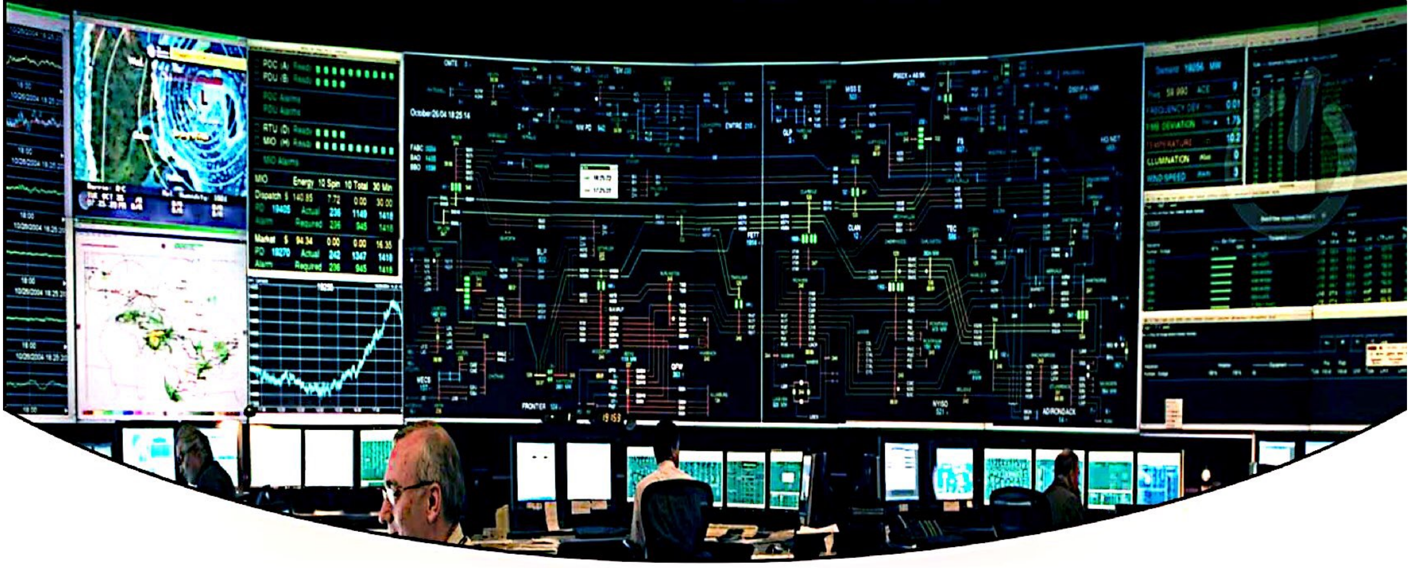

Control

Summer017

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Control Systems.

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Summer Semester 2017.

Notebook.

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* * *

Control Systems:

* Review of the laplace Transform (LT):

Def. : $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) \cdot e^{-st} dt$

⇒ any $f(t)$ has a laplace Transform if the value $\int_0^\infty f(t) e^{-st} dt$ is integrable.

$s = \sigma + j\omega$

* The LT is a Linear operator $\xrightarrow{\text{means}}$ it follows super position principle.

* The LT of certain functions:

① $\mathcal{L}\{u(t)\} = \frac{1}{s}$

② $\mathcal{L}\{t u(t)\} = \frac{1}{s^2}$ ⇒ in general: $\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}$

③ $\mathcal{L}\{e^{at} u(t)\} = \frac{1}{s-a}$

④ $\mathcal{L}\{e^{j\omega t}\} = \frac{1}{s-j\omega}$

⑤ ↳ $\mathcal{L}\{\cos\omega t\} = \mathcal{L}\left\{\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right\} = \frac{S}{S^2+\omega^2}$

⑥ $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$

* proparties of LT:

① $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

Ex. $\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$

$\mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}$

Ex. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+13}\right\}$

$= \frac{s+2-2}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} - \frac{2}{(s+2)^2+9}$

$\Rightarrow \mathcal{L}^{-1} = e^{-2t}\cos 3t - \frac{2}{3} e^{-2t} \sin 3t$

② $\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)$ → i.e multiplication by S in the S-domain is equivalent to differentiation in time domain.

→ more general:

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

→ in general:

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

③ $\mathcal{L}\{t f(t)\} = -\frac{d}{ds}F(s) = -F'(s)$

④ $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s)$ → i.e multiplication by $\frac{1}{S}$ in S-domain is equivalent to Integration in time domain.

⑤ Convelution:

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{\int_0^t f(\tau)g(t-\tau)d\tau\} = F(S)\cdot G(S)$$

$\Rightarrow \mathcal{L}\{f(t)\cdot g(t)\} = ?$ not a convelution (see table).

⑥ $\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$

proof:

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(at)\,e^{-st}dt = \frac{1}{a}\int_0^\infty f(at)\,e^{-\frac{s}{a}at}\,d(at) \quad \text{, Take } at = \lambda$$

$$= \frac{1}{a}\int_0^\infty f(\lambda)\,e^{-(\frac{s}{a})\lambda}\,d\lambda = \frac{1}{a}F\left(\frac{s}{a}\right) \quad \#$$

※ **Facts:**

$$** \quad \int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$$

proof: $F(s) = \int_0^\infty f(t)\,e^{-st}dt$.

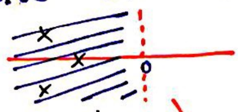
$$\Rightarrow \lim_{s\to 0} F(s) = \lim_{s\to 0}\int_0^\infty f(t)\,e^{-st}dt$$

$$\Rightarrow \lim_{s\to 0} F(s) = \int_0^\infty f(t)\,dt \quad \#$$

** $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$ ⇒ Known as:
The Initial Value Theorem.

** $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ ⇒ Known as:
The Final Value Theorem.

↳ Gives a correct answer as long as $sF(s)$ doesn't have poles in the right half of the S-plane or on the imaginary axis.



(i.e correct as long as the system is stable)

※ Inverse LT:

⇒ Illustrated by Numerical examples:

Ex. $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)(s+3)(s+4)}\right\}$ using cover-up rule.

must be distinct

$$\mathcal{L}^{-1} = \frac{\frac{1}{6}}{s+1} + \frac{\frac{1}{2}}{s+3} + \frac{-2/3}{s+4} = \frac{1}{6}e^{-t} + \frac{1}{2}e^{-3} - \frac{2}{3}e^{-4t}$$

Ex. $\mathcal{L}^{-1}\left\{\frac{s+4}{(s-1)^3(s+2)}\right\}$ ⇒ $\frac{A}{s+2} + \frac{B}{(s-1)^3} + \frac{C}{(s-1)^2} + \frac{D}{s-1}$

A & B evaluated using cover-up rule. ⇒ $A = \frac{-2}{27}$, $B = \frac{5}{3}$

first derivative: $C = \frac{d}{ds}(s-1)^3 \frac{s+4}{(s-1)^3(s+2)}\Big|_{s=1}$

second derivative: $D = \frac{1}{2!}\frac{d}{ds^2}\frac{s+4}{s+2}\Big|_{s=1}$ ⇒ you will get same answers.

(power − 1)!

OR by using Partial fraction ⇒ $C = \frac{-2}{9}$, $D = \frac{2}{27}$

$$\mathcal{L}^{-1} = \frac{-2}{27}e^{-2t} + \frac{1}{2} * \frac{5}{3}t^2 e^{t} + \frac{-2}{9}t e^{t} + \frac{2}{27}e^{t}$$

* Exercise:

1] Determine $\mathcal{L}\{tu(t)\}$, $\mathcal{L}\{te^{-2t}\}$ using as many methods as possible?

2] Evaluate $\mathcal{L}\left\{\int_0^t \left(\tau e^{-4\tau}\cos 5\tau \star \tau u(\tau)\right) d\tau\right\}$?

3] Determine $\mathcal{L}^{-1}\left\{\frac{58}{s(s^2+4s+29)}\right\}$?

4] Show That:

$$\mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\right\} = \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\cos^{-1}\zeta)\right]u(t)$$

or $\sin(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta})$

$\zeta < 1$
$\Rightarrow$ poles are complex.

where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Hence, $\mathcal{L}^{-1}\left\{\frac{5}{s^2+4s+5}\right\} = ?$

$\omega_d \equiv$ is underdamped natural frequancy.
$\omega_n \equiv$ is undamped natural frequancy.

5] Use LT concepts to evaluate: $\int_0^\infty t e^{-4t}\cos 8t \, dt$

Solutions:

① *using a direct method: $\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$, $\mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2}$

*using general rule:

$$\mathcal{L}\{tu(t)\} = \int_0^\infty t e^{-st} dt \Rightarrow (\text{By parts}) = -\frac{t e^{-st}}{s} - \frac{e^{-st}}{s^2}\Big|_0^\infty = -0-0+0+\frac{1}{s^2} = \frac{1}{s^2}$$

$$\mathcal{L}\{te^{-2t}\} = \int_0^\infty t e^{-t(s+2)} dt = \frac{t e^{-t(s+2)}}{s+2} - \frac{e^{-t(s+2)}}{(s+2)^2} = 0-0-0+\frac{1}{(s+2)^2} = \frac{1}{(s+2)^2}$$

*using differentiation: $\mathcal{L}\{tf(t)\} = -F'(s)$

- Let $f(t) = u(t) \Rightarrow F(s) = \frac{1}{s} \Rightarrow F'(s) = \frac{-1}{s^2} \Rightarrow \mathcal{L}\{tf(t)\} = -\left(\frac{-1}{s^2}\right) = \frac{1}{s^2}$

- Let $f(t) = e^{-2t} \Rightarrow F(s) = \frac{1}{s+2} \Rightarrow F'(s) = \frac{-1}{(s+2)^2} \Rightarrow \mathcal{L}\{te^{-2t}\} = -\left(\frac{-1}{(s+2)^2}\right) = \frac{1}{(s+2)^2}$

② $\mathcal{L}\left\{\int_0^t \left(\tau e^{-4\tau}\cos 5\tau \star \tau u(\tau)\right)d\tau\right\}$

let $g(t) = t\,e^{-4t}\cos 5t$

$m(t) = t\,u(t)$

$f(t) = g(t) \star m(t)$

$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$

where:

$F(s) = \mathcal{L}\{g(t) \star m(t)\}$

$= G(s) \cdot M(s)$

for $M(s)$:

$M(s) = \mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}$

for $G(s)$:

$G(s) = \mathcal{L}\{t\,e^{-4t}\cos 5t\} = -\left[\mathcal{L}\{e^{-4t}\cos 5t\}\right]' = -\left(\frac{s+4}{(s+4)^2+25}\right)'$

solve it.

$\Rightarrow \quad G(s) = \frac{(s^2+8s-9)}{((s+4)^2+25)^2}$

finally:

$\Rightarrow \quad \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}G(s)\cdot M(s) = \boxed{\frac{(s^2+8s-9)}{s^3((s+4)^2+25)^2}}$

---

③ $\Rightarrow \frac{58}{s((s+2)^2+25)} = \frac{A}{s} + \frac{Bs+C}{(s+2)^2+25}$

using partial fraction:

$A=2$

$B=-2$

$C=-8$

$\mathcal{L}^{-1}\left\{\frac{A}{s}\right\} = 2\,u(t) \quad \ldots ①$

$\frac{Bs+C}{(s+2)^2+25} = \frac{B(s+2)}{(s+2)^2+25} + \frac{-2B+C}{(s+2)^2+25}$ (−4) $\Rightarrow \mathcal{L}^{-1} = -2e^{-2t}\cos 5t + \frac{-4}{5}e^{-2t}\sin 5t \quad \ldots ②$

$\mathcal{L}^{-1}\left\{\frac{58}{s(s^2+4s+29)}\right\} = ① + ②.$

---

④ $\mathcal{L}^{-1}\left[\dfrac{\omega_n^2}{S(S^2+2\zeta\omega_n S+\omega_n^2 \pm \zeta^2\omega_n^2)}\right] \rightarrow \dfrac{A}{S} + \dfrac{BS+C}{(S+\zeta\omega_n)^2+\omega_n^2-\zeta^2\omega_n^2}$

$\Rightarrow A(S+\zeta\omega_n)^2+\omega_n^2-\zeta^2\omega_n^2+S(BS+C)=\omega_n^2$; Take $S=0 \Rightarrow \boxed{A=1}$

Take $S=1 \Rightarrow \boxed{B+C=-2\zeta\omega_n-1}$ --- [1]   Take $S=-1 \Rightarrow \boxed{B-C=2\zeta\omega_n-1}$: [2]

from [1] & [2] : $\boxed{B=-1}$ & $\boxed{C=-2\zeta\omega_n}$

$\Rightarrow \mathcal{L}^{-1}\left[\dfrac{1}{S} + \dfrac{-S-2\zeta\omega_n}{(S+\zeta\omega_n)^2+\omega_n^2-\zeta^2\omega_n^2}\right]$ $\omega_n^2-\zeta^2\omega_n^2 = \omega_d^2$

$\dfrac{1}{S} - \dfrac{S+2\zeta\omega_n}{(S+\zeta\omega_n)^2+\omega_d^2}$

$\rightarrow \dfrac{S+\zeta\omega_n}{(S+\zeta\omega_n)^2+\omega_d^2} + \dfrac{(-\zeta\omega_n+2\zeta\omega_n)}{(S+\zeta\omega_n)^2+\omega_d^2}$ $(-\zeta\omega_n+2\zeta\omega_n) = \zeta\omega_n$

$\dfrac{\omega_n \zeta}{\omega_d} = \dfrac{\omega_n\zeta}{\omega_n\sqrt{1-\zeta^2}}$

$\Rightarrow \mathcal{L}^{-1} = u(t) - \left[u(t)\, e^{-\zeta\omega_n t}\cos(\omega_d t) + \dfrac{\omega_n\zeta}{\omega_d} e^{-\zeta\omega_n t} u(t)\sin(\omega_d t)\right]$

$= u(t) - u(t)\, e^{-\zeta\omega_n t}\left[\sin(\omega_d t+90) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]$

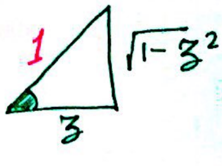
$= u(t) - u(t)\, e^{-\zeta\omega_n t}\left[\angle 90^\circ + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\angle 0^\circ\right]$

magnitude: $\Rightarrow \sqrt{1+\dfrac{\zeta^2}{1-\zeta^2}} = \dfrac{1}{\sqrt{1-\zeta^2}}$

phase : $\Rightarrow \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Now $\mathcal{L}^{-1}$ Become:

$$\mathcal{L}^{-1} = u(t) - \dfrac{u(t)\, e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \cos^{-1}(\zeta))$$



$\Rightarrow \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) = \cos^{-1}(\zeta)$

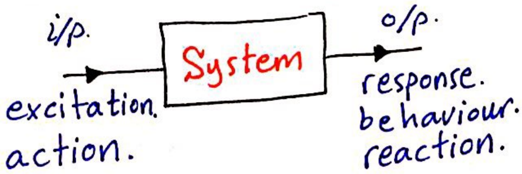
you can solve $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+4s+5}\right\}$ on this general rule.   ###

※ Control systems:

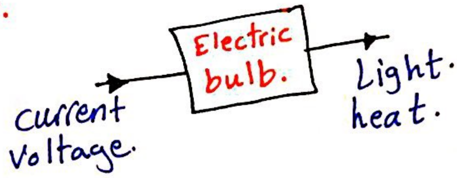
* **A system:** is a collection of objects that act (work) together to perform a certain objective.

* Systems are to be controlled using the concept of feedback. (FB).

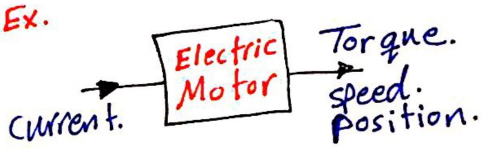
* i/p-o/p (cause-effect) description of systems:



Ex.



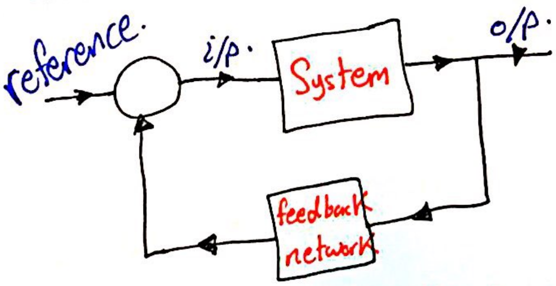
Ex.



Ex.

* All previous examples doesn't involve feedback ⇒ Called "open loop" systems. "Non-FeedBack systems"

⁂. with feedBack:

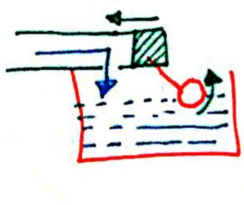


⇒ This system Called: FeedBack system (closed loop system).

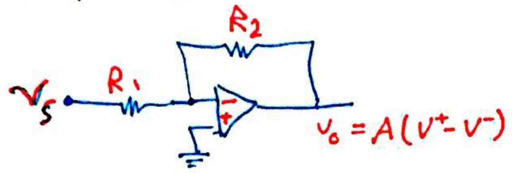
* **Examples on Systems with FeedBack:**

1] Biological Systems: eye pupil area, temprature control, blood sugar regulation, etc...

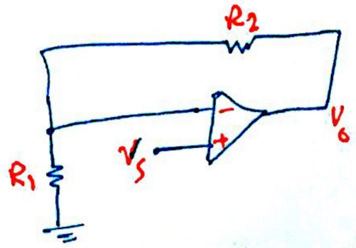
2] Water Level Control in domestic tanks.



3] (i)* Op- amplifiers.



Inverting Amplifier.

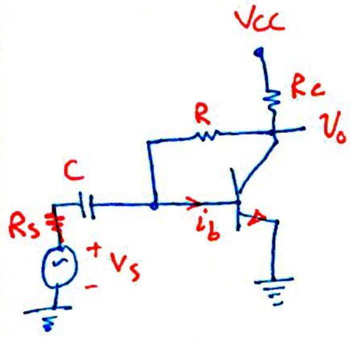


Non-inverting amplifier.

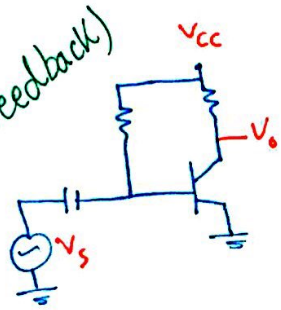
$$V_o = A(V^+ - V^-)$$
$$= A\left(V_s - V_o \frac{R_1}{R_1+R_2}\right)$$

⇒ feedback:
when ① $R_1 = \infty$
② $R_2 = 0$

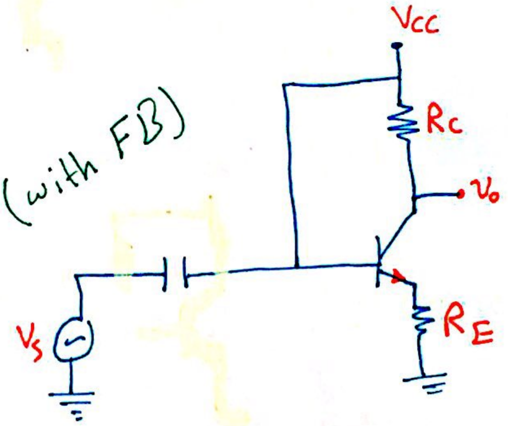
(ii) * Transistors with feedback:



(No feedback)

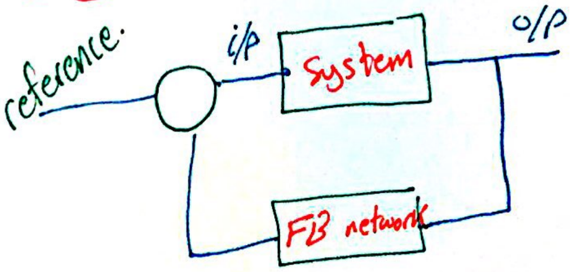


(with FB)



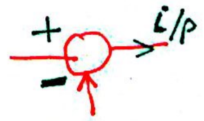
* close loop systems is Better than open loop systems.

* Types of feedback:



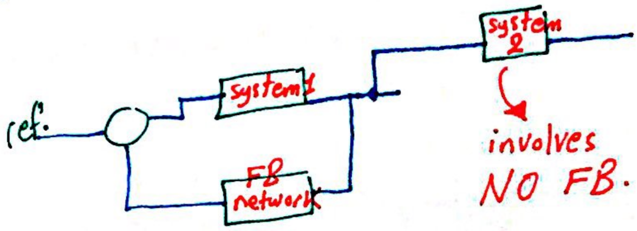
* Positive FB: the o/p is feedback & added to the reference.

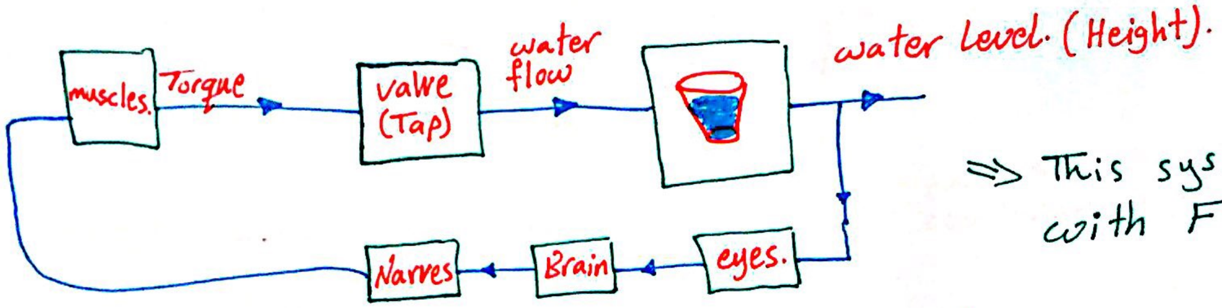
* Negative FB: the o/p is fedback & subtracted from the reference.



* Using (+, +) or (+, −) is artificial. The important is really to the employment of the feedback concept.

* **Examples:**





⇒ This system with FB.

↳ This Called: "Schematic Diagram."

⇒ it is a negative FB Control System.

* This system become open loop:

⇒ in case that you can't see (the brain doesn't receive signals)

⇒ in case that the brain stopped.

⇒ in case that the Nerves doesn't work.

* **Exercise:**

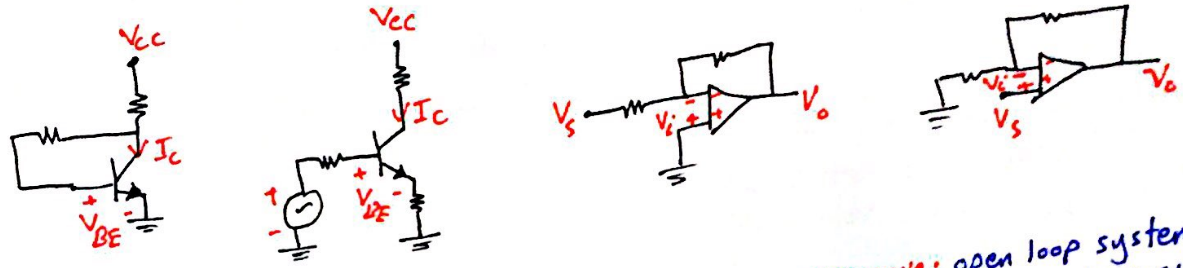
sketch a schematic diagram to represent:

- The process of driving a car (bicycle).
- The process of getting a sound quality.
- Firing a bullet.

& state if there is a FB or NOT.

Example: The most read article, the most seen picture involve positive FB concepts.

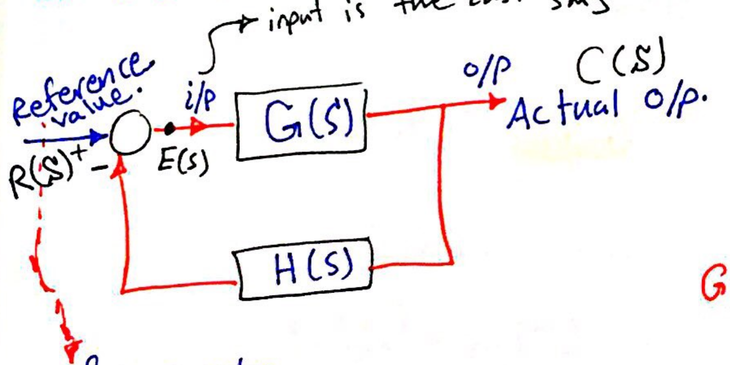
Example: The following circuits involve negative FB.



*microwave: open loop system.
*washermachine: open Loop system

Example: * Anger: +ve FB.
* Humbleness: +ve FB.

※ The Basic Feedback System:



reference value.
reference point.
set point.
set value.
Desired value.

$G(S) \equiv$ is the forward. T.F.

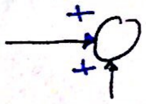
$E(S) \equiv$ is the Error T.F.

$G(S)\,H(S) \equiv$ is the open loop T.F.

$\frac{C(S)}{R(S)} \equiv$ is the closed loop T.F.

* It can be shown (prove):

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)\,H(S)}$$

⇒ if the FB  ⇒ $\frac{C(S)}{R(S)} = \frac{G(S)}{1-G(S)\,H(S)}$

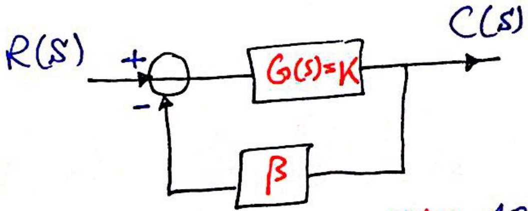
Example: "Advantage of FB systems when it comes to Parameter variation's"

$$\text{gain} = \frac{C(s)}{R(s)} = K$$

$$\frac{d}{dK}\text{gain} = 1$$

i.e a certain percentage change in $G(s)=K$ say 100% change results in the same percentage change in the gain. ⇒ In This Case: we say that the system is Sensetive to parameter variations.

** Consider Now the same figure with FB around it:



$$\Rightarrow \text{gain} = \frac{K}{1+K\beta}$$

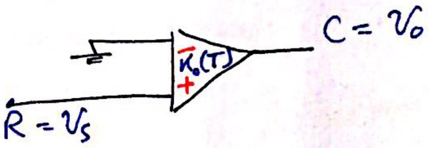
$$\frac{d}{dK}\text{gain} = \frac{1(1+K\beta)-\beta K}{(1+K\beta)^2} = \frac{1}{(1+K\beta)^2}$$

Note: $K\beta > 0$
K & β positive.

* To be insensetive we require $K\beta >> 1$

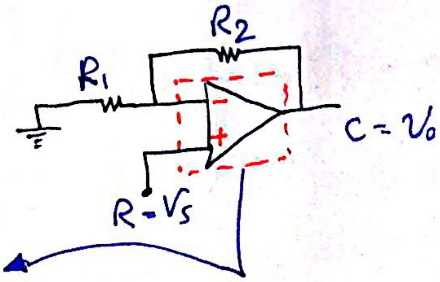
*** An electronic example:

$$\frac{C}{R} = \frac{v_o}{v_s} = \frac{K_o(T)\, v_s}{v_s} = K_o(T)$$

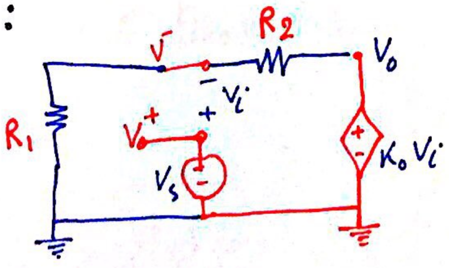


⇒ Sensitive.

Introduce FB:



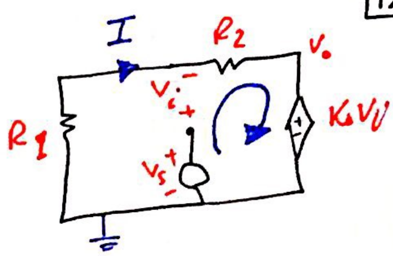
This Non-inverting Op-amp can be represented in the following model:



Continue.

⇒ Analysis: $V_o = K\,V_i$

⇒ KVL: $-V_s + V_i + R_2 I + V_o = 0$



$$\Rightarrow \quad -V_s + \frac{V_o}{K} + R_2 I + V_o = 0$$

$$V_o\left(\frac{1}{K} - \frac{R_2}{R_1+R_2} + 1\right) = V_s$$

$$V_o \; \frac{R_1+R_2-\cancel{KR_2}+KR_1+\cancel{KR_2}}{K(R_1+R_2)} = V_s \;\Rightarrow\; \frac{V_o}{V_s} = \frac{K(R_1+R_2)}{(1+K)R_1+R_2}$$

$$\Rightarrow \boxed{\frac{V_o}{V_s} = \frac{\frac{K}{1+K}(R_1+R_2)}{R_1 + \frac{R_2}{1+K}}}$$

$$\lim_{K\to\infty}\frac{V_o}{V_s} = \frac{1*(R_1+R_2)}{R_1+0} = \left(1+\frac{R_2}{R_1}\right)$$

* insensitive to K when $K \to \infty$.

Check, using $\frac{d}{dK}(\text{gain}) = \frac{d}{dK}\left(\frac{V_o}{V_s}\right) = \frac{d}{dK}\left(\frac{K(R_1+R_2)}{(1+K)R_1+R_2}\right)$

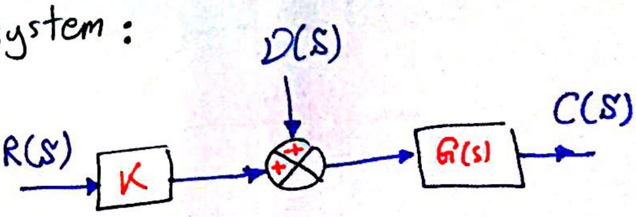
*important word: appreciable ⇒. مقدار محسوس

Examples on variation of parameters:

* Rocket in the situation of losing the fuel.
* A car full of military soldiers jumping through its movement.

※ Insensitivity of FB systems to External Disturbance:

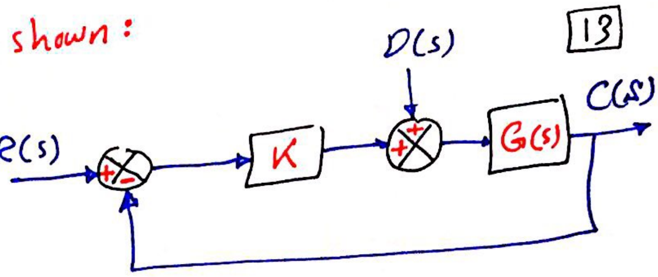
⇒ Consider the following open loop system:



$$\Rightarrow \quad C(S) = K\,G(S)\,R(S) + G(S)\,D(S)$$

* The effect of the disturbance is direct. It only diminishes if D(S) diminishes.

⇒ Introduce negative FB as shown:



$$C(s) = \underbrace{\frac{KG(s)}{1+KG(s)} R(s)}_{\text{we need this term.}} + \underbrace{\frac{G(s)}{1+KG(s)} D(s)}_{\text{reduces}}$$

reduces if $1+KG(s) >> G(s)$

if $K >> 1$

* in this case:

$$C(s) = \lim_{K>>1} R(s) \frac{KG(s)}{1+KG(s)} = R(s) \quad \text{as required (desired).}$$

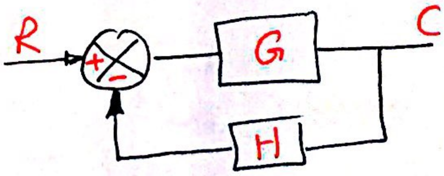
※ Sensitivity of Control Systems:

systems undergo variation in performance due to parameter variations, external disturbances, aging.

* A sensitivity measure may be defined as:

$$S_G^T = \lim_{\Delta G \to 0} \frac{\Delta T/T}{\Delta G/G} \Rightarrow = \frac{G}{T} \lim_{\Delta G \to 0} \frac{\Delta T}{\Delta G} = \frac{G}{T}\frac{dT}{dG}$$

⇒ Consider the following System:



Let: $T = \frac{C}{R} = \frac{G}{1+GH}$

⇒ is it sensitive or Not !!

⇒ we Know that by evaluating S_G^T:

$$S_G^T = \frac{G}{T}\frac{dT}{dG} = \frac{G}{\frac{G}{1+GH}} \cdot \left(\frac{1*(1+GH) - HG}{(1+GH)^2} \right) = \frac{1}{1+GH}$$

Note: The variation is in G. ⇒

So if $GH >> 1$ Then. $S_G^T = \frac{1}{1+GH} << 1$ (i.e insensitive to Variation in G).

⇒ However, $S_H^T = \frac{H}{\frac{G}{1+GH}} \frac{dT}{dH} = \frac{H}{\frac{G}{1+GH}} \left(\frac{-G^2}{(1+GH)^2} \right)$

⇒ $S_H^T = \frac{-GH}{1+GH}$ we use the same condition $(GH >> 1)$

sticking to $GH >> 1$ ⇒ $S_H^T \simeq -1$

* To know if sensitive: we compare $|S_H^T|$ with 1.

since $|S_H^T|$ is NOT $<< 1$

** So T is sensitive to variation in H.

⇒ we could use this example To represent an Inverting Amplifier:

Gain $= \frac{-R_2}{R_1}$ doesn't depend on G.

insensitive to G.

sensitive to H. (since H in fact is represented by R_1 & R_2).

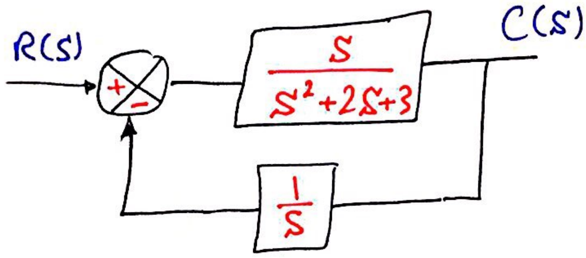
Exercise(1):

Investigate the following Properties:

- $S_G^{\frac{1}{T}} = - S_G^T$
- $S_G^{MN} = S_G^M + S_G^N$, hence or otherwise determine: $S_G^{\frac{M}{N}} = S_G^M - S_G^N$
- $S_K^{M(N(K))} = S_N^M S_K^N$ (function of a function.)
- $S_K^{constant} = 0$
- $S_G^{\alpha T} = S_G^T$ (α: constant.)

Exercise (2):

Given the following:



Let $T = \frac{C(s)}{R(s)}$

Determine:

$S_s^T \Big|_{s=5}$ using two methods.

Solution for exercise (1):

- for $S_G^{\frac{1}{T}} = -S_G^T \Rightarrow S_G^{\frac{1}{T}} = \frac{G}{\frac{1}{T}} \frac{d(\frac{1}{T})}{dG} = GT\left(\frac{0 - \frac{dT}{dG}}{T^2}\right)$

$$= -\frac{G}{T}\frac{dT}{dG} = -S_G^T \ \#$$

- for $S_G^{MN} = S_G^M + S_G^N$

$$\Rightarrow S_G^{MN} = \frac{G}{MN}\frac{d(MN)}{dG} = \frac{G}{MN}\left[\frac{dM}{dG}\cdot N + \frac{dN}{dG}\cdot M\right] = \frac{G}{M}\frac{dM}{dG} + \frac{G}{N}\frac{dN}{dG}$$

$$= S_G^M + S_G^N \ \#$$

- for $S_G^{\frac{M}{N}} = S_G^M - S_G^N \Rightarrow S_G^{M/N} = \frac{G}{\frac{M}{N}}\frac{d(\frac{M}{N})}{dG} = \frac{GN}{M}\left[\frac{\frac{dM}{dG}\cdot N - \frac{dN}{dG}\cdot M}{N^2}\right]$

$$= \frac{G}{M}\frac{dM}{dG} - \frac{G}{N}\frac{dN}{dG} = S_G^M - S_G^N \ \#$$

- for $S_K^{const} = Zero$

$$\Rightarrow S_K^{const} = \frac{K}{const}\frac{d(const)}{dK} = Zero \ \#$$

- for $S_G^{\alpha T} = S_G^T$

$$\Rightarrow S_G^{\alpha T} = \frac{G}{\alpha T}\frac{d(\alpha T)}{dG} = \frac{G}{T}\frac{dT}{dG} = S_G^T \ \#$$

- for $S_K^{M(N(K))} = S_N^M S_K^N$ — chain Rule.

$$\Rightarrow S_K^{M(N(K))} = \frac{K}{M}\frac{d(M(N(K)))}{dK} = \frac{K}{M}\left[\frac{dM}{dN}\cdot\frac{dN}{dK}\right] * \frac{N}{N}$$

$$= \frac{N}{M}\frac{dM}{dN}\cdot\frac{K}{N}\frac{dN}{dK} = S_N^M \cdot S_K^N \ \#$$

Solution for exercise (2):

※ Method(1):

$$T = \frac{C(s)}{R(s)} = \frac{\frac{S}{S^2+2S+3}}{1 + \frac{1}{S^2+2S+3}} = \boxed{\frac{S}{S^2+2S+4}}$$

$$S_S^T = \frac{S}{T}\frac{dT}{dS} \Rightarrow \frac{dT}{dS} = \frac{S^2+2S+4 - S(2S+2)}{(S^2+2S+4)^2} = \frac{-S^2+4}{(S^2+2S+4)^2}$$

$\Downarrow$

$$S_S^T = \frac{S}{\frac{S}{S^2+2S+4}} * \frac{-S^2+4}{(S^2+2S+4)^2}$$

$$\text{so } S_S^T\Big|_{S=5} = \frac{-S^2+4}{S^2+2S+4} = \frac{-25+4}{39} = \boxed{\frac{-21}{39}}$$ #

※ Method(2):

using the property: $S_G^{M/N} = S_G^M - S_G^N$

$$T = \frac{S}{S^2+2S+4} \Rightarrow S_S^T = S_S^S - S_S^{S^2+2S+4}$$

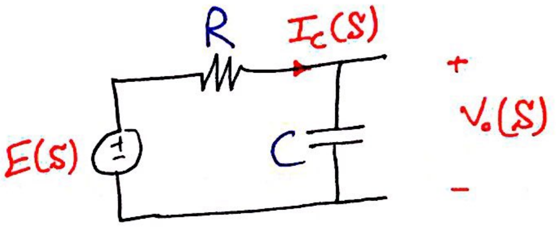
$$S_S^T = \frac{S}{S}\frac{dS}{dS} - \frac{S}{S^2+2S+4}\frac{d(S^2+2S+4)}{dS}$$

$$= 1 - \frac{S(2S+2)}{S^2+2S+4}\Bigg|_{S=5}$$

$$= 1 - \frac{5(12)}{39} = \frac{39-60}{39} = \boxed{\frac{-21}{39}}$$ #

Modeling of systems:

Consider the following circuit:



⇒ In order to obtain a detailed block diagram: (BD)

i) Assign necessary variables.

ii) Write down equations governing these variables.

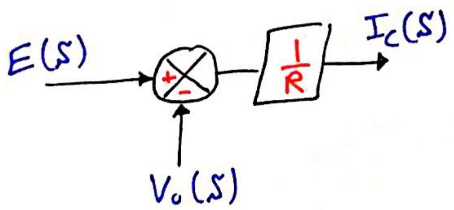
iii) Use the LT to obtain simple (BD).

iv) Assemble the BD's to end up with an overall BD relating a variable to another.

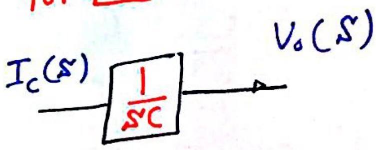
$$i_c = \frac{e(t) - v_o(t)}{R} \Rightarrow I_c(S) = \frac{E(S) - V_o(S)}{R} \quad \cdots\cdots [1]$$

$$v_o(t) = \frac{1}{C}\int_0^t i_c\,dt \Rightarrow V_o(S) = \frac{1}{SC} I_c(S) \quad \cdots\cdots [2]$$

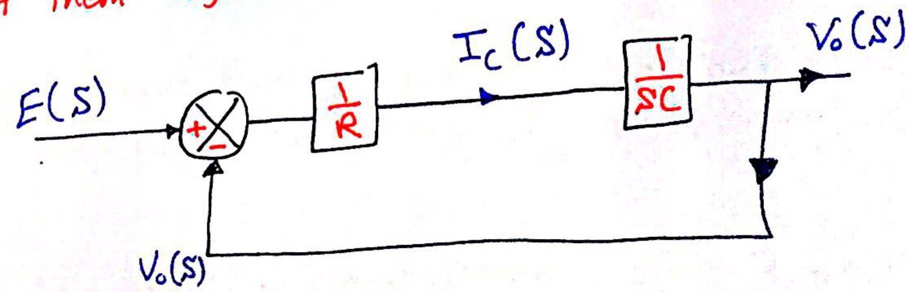
for [1]:



for [2]:



Now put them together:



Example: A control system is described by the following Equations:

$$\frac{dx_1}{dt} = -2x_1 - 4x_2 + u$$

$$\frac{dx_2}{dt} = -5x_1 - 6x_2 + 8u$$

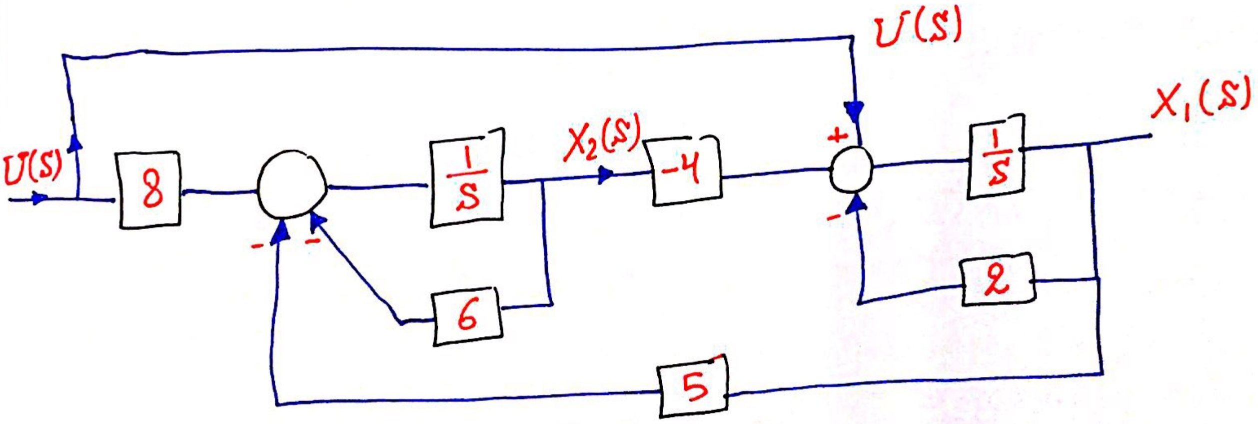
Obtain a block diagram involving integrators $\left(\frac{1}{S}\right)$ where $X_1(S)$ is the o/p & $U(S)$ is the set value.

* Always assume initial condions are equal Zero.

$$S\,X_1(S) = -2X_1(S) - 4X_2(S) + U(S)$$

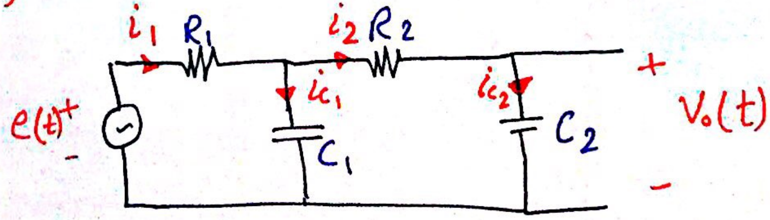
$$\Rightarrow X_1(S) = \frac{1}{S}\left[-2X_1(S) - 4X_2(S) + U(S)\right]$$

$$X_2(S) = \frac{1}{S}\left[-5X_1(S) - 6X_2(S) + 8U(S)\right]$$



Note: There is more than one way to solve, but stick to the conditions that determined in the question.

Exercise: Given the following circuit, Obtain a block diagram involves integrators only with $V_o(S)$ as o/p & $E(S)$ as set value.



Exercise: study the modeling an armature controlled DC motor. (see the Text Book).

※ Solution for Exercise of the electrical circuit in page (18):

⇒ Obtain Equations & take LT, it would be as follows:

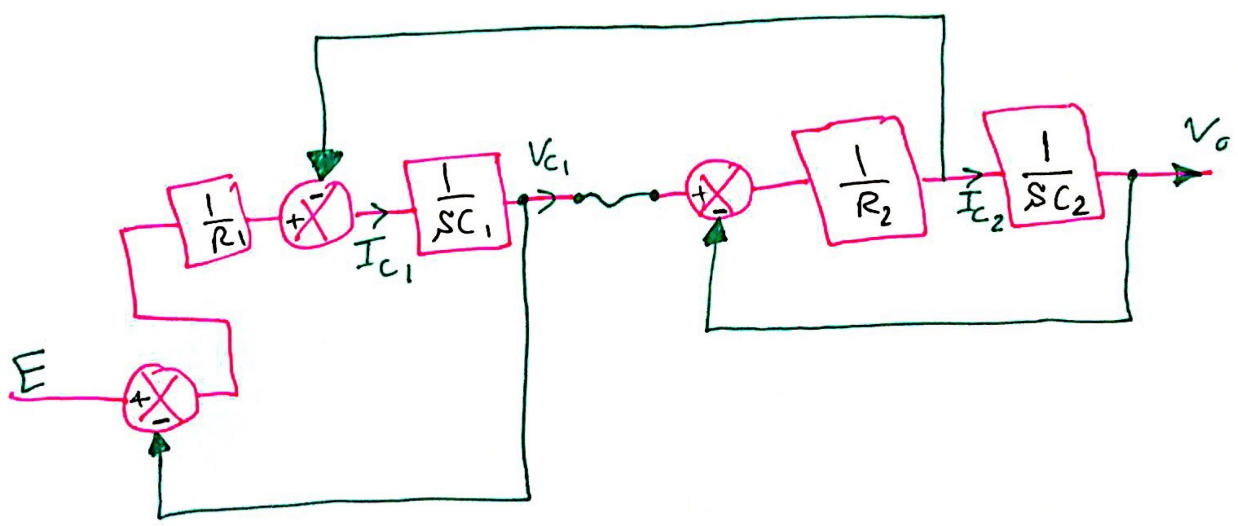
$$V_o = \frac{1}{SC_2} I_{C_2} \quad \cdots (1)$$

$$I_{C_2} = I_1 - I_{C_1} \quad \cdots (2)$$

$$V_{C_1} = \frac{1}{SC_1} I_{C_1} \quad \cdots (3)$$

$$I_1 = \frac{E - V_{C_1}}{R_1} \quad \cdots (4)$$

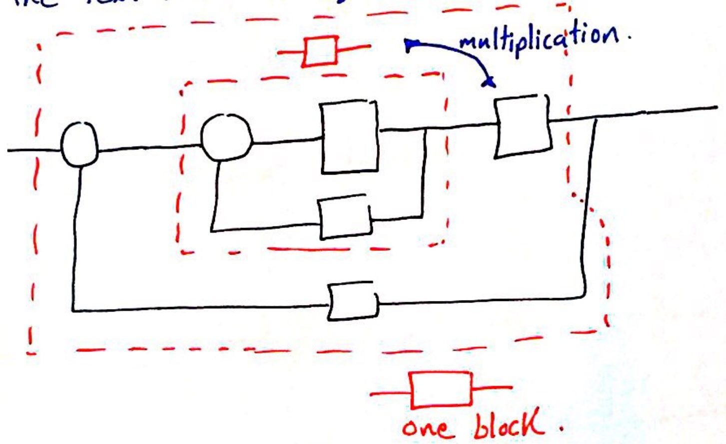
$$I_2 = I_{C_2} = \frac{V_{C_1} - V_o}{R_2} \quad \cdots (5)$$



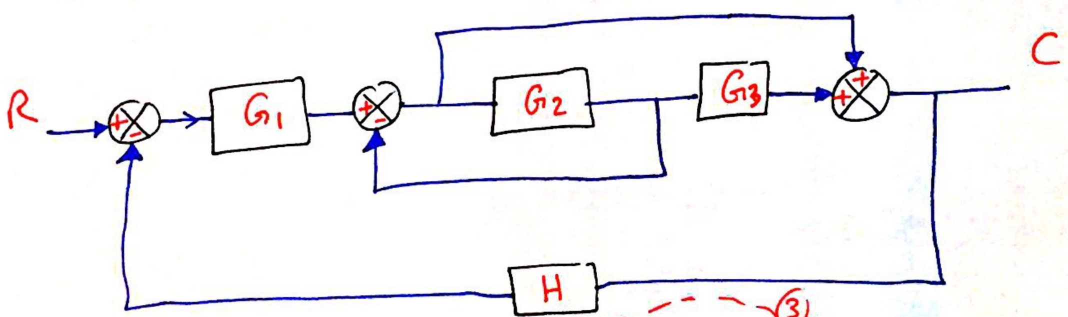
* Block Diagram Reduction Techniques:

Rearrange blocks & maneplulate some how to get another block diagram with basic feedback blocks within a much bigger one

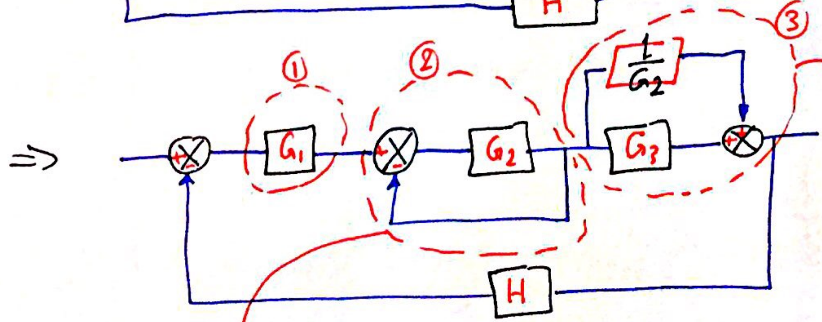
Exercise: see the Text book for equivalent block diagrams & signals.



Example: Reduce the following diagram:



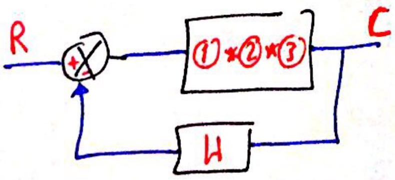
⇒



parallel gives: $G_3 + \frac{1}{G_2}$

Basic FB gives: $\frac{G_2}{1+G_2}$

①, ② & ③: are cascaded ⇒ we multiply them by eachother.

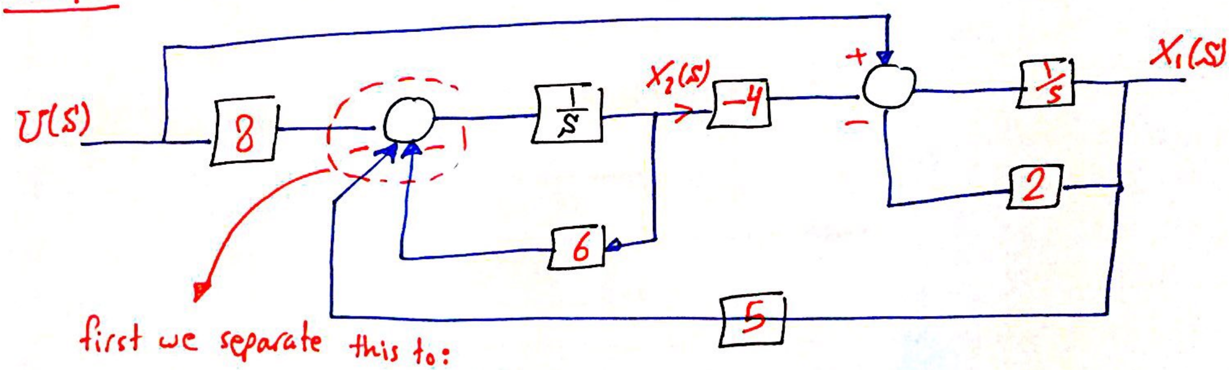


so it becomes Basic FB ⟹

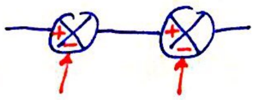
$$\Rightarrow \frac{C}{R} = \frac{G_1 \frac{\cancel{G_2}}{1+G_2} \cdot \frac{1+G_2G_3}{\cancel{G_2}}}{1 + H\left(G_1 \frac{\cancel{G_2}}{1+G_2} \cdot \frac{(1+G_2G_3)}{\cancel{G_2}}\right)}$$

$$\boxed{\frac{C}{R} = \frac{G_1(1+G_2G_3)}{1+G_2+(G_1(1+G_2G_3))H}}$$ ##

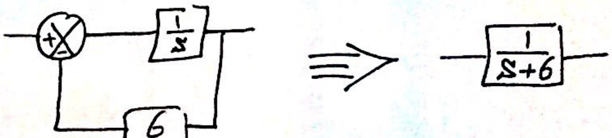
Example: Reduce the following diagram:



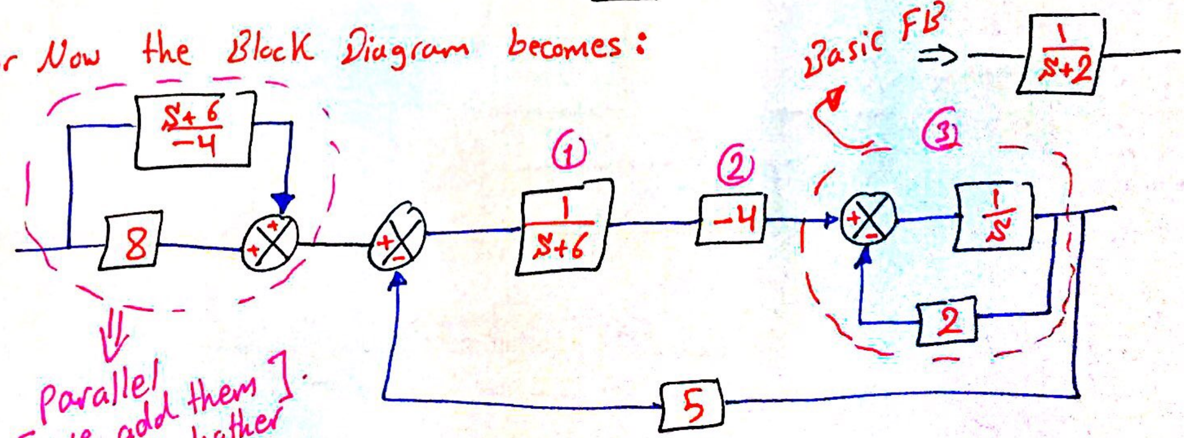
first we separate this to:



⇒ Then we have Basic FB:



for Now the Block Diagram becomes:



⇓ Parallel [we add them to eachother]

①, ② & ③ are cascaded : ① * ② * ③ ⇒ with [5]

⇒ They make Basic FB.

⟹

$$\Rightarrow \frac{1}{S+6} * -4 * \frac{1}{S+2} = \frac{-4}{S^2+8S+12} \equiv G(S)$$

$$H(S) \equiv 5$$

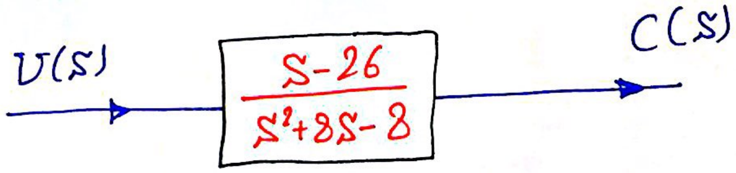
This Basic FB gives:

$$\Rightarrow \frac{G}{1+GH} = \frac{\frac{-4}{S^2+8S+12}}{1+\frac{-20}{S^2+8S+12}} = \frac{-4}{S^2+8S-8}$$

so Now:

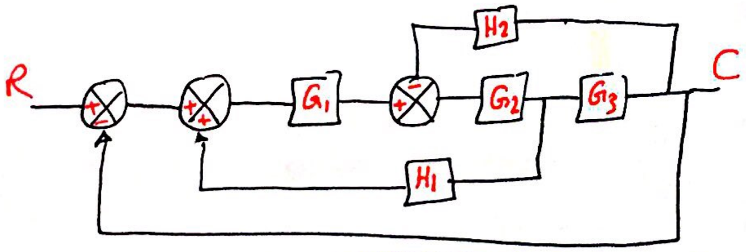
$$\frac{C}{R} = \frac{C}{U} = \frac{-4}{S^2+8S-8} * \left(8 + \frac{S+6}{-4}\right) = \frac{S-26}{S^2+8S-8}$$

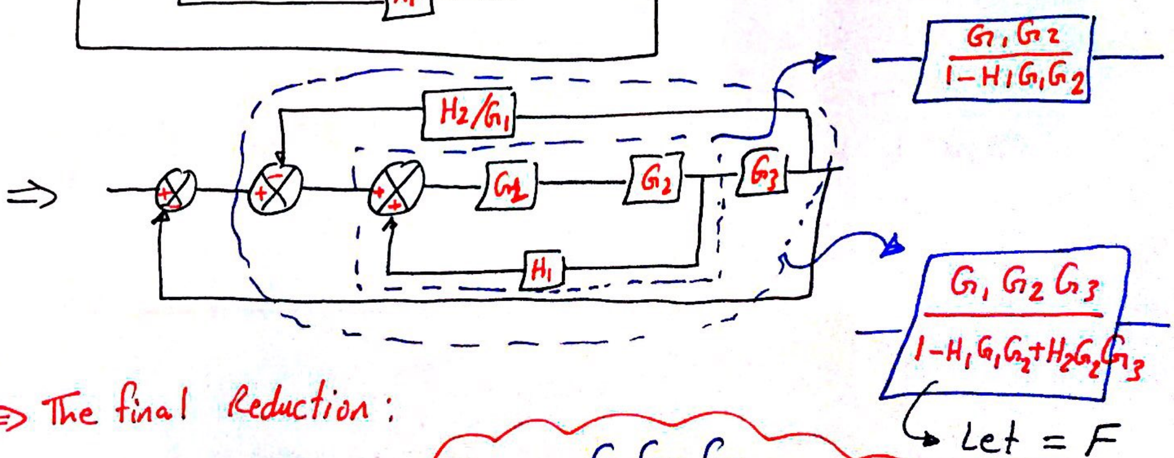
$\Rightarrow$ The simplified Block diagram become:



Example (2-1) in the Text Book:

Reduce the following diagram:



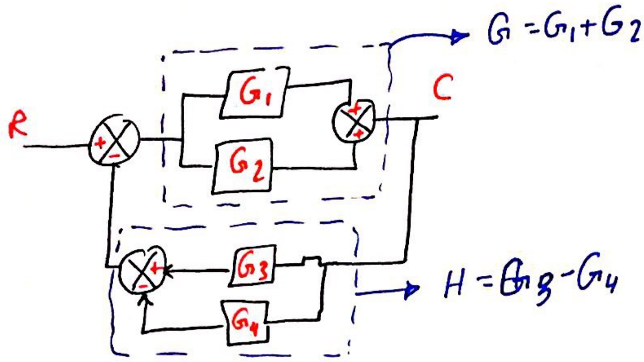


$\Rightarrow$ The final Reduction:

$$\frac{C}{R} = \frac{F}{1+F} \text{ (simplify)} \Rightarrow \frac{C}{R} = \frac{G_1 G_2 G_3}{1 - H_1 G_1 G_2 + H_2 G_2 G_3 + G_1 G_2 G_3}$$

Problem (2-1):

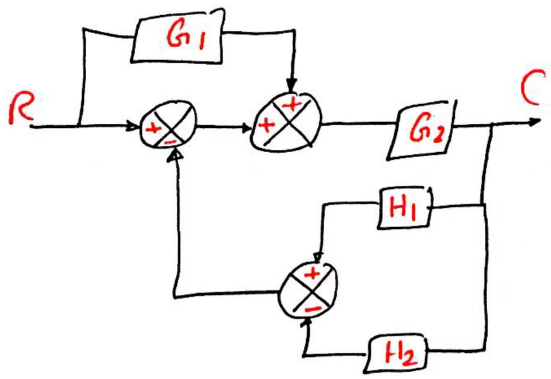
Reduce the following To obtain $\frac{C(S)}{R(S)}$:



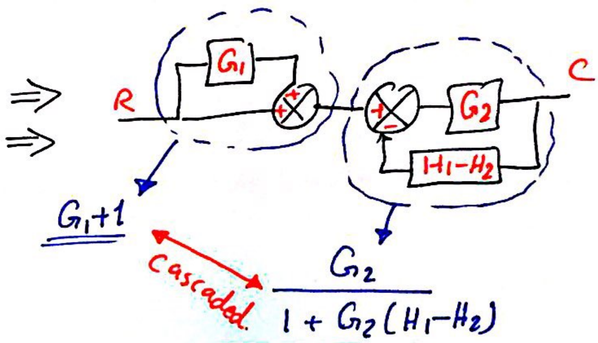
$$\frac{C}{R} = \frac{G}{1+GH}$$

$$\frac{C}{R} = \frac{G_1 + G_2}{1 + (G_1+G_2)*(G_3-G_4)}$$

Problem (2-2): simplify To obtain $\frac{C(S)}{R(S)}$:

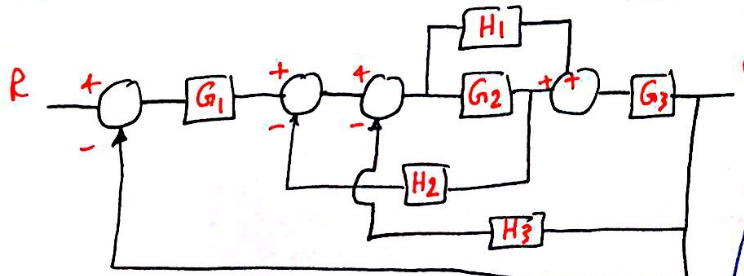


⇒

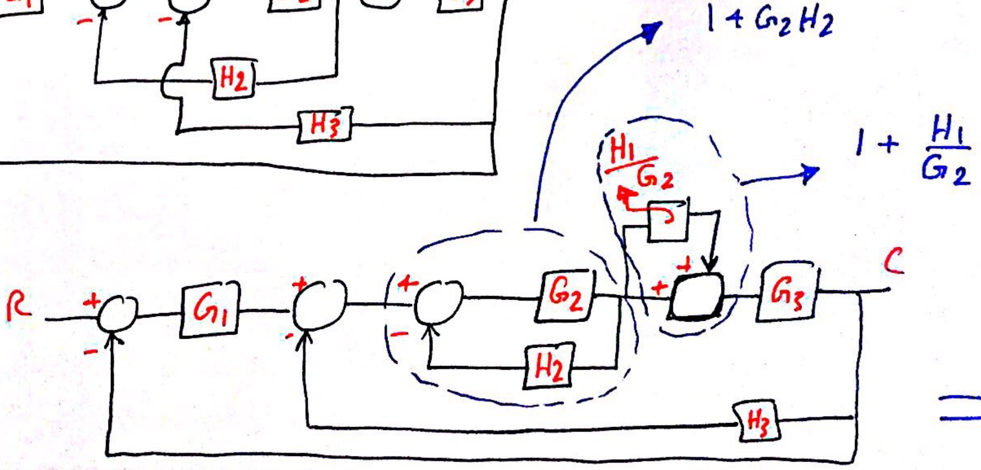


$$\frac{C(S)}{R(S)} = \frac{G_2(G_1+1)}{1+G_2(H_1-H_2)}$$

Problem (2-3): simplify To obtain $\frac{C(S)}{R(S)}$:



⇒

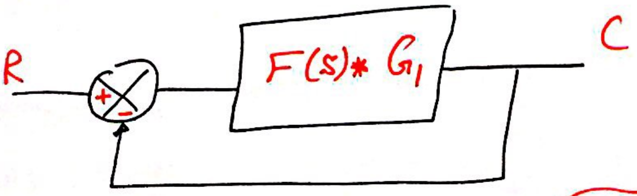


⇒

⇒ cascaded:

$$\frac{\left(\frac{G_2G_3}{1+G_2H_2}\right)\cdot\left(1+\frac{H_1}{G_2}\right)}{1+H_3\,\frac{G_2G_3}{1+G_2H_2}\cdot\left(1+\frac{H_1}{G_2}\right)}$$

$$=\frac{\frac{G_3(G_2+H_1)}{1+G_2H_2}}{1+\frac{G_3H_3(G_2+H_1)}{1+G_2H_2}}=\frac{G_3(G_2+H_1)}{1+G_2H_2+G_2G_3H_3+H_1H_3G_3}\equiv F(S)$$



$$\Rightarrow\ \frac{C(S)}{R(S)}=\frac{F(S)\,G_1(S)}{1+F(S)\,G_1(S)}=\frac{G_1G_2G_3+G_1G_3H_1}{H_1H_3G_3+G_1G_2G_3+G_1G_3H_1+1+G_2H_2+G_3G_2H_3}$$

※ Time Response & Specification of systems:

• First Order Systems:

First order systems characterized by a Transfer function having a highest power of one in the denominator.

A particular on has the form:

$$G(S)=\frac{K}{Ts+1}$$

$T\equiv$ Time Constant.

$K\equiv$ Gain.

* The Response of this system to a Unit step is:

$$C(S)=\frac{K}{S(TS+1)}$$

↳ $C(S)=G(S)\cdot R(S)$ ↳ $1/S$

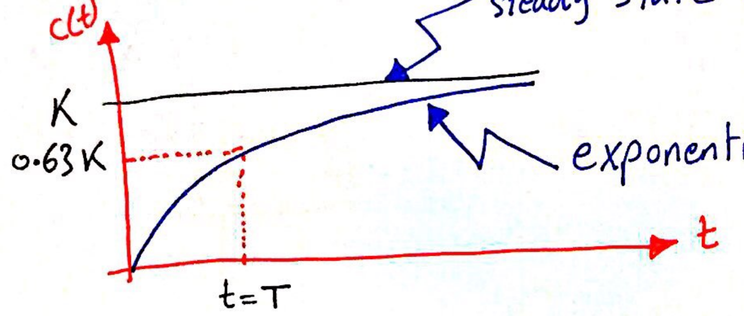
$$C(S)=\frac{A}{S}+\frac{B}{TS+1}$$

using cover-up rule:

$$C(S)=\frac{K}{S}+\frac{-KT}{TS+1}=\frac{K}{S}-\frac{K}{S+1/T}$$

$$c(t)=K\left(1-e^{-t/T}\right)u(t)$$

⇒ graph of c(t) as follows :



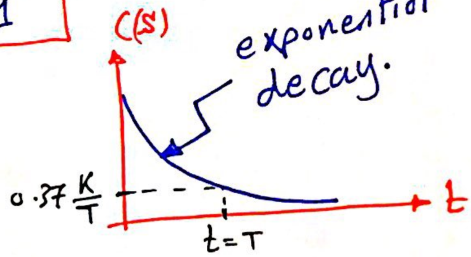
@ $t = T \Rightarrow 0.63K$. , after $t = 5T \Rightarrow c(t) = 0.99K$.

* The Response C(t) due to an impulse is:

$C(s) = G(s) \cdot R(s)$, where $R(s) = 1$

$$C(s) = \frac{K}{Ts+1}$$

$$\Rightarrow C(t) = \frac{K}{T} e^{-t/T} u(t)$$



* The Response C(t) due to a ramp input is:

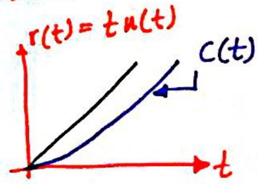
$C(s) = G(s) \cdot R(s)$, where $R(s) = \frac{1}{s^2}$

$$C(s) = \frac{K}{s^2(Ts+1)}$$

⇒ c(t) = find it in the exercise.

<u>Exercise</u>: Determine the o/p of that particular first order system due to a unit ramp, using at least two methods.

$$C(s) = \frac{K}{s^2(Ts+1)}$$



Take $\mathcal{L}^{-1} \Rightarrow$ $C(t) = Kt - TK\left(1 - e^{-t/T}\right)$

Second Order Systems:

Systems where the highest power of S in the denominator is 2.

i.e $G(S) = \dfrac{?}{AS^2+BS+C}$; $A \neq 0$

⇒ Typically, we study the following order system:

$$G(S) = \frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2}$$

ex. $\dfrac{9}{S^2+3S+9} = \dfrac{9}{S^2+2*\frac{1}{2}*3S+3^2}$ ⇒ $\omega_n = 3$, $\zeta = \frac{1}{2}$

where $\omega_n \equiv$ Undamped Natural Frequancy.

$\zeta \equiv$ Damping Ratio. $0 \leq \zeta < \infty$ & $\omega_n > 0$

* The Response of this system due to a unit for $\zeta < 1$ is:

$$c(t) = \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)\right] u(t)$$

where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$\omega_d \equiv$ Underdamped Natural Frequancy.

* * *

First Material.

* A Second Order System of a Particular Form: (with specification):

⇒ Consider the unit step response of the following second order system:

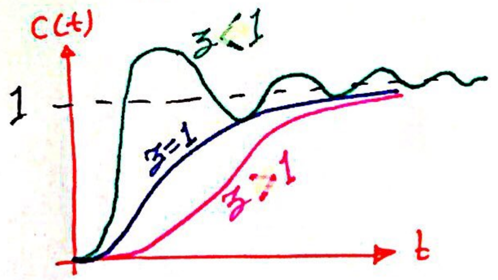
$$G(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

*Note: the factor of S^2 must be equaled to 1.

if $\zeta < 1$: under damping.
$\zeta = 1$: critical damping.
$\zeta > 1$: over damping.

* find C(S) then c(t), then draw it.

* in case $\zeta = 1, \zeta < 1, \zeta > 1$:



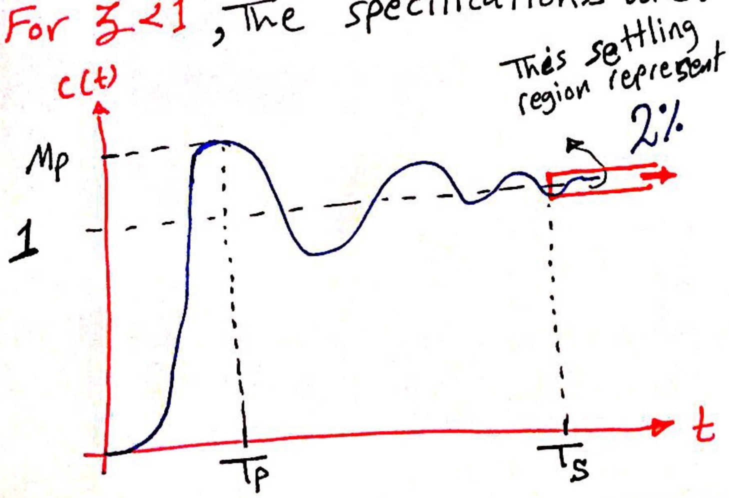
As long as ζ bigger
As long as the curve slower.

* physical example on $\zeta < 1$:
- → Elevator.
- → Number of Livings in a certain environment.
- → An Economic Situation for a certain country.

* under damped faster than critical damped faster than over damped.

for $\zeta < 1$ (under damped) ⇒ $\omega_d = \omega_n \sqrt{1 - \zeta^2}$.

* For $\zeta < 1$, The specifications are:



T_s ≡ settling Time
زمن الاستقرار

$$T_p = \frac{\pi}{\omega_d}$$

exactly, through differentiation of C(t).

⟹

$\Rightarrow$ $\boxed{T_p = \frac{\pi}{\omega_d}}$, $M_p = \left[1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}\right] C_{ss}$

$C(t)$ exactly when $t = \frac{\pi}{\omega_d}$ — C_{ss}: steady state. in our case $C_{ss} = 1$.

* $C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \cos^{-1}\zeta)$.

* M_p just depend on the value of ζ.

* The Highest value for M_p when $\zeta = 0 \Rightarrow M_p = 2C_{ss} = 2$

As long as ζ increasing $\Rightarrow$ M_p decrease.

* for T_s : $\boxed{T_s = \frac{4}{\zeta\omega_n}}$ settling time valid for $\zeta < 0.8$

* N.B: when dealing with critically & overdamped systems we specified a Rise Time.

$$T_r = t\Big|_{c(t)=0.9C_{ss}} - t\Big|_{c(t)=0.1C_{ss}}$$

---

* Matlab:

```
>> n=25; d=[1 2 25]; sys = tf(n,d), step(sys).
```

Example: What is the unit response of a system given by the TF : $G(s) = \frac{2s+5}{s^2+2s+25}$

$\Rightarrow C(s) = \frac{2s+5}{s(s^2+2s+25)} = C_1(s) + C_2(s)$

$\omega_n = \sqrt{25} = 5$ rad/sec.

$\zeta = \frac{2}{2 \times 5} = 0.2$.

$\zeta\omega_n = 1$

$\omega_d = 5\sqrt{0.96}$

$= 4.9$ rad/sec. $\Longrightarrow$

$$\Rightarrow C(s) = \frac{1}{5}\left(1 - \frac{e^{-t}}{\sqrt{0.96}}\sin(4.9t+78^\circ)\right) + \mathcal{L}^{-1}\, 2 * \frac{s}{5}\, C_1(s)$$

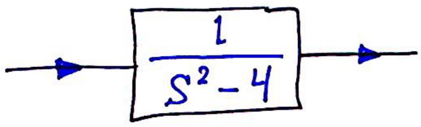
$$\Rightarrow \mathcal{L}^{-1}\, 2 * \frac{s}{5} C_1(s) = 0.4\,\frac{d}{dt}\left[\frac{1}{5}\left(1 - \frac{e^{-t}}{\sqrt{0.96}}\sin(4.9t+78^\circ)\right)\right]$$

---

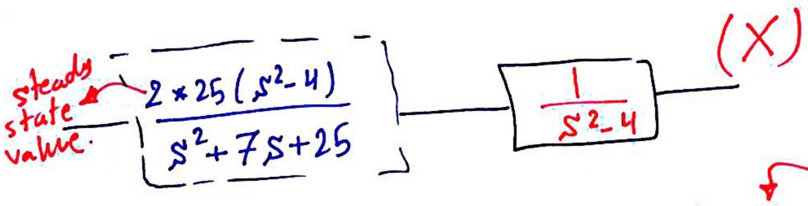
* Design Through Specifications:

⇒ Consider the following system which may represent a bicycle dynamics:

* $\frac{1}{s^2+4}$ ⇒ stable, bounded.



* Design a controller to end up with a system having $\zeta = 0.7$ & $\omega_n = 5$ rad/sec. and a steady state o/p of 2.

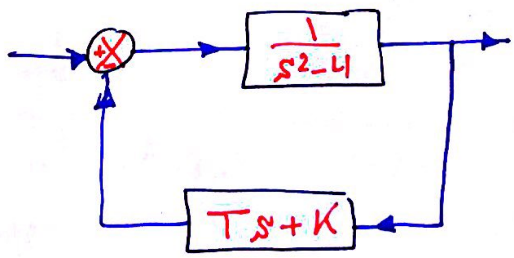


This Design is wrong. due to something called: Poles-Zero Cancellation.

⇒ Such design is rejected due to the practical disadvantage of pole-zero Cancellation.

** The Desired C.L.T.F (closed-loop Transfer Function) is:

$$\frac{50}{s^2 + 2*5*0.7s + 5^2}$$

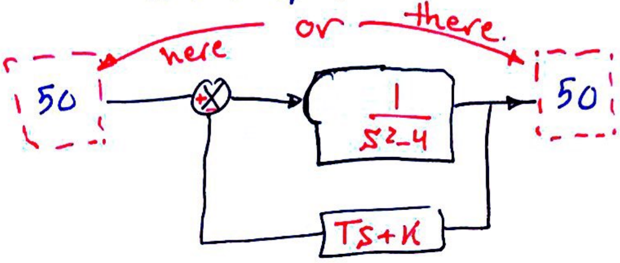


⇒

⇒ i.e we require:

$$\frac{1}{S^2+TS+(K-4)} * M$$

→ this M existed at the reference or at the output.



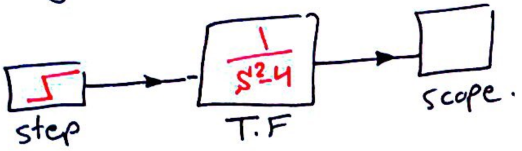
$$= \frac{50}{S^2+7S+25}$$

so we can find: $M=50$, $T=7$, $K=29$

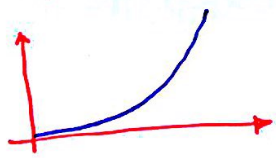
* * * * *

SimuLink:

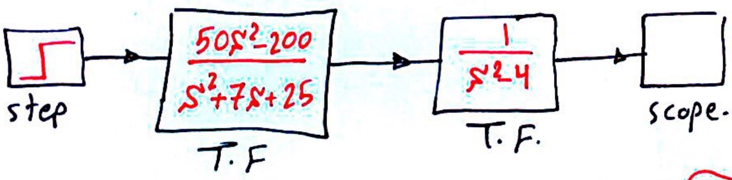
using simulink :



The output as follows:

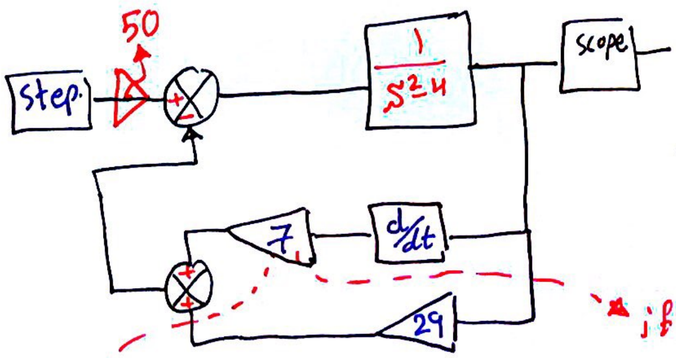


*using the first solution :

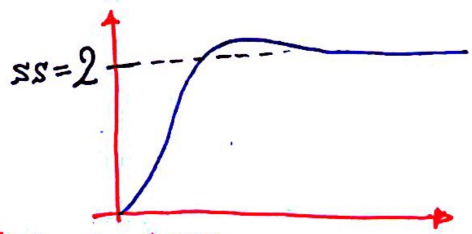


[50 0 -200] ⇒ [50 0 -202] → small change will change the output "unstable".

* using the correct solution:



The output as follows:



* if it was (-ve) ⇒ Unstable.

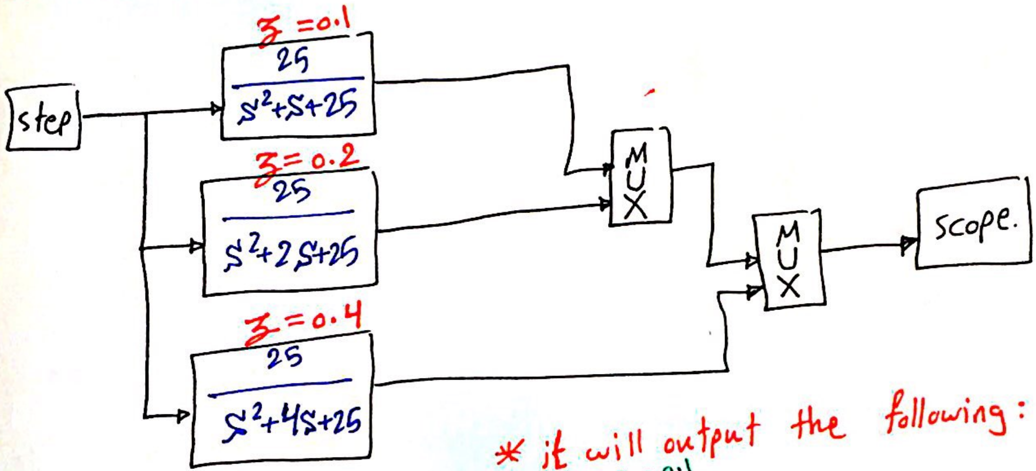
* @ 10 critical damping.

$\frac{50}{S^2+7S+25}$ Stable.

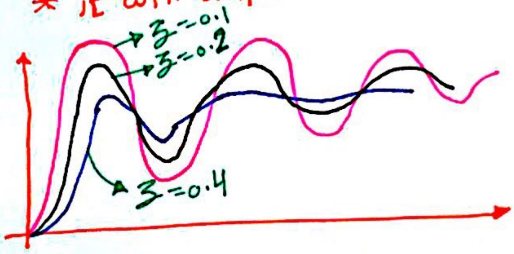
$\frac{50}{S^2-7S+25}$ Unstable.

$\frac{50}{-S^2-7S-25}$ Stable. → steady state (-ve).

* we can use the following 3 systems to see the difference for ζ :



* it will output the following :



* Stability of Linear Systems :

• Fact : Stability of linear systems doesn't depend on the magnitude and nature of forcing function or the initial conditions.

⇒ This fact simplifies the study of stability by considering the stabilities of an Unforced system. i.e $f(t)=0$

$$\frac{d^3c}{dt^3} + \alpha \frac{d^2c}{dt^2} + \beta \frac{dc}{dt} + \gamma c = f(t)$$

$c(t_0)=A$, $c'(t_0)=B$, $c''(t_0)=C$

* Definitions:

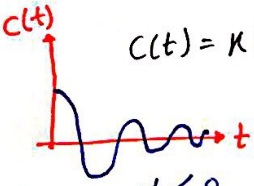
Consider a system of output $c(t)$.

i) If $\lim_{t \to \infty} c(t) = 0$ then the system is asymptotically stable.

Examples:

- A marvel in a bowl.

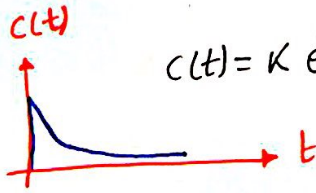
$c(t) = K e^{\alpha t} \cos\omega t$



$\alpha < 0$

- A man swing.

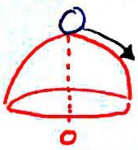
$c(t) = K e^{\alpha t}$



$\alpha < 0$

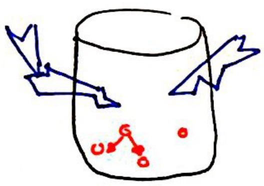
ii) If $\lim_{t \to \infty} c(t) = \infty$ then the system is unstable.

Examples:



Ball on bowl.

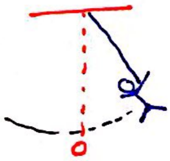
Undriven Bicycle.



Nuclear Reacter.

iii) If $\lim_{t \to \infty} c(t) =$ Bounded Value. then the system is stable.

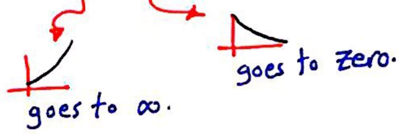
Example: swing of zero bearing friction and/or zero air resistance.



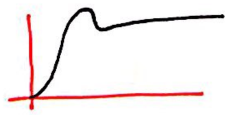
Example: pendulum bob.

Example: $G(S) = \frac{1}{S^2+1}$ is a stable function.

$G(S) = \frac{1}{S^2-1}$ → it will give e^{t}, e^{-t} ⇒ Unstable



$G(S) = \frac{1}{S^2+S+1}$ ⇒ stable.



$G(S) = \frac{1}{S^2+S+16}$ ⇒ stable. (more oscillation).

※ Judging Stability:

- First Method (the hardest): Calculate $c(t)$ then evaluate $\lim_{t \to \infty} c(t)$ ⇒ $c(t) = \underbrace{C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \dots + C_n e^{\lambda_n t}}_{\text{shouldn't approach } \infty} + P(t)$

- Second Method: Determination of λ_i is sufficient to judge stability if at least one λ_i has positive real part the system is unstable.

  ⇒ This method is easier than the first, BUT we still have difficulty in calculating λ_i.

- Third Method: using Routh's stability Criterian (the easiest).

  ⇒ for a closed loop system given by: $\frac{G(S)}{1 + G(S)\,H(S)}$

  The poles (the λ_i) are the zeros of $1+G(S)\,H(S)$ → CE.

  ⇒ Known as the characteristic equation CE or the characteristic polynomial.

  i.e $1+G(S)\,H(S) = 0$

⇒ the method is best illustrated by an example:

Consider the following system:

it is open loop system so:

$$\longrightarrow \boxed{\frac{s-8}{s^3+6s^2+11s+6}} \longrightarrow$$

$$CE \Rightarrow s^3+6s^2+11s+6=0$$

s^3	1	11
s^2	6	6
s^1	10	0
s^0	6	

$$\frac{11*6-1*6}{6}=10$$

$$\frac{10*6-6*0}{10}=6$$

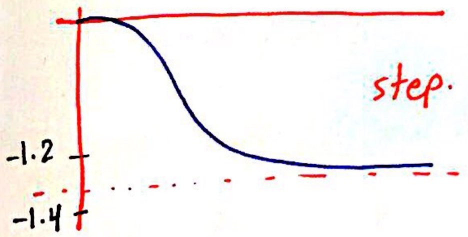
↑ we care for this column.

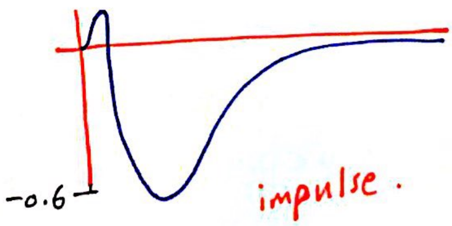
* No change in sign in the first column hence, the system is not unstable (stable).

⇒ asym. stable.

using matlab:

```
>> n=[1 -8]; d=[1 6 11 6]; sys=tf(n,d), step(sys)
                                        or impulse(sys)
```






```
>> r=roots(d)
   r=  -3
       -2
       -1
```

No change in sign ⇒ stable.

⇒ for the same example with:

$$\frac{S-8}{S^3+6S^2+11S+72}$$

S^3	1	11
S^2	6	72
S^1	-1	0
S^0	72	

⇒ There is a change in sign so it is **unstable**.

* since it change the sign two times ⇒ two positive real in the roots.

```
>> roots(d)
```

-6.1237 + 0.0j
0.0619 + 3.4284j
0.0619 − 3.4284j

two real positive. ⇐ (0.0619)

↪ if change just one time in sign ⇒ one positive real.

⇒ for the same example with:

$$\frac{S-8}{S^3+6S^2+15S+72}$$

S^3	1	15
S^2	6	72
S^1	3	0
S^0	72	

⇒ **Stable**.

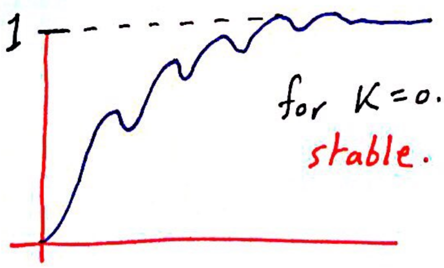
Example: Let $CE = 2S^4 + 4S^3 + 3S^2 + 4S + K = 0$

S^4	2	3	K
S^3	4	4	0
S^2	1	K	
S^1	4-4K	0	
S^0	K		

$K > 0$ & $4 - 4K > 0 \Rightarrow K < 1$

so To be stable it must be:

$$0 < K < 1$$



for K=0.5. stable.

* for a stable system:

$$\frac{K}{S^7 + \cdots + M}$$

→ steady state value $= \frac{K}{M}$

* The case where the first element in the first column is Zero. i.e

s^4	2	3	K
s^3	(0)	4	0
s^2	1	K	
s^1	4-4K		
s^0	K		

Example:

Let $CE = 2s^4 + 3s^2 + 4s + K = 0$

we replace the zero by (ϵ).

s^4	2	3	K
s^3	ϵ	4	0
s^2	$\frac{3\epsilon-8}{\epsilon}$	K	
s^1			
s^0			

→ multiply this row by ϵ for easier calculations.

s^4	2	3	K
s^3	ϵ	4	0
s^2	$3\epsilon-8$	$K\epsilon$	
s^1	$\frac{-32+12\epsilon-K\epsilon^2}{3\epsilon-8}$	0	
s^0	ϵK		

Take limits as $\epsilon \to 0$

$\lim_{\epsilon \to 0} (3\epsilon - 8) = -8$ system is unstable.

⇒⇒ Alternative approches to division by zero are:

1) Apply Routh's to $[(s+1) * C.E.]$

2) Replace s by $\frac{1}{s}$ in the C.E then apply Routh's.

Example: Let $C.E = 2s^4 + 5s^3 + 4s^2 + 10s + 10$

Apply the three methods?

```
>> roots([2 5 4 10 10])
```

⇒ 0.3806 + j1.3951
0.3806 − j1.3951
−2.1481
−1.1130

There is a real positive part so it is unstable.

※ The case of all elements of a row are Zeros:

This best illustrated by an example:

Example: Let $C.E. = 4S^5+2S^4+4S^3+2S^2+10S+5=0$

S^5	4	4	10
S^4	2	2	5
S^3	~~0~~ 8	~~0~~ 4	0
S^2	1	5	
S^1	-36	0	
S^0	5		

⇒ we look to the row above the row contain Zeros.

$A(S) = 2S^4+2S^2+5S^0$

→ Auxiallary equaition

$\frac{dA(S)}{dS} = 8S^3+4S$

a change in the sign

⇒ Unstable. ⇒ 2 poles with real positive part.

N.B:

{1} If the C.E has a missing power then the system is unstable.

{2} If the C.E has a change in sign then the system is unstable.

* Stability of Second order system: "special case"

$C.E = aS^2+bS+C$

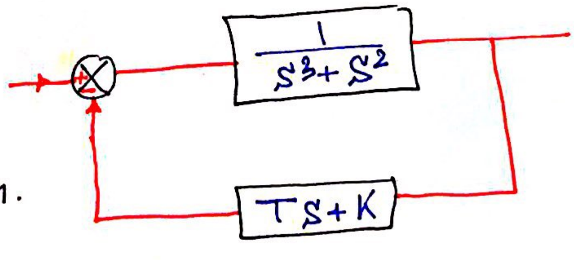
S^2	a	c
S^1	b	0
S^0	c	

if a,b,c together (+ve) or (-ve)

⇒ stable , otherwise unstable.

a,b,c (Non Zero number).

Exercise:
study the stability of the open loop & closed loop system.



⇒

in case OL : unstable due to missing power.

in case CL : $CL \Rightarrow \dfrac{1}{S^3+S^2+TS+K}$

as a necessary condition for stability is $T, K > 0$

S^3	1	T
S^2	1	K
S^1	T−K	
S^0	K	

$\Rightarrow K>0$ & $T-K>0 \Rightarrow T>K$ & $T>0$

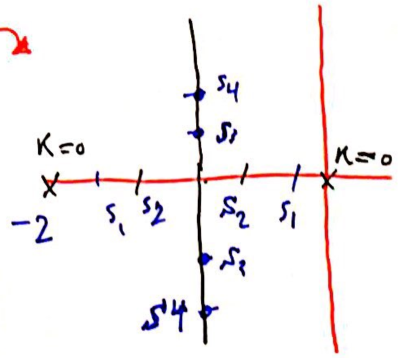
it is stable for $T>K$.

* * *

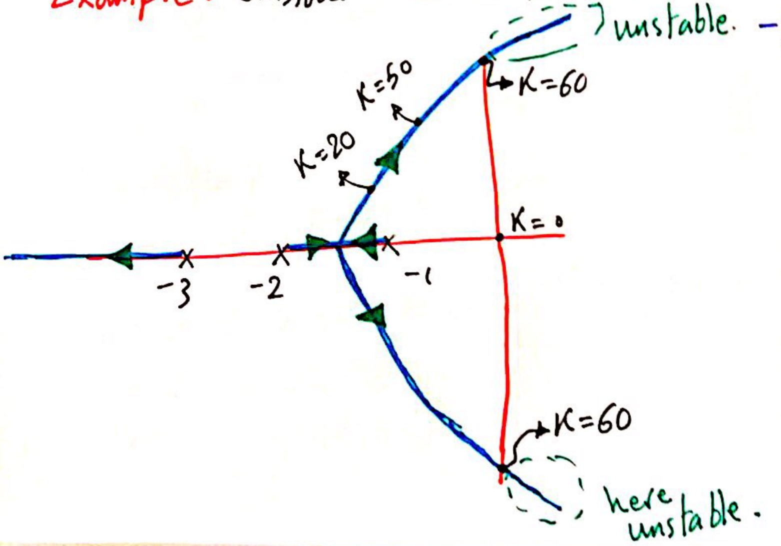
※ Root Locus : (RL)

The RL is a pictorial depecition of the roots (zeros) of a polynomial in terms of a certain parameter (say, the gain K).

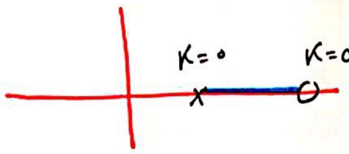
Example: Let $C.E = S^2+2S+K=0$



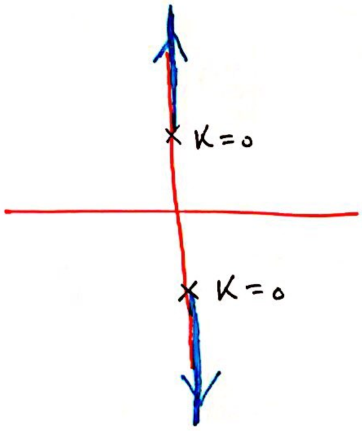
Example: Consider $CE = S^3+6S^2+11S+6+K=0$



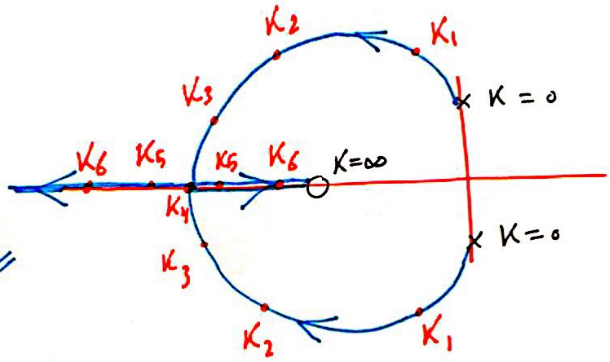
Example:



$\Rightarrow$ here the system is unstable for all values of K.



$\Rightarrow$ here the system is stable.



stable for all K. $\Downarrow$

※ Basis of the RL method:

start with the CE ($\rightarrow 1 + G(s)H(s)$) and arrange it in the form:

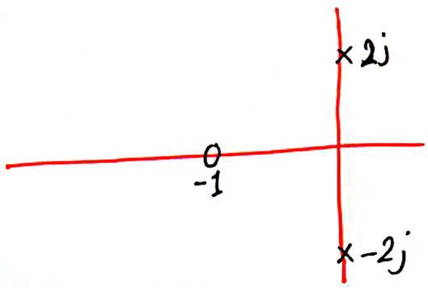
$$1 + K\frac{Z(s)}{P(s)} = 0$$

K is the parameter that you want to change.

e.g let $C.E = s^2 + 4 + Ks + K = 0$

$$\Rightarrow 1 + K\frac{s+1}{s^2+4} = 0$$

we have Zero: -1
poles: $\pm 2j$



Using $1 + G(s)H(s) = 0$ which makes $CE = 0$ are the closed Loop poles

$\rightarrow G(s)H(s) = -1 + j0$

or

$|G(s)H(s)| = 1$

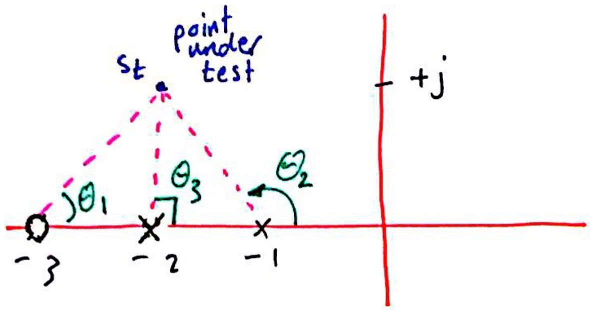
$\angle G(s)H(s) = (1 \pm 2h)180^\circ$

$\hookrightarrow$ used to sketch RL.

$\Rightarrow$

i.e the magnitude of $G(s)H(s)$ when s is a CL poles is 1. "magnitude condition".

the sum of angles between the CL pole & the starting poles & zeros is an odd multiple of 180°. "angle condition"



angle condition:

$$(\theta_1) - (\theta_2 + \theta_3) \overset{?}{=} \pm 180°$$

$$45 - 135 - 90 \overset{?}{=} \pm 180$$

$$\underline{\underline{-180}} \checkmark$$

magnitude condition:

$$K \frac{\ell_z}{\ell_{p_1} \ell_{p_2}} = 1 \Rightarrow K \frac{\sqrt{2}}{1 * \sqrt{2}} = 1 \Rightarrow \boxed{K = 1}$$

s_t on the root Locus & $K = 1$.

* * *

second Material second Material

* * *

*** Rules for Drawing the RL:**

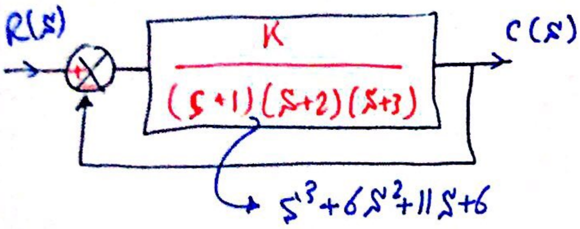
starting with $1 + G(s)H(s) = 0$, we get $1 + K \frac{Z(s)}{P(s)} = 0$

$Z(s)$ & $P(s)$ are independent on K.

The Rules are illustrated by considering certain systems:

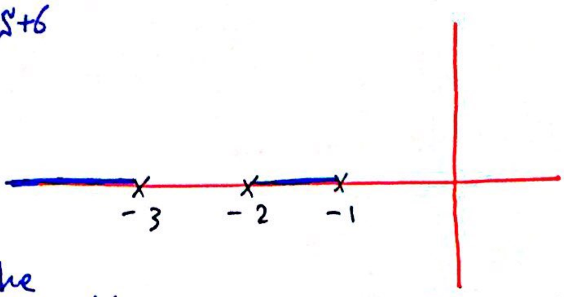
⟹

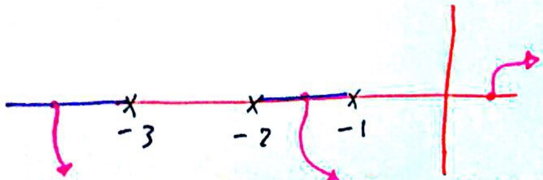
(K > 0)



$$\Rightarrow \quad 1 + K\frac{1}{(1+s)(s+2)(s+3)} = 0$$

* We have a RL on the real part axis if the sum of poles & zeros to the right of a test point is odd.





for a point here: $\sum\theta_z - \sum\theta_p$ = Zero (No RL).

$\sum\theta_z - \sum\theta_p = 180°$ (there is RL).

$\sum\theta_z - \sum\theta_p = 3 * 180°$ (there is RL)

※ Asymptotes:

$$\theta = \frac{(1+2h)\,180°}{n_p - n_z}$$

$h = 0, 1, n_p - n_z$

$\theta = 60°, 180°, 300°$

※ Intersection:

$$\sigma = \frac{\text{sum of poles} - \text{sum of zeros}}{n_p - n_z}$$

in our example:

$$\sigma = \frac{-6-0}{3} \Rightarrow \boxed{\sigma = -2}$$

※ Break-away point:

given by:

$$\frac{dK}{ds} = 0$$

in our case:

$-K = (1+s)(s+2)(s+3)$

$\Rightarrow s^3 + 6s^2 + 11s + 6 = -K$

$$\frac{dK}{ds} = 3s^2 + 12s + 11 = 0$$

solving: $s = -1.42, -2.57$.

-1.42 → accepted

-2.57 → rejected, doesn't locate at RL.

$$\boxed{s = -1.42}$$

⟹

Need to find intersection with the imaginary axis ?!

※ Point of Intersection with the imaginary axis :

Use Routh's to the CE.

$CE = s^3 + 6s^2 + 11s + 6 + K = 0$

s^3	1	11
s^2	6	6+K
s^1	$\frac{60-K}{6}$	0
s^0	6+K	

we look at the first column & see which element can be zero.

$\Rightarrow$ row 3 @ $K = 60$

then back to row 2 :

$6s^2 + 66 = 0 \Rightarrow s = \pm j\sqrt{11}$

$\Rightarrow$ Now we take a test point :

Test for the angles : $\sum \theta_z - \sum \theta_p \overset{?}{=} \pm 180°$

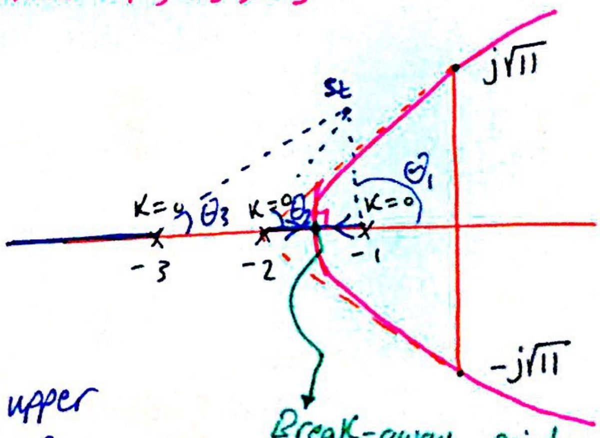
$\Rightarrow$ for our case measure $\theta_1, \theta_2, \theta_3$.

also Test for $j\sqrt{11}$?!

$\Rightarrow$ it must locate at RL.

$0 - \left[\tan^{-1}\frac{\sqrt{11}}{1} + \tan^{-1}\frac{\sqrt{11}}{2} + \tan^{-1}\frac{\sqrt{11}}{3}\right]$

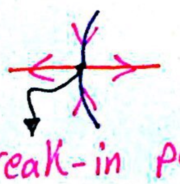
$= -180°$



** we just draw the upper half, & the lower half is reflected to the upper half.

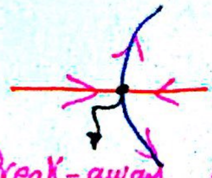
السهم يشيرإلى اتجاه زيادة K.

*



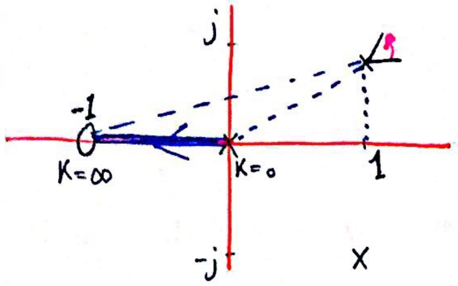
Break-in point.

*



Break-away point.

* In case of complex pole ⇒ Departure Angle.



$$(26^\circ) - (45^\circ + 90^\circ + \theta_d) = 180^\circ$$

$$\Rightarrow \theta_d = 26 - 90 - 45 - 180$$

$$\Rightarrow \boxed{\theta_d = -289^\circ}$$

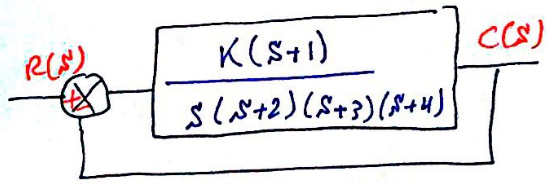
or $\theta_d = 71^\circ$

* In case of complex zero ⇒ Arraival Angle.

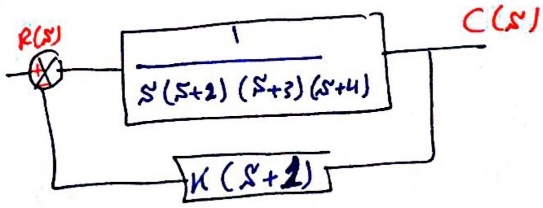
※ Design Using the RL:

Consider the systems:

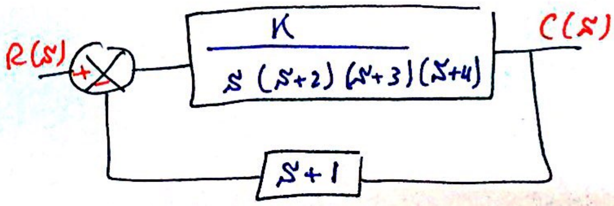
They all have the same C.E however, the time response is different.



⇒ They all have same RL.



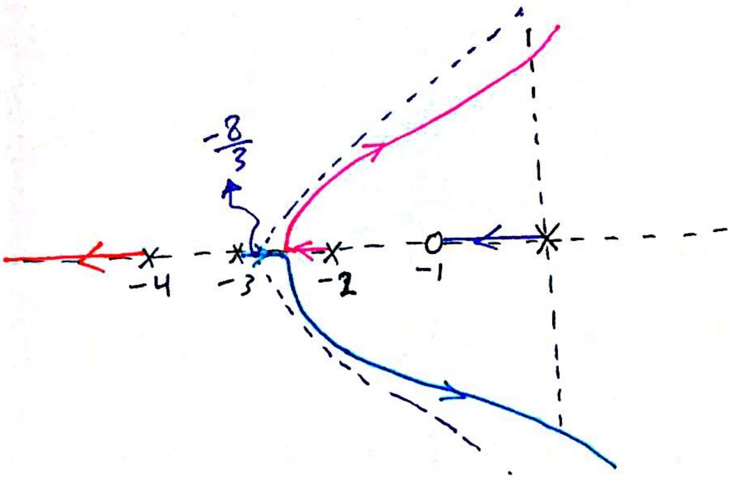
$$C.E = S(S+2)(S+3)(S+4) + K(S+1) = 0$$



Exercise:

Determine $\frac{C(S)}{R(S)}$ for the three systems.

>> n = [1 1]; d = [1 9 26 24 0]; sys = tf(n,d); rlocus(sys)



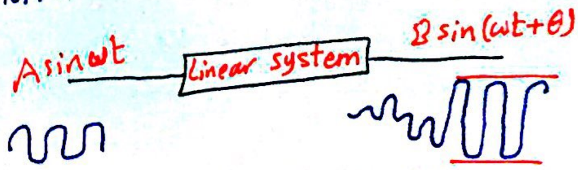
>> n = [1 1]; d = [1 9 26 24 0]; sys = tf(n,d); figure(1), rlocus(sys), K = rlocfind(sys),
sync = feedback(K * sys, 1), figure(2), step(sysc)

if we want the step response at a certain K:
remove(K = rlocfind(sys)) & use sync = feedback(K * sys, 1)
needed value.

* * *

※ Frequancy Response: (F.R)

<u>Definition</u>: FR is the steady state response of a linear system due to sinusoidal excitation.



Intersted in the gain = $\frac{B}{A}$
& the phase shift.

Gain & phase shift can be obtained either by calculation if the T.F is given or by practical measurement.

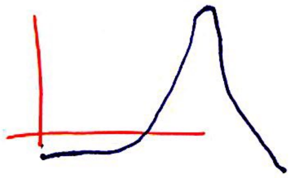
ω	gain	phase shift.
ω_1	...	...
ω_2	...	...
ω_3	...	...
⋮		

※ Graphing Methods for Representing the Freq. Response:

- **The Bode Diagram: (BD)**

```
>> n = 10*[1 1]; d = [1 1 20] ; bode(n,d)
```

↳ underdamped system with low damping.

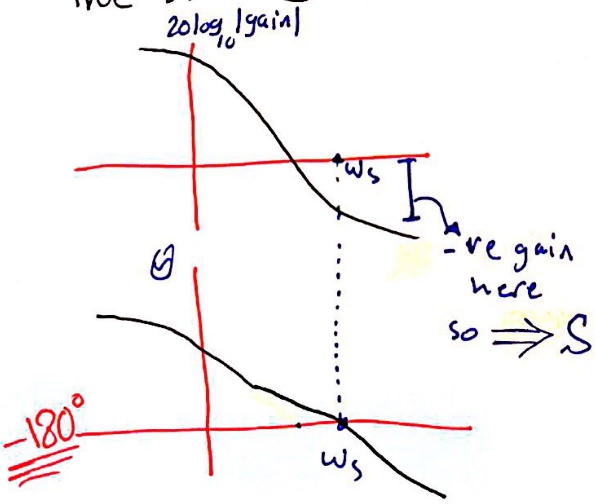


* Stability Using Bode Diagram:

Having the BD of the open loop system as given:

$G(j\omega)\,H(j\omega)$

The stability of the CL is determined as follows:



so ⇒ STABLE.

we see where the phase shift cut the (-180°) line at ω_s, and go to that value on the magnitude plot, and we test if the value their

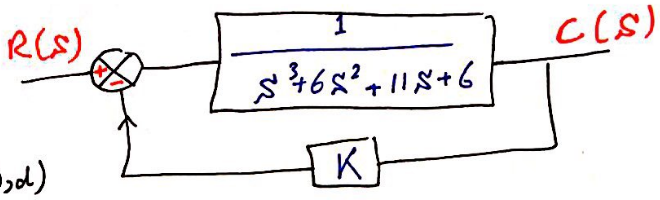
(-ve) ⇒ stable.

(+ve) ⇒ unstable.

※ Gain & Phase Margins:

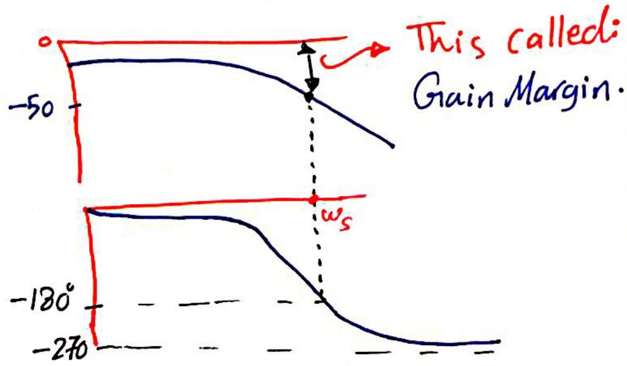
Consider the following system:

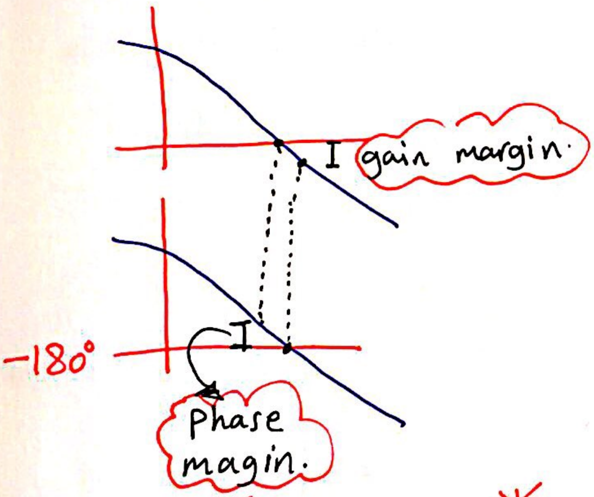
$$G(s)H(s) = \frac{K}{s^3+6s^2+11s+6}$$




```
>> K=1;n=K;d=[1 6 11 6];sys=tf(n,d)
, bode(sys)
```

−ve gain @ ω_s ⇐
so it is
stable.





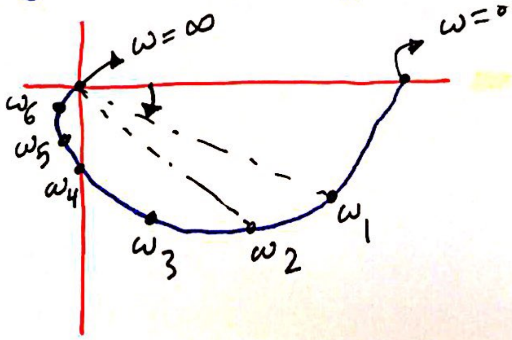
* if the gain equal Zero @ ω_s (value when the curve cut the −180° line) then it is called Marginally stable.

* * * *

• The Nyquist Diagram: (ND)

The ND is a plor plot, where magnitude and phase are represented on the same diagram, considering ω as a parameter.

This for second order system or more.



Example: Let $G(s) = \frac{1}{s+1}$

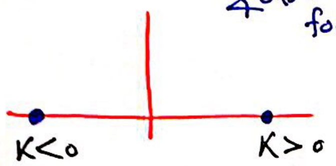
$$\Rightarrow G(j\omega) = \frac{1}{j\omega+1} \Rightarrow |G| = \frac{1}{\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = 0 - \tan^{-1}\left(\frac{\omega}{1}\right)$$

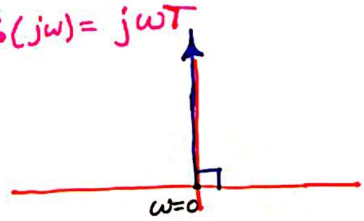
※ ND of a certain T.F:

• $G(j\omega) = K$

$\measuredangle G(j\omega) = 180^\circ$ for $K<0$



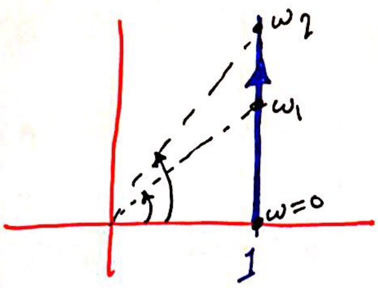
• $G(j\omega) = j\omega T$



• $G(j\omega) = 1 + j\omega T$

$\measuredangle G(j\omega) = \tan^{-1}\omega T$

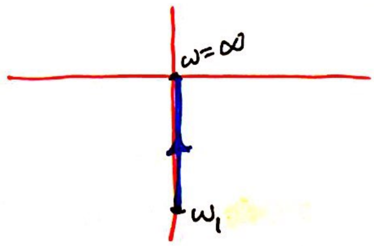
$|G(j\omega)| = \sqrt{1+\omega^2 T^2}$



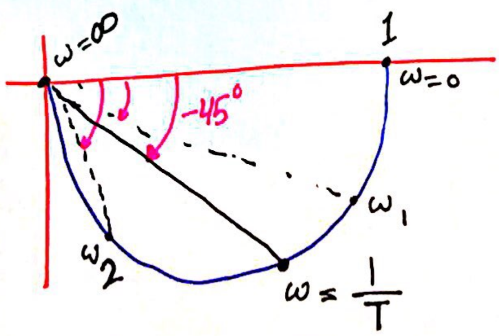
• $G(j\omega) = \frac{1}{j\omega T}$

$|G(j\omega)| = \frac{1}{\omega T}$

$\measuredangle G(j\omega) = -90^\circ$



• $G(j\omega) = \frac{1}{1+j\omega T}$



These all 5 NDs for certain T.F.

※ ND of $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, $\zeta < 1$:

$$G(j\omega) = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\zeta\omega\omega_n}$$

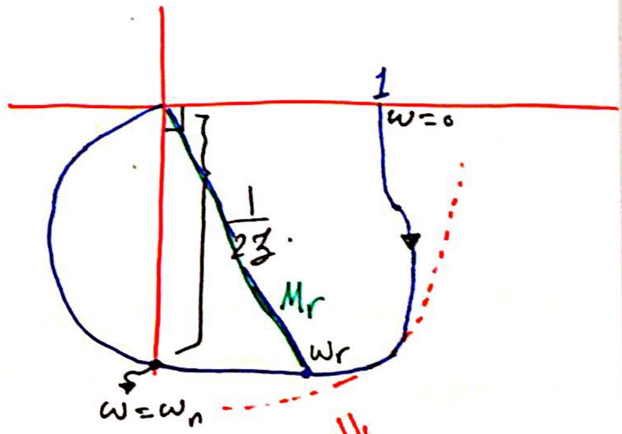
@ $\omega = \omega_n$, phase-shift $= -90°$ & magnitude $= \frac{1}{2\zeta}$

* resonant frequancy (ω_r):

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

↳ True for $\zeta < \frac{1}{\sqrt{2}}$

$$M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

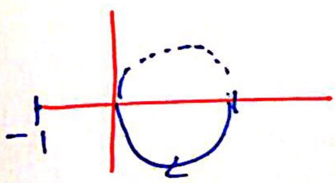


بنستخدم الفرجار لقياس Mr التي قمنا بحسابها.

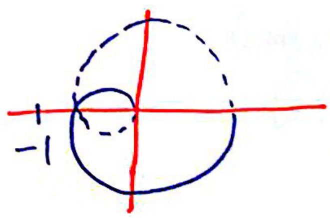
※ Stability Using ND:

* Plot the ND for $-\infty < \omega < \infty$.

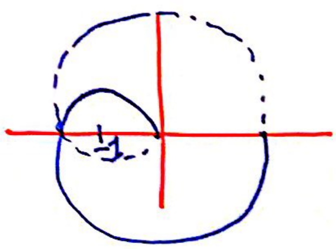
If $G(j\omega)H(j\omega)$ for $-\infty < \omega < \infty$ doesn't encircle (enclose) the -1 point then the system is stable.



closed system & doesn't enclose the -1 (stable).



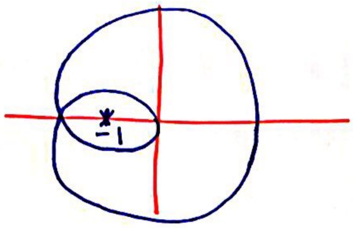
(stable).



(Unstable).

* one of the advantages of ND:
Determine the number of real positive part for the poles.

Example:

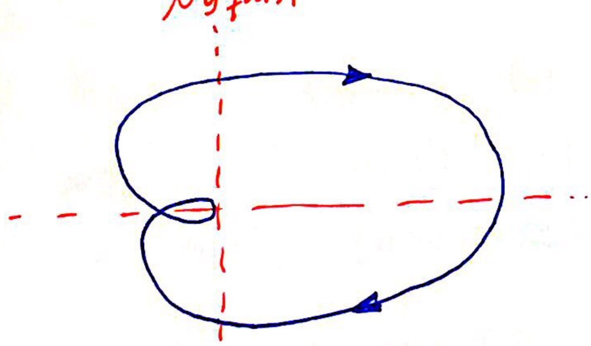


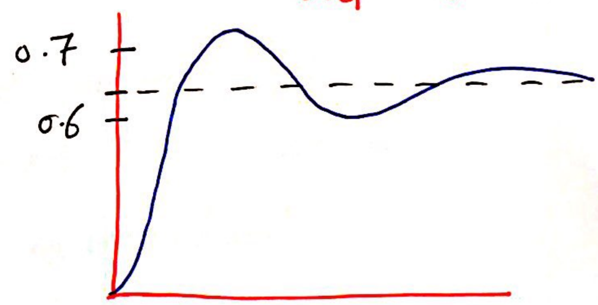
⇒ (−1) encircled two times so we have two real positive part.

Rule:

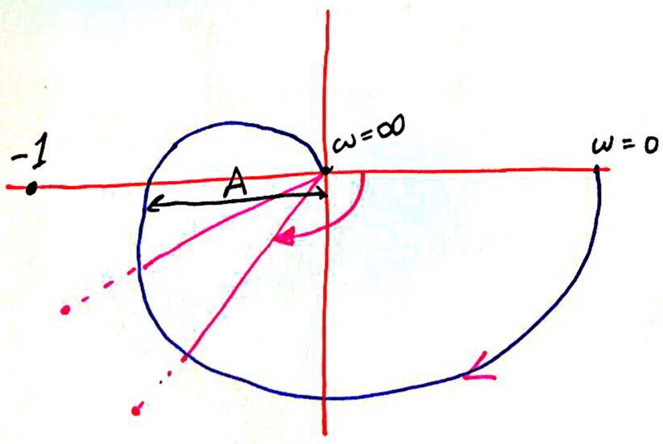
Number of real positive part = Number of times that (−1) encircled

```
>> n=10; d=[1 6 11 6]; sys=tf(n,d), figure(1), nyquist(sys),
   sync = feedback(sys,1), figure(2), step(sync).
```



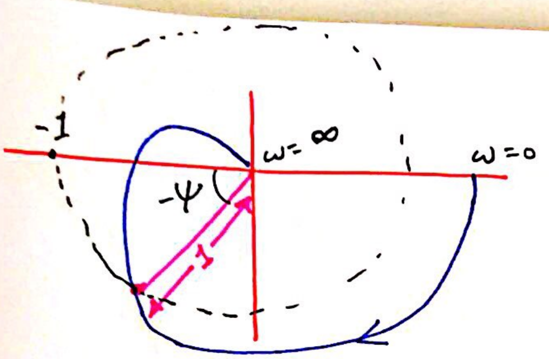


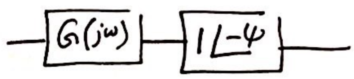
※ Gain & Phase Margins Using the ND:



Gain Margin:

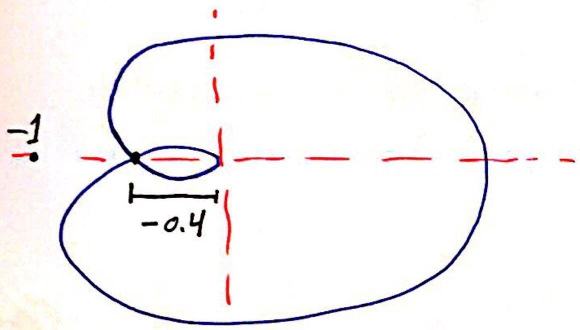
$$G.M = \frac{1}{A}.$$





phase. margin (P.M) = ψ

```
>> n=24; d=[1 6 11 6]; sys=tf(n,d), figure(1); nyquist(sys); sysc=feedback(sys,1)
, figure(2), step(sysc)
```



⇒ This system is stable.

if we take:

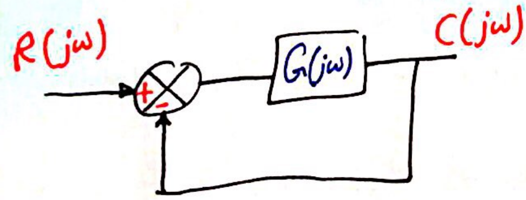
$$G.M = \frac{1}{0.4} = 2.5$$

⇒ multiply by it

⟹ Unstable in this case.

** The Gain Margin & the Phase Margin evaluations are valid provided the closed loop system is stable. (otherwise is meaningless).

※ Determination of the CL FR:



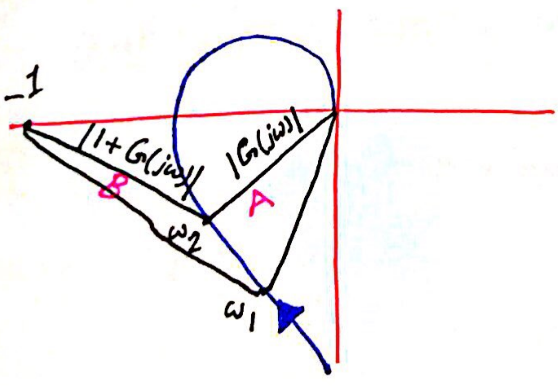
$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

$$\left|\frac{C(j\omega)}{R(j\omega)}\right| = \frac{|G(j\omega)|}{|1+G(j\omega)|}$$

$r(t) = A\sin\omega t$.

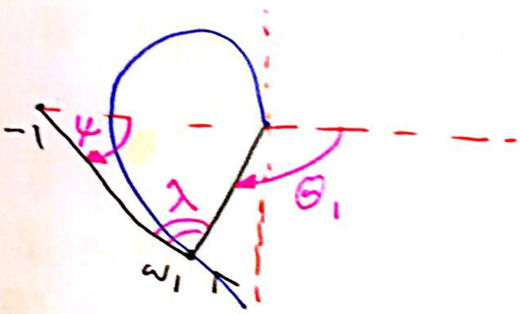
$C(t) = B\sin(\omega t+\theta)$

$$= \frac{A}{B}$$



$$\measuredangle\frac{C(j\omega)}{R(j\omega)} = \measuredangle G(j\omega) - \measuredangle 1+G(j\omega)$$

⟹



$\measuredangle G(j\omega) = \theta$

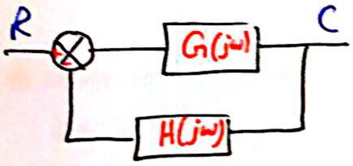
$\measuredangle 1+G(j\omega) = \psi$

$\Rightarrow \measuredangle \frac{C(j\omega)}{R(j\omega)} = \theta - \psi = \lambda$

we can measure θ & ψ & subtract them, or simply measure λ directly.

* such kind of measurements are valid provided the CL system is Stable.

Otherwise, if the system Unstable $\Rightarrow$ any value obtained by this method will be wrong.



$$\Rightarrow \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)\,H(j\omega)}{1+G(j\omega)\,H(j\omega)} \cdot \frac{1}{H(j\omega)}$$

we work on this as the unity FB But the results will be multiplied by $\frac{1}{H(j\omega)}$

** Advantages of ND on the BD: ☺

1] ND represented by one figure (mag. & phase), whereas the BD represented by two figures.

2] ND has the ability to determine the number of real positive part of the poles, while the BD Not.

3] Graphical Determination for ND which is NOT provided by BD.

※ State Space Representation :

The need for this representation arises when :

i] The system has a Non-Zero initial conditions.

ii] The system is Non-linear.

iii] The system is Time-varying.

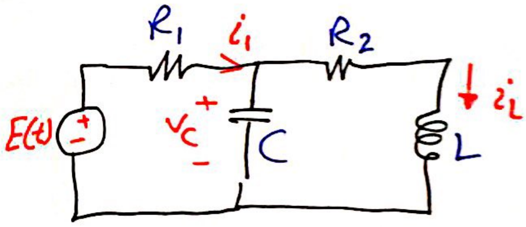
iv] The system has Time delays.

v] The system is require to act optimally.

vi] The system is multi-input multi-output.

* Modeling Using State Space Representation:

Consider the following circuit:



* usually choose the Number of variables equal to number of independent storage elements.

Here we have 2 indep. storage elements ⇒ 2 variables.

$X_1 = V_C$

$X_2 = i_L$

⇒ we have to determine the output :

$y = i_1$

$\dot{X}_1 = \frac{dX_1}{dt}$

By Nodal Analysis:

$$\frac{E - X_1}{R_1} = C\dot{X}_1 + X_2$$

By the Right mesh: (KVL)

$$X_1 = R_2 X_2 + L\dot{X}_2$$

$$\underline{\dot{X}} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_1 C} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R_2}{L} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C} \\ 0 \end{bmatrix} E$$

↳ These called: state Equations.

they are: "differential equations"

⇒

⇒ for the output: $y = i_1 = \frac{E - X_1}{R_1} = -\frac{1}{R_1}X_1 + \frac{E}{R_1}$

$y = \begin{bmatrix} \frac{-1}{R_1} & 0 \end{bmatrix} \underline{x} + \frac{1}{R_1}E$ ⇒ This called: Output Equations.

Generally we deal with a system described by:

$\underline{x}^{\circ} = A\underline{x} + B\underline{u}$ → differential equation.

$\underline{y} = C\underline{x} + D\underline{u}$ → algebraic equation.

$x \in R^n$; n number of states.

$u \in R^m$; m number of inputs.

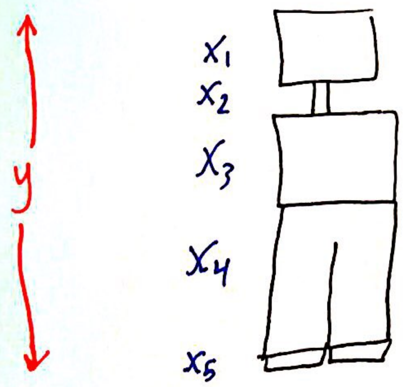
$y \in R^p$; p number of outputs.

$p < n$.
→ otherwise, Redundancy will occur.

* The output are Not the states.

⇒ They are a combination of states, and they can be the states.

Example: The height of a person.



states: x_1, x_2, x_3, x_4, x_5
⇒ Gives more details.

y ⇒ Gives a summary.

- The state need NOT be measurable, or accessible or even real.
- The outputs are nessesarly measurable.

※ stability:

$$x^{\circ} = AX + Bu$$
$$y = CX + Du$$

** Stability is determined by The eigen values of A.

example: $x^{\circ} = \begin{bmatrix} 0 & 1 \\ 4 & 4 \end{bmatrix} x + \begin{bmatrix} * \\ * \end{bmatrix} u$

$y = [* \;\; *] x + * u$

we don't care for B, C & D for stability.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 4 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 4 = 0 \Rightarrow \boxed{\begin{matrix} \lambda_1 = 4.8 \\ \lambda_2 = -0.8 \end{matrix}} \Rightarrow \text{Unstable.}$$

* For the system To be Stable:
All eigen values should have negative real parts.

※ Time Response Solution of The State Equations:

It Can be shown that:

$$x(t) = e^{At} \underline{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

general for

C.f (compare with) $\frac{dx}{dt} = ax + bf(t) \Rightarrow x(t) = e^{at} x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} f(\tau) d\tau$

more simple for calculations:

$$x(t) = e^{At} \underline{x}(t_0) + e^{At} \int_{t_0}^{t} e^{A(-\tau)} Bu(\tau) d\tau$$

** where e^{At} is Known as The Exponential Matrix.
Fundemental Matrix.
Transition Matrix.

⇒

⟹ Evaluated either as :

i] $e^{At} = \mathcal{L}^{-1}\{[sI_n - A]^{-1}\}$ gives a closed form solution.

ii] By definition :

$$e^{At} = I_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots + \frac{(At)^n}{n!} + \cdots$$

↳ Suitable Numerically.

** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

* Radius of Convergence for e^x is: ∞.

Exercise: Given $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$.

Evaluate e^{At} using the two methods ?

* Proparties of the Exponential Matrix :

- $e^{At}\big|_{t=0} = I_n$. ⟹ use as a check.
- $e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$
- $e^{(A+B)t} = e^{At} \cdot e^{Bt}$ ⟹ This True only if: $AB = BA$
- $[e^{At}]^{-1} = e^{A(-t)}$ ⟹ Just replace every (t) by $(-t)$.
- $\frac{d}{dt}[e^{At}] = A e^{At} = e^{At} A$ ⟹ A & e^{At} commute.

※ The Transfer Function Matrix $G(s)$:

It can be shown that :

$$G(s) = C\,[sI_n - A]^{-1} B + D$$

Example: Given: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

; $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$e^{At} = \mathcal{L}^{-1}\{[SI - A]^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s-4}{(s-2)^2} & \frac{1}{(s-2)^2} \\ \frac{-4}{(s-2)^2} & \frac{s}{(s-2)^2} \end{bmatrix}\right\} = \begin{bmatrix} e^{2t} - 2te^{2t} & te^{2t} \\ -4te^{2t} & e^{2t} + 2te^{2t} \end{bmatrix}$$

($s-4 \rightarrow s-2-2$)

You can check: $e^{At}\Big|_{t=0} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (yes) ✓

$x(t)$ when $u(t)=0$ $\{t_0 = 0\}$

$$x(t) = e^{At} x(0) + \int^{t} \dots * 0 \, d\tau$$

$$= e^{At} x(0) = \begin{bmatrix} e^{2t} - 2te^{2t} + 2te^{2t} \\ -4te^{2t} + 2e^{2t} + 4te^{2t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix} \begin{matrix} = x_1(t) \\ = x_2(t) \end{matrix}$$

⇒ The system is Unstable since e^{2t} always +ve so there is a real positive part.

Exercise: Determine $x(t)$ due to unit response.

※ **Steady state Value due to a steady state input:**

If the system is asymptotically stable & A^{-1} exist then due to a unit step:

$$\underline{x}_{ss} = \lim_{t \to \infty} x(t) = -A^{-1}B$$

($x(t) \rightarrow x_1(t), x_2(t), \dots$)

$$y_{ss} = C x_{ss} + D u_{ss}$$

* * * * * * *

End of Material

Best of luck.