text{lower triangular} \\ \text{matrix}\end{bmatrix}$

are the elements of the diagonal

Example:

$$A = \left[\begin{array}{c:c:c:c} 1 & 2 & 3 & 4 \\ \hdashline 0 & 5 & 6 & 7 \\ \hdashline 0 & 0 & 8 & 9 \\ \hdashline 0 & 0 & 0 & 10 \end{array}\right]$$

eigen value: 1, 5, 8, 10

$\lambda[A] = \{1, 5, 8, 10\}$

$$\lambda[A] = \left[\begin{array}{cc:c} -1 & 2 & 0 \\ 3 & 4 & 0 \\ \hline 10^6 & 1734 & 5 \end{array}\right] = \{5\} \,\underline{\cup}\, \lambda\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$$

use $|B - \lambda I_2| = 0$

$\rightarrow |A| = 5 * -10 = -50$

$\{5\} \,\underline{\cup}\, \{-2, 5\}$

$= \{-2, 5, 5\}$.

stair case

$$* \text{ If } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \qquad a_1 \neq 0$$

$$|A - \lambda I_n| = \lambda^4 - a_4\lambda^3 - a_3\lambda^2 - a_2\lambda - a_1 = 0$$

Known as characteristic equation (polynomial) C.E

28/2/2017

→ The calculation of the eigen vector for this Spiecl stair case matrix are

$$\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$$

→ if the eigen values are identical, say 3 eigen values then the three eigen vectors are

$$\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2\lambda \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ \vdots \end{bmatrix}$$

المشتقة

→ Example: calculate the eigen values and eigen vector of

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix}$$

(1) using Gauss elimination method
(2) adjoint method for two eigenvector
(3) using the stair case property

5/3/2017

$\lambda \leq \|A\|$ for any matrix norm

Example: let $A = \left[\begin{array}{cc:c} -2 & -1 & 0 \\ 1 & -2 & 0 \\ \hdashline 3 & 4 & 5 \end{array}\right]$

$$\lambda[A] = \lambda\left[\begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}\right] \cup \lambda[5]$$

$= \{-2 \pm j, 5\}$ → Used as a check to show that $|\lambda| < \min\{7.5, 5\}$

$\|A\|_1 = \max\{6, 7, 5\} = 7$ ⇒ Column

$\|A\|_\infty = \max\{3, 3, 12\} = 12$ ⇒ row

$\|A\|_2 = 7.25 = \max\{\lambda[A^T A]\}^{\frac{1}{2}}$

$|-2 \pm j| = \sqrt{(-2)^2 + (\pm 1)^2} = \sqrt{5}$

$|5| = 5$

⇒ Calculating e^{At} using the Eigen vectors.

Given $\dot{x} = Ax + Bu$

Assuming A has distinct eigenvalues λ_i $i = 1, 2, \ldots, n$ then the n eigenvector v_i are independent

leading to $W = [w_1\ w_2 \ldots w_n]$ being nonsingular i.e. W^{-1} exists.

Using $Aw_i = \lambda_i w_i$

$$A[w_1 \; w_2 \cdots w_n] = [w_1 \; w_2 \cdots w_n] \begin{bmatrix} \lambda & 0 & \\ 0 & \lambda_2 & \\ & & \ddots \; \lambda_n \end{bmatrix}$$

$$AW = W\Lambda \; ; \; |W| \neq 0$$

$$A = W\Lambda W^{-1} \quad \text{order important}$$

$$e^{At} = e^{W\Lambda W^{-1}t}$$

$$= e^{W\Lambda t W^{-1}}$$

It can be shown that $\rightarrow$

$$e^{At} = W e^{At} W^{-1}$$

$$= W \begin{bmatrix} e^{\lambda_1 t} & & O \\ 0 & e^{\lambda_2 t} & \\ 0 & 0 & \ddots \; e^{\lambda_n t} \end{bmatrix} W^{-1}$$

Examples: Given $\dot{x} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Calculate e^{At}

$\lambda^2 + 9\lambda + 20 = 0$

$(\lambda+4)(\lambda+5) = 0$

$\lambda_1 = -4 \Rightarrow$

$\lambda_2 = -5$

eigen vector

$w_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ or $\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$

$w_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

$$e^{At} = W e^{At} W^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix}$$ $* -1 \leftarrow$ determinante.

$$\begin{bmatrix} e^{4t} & e^{-5t} \\ -4e^{-4t} & -5e^{-5t} \end{bmatrix}$$

$$\begin{bmatrix} 5e^{-4t} - 4e^{-5t} & e^{-4t} - e^{-5t} \\ -20e^{-4t} + 20e^{-5t} & -4e^{-4t} + 5e^{-5t} \end{bmatrix}$$

← لما نعوض $t=0$
لازم تطلع ال I_n
عشان أتأكد انه
حلنا صح

* Ex: Confirm the result using i) $\mathcal{L}^{-1}[SI - A]^{-1}$

ii) $e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -20t & -9t \end{bmatrix} + \begin{bmatrix} -20t^2 & -9t^2 \\ 180t^2 & 61t^2 \end{bmatrix} * \frac{1}{2} + \dots$$

→ power series expansion method

using matlab

```
>> A=[0 1 ; -20 -9] , expm(A)
```

→ Steady State Value Due to a unit step 80.

Given $\dot{X} = AX + Bu$ Where the System is stable

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau, \quad X(0) \neq 0$$

Compare with $\frac{\partial x}{\partial t} = ax + f(t)$

$$X_{ss} = \lim_{t\to\infty} X(t) = \lim_{t\to\infty} e^{At} X(0) + \lim_{t\to\infty} e^{At} \int_0^t e^{-A\tau} Bu(\tau)\, d\tau$$

Zero

$$X_{ss} = e^{At} \left[\lim_{t\to\infty} \int_0^t e^{-A\tau} \times 1 \times d\tau \right] B$$

evaluate $\int_0^t e^{A\tau}\, d\tau = \int_0^t I + At + \frac{A^2t^2}{2!} + \dots\, d\tau$

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$$= \left[I\tau + \frac{A\tau^2}{2!} + \frac{A^2\tau^3}{3!} + \cdots \right] \Big|_0^t$$

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$$\int_0^t e^{-At}\, d\tau = \int_0^t \left[I_n - A\tau + \frac{A^2\tau^2}{2!} - \frac{A^3\tau^3}{3!} + \cdots \right] d\tau$$

$$= \left[I_n\tau - \frac{A\tau^2}{2!} + \frac{A^2\tau^3}{3!} - \cdots \right] A A^{-1} \Big|_0^t$$

$$= \left[A\tau - \frac{A^2\tau^2}{2!} + \frac{A^3\tau^3}{3!} - \cdots \right] A^{-1} \Big|_0^t$$

$$= \left[I_n - e^{-A\tau} \right] A^{-1} \Big|_0^t$$

$$= \left[I_n - e^{-At} - I_n + I_n \right] A^{-1} = \left[I_n - e^{-At} \right] A^{-1}$$

$$X_{ss} = \lim_{t \to \infty} e^{At} \left[I_n - e^{-At} \right] A^{-1} B$$

$$X(t) = \lim_{t \to \infty} \left[e^{At} - I_n \right] A^{-1} B$$

in stable System $e^{At} = 0$

$$= -A^{-1} B$$

Ex: Use the transfer function matrix to confirm this result

$$G(s) = C[SI - A]^{-1} B + D$$

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Hint: Consider a suitable C & D together with any limitting theorems in the S-domain

Ex: Use $\dot{x} = Ax + Bu$ to confirm the previous result ; u is a unit system is stable A^{-1} exists.

Theorem: for linear time-invariant systems.

If $u(t) \longrightarrow x(t)$ then

$$\frac{\partial u(t)}{\partial t} \longrightarrow \frac{\partial x(t)}{\partial t}$$

$$\int u(t)\, dt \longrightarrow \int x(t)\, dt$$

Ex: Given $\dot{x} = Ax + Bu$

Use $x(t)$ for a unit step (see notes and book)

Determine $x(t)$ for a unit ramp input (c.f. with book)

12/3/2017
Sunday

Diagonalization of a Matrix

Consider $\dot{x} = Ax + Bu$

$$\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$\lambda_1 + \lambda_2 = -1 + 4 = 3$

$\lambda_1 \lambda_2 = -10$

Solving $\lambda_1 = -2$, $\lambda_2 = 5$

$w_1 = adj \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$

$w_2 = adj \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\Lambda = W^{-1} A W$

λ in caps.

$$= \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} * -\frac{1}{7} = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

This is adiagonal Matrix

* example:- Diagonaliz $A = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix}$ $\Rightarrow$ Sol.

Q

$\rightarrow$ eigen(A) ans. $\rightarrow [1 \quad -2 \quad 1]$

write down the Matlab commands which Determine the eign vector. $\rightarrow$ eig(A)

the Trace

Sol. For previous example.

```
>> [w e] = eig(A)
lamda = inv(w) * A * w
```

ans.

$$\begin{matrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{matrix}$$

→ check the answer By hand calculations.

* System invarient Under Similarity Transformation.

given $\dot{x} = Ax + Bu$

$y = Cx + Du$

let $x = P\bar{x}$

$P\dot{\bar{x}} = AP\bar{x} + Bu$

$\dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu = \bar{A}\bar{x} + \bar{B}u$

$y = Cx + Du$

$y = C(P\bar{x}) + Du$

$x = P\bar{x}$

Identical $\neq$

Def: Given matrices A and $\bar{A}$ they are similar iff their exists a transformation P such that

$$\bar{A} = P^{-1}AP$$

$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ is similar to $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

where $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

eigenvector of A

or

$$A = P\bar{A}P^{-1}$$

* System invarients

$$\text{trace}(P^{-1}AP) = \text{trace}(A) = \text{trace}(\bar{A})$$

$$|\bar{A}| = |P^{-1}AP| = |A|$$

Note: Similarity doesn't mean identically

Proof: Using $|A||B| = |BA| = |A||B|$

- $|CB| = |BC| = |C||B| = |B||C|$

let $C = P^{-1}A$, $B = P$

$$|P^{-1}AP| = |PP^{-1}A| = |I_n A| = |A| \quad \#$$

- $\lambda[\bar{A}] = \lambda[A]$

Proof: using $\lambda[CB] = \lambda[BC]$

$$\lambda[\bar{A}] = \lambda[P^{-1}AP]$$
$$\lambda[PP^{-1}A] = \lambda[I_n A] = \lambda[A]$$

N.B: Similarity Transformation (ST) doesn't change the eigenvalues. It changes the eigenvectors (not invariant under ST)

- The TFM is invariant under ST

$$\bar{G}(s) = \bar{C}[SI-\bar{A}]^{-1}\bar{B}+\bar{D}$$

$$= CP[SI-P^{-1}AP]^{-1}P^{-1}B + D$$

$$= C[SIP^{-1} - P^{-1}APP^{-1}]^{-1}P^{-1}B + D$$

Using $(CD)^{-1} = D^{-1}C^{-1}$

$$= C[PSIP^{-1} - P^{-1}PAPP^{-1}]^{-1}B+D$$

$$= C[SI-A]^{-1}B+D$$

$$= G(s)$$

Theorem: every matrix satisfies its C.E. (C. equation) i.e

if $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n = 0$

is the C.E of A then

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI_n = 0$$

→ This is known as Cayley-Hamilton Theorem

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Ex: let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

i) determine C.E

ii) verify C.H theorem

iii) use the theorem to determine A^2 without squence A

iv) " " " " " A^{-1}.

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» Fifteen → لعبة
للتسلية ^^

* Jordan Blocks (Forms)

Result when there is insufficient number of independent eigen vectors associated with a repeated eigen value.

i.e if a matrix A has 1, 2, 3, 3, 3 eigen values then possible Jordan block maybe

$$J_1 = \left[\begin{array}{ccc|cc} 3 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{or} \quad J_2 = \left[\begin{array}{ccc|cc} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

* Generalized Eigen Vectors

Consider a matrix A with 3 identical eigen values (λ) then, at least an eigen vector ω_1 exists

$$[A - \lambda I]\,\omega_1 = 0$$

To get the other two eigen vectors (called generalized eigen vector), ω_2, ω_3

Solve $[A - \lambda I]\,\omega_2 = \omega_1$; $[A - \lambda I] = 0$, no inverse

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& $[A-\lambda I]\,\omega_3 = \omega_2$

Example:: let $A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ $-\lambda^2 - 4\lambda + \lambda = 0$

$(\lambda-2)^2 = 0$

$\lambda_1 = 2,\ \lambda_2 = 2$

$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \omega_1 = 0$

$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \omega_1 = 0 \Rightarrow \omega_1 = \begin{bmatrix} +1/2 \\ 1 \end{bmatrix}$

Solve ⇒

$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \omega_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \omega_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

$\omega_{22} = 1$

$-2\omega_{11} + 1 = \frac{1}{2} \Rightarrow \omega_{11} = \frac{1}{4}$

$\omega_2 = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$

⇒ ω_1 & ω_2 are independent i.e $|\omega_1 \omega_2| \neq 0$

To check answer

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$A_J = W^{-1} A W$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

→ another set of vector

$$W_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_J = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

* The exponential matrix e^{Jt}

let $J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ or $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

Simple structure.

it can be shown that « see book »

$$e^{Jt} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} & t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & t e^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

Example 20

let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & t e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

to check it at $t=0$ must give us the Identity

$\Rightarrow \left[e^{At} \right]^{-1} = e^{A(-t)}$

$$\left[e^{At} \right]^{-1} = \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{-2t} & -t e^{2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

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A and e^{At} Commute, i.e

$$A e^{At} = e^{At} A$$

$$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$$

$$e^{(a+b)t} = e^{at} \cdot e^{bt}$$ a, b Scalor.

$$e^{(A+B)t} \neq e^{At} \cdot e^{Bt}$$ A, B matrix

$=$ if A, B Commute.

$\rightarrow$ $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ $J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $\Rightarrow$ trace(9)

$|J| = 27$

A = P * J * inv(P)

Ex: Given $A = \begin{bmatrix} 5 & 1 & -1 \\ -2 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ $\Rightarrow$ trace(9)

$|A| = 9$

obtain the Jordan block for A i.e get the eigenvectores with associated eigenvectors and generalized eigenvectors, then compute

$P^{-1} A P \rightarrow J$, Check using matlab [P e] = eig(A),

J = inv(P) * (A) * P

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Basis of Jordan Blocks

To diagonalize a matrix A, we use a similarity transformation W consisting of the eigenvectors as columns.

matrix

i.e $A[w_1\ w_2 \cdots w_n] = [w_1\ w_2 \cdots w_n]\begin{bmatrix}\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots \\ 0 & 0 & 0 & \lambda_n\end{bmatrix}$

$\rightarrow AW = W\Delta$

$\rightarrow A = W\Delta W^{-1}$

$\rightarrow W^{-1}AW = \Delta$

If W are the eigenvector of A corresponding to distinct eigen values then W is always nonsingular

hence $\Delta = W^{-1}AW$ where Δ is diagonal.

→ Suppose now, we have repeated eigenvalues, then a possible form resembling that of a diagonal form is the following

$$A[w_1\ w_2\ w_3] = [w_1\ w_2\ w_3]\begin{bmatrix}\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda\end{bmatrix}$$

$Aw_1 = \lambda w_1 \Rightarrow [A-\lambda I]w_1 = 0$

$Aw_2 = w_1 + \lambda w_2 \Rightarrow [A-\lambda I]w_2 = w_1$

$Aw_3 = w_2 + \lambda w_3 \Rightarrow [A-\lambda I]w_3 = w_2$

→ eigen vector $\neq 0$

$$[A-\lambda I]\,w_1 = 0$$

$$[A-\lambda I]\,w_2 = w_1 \Rightarrow [A-\lambda I]^2 w_2 = 0$$

$$[A-\lambda I]\,w_3 = w_2 \Rightarrow [A-\lambda I]^3 w_3 = 0$$

Ex: $A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$, $\lambda_1 = 2$, $\lambda_2 = 2$

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} w_1 = 0 \Rightarrow w_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} w_2 = 0$$, many Choices for w_2: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dots$

Check: let $w_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$AW = W\Delta$$

$$\Delta = W^{-1} A W$$

$$\Delta = \begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

When we chose $w_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Delta = W^{-1} A W$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

The importance of Jordan block is getting them as upper triangular matrices

26/3/2017

* Independance of vectors :

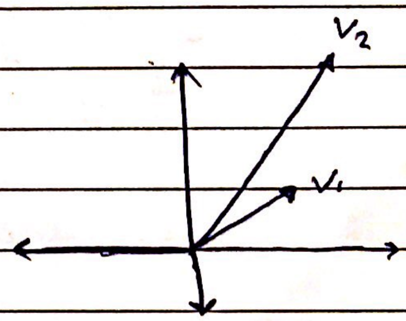
Def: Vectors $v_1, v_2, \dots v_n$ are independant iff

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

implies $c_1 = c_2 = \dots = c_n = 0$

Examples: let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1 + \begin{bmatrix} 2 \\ 5 \end{bmatrix} c_2 = 0$$



$$c_1 + 2c_2 = 0$$
$$2c_1 + 5c_2 = 0$$

$$0 + c_2 = 0 \Rightarrow c_2 = 0$$
$$c_1 = -2c_2 = 0$$

hence v_1 and v_2 are Independent

Example: let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$

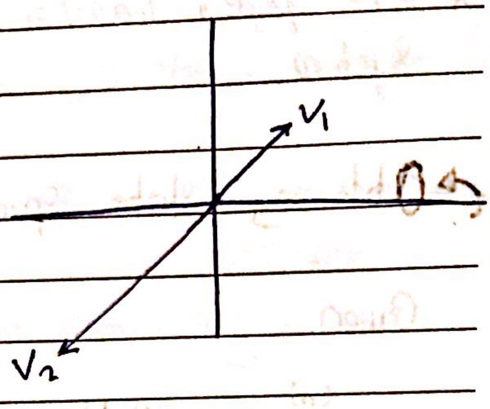
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1 + \begin{bmatrix} -4 \\ -8 \end{bmatrix} c_2 = 0$$

$$c_1 - 4c_2 = 0 \qquad \Rightarrow$$
$$2c_1 - 8c_2 = 0 \qquad \text{using Gauss-elimination}$$

26/3/2017

$$\begin{bmatrix} 1 & -4 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$0 . c_2 = 0 \Rightarrow c_2 = r \neq 0$ arbitrary

$c_1 = 4c_2 \Rightarrow c_1 = 4r$

* Implications: Consider a 3 dimensional vectors.

- Two ID vectors span a plane.
- Three ID vectors span a volume (space).
- If 3 vectors constitute a square matrix then the determinant is non-zero.
  This can be used as a check of indepency.
- Any forth vector can't be independent of the other 3 ID vectors.
- The number of ID vectors in a matrix (as columns of a matrix) is the rank of that matrix
- If A is $m \times n$ matrix then
  $\operatorname{rank}(A) \leq \min(m, n)$
- If A is square then if $\operatorname{rank}(A) = n$ then $|A| \neq 0$
  $\Rightarrow$ A is nonsingular

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28/3/2017

* State Space, Transfer functions, and Block Diagram Models of System

Obtaining state space models from Differential equations

Given

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = \underset{\text{constant}}{b}\, u(t)$$

let $y = x_1$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \dot{x}_2 = x_3$

$\vdots$

$y^{(n-1)} = \dot{x}_{n-1} = x_n$

$$\dot{x}_n = y^{(n)} = -a_1 y^{(n-1)} - a_2 y^{(n-2)} - \cdots - a_n y + b\,u(t)$$

$$= -a_1 x_n - a_2 x_{n-1} - \cdots - a_n x_1 + b\,u(t)$$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & & & & 1 \\ -a_n & \cdots & & -a_2 & -a_1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b \end{bmatrix} u \quad \rightarrow b \neq 0$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} X + 0 \cdot u$$

This form is known as a controllable Companion Form.

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→ G(s) can be obtain either i) Through the LT of the DE
ii) using
$G(s) = C[SI - A]^{-1}B + D$

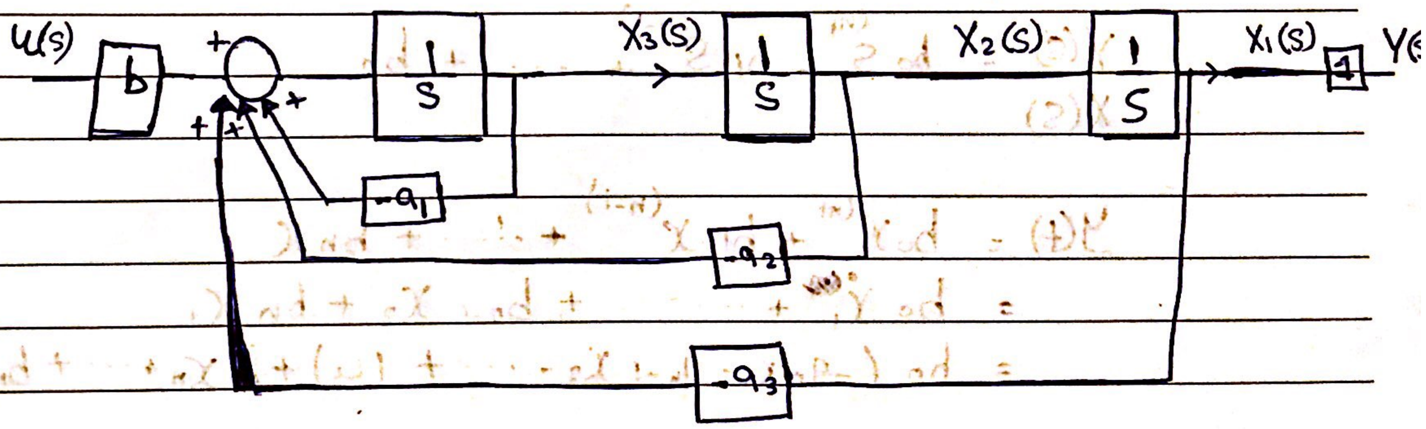
→ To obtain block diagram presentation consider the 3rd order system for simplicity.

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\, X$$

$\Rightarrow \dot{X}_1 = X_2 \Rightarrow X_1(S) = \frac{1}{S} X_2(S)$

$\dot{X}_2 = X_3 \Rightarrow X_2(S) = \frac{1}{S} X_3(S)$ integrator.



The case when derivatives of $u(t)$ are present

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_n u$$

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$\frac{Y(s)}{X(s)} \frac{X(s)}{U(s)} = (b_0 s^n + b_1 s^{n-1} + \cdots + b_n) * \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n = u$$, let $x = x_1$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ -a_n & \cdots & & -a_2 & -a_1 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\frac{Y(s)}{X(s)} = b_0 s^{(n)} + b_1 s^{(n-1)} + \cdots + b_n$$

$$y(t) = b_0 x^{(n)} + b_1 x^{(n-1)} + \cdots + b_n x$$
$$= b_0 \dot{x}_1 + \cdots + b_{n-1} x_2 + b_n x_1$$
$$= b_0 (-a_n x_1 - a_{n-1} x_2 - \cdots + 1u) + b_1 x_n + \cdots + b_n x_1$$

$$\Rightarrow C = [\, b_n - b_0 a_n \quad b_{n-1} - b_0 a_{n-1} \quad \cdots \,] ; \quad D = b_0$$

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Examples

$$G(s) = \frac{2s^2 + 3s + 4}{s^3 + 5s^2 + 6} \Rightarrow \frac{Y(s)}{X(s)} \cdot \frac{X(s)}{U(s)} = \frac{Y(s)}{U(s)} \frac{\underline{\text{out}}}{\text{input}}$$

$$\frac{X(s)}{U(s)} = \frac{1}{1 \cdot s^3 + 5s^2 + 6}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 0 & -5 \end{bmatrix}$$

Poles in T.F $\Rightarrow$ eigen value for the matrix A

4 poles $\Longrightarrow$ $A \rightarrow 4 \times 4$

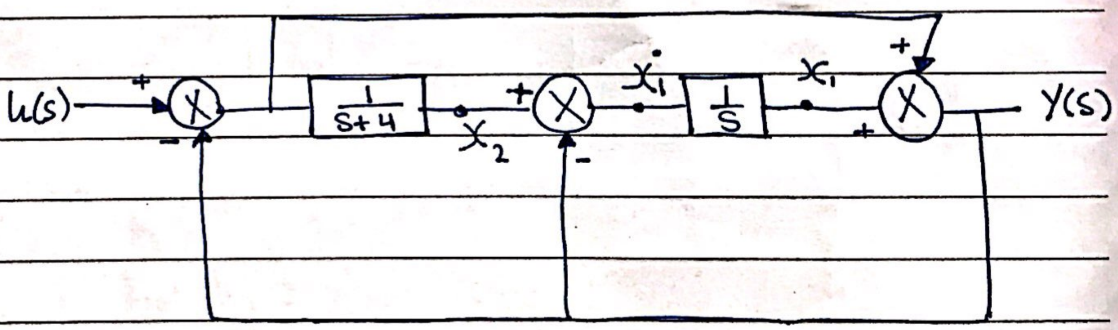
EX: Convert $y''' + 4y'' - 5y' + 2y = 3u'' + 6u' - 8u$

to a transfer function $G(s) = \frac{Y(s)}{U(s)}$

→ all initial conditions = 0

* Moving from Block Diagram to SS model

Introduce States at the output of integrators and Lags



$$\dot{x}_1 = x_2 - y$$

$$y = x_1 + u - y \quad \Rightarrow \quad y = \frac{1}{2}x_1 + \frac{1}{2}$$

hence $\dot{x}_1 = -\frac{1}{2}x_1 + x_2 - \frac{1}{2}u \quad \ldots\ldots (1)$

$$x_2 = \frac{1}{s+4}(u - y)$$

$$\dot{x}_2 + 4x_2 = u - \frac{1}{2}x_1 - \frac{1}{2}u$$

2/4/2017

$$\dot{x}_2 = -\frac{1}{2}x_1 - 4x_2 + \frac{1}{2}u \quad \text{-- ((2))}$$

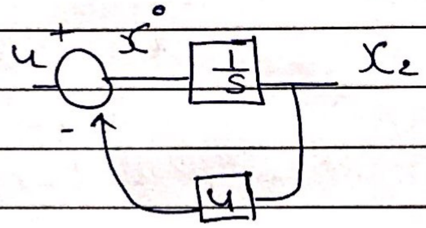
$$\dot{x} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & -4 \end{bmatrix} x + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} x + \frac{1}{2} u$$

Check: evaluate $G(s) = C[SI - A]^{-1}B + D$

do it at home

variation $\boxed{\dfrac{1}{S+4}}$ =



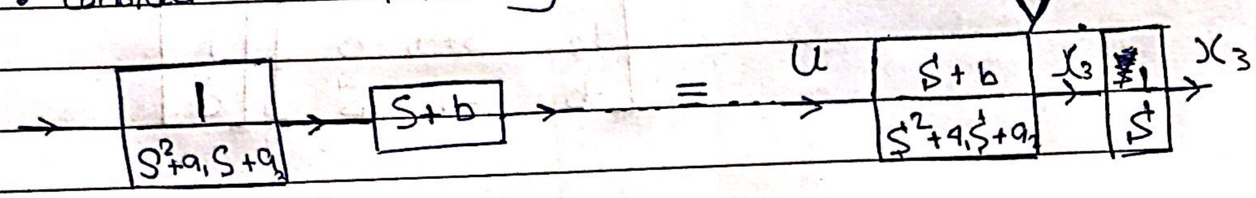
$$\frac{S+2}{S+5} = 1 + \frac{-3}{S+5}$$

→ See aditional Example in the book Ch9

$$\frac{S+10}{S} = 1 + \frac{10}{S}$$

4/4/2017

Example: Consider the following case



$$\frac{Y}{u} = \frac{Y}{X}\,\frac{X}{u} = (s+b)\,\frac{1}{S^2+a_1S+a_2}$$

$\Rightarrow Y = X \quad , \quad X_1 = X \Rightarrow y = x_1 \Rightarrow \dot{x}_1 = x_2$

$$\dot{x}_2 = -a_2x_1 - a_1x_2 + u$$

$$y = \dot{x}_1 + bx_1 = bx_1 + x_2$$

$$\dot{x}_3 = y = \dot{x}_1 + bx_1 = bx_1 + x_2$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -a_2 & -a_1 & 0 \\ b & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

if the output z is that of the integrator

then $\Rightarrow z = x_3$

$C = [0 \quad 0 \quad 1]$

eigen Values $\Rightarrow -a_1 , 0 , 0$

Check the T.F $\Rightarrow G(s) = C[SI - A]^{-1}B + \cancel{D}$

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$$G(s) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ a_2 & s+a_1 & 0 \\ -b & -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & \text{كل هدول مضروبين بـ zero ما بهمني الهم} & \\ b(s+a_1)-a_2 & s+b & s(s+a_1)+a_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * \frac{1}{\Delta}$$

inverse هنا اللي بهمنا اللي با ←

هذا اللي بهمني لانه رح ينضرب بـ 1

$$G(s) = \frac{s+b}{s(s(s+a_1)+a_2)}$$

$$G(s) = \frac{s+b}{s(s^2+a_1 s+a_2)}$$

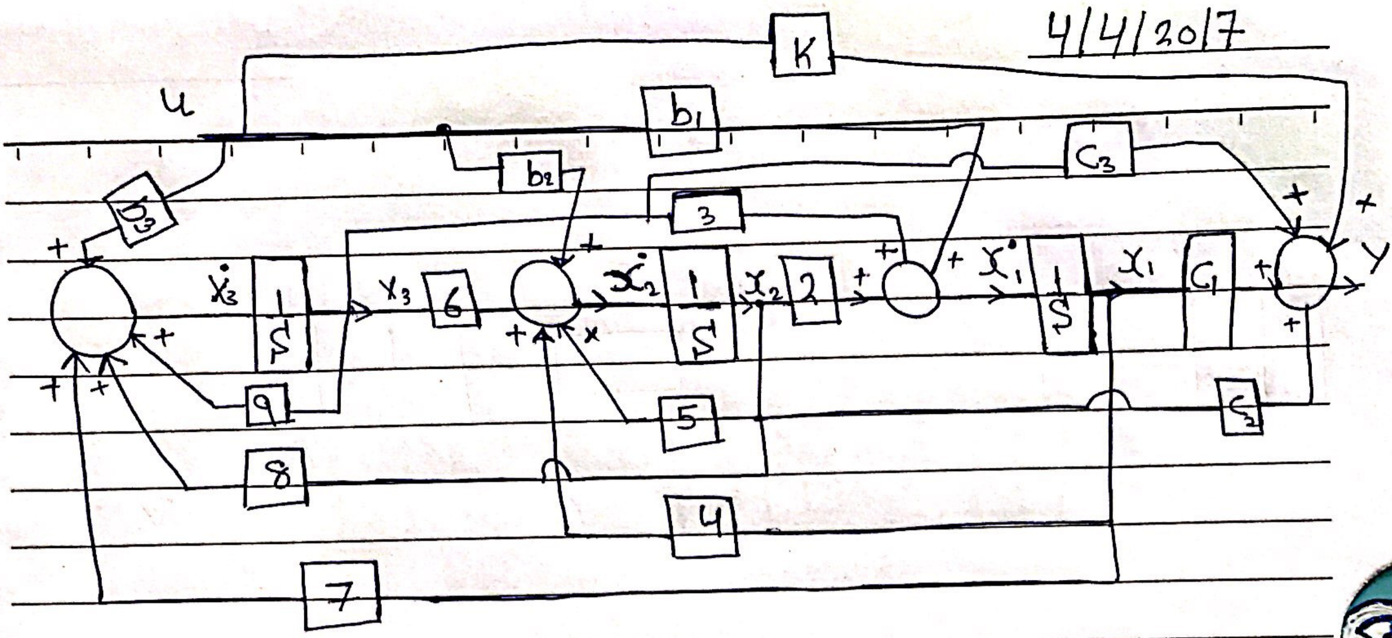
* Obtaining a BD using SS Representation

Given
$$\dot{x} = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x + \begin{bmatrix} K \end{bmatrix} u$$

⇒ Since System is Third order use 3 integrator

4/4/2017



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*Obtaning BD and SS model Using a T.F

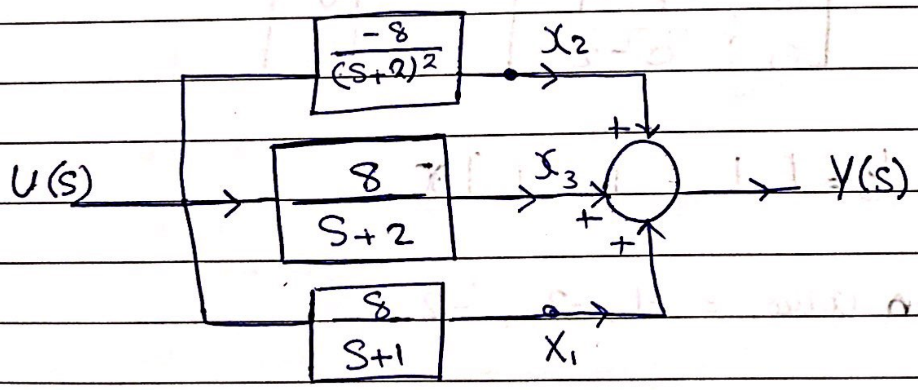
Example: let $\frac{Y(s)}{U(s)} = \frac{8}{(s+1)(s+2)^2}$

$$= \frac{8}{s+1} + \frac{-8}{(s+3)^3} + \frac{C}{s+2}$$

$$C = \frac{d}{ds}\left(\frac{8}{s+1}\right)\Bigg|_{s=-2}$$

$$Y(s) = \frac{8\,U(s)}{s+1} + \frac{-8\,U(s)}{(s+2)^2} + \frac{\overset{-8}{\cancel{C}}\,U(s)}{s+2}$$

Giving the following B.D.



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* Solving also the SS model

$$X_1(s) = \frac{8U(s)}{s+1} \Rightarrow s X_1(s) = -X_1(s) + 8U(s)$$

$$\dot{x}_1 = -x_1 + 8u(t)$$

$$X_2(s) = \frac{-1}{s+2} \frac{8U(s)}{(s+2)} \Rightarrow \frac{-1 * X_3(s)}{s+2}$$

$$\dot{x}_3 = -2x_3 + 8u(t)$$

$$\dot{x}_2 = -2x_2 - x_3$$

$$Y(s) = X_1(s) + X_2(s) + X_3(s)$$

$$y = x_1 + x_2 + x_3$$

$$x' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 8 \\ 0 \\ 8 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

eigen value = -1, -2, -2

Ex: obtain the B.D and SS models of

i) $$\frac{Y(s)}{U(s)} = \frac{640(s+5)}{(s+2)(s+4)^3}$$

ii) $\frac{Y(s)}{U(s)} = \frac{58}{(s+1)(s^2+4s+29)}$

```
>> syms s , g = C*inv(s*eye(size(A))-A)*B+D ,
                                       simplify
                                       or, simple(g)
                                       or, pretty(g)
```

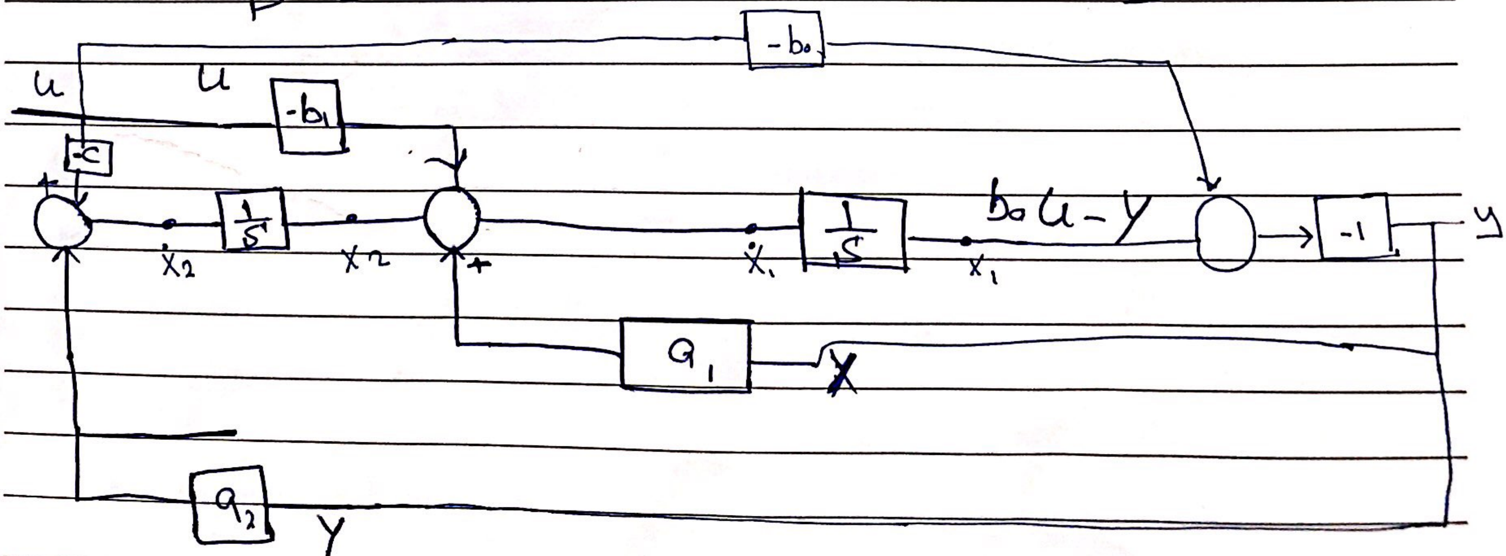
Giving also the SS model

Example: let $\frac{Y(s)}{U(s)} = \frac{b s^2 + b_1 s + c}{s^2 + a_1 s + a_2}$

$$Y(s)\left(s^2 + a_1 \frac{1}{s} + a_2 \frac{1}{s^2}\right) = \left(b_0 + b_1 \frac{1}{s} + c \frac{1}{s^2}\right) U(s)$$

$$= (a_2 Y - C U)\frac{1}{s^2} + (a_1 Y - b_1 U)\frac{1}{s} = b_0 U - Y$$

$$= \frac{1}{s}\left[(a_2 Y - C U)\frac{1}{s} + (a_1 Y - b_1 U)\right] = b_0 U - Y$$



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$X_1 = b_0 u - y$ $\qquad\qquad y = b_0 u - X_1$

$\dot{X}_1 = X_2 + a_1(-X_1 + b_0 u) - b_1 u$

$-a_1 X_1 + X_2 + (a_1 b_0 - b_1) u$

$\dot{X}_2 = a_2(-X_1 + b_0 u) - CV$

$= -a_2 X_1 + X(a_2 b_0 - C) u$

$$\dot{X} = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} X + \begin{bmatrix} a_1 b_0 - b_1 \\ a_2 b_0 - C \end{bmatrix}$$

$$y = [-1 \quad 0] X + b_0 u$$

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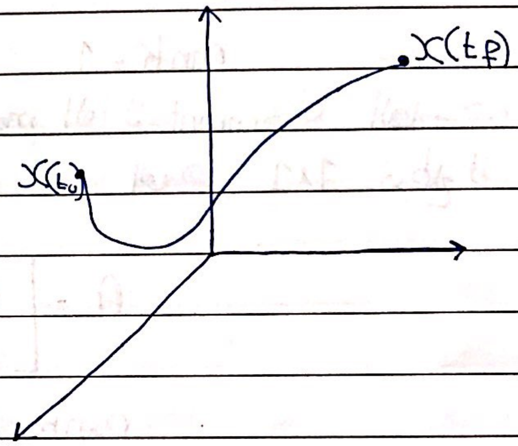
→ Structural probarties of systems

They Give qualitative information about Systems

→ Controlability :

Def:

The system is completly Controlabil if it is possible to move it from an intial state $X(t_0)$ to any arbitrary final state $X(t_f)$ in a finit time using an un Constrained control law



* A Test of Controlability (CC):

→ A system $\dot{x} = Ax + Bu$ is completely CC iff

$rank(M_c) = rank([B \quad AB \quad A^2B \cdots A^{n-1}B]) = n$

Def: rank of amatrix is the number of independent Columns

If A is $m \times n$ matrix then $rank(A) \leq \min(m, n)$

$rank = 0 \rightarrow matrix = 0$

11/4/2017

→ For a square matrix A if $|A| \neq 0$ then the rank is n.

Generally for a non square matrix the rank is the highest dimention of a sub matrix having a non zero determinant.

Example: let $A = \begin{bmatrix} 1 & 4 & -3 & 5 \\ 2 & 8 & -6 & 10 \end{bmatrix}_{2\times4}$ ≤ 2

بناخذ Sub matrix وبنطلع determinant ← rank = 1

وبتطلع 0 فباخذ ال matrix الي ال determinant الها مش Zero الي هي 1x1 فبطلع ال rank = 1

$A = \begin{bmatrix} 1 & 4 & -3 & 5 \\ 2 & 8 & 6 & 10 \end{bmatrix}$

rank = 2

Sub matrix

$|2\times2| \neq 0$

$A = \begin{bmatrix} 1 & 4 & -3 & 5 \\ 2 & 8 & 6 & 10 \\ 4 & 16 & 0 & 20 \end{bmatrix}$

بحسب |3x3| لأي matrix داخل A اذا = 0

بحسب |2x2| لأي matrix داخل A اذا = 0

بحسب |1x1|

كل هذا عشان احسب ال rank.

11/4/2017

Example : $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$M_c = [\, B \;\; AB \,]$

$M_c = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\qquad |M_c| \leq 2$

$\text{rank}(M_c) = 1 < 2 \Rightarrow$ uncontrollable

Ex: $\dot{x} = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$

$\Rightarrow$ Uncontrolabilty results in pole Zero Cancelation in the T.F

Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$

$y = [\, 1 \;\; 2 \,]\, x$

Calculate $G(s)$ then judge Controllability Confirm using the rank test

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→ The rank of an m×n matrix is the number of maximum number of independent columns (rows)

Column rank = row rank

Example :
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$
$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ → un Controlable

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → any number Controlable.

→ stair

$$Y = [1 \quad 2]x + [0]u$$
always Controlable

$$G(s) = C[SI - A]^{-1}B + D$$

$$G(s) = [1 \quad 2]\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} * \frac{1}{(s+1)(s+2)}$$

$$= [s+1 \quad 1+2s]\begin{bmatrix} 1 \\ -1 \end{bmatrix} * \frac{1}{(s+1)(s+2)}$$

$$= \frac{-s-2}{(s+1)(s+2)} = \frac{-\cancel{(s+2)}}{(s+1)\cancel{(s+2)}} = \frac{-1}{s+1}$$
cancelation un Controlable.

Controlability Based on Diagonalization :

1) Distinct eigen values :

There exists a transformation according to $x = P\bar{x}$ giving

$$\dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}B\,u$$

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\bar{x} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

→ The system is completly controllable iff all $\beta_i \neq 0$

Ex : $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$ has eigen Values $-1, -2$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \bar{B} = P^{-1}B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

⇒ eigen value -2 is uncontrollable

suppose $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $P^{-1}B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

⇒ system controllable

18/4/2017

Ex: Given $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x & x & x & x \end{bmatrix}$ & $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x \end{bmatrix}$

Prove that the system is cc irrespective of x ?!

* Controllability of Jordan Form :

Given

$$\dot{x} = \left[\begin{array}{cc:c} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ \hdashline 0 & 0 & \lambda \end{array}\right] x + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

$$M_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \qquad A^2B = A \cdot AB$$

$$M_c = \begin{bmatrix} \beta_1 & \beta_1\lambda + \beta_2 & \beta_1\lambda^2 + \beta_2\lambda + \beta_2\lambda \\ \beta_2 & \beta_2\lambda & \beta_2\lambda^2 \\ \beta_3 & \beta_3\lambda & \beta_3\lambda^2 \end{bmatrix}$$

For Controllability $\beta_2 \neq 0$

$\beta_3 \neq 0$

18/4/2017

eg $\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

rank $(M_c) = \left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \right) = 1 < n = 2 \Rightarrow$ un controlable

إذا اخر قيمة ل $B = 0 \leftarrow$ un controlable
في JB

```
>> A = [0 1 0
        0 0 1
        27 -27 9 ]

   eig(A)

>> B=rand(3,1) , rank([B  A*B  A^2*B])

>> Mc = ctrb(A,B) , r=rank(Mc)
```

20/4/2017

* Observability

Def: A system is completely observable if it is possible to retrieve all the states from measurments of the output over a finite time period.

* A Test of observability

A system is completly observable iff $\text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$

* If a system is unobservable, then we have pole-zero cancellation in the T.F.

Ex: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ → controlable by incpection

Staircase and $B = \begin{bmatrix} 0 \\ x \end{bmatrix}$

$y = [1 \quad -1] x$

$\text{rank}\left(\begin{bmatrix} C \\ CA \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\right) = 2$ observable.

If $C = [1 \quad 1]$

$\text{rank}\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} = 1 \Rightarrow$ unobservable.

If $C = [1 \quad d]$

$$G(S) = \begin{bmatrix} 1 & \alpha \end{bmatrix} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} * \frac{1}{(S+1)(S+2)}$$

$$G(S) = \frac{1+\alpha S}{(S+1)(S+2)}$$

we have pole-zero concelation when $\alpha = 1$

Given $G(S) = \frac{1}{S+2}$

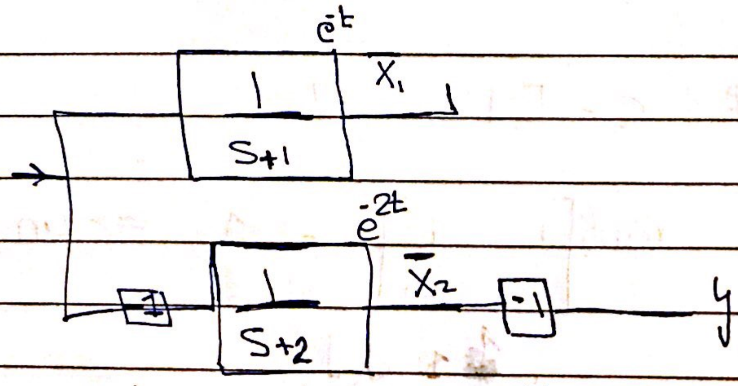
Diagonalize the system : $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$; $P^{-1} = \begin{bmatrix} +2 & 1 \\ -1 & -1 \end{bmatrix}$

$$\dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu$$
$$y = CP\bar{x}$$

$$\dot{\bar{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightsquigarrow \text{Controlable}$$

$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} \bar{x} = -\bar{x}_2$$

input can control all state «controlable» u



by insepection unobeservable, controlable.

i.e e^{-t} cannot be obtained from y

23/4/2017

N.B: Stability (s), Controllable (cc), and Observability (oo) don't imply each other i.e

S	cc	oo
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Theorem: For any uncontrollable system there exists a transformation P where $x = P\bar{x}$

if $\dot{x} = Ax + Bu$; $y = Cx + Du$

then $\dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu \Rightarrow \bar{A}\bar{x} + \bar{B}u$

$$\bar{y} = CP\bar{x} + Du \rightarrow \bar{C}\bar{x} + Du$$

$$\dot{\bar{x}} = \begin{bmatrix} A_c & A_2 \\ 0 & A_u \end{bmatrix} \bar{x} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

The cc eigenvalues are those of A_c

The uc " " " " A_u

Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} x + \begin{bmatrix} 1 \\ -5 \end{bmatrix} u$

chose $P = \left[B \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$

$$\dot{\bar{x}} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$= \begin{bmatrix} 0 & 1 \\ -20 & -4 \end{bmatrix} *$$

$$= \left[\begin{array}{c:c} -5 & 1 \\ \hdashline 0 & -4 \end{array} \right] \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad v$$

-4 is uncontrollable eigen value

Check rank $[B \quad AB]$

$$\text{rank}\left(\begin{bmatrix} 1 & -5 \\ -5 & 25 \end{bmatrix} \right) = 1 < n = 2 \Rightarrow \text{uncontrollable}$$

Ex: consider the previous system where $P = \begin{bmatrix} 1 & \alpha \\ -5 & 1 \end{bmatrix}$ prove uncontrollability for any value of α

Ex: given $\dot{x} = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$

chose $P = \left[B \quad \overset{AB}{[\;]} \quad [\;] \right]$ must be invertable (non-singular)

Investigate the controllability.

* Duality of linear systems:

For a system of the form

$$\dot{x} = Ax + Bu \qquad \text{Oregenal system (OS)}$$
$$y = Cx + Du$$

Anther System associated with it and called the dual system is of the form

$$\dot{z} = A^T z + C^T u \qquad \text{Dual System (DS)}$$
$$w = B^T z + D^T u$$

* The OS and DS have the same Stability Propertyies

proof: eigen value (A) = eigen valu (A^T)

$$\lambda[A] = \lambda[A^T]$$

* The two Systems have the same C.E

$$CE = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots\cdots(\lambda-\lambda_n) = 0 = |A^T - \lambda I_n|$$

➔ Theorem: A system is cc (oo) iff its dual is oo(cc)

» ctrb → Test controllability
» obsv → Test observability

➔ Theorem: An unobservable System can be decomposed as follows:

25/4/2017

$$\dot{\bar{x}} = \left[\begin{array}{c:c} A_0 & 0 \\ \hdashline A_2 & A_{\bar{o}} \end{array}\right] \bar{x} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = [\, C \;\vdots\; 0 \,] \bar{x}$$

Proof: Using Duality

* Another Proof : Using

$$OO = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} C_o & 0 \\ C_o A_o & 0 \\ C_o A_o^2 & 0 \\ \vdots & 0 \\ C_o A_o^{n-1} & 0 \end{bmatrix}$$

$$\text{rank}(OO) < n \Rightarrow \text{unobservable.}$$

27/4/2017

* CC & OO by Inspection :

There exists a transformation P which puts S(A,B,C,D) in a Jordan Form. In which case CC & OO can be determined by inspection.

(state space)

This is best illustrated by Example (P 680)

$$
\dot{x} = \left[\begin{array}{cc:cc:ccc:c:c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6
\end{array}\right] x + \left[\begin{array}{cc}
NZ & IR \\
IR & NZ \\
0 & 0 \\
NZ & IR \\
NZ & NZ \\
NZ & NZ \\
0 & 0 \\
0 & 0 \\
IR & NZ
\end{array}\right] u
\begin{array}{l}
1 \to cc \\
2 \to cc \\
\leftarrow JB \\
3 \to cc \\
 \\
 \\
4 \text{ is uc} \\
5 \text{ is uc} \\
6 \text{ is cc}
\end{array}
$$

NZ : non-Zero

IR : irrelevant

اخر سطر لازم

يكون عناصر Non

تكون Zero

CC

$$
y = \left[\begin{array}{c:c:cc:ccc:c:c}
0 & IR & 0 & NZ & NZ & 0 & 0 & NZ & 0 \\
0 & NZ & 0 & NZ & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

يعني اول عمود

برانق لل Block

إذا كان oo

un observable

1 is uo   2 is oo   3 is uo   4 is oo   5 is oo   6 is u[illegible]

For additional example See book

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* pole (Eigenvalue) placement (assignment)

- Suppose a system $S(A, B, C, D)$ response is deemed unsatisfactory.
- One way to change its behaviour is to employ state feedback.

i.e

$$\underline{u} = \underline{r} - K \underline{x}$$

r is reference point

$\dot{x} = Ax + Bu$; $y = Cx + Du$ → output لتغذية state راجعة feedback

$u = r - Ku$

$\dot{x} = Ax + B(r - Kx)$

$\dot{x} = (A - BK)x + Br$

* So choose a K to end up with a desired set of eigenvalues.

* Ackermann's Method

1. let $\lambda_1, \lambda_2, \dots \lambda_n$ be the desired eigenvalues to be assigned.
2. generate $P(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_2) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_n$

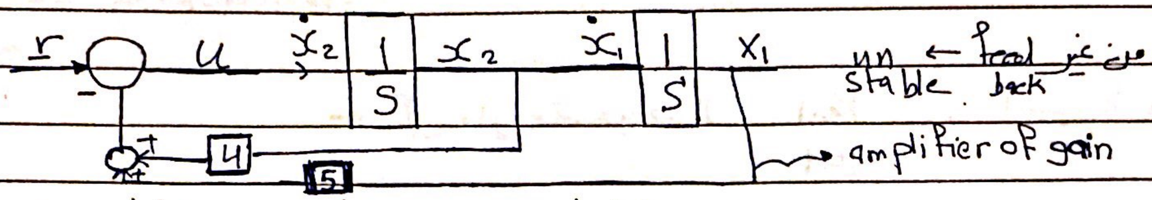
then

$$P(A) = A^n + \alpha_1 A^{n-1} + \dots + \alpha_n I_n$$

3. $K = [0 \; 0 \dots 1][B \; AB \dots A^{n-1}B]^{-1} P(A)$

27/4/2017

Example: Given $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$



un stable so it needs stabilization

Assign $-2 \pm j$.

$P(\lambda) = (\lambda + 2 - j)(\lambda + 2 + j) = \lambda^2 + 4\lambda + 5$

$P(A) = A^2 + 4A + 5 I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 0 & 5 \end{bmatrix}$

$K = [0 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 4 \\ 0 & 5 \end{bmatrix}$

$\rightarrow K = [5 \quad 4]$

Check $A - BK$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}$

$\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda_1 = -2 + j, \ \lambda_2 = -2 - j$

```
>> a=[0 1; 0 0] ; b=[0; 1] ; K=acker(a,b,[-2+i  -2-i])
```

N.B: K calculated for a single input system is uniqe.

إذا كان في الترمز input ← K is not unique.

≫ a=[0 1 ; 0 0]; b=[0 ; 1] ; K=acker(a, b, [-2+j -2-j]);
figure(1), step(a, b, eye(2), 0), 0), figure(2), step(a-b*k, b, eye(2), 0)

← كل مازادت قيمة الـ Real Part بصير أسرع الـ System

← بس قيمة الـ gain بتزيد more power Consumption

Ex: Given $\dot{x} = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$K = [0 \cdots 1][B \quad AB \quad A^2B]^{-1} P(A)$

N.B: ① a disadvantage of the Ackerman's method is that it doesn't apply to uncontrollable System.

② another disadvantage it doesn't work at multi input.

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* The entire-Eigenstructure Assignment Method (ENT)

Also deals with multi-input systems and uncontrollable systems.

Given $\dot{x} = Ax + Bu$

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues to be assigned.

1. Solve $[A - \lambda_i I_n] w_i = B z_i$, for w_i given a particular z_i for each λ_i

2. $K = [z_1 \; z_2 \dots z_n][w_1 \; w_2 \dots w_n]^{-1}$

N.B: If an λ_i is a re-assigned or uncontrollable eigenvalue then

$$|A - \lambda I_n| = 0$$

Example: Given $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$

assign -4 and -2

re-assigned because it is uncontrollable

For -4;

$$\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} w_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} * 6$$

z_i بنفرضها اي اشي

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$$w_1 = \begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} 6 * \frac{1}{6} = \begin{bmatrix}2\\ -2\end{bmatrix}$$

For -2 ;

مالها inverse $\quad \begin{bmatrix}2 & 1\\ -2 & -1\end{bmatrix} w_1 = \begin{bmatrix}1\\ -1\end{bmatrix} * 1$

Now solve using gauss-elimination , one solution is

$$\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}w_2'\\ w_2''\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} \Rightarrow w_2 = \begin{bmatrix}0\\ 1\end{bmatrix}$$

$$K = \begin{bmatrix}6 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ -2 & 1\end{bmatrix}^{-1}$$

$$= \begin{bmatrix}6 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix} 1/2 = \begin{bmatrix}4 & 1\end{bmatrix}$$

Check $A - BK$:

$$A - \begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}4 & 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} - \begin{bmatrix}4 & 1\\ -4 & -1\end{bmatrix} = \begin{bmatrix}-4 & 0\\ 2 & -2\end{bmatrix} \#$$

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$$\dot{x} = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} * \begin{array}{c} z_i \\ \begin{bmatrix} * \\ * \end{bmatrix} \end{array}$$

This system needs a controler since it is unstable due to A having an eigenvalue of +1.

assign −3, −4 & −5 negative. System stable.

Solve $[A - \lambda_i I_n]\, \omega_i = B z_i$

, for $\lambda_1 = 3$, $\lambda_2 = -4$, $\lambda_3 = -5$

Choosing $z_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $z_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $z_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

\>>

\>> B=[1 0 ; 1 1 ; 0 0] ; z1 = [1 ; 0] ; z2 = [0 ; 1] ; z3 = [1 ; 1]

\>> lam = -3 ; A1 = A - lam * eye(3) ; w1 = A1\B*z1

$$\omega_1 = \begin{bmatrix} 1 \\ 1.5 \\ -0.5 \end{bmatrix}$$

\>> lam = -4 ; A1 = A - lam * eye(3) ; w2 = A1\B*z2

$$\omega_2 = \begin{bmatrix} -0.3 \\ -0.1\bar{3} \\ 0.0\bar{3} \end{bmatrix}$$

\>> " " " " ; w3 = A1\B*z3

$$\omega_3 = \begin{bmatrix} 0.1\bar{6} \\ 0.41\bar{6} \\ -0.08\bar{3} \end{bmatrix}$$

$\Rightarrow Z = [z_1 \quad z_2 \quad z_3]; \; W = [w_1 \quad w_2 \quad w_3];$

$K = Z * inv(W), \; eig(A - B * K)$

K =	-1	5	11
	-5.1429	6	7.7143

ans = -5
-4
-3

Ex: Given $\dot{x} = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 4 & 2 & 1 \\ \hdashline 0 & 0 & 1 \end{array}\right] x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$

unstable
positive eign value.

Assign -1, -2, -3 using the two method

N.B: k is uniqe for single input systems irrespective of the method used.

* Observers :

Used to estimate the states of a system which are unmeasurable (either inaccessible or sensors are unavailable or they are expensive)

- consider a system described by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Where A, B and C are known.

and an observer described by

$$K_e = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$$

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - C\tilde{x})$$

let error e be defined as $e = \overset{?}{x} - \tilde{x}$ ✓

$$\Rightarrow \dot{e} = \dot{x} - \dot{\tilde{x}} = Ax + \cancel{Bu} - A\tilde{x} - \cancel{Bu} - K_eCx + K_eC\tilde{x}$$
$$= Ae - K_eCe$$
$$\dot{e} = (A - K_eC)e$$

to get $\lim_{t\to\infty} e(t) = 0$, the eigenvalues of $A - K_eC$ should have negative real parts.

So a K_e is chosen to fullfill this requirement

resulting $\lim_{t\to\infty} x = \lim_{t\to\infty} \tilde{x}$

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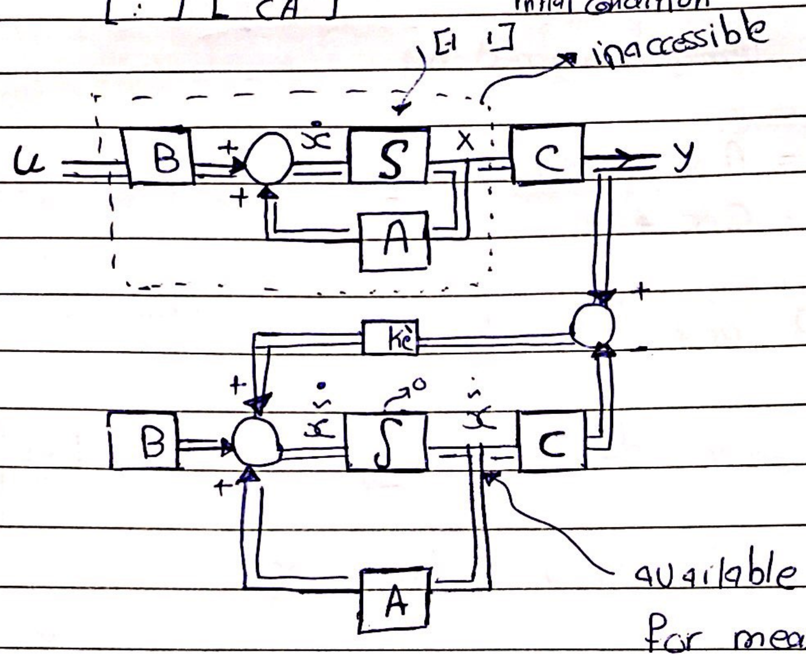
* Using Ackermann's method, it can be shown (prove that) that

$$K_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} P(A)$$



>> A=[0 1; -2 -2]; B=[0; 2]; C=[1 2]; ke=[-8 14]'

Ke = acker (A', C', [-4 -4])

* The Separation Principle :

Suppose $u = r - K\hat{x}$ $\quad K = [\quad]$ while $K_e = [\quad]$

So $\dot{x} = Ax + B(r - K\hat{x})$

$= Ax - BK\hat{x} + Br$

Recall

$e = x - \hat{x}$ & $\dot{e} = (A - K_e C)e$

$\hat{x} = x - e$

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$\therefore \dot{x} = Ax - BKx + BKe + B\bar{r}$

$\dot{x} = (A-BK)x + BKe + Br \quad \ldots (2)$

i.e $\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \left[\begin{array}{c:c} A-BK & BK \\ \hdashline 0 & A-K_eC \end{array}\right] \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$

adu → eigen values بالطريقة
ع بتاعة

Ex

$K_e = \begin{bmatrix} -0.8 \\ 10.8 \end{bmatrix}$ Choose K to get eigenvalue -2 & -2 using A & B

$\gg K = \text{acker}(A, B, [-2 \quad -2]) \Rightarrow K = [1 \quad 3]$

Choice of K & K_e } See book.

* Determination of System Zeros :-

Given $\dot{x} = Ax + Bu$ $\quad u \in R^m$

$y = Cx$ $\quad$ Where $|CB| \neq 0$

→ One method to determine the System Zeros is to calculate $G(s)$

→ Another method is as follows

i) Determine the eigen values of $A_z = [I_n - B(CB)^{-1}C]A$

ii) The zero's are the eigen value of A_z excluding m eigen value of value 0.

i.e $\{\text{Zero's of system}\} = \{\lambda[A_z] - \{\underbrace{0, 0, 0}_{m}\}\}$

Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = [1 \quad 1] x$

$A_z = [I - B(CB)^{-1}C]A$

$= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad \lambda[A_z] = \{0, -1\}$

Zero of system $= \{0, -1\} - \{0\} = \{-1\}$

Check answer determing $G(s)$

Ex: $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$y = [20 \quad 9 \quad 1]$