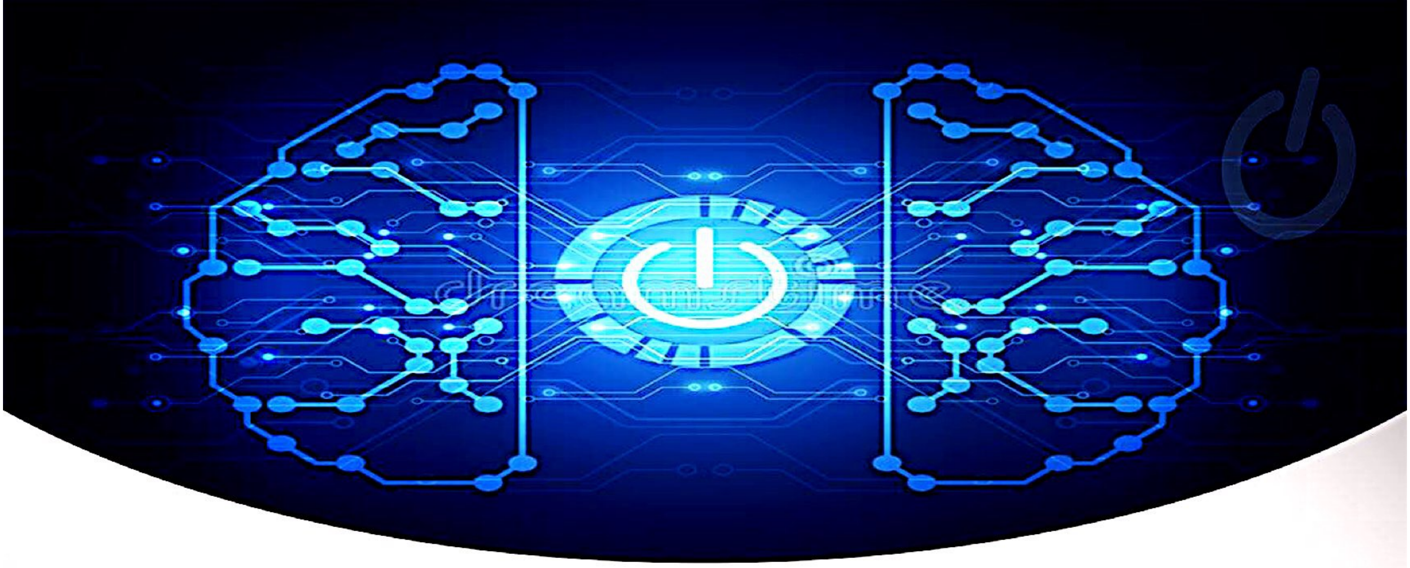

Topics in Control

Fall017

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Selected Topics in Control

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Notebook

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* * *

※ Review of state Space Representation (SSR):

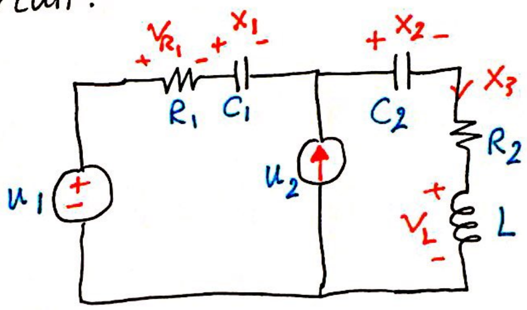
The need for this representation arises when:

i) The system has a Non-zero initial conditions.
ii) The system is Non-linear.
iii) The system is Time-varying.
iv) The system has Time delays.
v) The system is require to act optimally.
vi) The system is multi-input multi-output.

Example: Model the following circuit:

with two outputs:

$y_1 = V_{R_1}$, $y_2 = V_L$



Note:

- Electrical Energy Storage: capacitor, Inductor.
- Mechanical Energy Storage: Mass, Spring.
- If we remove the current source C_1 & C_2 become one state. we find C_{eq} of C_1 & C_2.

Solution: Note that $\dot{x} = \dfrac{dx}{dt}$

- We choose the voltage of the capacitor & the current of inductor as states for differential equations & easy modeling.

KCL @ the middle node:

$\boxed{C_2 \dot{x}_2 = x_3}$ ... (1) $\boxed{C_1 \dot{x}_1 = x_3 - u_2}$ ... (2)

KVL for the outside loop: $\boxed{-u_1 + R_1(x_3 - u_2) + x_1 + x_2 + R_2 x_3 + L\dot{x}_3 = 0}$ ... (3)

- Note: All the components of the cct must be shown in our equations.

Continue.

⇒ Now for the output:

$$y_1 = R_1(x_3 - u_2)$$
$$y_2 = -x_1 - x_2 - (R_1+R_2)x_3 + u_1 + R_1 u_2$$

Note: No derivative at all in the output states equations.
"Algebric Equations."

$$\dot{x} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{-1}{L} & \frac{-1}{L} & \frac{-(R_1+R_2)}{L} \end{bmatrix} x + \begin{bmatrix} 0 & \frac{-1}{C_1} \\ 0 & 0 \\ \frac{1}{L} & \frac{R_1}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & R_1 \\ -1 & -1 & -(R_1+R_2) \end{bmatrix} x + \begin{bmatrix} 0 & -R_1 \\ 1 & R_1 \end{bmatrix} u$$

i.e a MIMO system → two inputs, two outputs.

• **Note:** The general Form is: $\dot{x} = A\underline{x} + B\underline{u}$
$y = C\underline{x} + D\underline{u}$

* **N.B:** For a certain system A, B, C and D are NOT unique.
However, the transfer function matrix is unique.

* A person's Weight could be represented by multiple different units
stones, pounds, Kgs.
also for the Height foots, meters.

* **Analysis of Systems:**

Given a system $S(A,B,C,D)$, LTI system.

i.e

$$\underset{n\times1}{\dot{x}} = \underset{n\times n}{A}\ \underset{n\times1}{x} + \underset{n\times m}{B}\ \underset{m\times1}{u}$$

$$\underset{p\times1}{y} = \underset{p\times n}{C}\ \underset{n\times1}{x} + \underset{p\times m}{D}\ \underset{m\times1}{u}$$

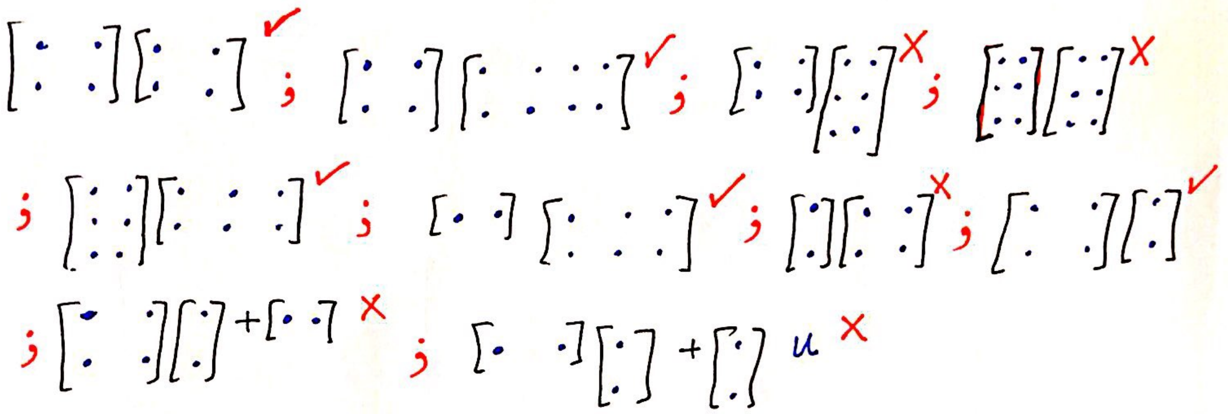
$x \in R^n$ → number of states. x is a vector.

$u \in R^m$

$m \leq n$

$y \in R^p$

* Visual Reminder of Matrix-vector Operations:



- Addition, subtraction, ... of vectors is defined.
- vector divide by a vector we say NON-DEFINED.

** $$x(t) = e^{At}x(t_0) + e^{At}\int_{t_0}^{t} e^{A(-\tau)} Bu(\tau)\,d\tau$$

- Recall: $\frac{dx}{dt} = ax + f(t)$.

* Asides:

- Remember: $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

radius of convergence: from $0 \to \infty$ "Always" Converges.

$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\dots$

↳ its Called: Harmonic Series. ⇒ "Divergent".

- prove collatz conjecture: if n is even get $n/2$
  if n is odd get $3n+1$

⇒ conjecture: it always ends with 4 2 1.

e.g. 17 52 26 13 40
20 10 5 16 8
4 2 1

Exponential Matrix.

* where: $e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$ ⇒ it's a closed form solution.

OR

$e^{At} = I_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$ ⇒ Numerically Convenient.

Exercise: given $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Calculate: $e^{A_1 t}$, $e^{A_2 t}$ using two methods?

Solution:

* for A_1:

- method (1): $[SI-A] = \begin{bmatrix} S & -1 \\ 0 & S \end{bmatrix} \Rightarrow [SI-A]^{-1} = \begin{bmatrix} \frac{1}{S} & \frac{1}{S^2} \\ 0 & \frac{1}{S} \end{bmatrix}$

$$\mathcal{L}^{-1}\{[SI-A]^{-1}\} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = e^{A_1 t} \quad \#$$

- method (2): $e^{A_1 t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\frac{t^2}{2!} + \cdots = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \#$

* for A_2:

- method (1): $[SI-A] = \begin{bmatrix} S & -1 \\ 0 & S-1 \end{bmatrix} \Rightarrow [SI-A]^{-1} = \begin{bmatrix} \frac{1}{S} & \frac{1}{S(S-1)} \\ 0 & \frac{1}{S-1} \end{bmatrix}$

$$\mathcal{L}^{-1} = \begin{bmatrix} 1 & -1+e^{t} \\ 0 & e^{t} \end{bmatrix} = e^{A_2 t} \quad \#$$

- method (2): $e^{A_2 t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\frac{t^2}{2!} + \cdots$

$$\Rightarrow e^{A_2 t} = \begin{bmatrix} 1 & t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots \\ 0 & 1+t+\frac{t^2}{2!}+\cdots \end{bmatrix} = \begin{bmatrix} 1 & e^{t}-1 \\ 0 & e^{t} \end{bmatrix} \quad \#$$

※ Proparties of the Exponential Matrix:

- $e^{At}\big|_{@\,t=0} = I_n$
- $e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$
- $e^{(A+B)t} = e^{At} \cdot e^{Bt}$ only if $AB = BA$
- $[e^{At}]^{-1} = e^{A(-t)}$ just replace each (t) by $(-t)$
- $\text{trace}(e^{At}) = e^{\text{trace}(At)}$ = a scalar.

  trace: summation of diagonal elements.

  Tracing At easier than e^{At}.
- if $A^2 = 0$, then $e^{At} = I + At$
- $e^{A^T t} = [e^{At}]^T$
- $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$
- $\int e^{At}\,dt = (e^{At} - I_n)A^{-1}$ only if A^{-1} is exist.

Exercise(1): Determine : $e^{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} t}$

Exercise (2): if $A^2 = A$, Then find e^{At} ?

Exercise (3): Consider $x(t) = e^{At}x(0) + e^{At}\int_{t_0=0}^{t} e^{A(-\tau)} Bu(\tau)d\tau$ → memorize it

, when $u(t)$ is a unit step.

⇒ prove that: (or show that / obtain) i) $x(t) = e^{At}x(0) + [-I + e^{At}]A^{-1}B$
$= e^{At}x(0) + A^{-1}[-I + e^{At}]B$

ii) ① Reason why A should have an inverse.

② $x_{ss} = \lim_{t\to\infty} x(t) = -A^{-1}B$ when the system is asymptotically stable. $(real(\lambda_i) < 0)$

Solutions:

Ex.(1): $[sI - A] = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \Rightarrow [sI-A]^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$

$$\Rightarrow \mathcal{L}^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} = e^{At} \quad \#$$

Ex.(2): since $A^2 = A$, you can reach that $A = A^2 = A^3 = A^4 = \cdots$

$$\text{so} \quad e^{At} = I_n + At + A\frac{t^2}{2!} + A\frac{t^3}{3!} + \cdots = I_n + A(e^t - 1) \quad \#$$

remember: $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

Ex.(3):

i) remember that: $\int e^{At}dt = (e^{At} - I_n)A^{-1}$

$$\text{so} \quad x(t) = e^{At}x(0) + e^{At}(-1)\int_0^t e^{A(-\tau)} d(-\tau)\, B$$

$$= e^{At}x(0) - e^{At}\left[e^{A(-\tau)} - I_n\right]_0^t A^{-1}B$$

$$= e^{At}x(0) - e^{At}\left(e^{-At} - I_n - \cancel{I_n} + \cancel{I_n}\right)A^{-1}B$$

$$x(t) = e^{At}x(0) + \left[e^{At} - I_n\right]A^{-1}B \quad \#$$

Continue.

ii)

1 Reasons for A to have A^{-1}:

- Since one of the proparties of the exponential matrix is $\int e^{At} dt = [e^{At} - I_n] A^{-1}$ so A^{-1} must be exist to satisfy this proparty.

- Also one of the conditions on the system to be STABLE is to have A^{-1}, and we care for the stability of the system, so need A^{-1} to be exist.

(i.e $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, this system is NOT stable).

2 From the Definition of the systems asymptotically stable is that: $\lim_{t \to \infty} e^{At} = \text{Zero}$

$$x(t) = e^{At} x(0) + [e^{At} - I_n] A^{-1} B \quad \text{as proved before.}$$

$\Rightarrow$ Taking $\lim_{t\to\infty}$ for Both sides:

$$X_{ss} = \lim_{t\to\infty} x(t) = \cancel{\lim_{t\to\infty} e^{At}}^{\,0} \cdot x(0) + \lim_{t\to\infty} [e^{At} - I_n] A^{-1} B$$

$$= 0 + (0 - I_n) A^{-1} B$$

$$\Rightarrow \boxed{X_{ss} = -A^{-1} B} \ \#$$

$\hookrightarrow$ when the system is asymptotically stable.

* * * * *

* Matlab Simulation:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -29 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 58 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

To see the output of x_1

$y = [1 \quad 0] x$

To see the output of x_2

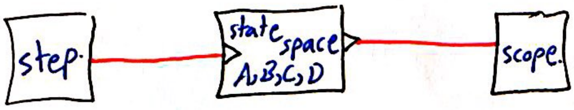
$y = [0 \quad 1] x$

Both of them

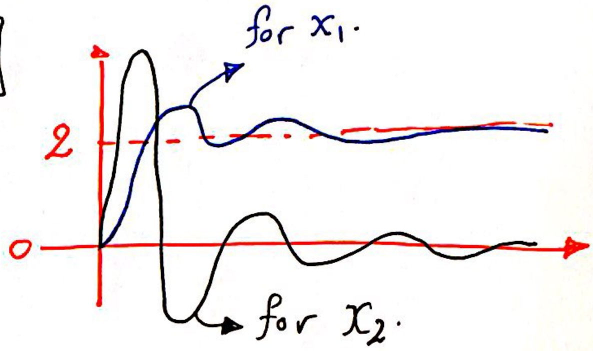
$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$

```
>> A=[0 1;-29 -4]; B=[0;58]; C=eye(length(A)); D=[0;0];
```

simulink → Continuous → state space.



Note: $x_{ss} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$



- if we change -4 to +4.

  ⇒ Unstable system.

```
>> syms t ; E=expm(A*t)
>> pretty(E)
>> simplify(E)
```

pretty(E) → it will give the answer more nicer.

※ The Transfer Function Matrix:

Given $S(A,B,C,D)$ ⇒ $G(S) = C[SI_n - A]^{-1}B + D$

After possible Cancellations,
G(s) gives the intrinsic characteristics of a system.

e.g $G(S) = \dfrac{(S^2+5S+6)}{S^3+6S^2+11S+6}$ ⇒ it is NOT a 3rd order.

After pole-zero cancellation $G(S) = \dfrac{(S+2)(S+3)}{(S+1)(S+2)(S+3)} = \dfrac{1}{S+1}$

a first order system. it is 3rd if G(s) was: $G(S) = \dfrac{S^2-5S+6}{S^3+6S^2+11S+6}$

$Y(s) = \overset{\text{square matrix.}}{G(s)}\, U(s)$ for MIMO.

OR $G(s) = \dfrac{Y(s)}{U(s)}$ only when the system is SISO

→ so sometimes we write it as $g(s)$ in this case.

* Revision:

$$A = \begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow |A| = -1*(-5*4 - 3*-6) = \underline{\underline{2}}$$

choose the row with more zeros.

OR $|A| = -5(-4) - 3(0) + 3(-6) = \underline{\underline{2}}$

OR $|A| = -5(-4) - (-6)(-3) = \underline{\underline{2}}$

For A^{-1}: $A^{-1} = \dfrac{\begin{bmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ -6 & 5 & 3 \end{bmatrix}}{2}$

* Proparties of Matrices:

```
>> n=3; a=round(10*rand(n)-5), b=round(10*rand(n)-5), det(a), det(a'), det(a*b), det(b*a)
```

Let $\lambda[\cdot]$ denotes the eigen value of $[\cdot]$.

```
>> E = eig(a); trace(a)
```

- $|A| = |A^T|$ • $|I_n| = 1$ • $|\alpha A| = \alpha^n |A|$
- $|AB| = |A||B| = |B||A| = |BA| \Rightarrow |ABCD| = |BDAC| = |\cdots|$

*** Note:** if you dealing with complex number, then to find |transpose|: use det(a.')

- $|A^{-1}| = \dfrac{1}{|A|} \Rightarrow$ prove using $|I_n| = 1$
- $trace(AB) = trace(BA)$
- $\sum_{i=1}^{n} \lambda_i[A] = trace(A) = \sum_{i=1}^{n} a_{ii} \longrightarrow$ if it is +ve ⇒ Unstable.

$\begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix}$ → Need to find eigen values to know if stable or NOT.

$\begin{bmatrix} -5 & 3 & 3 \\ -6 & 3 & 4 \\ 0 & 1 & 10 \end{bmatrix}$ → Directly from the trace, we knew it is unstable.

- $\prod_{i=1}^{n} \lambda_i[A] = |A|$
- $\lambda[A^{-1}] = \dfrac{1}{\lambda[A]}$ if A has an inverse it doesn't have a $\underline{\underline{0}}$ eigenvalue.
- $\lambda[AB] = \lambda[BA] \Rightarrow \lambda[ABCD] = \lambda[CDAB] = \cdots$

- $\lambda[D] = \lambda[UT] = \lambda[LT] = \{\text{diagonal elements}\}$

  $\underset{\text{Diagonal}}{D}$, $\underset{\text{Upper Triangular}}{UT}$, $\underset{\text{Lower Triangular.}}{LT}$

- $\lambda[T^{-1}AT] = \lambda[A] = \lambda[TAT^{-1}]$, T any nonsingular matrix.

  Later we will call it: Similarity Transformation.

- $\lambda[A^T] = \lambda[A]$
- $\lambda[A + K I_n] = \lambda[A] + K$, $\forall K \in \mathbb{C}^1$ — stand for field of complex scalar numbers.

  add K to the diagonal elements.

e.g. given a matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ with eigen values $-1, -2$:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}}_{T^{-1}} \Rightarrow$$

it will result a matrix with eigen values $-1, -2$.

on MATLAB:

```
>> a = [0 1; -2 -3], trace(a); det(a); E = eig(a), T=[1 2;3 4] ...
... ; Et = eig (T*a *inv(T)), EK = eig (a+4*eye(length(a)))
```

$E_t = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ $\quad E_K = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Note: $\begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} \Rightarrow$ always has eigenvalues $\sigma \pm j\omega$

e.g. $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow$ eigen values $2 \pm j$

*** $\lambda[A]$ are always real if A is symmetric. (i.e if $A = A^T$)

* Suppose: $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \Rightarrow |A - \lambda I_n| = 0 \equiv C.E \equiv$ Characteristic Equation/Polynomial.

$C.E = \lambda^4 - a_4\lambda^3 - a_3\lambda^2 - a_2\lambda - a_1 = 0$

The EigenVectors associated with $\lambda = \lambda_1$ is: $\begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \\ \lambda_1^3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}$ if $\lambda_1 = -2$

Example: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \Rightarrow C.E = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$= (\lambda+1)(\lambda+2)(\lambda+3) = 0 \qquad \begin{Bmatrix} \lambda_1 = -1 \\ \lambda_2 = -2 \\ \lambda_3 = -3 \end{Bmatrix}$

Eigenvectors are: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$

- $\lambda\left(\left[\begin{array}{c:c} \underset{n\times n}{A_1} & \underset{n\times m}{0} \\ \hdashline \underset{m\times n}{A_2} & \underset{m\times m}{A_3} \end{array}\right]\right) = \lambda[A_1] \, \underline{U} \, \lambda[A_3]$

  we partition it.

  $\underline{U}$ stand for union & here we take the repeated eginvalues unlike the U.

- $\lambda\left(\left[\begin{array}{c:c} A_{n\times n} & B_{n\times m} \\ \hdashline 0_{m\times n} & C_{m\times m} \end{array}\right]\right) = \lambda[A] \, \underline{U} \, \lambda[C] \Rightarrow$ prove: since $\lambda[A^T] = \lambda[A]$

  Note $\Rightarrow$ Square matrices are on the diagonal.

Example: let $A = \left[\begin{array}{cc:c} 2 & 1 & 700 \\ -1 & 2 & \frac{1}{1003} \\ \hdashline 0 & 0 & -5 \end{array}\right] \Rightarrow \lambda[A] = \lambda\left(\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\right) \, \underline{U} \, -5$

$= \{2+j, 2-j, -5\}$

***Note:** the following proparties is HARD To Prove:

- $|A^T| = |A|$ • $\lambda[AB] = \lambda[BA]$ • $trace(AB) = trace(BA)$

BUT, we can use them in proving other proparties.

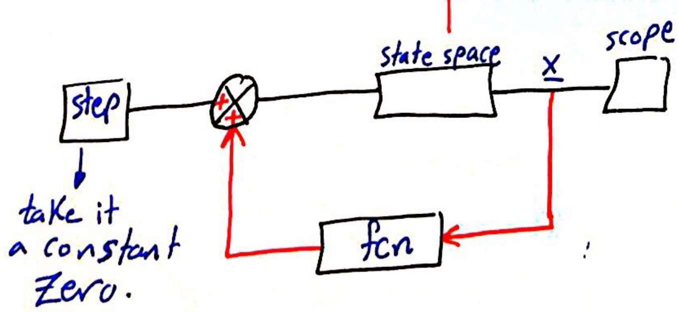
e.g. $\lambda[T^{-1}AT] = \lambda[A] \Rightarrow \lambda[T^{-1}AT] = \lambda[ATT^{-1}] = \lambda[AI_n] = \lambda[A]$

Simulink Example:

Consider $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \; , \; y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$

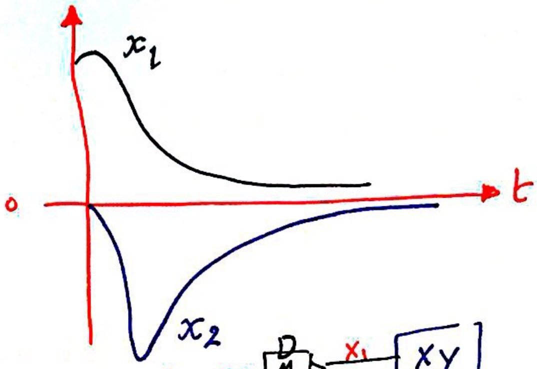
$\Rightarrow$ This is unstable system.

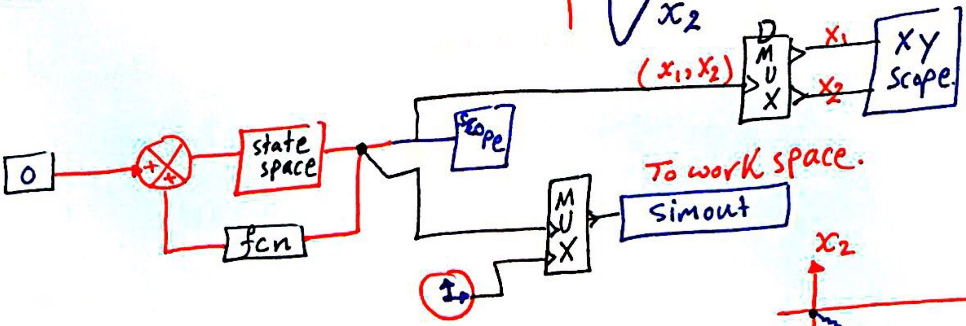
initial conditions: [1 0]

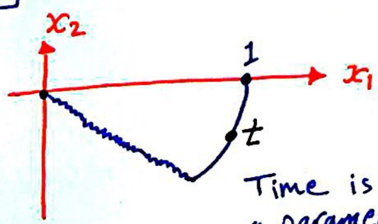


$\Longrightarrow$

```
⇒ function u = fcn(x)
S = x(1)+ x(2);
if S>0
  u =-1;
else
  u=1;
end
```





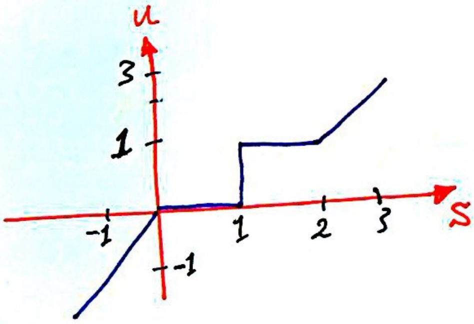


Time is a parameter.

>> plot3 (y(:,1), y(:,2), y(:,3))

Exercise: Consider the previous system with the following controller:

$S = x_1 + x_2$ and u as shown:



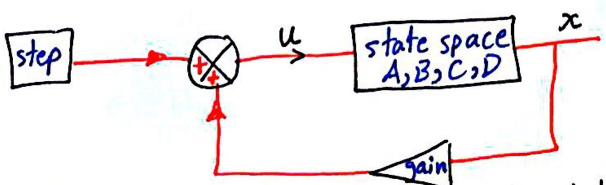
Example: Consider the previous system with the following controller:

$u = u(t) + Kx$, where: $K = [5 \ -2]$

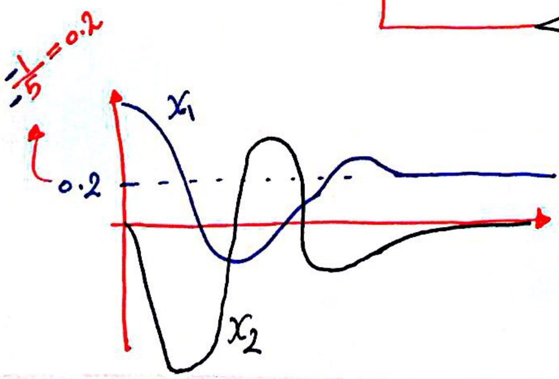
if $K = [-5 \ 2]$
new system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} * 1$$

⇒ Unstable.



↳ use Multiplication (Matrix K * u)



stable system.

※ The Exponential Matrix Using Eigenvectors:

For a transformation (T), does there exists a non-zero vector (x) which remains in the same direction after being operated on by that transformation.

$T(x) = \lambda x$; where λ is a scalar & $x \neq 0$

- A Linear operator can be represented by a Matrix.

e.g $T\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \\ \ln 4 \end{bmatrix}$ $\Rightarrow$ This is a Non-linear operator.

$\overset{\checkmark}{A}\overset{?}{x} = \overset{?}{\lambda}\overset{?}{x} \Rightarrow [A - \lambda I_n]\, x = 0.$

for $x \neq 0$ $A - \lambda I_n$ shouldn't have an inverse (i.e $|A - \lambda I_n| = 0$)

(Why??) Because if there is an inverse it will result the following:

$[A - \lambda I_n]^{-1} [A - \lambda I_n]\, x = 0$; this will give $x = 0$, and we Know that $x \neq 0$, so it shouldn't have an inverse.

$|A - \lambda I| = 0 = C.E$ "C.E is Known as the characteristic equation (polynomial)"

↳ Solve for the root of C.E giving the eigenvalues.

* For a certain eigenvalue $\lambda = x_1$ solve: $[A - \lambda_1 I_n]\, x_1 = 0$

$\Rightarrow$ Use Gauss-Elimination To determine x_1.

Example: Let $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow [A - \lambda I_n] = \begin{bmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$

$|A - \lambda I_n| = (-1-\lambda)(4-\lambda) - 6 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow (\lambda+2)(\lambda-5) = 0$

$\boxed{\begin{matrix} \lambda_1 = -2 \\ \lambda_2 = 5 \end{matrix}}$

- you can check the answer by the learned proparties.

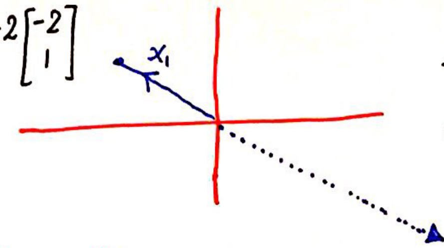
- The eigenvectors associated with $\lambda = -2$:

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $x_1^2 = r$, $x_1^1 + 2r = 0$, $x_1^1 = -2r$

$\Rightarrow x_1 = \begin{bmatrix} -2r \\ r \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix} \equiv$ eigen vector.

Continue ⟹

check: $A\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix} = -2\begin{bmatrix}-2\\1\end{bmatrix}$



They are in the same direction by different by (-2).

this (-2) represent the eigenvalue.

we could find eigen value from $A\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$

$\Rightarrow A\begin{bmatrix}-2\\1\end{bmatrix} = (-2)\begin{bmatrix}-2\\1\end{bmatrix}$ → eigen value.

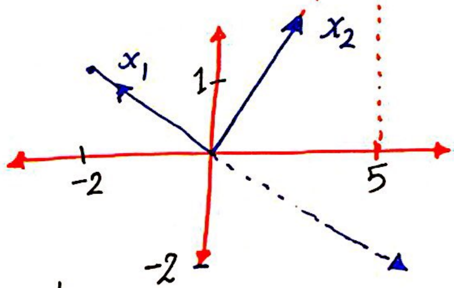
same

• **The eigenvector associated with $\lambda_2 = 5$:**

$\begin{bmatrix}-6 & 2\\3 & -1\end{bmatrix} x = 0 \Rightarrow \begin{bmatrix}-6 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x_2^1\\x_2^2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{matrix}x_2^2 = r\\ -6x_2^1 + 2r = 0 \Rightarrow x_2^1 = \frac{1}{3}r\end{matrix}$

$x_1 = \begin{bmatrix}\frac{1}{3}r\\r\end{bmatrix} \Rightarrow \begin{bmatrix}1\\3\end{bmatrix}$ ← eigenvector.

check: $A\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}5\\15\end{bmatrix} = 5\begin{bmatrix}1\\3\end{bmatrix}$ → eigenvalue.



Exercise: Determine the eigenvaleses & eigen vectors of A where:

$$A = \begin{bmatrix}-5 & 3 & 3\\-6 & 3 & 4\\0 & 1 & 0\end{bmatrix}, \quad A = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-6 & -11 & -6\end{bmatrix}$$

↳ solved before in page (10) using proparties.

Answers:

eigen values: $\lambda = -1, 1, -2$

eigen vectors: $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-2\\1\end{bmatrix}$

*** The Adjoint Method for Determining the Eigenvector:**

Best illustrated by an example: eigenvector = adj([A − λIn])

let $A = \begin{bmatrix}-3 & 3 & 3\\-6 & 5 & 4\\0 & 1 & 2\end{bmatrix}$ which has eigenvalues 0, 3, 1

Calculate the Eigenvector associated with 3 ?!

⇛

$$\Rightarrow \operatorname{adj}(A-(3)I_3) = \operatorname{adj}\left(\begin{bmatrix} -6 & 3 & 3 \\ -6 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} -6 & 6 & 6 \\ -6 & 6 & 6 \\ -6 & 6 & 6 \end{bmatrix}$$

* Note that all the resultant columns are the same just different by a factor so you can find the first column & find the eigenvector directly.

$$\text{Eigenvector} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Exercise: Determine the other two eigenvectors for previous example?

Solution:

for $\lambda = 0$:

$$\operatorname{adj}(A-(0)I) = \operatorname{adj}\begin{bmatrix} -3 & 3 & 3 \\ -6 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 & -3 \\ 12 & -6 & -6 \\ -6 & 3 & 3 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

for $\lambda = 1$:

$$\operatorname{adj}(A-(1)I) = \operatorname{adj}\begin{bmatrix} -4 & 3 & 3 \\ -6 & 4 & 4 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 6 & -4 & -2 \\ -6 & 4 & 2 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

※ Determination of e^{At} using the eigenvectors:

Suppose A has a distinct (different) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with associated eigenvectors $V_1, V_2, \dots, V_n$

Let $\underset{\text{square matrix}}{V} = [V_1 \; V_2 \; \cdots \; V_n]$, $|V| \neq 0$ in our case.

$$e^{At} = V e^{\Delta t} V^{-1} = V \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} V^{-1}$$

where $\Delta \equiv$ upper case Letter for λ.

Exercise: Given $A = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}$

i) Determine the eigenvalues ?

ii) Determine the eigenvectors using three methods ?

iii) Determine e^{At} using two methods ?

Solution: i) $[A-\lambda I] = \begin{bmatrix} -\lambda & 1 \\ -20 & -9-\lambda \end{bmatrix} \Rightarrow |A-\lambda I| = \lambda^2+9\lambda+20=0$ so eigenvalues : $\lambda_1=-4$, $\lambda_2=-5$

ii) • By stair property method :

$\begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \Rightarrow \lambda^2+9\lambda+20=0$, $\lambda=-4,-5$ so eigen vectors are : $\begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ #

• By the standard method :

for $\lambda=-4$:

$\begin{bmatrix} 4 & 1 \\ -20 & -5 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \Rightarrow$ eigenvector : $\begin{bmatrix} \frac{-r}{4} \\ r \end{bmatrix}$

$x_2 = r$

$x_1 = \frac{-r}{4}$

$\Rightarrow \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ #

for $\lambda=-5$:

$\begin{bmatrix} 5 & 1 \\ -20 & -4 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \Rightarrow$ eigenvector: $\begin{bmatrix} \frac{-r}{5} \\ r \end{bmatrix}$

$x_2 = r$

$x_1 = \frac{-r}{5}$

$\Rightarrow \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ #

• By Adjoint Method :

eigenvector $= adj(A-\lambda I_n)$

for $\lambda=-4$: $adj\begin{bmatrix} 4 & 1 \\ -20 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 20 & -4 \end{bmatrix} \Rightarrow$ eigenvector is : $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ #

for $\lambda=-5$: $adj\begin{bmatrix} 5 & 1 \\ -20 & -4 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 20 & -5 \end{bmatrix} \Rightarrow$ eigenvector is : $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$ #

Continue

iii) • Method(1):

By using $e^{At} = \mathcal{L}^{-1}[SI-A]^{-1}$

$$[SI-A]^{-1} = \begin{bmatrix} S & -1 \\ 20 & S+9 \end{bmatrix}^{-1} = \frac{1}{(S+4)(S+5)} \begin{bmatrix} S+9 & 1 \\ -20 & S \end{bmatrix}$$

Now for $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1} \frac{1}{(S+4)(S+5)} = \mathcal{L}^{-1}\left(\frac{1}{S+4} - \frac{1}{S+5}\right) = e^{-4t} - e^{-5t}$$

$$\mathcal{L}^{-1} \frac{-20}{(S+4)(S+5)} = \mathcal{L}^{-1}\left(\frac{1}{S+4} - \frac{1}{S+5}\right) = 20e^{-5t} - 20e^{-4t}$$

$$\mathcal{L}^{-1} \frac{S}{(S+4)(S+5)} = \mathcal{L}^{-1}\left(\frac{-4}{S+4} + \frac{5}{S+5}\right) = -4e^{-4t} + 5e^{-5t}$$

$$\mathcal{L}^{-1} \frac{S+9}{(S+4)(S+5)} = \mathcal{L}^{-1}\left(\frac{5}{S+4} - \frac{4}{S+5}\right) = 5e^{-4t} - 4e^{-5t}$$

$$** \quad e^{At} = \begin{bmatrix} 5e^{-4t} - 4e^{-5t} & e^{-4t} - e^{-5t} \\ 20e^{-5t} - 20e^{-4t} & 5e^{-5t} - 4e^{-4t} \end{bmatrix} \quad \#$$

• Method(2):

By using eigenvectors:

$$V = \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix}, \quad e^{\Lambda t} = \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-5t} \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix}$$

e^{At} given by: $e^{At} = V \cdot e^{\Lambda t} \cdot V^{-1}$

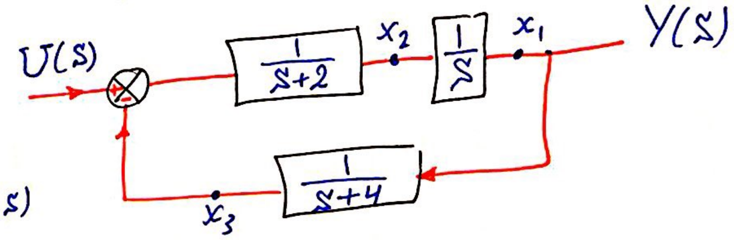
$$\Rightarrow e^{At} = \begin{bmatrix} e^{-4t} & e^{-5t} \\ -4e^{-4t} & -5e^{-5t} \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} 5e^{-4t} - 4e^{-5t} & e^{-4t} - e^{-5t} \\ 20e^{-5t} - 20e^{-4t} & 5e^{-5t} - 4e^{-4t} \end{bmatrix} \quad \#$$

* * *

※ State-Space Models Using the Block Diagram:

- Review Block Diagram Reduction Techniques.

* Consider the following Block Diagram:



- Multiplication by S in the S-domain is differentiation in time-domain.

$$X_1(S) = \frac{1}{S} X_2(S) \Rightarrow S X_1(S) = X_2(S)$$

$$\Rightarrow \underline{\dot{x}_1 = x_2}$$

$$X_2(S) = \frac{1}{S+2}\left(U(S) - X_3(S)\right) \Rightarrow \underline{\dot{x}_2 = -2x_2 - x_3 + u}$$

$$X_3(S) = \frac{1}{S+4} X_1(S) \Rightarrow \underline{\dot{x}_3 = x_1 - 4x_3} \qquad \underline{y = x_1}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

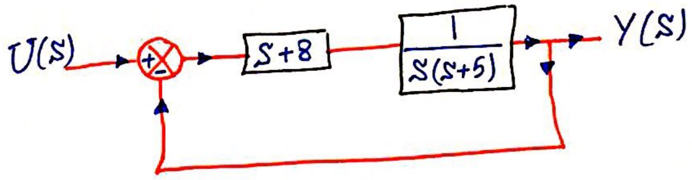
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0 \cdot u$$

This is the state-space representation of this system.

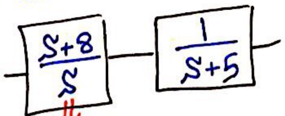
- Check: using the T.F obtained using the Block Diagram.

$$G(S) = \frac{S+4}{S^3 + 6S^2 + 8S + 1}$$

Example: Obtain an A, B, C, D for the following system:



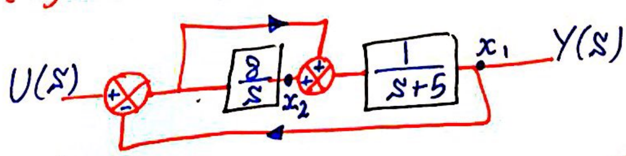
here will face a problem, so we edit on the system. ⇐



$$\Downarrow$$

$$1 + \frac{8}{S}$$

The system becomes as follows:



$$X_1(S) = \frac{1}{S+5}\left(X_2(S) + U(S) - X_1(S)\right) \Rightarrow \dot{x}_1 = -6x_1 + x_2 + u \quad \cdots (1)$$

$$X_2(S) = \frac{8}{S}\left(U(S) - X_1(S)\right) \Rightarrow \dot{x}_2 = -8x_1 + 8u \quad \cdots (2)$$

$$\underline{y = x_1}$$

$$\dot{x} = \begin{bmatrix} -6 & 1 \\ -8 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 8 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0 \cdot u$$

⟹ check.

• check: using $G(S)$ as obtained from the block diagram and as obtained from $G(S) = C\,[SI-A]^{-1}B + D$

from B.D $\Rightarrow$ $G(S) = \dfrac{\dfrac{S+8}{S(S+5)}}{1+\dfrac{S+8}{S(S+5)} * 1} = \dfrac{S+8}{S^2+6S+8}$

$$G(S) = [1 \quad 0] \begin{bmatrix} S+6 & -1 \\ 8 & S \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 8 \end{bmatrix} + 0 \qquad \frac{1}{S^2+6S+8}\begin{bmatrix} S & 1 \\ -8 & S+6 \end{bmatrix}$$

$$= [S \quad 1] \begin{bmatrix} 1 \\ 8 \end{bmatrix} \cdot \frac{1}{S^2+6S+8} = \frac{S+8}{S^2+6S+8}$$

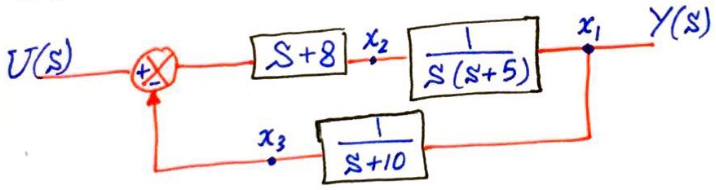
*Note:
for any polynomial $\sum$ roots $= -1 *$ second highest order. $= \sum$ eigenvalues.

e.g: $S^2+6S+8=0 \Rightarrow$ roots $= -2, -4 \Rightarrow \sum$ roots $= -1*6 = \underline{-6}$

e.g: $S^{17}+S^{15}+\cdots = 0 \Rightarrow \sum$ roots $= -1 * 0 =$ Zero.

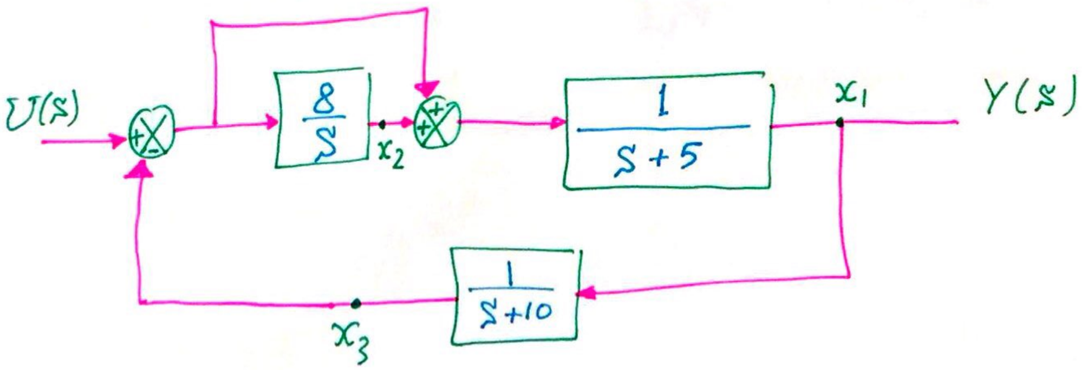
Exercise(1): Utilize (use) another re-arrangement of blocks to get another state space for the system ?

Exercise(2): Determine A, B, C, D for the following system:



solution for Ex.(2):

The system becomes as follows:



$$X_1(s) = \frac{1}{s+5}\left[X_2(s) + U(s) - X_3(s)\right]$$

$$\Rightarrow \quad \dot{x}_1 = -5x_1 + x_2 - x_3 + u \quad \ldots (1)$$

$$X_2(s) = \frac{8}{s}\left[U(s) - X_3(s)\right] \qquad y = x_1$$

$$\Rightarrow \quad \dot{x}_2 = -8x_3 + 8u \quad \ldots (2)$$

$$X_3(s) = \frac{1}{s+10} X_1(s) \Rightarrow \dot{x}_3 = x_1 - 10x_3 \quad \ldots (3)$$

The A, B, C, D parameter given by:

$$\dot{x} = \begin{bmatrix} -5 & 1 & -1 \\ 0 & 0 & -8 \\ 1 & 0 & -10 \end{bmatrix} x + \begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0 \cdot u$$

* * * *

※Obtaining S(A, B, C, D) From a T.F:

i) Case(1): No derivative of u.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}, \quad y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = u$$

Let
$x_1 = y$
$x_2 = \dot{y} = \dot{x}_1$
$x_3 = \ddot{y} = \dot{x}_2$
$x_4 = \dddot{y} = \dot{x}_3$
$\vdots$
$x_{n+1} = y^{(n)} = \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \cdots - a_1 x_n + u$

Example: Given $G(s) = \dfrac{5}{s^3 - 4s^2 + 6s}$ $\Rightarrow$ so three x's suffice.

$x_1 = y$
$x_2 = \dot{y} = \dot{x}_1$
$x_3 = \ddot{y}$
$x_4 = \dddot{y}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0 \cdot u$$

ii) Case(2): When the Numerator involves non-zero powers of S.

Best illustrated by an example:

Let $G(S) = \dfrac{3S^3+4S^2+5S+6}{S^3+2S^2+8S+9}$

Note: $|A| = (-1)^{\text{highest power}} * a_n S^0$

here $|A| = (-1)^3 * 9 = \underline{\underline{-9}}$

$\sum$ eigenvalues $= -2$.

$$\frac{Y(S)}{U(S)} = \frac{Y(S)}{Z(S)} \cdot \frac{Z(S)}{U(S)} = (3S^3+4S^2+5S+6) \cdot \frac{1}{S^3+2S^2+8S+9}$$

* let $\dfrac{Z(S)}{U(S)} = \dfrac{1}{S^3+2S^2+8S+9}$

let: $x_1 = Z$

$x_2 = \dot{Z} = \dot{x}_1$

$x_3 = \ddot{Z} = \dot{x}_2$

$\dddot{Z} = \dot{x}_3 = -9x_1 - 8x_2 - 2x_3 + u$

* Use $\dfrac{Y(S)}{Z(S)} = 3S^3+4S^2+5S+6$

$y(t) = 3\dddot{Z} + 4\ddot{Z} + 5\dot{Z} + 6Z$

$= 3(-9x_1 - 8x_2 - 2x_3 + u) + 4x_3 + 5x_2 + 6x_1$

$\underline{y(t) = -21x_1 - 19x_2 - 2x_3 + 3u}$

$$\therefore \quad \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -8 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} -21 & -19 & -2 \end{bmatrix} x + 3 \cdot u$$

* You Can check: trace $= \sum$ eigen values $= -2$ #

$|A| = (-1)(0+9) = -9$ #

iii) Case(3): Obtaining Diagonal State Space Representation.

Best Illustrated by an example:

→ By Partial Fraction:

$$\text{Given: } G(S) = \frac{S^2+9S+20}{S^3+6S^2+11S+6} = \frac{S^2+9S+20}{(S+1)(S+2)(S+3)} = \frac{6}{S+1} - \frac{6}{S+2} + \frac{1}{S+3}$$

$$Y(S) = \underbrace{\frac{6U(S)}{S+1}}_{X_1(S)} - \underbrace{\frac{6U(S)}{S+2}}_{X_2(S)} + \underbrace{\frac{U(S)}{S+3}}_{X_3(S)} = X_1(S) + X_2(S) + X_3(S)$$

$\Rightarrow y = x_1 + x_2 + x_3$

$X_1(S) = \dfrac{6U(S)}{S+1} \Rightarrow \dot{x}_1 = -x_1 + 6u \quad \ldots (1)$

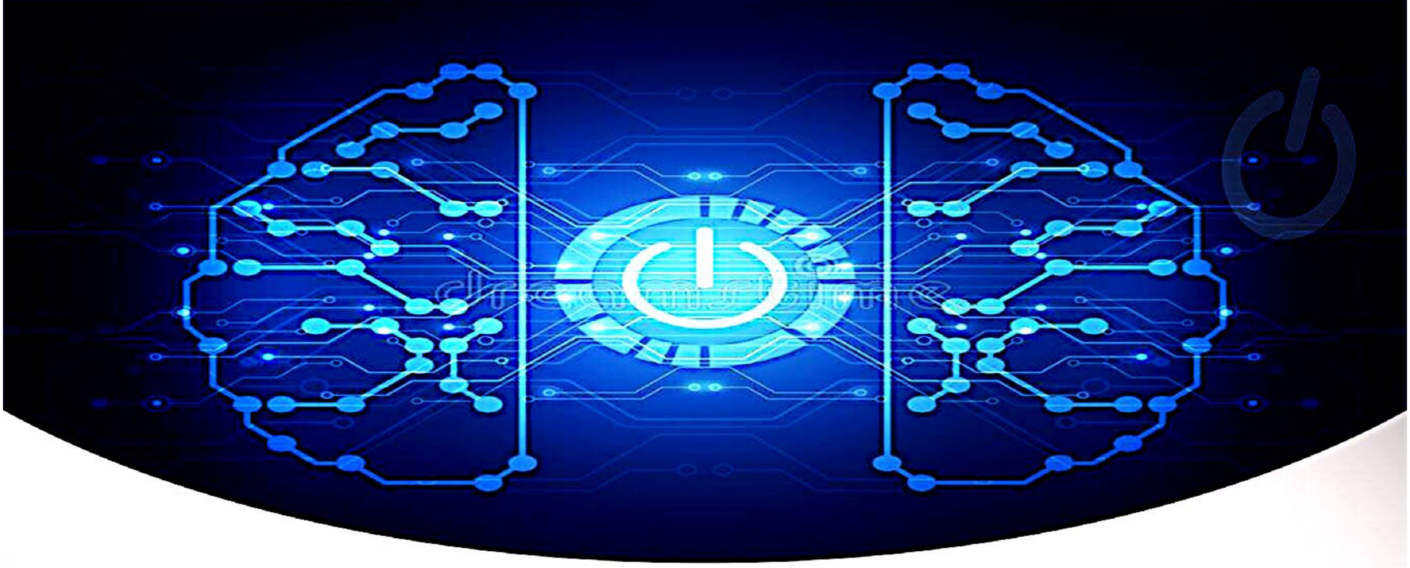
$X_2(S) = \dfrac{-6U(S)}{S+2} \Rightarrow \dot{x}_2 = -2x_2 + 6u \quad \ldots (2)$

$X_3(S) = \dfrac{U(S)}{S+3} \Rightarrow \dot{x}_3 = -3x_3 + u \quad \ldots (3)$

$$\therefore \quad \dot{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} X + \begin{bmatrix} 6 \\ -6 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} X + 0 \cdot u$$

* * * *

End of First Material.



Topics in Control

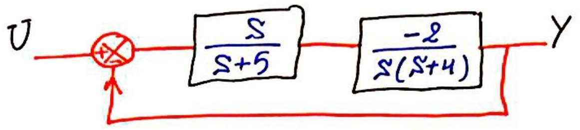
Fall017

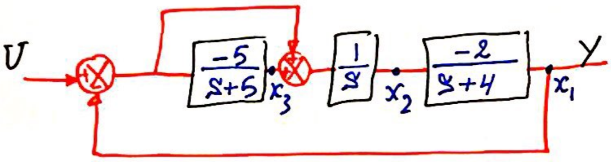
Dr. Omar Ghzawi

By: Mhmd Abuhashya

***** Solution for Q4 in first exam:** find $S(A,B,C,D)$ using 3 states?

Re-Build the B.D $\Leftarrow$





$\dot{x}_1 = -4x_1 - 2x_2 \quad \cdots (1)$

$\dot{x}_2 = -x_1 + x_3 + u \quad \cdots (2)$

$\dot{x}_3 = 5x_1 - 5x_3 - 5u \quad \cdots (3)$

$y = x_1 \quad \cdots (4)$

$$\dot{x} = \begin{bmatrix} -4 & -2 & 0 \\ -1 & 0 & 1 \\ 5 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0 \cdot u$$

*Reduce the B.D: $\frac{Y(S)}{U(S)} = \frac{-2}{S^2+9S+18}$

By partial fraction:

$$Y(S) = \underbrace{\frac{-2/3\, U(S)}{S+3}}_{X_1(S)} + \underbrace{\frac{2/3\, U(S)}{S+6}}_{X_2(S)}$$

$\Rightarrow X_1(S) = \frac{-2/3\, U}{S+3} \Rightarrow \dot{x}_1 = -3x_1 - \frac{2}{3}u$

$X_2(S) = \frac{2/3\, U}{S+6} \Rightarrow \dot{x}_2 = -6x_2 + \frac{2}{3}u$

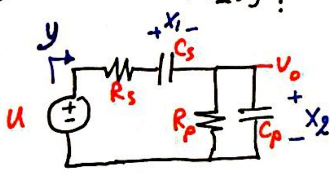
$y = x_1 + x_2$

$$\Rightarrow \dot{x} = \begin{bmatrix} -3 & 0 \\ 0 & -6 \end{bmatrix} x + \begin{bmatrix} -2/3 \\ 2/3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u.$$

***Note:** The two systems of State Space will give the same $G(S)$.

***** Solution for Q1 in first exam:** find $S(A,B,C,D)$ with states x_1, x_2, y?



(By KVL on outside loop): $-u + R_s C_s \dot{x}_1 + x_1 + x_2 = 0$

(By KCL at the right Node): $C_p \dot{x}_2 = \frac{u - x_1 - x_2}{R_s} - \frac{x_2}{R_p}$

for y: $y = \frac{u - x_1 - x_2}{R_s}$

$$\dot{x} = \begin{bmatrix} \frac{-1}{C_s R_s} & \frac{-1}{R_s C_s} \\ \frac{-1}{R_s C_p} & -\left(\frac{1}{R_s C_p} + \frac{1}{R_p C_p}\right) \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_s C_s} \\ \frac{1}{R_s C_p} \end{bmatrix} u \quad , \quad y = \begin{bmatrix} \frac{-1}{R_s} & \frac{-1}{R_s} \end{bmatrix} x + \frac{1}{R_s} u$$

* T.F could be obtained by two methods: $G(S) = \frac{I(S)}{U(S)}$ OR $G(S) = C[SI-A]^{-1}B + D$

The Case of Identical (Repeated) poles:

Suppose $G(S) = \frac{Y(S)}{U(S)} = \frac{A.}{(S+P)^3} \Rightarrow Y(S) = A. \frac{U}{(S+P)^3}$

Let $X_3(S) = \frac{U}{S+P} \Rightarrow \dot{x}_3 = -Px_3 + u \cdots (1)$ for y: $y = A x_1 \cdots (4)$

$X_2(S) = \frac{U}{(S+P)^2} \Rightarrow \frac{1}{S+P} \cdot \frac{U}{S+P} = \frac{1}{S+P} \cdot X_3(S) = X_2(S) \Rightarrow \dot{x}_2 = -Px_2 + x_3 \cdots (2)$

$X_1(S) = \frac{U}{(S+P)^3} \Rightarrow X_1(S) = \frac{1}{S+P} \cdot \frac{U}{(S+P)^2} = \frac{1}{S+P} \cdot X_2(S) \Rightarrow \dot{x}_1 = -Px_1 + x_2 \cdots (3)$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} -p & 1 & 0 \\ 0 & -p & 1 \\ 0 & 0 & -p \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad , \quad y = [A \ \ 0 \ \ 0]\, x + 0 \cdot u$$

eigen values: $-p, -p, -p$, $|A| = -p^3$

* The form of Matrix A is Known as a "Jordan form" OR "Jordan Block".

Exercise: Get A, B, C, D when i) $G(s) = \frac{6}{(s+3)^3} + \frac{4}{s+4} + \frac{5}{s+5}$

ii) $G(s) = \frac{s^4 + s^2 + 96}{(s^2+4s+4)(s^2+7s+12)}$

Solution:

i)

$$\Rightarrow Y(s) = \frac{6U}{(s+3)^3} + \frac{4U}{s+4} + \frac{5U}{s+5}$$

Let $X_1 = \frac{4U}{s+4} \Longrightarrow \dot{x}_1 = -4x_1 + 4u \quad \cdots (1)$

let $X_5 = \frac{5U}{s+5} \Longrightarrow \dot{x}_5 = -5x_5 + 5u \quad \cdots (2)$

Let $X_2 = \frac{U}{s+3} \Longrightarrow \dot{x}_2 = -3x_2 + u \quad \cdots (3)$

Let $X_3 = \frac{U}{(s+3)^2} = \frac{1}{s+3} \cdot x_2 \Rightarrow \dot{x}_3 = -3x_3 + x_2 \quad \cdots (4)$

Let $X_4 = \frac{U}{(s+3)^3} = \frac{1}{s+3} \cdot X_3 \Rightarrow \dot{x}_4 = -3x_4 + x_3 \quad \cdots (5)$

Now for y: $\quad y = 6x_4 + x_1 + x_5 \quad \cdots (6)$

* S(A, B, C, D) becomes as follows:

$$\dot{x} = \begin{bmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 5 \end{bmatrix} u$$

$$y = [1 \ \ 0 \ \ 0 \ \ 6 \ \ 1]\, x + 0 \cdot u$$

(ii) $G(S) = \dfrac{S^4 + S^2 + 96}{(S^2+4S+4)(S^2+7S+12)}$

Since the Numerator & the Denominator have the same power, Need to do a Division, and it will result the following:

$$G(S) = 1 - \frac{11S^3 + 43S^2 + 76S - 48}{(S+2)^2 * (S+4) * (S+3)}$$

$$= 1 - \left[\frac{A}{S+4} + \frac{B}{S+3} + \frac{C}{(S+2)^2} + \frac{D}{S+2}\right]$$

* By using Cover-up Rule it can be found that:

$\boxed{A = +92}$, $\boxed{B = -186}$, $\boxed{C = -58}$

* By Partial fraction: $A(S+3)(S+2)^2 + B(S+4)(S+2)^2 + C(S+3)(S+4) + D(S+2)(S+3)(S+4)$
$= 11S^3 + 43S^2 + 76S - 48$

@ $S = 0$ it can be found that: $\boxed{D = 105}$

Now: $$Y = U + \frac{-92U}{S+4} + \frac{186U}{S+3} + \frac{58U}{(S+2)^2} + \frac{-105U}{S+2}$$

Let $X_1 = \dfrac{-92U}{S+4} \Rightarrow \dot{x}_1 = -4x_1 - 92u$ ... (1)

let $X_2 = \dfrac{186U}{S+3} \Rightarrow \dot{x}_2 = -3x_2 + 186u$ ... (2)

let $X_3 = \dfrac{U}{S+2} \Rightarrow \dot{x}_3 = -2x_3 + u$ ... (3)

let $X_4 = \dfrac{U}{(S+2)^2} = \dfrac{X_3}{S+2} \Rightarrow \dot{x}_4 = -2x_4 + x_3$ ... (4)

for y: $y = x_1 + x_2 + -105x_3 + 58x_4 + u$ ... (5)

$$\dot{x} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} -92 \\ 186 \\ 1 \\ 0 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 1 & 1 & -105 & 58 \end{bmatrix} x + 1 \cdot u$$

* * *

** Writing T.F on MATLAB:

```
>> A=[...]; B=[...]; C=[...]; D=[...];
>> syms s
>> G = C * inv(s * eye(length(A)) - A) * B + D
>> simple(G)
   OR simplify(G)
   OR pretty(G)
```

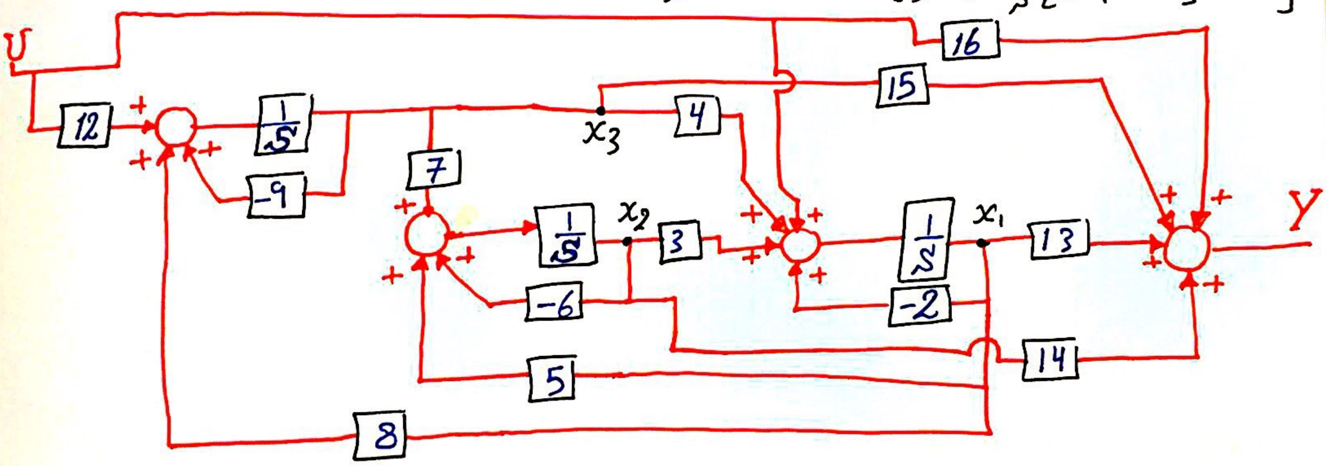
* Obtaining a B.D from SSR:

Best illustrated by a Numerical Example:

Given: $\dot{x} = \begin{bmatrix} -2 & 3 & 4 \\ 5 & -6 & 7 \\ 8 & 0 & -9 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 12 \end{bmatrix} u$, $y = [13 \quad 14 \quad 15] x + 16\, u$

$\dot{x}_1 = -2x_1 + 3x_2 + 4x_3 + u$, do the same for $\dot{x}_2$ & $\dot{x}_3$.

$\Rightarrow \quad X_1 = \frac{1}{s}[-2X_1 + 3X_2 + 4X_3 + U]$, $X_2 = \frac{1}{s}[5X_1 - 6X_2 + 7X_3]$, $X_3 = \frac{1}{s}[8X_1 - 9X_3 + 12U]$



Exercise: Given $\dot{x} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u$, $y = [7 \quad 8] x + 9\, u$

i) Obtain a BD representing the system ?

ii) Reduce the BD and determine the Poles ?

iii) Confirm the poles using another method ?

iv) Determine the TF using two methods?

v) Can you determine the steady state value ? why ?

vi) If the system were stable, now can you calculate x_{ss}, y_{ss} using two methods ?

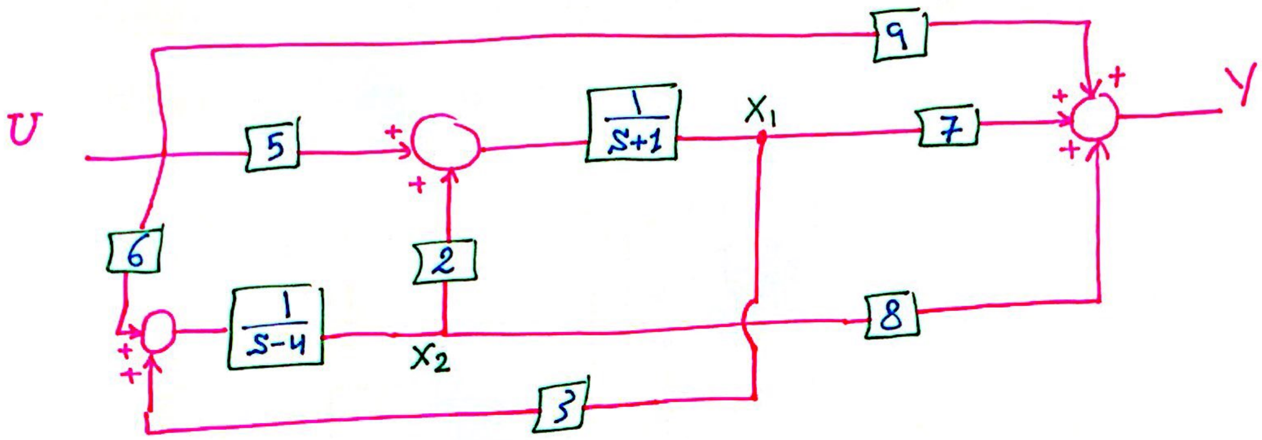
vii) Remind yourself with the final value theorem?

Solution:

(i) $\dot{x}_1 = -x_1 + 2x_2 + 5u \Rightarrow X_1 = \frac{1}{S+1}[2x_2 + 5u]$

$\dot{x}_2 = 3x_1 + 4x_2 + 6u \Rightarrow X_2 = \frac{1}{S-4}[3x_1 + 6u]$

$y = 7x_1 + 8x_2 + 9u$



(ii) This Block Diagram Can't be Reduced easily it will take too much time.

* It could be solved by equations not included in this course.

* OR By using MATLAB, Either way you will obtain the following Transfer function:

$$G(S) = \frac{9S^2 + 56S + 22}{S^2 - 3S - 10} \quad \#$$

(iii) The poles Represent the eigenvalues, so we find λ's:

$$[A-\lambda I] = \begin{bmatrix}-1-\lambda & 2\\ 3 & 4-\lambda\end{bmatrix} \Rightarrow |A-\lambda I| = 0 = (-1-\lambda)(4-\lambda)-6$$

$$\lambda^2-3\lambda-10=0 \qquad \lambda=-2, 5$$

(iv) Method (1): By Reducing the B.D we find $G(s)=\frac{Y(s)}{U(s)}$

Method (2): By using $G(s) = C[sI-A]^{-1}B + D$

$$G(s) = [7 \quad 8]\begin{bmatrix}s-4 & 2\\ 3 & s+1\end{bmatrix}\begin{bmatrix}5\\ 6\end{bmatrix}\cdot\frac{1}{s^2-3s-10} + 9$$

$$[7s-28+24 \quad 14+8s+8] \rightarrow 35s-20+48s+132 = 83s+112$$

$$\Rightarrow G(s) = \frac{83s+112 + 9(s^2-3s-10)}{s^2-3s-10} = \frac{9s^2+56s+22}{s^2-3s-10} \quad \#$$

(v) No / since the system is unstable, you can see that from the trace; trace $= -1+4 = 3 > 0$ (for sure there is a +ve real part in the eigenvalues, so unstable system).

(vi) Final Value Theorem (F.V.T) states that: $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = X_{ss}$

(vi) Method(1): By using $X_{ss} = -A^{-1}B = 0.1\begin{bmatrix}4 & -2\\ -3 & -1\end{bmatrix}\begin{bmatrix}5\\ 6\end{bmatrix} = \begin{bmatrix}0.8\\ -2.1\end{bmatrix}$ $\boxed{X_{ss_1}=0.8,\ X_{ss_2}=-2.1}$ #

$y_{ss} = [7 \quad 8]\begin{bmatrix}0.8\\ -2.1\end{bmatrix} + 9 = -2.2$ $\boxed{y_{ss}=-2.2}$ # • Remember $\mathcal{L}\, u(t) = \frac{1}{s}$ & $\lim_{t\to\infty} u(t) = 1$

Method(2): $G(s)=\frac{Y(s)}{U(s)} \Rightarrow Y(s) = U(s)\cdot\frac{9s^2+56s+22}{s^2-3s-10} \Rightarrow y_{ss} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\cdot\frac{1}{s}\cdot\frac{9s^2+56s+22}{s^2-3s-10} \Rightarrow \boxed{y_{ss}=-2.2}$ #

$X_1 = \frac{1}{s+1}[2X_2+5u] \cdots (1)$

$X_2 = \frac{1}{s-4}[3X_1+6u] \cdots (2)$

$\Rightarrow$ • Sub. (2) in (1) you will observe:

$$X_1 = \frac{-8U+5US}{s^2-3s-10} \Rightarrow X_{1_{ss}} = \lim_{s\to 0} sX_1 = \lim_{s\to 0}\frac{-8\cancel{s}\cdot\frac{1}{\cancel{s}}+5s^{\cancel{2}}\cdot\frac{1}{\cancel{s}}}{s^2-3s-10}$$

$$\Rightarrow \boxed{X_{1_{ss}} = 0.8} \ \#$$

• Sub. (1) in (2) you will observe:

$$X_2 = \frac{21U+6US}{s^2-3s-10} \Rightarrow X_{2_{ss}} = \lim_{s\to 0} sX_2 = \lim_{s\to 0}\frac{21\cancel{s}\cdot\frac{1}{\cancel{s}}+6s^{\cancel{2}}\cdot\frac{1}{\cancel{s}}}{s^2-3s-10} \Rightarrow \boxed{X_{2_{ss}} = -2.1} \ \#$$

* * * * * *

Example: Given $\dot{x} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u$

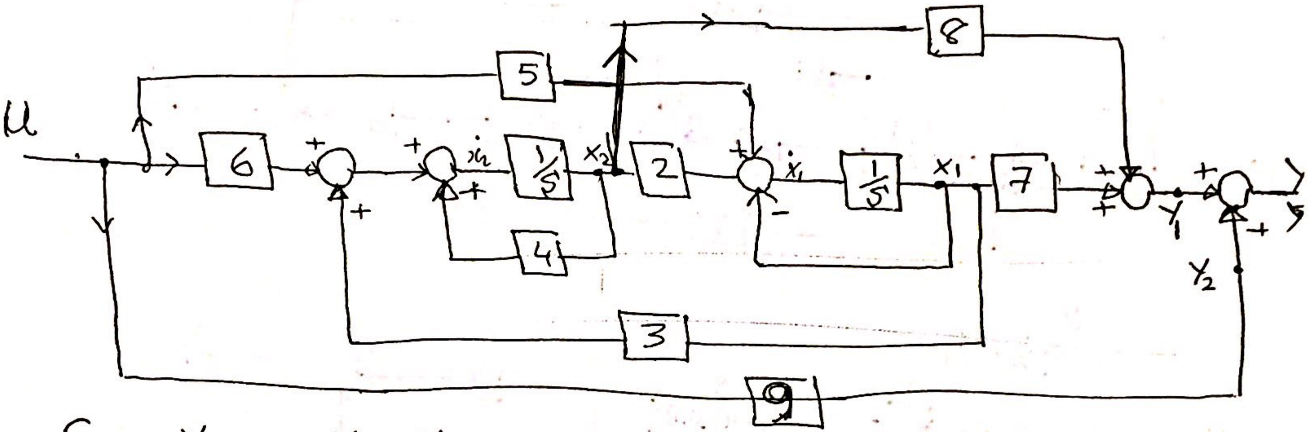
$$y = \begin{bmatrix} 7 & 8 \end{bmatrix} x + 9u$$

i) obtain a block diagram representation

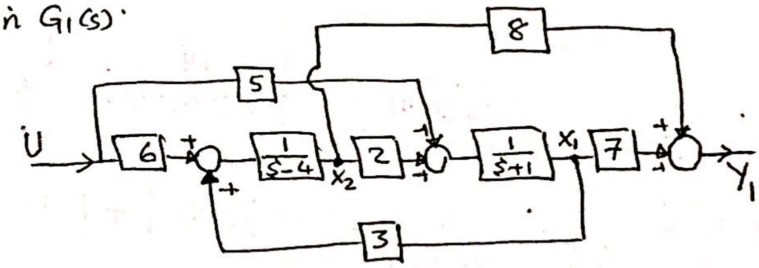
ii) Use the block diagram to obtain the T.F

iii) Obtain the T.F. using the state space representation together with the matlab ss2tf function and/or $C(sI-A)^{-1}B$

Solution: A convenient well-drawn tidy B.D may ease the reduction process, such as



$G(s) = \frac{Y}{U} = \frac{Y_1 + Y_2}{U} = \frac{Y_1}{U} + \frac{9U}{U} = G_1(s) + 9$, so determine $G_1(s)$. This proves very difficult using reduction techniques. Instead the following has been done to obtain $G_1(s)$.



$$(s-4)X_2 = 6U + 3X_1$$

$$(s+1)X_1 = 5U + 2X_2$$

$$Y_1 = 7X_1 + 8X_2$$

$$\Rightarrow \begin{bmatrix} -3 & s-4 & 0 \\ s+1 & -2 & 0 \\ -7 & -8 & 1 \end{bmatrix} \begin{bmatrix} X_1/U \\ X_2/U \\ Y_1(s) \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} \Rightarrow Ax = b$$

$$|A| = -s^2 + 3s + 10$$

Using Cramer's rule

$$G_1(s) = \frac{Y_1(s)}{U} = \frac{\begin{vmatrix} -3 & s-4 & 6 \\ s+1 & -2 & 5 \\ -7 & -8 & 0 \end{vmatrix}}{-s^2+3s+10} = \frac{6(-8s\cancel{-8}-22) - 5(\cancel{24}-4+7s) \\ -48s - 132 + 20 - 35s}{-s^2+3s+10} = \frac{83s + 112}{s^2 - 3s - 10}$$

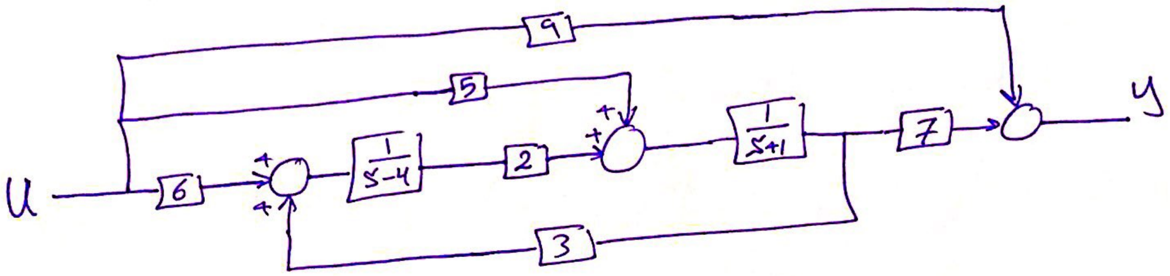
$$G(s) = G_1(s) + 9 = \frac{83s + 112 + 9s^2 - 27s - 90}{s^2 - 3s - 10} = \frac{9s^2 + 56s + 22}{s^2 - 3s - 10}$$

Exercise: Obtain BD Reduction, Then Reduce it, Confirm answers using three methods:

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u$$

$$y = [7 \quad 0] x + 9u$$

$$x_1 = \frac{1}{s+1}[2x_2 + 5u], \quad x_2 = \frac{1}{s-4}[3x_1 + 6u], \quad y = 7x_1 + 9u$$



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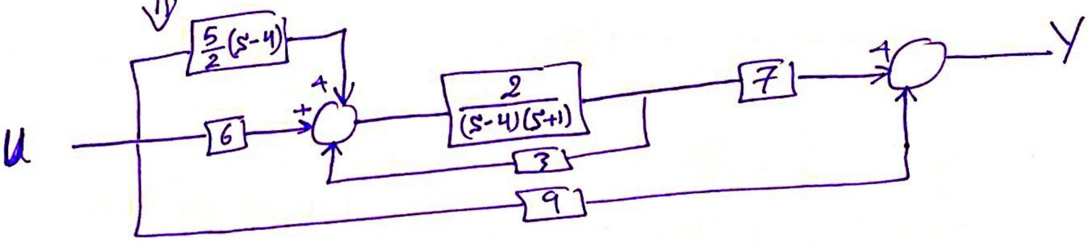

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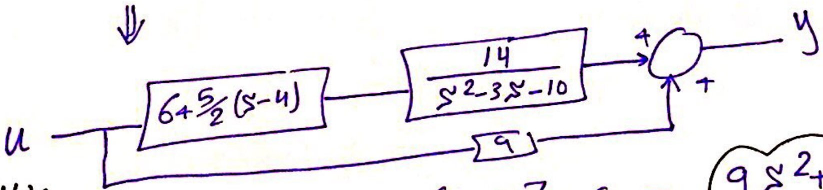


• **Method (1):**

$$G(s) = \frac{Y(s)}{U(s)} = \frac{14}{s^2-3s-10} \cdot \left[6 + \tfrac{5}{2}s - 10\right] + 9 = \boxed{\frac{9s^2+8s-146}{s^2-3s-10}} \#$$

• **Method (2):**

$$G(s) = C[sI-A]^{-1}B + D = [7 \quad 0]\frac{1}{s^2-3s-10}\begin{bmatrix} s-4 & 2 \\ 3 & s+1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} + 9 = \boxed{\frac{9s^2+8s-146}{s^2-3s-10}} \#$$

• **Method (3):**

By using MATLAB command:

```
>> A=[-1 2;3 4]; B=[5;6]; C=[7 0]; D=9;
>> [nn,dd]=ss2tf(A,B,C,D)
Answers:  nn =  9   8   -146
          dd =  1  -3   -10
```

$$\Rightarrow \boxed{G(s) = \frac{9s^2+8s-146}{s^2-3s-10}} \#$$

※ Converting a T.F to a B.D:

Given $G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}$

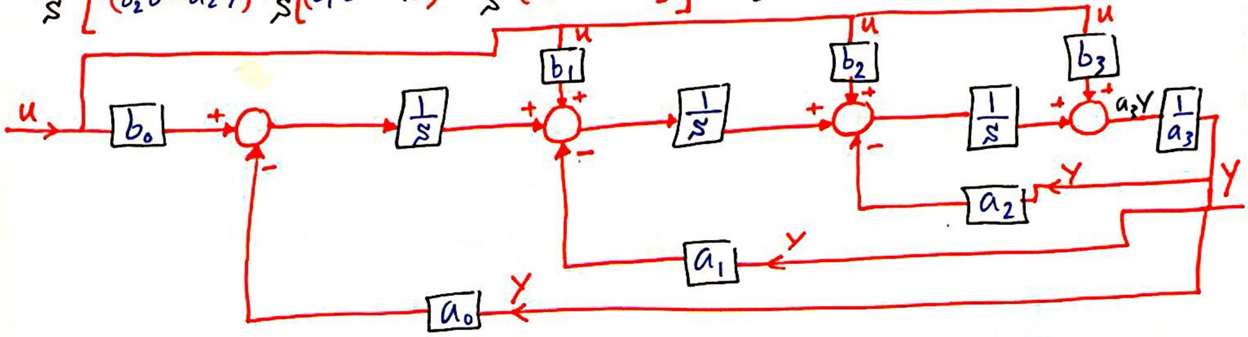
Using Integrators: (divide By s^3)

*N.B: if the Numerator and the Denominator have the same highest power then $D \neq 0$ in the SS model.

$$\Rightarrow G(s) = \frac{b_3 + b_2\frac{1}{s} + b_1\frac{1}{s^2} + b_0\frac{1}{s^3}}{a_3 + a_2\frac{1}{s} + a_1\frac{1}{s^2} + a_0\frac{1}{s^3}} = \frac{Y(s)}{U(s)}$$

$$\Rightarrow (b_3 U - a_3 Y) + (b_2 U - a_2 Y)\frac{1}{s} + (b_1 U - a_1 Y)\frac{1}{s^2} + (b_0 U - a_0 Y)\frac{1}{s^3} = 0$$

$$\frac{1}{s}\left[(b_2 U - a_2 Y) + \frac{1}{s}\left[(b_1 U - a_1 Y) + \frac{1}{s}(b_0 U - a_0 Y)\right]\right] + b_3 U = a_3 Y$$

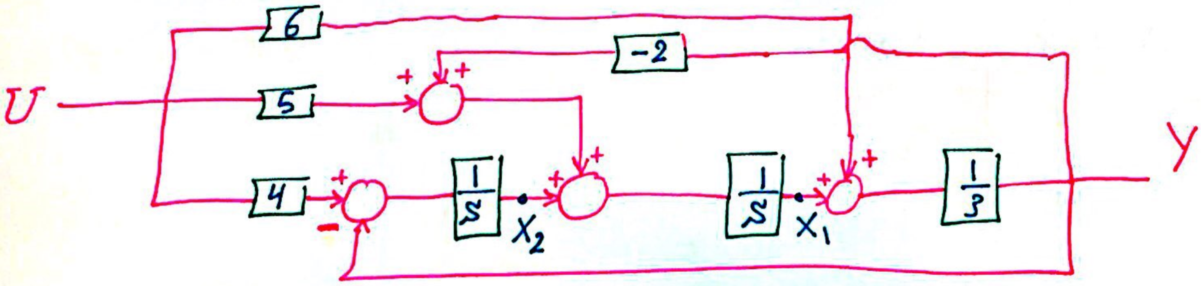


Exercise: Convert $G(s) = \frac{Y(s)}{U(s)} = \frac{6s^2+5s+4}{3s^2+2s+1}$ to a B.D. Then to a SS Model.

Then Confirm the T.F using Two methods.

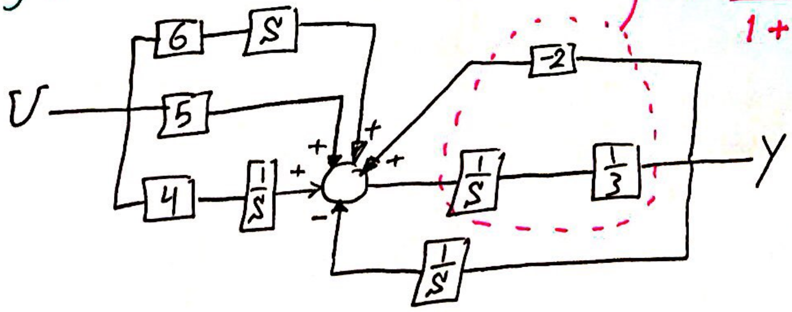
Solution: After dividing by s^2 you will obtain the following:

$$G(s) = \frac{Y(s)}{U(s)} \Rightarrow \frac{1}{s}\left[(5U-2Y) + \frac{1}{s}(4U-Y)\right] + 6U = 3Y$$



• **Method(1):**

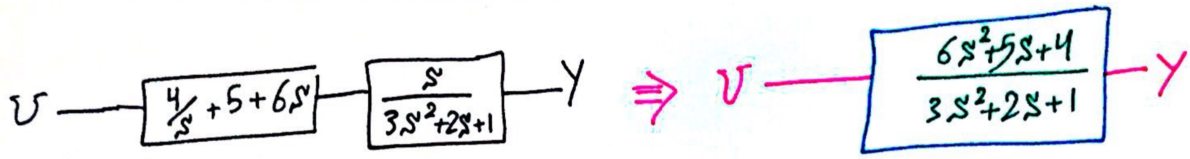
By B.D Reduction:



$\frac{\frac{1}{3s}}{1+\frac{2}{3s}} = \frac{1}{3s+2}$ feed back with $\frac{1}{s}$

$$\Rightarrow \frac{\frac{1}{3s+2}}{1+\frac{1}{3s^2+2s}} = \frac{s}{3s^2+2s+1}$$

Continue.



$$G(s)=\frac{Y(s)}{U(s)}=\frac{6s^2+5s+4}{3s^2+2s+1} \quad \#$$

Method (2): By obtaining SSR:

$$x_1=\frac{1}{s}\left[x_2+5u-\frac{2}{3}(6u+x_1)\right] \quad \Rightarrow \dot{x}_1=-\frac{2}{3}x_1+x_2+u \quad \cdots(1)$$

$$x_2=\frac{1}{s}\left[4u-\frac{1}{3}(x_1+6u)\right] \quad \Rightarrow \dot{x}_2=-\frac{1}{3}x_1+2u \quad \cdots(2)$$

$$y=\frac{1}{3}\left[x_1+6u\right] \qquad y=\frac{1}{3}x_1+2u \quad \cdots(3)$$

$$\dot{x}=\underbrace{\begin{bmatrix}-\frac{2}{3} & 1\\ -\frac{1}{3} & 0\end{bmatrix}}_{A}x+\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{B}u \quad , \quad y=\underbrace{\begin{bmatrix}\frac{1}{3} & 0\end{bmatrix}}_{C}x+\underbrace{2}_{D}\cdot u$$

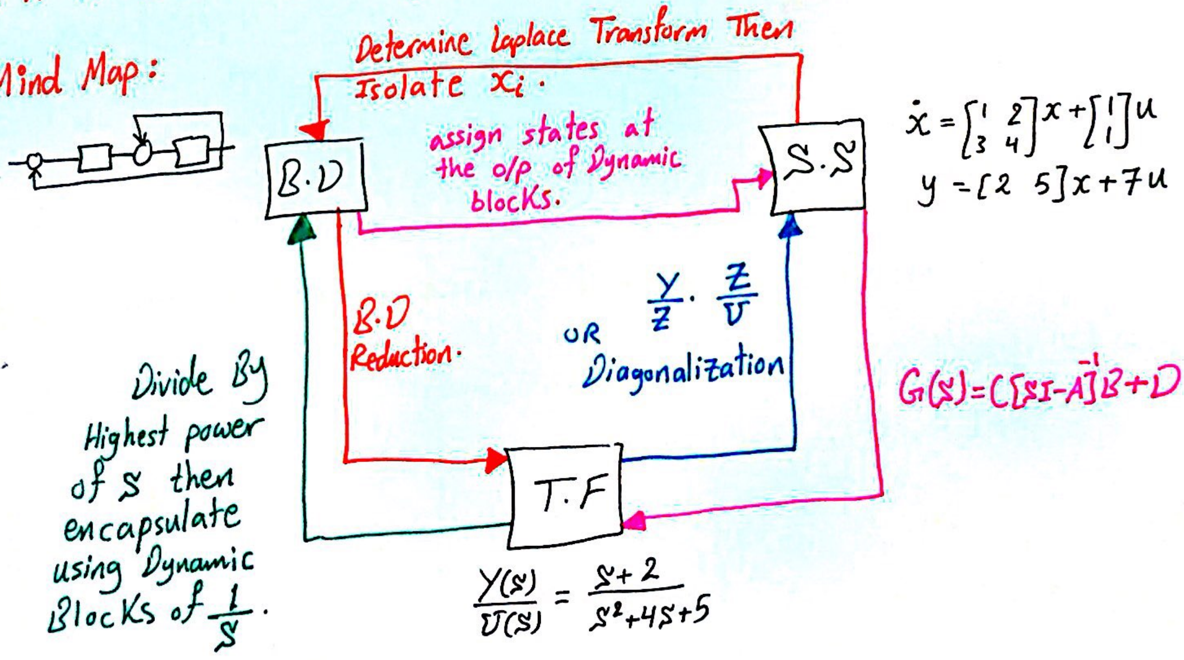
from $G(s)=C[sI-A]^{-1}B+D$ we found T.F:

$$G(s)=\begin{bmatrix}\frac{1}{3} & 0\end{bmatrix}\begin{bmatrix}s & 1\\ -\frac{1}{3} & s+\frac{2}{3}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\cdot\frac{1}{s^2+\frac{2}{3}s+\frac{1}{3}}+2=\frac{6s^2+5s+4}{3s^2+2s+1} \quad \#$$

* * *

- If it is hard to solve using B.D $\longrightarrow$ go to SSR.

※ **Mind Map:**



* Matlab Aside:

```
n = [1   1]            % s+1
d = [2   4   8]        % 2s^2+4s+8
G = tf(n,d)            % Transfer Function
[A,B,C,D] = tf2ss(n,d)
[nn,dd] = ss2tf(A,B,C,D)
```

`[A,B,C,D] = tf2ss(n,d)` ⇒ this command converts from the transfer function to state space.

`[nn,dd] = ss2tf(A,B,C,D)` ⇒ for SISO systems.

* Back To Matrix Properties:

- **The Cayley-Hamilton Theorem:**

Given that $\Delta(\lambda)$ is the characterstic polynomial/Equation CP or CE of a square matrix $A_{n\times n}$, Then: $\Delta(A) = 0$

i.e A satisfies its CP/CE.

Example: Let $A = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}$ ⇒ $\Delta(\lambda) = \lambda^2 + 9\lambda + 20 = 0$

$\Delta(A) = A^2 + 9A + 20I_n = 0$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} = \begin{bmatrix} -20 & -9 \\ 180 & +61 \end{bmatrix}, \quad 9A = \begin{bmatrix} 0 & 9 \\ -180 & -81 \end{bmatrix}, \quad 20I_n = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$

$$\Rightarrow \Delta A = \begin{bmatrix} -20+0+20 & -9+9+0 \\ 180-180+0 & 61-81+20 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \#$$

Now To Evaluate A^3:

$$A^3 = -9A^2 - 20A = -9(-9A - 20I_n) - 20A = \underline{\underline{61A + 180I_n}}$$

Exercise: Use the Cayley-Hamelton theorem to Calculate A^{-1} ?

Solution: $\Delta(A) = A^2 + 9A + 20I_n = 0$

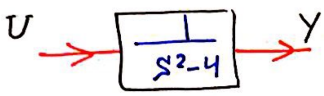
$$\Rightarrow A^2 \cdot A^{-1} + 9A \cdot A^{-1} + 20I_n \cdot A^{-1} = 0 \Rightarrow A + 9I_n + 20A^{-1} = 0$$

$$\Rightarrow A^{-1} = [-A - 9I_n] * \frac{1}{20} = \begin{bmatrix} 0-9 & -1 \\ 20 & 9-9 \end{bmatrix} * \frac{1}{20} = \begin{bmatrix} \frac{-9}{20} & \frac{-1}{20} \\ 1 & 0 \end{bmatrix} \quad \#$$

* * *

* **Bicycle:** is to be decided to be stable or Not depending on the Controller.

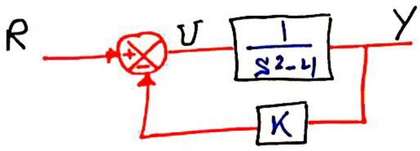
Example: Consider a simplified model of a Bicycle given by $G(S) = \frac{1}{S^2-4}$



it is UNSTABLE, since $\left(\frac{1}{S^2-4}\right)$ changing its sign.

- Test it By MATLAB.

* **By Using Feedback:**

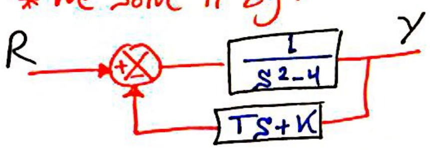


$$\frac{Y}{R} = \frac{1}{S^2+K-4}$$

$\Rightarrow$ At its best if $K > 4$ Then we get Sinusoidal Response.

Marginally Unstable.

* **We Solve it By:**



$$\Rightarrow \frac{Y}{R} = \frac{1}{S^2+TS+K-4}$$

- To Be Stable: $K-4>0 \Rightarrow K>4$ for any positive T.

\# **Analysis Using SS Representation:**

Systems are represented as:

$$\dot{x} = Ax + Bu \qquad x \in R^n,\ u \in R^m$$
$$y = Cx + Du \qquad y \in R^p$$

* output: must be measurable.

* input: Need Not to be always measurable.

* **General Representation of systems Can be put on the following forms:**

1) Controllable Form: Given: $G(S) = \frac{b_0 S^3 + b_1 S^2 + b_2 S + b_3}{S^3 + a_1 S^2 + a_2 S + a_3}$

$\dot{x} \Leftrightarrow G(S)$

factor of S^3 in the Denominator must be $=1$.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} b_3 - b_0 a_3 & b_2 - b_0 a_2 & b_1 - b_0 a_1 \end{bmatrix} x + b_0 u$$

* if Last row of A was: $[-1 \quad 2 \quad -3] \Rightarrow$ system Unstable (due to changing in sign)

Example: Given $G(S) = \frac{Y(S)}{U(S)} = \frac{S^2 + 9S + 20}{S^3 + 6S^2 + 11S + 6}$

Memorize the General Form.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 20 & 9 & 1 \end{bmatrix} x + 0 \cdot u$$

2] Observable Form: It is Obtained by letting $A_o = A_c^T$ (A_o → observable, A_c → Controllable.), $B_o = C_c^T$, $C_o = B_c^T$, $D_o = D_c$

i.e: $\dot{x} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} x + \begin{bmatrix} b_3 - b_0 a_3 \\ b_2 - b_0 a_2 \\ b_1 - b_0 a_1 \end{bmatrix} u$, $y = [0 \quad 0 \quad 1]\, x + b_0 u$

3] Jordan Form: given:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{C_3}{(s+P_3)^3} + \frac{C_2}{(s+P_3)^2} + \frac{C_1}{(s+P_3)} + \frac{C_4}{(s+P_2)} + \frac{C_5}{(s+P_1)}; n=5$$

The Jordan Form is as depicted below:

$$\dot{x} = \left[\begin{array}{ccc:c:c} -P_3 & 1 & 0 & 0 & 0 \\ 0 & -P_3 & 1 & 0 & 0 \\ 0 & 0 & -P_3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -P_1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -P_2 \end{array}\right] x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u, \quad y = [C_3 \quad C_2 \quad C_1 \quad C_5 \quad C_4]\, x + b_0 u$$

*Advantage: The eigenvalues are obtained by inspection, hence stability is determined.

*Disadvantage: Need to do partial fraction.

Exercise: obtain a state space representation in diagonal form given:

i) $G(s) = \dfrac{4s+5}{s^3 + 6s^2 + 11s + 6}$ ii) $G(s) = \dfrac{s^2 + 5s + 6}{(s+2)^2 (s+3)(s+4)}$ iii) $G(s) = \dfrac{s^3 + 2s^2 + 4s + 5}{s(s+3)^3}$

Solution:

i) Rewriting $G(s)$ as follows: $G(s) = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{s+3}$

By using The Cover Up Rule you can obtain that: $\boxed{A = \frac{1}{2}}$, $\boxed{B = 3}$, $\boxed{C = \frac{-7}{2}}$

Now $G(s)$ Becomes: $G(s) = \dfrac{1/2}{s+1} + \dfrac{3}{s+2} + \dfrac{-7/2}{s+3}$

SSR is Given By:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} \frac{1}{2} & 3 & \frac{-7}{2} \end{bmatrix} x + 0 \cdot u$$

ii) $G(s) = \dfrac{(s+2)(s+3)}{(s+2)^2 (s+3)(s+4)} = \dfrac{1}{(s+2)(s+4)} = \dfrac{A}{s+2} + \dfrac{B}{s+4}$

By Cover Up Rule you can obtain that: $\boxed{A = 1/2}$, $\boxed{B = -1/2}$

SSR is Given By:

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \end{bmatrix} x + 0 \cdot u$$

(iii) Rewriting G(s) as follows: $G(s) = \frac{C_3}{(s+3)^3} + \frac{C_2}{(s+3)^2} + \frac{C_1}{(s+3)} + \frac{C_4}{s}$

By Using Cover Up Rule you Can Obtain that: $\boxed{C_3 = \frac{16}{3}}$, $\boxed{C_4 = \frac{5}{27}}$

Now By using this equation: $C_3 s + C_2 s(s+3) + C_1 s(s+3)^2 + C_4 (s+3)^3 = s^3 + 2s^2 + 4s + 5$

& Substituting $s=1$ & $s=-1$ you can observe these two equations:

$$4C_2 + 16C_1 = \frac{-140}{27} \quad \dots (1)$$
$$-2C_2 - 4C_1 = \frac{158}{27} \quad \dots (2)$$

solving: $\boxed{C_1 = \frac{22}{27}}$, $\boxed{C_2 = \frac{-41}{9}}$

SSR is Given By:

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} \frac{16}{3} & \frac{-41}{9} & \frac{22}{27} & \frac{5}{27} \end{bmatrix} x + 0u$$

* * *

*** Diagonalization of a Matrix:**

Let P represents the eigenvectors matrix:

$$A P_1 = \lambda_1 P_1 \qquad P_1 \neq 0$$
$$A P_2 = \lambda_2 P_2 \qquad P_2 \neq 0$$

$$\underset{n\times n}{A}\ \underset{n\times q}{[P_1\ P_2]} = [P_1\ P_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} , \text{ not } \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [P_1\ P_2]$$

So $AP = P\Delta$, to get A; use: $A = P\Delta P^{-1}$, to get Δ; use: $\Delta = P^{-1}AP$

* If the eigenvalues are distinct (different, non-equal) then: the eigen vectors are independent, hence $|P| \neq 0 \Rightarrow P^{-1}$ exists, hence: $\Delta = P^{-1}AP$

Exercise: Diagonalize: i) $A = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$ ii) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ iii) $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ iv) $A = \begin{bmatrix} -1 & 3 & 3 \\ -6 & 7 & 4 \\ 0 & 1 & 4 \end{bmatrix}$

Solution: (i) $\lambda^2 + 7\lambda + 12\lambda = 0 \Rightarrow \lambda = -3, -4 \Rightarrow P = \begin{bmatrix} 1 & 1 \\ -3 & -4 \end{bmatrix} \Rightarrow \Delta = \begin{bmatrix} 1 & 1 \\ -3 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$ #

(ii) $\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$
$\Rightarrow \lambda = -1, -2, -3$ $\Rightarrow P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \Rightarrow \Delta = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ #

(iii) $[A - \lambda I] = \begin{bmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$

$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$
$\lambda = -2, 5$

• for $\lambda = -2$: eigenvector $= \text{adj}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

• for $\lambda = 5$: eigenvector $= \text{adj}\begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\Delta = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}$ #

continue

(iv) $A-\lambda I = \begin{bmatrix} -1-\lambda & 3 & 3 \\ -6 & 7-\lambda & 4 \\ 0 & 1 & 4-\lambda \end{bmatrix} \Rightarrow -\lambda^3+10\lambda^2-31\lambda+30=0$

$\Rightarrow \lambda = 2, 3, 5$

- for $\lambda=2$: eigenvector $=\text{adj}\begin{bmatrix} -3 & 3 & 3 \\ -6 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 \\ 12 \\ -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ #
- for $\lambda=3$: eigenvector $=\text{adj}\begin{bmatrix} -4 & 3 & 3 \\ -6 & 4 & 4 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ #
- for $\lambda=5$: eigenvector $=\text{adj}\begin{bmatrix} -6 & 3 & 3 \\ -6 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 \\ -6 \\ -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ #

$$\Rightarrow \Delta = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 3 & 3 \\ -6 & 7 & 4 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \#$$

* If the eigenvalues are identical (repeated – the same) say suppose we have three identical eigenvalues (λ_1), then get three eigenvectors:

- $[A-\lambda_1 I_n]P_1 = 0$ → Use G.E or adjoint method.
- $[A-\lambda_1 I_n]P_2 = P_1$ → Use G.E.
- $[A-\lambda_1 I_n]P_3 = P_2$

OR:

- $[A-\lambda_1 I_n]\,P_1 = 0$
- $[A-\lambda_1 I_n]^2 P_2 = 0$
- $[A-\lambda_1 I_n]^3 P_3 = 0$

P_1, P_2, P_3 are Known as: "Generalized Eigenvectors."

* Structural Properties of systems:

1] Controllability (CC):

Def.: a system is completely state Controllable if it is possible to move its states from any arbitrary points to any final arbitrary points on a finite time using an unconstrained input.

- A Test for Controllability: consider $\dot{x} = Ax + Bu$; $x \in R^n$, $u \in R^m$

$\Rightarrow$ A system is CC iff: $\text{rank}([B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B]) = n$

Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ is it CC ?

* • if $|M_{n\times n}| = 0$, Then: $\text{rank}(M) < n$. *

solution: $\text{rank}(\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & \beta \end{bmatrix}}_{M_{2\times 2}}) = n$

in our case: $|M| = -1 \neq 0 \Rightarrow$ so $\text{rank} = n$ ∴ it is CC.

Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u$ is it CC?

Solution:

$\text{rank}([B \ \ AB]) = \text{rank}\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\right) \Rightarrow |M| = 0$ so $\text{rank}(M) < n$

$\therefore \text{rank} = 1 \quad \therefore$ Uncontrollable.

same to each other just different by factor -2.

Example: $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} u \Rightarrow M = [B \ \ AB \ \ A^2B]$

$= \begin{bmatrix} 1 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -8 & 16 \end{bmatrix}$ — you can deal with it as: $A \cdot AB$

$\Rightarrow \text{rank}(M) = 1$

By Using "Sylvester Method". $\Rightarrow \text{rank}(M) = 1 < n = 3 \quad \therefore$ Uncontrollable.

- **Note:** if there is 2x2 matrix in this 3x3 matrix, has determenant $\neq 0$ the rank will be 2 in this case, BUT since All submatrices 2x2 gives det. $= 0$; rank $= 1$.

\# Controllability By Inspection:

Given $\dot{x} = Ax + Bu$. There exists a similarity transformation $x = P\bar{x}$

where: $\dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu$

$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$;

$x = P\bar{x}$

$AP = P\bar{A}$

$A = P\bar{A}P^{-1}$

$\bar{A} = P^{-1}AP$

- Judge CC by inspecting $\bar{A}$ together with $\bar{B}$:

if $\bar{A} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{bmatrix}$ Then the system is CC only if all $\bar{b}_i$ are Non-Zero.

if $\bar{b}_2$ is zero, then the system is uncontrollable.

Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} x + \begin{bmatrix} 1 \\ -5 \end{bmatrix} u$ is it CC?

Solution: • **Note:** the columns of Matrix P are the eigenvectors.

$\therefore$ let $P = \begin{bmatrix} 1 & 1 \\ -5 & -4 \end{bmatrix} \Rightarrow \bar{A} = P^{-1}AP = \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}$, $P^{-1}B = \begin{bmatrix} -4 & -1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The system is:

$\dot{\bar{x}} = \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$\therefore$ Uncontrollable.

* Physically CC means that the e^{-4t} mode Cannot be affected by u.

2) Observability:

A system is state observable if it is possible to get the states out of the outputs in a finite time.

- A Test for Observability (OO): Given $\dot{x} = Ax + Bu$, $y = Cx + Du$

A system is Completely OO iff: $\text{rank}(N) = \text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$

where $\underline{D = 0}$.

※ Implication of Controllability:

1) Ability to change <u>all</u> of the eigenvalues using <u>state feedback</u>.
2) Ability to design <u>optimal</u> controllers.

* There is NO Relation between stability & Controllability.
↳ we could have 4-states: stable-controllable, stable-uncontrollable, unstable-controllable, unstable-uncontrollable.

* Fire-Fighting Planes are: Naturally Unstable.

* <u>N.B</u>: Controllability & stability DO NOT imply each other (i.e CC & S, CC & $\overline{S}$, $\overline{CC}$ & S, $\overline{CC}$ & $\overline{S}$) ; where $\underline{S}$ stand for stable. $\overline{S}$ stand for unstable.

<u>Example</u>: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x$, $y = [4 \quad 1] x$ is it OO?

<u>solution</u>: $N = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -8 & -2 \end{bmatrix}$ the two colums depend on each other. $|N| = 0$, so $\text{rank}(N) < 2$

$\therefore \text{rank}(N) = 1$ $\therefore$ system is unobservable ($\overline{OO}$).

* $\overline{CC}$ & $\overline{OO}$ are reflected by Pole-Zero Cancellation in the T.F. (i.e if the T.F doesn't have the order of the system (i.e n) Then we have Pole-Zero Cancellation).

in this case the system could be $\overline{CC}$, or $\overline{OO}$, or $\overline{CC}$ & $\overline{OO}$.

so need to check on the system to Know which one of these 3-cases.

Topics in Control

Fall017

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※ Observability by Inspection:

If a system is diagonalized then for it to OO, all elements of output matrix C should be Non-Zero.

Example: $\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 13 \\ -4 \end{bmatrix} u \;,\; y = [-7 \;\; 0 \;\; 20] x$

⇒ the (-1) eigenvalue is $\overline{CC}$. → we Knew since it face zero in matrix B.

⇒ the (-2) eigenvalue is $\overline{OO}$. → we Knew since it face zero in matrix C.

Example: $\dot{x} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 5 \end{bmatrix} u \;,\; y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} x$

Jordan Form.

⇒ the (2) eigenvalue is $\overline{CC}$ & OO. → since the row in matrix B [0 0].

⇒ " (3) " is CC & OO.

* end of second Material *

Example: Consider $\dot{x} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \;;\; y = [-3 \;\; 1] x \;;\; x(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

Find $e^{At} = P e^{\lambda t} P^{-1}$?

Solution: $|A - \lambda I_2| = \lambda^2 - 3\lambda - 10 = 0 \Rightarrow \boxed{\lambda_1 = 5}, \boxed{\lambda_2 = -2}$

- eigenvector associated with $\lambda_1 = 5$: v_1 = a column of adj $\begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & -6 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

  - **Note:** always the matrix that we found the eigenvector from it, has determinant = 0.

- eigenvector associated with $\lambda_2 = -2$: v_2 = a column of adj $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

  - **Note:** always each eigenvalue has one eigenvector.

$\therefore P = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \quad \therefore e^{At} = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \cdot \frac{1}{7} = \begin{bmatrix} e^{5t} & -2e^{-2t} \\ 3e^{5t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \cdot \frac{1}{7}$

$e^{At} = \frac{1}{7} \begin{bmatrix} e^{5t} + 6e^{-2t} & 2e^{5t} - 2e^{-2t} \\ 3e^{5t} - 3e^{-2t} & 6e^{5t} + e^{-2t} \end{bmatrix}$ • **check:** it must give $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ @ $t = 0$.

• **Note:** always the matrix e^{At} must have +ve trace. ⇒ trace $= 7e^{5t} + 7e^{-2t} > 0$.

⟹ Continue.

$\Rightarrow$ given $u(t)=0$ for $t>0$.

$$x(t) = e^{At}x(0) + \underbrace{e^{At}\int_0^t e^{A(-\tau)} \cdot Bu(\tau)\,d\tau}_{\text{equal Zero}} \Rightarrow x(t) = e^{At}\cdot X(0).$$

$$\Rightarrow x(t) = \begin{bmatrix} 3e^{5t}+4e^{-2t} \\ 9e^{5t}-2e^{-2t} \end{bmatrix} \Rightarrow y(t) = C\,x(t) = [-3 \quad 1]\,x(t) = \underline{\underline{-14e^{-2t}}}$$

- We notice a missing solution (mode) $\Rightarrow$ i.e e^{5t} is missing in the o/p.

This is due to Unobservability of the system.

$\Rightarrow$ To check: $N = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \Rightarrow \text{rank}(N) = 1$ "<u>Unobservable</u>".

Exercise: Calculate the T.F for previous example & use it to judge the Unobservability ?

Solution: $G(s) = C[sI-A]^{-1}B + D = \dfrac{1}{s^2-3s-10}[-3 \quad 1]\begin{bmatrix} s-4 & 2 \\ 3 & s+1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \dfrac{s-5}{s^2-3s-10}$

$G(s) = \dfrac{1}{s+2}$; Pole-Zero Cancellation occured which confirms the Unobservability.

※ Continue for CC & OO By inspection:

- Best illustrated by an example: Using $x = P\bar{x}$ where P contains the eigenvectors as columns.

$$\dot{\bar{x}} = \left[\begin{array}{ccc:cc} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]\bar{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ \hdashline 1 & 0 \\ \hdashline 0 & 1 \end{array}\right]u \quad , \quad y = \left[\begin{array}{ccc:cc} 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right]\bar{x}$$

- If you have two Jordan blocks associated with the same eigen value. Then the system is Uncontrollable. [in case we replace the element (4) in Matrix A with new element (3)].

- eigenvalue (3) $\Rightarrow$ CC & $\overline{OO}$.
- eigenvalue (4) $\Rightarrow$ CC & $\overline{OO}$.
- eigenvalue (5) $\Rightarrow$ CC & OO.

$$\dot{\bar{x}} = \left[\begin{array}{cc:ccc} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]\bar{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \\ \hdashline 17 & -31 \\ 0 & 0 \end{array}\right]u$$

- eigen value (3) $\Rightarrow$ CC & $\overline{OO}$.
- eigen value (4) $\Rightarrow$ $\overline{CC}$ & $\overline{OO}$.
- eigenvalue (5) $\Rightarrow$ $\overline{CC}$ & OO.

$$y = \left[\begin{array}{cc:c:cc} 0 & -3 & 0 & 1 & -2 \\ 0 & 53 & 0 & 0 & 8 \end{array}\right]\bar{x}$$

※ Duality of LTI Systems :

For each primal system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ is associated with another dual system $\begin{cases} \dot{z} = A^T z + C^T u \\ w = B^T z + Du \end{cases}$

- If the primal system is CC OR OO, Then the dual system is OO OR CC

primal		dual
CC	$\longrightarrow$	OO
OO	$\longrightarrow$	CC

- ON MATLAB :

Note: $ctrb = [B \ \ AB \cdots A^{n-1}B]$

```
>> a=[-1 2;3 4]; b = [-2;1]; C=[1 1]; d=0;
>> CC = ctrb(a,b), r=rank(CC)
>> if r==length(a)
       disp ('system is Controllable')
   else
       disp('system is Uncontrollable').
   end.
>> ro=rank(ctrb(a',c'))      % determines Observability using duality.
```

↳ if the dual is CC ⟹ Then primal is OO.

```
>> ro = rank(obsv(a,c))
```

※ System Invarient :

i) Eigenvalues of the dual system are those of the primal.

ii) Trace, Determinant are also invarient for the pirmal & dual systems.

iii) Both systems have the same Transfer Function. (prove !?).

iv) Both systems have the same Zeros.

⁂ Zero Determination :

Given $\dot{x} = \underset{n\times n}{A}x + \underset{n\times m}{B}u$ $\quad x \in R^n, u \in R^m$

$y = \underset{m\times n}{C}x$ $\quad y \in R^m$

⟹ The $(n-m)$ Zeros are given By: $\lambda(AZ) = \lambda\left(\left[A - B(CB)^{-1}CA\right]\right)$, excluding m zero valued eigenvalues.

for e.g: if $m=1$, $\lambda = 2, 0, 0$

Then the Zeros are : 2 & 0.

Example: $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$, $y = [20 \;\; 9 \;\; 1]\, x$ Find Zeros?

Note: This system is CC since it is on the Controllable Form.

Solution:

$\boxed{\begin{matrix} n=3 \\ m=1 \end{matrix}}$ $\lambda(AZ) = -4, -5, 0 \Rightarrow$ The Zeros are: -4 & -5.

Try for the same system with $C = [0 \;\; 9 \;\; 1] \Rightarrow \lambda(AZ) = 0, 0, -9$

The Zeros are: 0 & -9.

Exercise: $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$, $y = \begin{bmatrix} 1 & 10 & 0 \\ 0 & 20 & 1 \end{bmatrix} x$ find poles & Zeros?

Solution: poles are: $-1, -2, -3$.

so we will have one Zero.

Note that: $m = 2$, $n = 3$

$\lambda(AZ) = 0, 0, -20$

The Zeros are: $\underline{\underline{-20}}$.

Exercise: $\dot{x} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u$, $y = [0 \;\; 1]\, x$ find Zeros?

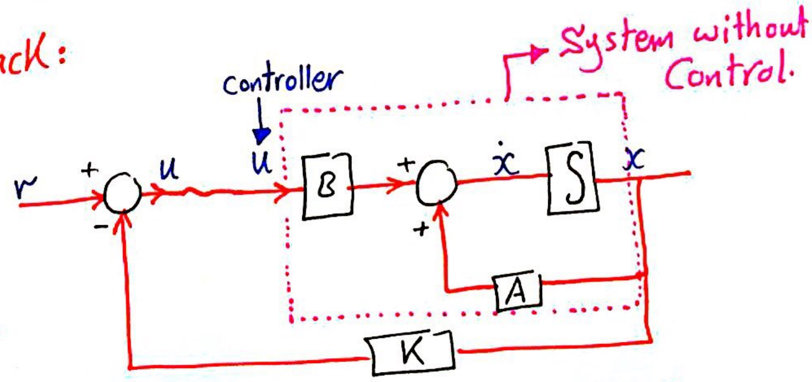
Note that: we can check our answers by Matlab or By finding T.F.

Solution: $\lambda(AZ) = 5, 0$.   The Zeros are: $\underline{\underline{5}}$.

* Design Using State Feedback:

- Consider $\dot{x} = Ax + Bu$
  $x \in R^n, u \in R^m$

$\Rightarrow$ Let $u = -Kx + r$ be the state feedback controller



The system with such controller.

System with state feedback.

$\dot{x} = Ax + B(r - Kx)$

$\Rightarrow \dot{x} = (A - BK)x + Br$

$(A-BK)$ will determine the stability of the system.

- Properties of the controlled system is now determined by: $A-BK$ & not by A alone.

* Choosing a suitable K:

⇒ Suppose A is in Controllable Form : i.e $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$

if $K = [K_3 \quad K_2 \quad K_1]$

$$\dot{x} = A - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[K_3 \quad K_2 \quad K_1]x + Br = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_3 - K_3 & \alpha_2 - K_2 & \alpha_1 - K_1 \end{bmatrix} x + Br$$

$$C.E = \lambda^3 - (\alpha_1 - K_1)\lambda^2 - (\alpha_2 - K_2)\lambda - (\alpha_3 - K_3) = 0$$

- Suppose $-2 \pm j, -5$ are eigenvalues to be assigned, Then the closed Loop C.E will be $(\lambda^2 + 4\lambda + 5)(\lambda + 5) = 0$

OR By MATLAB: >> conv([1 4 5],[1 5]) = [1 9 25 25]

$$C.E = \lambda^3 + 9\lambda^2 + 25\lambda + 25$$

→ $K_1 - \alpha_1 = 9 \Rightarrow K_1 = 9 + \alpha_1$

→ $K_2 - \alpha_2 = 25 \Rightarrow K_2 = 25 + \alpha_2$

→ $K_3 - \alpha_3 = 25 \Rightarrow K_3 = 25 + \alpha_3$

For example:

- let $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (system unstable) $\Rightarrow K = [25 \quad 25 \quad 9]$

↳ system stable.

* To check:

```
>> a = [0 1 0;0 0 1; 0 0 0]; b = [0;0;1]
>> K = [25 25 9]; aK = a - b*K, eig(aK)
ANS=  -2+j
      -2-j
      -5
```

- This Method is Suitable when the system is in the Controllable Form. Besides it applies to SISO systems.

- Method(2):

※ Ackermann's Method:

- Doesn't require the system to be in CC form.

1) let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues to be assigned.

2) Generate $\phi(\lambda) = (\lambda - \mu_1)(\lambda - \mu_2) \cdots (\lambda - \mu_n)$

3) $K = [0 \ 0 \ \cdots \ 1] * [B \quad AB \quad A^2B \ \cdots \ A^{n-1}B]^{-1} * \phi(A)$

*if the system is SISO, Then it has a Unique feedback matrix "K".

Example: Given: $\dot{x} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 3 & 4 & 2 \end{bmatrix} x + \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} u$, assign $-2 \pm j, 5$ find K?

Solution:

$$\phi(\lambda) = \lambda^3 + 9\lambda^2 + 25\lambda + 25$$

using Matlab:

```
>> A = [0  1  0; 0  0  1; 3  4  2]; B = [4;2;1];
>> K = [0  0  1] * inv([B  A*B  A^2*B]) * [A^3+9*A^2+25*A +25*eye(3)]
>> format rat
```

ANS. ⇒ $K = \begin{bmatrix} \frac{506}{995} & \frac{670}{199} & \frac{2221}{995} \end{bmatrix}$

Exercise: Given: $\dot{x} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, assign $-1 \pm j2$, find K?

Solution: $\phi(\lambda) = \lambda^2 + 2\lambda + 5$

$$K = [0 \quad 1] \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}^{-1} [A^2 + 2A + 5I_2] = [0 \quad 1] \begin{bmatrix} -2 & 1 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} 10 & 10 \\ 15 & 35 \end{bmatrix}$$

By doing the multiplication you will observe that: $K = [5 \quad 5]$ #

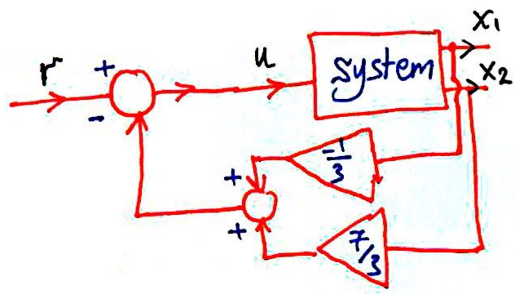
Example: $\dot{x} = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$, which has -5 & 2 as eigenvalues indicating an unstable system. Hence, a need to change the eigenvalues to say -2 & -3.

$$\phi(\lambda) = (\lambda+2)(\lambda+3) = \lambda^2 + 5\lambda + 6$$

$$\phi(A) = A^2 + 5A + 6I_2 = \begin{bmatrix} 22-20+6 & -6+10 \\ -9+15 & 7+5+6 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 6 & 18 \end{bmatrix}$$

$$K = [0 \quad 1] \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 8 & 4 \\ 6 & 18 \end{bmatrix} = [0 \quad 1] \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 8 & 4 \\ 6 & 18 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & \frac{7}{3} \end{bmatrix}$$

$$\Rightarrow u = r - Kx = r - \left(-\tfrac{1}{3} x_1 + \tfrac{7}{3} x_2\right)$$



⇒ **CHECK!!**

$$A - BK = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{3} & \frac{7}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -4 + \frac{1}{3} & 2 - 7/3 \\ 3 + \frac{1}{3} & 1 - 7/3 \end{bmatrix}$$

- trace $(A-BK) = -5$
- det $(A-BK) = 6$

```
>> format rat
>> K = acker(A, B, [-2 -3])
```

* **N.B:** The ackermann's Method doesn't apply if the system is Uncontrollable.

- Method(s):

※ The Eigenstructure method:

Given $\dot{x} = Ax + Bu$ $x \in R^n, u \in R^m$. To assign $\lambda_1, \lambda_2, \dots \lambda_n$:

i) Solve $\underset{n\times n}{[A - \lambda_i I_n]} \underset{n\times 1}{w_i} = \underset{n\times m}{B} \underset{m\times 1}{z_i}$ $i = 1, 2, \dots, n$ you can take $z_i = 1$

ii) $K = [z_1 \; z_2 \; \dots \; z_n][w_1 \; w_2 \; \dots \; w_n]^{-1}$

Example: $\dot{x} = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$, assign $-2, -3$ using the eigenstructure Method?

Solution:

for $\lambda = -2 \Rightarrow \begin{bmatrix} -2 & 2 \\ 3 & 3 \end{bmatrix} w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} * 1 \Rightarrow w_1 = \begin{bmatrix} \frac{-1}{12} \\ \frac{5}{12} \end{bmatrix}$

for $\lambda = -3 \Rightarrow \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times 1 \Rightarrow w_2 = \begin{bmatrix} -2/10 \\ 4/10 \end{bmatrix}$

$$K = [1 \;\; 1] \begin{bmatrix} -1/12 & -2/10 \\ 5/12 & 4/10 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{-1}{3} & \frac{7}{3} \end{bmatrix}$$

```
>> K = place(a,b,[-2 -3])
```

- K is the same as that otained using Ackermann's Method.
- A Fact: for Controllable SISO systems K is unique irrespective of the method used.

CHAPTER(10):

※ Full-Order Observer:

Given an observable system $\dot{x} = Ax + Bu$, $y = Cx$, The state are assumed inaccessible.

To estimate the states we build an observer which uses the outputs and the inputs of the system.

- The observer is described by:

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - \tilde{y})$$

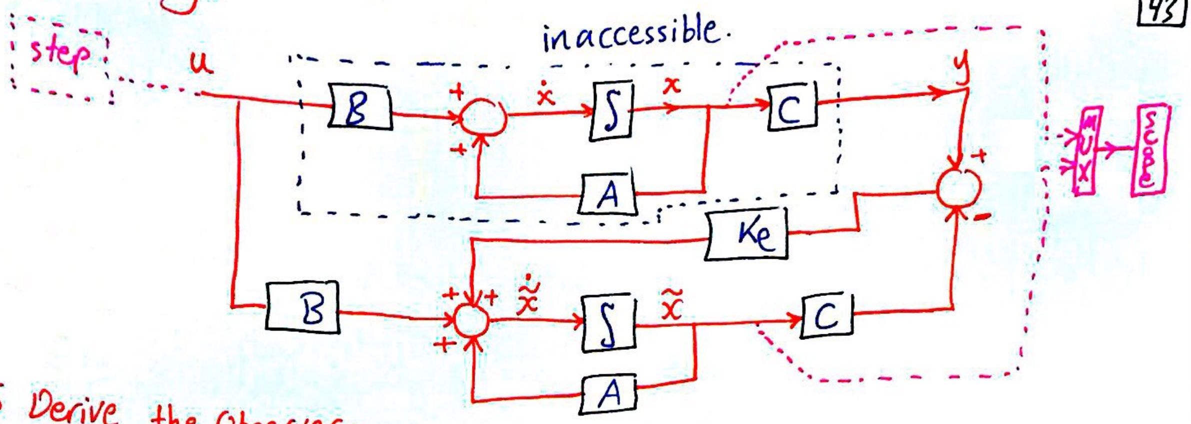
$y \equiv$ actual o/p.

$\tilde{y} \equiv$ observer o/p.

$\tilde{x} \equiv$ the estimates of the states and the states of the observer

- The schematic representation of the observer shown NEXT PAGE.

• Schematically:



• To Derive the Observer:

we assume $e = x - \tilde{x}$

actual. "unavailable" — estimate "available"

$\Rightarrow$ So to get x as $\tilde{x}$ we should make the error (e) zero.

It can be shown that: $\dot{e} = (A - K_eC)e$

for $\lim_{t\to\infty} e(t) = 0$ the eigenvalues of $A-K_eC$ should lie in LHS of S-plane.

* The design of the observer boils down to a state feedback design problem where a K_e is chosen to specify desired eigenvalue for the observer.

* The Ackermann's Method can be adapted to the selection of K_e.
It can be shown that:

$$K_e = \phi(A) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

• Note: when we want to design an observer deal with $\dot{e}$ & find K_e.

Example: Design an observer with eigenvalues $-5, -4$ for the following system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 2]\, x$$

solution: $A-K_eC$ should have $-4, -5$ as eigenvalues.

$$\phi(\lambda) = \lambda^2 + 9\lambda + 20$$

$$\Rightarrow K_e = [A^2 + 9A + 20I_2] \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \end{bmatrix}$$

```
>> Ke = (acker(a',c',[-4 -5]))'
```

• for previous example: it is a stable system. if we change A to $\begin{bmatrix} 0 & 1 \\ -2 & +3 \end{bmatrix}$ the system is Unstable.

※ Design of a system Controller (K):

$u = r - K\tilde{x}$ [i.e the states of the observer [Not of the system] are fedback].

so $\dot{x} = Ax + B(r - K\tilde{x}) = Ax - BK\tilde{x} + Br$; $e = x - \tilde{x}$

$\Rightarrow \dot{x} = (A - BK)x + BKe + Br \quad \cdots (1)$

$\dot{e} = (A - K_e C)e \quad \cdots (2)$ • Now Re-write (1) & (2) in Matrix Form.

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \left[\begin{array}{c:c} A-BK & BK \\ \hdashline 0 & A-K_eC \end{array}\right] \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

i.e the eigenvalues of the controller system (through K) are assigned separately from those of the observer system (through K_e) this is Known as: "the separation principle".

Example: Given $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $y = [1 \;\; 2]x$

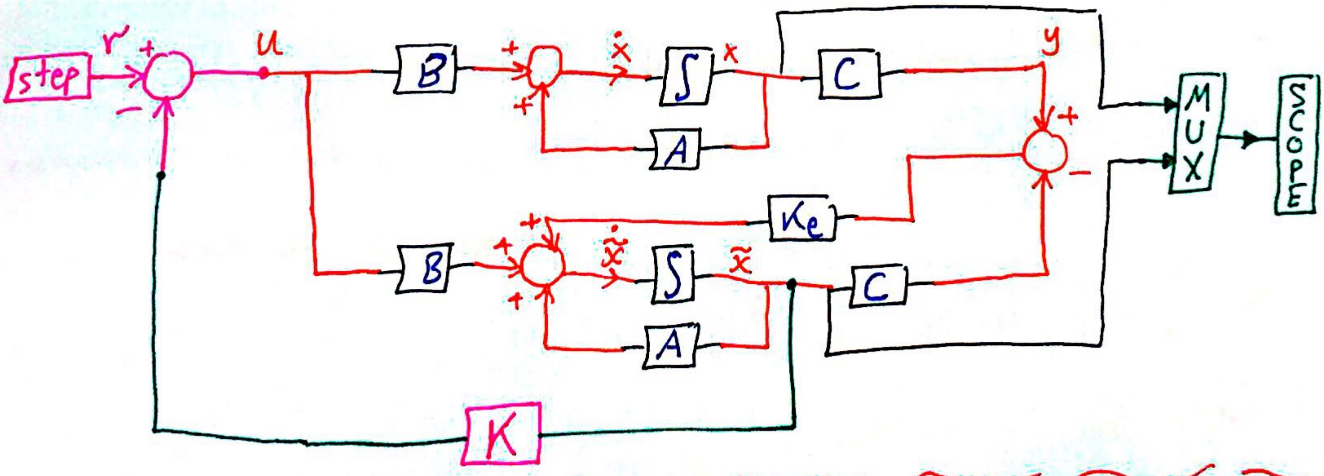
• Design an observer with eigenvalues $-5, -5$. • Design a Controller to get a system with eigenvalues $-3, -4$?

Solution:

```
>> A=[0 1;-2 3]; B=[0;1]; C=[1 2];
>> Ke=(acker(A',C',[-5 -5]))'
>> K=acker(A,B,[-3 -4])
```

$\Rightarrow K_e = \begin{bmatrix} -2.2 \\ 7.6 \end{bmatrix}$

$\Rightarrow K = [10 \quad 10]$



~ ~ ~ ~ ~ ~ ~ ~ ~

* * * End of Material * * *

* * * Best of Luck * * *