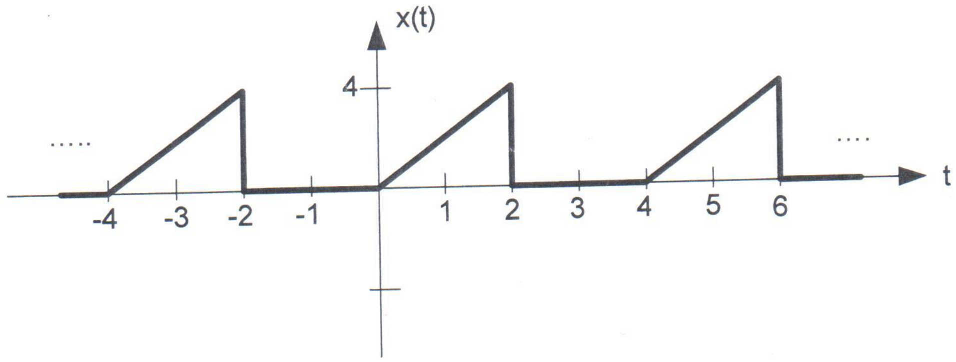
(25 marks) Q(1): Consider the following period signal.

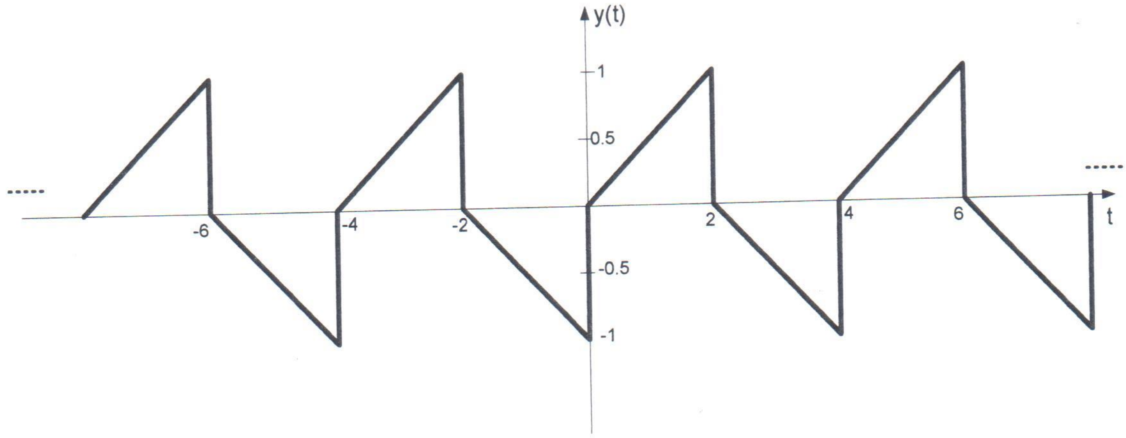


(a) **(10 marks)** By direct integration find and plot the double-sided **Magnitude** and **Phase** spectrum (up to the fourth harmonic).

Hint: Note that

$$\int (at+b)e^{ct}dt = \frac{(at+b)e^{ct}}{c} + \frac{ae^{ct}}{c^2} + K$$

(b) **(10 marks)** Use this result to obtain the Fourier series for the signal $y(t)$ shown in the Figure below. **(Do not use integration)**



(c) **(5 marks)** Determine the power in the third harmonic.

$T_0 = 4, \quad \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}, \quad x(t) = 2t, \quad 0 < t < 2$

$$C_n = \frac{1}{4}\int_0^4 x(t)\, e^{-j\frac{\pi}{2}nt}\, dt$$

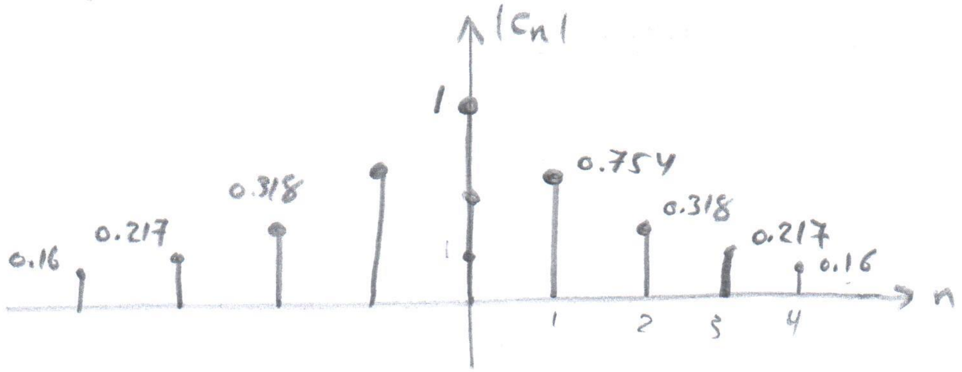
$$C_n = \frac{1}{4}\int_0^2 2t\, e^{-j\frac{\pi}{2}nt} dt = \frac{1}{2}\int_0^2 t\, e^{-j\frac{\pi}{2}nt} dt = \frac{1}{2}\left[\frac{2t\, e^{-j\frac{\pi}{2}nt}}{-j\pi n} + \frac{4e^{-j\frac{\pi}{2}nt}}{\pi^2 n^2}\right]_0^2$$

$$= \left[\frac{jt}{\pi n} + \frac{2}{\pi^2 n^2}\right] e^{-j\frac{\pi}{2}nt}\Bigg|_0^2 = \left[\frac{2j}{\pi n} + \frac{2}{\pi^2 n^2}\right] e^{-j\pi n} - \left[0 + \frac{2}{\pi^2 n^2}\right]$$

$$= \frac{2}{\pi n}\left(\left[j + \frac{1}{\pi n}\right] e^{-j\pi n} - \frac{1}{\pi n}\right) = \begin{cases} \frac{2}{\pi n}\left[-j - \frac{1}{\pi n} - \frac{1}{\pi n}\right], & n \text{ odd} \\ \frac{2}{\pi n}\left[j + \frac{1}{\pi n} - \frac{1}{\pi n}\right], & n \text{ even} \end{cases}$$

$$c_n = \begin{cases} \dfrac{-2}{\pi n}\left(+j + \dfrac{2}{\pi n}\right), & n: odd \\[2ex] \dfrac{2j}{\pi n}, & n: even \end{cases}$$

$$c_0 = \frac{1}{4}\int_0^2 2t\,dt = \frac{1}{2}\int_0^2 t\,dt = \frac{1}{2}\,\frac{t^2}{2}\Big|_0^2 = \frac{1}{4}[4] = 1$$



$$c_1 = \frac{-2}{\pi}\left(j + \frac{2}{\pi}\right) = \frac{2}{\pi}\sqrt{1+\frac{4}{\pi^2}}\ \angle \tan^{-1}\left(\frac{\pi}{2}\right) = 0.754\ \angle -122.48^\circ$$

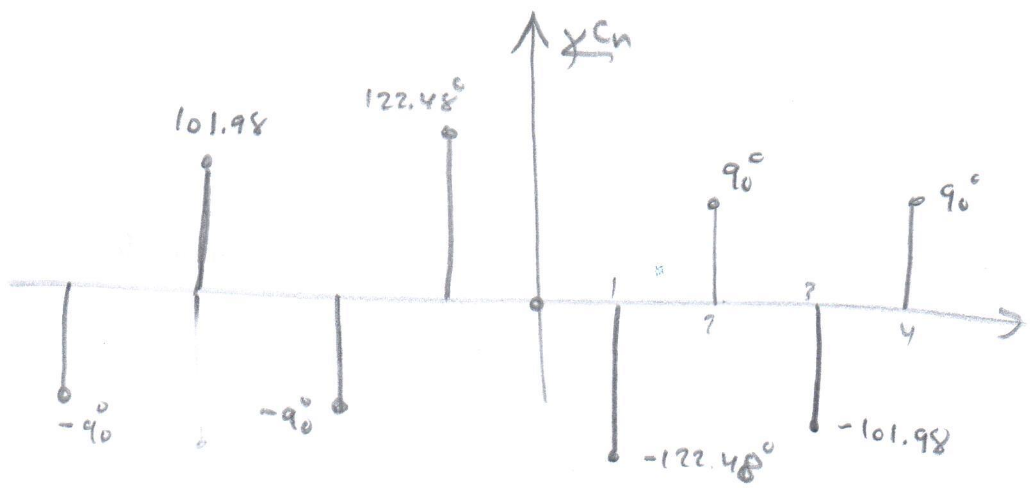
$$c_{-1} = 0.754\ \angle -122.48^\circ$$

$$c_2 = \frac{2j}{\pi 2} = \frac{j}{\pi} = 0.318\ \angle 90^\circ, \quad c_{-2} = 0.318\ \angle -90^\circ$$

$$c_3 = \frac{-2}{3\pi}\left(j + \frac{2}{3\pi}\right) = \frac{2}{3\pi}\sqrt{1+\frac{4}{9\pi^2}}\ \angle \tan^{-1}\left(\frac{3\pi}{2}\right) = 0.217\ \angle -101.98$$

$$c_{-3} = 0.217\ \angle 101.98$$

$$c_4 = \frac{2j}{4\pi} = \frac{j}{2\pi} = 0.16\ \angle 90^\circ, \quad c_{-4} = 0.16\ \angle -90^\circ$$



(b)

$$y(t) = \frac{1}{4}x(t) - \frac{1}{4}x(t-2)$$

$$\beta_n = \frac{1}{4}\left[C_n - C_n e^{-jn\frac{\pi}{2}(2)}\right] = \frac{1}{4}\left[C_n - C_n e^{-jn\pi}\right]$$

$$\beta_n = \begin{cases} \frac{1}{2}C_n , & n \text{ odd} \\ 0 , & n \text{ even} \end{cases}$$

(c)

$$2|C_3|^2 = 2(0.217)^2 = 0.094 \text{ Watt}$$

(25 marks) Q(2): For the systems shown below.

$$x(t) \rightarrow \boxed{h_1(t) = 4\delta(t+2)} \xrightarrow{z(t)} \boxed{h_2(t) = \frac{1}{2}e^{-2t}u(t-2)} \rightarrow y(t)$$

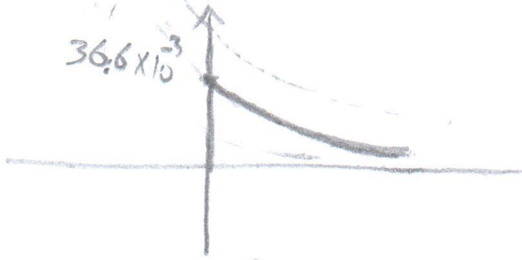
(a) **(10 marks)** Determine and plot the impulse response $h(t)$ between input $x(t)$ and output $y(t)$.

(b) **(10 marks)** If $x(t) = rect\left(\frac{t-1}{2}\right) - rect\left(\frac{t+1}{2}\right)$ then get $y(t)$ graphically.

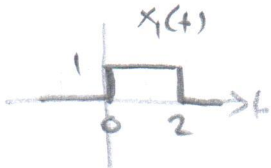
(c) **(5 marks)** Given that $z(t) = 0.5tri\left(\frac{t-2}{4}\right)$, then find $x(t)$.

ⓐ $h(t) = h_1(t) * h_2(t) = 4\,\delta(t+2) * \frac{1}{2} e^{-2t} u(t-2) = 2\, e^{-2(t+2)} u(t+2-2)$

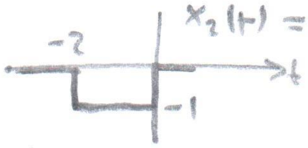
$\Rightarrow h(t) = 2\, e^{-2(t+2)} u(t)$ 10

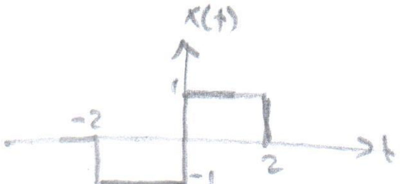


ⓑ $x(t) = x_1(t) + x_2(t) \Rightarrow x_1(t) = rect\left(\frac{t-1}{2}\right)$

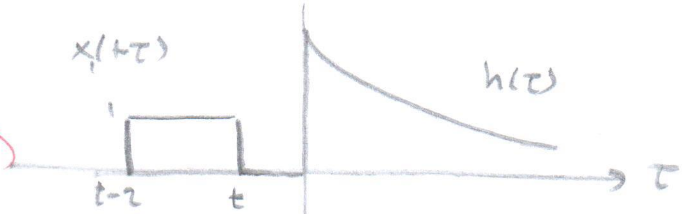


$x_2(t) = rect\left(\frac{t+1}{2}\right)$





$y(t) = h(t) * [x_1(t) + x_2(t)] = h(t) * x_1(t) + h(t) * x_2(t) = y_1(t) + y_2(t)$



5

$* \; t < 0 \Rightarrow y_1(t) = 0$

$* \; t > 0 \;\&\; t-2 < 0 \Rightarrow t < 2$

$0 < t < 2$

$y_1(t) = 2\int_0^t e^{-2(\tau+2)} d\tau = \frac{2}{-2} e^{-2(\tau+2)} \Big|_0^t = -\left[e^{-2(t+2)} - e^{-4} \right]$

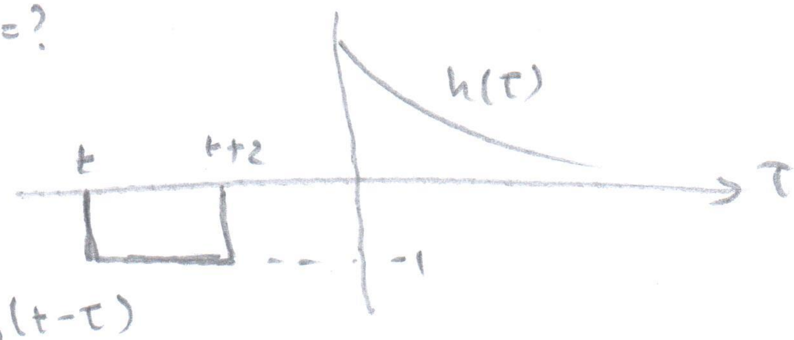
$* \; t > 2$

$y_1(t) = 2\int_{t-2}^{t} e^{-2(\tau+2)} d\tau = -\left[e^{-2(t+2)} - e^{-2(t-2+2)} \right] = -\left[e^{-2t} . e^{-4} - e^{-2t} \right]$

$y_1(t) = e^{-2t}\left[1 - e^{-4} \right]$

$$y_1(t) = \begin{cases} 0, & t < 0 \\ -\left[e^{-2(t+2)} - e^{-4} \right], & 0 < t < 2 \\ e^{-2t}\left[1 - e^{-4}\right], & t > 2 \end{cases}$$

$y_2(t) = ?$



$$x_2(t) = -x_1(t+2)$$

$$\Rightarrow y_2(t) = -x_1(t+2) * h(t) = -y_1(t+2)$$

$$y_2(t) = \begin{cases} 0, & t < -2 \\ +\left(e^{-2(t+4)} - e^{-4}\right), & -2 < t < 0 \\ -e^{-2(t+2)}\left(1 - e^{-4}\right), & t > 0 \end{cases}$$

5

$$y(t) = y_1(t) - y_1(t+2)$$

(c) $$z(t) = x(t) * h_1(t) = x(t) * 4\delta(t+2) = 4x(t+2)$$

$$z(t) = 4x(t+2) = \frac{1}{2}\text{tri}\left(\frac{t-2}{4}\right)$$

5

$$x(t+2) = \frac{1}{8}\text{tri}\left(\frac{t-2}{4}\right)$$

$$\boxed{x(t) = \frac{1}{8}\text{tri}\left(\frac{t-4}{4}\right)}$$

(**25 marks**) Q(3): Assume the periodic signal $x(t)$ with the Fourier series representation

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_o t}$$

is the input to an LTI system described by the differential equation

$$\dot{y}(t) + ay(t) = bx(t-c)$$

Since the system is LTI the output will be periodic with Fourier series representation

$$y(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega_o t}$$

12.5

12.5

(a) Determine an algebraic relationship between α_n and β_n.
(b) Determine the (continuous frequency) transfer function $H(\omega)$ relating the input and output.

(a) $$\sum_{n=-\infty}^{\infty} \beta_n jnw_o e^{jnw_o t} + a \sum_{n=-\infty}^{\infty} \beta_n e^{jnw_o t} = b \sum_{n=-\infty}^{\infty} \alpha_n e^{+jnw_o(t-c)}$$

$$\sum_{n=-\infty}^{\infty} (jnw_o \beta_n + a\beta_n) e^{jnw_o t} = \sum_{n=-\infty}^{\infty} b\alpha_n e^{-jnw_o c} \cdot e^{+jnw_o t}$$

$$\boxed{a\beta_n + jnw_o \beta_n = b\alpha_n e^{-jnw_o c}} \Rightarrow \boxed{\beta_n = \frac{b\alpha_n e^{-jnw_o c}}{a+jnw_o}}$$

(b) $x(t) = e^{jwt}$, $y(t) = e^{jwt} H(w)$

$$\Rightarrow jw e^{jwt} H(w) + a e^{jwt} H(w) = b e^{jw(t-c)} = b e^{-jwc} \cdot e^{jwt}$$

$$H(w)\left[a + jw\right] = b e^{-jwc}$$

$$\boxed{H(w) = \frac{b e^{-jwc}}{(a+jw)}}$$

(25 marks) Q(4): The Fourier coefficients of the periodic signal $x(t)$ with period T_o are given by $c_n = \frac{1}{n^2+1} e^{jn\frac{\pi}{2}}$, then:

(a) **(5 marks)** Is $x(t)$ complex-valued signal? (show your work)

(b) **(5 marks)** Is $x(t)$ even of odd signal? (show your work)

(c) **(5 marks)** Evaluate $\frac{1}{T_o}\int_{-T_o}^{T_o} x(t)\,dt$

(d) **(5 marks)** Find the Fourier series of $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

(e) **(5 marks)** If it is possible to express $x(t)$ in the trigonometric form, then find a_n, and b_n

(a) $c_{-n} = c_n^*$ Real

$$c_{-n} = \frac{1}{(-n)^2+1} e^{-jn\frac{\pi}{2}} \Rightarrow c_{-n}^* = \frac{1}{n^2+1} e^{jn\pi/2} = c_n$$

$\Rightarrow x(t)$ is real.

(b) $c_n = \frac{1}{(n^2+1)}\left[\cos(n\frac{\pi}{2}) + j\sin(n\frac{\pi}{2})\right]$ Neither.

for even $x(t) = x(-t)$ $\Rightarrow c_n = c_{-n}$
for odd $x(t) = -x(-t)$ $\Rightarrow c_n = -c_{-n}$
which does not held here

(c) $$\frac{1}{2T_o}\int_{-T_o}^{T_o} x(t)\,dt = c_0 \Rightarrow \frac{1}{T_o}\int_{-T_o}^{T_o} x(t)\,dt = 2c_0 = 2(1) = 2$$

$$c_0 = \frac{1}{0+1} e^{j0} = 1$$

(d) $c_0 = 1 \Rightarrow$ DC value $\neq 0 \Rightarrow y(t)$ Not periodic
(Fourier series does Not exist).

(e) $x(t)$ real

$$\Rightarrow c_n = \frac{a_n}{2} - j\frac{b_n}{2} = \frac{1}{(n^2+1)}\cos(n\frac{\pi}{2}) + j\frac{1}{(n^2+1)}\sin(n\frac{\pi}{2})$$

$$\Rightarrow a_n = \frac{2}{(n^2+1)}\cos(n\frac{\pi}{2})$$

$$b_n = -\frac{2}{(n^2+1)}\sin(n\frac{\pi}{2})$$

Good Luck