Signals Analysis and Systems.

Second Semester (2017)

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※ Signal Definition:

⇒ it is something give us information.

→ Mathematical Definition: it is a function of one or more variables.

$y = f(x)$.

dependent variable.

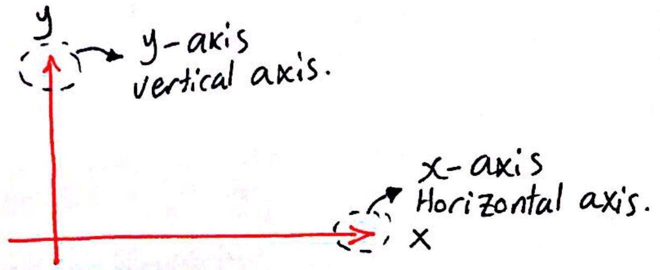
independent variable.

also the function could be:

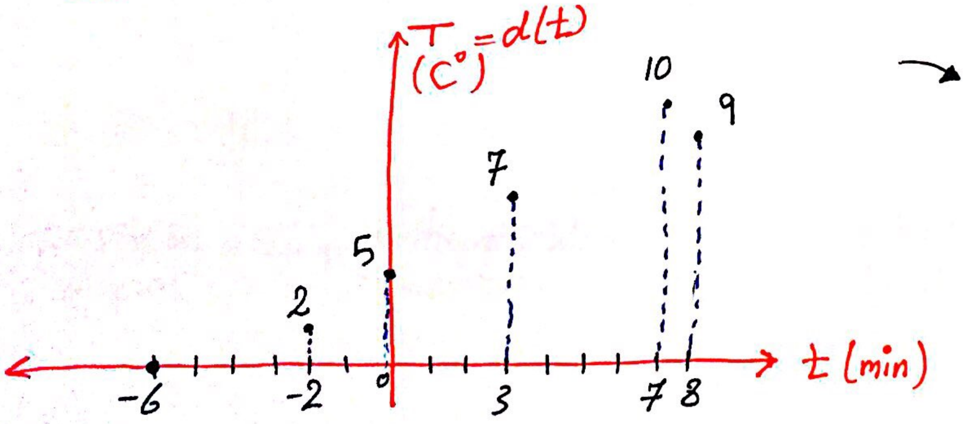
$g = f(x, y)$

$\ell = f(x, y, z)$ → This called static function.

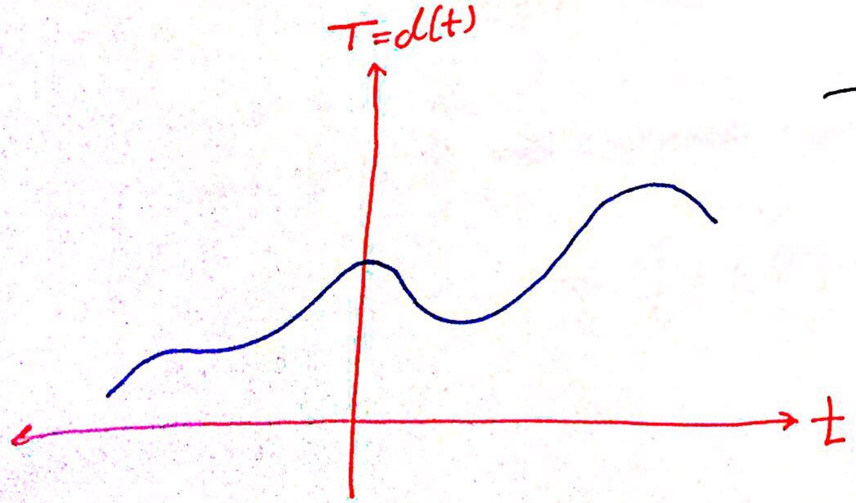
$d = f(x, y, z, t)$ → This called dynamic function.



Ex. Temp. vs. Time.



→ This Called:
"Discrete-Time"
(DT)



→ This Called:
"Continuous-Time"
(CT)

* Another Types of signals:

- voltage, current, electric field ⟹ Called electric signal.
- Speed ⟹ Called mechanical signal.
- Traffic Light.

※ System Definition:

⟹ process or iterational operations for input signal To give output signal with different proparties.

Inputs Called: [Excitation Signal.]

outputs called: [Response Signal.]

※ Modeling:

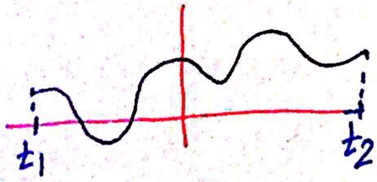
physical signal → Mathematical Function.

physical system → Mathematical Equations.

※ Continuous Time & Discrete Time:

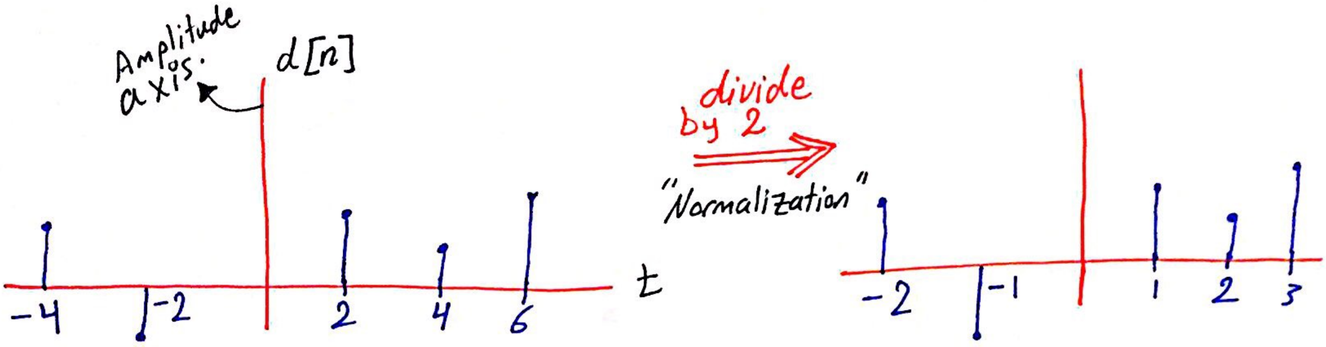
$DT \longrightarrow t \in \overline{Z}$

$CT \longrightarrow t \in \overline{R}$ ⟹ continuous time in interval $t_1 < t < t_2$



$$\Rightarrow \{ t \in R \mid t_1 < t < t_2 \}$$

Note: the continuous time is continuous even if the function was discontinuous or the function has a discontinuoty at some points.



* we can do this division if the reads were uniform.

$\Rightarrow \{n \in \mathbb{Z} ; -2 \leq n \leq 3\} = \{-2, -1, 0, 1, 2, 3\}$

* Analog & Digital:

* finite set is always countable.
* infinite set could be countable or uncountable.

$\{-2, -1, 0, 1, 2, ----\}$ ⇒ infinite but it is countable.

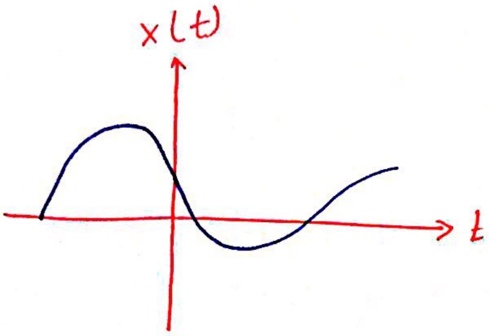
$\{\cdots, 1, 10, 20, 31, \cdots\}$ ⇒ infinite & countable.

$\{x \in R : -2 < x \leq 5\}$ ⇒ infinite since it is uncountable.

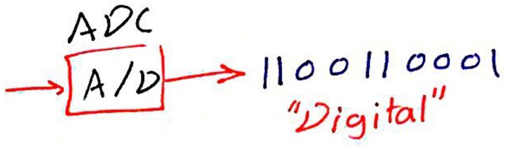
* if the amplitude → infinite → Analog.
→ finite → Digital.

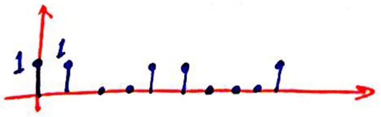
* **Note:** we determine that the signal discrete or continuous from the Horizontal axis, we determine that the signal Analog or digital from the vertical axis.

Ex.



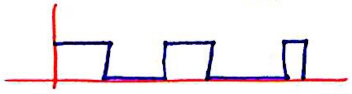
it is a Hyprid system
CT → DT.



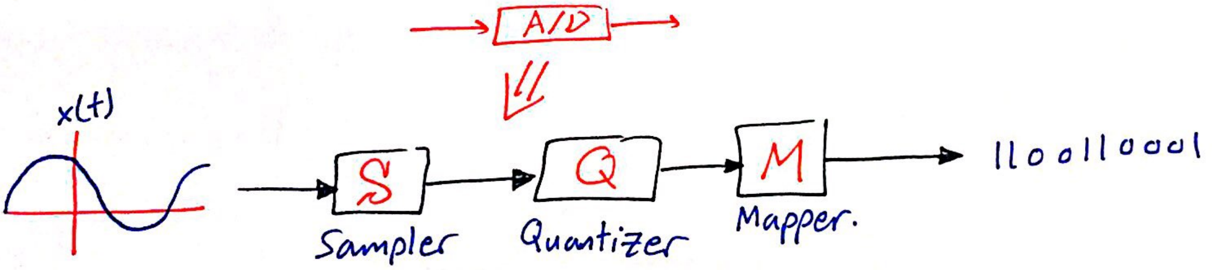


DT & Digital

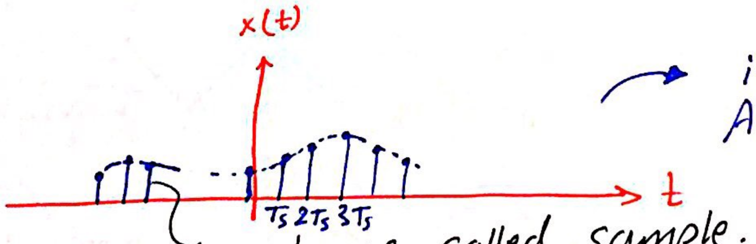
also it could be



CT & Digital.



* Sampler:



it is still Analog signal.

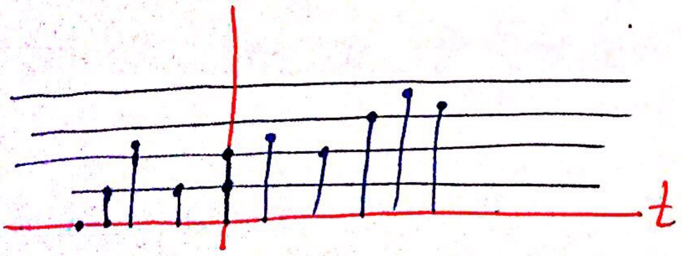
each one called sample.

$T_s \equiv$ Sampling Time.

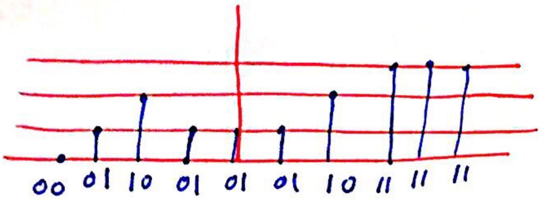
* * Quantizer:

⇒ it will convert from A → D.

Taking
4
levels.



→ we do approximate
take the floor.



→ Now this signal DT & Digital.

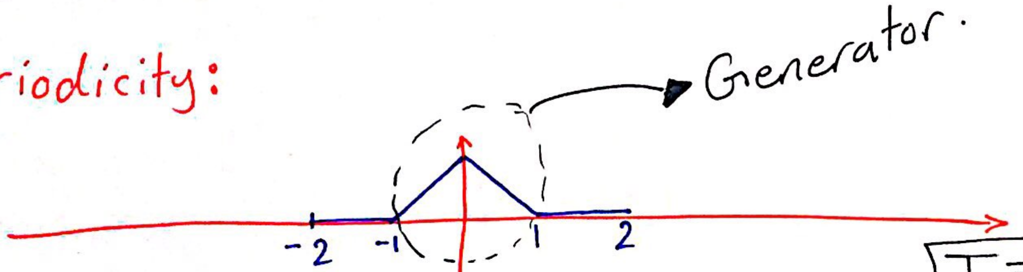
* **Note:** There will be error in the converting This error called quantization error "noise error" we decrease it by increasing # of Levels.

*** Mapper:

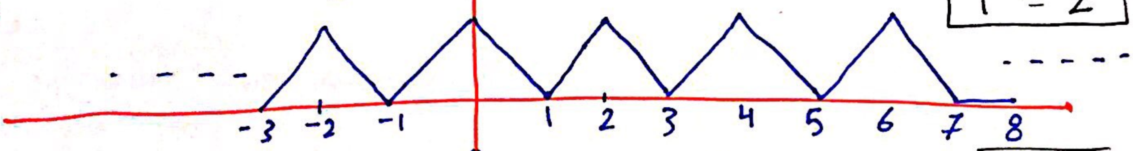
L_1	00
L_2	01
L_3	10
L_4	11

* Periodicity:

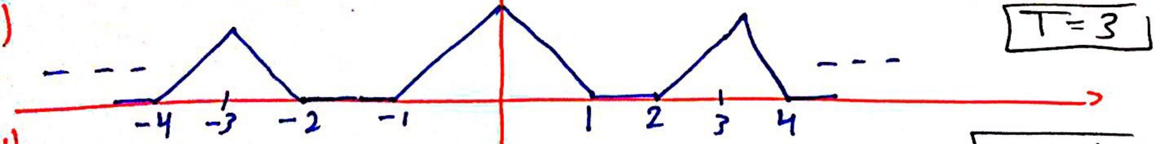
figure(1)



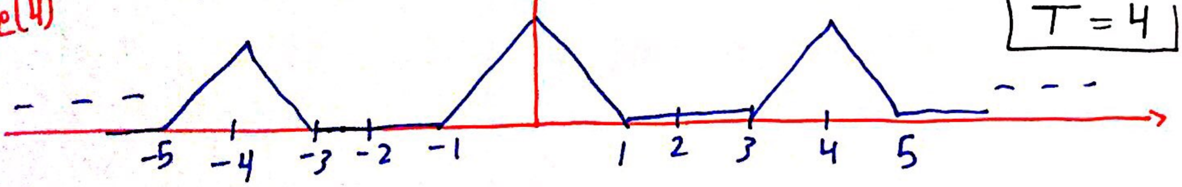
figure(2)



figure(3)



figure(4)



→ it is called Time unlimited signal,
if it was limited ⇒ Time limited signal.

* The signal could repeat itself every $1*T$ or $2*T$ or $3*T$ ---- or $K*T$; $K \in \mathbb{Z}$

⇒ so in figure(2): the smallest value $T_o = 2$

$T_o \equiv$ fundemental period.

$$x(t) = x(t+T) = x(t+2T) = \cdots$$

prove:

$$x(\underset{t+T}{t}) = x(\underset{t+T}{t}+T) \Rightarrow x(t+T) = x(t+2T)$$

and so on ....

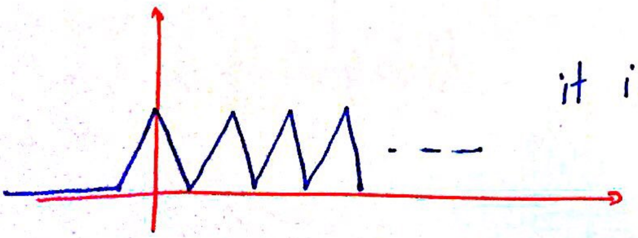
$$\Rightarrow \boxed{x(t) = x(t+KT_o)} \quad K \in \mathbb{Z}$$

Ex.

$T_o = 2 msec.$

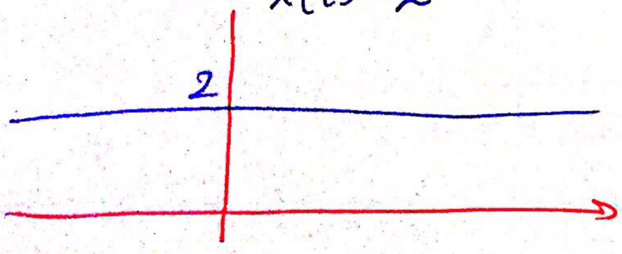
$$\Rightarrow f_o = \frac{1}{T_o} = \frac{1}{2m} \Rightarrow \underline{\underline{f_o = 500 Hz}}$$

⇒ fundemental angular freq. (ω_o) $\Rightarrow \omega_o = 2\pi f_o = \frac{2\pi}{T_o} \Rightarrow \omega_o = 1000\pi = \underline{\underline{3140}}$ rad/sec.



it is Not periodic.

$x(t) = 2$



DC signal

$$\boxed{\begin{matrix} T_o = \infty \\ f_o = 0 \end{matrix}}$$

* sinusoidal in CT:

$$x(t) = \left.\begin{matrix} \sin(\omega_0 t+\theta) \\ \cos(\omega_0 t+\phi) \end{matrix}\right] \rightarrow \text{always periodis signals.}$$

Ex. $\cos(t+\frac{\pi}{3}) \Rightarrow \omega_0 = 1 = \frac{2\pi}{T_0}$ so $\boxed{T_0 = 2\pi}$

* sinusoidal in DT:

$$x[n] = \left.\begin{matrix} \cos(\Omega n+\phi) \\ \sin(\Omega n+\phi) \end{matrix}\right] \rightarrow \text{Not always periodic}$$

"discussed Later"

* Is $x(t) = e^{\sin(t)}$ a periodic signal ?

$$x(t) \overset{?}{=} x(t+KT_0) \Rightarrow e^{\sin t} \overset{?}{=} e^{\sin(t+KT_0)}$$

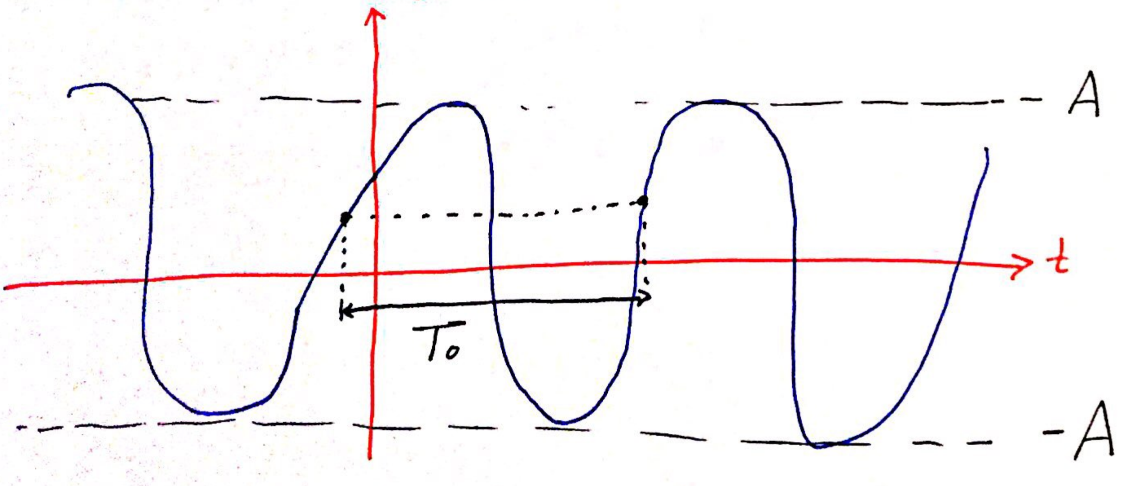
This true if: $\sin t = \sin(t+KT_0)$

since the sine periodic signal so $x(t) = e^{\sin(t)}$ is a periodic signal with $T_0 = 2\pi$.

$$\sin(t) \Rightarrow \omega_0 = 1 \Rightarrow \underline{\underline{T_0 = 2\pi}}$$

Ex. $A \begin{matrix}\cos \\ \sin\end{matrix} (\omega_0 t+\theta)$

$x(t) = A\sin(\omega_0 t+\theta)$



※ Addition & subtraction:

① Non-periodic ± Non-periodic ≡ Non-Periodic

② Non-periodic ± periodic ≡ Non-Periodic

③ Periodic ± DC ≡ Periodic

④ Periodic (T_0) ± Periodic (T_0) ≡ Periodic (T_0)

⑤ Periodic (T_1) ± Periodic (T_2) ≡ ? → periodic → if all the ratios were rational.

↓ aperiodic (Non-periodic)

↓ if it is periodic The period T_0 will be:

$$\boxed{T_0 = T_1 * LCM\begin{pmatrix}\text{denominator}\\ \text{of}\\ \text{the}\\ \text{ratioes}\end{pmatrix}}$$

* rational number $\equiv \frac{\text{integer}}{\text{integer}} = \frac{K}{L}$

$\hookrightarrow K, L \in \mathbb{Z}$

* irrational number: $\pi, e^1, e^2, \sqrt{2}, \ldots$

$$\underset{\downarrow \atop T_0}{x(t)} = \underset{\downarrow \atop T_1}{x_1(t)} \pm \underset{\downarrow \atop T_2}{x_2(t)} \pm \underset{\downarrow \atop T_3}{x_3(t)}$$

⇒ Ratio: $\frac{T_1}{T_2}$ & $\frac{T_1}{T_3}$

⇒ Number of ratioes = Number of periods − 1

※ LCM:

Ex. find LCM (90, 155, 75, 80) ?

90 : 2 * 3 * 3 * 5

155 : 5 * 31

75 : 3 * 5 * 5

80 : 2 * 2 * 2 * 2 * 5

بنعد أكثر مكان تكررت فيه ال 2

$LCM(90, 155, 75, 80) = 2^4 * 3^2 * 5^2 * 31^1$

Ex. if $x(t) = 2\cos(3.5t+30^\circ) - 5\sin(2t-\frac{\pi}{2}) + 2\cos(\frac{7}{6}t)$ then determine if x(t) periodic or Not and find T_0 if it is periodic ?

$\omega_1 = 3.5 = \frac{2\pi}{T_1} \Rightarrow \boxed{T_1 = \frac{4\pi}{7}}$

$\omega_2 = 2 = \frac{2\pi}{T_2} \Rightarrow \boxed{T_2 = \pi}$

$\omega_3 = \frac{7}{6} = \frac{2\pi}{T_3} \Rightarrow \boxed{T_3 = \frac{12\pi}{7}}$

since we have $T_1, T_2, T_3 \Rightarrow$ we have two ratios $\frac{T_1}{T_2}$ & $\frac{T_1}{T_3}$

$\frac{T_1}{T_2} = \frac{4\pi}{7} * \frac{1}{\pi} = \frac{4}{7}$ (rational)

$\frac{T_1}{T_3} = \frac{4\pi}{7} * \frac{7}{12\pi} = \frac{1}{3}$ (rational)

$\Rightarrow$ so it is periodic.

$\Rightarrow \frac{4}{⑦}, \frac{1}{③} \Rightarrow T_0 = T_1 * LCM(7,3)$

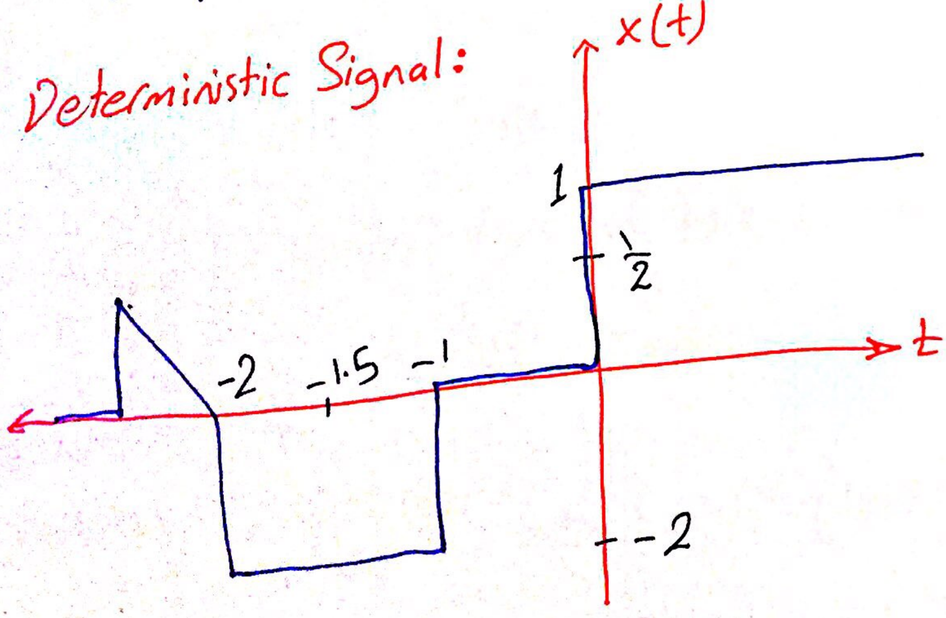
$= \frac{4\pi}{7} * (3*7) \Rightarrow \boxed{T_0 = 12\pi \text{ sec}}$

Note:
the ratio must be in the simplest form.

* Try for: $x_1(t) * x_2(t)$

```
>> t= -10:0.01:10;
>> x1 =5*cos(2*t+pi/2);
>> x2 = 2 * sin(pi*t - pi/3);
>> plot(    )
```

※ Deterministic Signal:



$x(2) = 1$
$x(-1.5) = -2$

Ex. $x(t) = e^{-2t^2}\cos(5t) + 1$

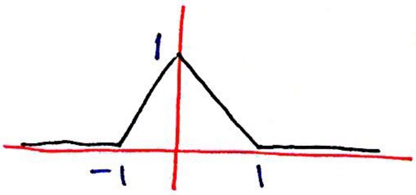
$\Downarrow$

it is Deterministic Signal.

$$tri(t) = \begin{cases} 1-|t|, & |t|<1 \\ 0, & \text{otherwise.} \end{cases}$$

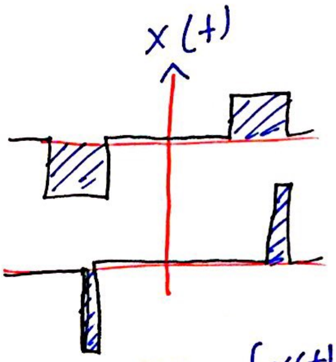
$\downarrow$

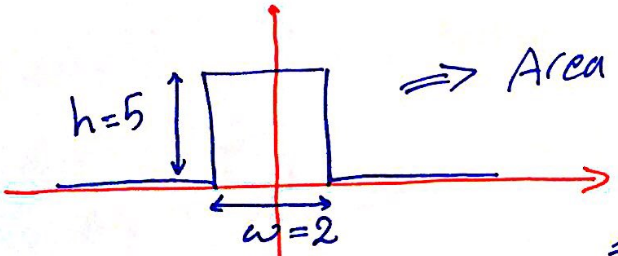
Δ



it is the unit triangular.

※ Energy vs. Power:

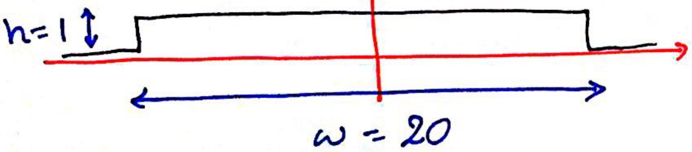


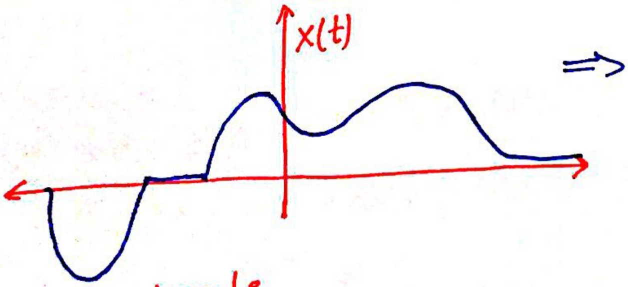


$\Rightarrow$ Area $= h*W$
$= \underline{\underline{10}}$

Area $= \int x(t)dt$
$=$ Zero.

$\Rightarrow$ $A = 1*20 = \underline{\underline{20}}$





$\Rightarrow$ Area $= \int |x(t)|dt.$

* for CT signal:

$$E_T = \int_{-\infty}^{\infty} x^2(t)dt \text{ Joule.}$$

$\Rightarrow$ for a general form if there is complex number:

$$E_T = \int_{-\infty}^{\infty} |x(t)|^2 dt \text{ Joule.}$$

i.e: $x(t) = \ln(t^2) + j\cos(\frac{t}{5})$ $\Longrightarrow$ $|x(t)|^2 = (\ln(t^2))^2 + (\cos(\frac{t}{5}))^2$

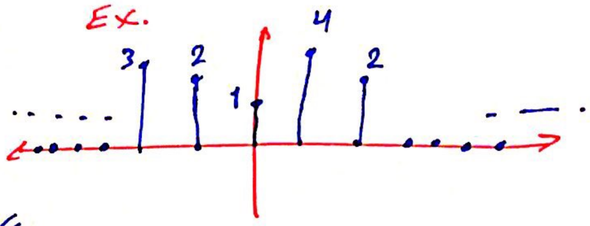
$Re\{x(t)\} = \ln(t^2)$

$Im\{x(t)\} = \cos(\frac{t}{5})$

* for DT signal:

$$E_T = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Ex.

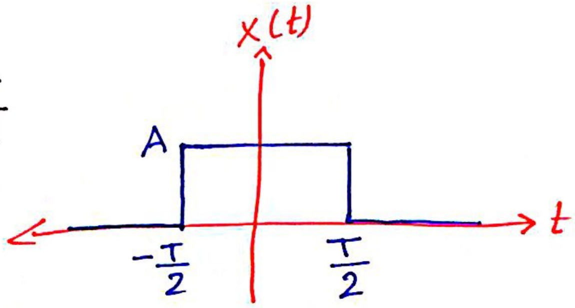


⇙

$$E_T = 3^2 + 2^2 + 1^2 + 4^2 + 2^2$$

Ex. for CT signal:

$$x(t) = \begin{cases} A, & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{otherwise.} \end{cases}$$



$$E_T = \int_{-\infty}^{\infty} x^2(t)\,dt$$

$$= \cancelto{0}{\int_{-\infty}^{-\frac{T}{2}} (0)^2 dt} + \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt + \cancelto{0}{\int_{-\frac{T}{2}}^{\infty} (0)^2 dt} = \boxed{A^2 T}$$

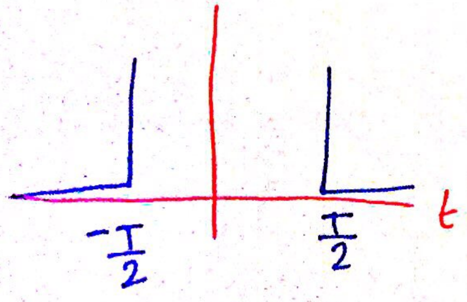
Joule.

* we evaluate the energy at first if it is finite we said energy signal.

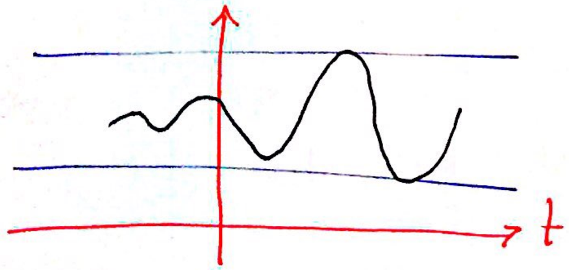
* if it is infinite ⇒ we calculate the power.

infinite Neither.

finite ⇒ power.



unbounded

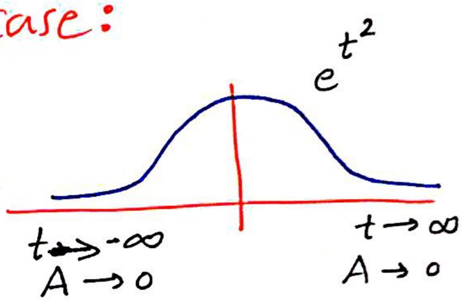


Bounded.

Bounded → Energy, power.
unbounded → Neither.

Bounded
- Time limited → Energy.
- Time unlimited → power.

*But there is special case:

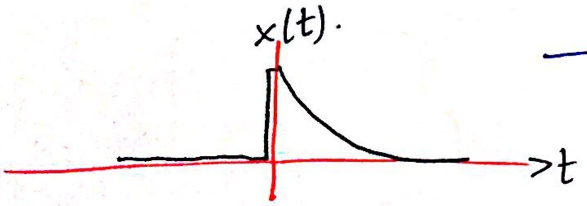


it is unlimited But it is Energy

under the condition:

$$\lim_{|t| \to \infty} x(t) = \text{Zero}$$

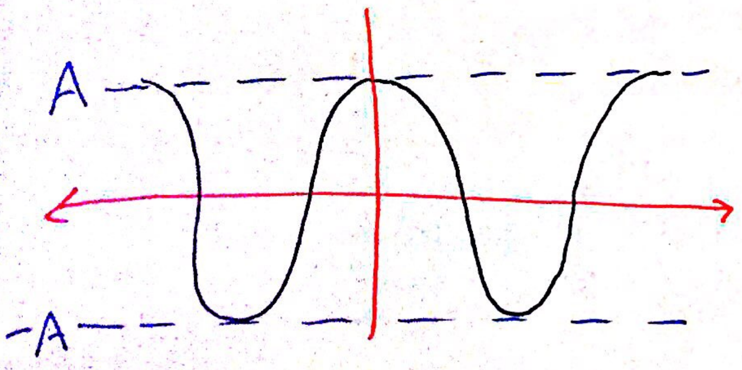
* Another Ex.:



→ unlimited time But it is Energy

since: $\lim_{|t| \to \infty} x(t) = \text{Zero}.$

Ex. $x(t) = A\cos(2\pi f_0 t)$

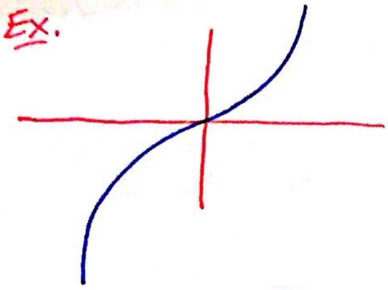


*it is Bounded.
⇒ & unlimited time.

but $\lim_{|t| \to 0} x(t) \neq 0$

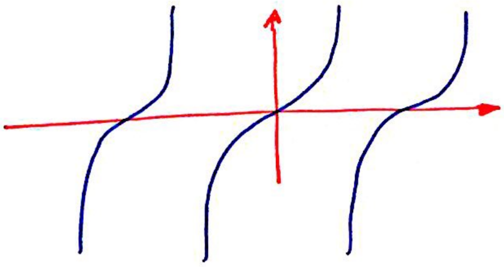
So it is Power signal.

Ex.



This unbounded so it is Neither.

⇒

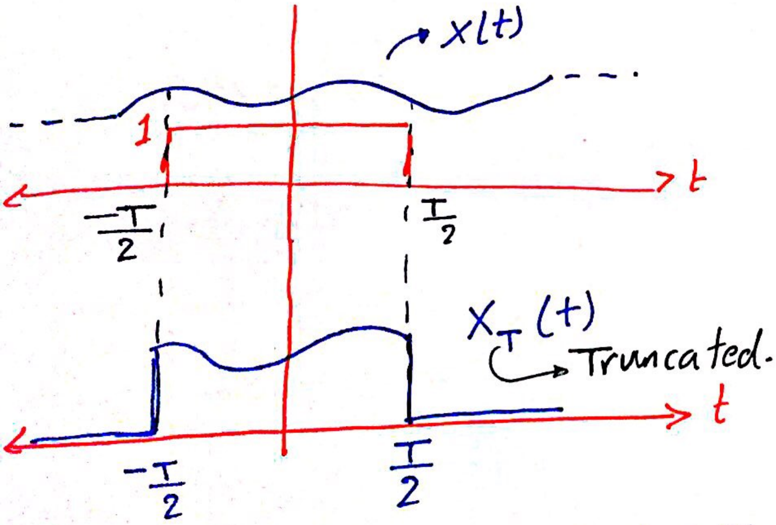


the same as above.

※ Power:

$$P = \frac{\Delta E}{\Delta t} \text{ Joule/sec} \Rightarrow [\text{Watt}].$$

$$= \frac{d\,e(t)}{\partial t}.$$



this called windowing OR truncation.

$$x_T(t) = \begin{cases} x(t), & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{P_T = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} |x(t)|^2 dt}$$

↳ Average Power.

we could use the interval

$-T < t < T$

↳ The difference will be

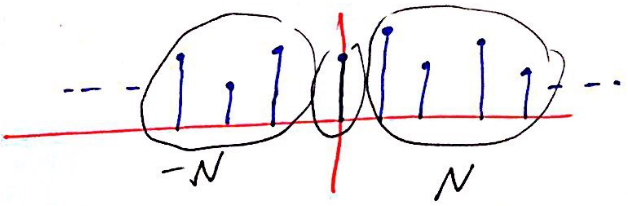
$$P_T = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

"the same"

for DT :

$$P_T = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2 \simeq \lim_{N\to\infty}\frac{1}{2N}\sum_{n=-N}^{N}|x[n]|^2$$

or we could take N-1 .



** mean (average) :

$$\Rightarrow \text{mean} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|\,dt$$ for power.

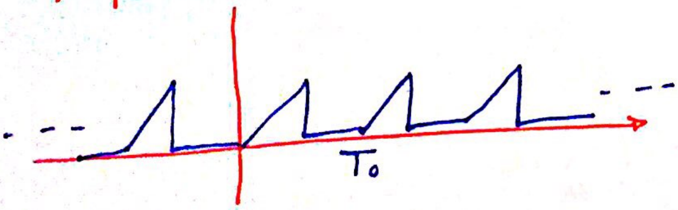
↳ we square it then take the root ⇒ RMS .

$$* \quad RMS = \sqrt{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt}$$

↳ P_T.

$$\boxed{RMS = \sqrt{P}}$$

* periodic CT :

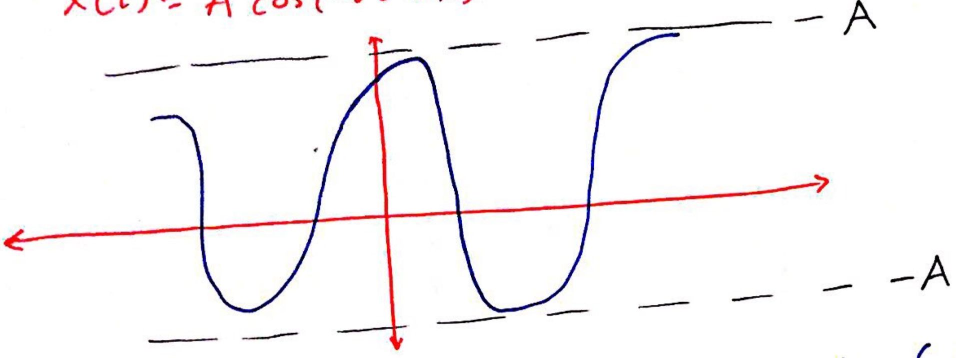


$$P = \frac{1}{T_0}\int_{T_0}|x(t)|^2\,dt$$

* periodic DT :

$$P = \frac{1}{N_0}\sum_{N_0}|x[n]|^2$$

$x(t) = A\cos(\omega_0 t + \emptyset)$



$$P = \frac{1}{T_0}\int_{T_0}\left[A\cos(\omega_0 t+\emptyset)\right]^2 dt = \frac{1}{T_0}\int_{T_0} A^2\cos^2(\omega_0 t+\emptyset)\,dt.$$

$$= \ldots \Rightarrow RMS = \frac{A}{\sqrt{2}}$$

so $P = \frac{A^2}{2}$

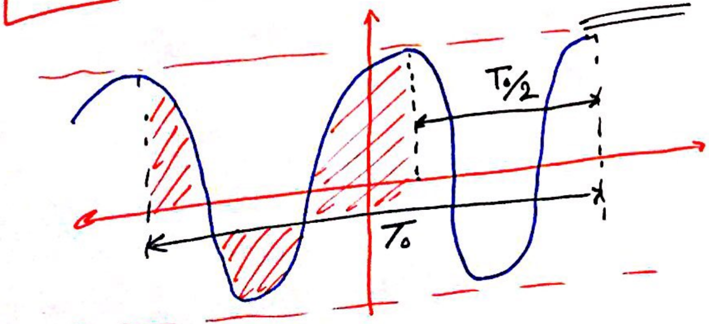
$$= \frac{A^2}{T_0}\int_{T_0}\frac{1}{2}\left(1+\cos(2\omega_0 + 2\emptyset)\right)dt$$

$$= \frac{A^2}{2T_0}\left[T_0 + \underbrace{\int_{T_0}\cos(\underbrace{2\omega_0}_{\omega=2\omega_0} t+2\emptyset)\,dt}_{0}\right]$$

$$= \boxed{\frac{A^2}{2}}$$

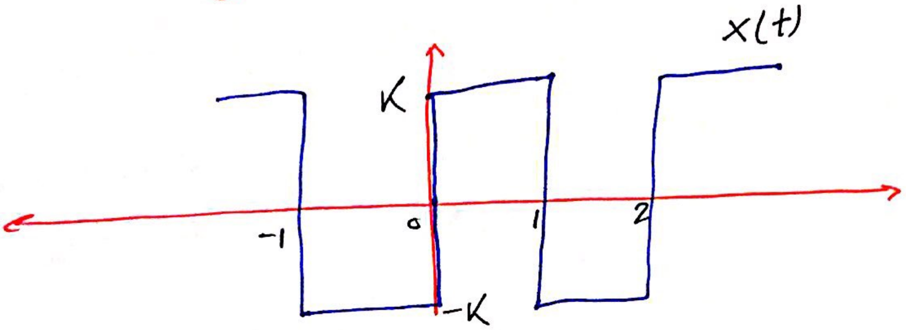
$\frac{2\pi}{T} = 2\ \frac{2\pi}{T_0}$

$T = \frac{T_0}{2}$



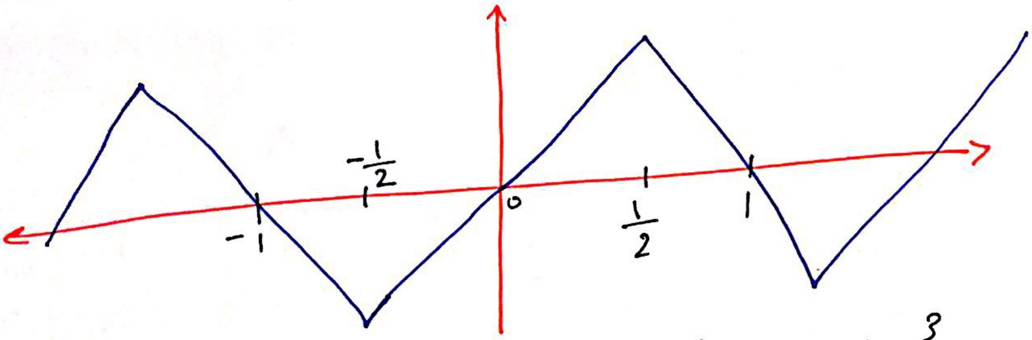
+ve & -ve cancel eachother so it will give Zero.

* TTL signal:



$$P = \frac{1}{2}\int_{2} x^2(t)\,dt$$

$$= \frac{1}{2}\left[\int_0^1 K^2 dt + \int_{-1}^{0} (-K)^2 dt\right] = \underline{\underline{K^2}}$$



* choose the best periode $-\frac{1}{2} < t < \frac{3}{2}$

Then find the power $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots + \int_{\frac{1}{2}}^{3/2} \cdots$

* The power for an Energy signal is zero.

* The Energy for a power signal is infinite.

* * Very Important:

$$\frac{E}{P} \propto M^2 \Rightarrow \frac{E}{P} = \underbrace{C}_{\text{Constant}}\,\underbrace{M^2}_{\text{Magnitude}}$$

$Z(t) = X(t) + y(t)$

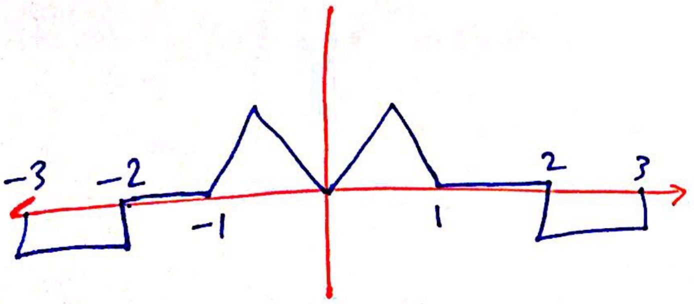
$\Rightarrow P_Z \neq P_X + P_y$

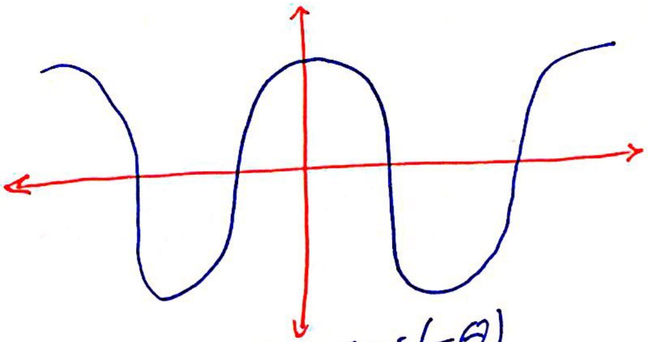
$P_Z = P_X + P_y$ just if $x(t) \perp y(t)$ and they must have different frequancy.

※ Signal Symmetry:

Even Signal:

$x(t) = x(-t)$



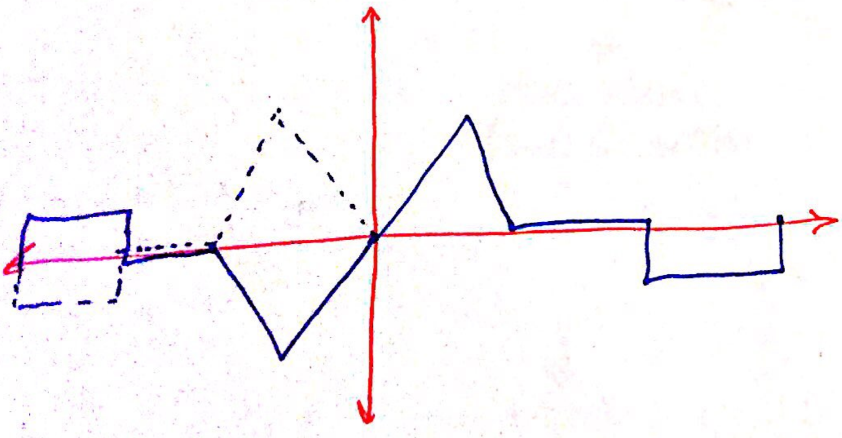


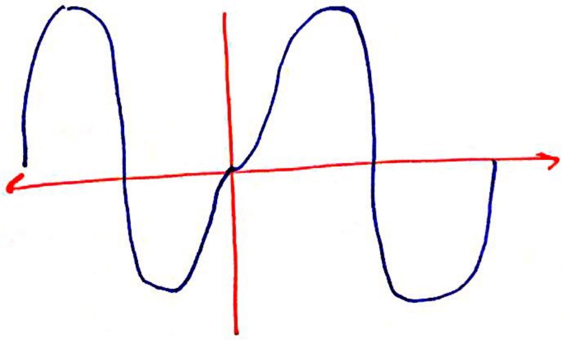
$\cos\theta = \cos(-\theta)$

$\cos(\omega_0 t) = \cos(-\omega_0 t)$

Odd signal:

$x(t) = -x(-t)$





$\sin\theta = -\sin(-\theta)$

$\sin(\omega_0 t) = -\sin(-\omega_0 t)$

**

$$x(t) = \underbrace{x_e(t)}_{\text{even}} + \underbrace{x_o(t)}_{\text{odd}} \quad ---(1)$$

$$x_e(t) = x_e(-t)$$
$$-x_o(t) = x_o(-t)$$
$$\Rightarrow x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t) \quad ---(2)$$

Add ① to ②:

$$\boxed{x_e(t) = \frac{x(t)+x(-t)}{2}}$$

sub. ② from ①:

$$\boxed{x_o(t) = \frac{x(t)-x(-t)}{2}}$$

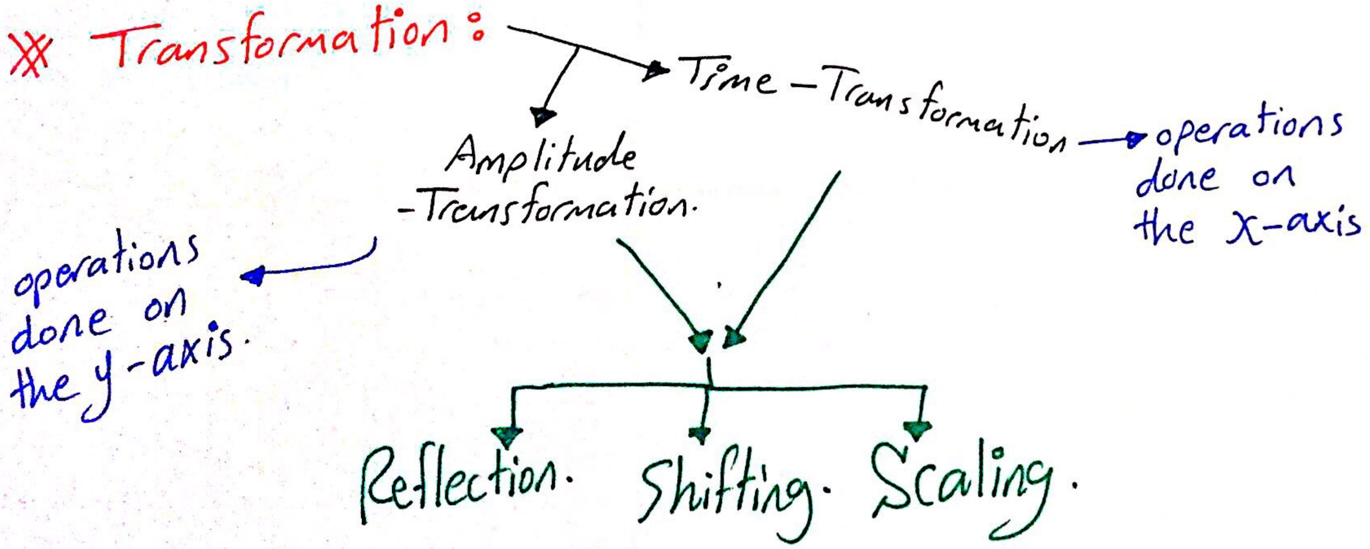
* for calculations:

$$x(t) = e^{-t} + \sin(5t)$$
$$\Rightarrow x(-t) = e^{t} + \sin(-5t) = e^{t} - \sin 5t$$
$$\Rightarrow x_e(t) = \frac{e^{-t}+e^{t}}{2}$$

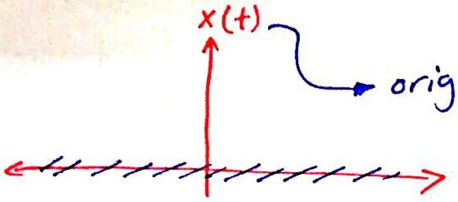
* for graphical:

discussed in the coming pages.

※ Transformation:



* Time - Transformation:



⇒ $y(t) = x(at+b)$

↳ new signal.

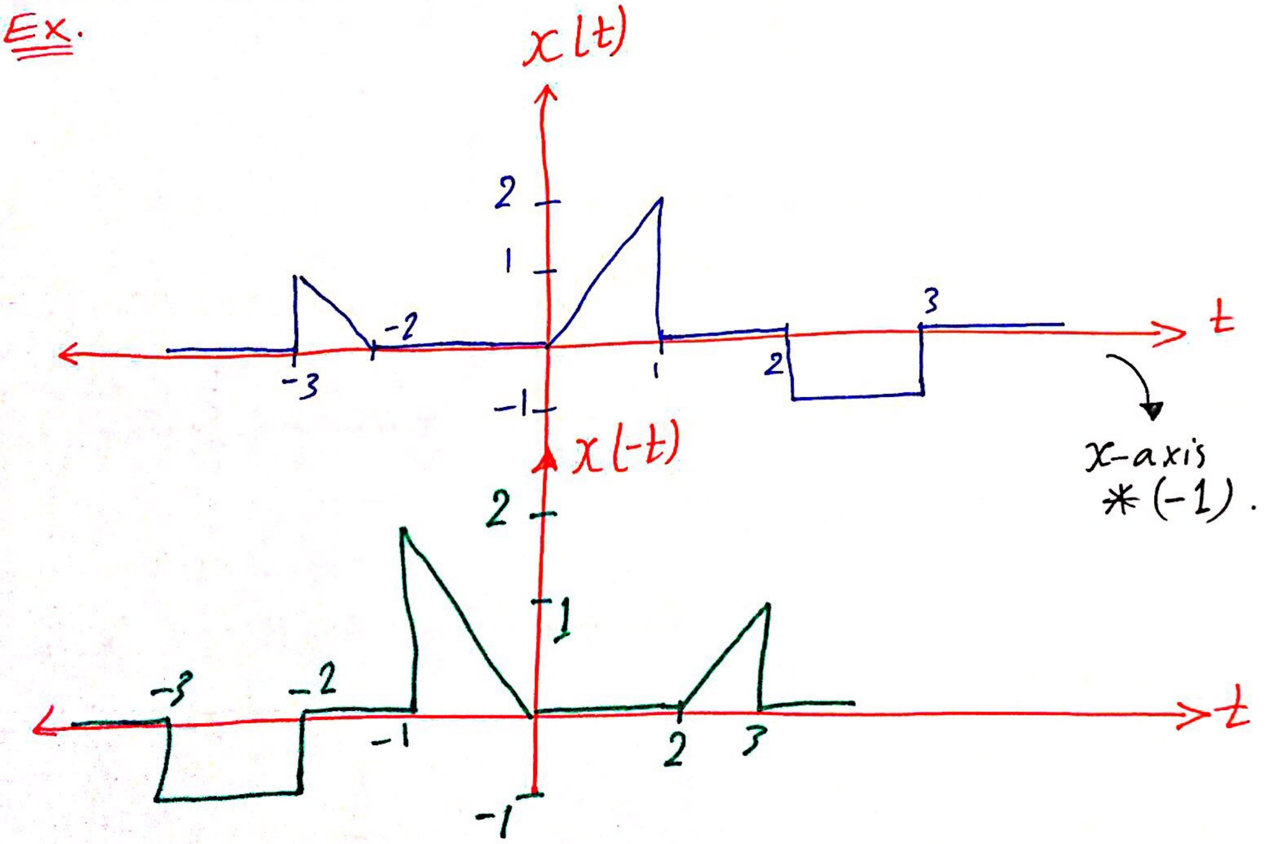
* Time Reflection: $a = -1 , b = 0$

⇒ it is also called [flipping/Reversal].

$y(t) = x(-t)$

↳ time reversal version of x.

Ex.



* Time Shifting: $a=1$, $b=-t_0$

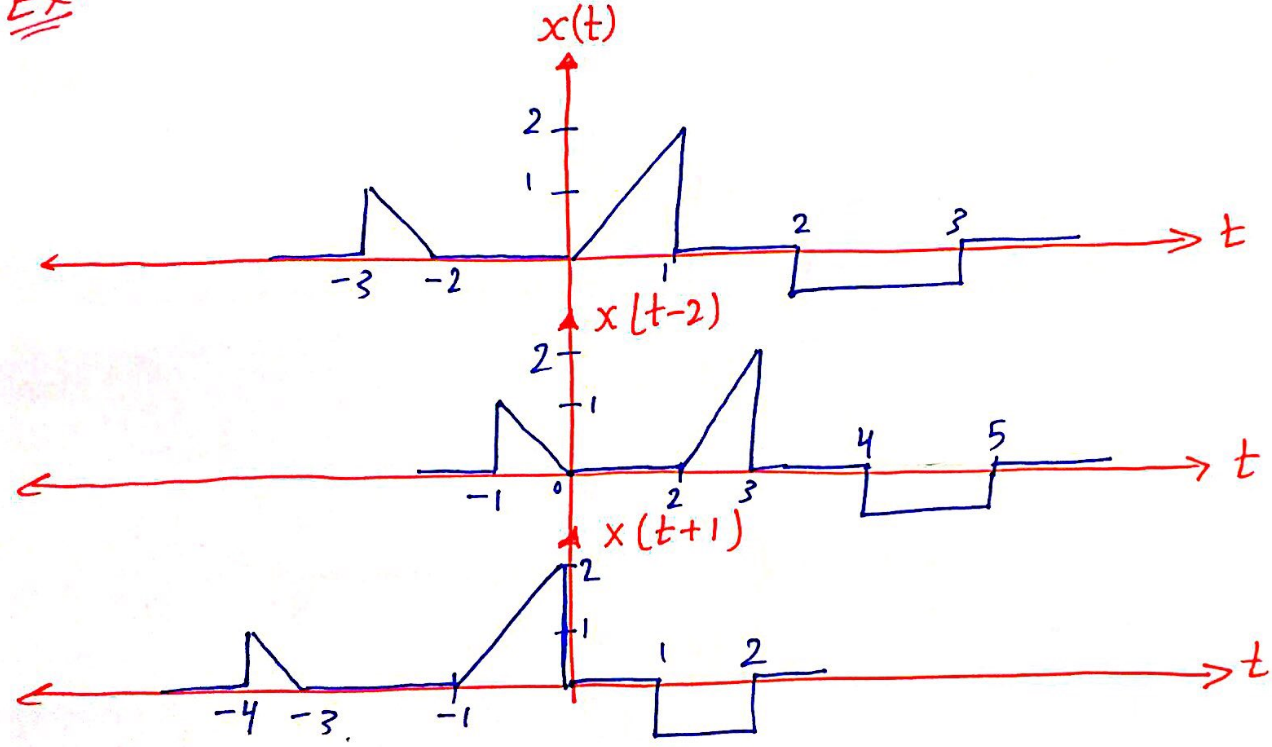
$y(t) = x(t-t_0)$

$\rightarrow t_0 \in \mathbb{R}$

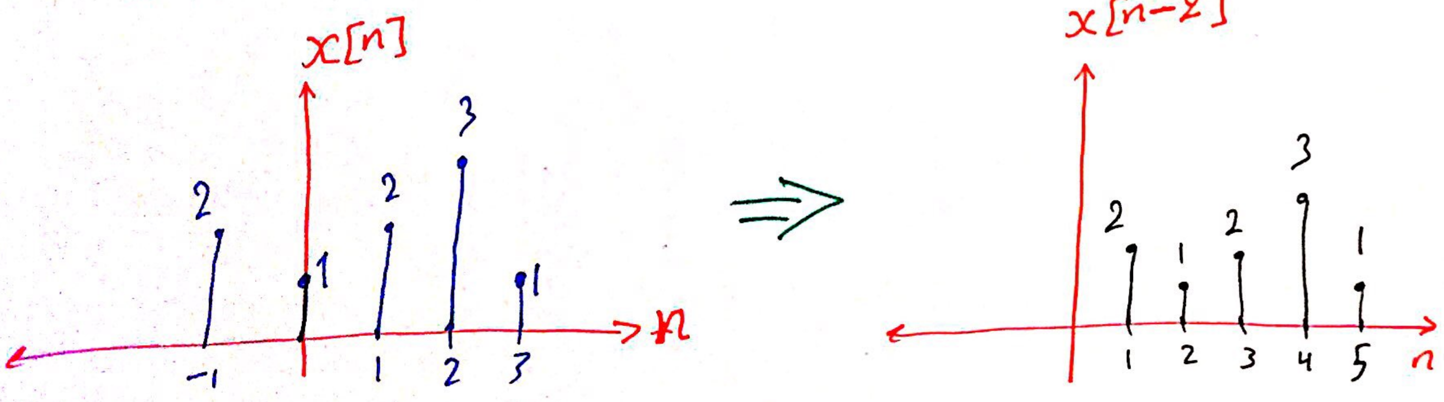
$\rightarrow t_0 \rightarrow +ve$ "Delay".

$\rightarrow -ve$ "Advance".

Ex:



* if we have DT signal:



* Time Scaling: $b=0$

$y(t) = x(at)$

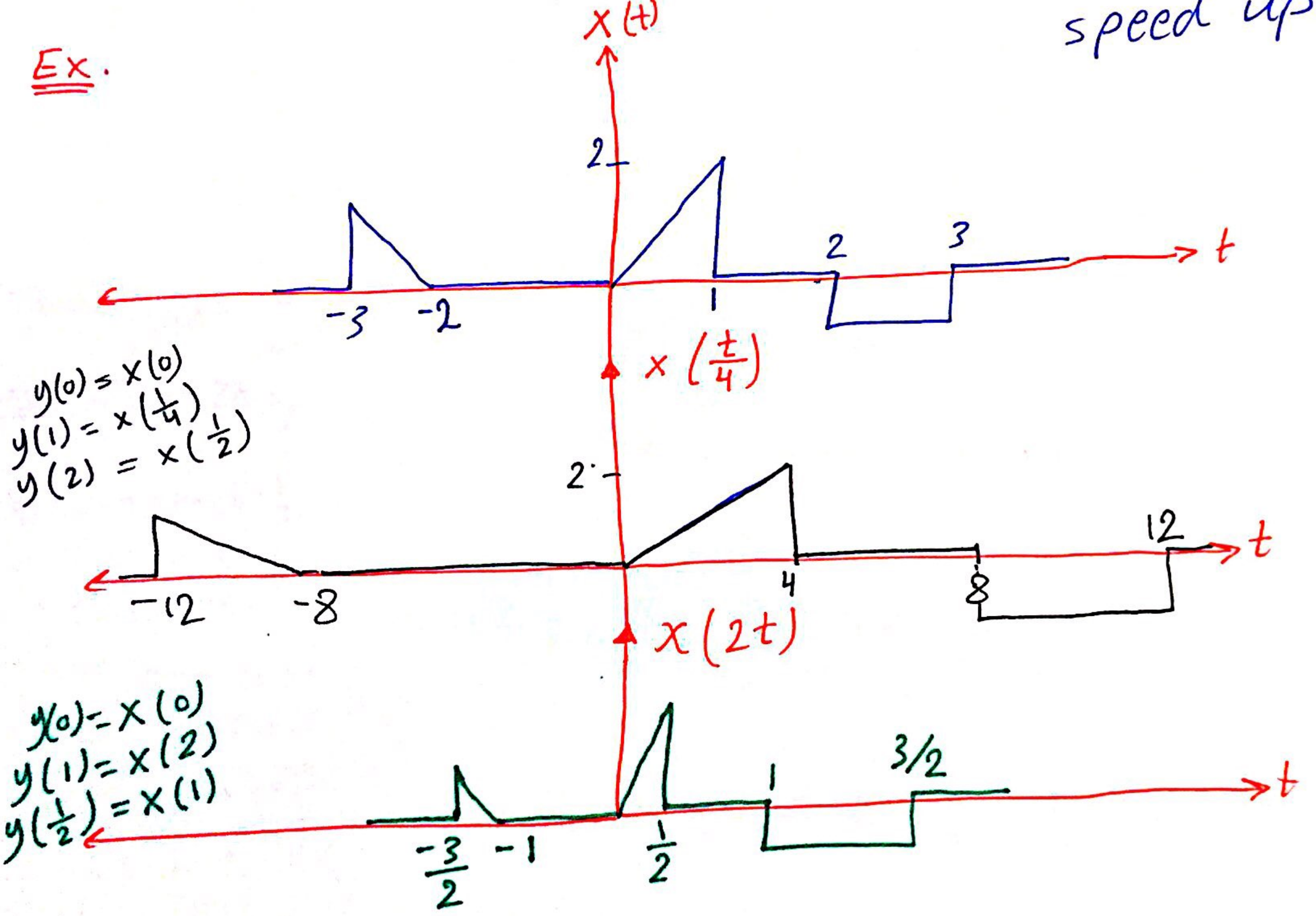
$\hookrightarrow a \in R^{+}$

Two cases:

$0<a<1 \rightarrow$ Expanssion.
streching.
slow down.

$a>1 \rightarrow$ Compression.
speed up.

Ex.



$y(0) = x(0)$
$y(1) = x(\frac{1}{4})$
$y(2) = x(\frac{1}{2})$

$y(0) = x(0)$
$y(1) = x(2)$
$y(\frac{1}{2}) = x(1)$

$*$

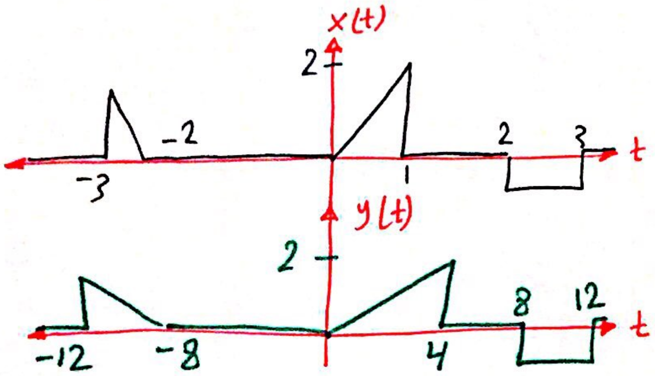
$0<a<1 \Rightarrow$ ex $y(\frac{1}{4}t)$ $\Rightarrow$ $\underline{\underline{b>a}}$.

$a = \frac{1}{4}$

$\boxed{b = \frac{1}{a}} = 4$

$y(at) = y(\frac{t}{b})$

Ex. given the figure: → find $y(t)$ in term of $x(t)$?



Solution:

No change in the amplitude & it is look like an expanssion

⇒ Time - shifting.

$y(t) = x(at+b)$

$\Rightarrow y(t) = x\left(\frac{t}{4}\right)$

Ex. Draw $y(t) = x(-2t+1)$

$\tau = -2t + 1$ (τ → old t, t → new t)

@ $\tau = 0 \longrightarrow t = \frac{1}{2}$

$\tau = 1 \longrightarrow t = 0$

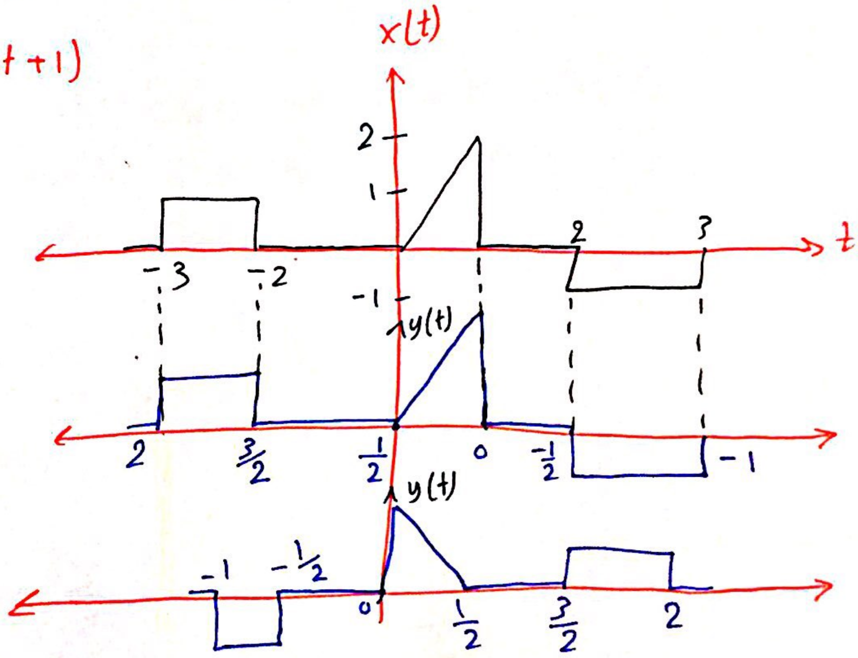
$\tau = 2 \longrightarrow t = -\frac{1}{2}$

$\tau = 3 \longrightarrow t = -1$

$\tau = -1 \longrightarrow t = 1$

$\tau = -2 \longrightarrow t = \frac{3}{2}$

$\tau = -3 \longrightarrow t = 2$



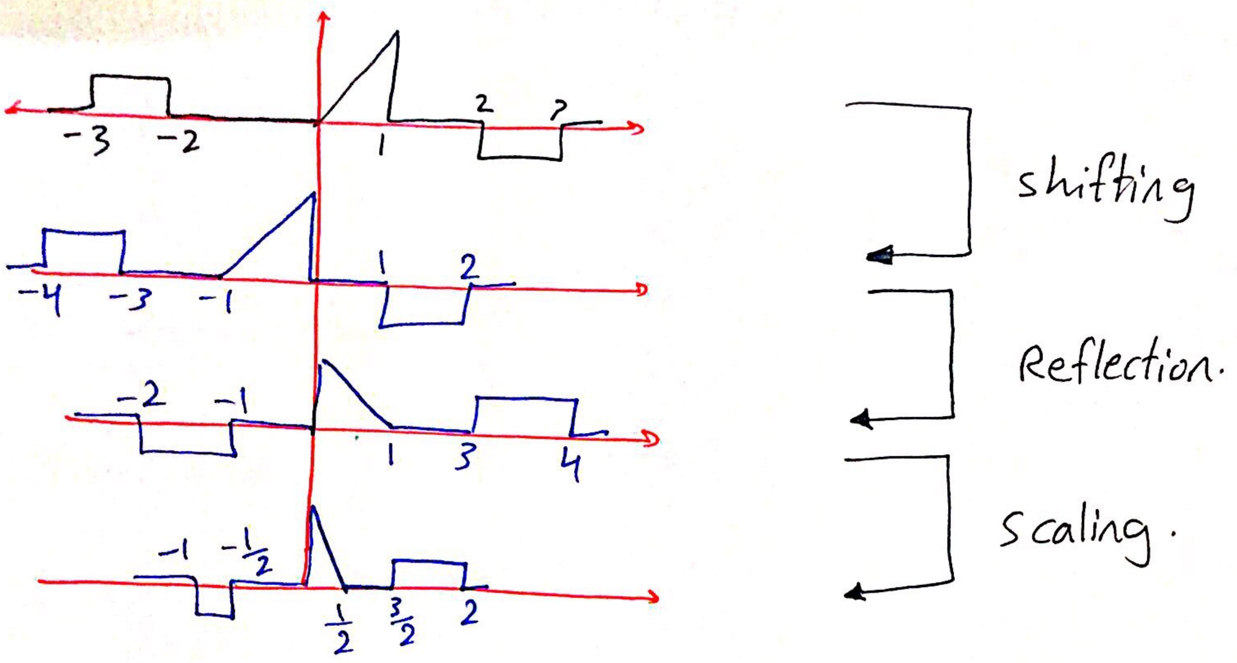
* Another method:

we could solve step by step.

start with shifting → Then Reflection → Then scaling.

* most important start with shift the do the others.

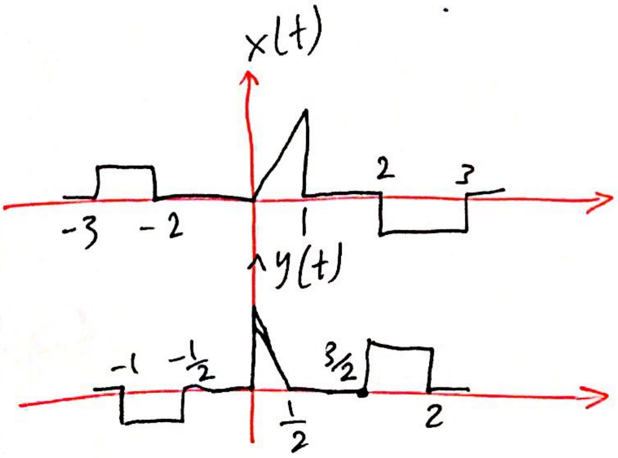
⇒



* if $x\left(\frac{t+c}{d}\right)$ ⇒ here we could start with scaling (d) then do the shift.

※ Now finding $y(t)$ in term of x:

put $y(t) = x(at+b)$



we start with the scaling.

there is a (-) & there is a compression ⇒ by 2

so $x(at+b)$

↳ $\underline{\underline{-2}}$

for b: $x(-2t+b) = x(2(t-\frac{1}{2}))$ → we did the shift on -2t not on t alone.

$$\boxed{y(t) = x(-2t+1)}$$

※ Amplitude - Transformation:

$y(t) = A\,x(t) + B$

$B \to$ $>0 \Rightarrow$ shift-up

$<0 \Rightarrow$ shift-down $\Rightarrow$ This called Biasing.

$A \to$ if $A > 1$

Amplification.

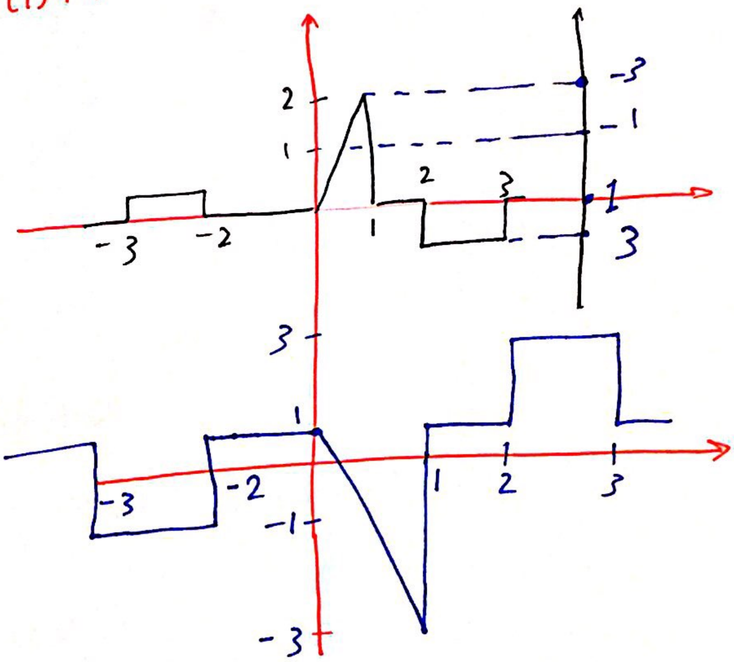
Ex. Draw $y(t) = -2\,x(t) + 1$

@ $x(t) = 2 \longrightarrow y(t) = -3$

$x(t) = 1 \longrightarrow y(t) = -1$

$x(t) = 0 \longrightarrow y(t) = 1$

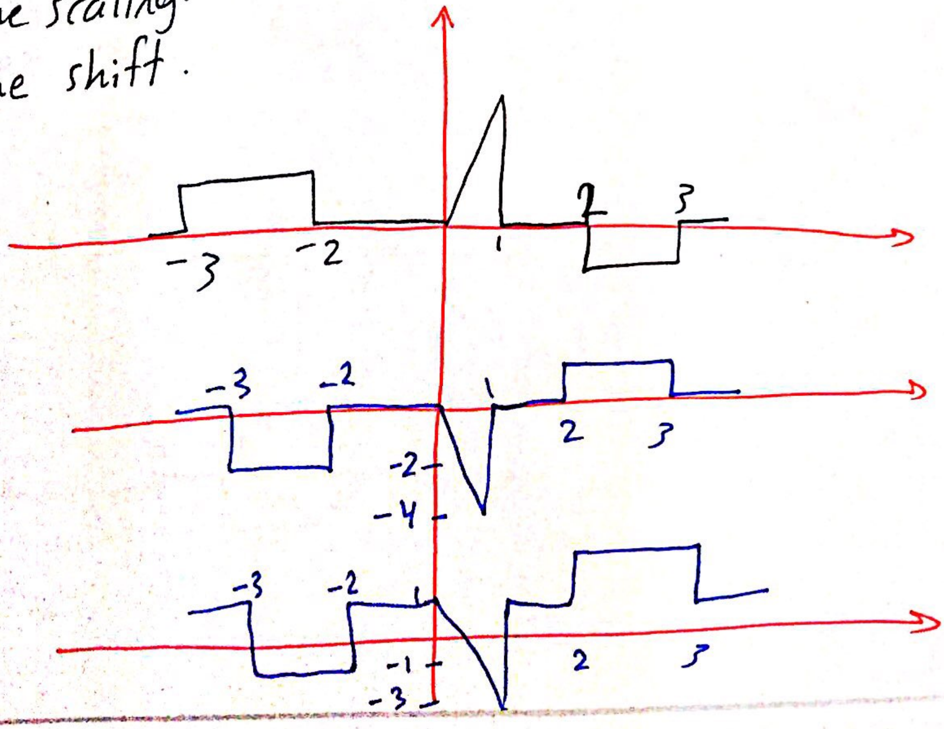
$x(t) = -1 \longrightarrow y(t) = 3$



∗ Another method step by step:

① do the scaling.

② do the shift.



* writing y(t) in term of x(t):

$$y(t) = A\,x(t) + B$$

For last ex.

(−) for ref.

2 → Amplification

so $\boxed{A = -2}$

original: 4
⋮
−2

y(t) ?
⋮
−3

it is look like shift down but Becarful since $A = \textcircled{-}2$

└→ $B = +1$

$y(t) = -2t+1$

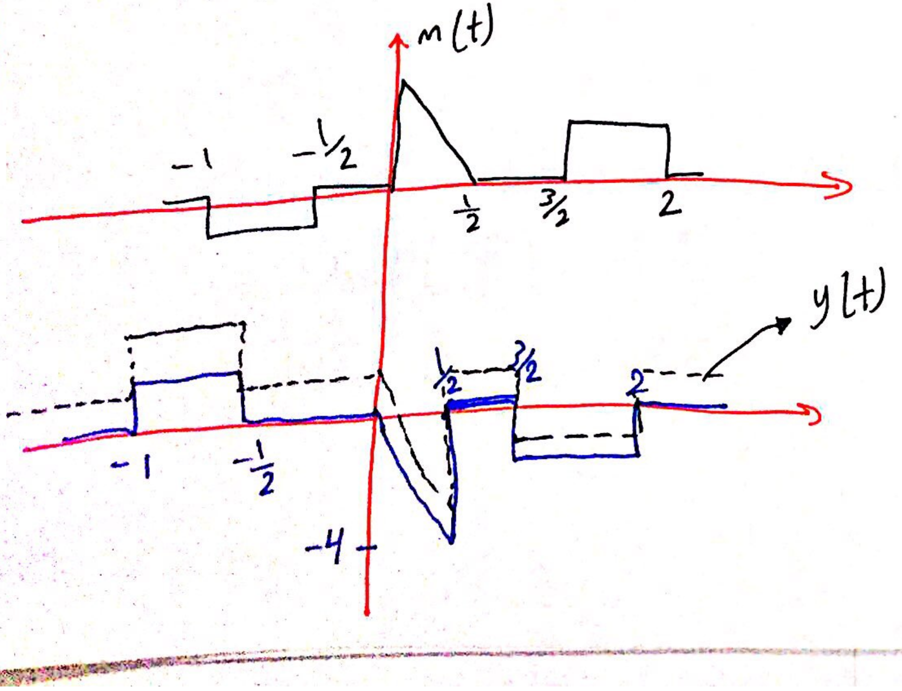
Now: $y(t) = A\,x(at+b) + B$

⇒ we assume $m(t) = x(at+b)$.

Ex. $y(t) = -2X(-2t+1) + 1$

⇒ $y(t) = -2\,m(t) + 1$

Assume previous figure:



Solve for Reverse operation.

※ The Exponential Function:

$$x(t) = f(t) + j\, g(t) \longrightarrow g(t) = Im\{x(t)\}$$

$f(t) = Re\{x(t)\}$

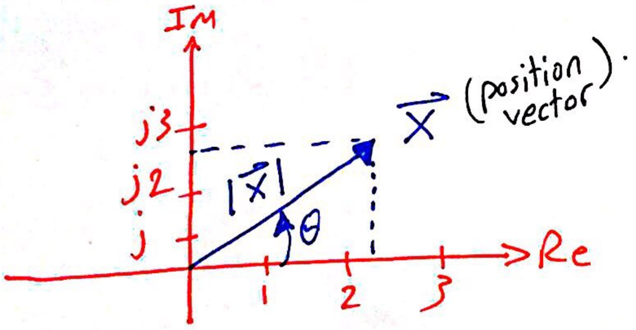
complex operator $j = \sqrt{-1}$

complex exponential function:

$$x(t) = c\, e^{at}.$$

$$\vec{X} = |\vec{X}|\, e^{j\theta} = |\vec{X}| \angle\theta = \overbrace{|\vec{X}|\cos\theta}^{Re\{\vec{X}\}} + j\overbrace{|\vec{X}|\sin\theta}^{Im\{\vec{X}\}}$$

means Complex. polar form phasor form. rectangular form.



This called complex-plane OR S-plane.

$$|\vec{X}| = \sqrt{(Re\{\vec{X}\})^2 + (Im\{\vec{X}\})^2} \qquad \theta = \tan^{-1}\left(\frac{Im\{\vec{X}\}}{Re\{\vec{X}\}}\right).$$

Ex. find θ & $|\vec{X}|$?

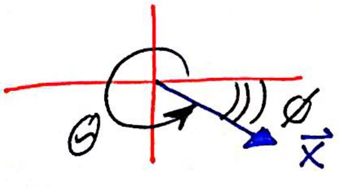
$\boxed{1}$ $\vec{X} = 2 - j3$

$|\vec{X}| = \sqrt{4+9} = \sqrt{13}$

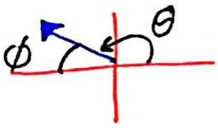
$\phi = \tan^{-1}\left(\frac{-3}{2}\right) = -57°$

$\Longrightarrow \theta = 360° - 57°$

$\boxed{\theta = 303°}$



$\boxed{2}$ $\vec{X} = -2 + j3$



$|\vec{X}| = \sqrt{13}$

$\theta = 180° + \tan^{-1}\left(\frac{-3}{2}\right)$

$\boxed{\theta = 123°}$

if we want $-\vec{x}$:

$$-\vec{x} = -|\vec{x}|\, e^{j\theta} = |\vec{x}|\, e^{j(\theta \pm 180^\circ)}$$

* Conjugate :

$$\vec{x}^* = |\vec{x}|\, e^{-j\theta} = |\vec{x}|\cos\theta - j|\vec{x}|\sin\theta$$

Ex. $\vec{x} = -2+j3 \Longrightarrow \vec{x}^* = -2-j3$

**

$0+j = 1\, e^{j90^\circ}$

$-j = e^{-j90^\circ}$

$\frac{1}{j} = -j$

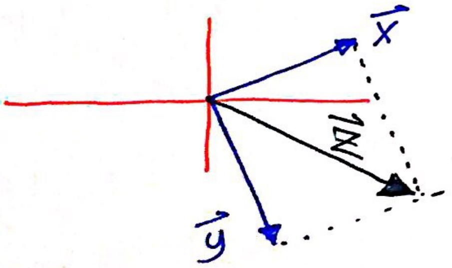
$j^* = -j$

* Addition :

$\vec{x} = a+jb = |\vec{x}|\, e^{j\theta}$

$\vec{y} = c+jd = |\vec{y}|\, e^{j\phi}$

$\vec{x}+\vec{y} = (a+c)+j(b+d) = \vec{Z}$

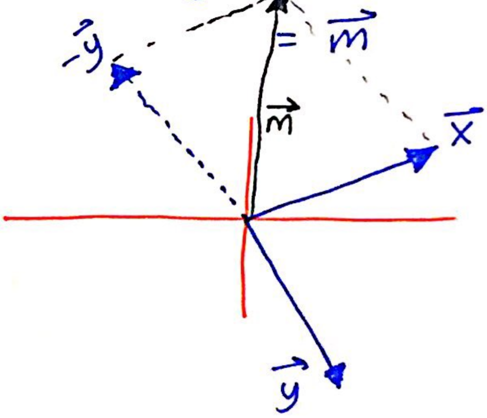


$\vec{x}+\vec{x}^* = 2\,\mathrm{Re}\{\vec{x}\}$

$\vec{x}-\vec{x}^* = 2\,\mathrm{Im}\{\vec{x}\}$

* Subtraction :

$\vec{x}-\vec{y} = (a-c)+j(b-d) = \vec{m}$



* Multiplication :

$$\vec{x}\cdot\vec{y} = |\vec{x}|\, e^{j\theta}\cdot|\vec{y}|\, e^{j\phi} = |\vec{x}||\vec{y}|\, e^{j(\theta+\phi)}$$

$$\vec{x}\cdot\vec{x}^* = |\vec{x}|^2$$

* Division :

$$\frac{\vec{x}}{\vec{y}} = \frac{|\vec{x}|\, e^{j\theta}}{|\vec{y}|\, e^{j\phi}} = \frac{|\vec{x}|}{|\vec{y}|}\, e^{j(\theta-\phi)}$$

$$\frac{\vec{x}}{\vec{x}^*} = 1\, e^{j2\theta}$$

$2 = 2e^{j0} = 2e^{j(2\pi n)}$ $n \in \mathbb{Z}$

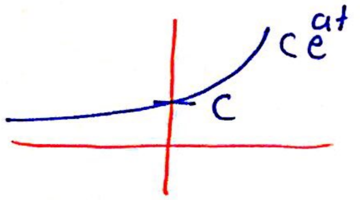
$-2 = 2e^{\pm j(180° + 360° n)}$

$0 = 0e^{jx}$ any angle.

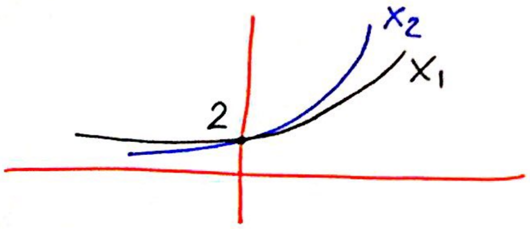
Case (1):

C & a Real.

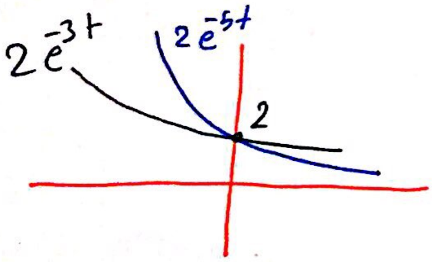
* take +ve c & +ve a



Ex. $x_1(t) = 2e^{3t}$
$x_2(t) = 2e^{5t}$



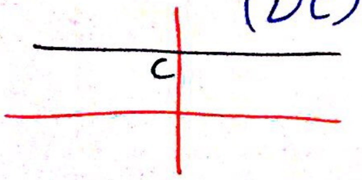
* take +ve c & -ve a



* take C +ve & a=0

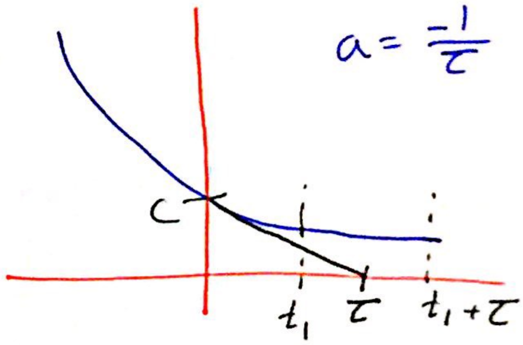
$x(t) = Ce^{0} = C$

(DC)



* $x(t) = Ce^{-t/\tau}$ → τ +ve.

$a = \frac{-1}{\tau}$



$x(\tau) = 0.37C$

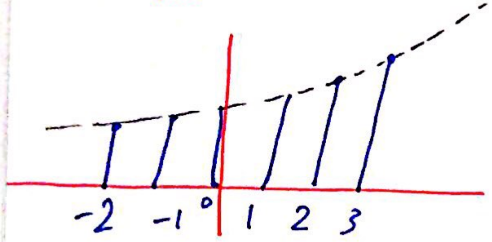
$\Rightarrow Ce^{-1} = \underline{\underline{0.37C}}$

$$\frac{d}{dt}x(t)\Big|_{t=0} = C\left(\frac{-1}{\tau}\right)e^{-t/\tau}\Big|_{t=0}$$

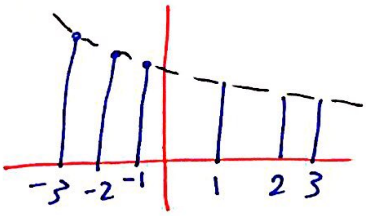
$$= \frac{-C}{\tau}.$$

$x(t) = c\,e^{an} \qquad n \in \mathbb{Z}$

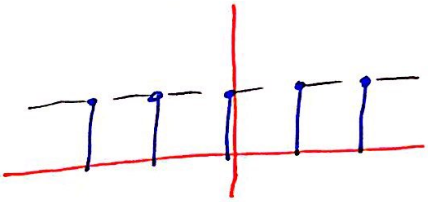
a^{+ve}



a^{-ve}



$a = 0$



Case (2):

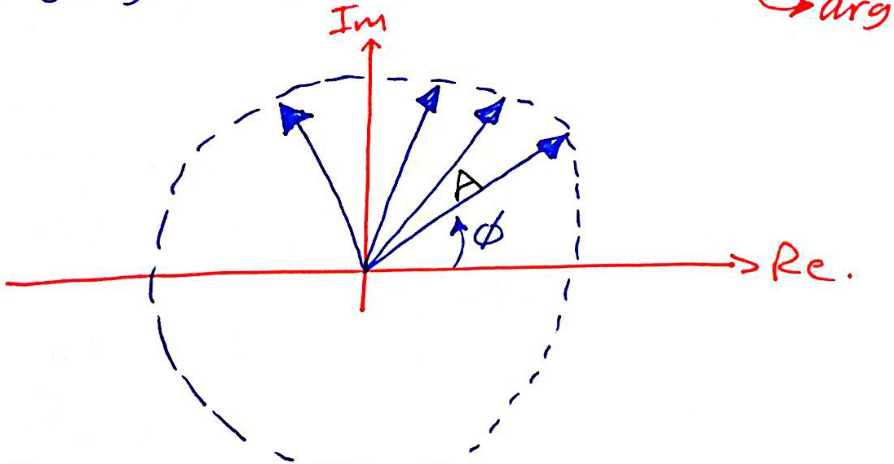
c → complex
a → pure imaginary.

$c = A\,e^{j\phi}$

$a = j\omega_0$

$$\Rightarrow \quad x(t) = A\,\overset{\text{phase shift.}}{e^{j\phi}} \cdot e^{j\omega_0 t}$$

$$= A\, e^{\underbrace{j(\omega_0 t + \phi)}_{\text{argument.}}}$$



$\Theta = \omega_0 t + \phi$

$\frac{d\theta}{dt} = \omega_0$ "angular frequancy"

$$x(t) = A\,e^{j(\omega_0 t + \phi)} = \underbrace{A\cos(\omega_0 t + \phi)}_{Re\{x(t)\}} + j\underbrace{A\sin(\omega_0 t + \phi)}_{Im\{x(t)\}}$$

⇒ we can draw each sine & cosine alone.

$x(t) = A\,e^{j(\omega t + \phi)}$

$\Rightarrow P = A^2 \quad$ or $\quad P = \overset{\text{cosine}}{\frac{A^2}{2}} + \overset{\text{sine}}{\frac{A^2}{2}} = A^2$

is $x(t) \overset{?}{=} x(t+nT_0)$

$\underbrace{A e^{j(\omega_0 t+\phi)}} = \underbrace{A e^{j(\omega_0 t+\phi)}} \cdot \boxed{e^{j\omega_0 n T_0}} \longrightarrow$ need to prove equal (1).

$\downarrow$

$T_0 = \frac{2\pi}{\omega_0}$

$\Rightarrow \quad e^{j2\pi n} = 1$

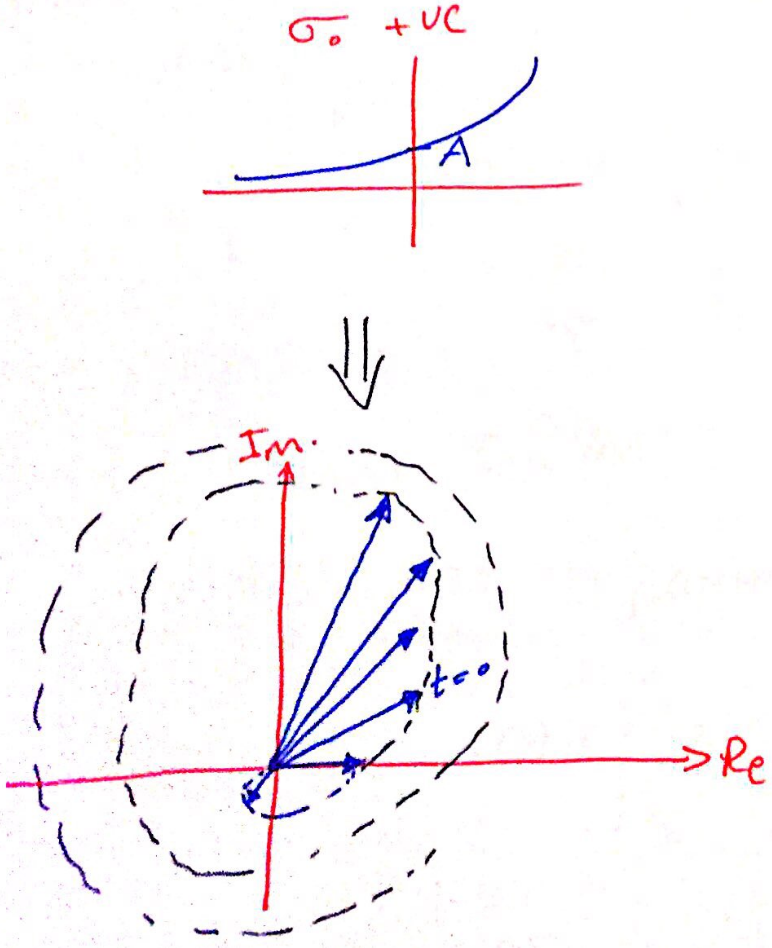
so it is periodic.

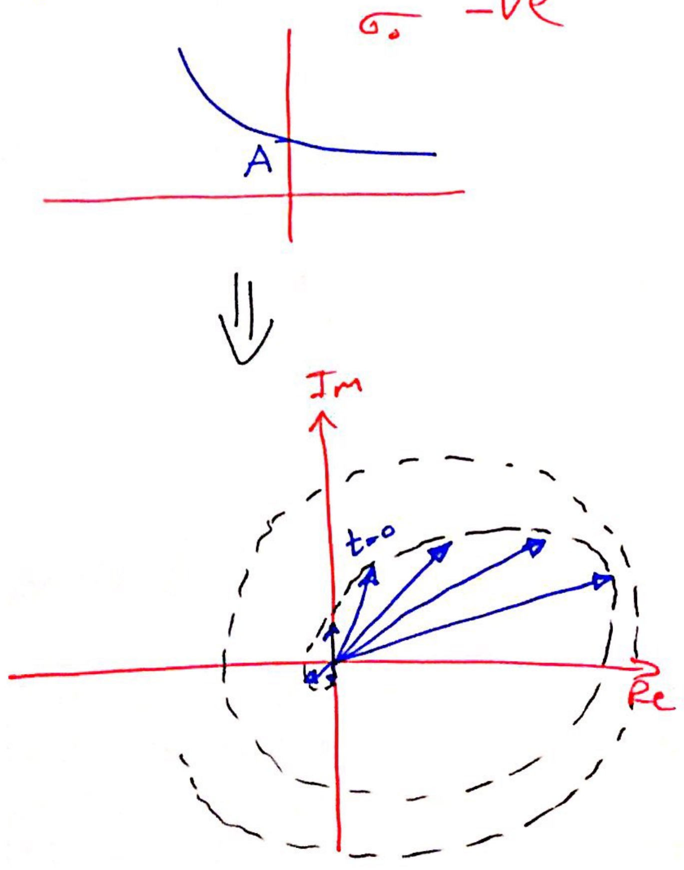
Case(3):

$x = A e^{j\phi a} \longrightarrow a = \sigma_0 + j\omega_0$

$\Rightarrow \quad x(t) = A e^{j\phi} \cdot e^{t(\sigma_0 + j\omega_0)}$

$= (A e^{\sigma_0 t}) \, e^{j(\omega_0 t + \phi)}$

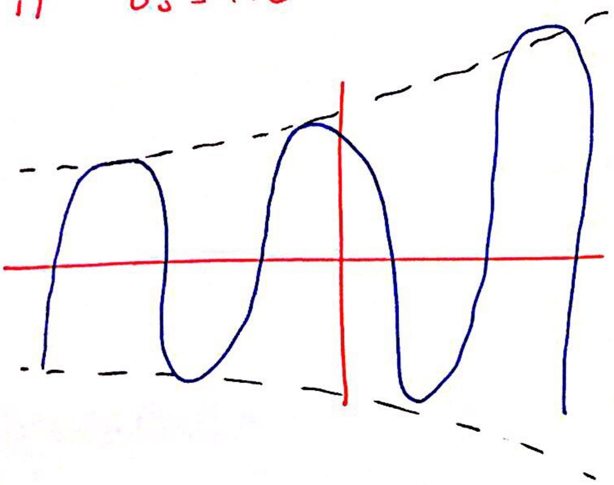




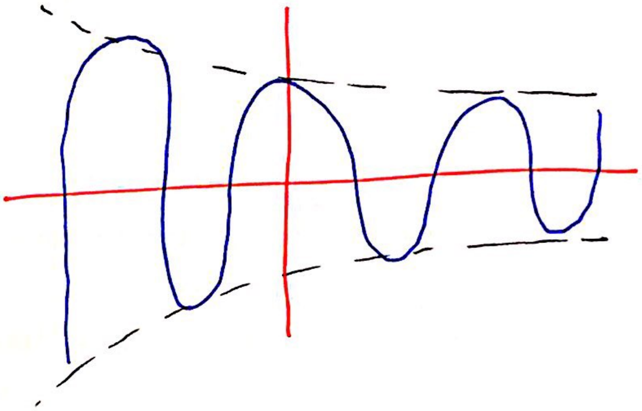
$x(t)$ in rectangular function:

$$x(t) = A e^{\sigma_0 t} \cos(\omega_0 t+\phi) + jA e^{\sigma_0 t} \sin(\omega_0 t+\phi)$$

if $\sigma_0 = +ve$



if $\sigma_0 = -ve$



※ Harmonic Complex exponential:

$$\{e^{jK\omega_0 t}\}, K \in \mathbb{Z}$$

$$= \{\ldots, e^{-j3\omega_0 t}, \underset{\substack{\downarrow \\ \frac{T_0}{2}}}{e^{-j2\omega_0 t}}, \underset{\substack{\downarrow \\ T_0}}{e^{-j\omega_0 t}}, 1, \underset{\substack{\downarrow \\ T_0}}{e^{j\omega_0 t}}, \underset{\substack{\downarrow \\ \frac{T_0}{2}}}{e^{j2\omega_0 t}}, \ldots\}$$

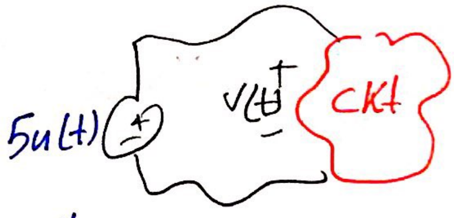
$\omega_1 = 2\omega_0 = 2 \frac{2\pi}{T_0} = \frac{2\pi}{T_1}$

so $T_1 = \frac{T_0}{2}$

※ Unit step Function:

$$u(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t < 0 \end{cases}$$; discontinuouty @ $t=0$

$$@\ t=0 \Rightarrow u(t) = \begin{cases} 1, \\ 0, \\ \frac{1}{2} \end{cases}$$



$\Downarrow$

$$V(t) = \begin{cases} 5, & t > 0 \\ 0, & t < 0 \end{cases}$$

*Proparties:

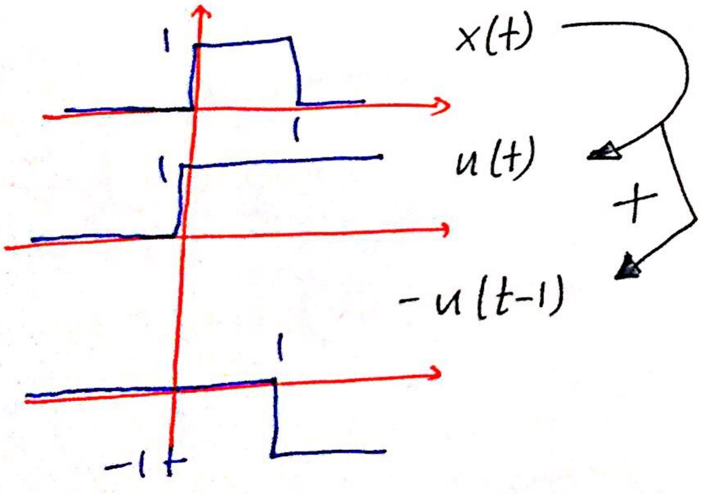
$$[u(t-t_0)]^2 = [u(t-t_0)]^K = u(t-t_0) \quad , K \in \mathbb{Z}^+$$

Ex. Draw $5e^{-2t}$ & remove -ve part of it ?

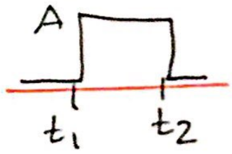
To remove +ve side $*[-u(t)]$

*Expressing any function using unit step:

$$x(t) = u(t) - u(t-1)$$

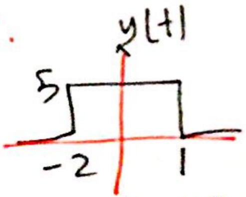


**



$y(t) = Au(t-t_1)$
$-Au(t-t_2)$

Ex.



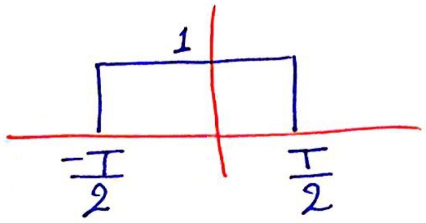
$y(t) = 5u(t+2)$
$-5u(t-1)$

* Descrete time unit step:

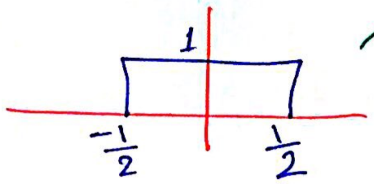
$$u[n] = \begin{cases} 1 & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$

* Rectangular Pulse:

$$\text{Rect}\left(\frac{t}{T}\right) = \begin{cases} 1 , & |t| < \frac{T}{2} \\ 0 , & \text{otherwise} \end{cases}$$

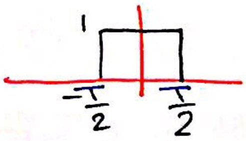


if $T = 1$



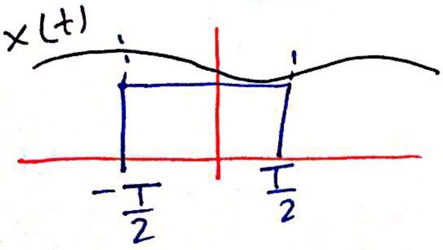
→ special case: square pulse.

* writing rect. in unit step:



$$\Rightarrow u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)$$

* Windowing:

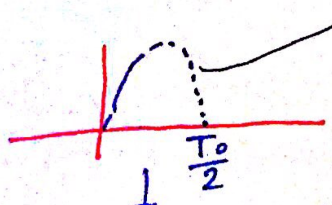


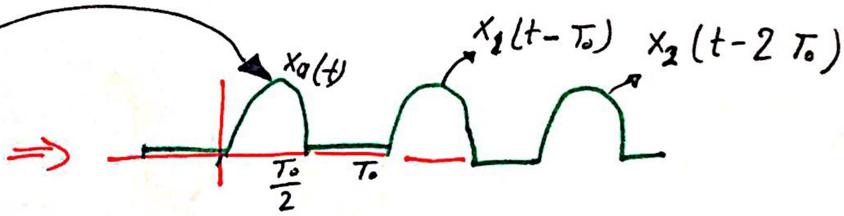
$$\Rightarrow x(t) * rect\left(\frac{t}{T}\right)$$

* Cosine: $(0<t<2\pi) \Rightarrow \cos(t)\, rect\left(\frac{t-\pi}{2\pi}\right)$

$$\Rightarrow \cos(t)\left(u(t) - u(t-2\pi)\right)$$

* $\sin(\omega_0 t)$:





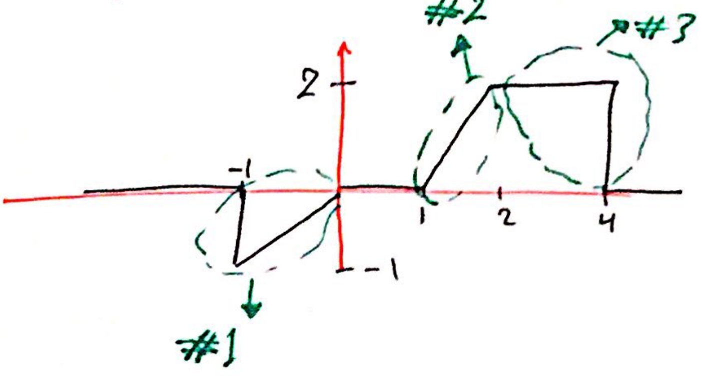
↓

$$\sin(\omega_0 t) * rect\left(\frac{t-T_0/4}{T_0/2}\right)$$

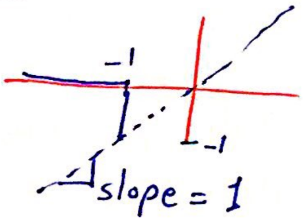
$$\Rightarrow \sin(\omega_0 t)\left[u(t) - u\left(t-\frac{T_0}{2}\right)\right]$$

↳ $$x(t) = \sum_{n=0}^{\infty} x(t-nT_0)$$

Ex. write the following using unit step function:

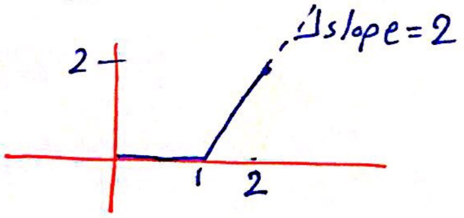


#1:



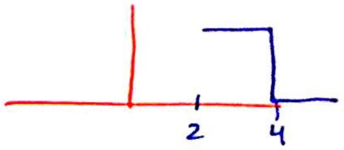
$\Rightarrow t(u(t+1)-u(t)) \cdots ①$

#2:



$\Rightarrow 2(t-1)[u(t-1)-u(t-2)] \; --- ②$

#3:

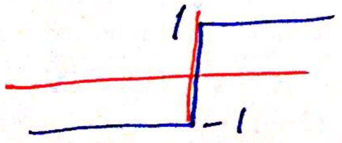


$\Rightarrow 2[u(t-2)-u(t-4)] \cdots ③$

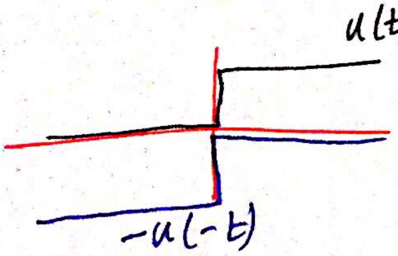
Add All of them together ①+②+③

※ Signum function:

$$sgn(t) = \begin{cases} 1 & , t>0 \\ -1 & , t<0 \end{cases}$$

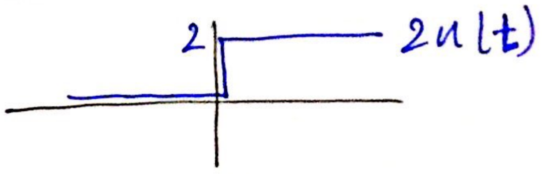


⇓

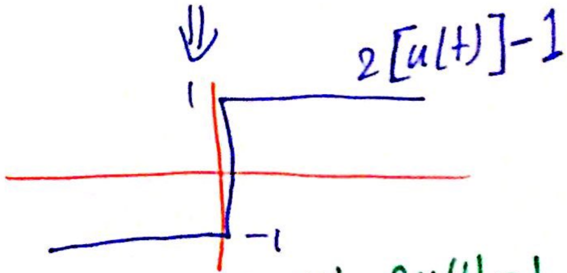


$\Rightarrow sgn(t) = u(t) - u(-t)$

another method using unit step:



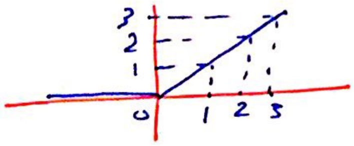
⇓

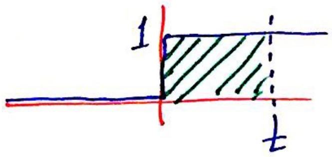


$sgn(t) = 2u(t) - 1$

※ Unit Ramp function:

$$\text{Ramp}(t) = \begin{cases} t, & t>0 \\ 0, & t\leq 0 \end{cases} = \underline{\underline{t\,u(t)}}$$





$$\rightarrow \int_{-\infty}^{t} u(t')\,dt' = \begin{cases} t, & t>0 \\ 0, & t<0 \end{cases} = \text{Ramp}(t)$$

Also $\frac{d\,\text{ramp}(t)}{dt} = u(t)$

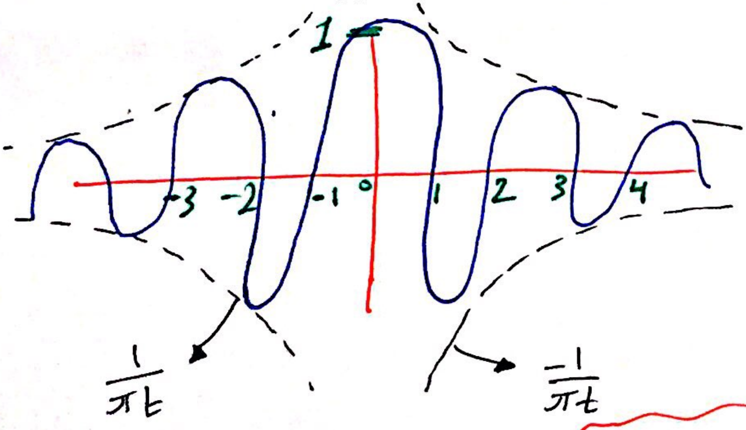
※ Unit sinc function:

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} = \frac{1}{\pi t}\sin(\pi t)$$

→ @ $t=0$:

using L'Hopital's Rule :

$$\lim_{t\to 0} \frac{\sin \pi t}{\pi t} = \lim_{t\to 0} \frac{\pi \cos \pi t}{\pi} = \underline{\underline{1}}$$

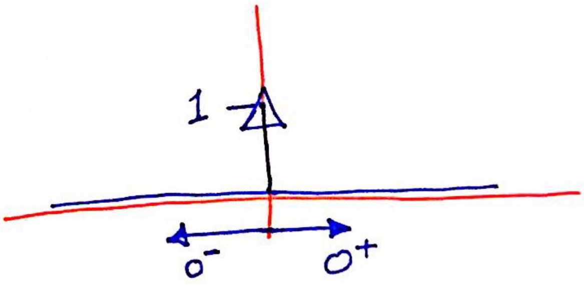


↳ it is even & Energy signal.

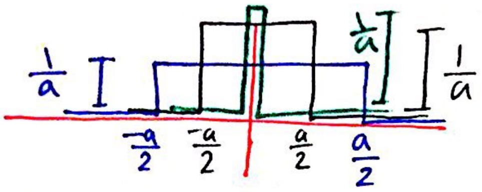
※ Unit Impulse Function:

$$\delta(t) = \begin{cases} 0 & , t\neq 0 \\ \infty & , t=0 \\ \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \end{cases}$$

↳ Area called "strength/weight".



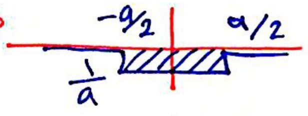
$\delta_a(t) = \frac{1}{a}\,\mathrm{rect}\left(\frac{t}{a}\right)$



$\Rightarrow$ Area to all of them $= 1$

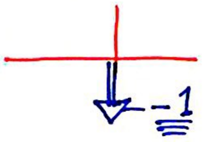
$$\Rightarrow \lim_{a \to 0} \delta_a(t) = \delta(t)$$

$-\delta(t) \Rightarrow$



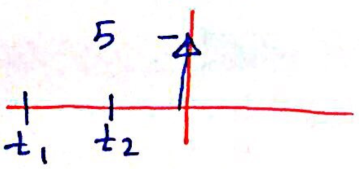
$\rightarrow$ area $= \underline{\underline{-1}}$

$\Downarrow$



$*$ we could do scaling on the impulse function.

$5\delta(t)$



$$\Rightarrow \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

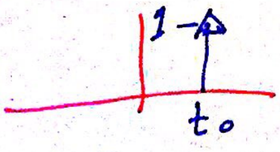
$$\int_{t_1}^{t_2} \delta(t)\,dt = 0$$

$$\int_{-\infty}^{0^-} \delta(t)\,dt = 0$$

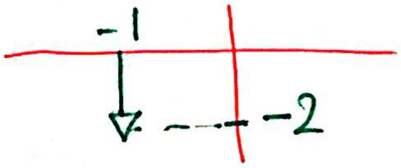
so $$\int_{t_1}^{t_2} A\,\delta(t)\,dt = \begin{cases} A, & t_1 < 0 < t_2 \\ 0, & \text{otherwise.} \end{cases}$$

$**$ shifting:

$\delta(t - t_0)$

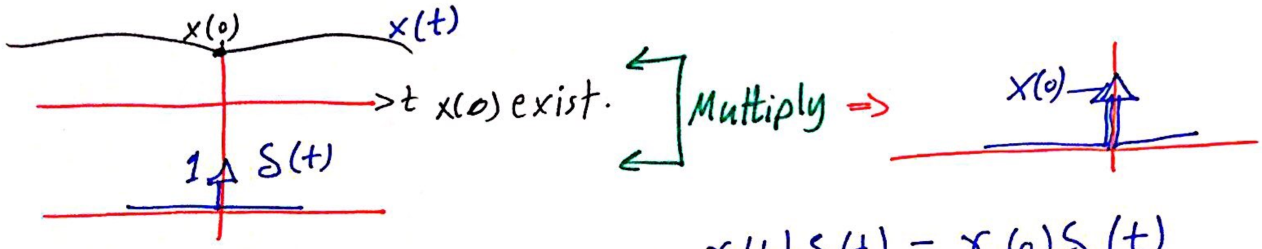


Ex. draw $-2\delta(t+1)$

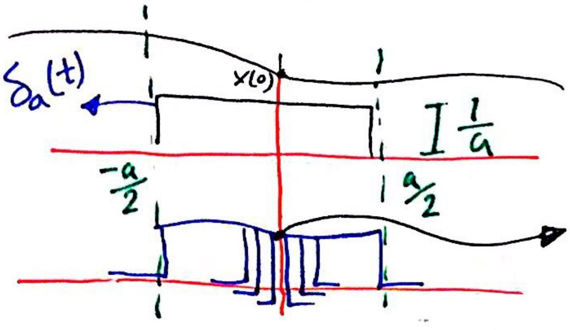


$$\int_{t_1}^{t_2} \delta(t - t_0)\,dt = \begin{cases} 1, & t_1 < t_0 < t_2 \\ 0, & \text{otherwise.} \end{cases}$$

* Proparties of $\delta(t)$:



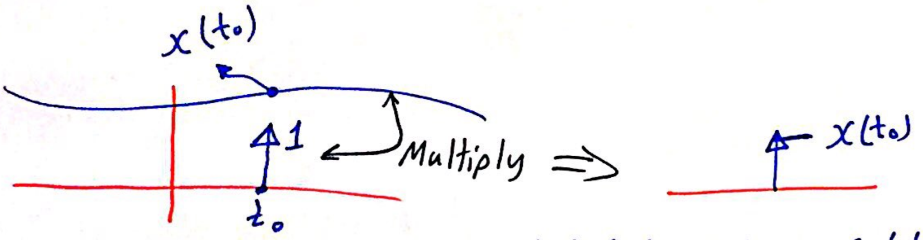
$$x(t).\delta(t) = \underline{\underline{x(0)\delta(t)}}$$



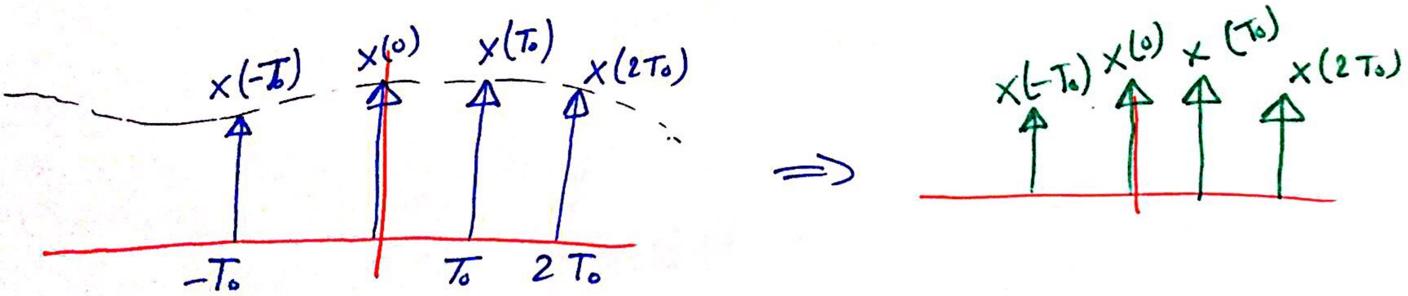
it will look Like a point as long as we get close to t=0.

$$\Rightarrow \lim_{a\to 0} \delta_a(t)x(t) = x(0)\delta(t)$$

This is sampling.



$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$



***

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = \int_{-\infty}^{\infty} x(0)\delta(t)dt = x(0)\underbrace{\int_{-\infty}^{\infty}\delta(t)dt}_{\underline{\underline{1}}}$$

impulse as pulse

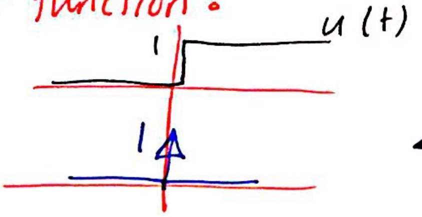
(38)

$$\int_{t_1}^{t_2} x(t)\,\delta(t)\,dt = \int_{t_1}^{t_2} x(0)\,\delta(t)\,dt = \begin{cases} x(0), & t_1<0<t_2 \\ 0 & , \text{otherwise.} \end{cases}$$

$$\int_{t_1}^{t_2} x(t)\,\delta(t-t_0)\,dt = \int_{t_1}^{t_2} x(t_0)\,\delta(t-t_0)\,dt = \begin{cases} x(t_0), & t_1<t_0<t_2 \\ 0 & , \text{otherwise.} \end{cases}$$

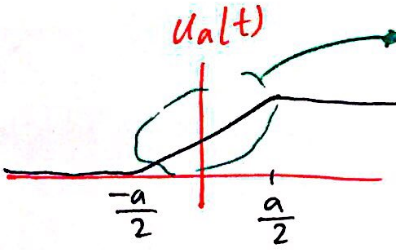
*writing impulse as unit function:

$$\boxed{\frac{du(t)}{dt} = \delta(t)}$$



derive it.

$$\Rightarrow \boxed{u(t) = \int_{-\infty}^{t} \delta(t')\,dt'}$$

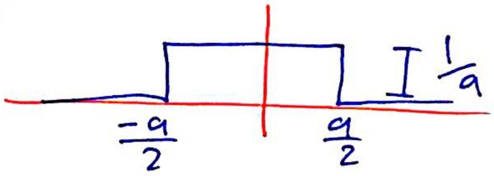


$$h = mt - t_0$$

$$\frac{dh}{dt} = m = \frac{1-0}{a} = \frac{1}{a}$$

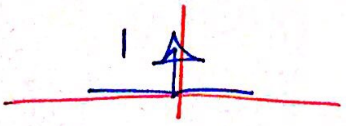
deriving

$\delta_a(t)$

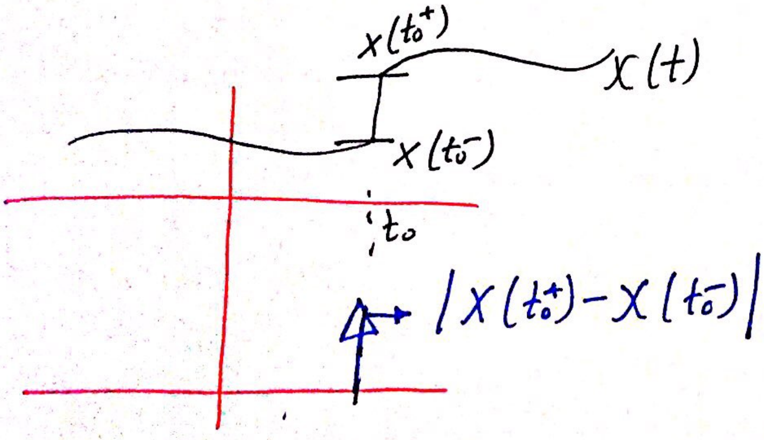


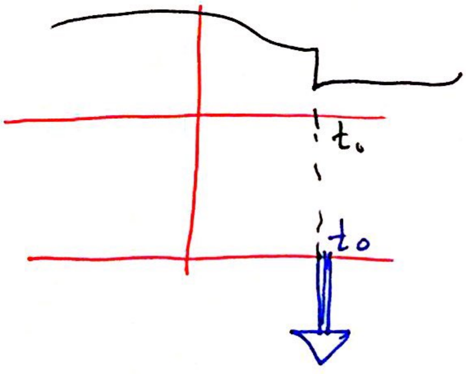
$$\lim_{a\to 0} u_a(t) = u(t)$$

so $$\underline{\underline{\frac{du_a(t)}{dt} = \delta_a(t)}}$$

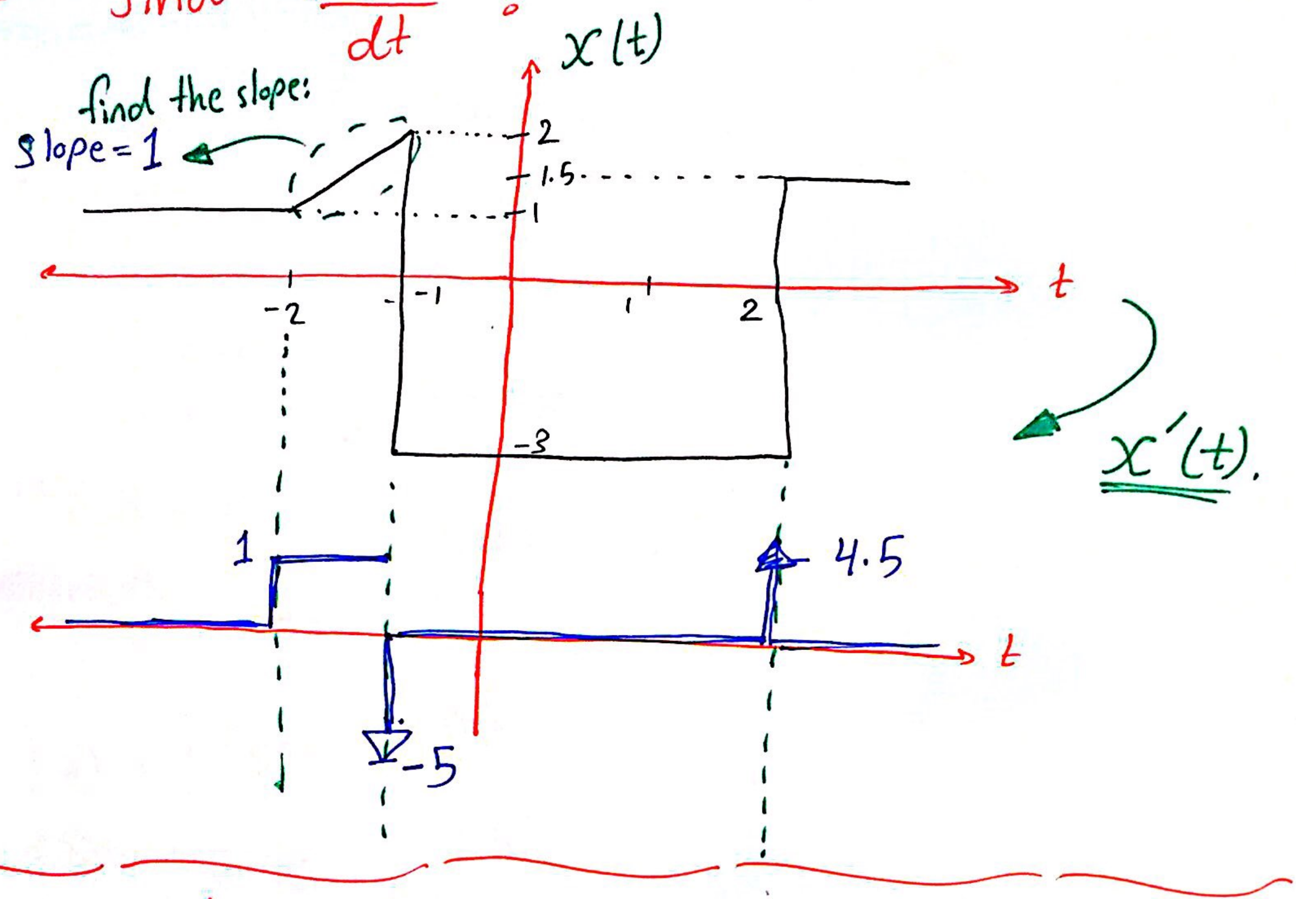


$$\Rightarrow \int_{-\infty}^{t} \delta(t)\,dt = \begin{cases} 1 & , t>0 \\ 0 & , t<0 \end{cases} \equiv u(t)$$





Ex. find $\frac{dx(t)}{dt}$:



*** Prove that:**

$$\int_{-\infty}^{\infty} x(t)\, u'(t)\,dt = x(0)$$

remember: $\int_a^b m\,dv = mv\Big|_a^b - \int_a^b v\,dm$

$$\int_{-\infty}^{\infty} x(t)\,\frac{du(t)}{\cancel{dt}}\cancel{dt} = x(t)\,u(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\frac{d\,x(t)}{dt}\,dt = x(\infty)\underbrace{u(\infty)}_{1} - x(-\infty)\cancel{u(-\infty)}^{0} - \int_{-\infty}^{\infty} u(t)\,x'(t)\,dt$$

$$= x(\infty) - \left[\int_{-\infty}^{0} 0\cdot x'(t)\,dt + \int_{0}^{\infty} 1\cdot x'(t)\,dt\right] = x(\infty) - 0 - x(t)\Big|_{0}^{\infty} = x(\infty) - x(\infty) + x(0)$$

$$= x(0) \ \#$$

* find $\int_{t_1}^{t_2} x(t)\,\delta^{(n)}\,dt$? "see lathi's Book"

 # Scaling Property:

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(at)\,dt = \int_{-\infty}^{\infty} \frac{1}{|a|}\,\delta(t)\,dt = \frac{1}{|a|}\overbrace{\int_{-\infty}^{\infty} \delta(t)\,dt}^{=1}$$

$$= \frac{1}{|a|}$$

Let $\underline{at} = \lambda$ $\Rightarrow$ $\frac{1}{a}\int \delta(\lambda)d\lambda \overset{?}{=} \frac{1}{|a|}$

$\Rightarrow adt = d\lambda$

if a +ve: $\frac{1}{a}\int_{-\infty}^{\infty}\delta(\lambda)d\lambda = \boxed{\frac{1}{a}}$

if a -ve: $\frac{1}{a}\int_{\infty}^{-\infty}\delta(\lambda)d\lambda = -\frac{1}{a}\int_{-\infty}^{\infty}\delta(\lambda)d\lambda = \boxed{-\frac{1}{a}}$ $\rightarrow$ $\frac{1}{|a|}$ #

* In General:

$$\delta(at+b) = \frac{1}{|a|}\delta(t+\frac{b}{a})$$

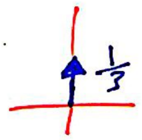
↳ To prove it:

$$\delta(a\underbrace{[t+\tfrac{b}{a}]}_{\tau}) = \delta(a\tau) = \frac{1}{|a|}\delta(\tau) \Rightarrow \underline{\underline{\delta(a[t+\tfrac{b}{a}]) = \frac{1}{|a|}\delta(t+\tfrac{b}{a})}}$$

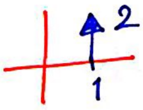
#

Ex.

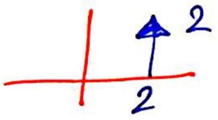
$\delta(3t) = \frac{1}{|3|}\delta(t)$



$\delta(\frac{t-1}{2}) = \frac{1}{\frac{1}{2}}\delta(t-1) = 2\,\delta(t-1)$



$\delta(\frac{t}{2}-1) = \delta(\frac{t-\textcircled{2}}{\textcircled{2}})$ → represent shift. → represent weight



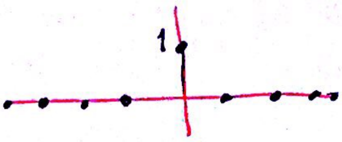
Ex evaluate:

$\boxed{1}$ $\int_{-2}^{4}\overset{x(t)}{(t+t^2)}\delta(t-3)dt = x(3) = 3+9 = \underline{\underline{12}}$

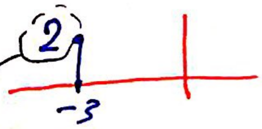
$\boxed{2}$ $\int_{-2}^{4}(t+t^2)\delta(\frac{1}{2}t-3)dt = 2\int_{-2}^{4}(t+t^2)\delta(t-6)dt =$ Zero since 6 is out of -2 → 4.

※ DT impulse:

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$



$\underline{\underline{2\delta[n+3]}}$

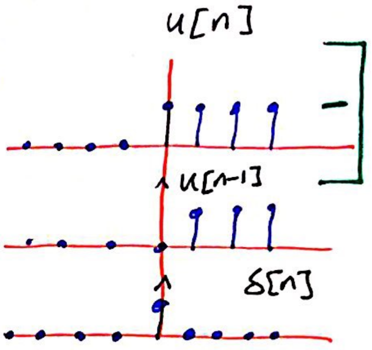


here it is just representing an amplitude Not weight.

* Proparties:

* even function: $\delta[-n] = \delta[n]$.

* $\frac{\partial u(t)}{\partial t} = \delta(t) \Rightarrow u[n] - u[n-1] = \delta[n]$



*

$$\int_{-\infty}^{t} \delta(t')dt' = u(t)$$

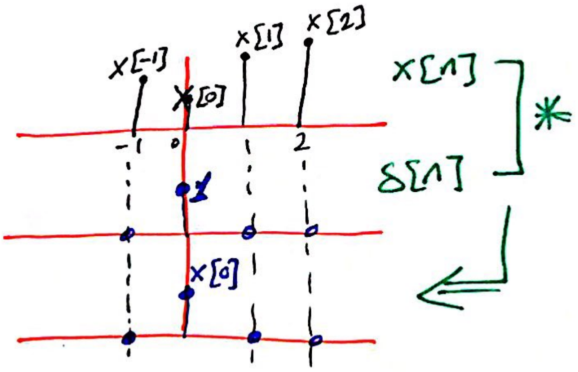
$$\Rightarrow \sum_{m=-\infty}^{n} \delta[m] = u[n]$$

* Sampling proparty:

$x(t)\,\delta(t) = x(0)\,\delta(t)$

$\Rightarrow x[n]\,\delta[n] = x[0]\,\delta[n]$

prove: $\Rightarrow$



also ingeneral:

$x[n]\,\delta[n-k] = x[k]\,\delta[n-k]$

* Sifting Proparty:

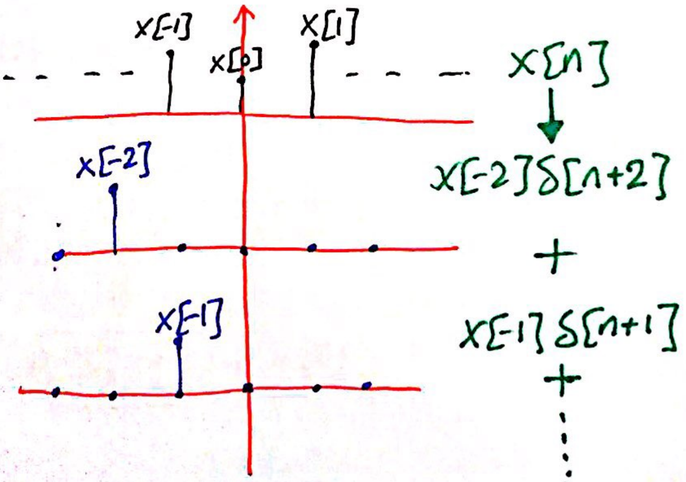
$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)dt = x(t_0)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} x[n]\,\delta[n-k] = x[k]$$

$$\Rightarrow x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

proving:

$= \cdots + x[-2]\,\delta[n+2]$
$+ x[-1]\,\delta[n+1] + \cdots$



and so on for other terms, the sum of all terms will give $x[n]$.

**

$$u[n] = \sum_{K=-\infty}^{\infty} u[K]\delta[n-K]$$

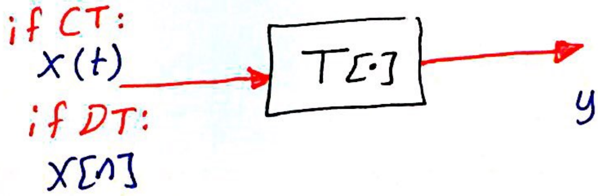
$$= \sum_{K=-\infty}^{-1} u[K]\cancel{\delta[n-K]}^{0} + \sum_{K=0}^{\infty} \overbrace{u[K]}^{1}\delta[n-K]$$

$$\Rightarrow u[n] = \sum_{K=0}^{\infty} \delta[n-K]$$ → you could also prove it by graph.

* Scaling:

$$\delta(at) = \frac{1}{|a|}\delta(t) \Longrightarrow \delta[an] = \delta[n]$$

System Description:

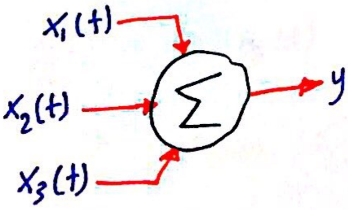


$y(t) = T[x(t)]$

$y[n] = T[x[n]]$

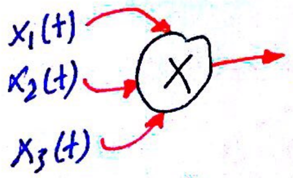
This called Transformation Notation.

* Sumer:



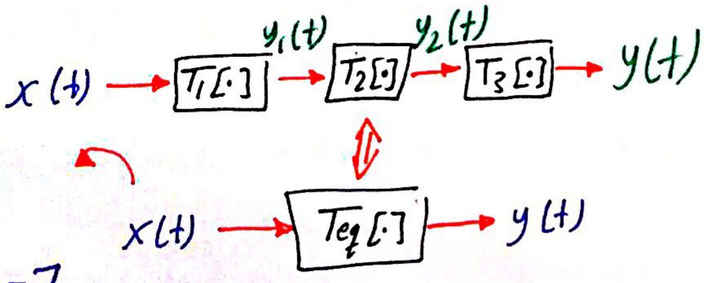
$$y(t) = \sum_{K=1}^{3} x_K(t) = x_1(t) + x_2(t) + x_3(t)$$

* product:



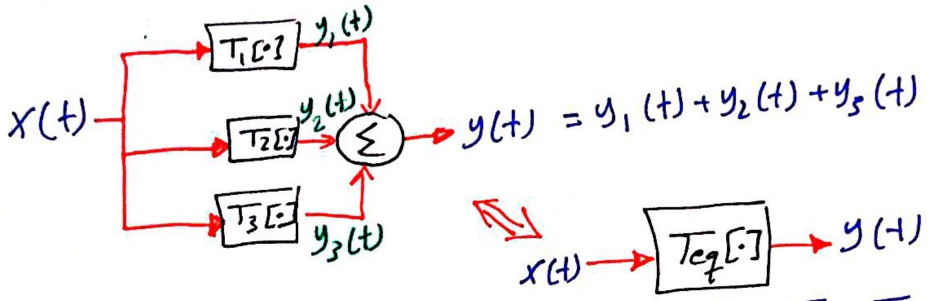
$$y(t) = \prod_{K=1}^{3} x_K(t) = x_1(t) \cdot x_2(t) \cdot x_3(t)$$

* Cascade connection:



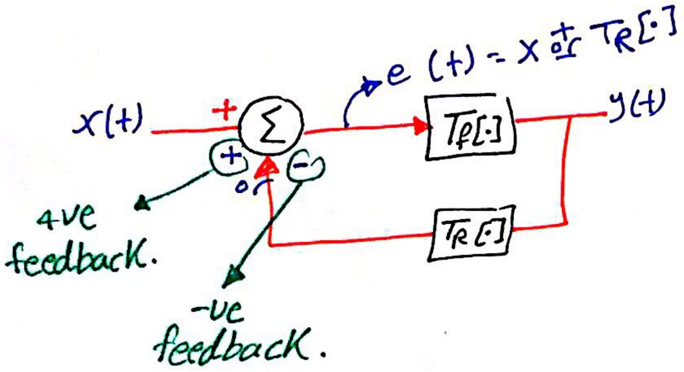
$$y(t) = T_3[y_2(t)]$$
$$= T_3[T_2[y_1(t)]]$$
$$= T_3\{T_2[T_1[x(t)]]\}$$

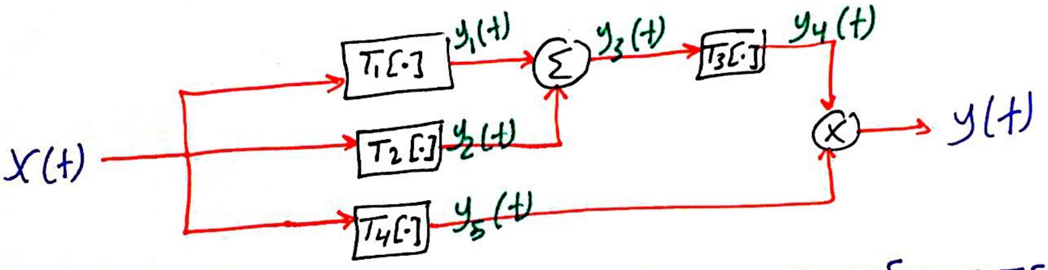
* paralle connection:



$y(t) = y_1(t) + y_2(t) + y_3(t)$

$y(t) = T_1[x(t)] + T_2[x(t)] + T_3[x(t)]$





$y(t) = y_4(t) \cdot y_5(t) = T_3[y_3(t)] \cdot T_4[x(t)] = T_3[y_1(t) + y_2(t)] \cdot T_4[x(t)] = T_3\big[T_1[x(t)] + T_2[x(t)]\big] + T_4[x]$

※ System Classification:

$y(t) = 5x(t) + 2$

This system is:

① static. ② Causal. ③ invertable. ④ Stable.

⑤ Time-invariant. ⑥ Non-linear.

⇒ we will discuss how to determine all of them.

※ Static / dynamic:

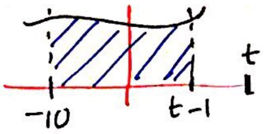
$y(t) = 5x^2(t)$ static

$y(t) = 5x^2(t-1)$ dynamic

$y(t) = 5x(t-5) - 3x(t-2)$ dynamic.

$$y(t) = \int_{-10}^{t-1} 5x(\tau)\,d\tau$$

Dynamic



※ Causality:

if the output need a future value [Non-causal system] otherwise [causal system].

$y(t) = 5x(t+3)$ Non-Causal.

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ Causal.

$y(t) = \int_{-\infty}^{t+1} x(\tau)\,d\tau$ Non-Causal.

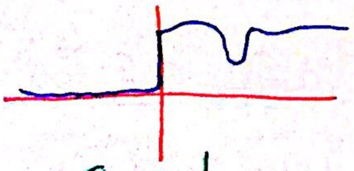
$y(t-1) = \int_{-10}^{t} 5x(\tau)\,d\tau$

Non-Causal.

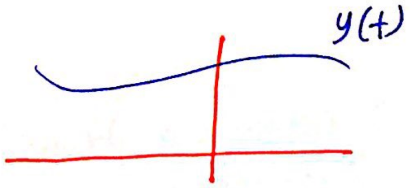
* Note: static system is always Causal.

⇒ for a signal:

$x(t) = 0 \quad (t<0) \Rightarrow$ Causal.



Causal.



Non-Causal.

⇓

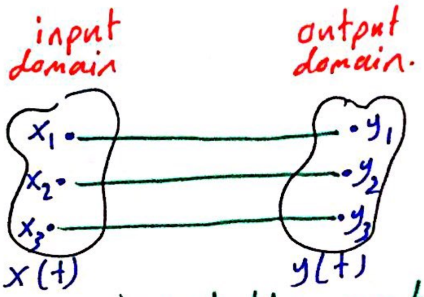
⇒ Causal.

* Note: Any signal multiplied by the unit step function is causal signal.

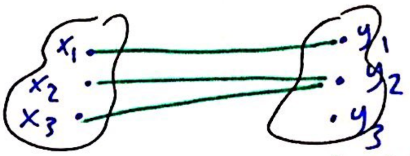
※ Invertability:

$$x(t) \rightarrow \boxed{T[\cdot]} \rightarrow y(t)$$

$$y(t) \rightarrow \boxed{T^{-1}[\cdot]} \rightarrow x(t)$$

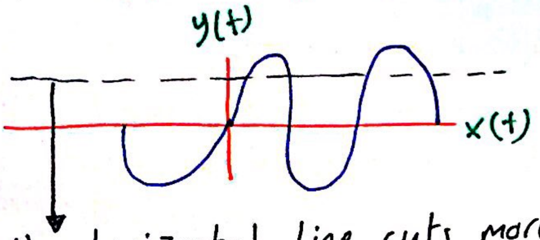


invertable system.



Non-invertable system.

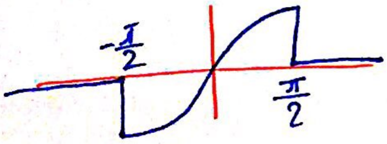
* is $y(t) = \sin(x(t))$ invertable ?!



the horizontal line cuts more than one point so Non-invertable.

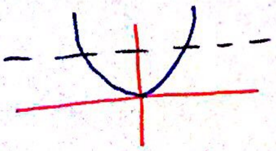
↳ But. if the function was defined on a certain period:

$y(t) = \sin(x(t)) \Rightarrow -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ & zero otherwise.



$\Longrightarrow$ Invertable.

ex. $y(t) = (x(t))^2$

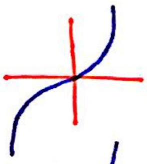


$\Downarrow$

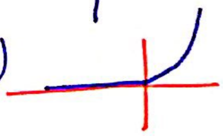
Non-invertable.

could be invertable in some cases like:

$y(t) = x^2(t)$ → * sgn(t)



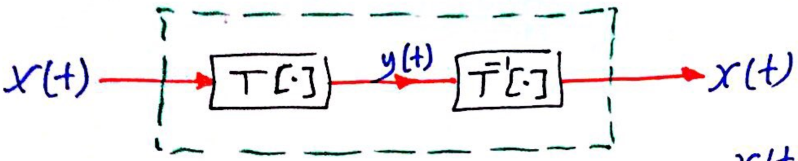
→ * u(t)



$\Downarrow$

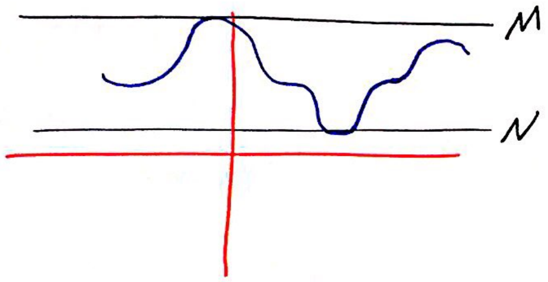
Invertable.

** Identity System:



$$x(t) = T^{-1}[y(t)] = T^{-1}[T[x(t)]]$$

※ Stability:



$\Rightarrow N < x(t) < M$

OR.

$|x(t)| \leq M$ → finite Number.

$|y(t)| \leq K$

* if the input bounded & the output bounded ⇒ " Stable system "

Example: is $y(t) = x^2(t)$ stable?

$y(t) = x^2(t)$

$T[\cdot] = (\cdot)^2$

Let us assume $|x(t)| \leq M$ where M finite Number.

$\Rightarrow$ is $|y(t)| \overset{?}{\leq} N$

* remember:
$|x^2| = |x|^2$

$|y(t)| = |x^2(t)| = |x(t)|^2 \Rightarrow |x(t)|^2 \leq M^2$

$\Rightarrow |y(t)| \leq M^2$ → assume this is N. ↳ finite number.

So it is bounded ⇒ stable system.

Example: is $i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau)\,d\tau$ stable?

Let us assume $|v(\tau)| \leq M$ where M is a finite Number.

$\Rightarrow$ is $|i(t)| \overset{?}{\leq} N$

$$\Rightarrow |i(t)| = \underbrace{\left|\frac{1}{L}\right|}_{\text{always +ve.}} \cdot \left| \int_{-\infty}^{t} v(\tau)\,d\tau \right|$$

*remember:

$$\boxed{\left|\int (\,)\,dt\right| \leq \int |(\,)|\,dt}$$

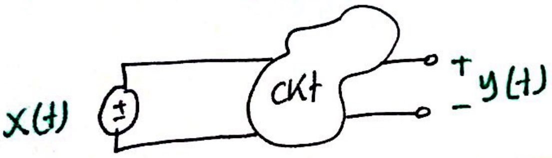
$$\Rightarrow |i(t)| \leq \frac{1}{L} \int_{-\infty}^{t} |v(\tau)|\,d\tau \Rightarrow |i(t)| \leq \frac{1}{L} \int_{-\infty}^{t} M\,d\tau$$

$$\Rightarrow |i(t)| \leq \frac{M}{L} \int_{-\infty}^{t} d\tau \qquad = \tau \Big|_{-\infty}^{t} = \underline{\underline{t + \infty}} = \infty$$

$\Rightarrow |i(t)| \leq \infty \Rightarrow$ Unbounded

$\Rightarrow$ Unstable system.

※ Time - Invariance:



$x(t) = 5V$

$x(t-2) = 5V$

$x(t-t_0) = 5V$

$y(t) = 2V$

$y(t-2) = 2V$

$y(t-t_0) = 2V$

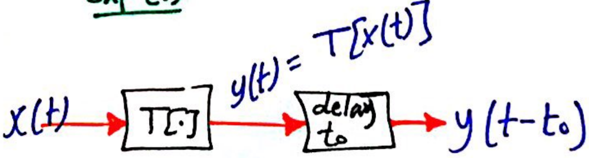
Doesn't change with time $\Rightarrow$ Time-invariant system.

if the value change with time

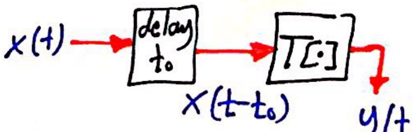
Time-variant.

*How to test for time-invariant?

exp(1):



exp(2):



$$\underset{\text{delay version.}}{y_d(t)} = T[x(t-t_0)]$$

ex. $y = e^{x(t)}$

first we write it in transformation Notation:

$\Rightarrow T[\cdot] = e^{(\cdot)}$

*replace (t) by $(t-t_0)$:

$y(t-t_0) = e^{x(t-t_0)} \Rightarrow y_d(t) = T[x(t-t_0)] = e^{x(t-t_0)}$

so $y(t-t_0) = y_d(t)$ Time inavariant.

ex. $y(t) = e^{-t}\, x(t)$

$\Rightarrow T[\cdot] = e^{-t}(\cdot) \Rightarrow y(t-t_0) = e^{-(t-t_0)} x(t-t_0)$

$\Rightarrow y_d(t) = e^{-t} x(t-t_0)$ since $y(t-t_0) \neq y_d(t)$

Time-variant.

ex. $y(t) = \cos(t) \cdot x(t)$

$T[\cdot] = \cos(t) \cdot (\cdot) \Rightarrow y_d(t) = \cos(t)\, x(t-t_0)$

$y(t-t_0) = \cos(t-t_0)\, x(t-t_0)$ since $y_d(t) \neq y(t-t_0)$

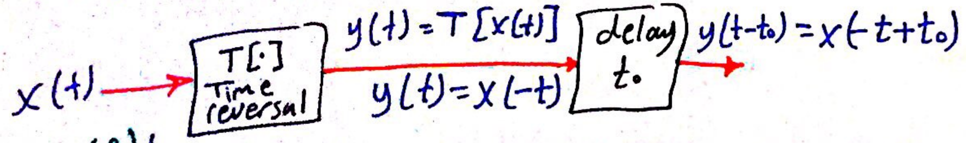
Time-variant.

** * Note: As a summary if there is a function of time between the output & the input ⇒ Time-variant.

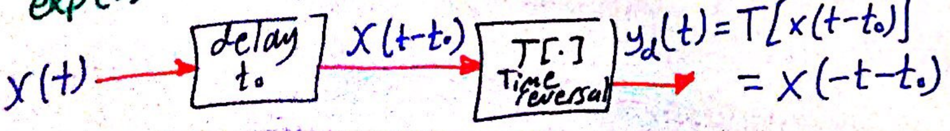
*Now if we can't write the Transformation Notation!?

ex. $y(t) = x(-t)$

exp(1):



exp(2):



Time varying.
(Non causal)

⇒ why Non-causal ?!

$y(t) = x(-t)$

$\Rightarrow y(-2) = x(2) \longrightarrow$ Need a future value so it is Non-causal.

※ Linearity:

→ we test for it by superposition.

There is Two conditions for a system to be Linear
⇒ both of them must be true to be linear if one Not ⇒ directly Non-linear system.

$\boxed{1}$ Scaling: (Homogeneity)

$x(t) \longrightarrow \boxed{T[\cdot]} \longrightarrow y(t) = T[x(t)]$

$x_a(t) = a\,x(t) \xrightarrow{T[\cdot]} y_a(t) = T[a\,x(t)]$
$= a\,T[x(t)] = a\,y(t)$

could be arbitrary "Complex Number"

$\boxed{2}$ Additivity:

$x_1(t) \longrightarrow \boxed{T[\cdot]} \longrightarrow y_1(t) = T_1[x_1(t)]$

$+$

$x_2(t) \longrightarrow \boxed{T[\cdot]} \longrightarrow y_2(t) = T_2[x_2(t)]$

$x_1(t) + x_2(t) \xrightarrow{T[\cdot]} y_4(t) = T[x_1(t) + x_2(t)]$
$= T_1[x_1(t)] + T_2[x_2(t)] = y_1(t) + y_2(t)$

* if the two test $\boxed{1}$ & $\boxed{2}$ true
we say Linear System.

ex. Is the following system linear:

$$y(t) = x^2(t)$$

$$\Rightarrow T[\cdot] = (\cdot)^2$$

$$y_a(t) = T[aX(t)] = (a\,x(t))^2 = a^2 x^2(t) \overset{?}{=} a y(t)$$

$$\overset{?}{=} a\,x^2(t)$$

NO

so it is Non-linear system & we don't have to test the additivity.

But if we Need to test:

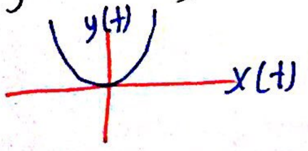
$$y_1(t) = x_1^2(t) \;,\; y_2(t) = x_2^2(t)$$

$$y_+(t) = T[x_1(t)+x_2(t)] = (x_1+x_2)^2$$

$$= x_1^2(t) + 2x_1(t)\,x_2(t) + x_2^2(t) \neq x_1^2(t) + x_2^2(t)$$

** Note:

By drawing $y(t) = x^2(t)$

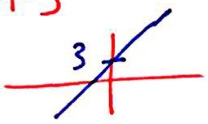


since the graph non linear shape we can say directly Non linear system.

*But be carful if the system gave a linear shape when you draw it ⇒ you should do the test.

ex. $y(t) = 2x(t) + 3$

Here if you draw



Linear shape.

⟹ but you will discover that it is Non linear system

$T[\cdot] = 2(\cdot) + 3$

$$y_a(t) = T[ax(t)] = 2(ax(t)) + 3 = 2ax(t) + 3 \overset{?}{=} ay(t) \overset{?}{=} 2ax(t) + 3a$$

Non-linear system.

NO

* if we want to test for additivity:

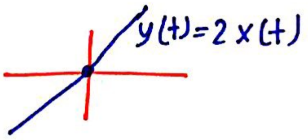
$$+\begin{array}{l} y_1(t) = 2x_1(t) + 3 \\ y_2(t) = 2x_2(t) + 3 \\ \hline \end{array}$$

$$y_+(t) = T[x_1 + x_2] = 2(x_1(t) + x_2(t)) + 6 \overset{?}{=} 2(x_1(t) + x_2(t)) + 3$$

NO.

* Note: if the system was representing a linear line pass through the origin ⟹ Linear system.

ex. $y(t) = 2x(t)$



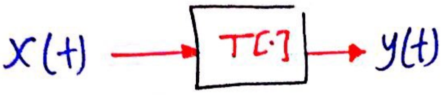
⟹ Linear system.

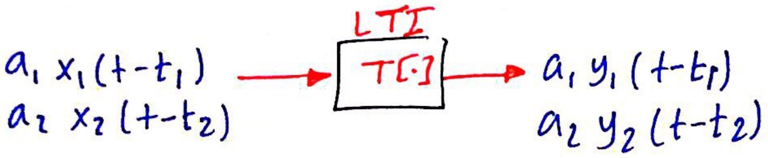
ex. Is the following system linear:

$$ay''(t) + by^2(t) = x(t)$$

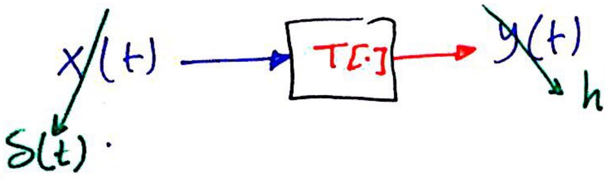
⟹ Non-linear system because the existing of the square y^2.

※ Linear Time Invariant (LTI) system:





* Impulse Response $\equiv h(t)$ ⇒ it is just applicable for LTI system.

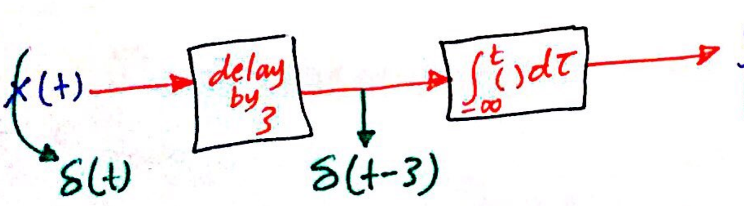


$h(t) = T[\delta(t)]$

if we have: $y(t) = 2x^2(t) - \sqrt{x(t+3)}$ write impulse response:

$$\Rightarrow \quad h(t) = 2\delta^2(t) - \sqrt{\delta(t+3)}$$

Ex.



$$h(t) = \int_{-\infty}^{t} \delta(\tau-3)\,d\tau$$

* prove: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$

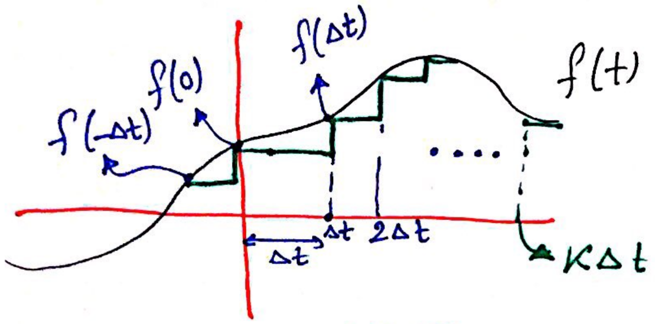
$$\Rightarrow \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,dt = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

$$\Rightarrow x(t) \underbrace{\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau}_{=1} = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(t-\tau)\,dt = \int_{-\infty}^{\infty} \delta(\tau-t)\,d\tau = \int_{-\infty}^{\infty} \delta(\lambda)\,d\lambda = \int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau = 1$$

so $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$. #

$$x(t)=\int_{-\infty}^{\infty}x(\tau)\,\delta(t-\tau)\,d\tau=\lim_{\Delta\tau\to 0}\sum_{K=-\infty}^{\infty}x(K\Delta\tau)\,\delta(t-K\Delta\tau)\,\Delta\tau$$



$$\int_{-\infty}^{\infty}f(t)\,dt\simeq\sum_{K=-\infty}^{\infty}f(K\Delta t)\,\Delta t$$

$$=\lim_{\Delta t\to 0}\sum_{K=-\infty}^{\infty}f(K\Delta t)\,\Delta t$$

$K\in\mathbb{Z}$

$$\underset{\delta(t)}{\cancel{x(t)}}\longrightarrow\overset{\text{LTI}}{\boxed{T[\cdot]}}\longrightarrow\cancel{y(t)}\rightarrow h(t)$$

$$a_K\,\delta(t-K\Delta\tau)\longrightarrow a_K\,h(t-K\Delta\tau)$$

$$a_1\,\delta(t-\Delta\tau)\longrightarrow a_1\,h(t-\Delta\tau)\;+$$

$$+\;a_2\,\delta(t-2\Delta\tau)\longrightarrow a_2\,h(t-2\Delta\tau)$$

$$\vdots\qquad\qquad\vdots$$

$$\lim_{\Delta\tau\to 0}\sum_{K=-\infty}^{\infty}x(K\Delta\tau)\,\delta(t-K\Delta\tau)\,\Delta\tau\longrightarrow\lim_{\Delta\tau\to 0}\sum_{K=-\infty}^{\infty}x(K\Delta\tau)\,h(t-K\Delta\tau)\,\Delta\tau$$

$$x(t)\longrightarrow y(t)=\int_{-\infty}^{\infty}x(\tau)\,h(t-\tau)\,d\tau$$

$$x(t)\longrightarrow\overset{\text{LTI}}{\boxed{h(t)}}\longrightarrow y(t)=\int_{-\infty}^{\infty}x(\tau)\,h(t-\tau)\,d\tau$$

This called Convelution.

$$y(t)=x(t)\overset{\text{convolve.}}{\bigstar}h(t)\;\Longrightarrow$$ continuous convelution

convelution Integral.

Discrete Convelution $\Leftarrow y[n]=x[n]\bigstar h[n]$

Convelution Summation.

* commutative proparty:

$$x(t) \star h(t) = h(t) \star x(t) \Rightarrow \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

see prove in slides.

* h(t) it is a system.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau \Rightarrow \underline{\underline{x(t) = x(t) \star \delta(t)}}$$

$$\underset{\downarrow \atop \delta(t)}{x(t)} \longrightarrow \overset{LTI}{\boxed{h(t)}} \longrightarrow \underset{\downarrow \atop h(t) = \delta(t) \star h(t)}{y(t) = x(t) \star h(t)}$$

* in general:

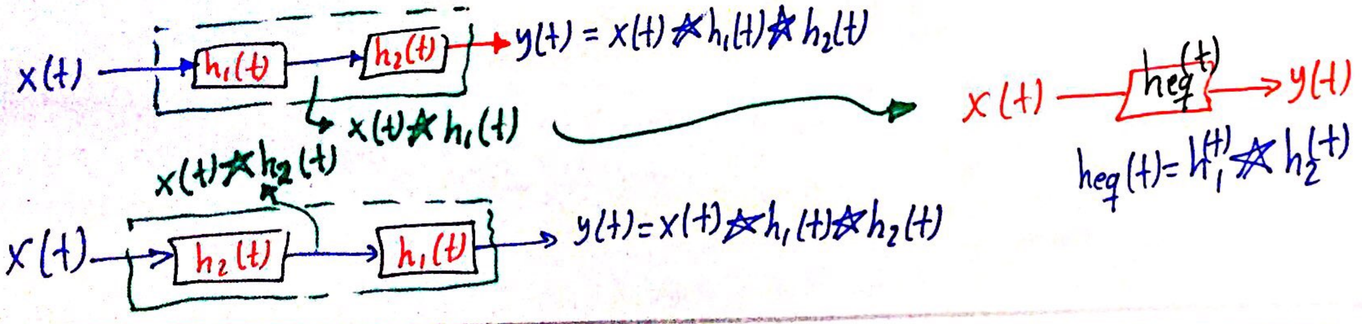
$$\delta(t-t_0) \longrightarrow \boxed{h(t)} \longrightarrow h(t-t_0) = \delta(t-t_0) \star h(t)$$

$$\Rightarrow \boxed{\begin{aligned} x(t) \star \delta(t-t_0) &= x(t-t_0) \\ x(t-t_0) \star \delta(t) &= x(t-t_0) \end{aligned}}$$

*Associative Proparty:

$$x(t) \star [h_1(t) \star h_2(t)] = [x(t) \star h_1(t)] \star h_2(t) = [x(t) \star h_2(t)] \star h_1(t)$$
$$= x(t) \star h_1(t) \star h_2(t)$$

* h_1 & h_2 as systems:

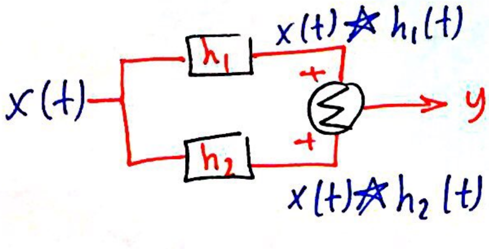


* Distributive Property:

$$x(t) \star [h_1(t) \pm h_2(t)] = x(t) \star h_1(t) \pm x(t) \star h_2(t)$$

⇒ in general:

$$x(t) \star \sum_{K=K_1}^{K_2} h_K(t) = \sum_{K=K_1}^{K_2} x(t) \star h_K(t)$$



$$\to y(t) = x(t) \star h_1(t) + x(t) \star h_2(t)$$
$$= x(t) \star \underbrace{[h_1(t) + h_2(t)]}_{h_{eq}(t)}$$

$$\Rightarrow x(t) \rightarrow \boxed{h_{eq}(t)} \rightarrow y(t)$$

※ Classification for LTI system:

※ Static/dynamic:

⇒ The only form for h(t) to be static (memoryless):

$$y(t) = K x(t) \Rightarrow \boxed{h(t) = K\delta(t)}$$

otherwise, Dynamic (memory).

※ Causality:

$$y(t) = x(t) \star h(t) = \int_{-\infty}^{\infty} x(\tau) \boxed{h(t-\tau)} d\tau$$

$t - \tau < 0$

To be causal: $h(t-\tau) = 0$ for $\tau > t$

Let $t - \tau = s \Rightarrow h(s) = 0, \; s < 0$

↳ $t - \tau < 0$

$\tau > t$

$$y(t) = x(t) \star h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= h(t) \star x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{\infty} h(t)\; x(t-\tau)\, d\tau$$

* Invertability:

$$x(t) \rightarrow \boxed{h(t)} \xrightarrow{x(t) \star h(t)} \boxed{h^{-1}(t)} \rightarrow x(t) = x(t) \star \left(h(t) \star h^{-1}(t)\right)$$

we have:

$$x(t) = x(t) \star \delta(t)$$

so $\Rightarrow$ $\boxed{h(t) \star h^{-1}(t) = \delta(t)}$

* Stability:

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) \star h(t) = h(t) \star x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$|x(\tau - t)| \leq M \Rightarrow \underline{\underline{|x(t-\tau)| \leq M}}$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right| \Rightarrow |y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau$$

$$\Rightarrow |y(t)| \leq \int_{-\infty}^{\infty} |h(t)| * M\, d\tau \Rightarrow |y(t)| \leq M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

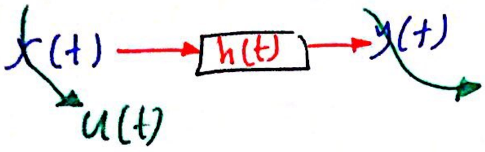
so to be a stable system:

$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau \Rightarrow$ must be finite Number.
↓
"Absolutly Integrable."

*Note: if the system is Causal:

just test for $\int_{0}^{\infty} |h(t)|\, dt$ since $h(t) = 0,\ t < 0$

* Unit Step Response:



$$S(t) = u(t) \star h(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} h(t-\tau)\, d\tau \qquad \text{let } t-\tau=\lambda$$

$$= -\int_{t}^{-\infty} h(\lambda)\, d\lambda = \int_{-\infty}^{t} h(\lambda)\, d\lambda$$

$$\Rightarrow \boxed{S(t) = \int_{-\infty}^{t} h(\tau)\, d\tau}$$

Ex. $\underline{h(t) = e^{-3t} u(t)}$, test for Invertability & causality and state if it is static or dynamic?

- * Dynamic ⇒ since h(t) not on the form $h(t) = K\delta(t)$.
- * Causal ⇒ since it is multiplied by u(t).
- * since causal system we test for $\int_0^{\infty} |h(t)|\, dt$

$$\int_0^{\infty} |h(t)|\, dt = \int_0^{\infty} |e^{-3t}|\, \underbrace{|u(t)|}_{=1}\, dt = \int_0^{\infty} e^{-3t}\, dt = -\frac{1}{3} e^{-3t} \Big|_0^{\infty}$$

always +ve.

$$= -\frac{1}{3}\left(e^{-\infty} - 1\right)$$

$$= \boxed{\frac{1}{3}}$$

⇒ since it gave a finite Number

⇒ Stable.

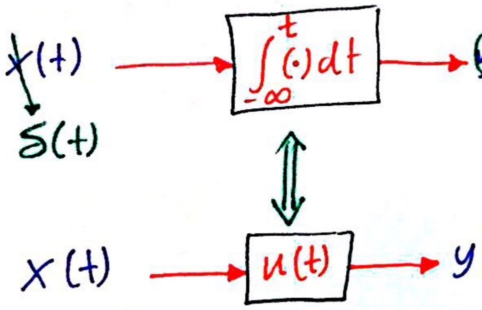
$$S(t) = \int_0^{\infty} h(\tau)\, d\tau = \int_0^{t} e^{-3\tau}\, d\tau = -\frac{1}{3} e^{-3\tau} \Big|_0^{t} = -\frac{1}{3}\left(e^{-3t} - 1\right)$$

$$\Rightarrow S(t) = \frac{1}{3} - \frac{1}{3} e^{-3t} \quad , \quad \underline{S(t) > 0}$$

$$\Rightarrow S(t) = \left(\frac{1}{3} - \frac{1}{3} e^{-3t}\right) u(t)$$

Example: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

① find $h(t)$?   ② if $x = tu(t)$, find $y(t)$?



$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$

$y(t) = x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$

$\Rightarrow \int_{-\infty}^{t} x(\tau)\, \underset{\downarrow\, 1}{u(t-\tau)}\, d\tau + \int_{t}^{\infty} x(\tau)\, \underset{\downarrow\, 0}{u(t-\tau)}\, d\tau$

$= \int_{-\infty}^{t} x(\tau)\, d\tau.$

$u(t-\tau) = \begin{cases} 0 & , t<\tau \\ 1 & , t>\tau \end{cases}$

** $\underline{x(t) = tu(t)}$

$y(t) = x(t) * u(t)$

$= (tu(t)) * u(t) = \int_{-\infty}^{\infty} \tau u(\tau) \cdot u(t-\tau)\, d\tau = \int_{0}^{\infty} \tau\, u(t-\tau)\, d\tau$

$u(t-\tau) = \begin{cases} 0 , & t-\tau<0 \quad (\tau>t) \\ 1 , & \tau<t \end{cases}$

$= \int_{0}^{t} \tau\, u(t-\tau)\, d\tau + \int_{t}^{\infty} \tau\, u(t-\tau)\, d\tau$

$= \int_{0}^{t} \tau\, d\tau + 0 = \frac{\tau^2}{2}\Big|_{0}^{t} = \boxed{\frac{t^2}{2}} * u(t)$

or easier way:

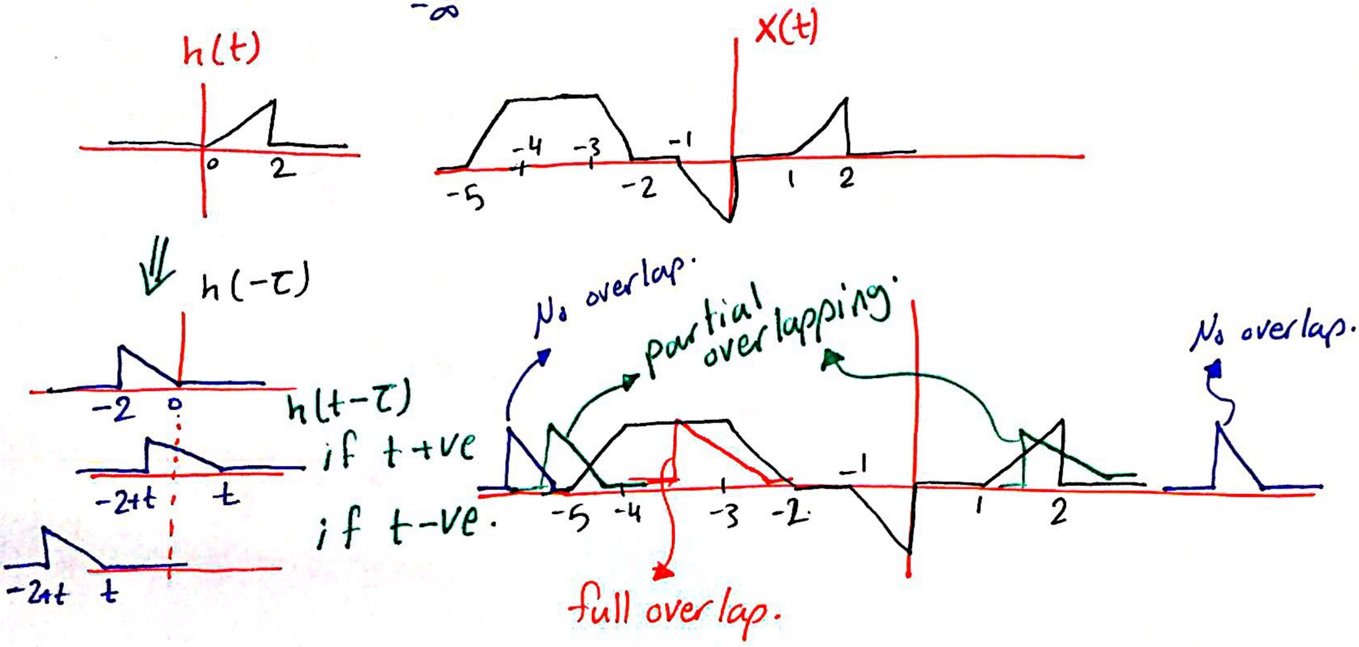
$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{t} \tau\, u(\tau)\, d\tau = \int_{0}^{t} \tau\, d\tau$

$= \boxed{\frac{t^2}{2}} * u(t)$

* graphically:

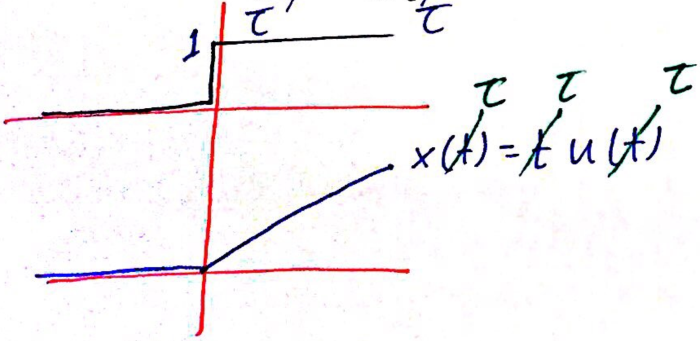
$$y(t) = x(t) \star h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

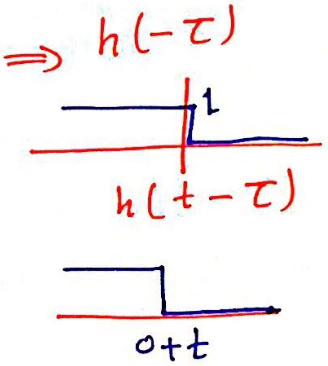
$$y(t) = h(t) \star x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

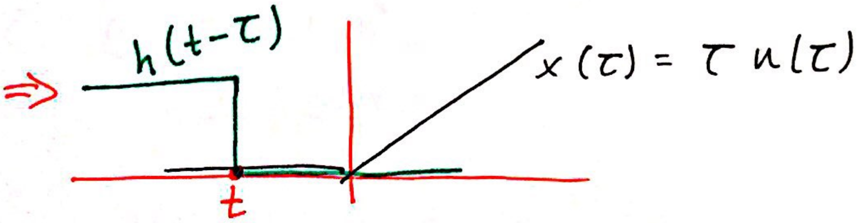


$h(\overset{\tau}{\cancel{t}}) = u(\overset{\tau}{\cancel{t}})$

$x(\overset{\tau}{\cancel{t}}) = \overset{\tau}{\cancel{t}}\, u(\overset{\tau}{\cancel{t}})$



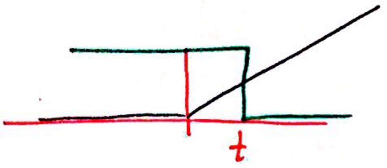




as long as $\underline{\underline{t<0}}$ No overlapping.

for $t<0$ $\quad y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau = 0$

continue. ⟹

⇒ 

for $t > 0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

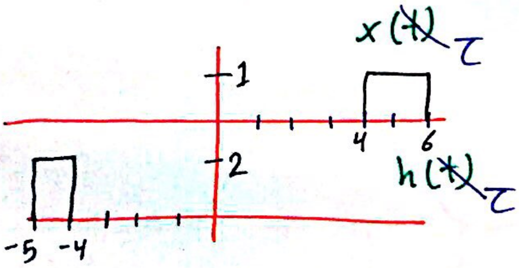
$$= \int_{-\infty}^{0} 0\, d\tau + \int_{0}^{t} (\tau \cdot 1)\, d\tau + \int_{t}^{\infty} 0\, d\tau$$

$$= \frac{t^2}{2}$$

so

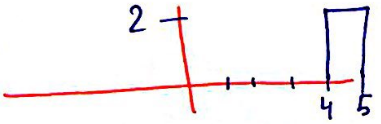
$$y(t) = \begin{cases} 0 & , t < 0 \\ \frac{t^2}{2} & , t > 0 \end{cases}$$

Ex. find $y(t) = x(t) \star h(t)$ for:



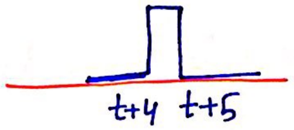
⟹

$h(-\tau)$

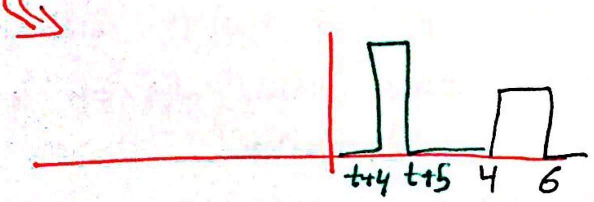


⇓

$h(t-\tau)$



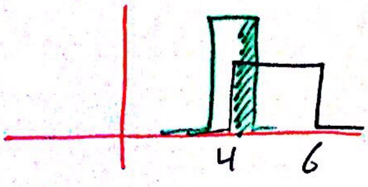
⇓



for $t+5 < 4$

$\Rightarrow \underline{t < -1}$

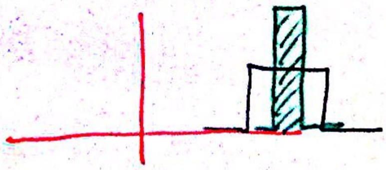
$y(t) = \int 0\, d\tau = \underline{\underline{0}}$



for $t+4 < 4$ & $t+5 > 4$

$\Rightarrow \underline{-1 < t < 0}$

$$y(t) = \int_{4}^{t+5} (1)(2)\, d\tau = \underline{2t+2}$$



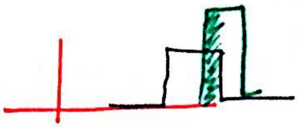
for $t+4 > 4$ & $t+5 < 6$

$\Rightarrow \underline{0 < t < 1}$

$$y(t) = \int_{t+4}^{t+5} (1)(2)\, d\tau = \underline{\underline{2}}$$

continue. ⟹

⇒



for $t+4<6$ & $t+5>6$

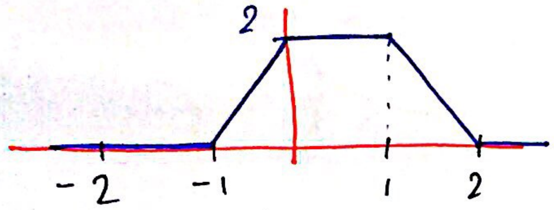
$\Rightarrow \quad 1<t<2$

$$y(t) = \int_{t+4}^{6} (1)(2)\,d\tau = (2)(2-t) = 4-2t$$

for $t+4>6 \Rightarrow t>2$

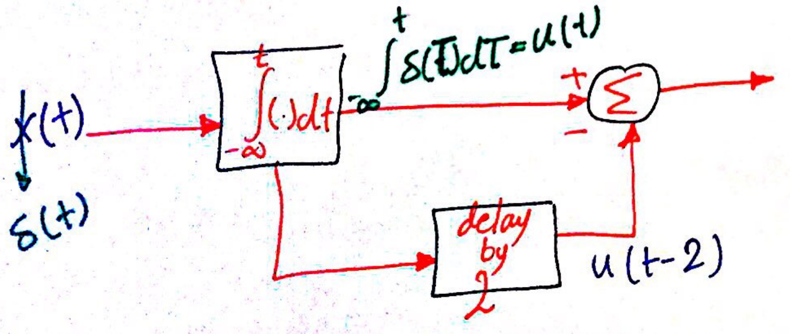
$y(t) =$ Zero.

so $$y(t) = \begin{cases} 0 & , t<-1 \\ 2t+2 & , -1<t<0 \\ 2 & , 0<t<1 \\ -2t+4 & , 1<t<2 \\ 0 & , t>2 \end{cases}$$

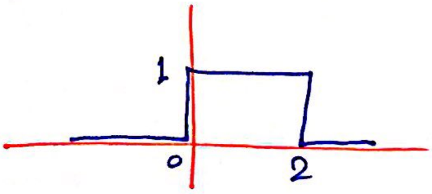


* **Note:** Convelution between two rect with different width result a trapazoidal.
if the two with same width (special case) result a triangle.

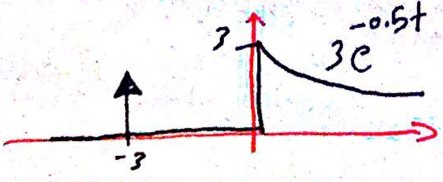
Example: determine $h(t)$ then $y(t)$:



y(t) ⟶ $h(t) = u(t) - u(t-2)$



for $x(t)$:



$$\Rightarrow x(t) = \underbrace{\delta(t+3)}_{x_1(t)} + \underbrace{3e^{-0.5t}u(t)}_{x_2(t)}$$

⟹

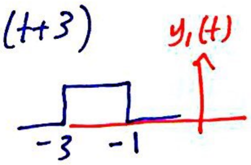
$$\Rightarrow \quad y(t) = x(t) \star h(t) = [x_1(t)+x_2(t)] \star h(t)$$
$$= \underbrace{x_1(t) \star h(t)}_{y_1(t)} + \underbrace{x_2(t) \star h(t)}_{y_2(t)}$$

$\Rightarrow$ find $y(t) = y_1(t) + y_2(t)$

$*$ for $y_1(t)$:

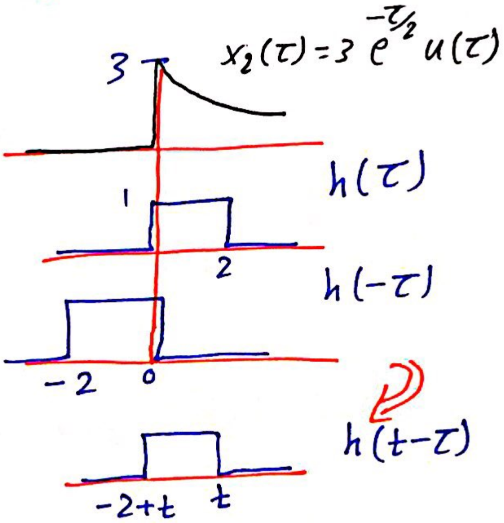
$y_1(t) = x_1(t) \star h(t) = \delta(t+3) \star h(t) \overset{\Rightarrow \text{property.}}{=} h(t+3)$

remember: $h(t) = h(t) \star \delta(t)$

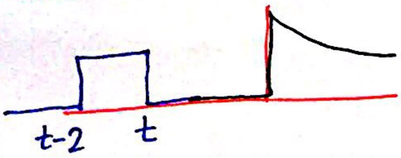


$*$ for $y_2(t)$:

$y_2(t) = x_2(t) \star h(t) = \int_{-\infty}^{\infty} x_2(\tau)\, h(t-\tau)\, d\tau$



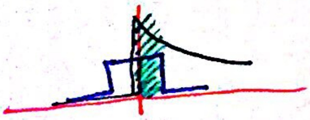
$*$ case (1):



for $t<0$ (No overlapping).

$\therefore \; y_2(t) = 0$

$*$ case (2):



$t>0$ & $t-2<0$

$\Rightarrow 0<t<2$

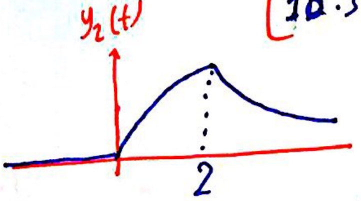
$$y_2(t) = \int_0^t 3e^{-\tau/2} u(\tau) \cdot 1\, d\tau$$
$$= -6e^{-\tau/2} \Big|_0^t = 6-6e^{-t/2}$$

$*$ Case (3):

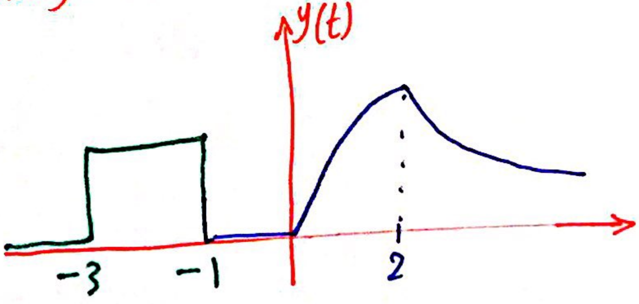
$t-2>0 \Rightarrow t>2$

$$y_2(t) = \int_{t-2}^{t} 3e^{-\tau/2} u(\tau)\, d\tau = -6(e^{-t/2} - e^{1-t/2})$$
$$= 10.31\, e^{-t/2}$$

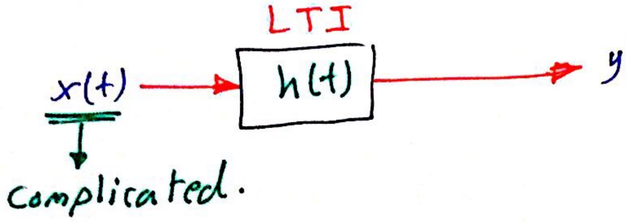
$$\Rightarrow y_2(t) = \begin{cases} 0 & , t<0 \\ 6-6e^{-t/2} & , 0<t<2 \\ 10.31\, e^{-t/2} & , t>2 \end{cases}$$



Finally: Add $y_1(t)$ to $y_2(t)$:



※ Fourier Series:



$y(t) = x(t) \star h(t)$

$y(t) = [a_1 x_1(t) + \cdots + a_N x_N(t)] \star h(t)$

$= \sum_{n=1}^{N} a_n x_n(t) \star h(t)$

$x(t) = a_1 x_1(t) + \cdots + a_N x_N(t)$

$= \sum_{n=1}^{N} a_n x_n(t)$

$x(t) \Rightarrow T_0 = \frac{2\pi}{\omega_0}$

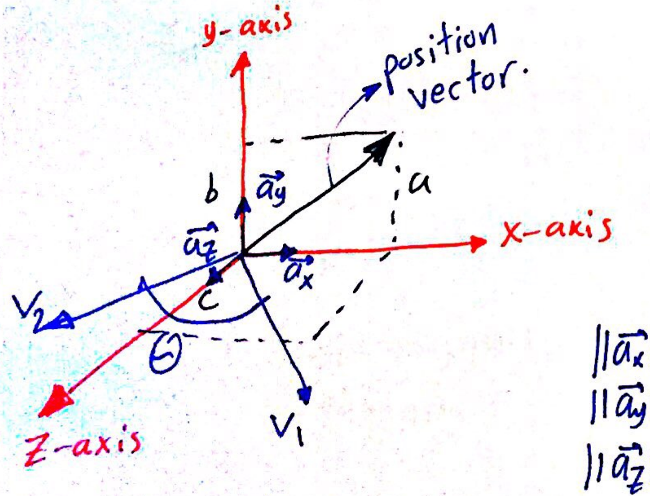
$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$, $\underset{\nearrow \text{weight or projection.}}{C_n} = C[n]$

$\{ e^{jn\omega_0 t} \}_{n \in \mathbb{Z}}$

$x(t) = \sum_{n=-\infty}^{\infty} C_n \phi_n(t)$

$\{\phi_n(t)\}_{n \in \mathbb{Z}} = \{ \ldots, \phi_{-2}(t), \phi_{-1}(t), \phi_0(t), \phi_1(t), \ldots \}$

※ vector space: $(\mathbb{R}^3)$



colomn vector

$\vec{V} = (a,b,c) = \underset{3 \times 1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \underline{\underline{a}}\, \vec{a_x} + \underline{\underline{b}}\, \vec{a_y} + \underline{\underline{c}}\, \vec{a_z}$

- scalars.
- weights.
- Coefficients.
- projections.

$\|\vec{a_x}\| = 1$ $\quad \vec{a_x} \perp \vec{a_y}$

$\|\vec{a_y}\| = 1$ $\quad \vec{a_y} \perp \vec{a_z}$

$\|\vec{a_z}\| = 1$ $\quad \vec{a_x} \perp \vec{a_z}$

if this satisfied: $\vec{a_x}, \vec{a_y}, \vec{a_z}$ are orthonormal set of basis vectors.

$\Longrightarrow$

Let: $\vec{V_1} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, \vec{V_2} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

inner product:

$<\vec{V_1},\vec{V_2}> = \vec{V_1}^T \cdot \vec{V_2}$

$= [a_1\ b_1\ c_1] \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = a_1 a_2 + b_1 b_2 + c_1 c_2$

$= ||\vec{V_1}||\,||\vec{V_2}|| \cos\theta$

$<\vec{V_1},\vec{V_1}> = ||\vec{V_1}||^2$

$<\vec{a_x},\vec{a_x}> = [1\ 0\ 0] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$

$<\vec{a_x},\vec{a_y}> = [1\ 0\ 0] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$

$a = <\vec{V},\vec{a_x}> = [a\ b\ c] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{\underline{a}}$

$b = <\vec{V},\vec{a_y}> = [a\ b\ c] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{\underline{b}}$

* $\underline{\underline{R^N}}$:

$\vec{V} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = c_1 \vec{a_1} + \cdots + c_N \vec{a_N} = \sum_{n=1}^{N} c_n \vec{a_n}$ $\{\vec{a_n}\}_{n=1}^{N}, ||\vec{a_n}|| = 1$

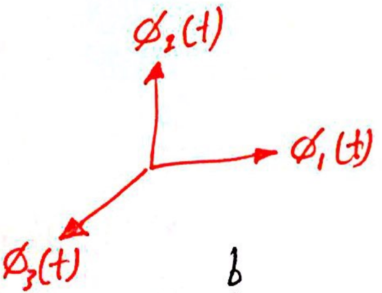
$<\vec{a_K},\vec{a_L}> = 0, \forall K \,\&\, L \in \{1,\ldots,N\} \Rightarrow \vec{a_K} \perp \vec{a_L}$

$c_n = <\vec{V},\vec{a_n}>$

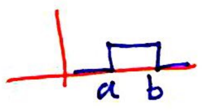
* for signals:

$\{\ldots,\phi_{-1}(t),\phi_0(t),\ldots\}$

$\{\phi_n(t)\}$ basis signals orthognal.



$<f(t),g(t)> = \int_a^b f(t)\cdot g^*(t)\,dt$



$\Rightarrow <f(t),g(t)> = 0 \Rightarrow f(t) \perp g(t)$

$<f(t),f(t)> = \int_a^b f(t)\cdot f^*(t)\,dt = \int_a^b |f(t)|^2\,dt \equiv \underline{\underline{E_f}}$

$\{\phi_n(t)\}$ basis signals orthognal. $\quad <\phi_K(t), \phi_L(t)> = 0$

$$\Rightarrow \phi_K(t) \perp \phi_L(t)$$

$$\forall K \,\&\, L \in \mathbb{Z}$$

$\swarrow$

if $E_K = 1$

Orthonormal.

* if $\{\phi_n(t)\}$ orthognal

$$\left\{\frac{\phi_n(t)}{\sqrt{E_n}}\right\} \Rrightarrow \text{Orthonormal.}$$

$$<\frac{\phi_n(t)}{\sqrt{E_n}}, \frac{\phi_n(t)}{\sqrt{E_n}}> = \int_a^b \frac{\phi_n(t)}{\sqrt{E_n}} \cdot \left(\frac{\phi_n^*(t)}{\sqrt{E_n}}\right) dt = \frac{1}{E_n}\int_a^b |\phi_n|^2 dt$$

$$= \frac{1}{E_n} * E_n = \boxed{1} \quad \#$$

$*$ given : $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$ orthognal

$E_1 = 2J \quad E_2 = 10J \quad E_3 = 7J$

$\Rightarrow$ To be orthonormal:

$$\left\{\frac{\phi_1(t)}{\sqrt{2}}, \frac{\phi_2(t)}{\sqrt{10}}, \frac{\phi_3(t)}{\sqrt{7}}\right\}$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, \phi_n(t) \qquad <x(t), \phi_K(t)>$$

$$= \int_a^b x(t) \cdot \phi_K^*(t)\, dt$$

$$= \int_a^b \sum_{n=-\infty}^{\infty} \phi_n(t)\, C_n \cdot \phi_K^*(t)\, dt$$

$$\Rightarrow = \sum_{n=-\infty}^{\infty} C_n \int_a^b \phi_n(t)\, \phi_K^*(t)\, dt = \sum_{n=-\infty}^{\infty} C_n <\phi_n(t), \phi_K(t)> = \sum_{n=-\infty}^{\infty} C_n \begin{cases} 0, & n \neq K \\ E_K, & n = K \end{cases}$$

$$\Rrightarrow$$

⇛ when:

$$n \neq K \Rightarrow \langle \phi_n(t), \phi_K(t) \rangle = 0$$

$$n = K \Rightarrow \langle \phi_n(t), \phi_K(t) \rangle = E_n$$

$$\langle x(t), \phi_K(t) \rangle = C_K E_K$$

$$\Rightarrow \boxed{C_K = \frac{1}{E_K} \langle x(t), \phi_K(t) \rangle}$$

as a summary:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \phi_n(t) \qquad \{\phi_n(t)\}_{n \in \mathbb{Z}} \qquad \phi_K(t) \perp \phi_L(t) \quad \forall K \,\&\, L \in \mathbb{Z}$$

$$\Rightarrow \underline{\underline{C_n = \frac{1}{E_n} \langle x(t), \phi_n(t) \rangle}}$$

$$\Rightarrow \langle \phi_K(t), \phi_L(t) \rangle = \int_a^b \phi_K(t) \cdot \phi_L^*(t) dt = \begin{cases} 0, & K \neq L \\ E_K, & K = L \end{cases}$$

Ex. show that $\phi_m(t) = \sin(mt)$, $m \in \mathbb{Z}^+$ it is orthognal over $[-\pi, \pi]$?

$\omega_0 = m$

$T_0 = \frac{2\pi}{m}$

define: $\phi_K(t) = \sin(Kt)$ & $\phi_L(t) = \sin(Lt)$ assume $K \neq L$

$$\Rightarrow \langle \phi_K(t), \phi_L(t) \rangle = \int_{-\pi}^{\pi} \sin(Kt)\sin(Lt) dt = \int_{-\pi}^{\pi} \frac{1}{2}\left(\cos((K-L)t) - \cos((K+L)t)\right) dt$$

$= 0 - 0 = 0$ (integral of cosine on its period zero)

so it is orthognal.

if we want to know it is orthonormal:

$$\langle \phi_K(t), \phi_K(t) \rangle \overset{?}{=} 1$$

$$= \int_{-\pi}^{\pi} \sin(Kt)\sin(Kt) dt = \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2Kt) dt \rightarrow \boxed{E_K = \pi}$$

Not orthonormal.

To make it orthonormal $\Rightarrow \left\{ \frac{\sin(mt)}{\sqrt{\pi}} \right\}_{m \in \mathbb{Z}^+}$

Ex. is $\phi_n(t) = e^{jn\omega_o t}$, $n \in \mathbb{Z}$ orthonormal over $(0, T_o)$?

$\{\ldots, e^{-j2\omega_o t}, e^{-j\omega_o t}, 1, e^{j\omega_o t}, \ldots\}$

DC

first harmonic.

$\omega = n\omega_o$

so $T = \frac{T_o}{n}$

Let $\phi_K(t) = e^{jK\omega_o t}$

$\phi_L(t) = e^{jL\omega_o t}$

$L \neq K$

$$<\phi_K(t), \phi_L(t)> = \int_0^{T_o} e^{jK\omega_o t} \cdot e^{-j\omega_o t L} \, dt = \int_0^{T_o} e^{j(K-L)\omega_o t} \, dt$$

$$= \frac{1}{j(K-L)\omega_o} e^{j(K-L)\omega_o t} \Big|_0^{T_o}$$

$$= \frac{1}{j(K-L)\omega_o} \left[e^{j(\overset{M}{K-L})\,\omega_o \overset{\frac{2\pi}{\omega_o}}{T_o}} - 1 \right] \quad \frac{e^{j2\pi M} - 1}{j(K-L)\omega_o} = 0$$

so orthognal.

is it orthonormal !? $<\phi_K(t), \phi_K(t)> = \int_0^{T_o} e^{j(K-K)\omega_o t} \, dt = T_o = E_K \neq 1$

Not orthonormal.

$\Longrightarrow$ To be orthonormal: $\left\{ \frac{e^{jn\omega_o t}}{\sqrt{T_o}} \right\}$

※ Exponential Fourier Series:

- $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_o t}$, $\omega_o = \frac{2\pi}{T_o}$ where: $C_n = \frac{1}{T_o} \int_0^{T_o} x(t) e^{-jn\omega_o t} \, dt$.

- C_n could be written as:

$C_n = C[\omega_o n] \longrightarrow$ it is a complex function:

discrete function of frequancy.

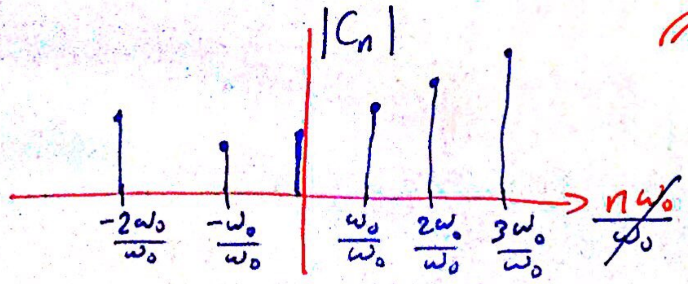
$C_n = a_n + j b_n$.

special case (DC): $C_0 = \frac{1}{T_o} \int_{T_o} x(t) \, dt$.

it is the average value.

$\{ \ldots, e^{-j\omega_o t}, 1, e^{j\omega_o t}, \ldots \}$

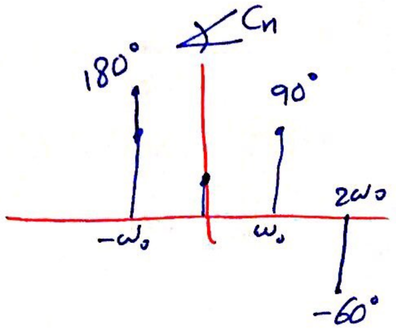
C_{-1} C_0 C_1



$\Rightarrow$ Magnitude Spectrum.

+ve & -ve : double sided magnitude spectrum.

just +ve : single sided mag. spec.



phase spectrum.

Magnitude & phase spectrum are called:

Frequancy spectra.

or line spectra.

if $x(t)$ real:

$$\underline{\underline{C_{-n}^* = C_n}}$$

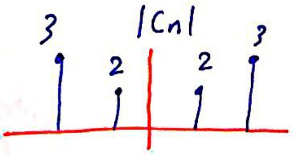
$$C_n = |C_n|\, e^{j\measuredangle C_n}$$
$$C_{-n}^* = |C_{-n}|\, e^{-j\measuredangle C_n}$$

$$|C_n| = |C_{-n}|$$
$$C[n] = C[-n]$$

even.

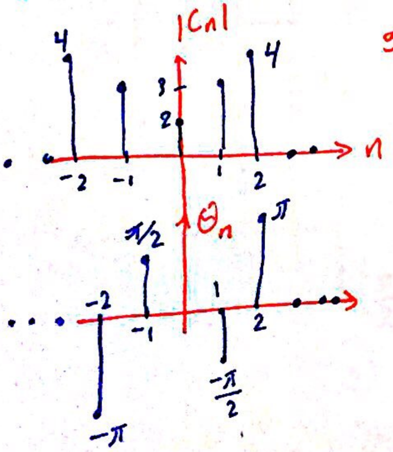
$$\measuredangle C_n = -\measuredangle C_n$$

odd.



$\Rightarrow$

Ex.



given:

$\underline{\underline{\omega_0 = 2}}$

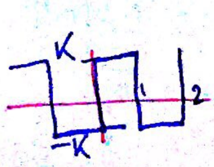
$$x(t) = \sum_{n=-2}^{2} C_n e^{jn(2)t}$$
$$= C_{-2} e^{-j4t} + C_{-1} e^{-j2t} + C_0 + C_1 e^{j2t} + C_2 e^{j4t}$$

$$\left.\begin{aligned} C_{-2} &= 4e^{-j\pi} = \underline{\underline{-4}} \\ C_2 &= 4e^{j\pi} = \underline{\underline{-4}} \end{aligned}\right\} \quad \begin{aligned} C_{-1} &= 3e^{j\pi/2} = \underline{\underline{j3}} \\ C_1 &= 3e^{-j\frac{\pi}{2}} = \underline{\underline{-j3}} \end{aligned} \quad \left\{ C_0 = 2 \right.$$

$$\Rightarrow x(t) = -4e^{-j4t} + 3e^{j\pi/2} e^{-j2t} + 2 + 3e^{-j\frac{\pi}{2}} e^{j2t} + -4e^{j4t}$$

you can simplify it more.

Ex. find fourier series of $x(t)$:



$T_0 = 2$

$\Rightarrow \omega_0 = \pi$

$$C_0 = \frac{1}{2} \int_2 x(t)\, dt = 0 \text{ (odd signal).}$$

$$C_n = \frac{1}{2} \int_2 x(t)\, e^{-jn\pi t}\, dt = \frac{1}{2}\left[\int_0^1 K e^{-jn\pi t} dt + \int_1^2 -K e^{-jn\pi t} dt\right]$$

$$= \frac{K}{2}\left[\frac{1}{-jn\pi} e^{-jn\pi t}\Big|_0^1 - \frac{1}{-jn\pi} e^{-jn\pi t}\Big|_1^2\right]$$

$$= \frac{-K}{2jn\pi}\left[e^{-jn\pi} - 1 - \overset{=1}{e^{-j2n\pi}} + e^{-jn\pi}\right] = \frac{-K}{j2n\pi}\left[2e^{-jn\pi} - 2\right] \quad \Rightarrow$$

$\Rightarrow C_n = \frac{K}{jn\pi}\left[1 - e^{-jn\pi}\right]$ where: $e^{-jn\pi} = \begin{cases} +1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$ $n \in \mathbb{Z}$

if n even: $\boxed{C_n = 0}$

if n odd: $\boxed{C_n = \frac{2K}{jn\pi}}$

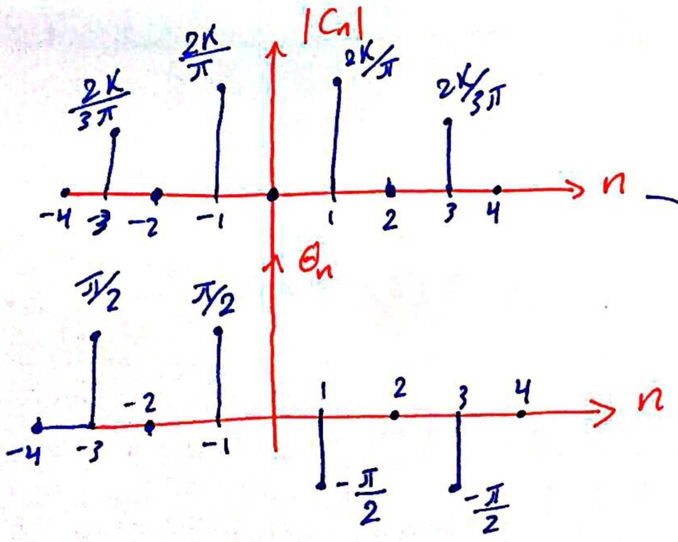
so $C_n = \begin{cases} 0, & n \text{ even} \\ -j\frac{2K}{n\pi}, & n \text{ odd} \end{cases}$

$= \frac{2K}{n\pi} e^{-j\frac{\pi}{2}}$

$\Rightarrow |C_n| = \begin{cases} 0, & n \text{ even} \\ \frac{2K}{|n|\pi}, & n \text{ odd} \end{cases}$

$\measuredangle C_n = \theta_n = \begin{cases} \text{any angle}, & n \text{ even.} \\ -\pi/2, & n(+ve) \,\&\, \text{odd.} \\ +\pi/2, & n(-ve) \,\&\, \text{odd.} \end{cases}$

Now Draw $|C_n|$ & $\measuredangle C_n$:



since the example real so as we expected that

$|C_n|$ even.

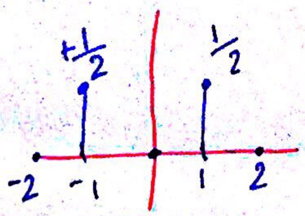
θ_n odd.

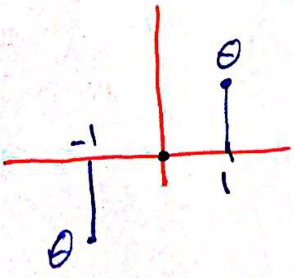
we could draw it as a single sided spectrum.

$A_n = 2|C_n|$.

Ex. find fourier series for: $x(t) = \cos(\omega_0 t + \theta)$

$$\underset{\text{even}}{x(t)} = \frac{1}{2} e^{j(\omega_0 t+\theta)} + \frac{1}{2} e^{-j(\omega_0 t+\theta)} = \underbrace{\frac{1}{2} e^{j\theta}}_{C_1} \underbrace{e^{j\omega_0 t}}_{n=1} + \underbrace{\frac{1}{2} e^{-j\theta}}_{C_{-1}} \underbrace{e^{-j\omega_0 t}}_{n=-1}$$





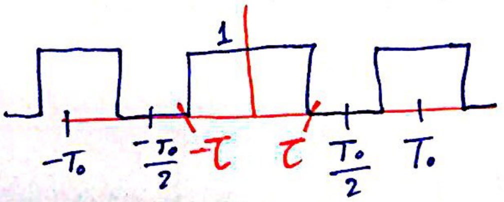
$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

* Note:

if we want to write C_n as a function:

$$C_n = \dots + \frac{2K}{3\pi} e^{j\frac{\pi}{2}} \delta[n+3] + \frac{2K}{\pi} e^{j\frac{\pi}{2}} \delta[n+1] + \frac{2K}{\pi} e^{-j\frac{\pi}{2}} \delta[n-1] + \dots$$

*Ex. find fourier series for:



$$C_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\int_{-\tau}^{\tau}(1)\,dt = \boxed{\frac{2\tau}{T_0}}$$

$$C_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-jn\omega_0 t}\,dt$$

$$= \frac{e^{-jn\omega_0 t}}{-T_0\, jn\omega_0}\Bigg|_{-\tau}^{\tau} = \frac{e^{jn\omega_0\tau} - e^{-jn\omega_0\tau}}{jn(2\pi)} * \frac{2j}{2j}$$

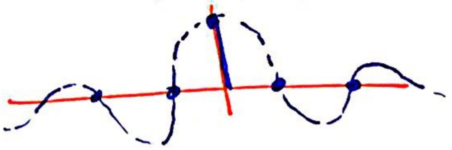
$$= \boxed{\frac{1}{n\pi}\sin(n\omega_0\tau)}$$

$$\Rightarrow \frac{1 \;{}^{*\omega_0\tau}}{\frac{\tau\omega_0 n\pi}{\pi}\;\pi}\sin\left(\overset{X}{\frac{n\omega_0\tau}{\pi}}\,\pi\right)$$

$$= \frac{\omega_0\tau}{\pi}\operatorname{sinc}\left(\frac{n\omega_0\tau}{\pi}\right)$$

$$= \frac{2\tau}{T_0}\operatorname{sinc}\left(\frac{n\omega_0\tau}{\pi}\right)$$

$$\operatorname{sinc}(n) = \frac{\sin(\pi n)}{\pi n}$$

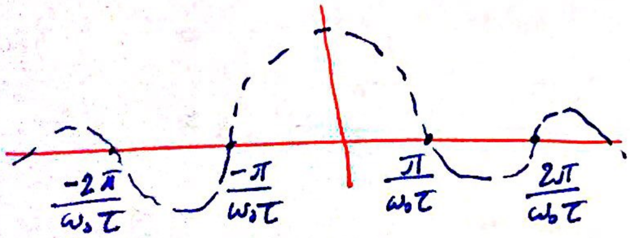


$$x[n] = \operatorname{sinc}(n) = \frac{\sin(\pi n)}{\pi n}$$

$\hookrightarrow x[an]$

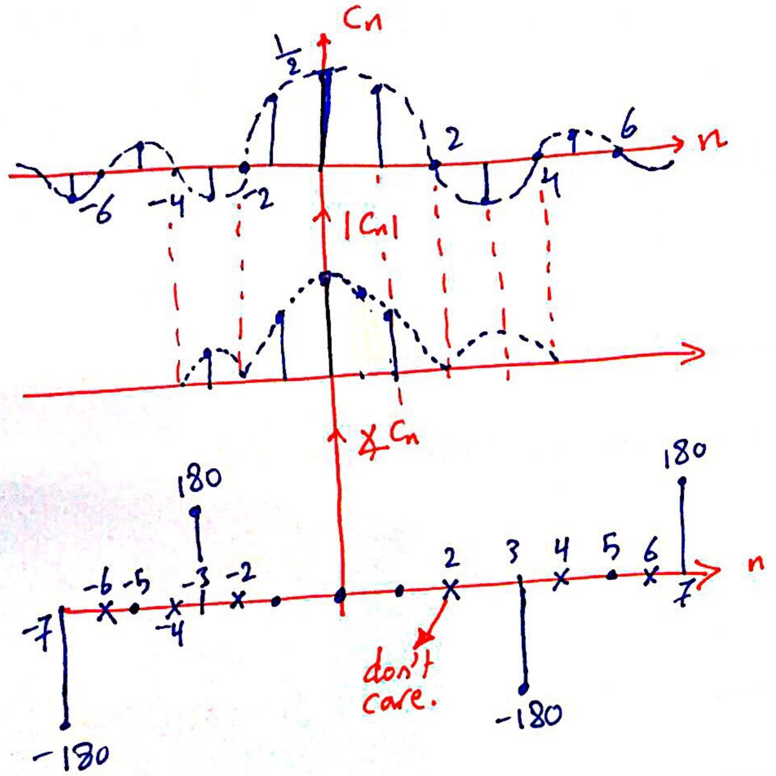
$\downarrow$

$\frac{\omega_0\tau}{\pi}$



Assume $\tau = \frac{T_0}{4}$

$C_0 = \frac{2T_0}{4T_0} = \frac{1}{2}$ $\qquad C_n = \frac{1}{2}\,\text{sinc}\left(\frac{n}{2}\right)$



$*$ Compact (combined) trigonometric form:

if $x(t) = x^*(t)$ [real]. $\longrightarrow$ $C_n^* = C_{-n}$ $\Rightarrow |C_n| = |C_{-n}|$

$\theta_n = -\theta_{-n}$

$$\Rightarrow x(t) = A_0 + \sum_{\underline{n=1}}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

$$\text{if } x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \dots + C_{-1} e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + \dots$$

$$= C_0 + \sum_{n=1}^{\infty} \left(\underbrace{C_{-n}}_{=C_n^*} e^{-jn\omega_0 t} + C_n e^{jn\omega_0 t}\right)$$

$C_n = |C_n| e^{j\theta_n}$

$$= C_0 + \sum_{n=1}^{\infty} \left(|C_n| e^{-j\theta_n} e^{-jn\omega_0 t} + |C_n| e^{j\theta_n} e^{-jn\omega_0 t}\right)$$

$$= C_0 + \sum_{n=1}^{\infty} |C_n| \left[\frac{e^{-j(n\omega_0 t + \theta_n)} + e^{j(n\omega_0 t + \theta_n)}}{2} \right] * 2 \quad \Rightarrow$$

$$\Rightarrow = C_0 + \sum_{n=1}^{\infty} 2|C_n| \cos(n\omega_0 t + \theta_n)$$ → Compact.

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

$$\Rightarrow \boxed{\begin{aligned} A_0 &= C_0 \\ A_n &= 2|C_n| \end{aligned}}$$

Now write $x(t)$ as:

$$x(t) = A_0 + A_1 \cos(\underbrace{\omega_0}_{\omega_1} t + \theta_1) + A_2 \cos(\underbrace{2\omega_0}_{\omega_2} t + \theta_2) + \cdots$$

remember: $\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$$\Rightarrow \cos(n\omega_0 t + \theta_n) = \cos(n\omega_0 t)\cos(\theta_n) - \sin(n\omega_0 t)\sin(\theta_n)$$

$$\Rightarrow x(t) = C_0 + \sum_{n=1}^{\infty} 2|C_n| \left[\cos(n\omega_0 t)\cos\theta_n - \sin(n\omega_0 t)\sin\theta_n\right]$$

$$\Rightarrow x(t) = C_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$\Rightarrow x(t) = A_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

→ "Combined"

$C_n = |C_n| e^{j\theta_n}$

$2C_n = \left(|C_n|\cos\theta_n + j|C_n|\sin\theta_n\right) \times 2$

Let $2C_n = a_n - j\,b_n$

$$\boxed{\begin{aligned} a_n &= 2|C_n|\cos\theta_n \\ b_n &= -2|C_n|\sin\theta_n \end{aligned}}$$

where:

$$C_n = \frac{a_n}{2} - j\,\frac{b_n}{2} \qquad A_n = 2|C_n| = \sqrt{a_n^2 + b_n^2}$$

$$|C_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \qquad \theta_n = \tan^{-1}\left(-b_n/a_n\right)$$

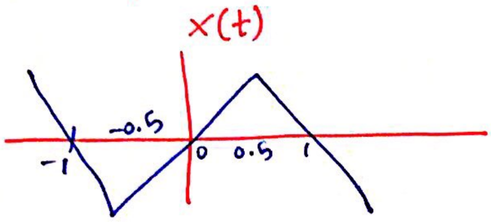
$$C_n = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$ → $= \cos(n\omega_0 t) - j\sin(n\omega_0 t)$

$$= \frac{1}{T_0} \int_{T_0} x(t)\cos(n\omega_0 t)\,dt - \frac{1}{T_0} j \int_{T_0} x(t)\sin(\omega_0 n t)\,dt = \frac{a_n}{2} - j\,\frac{b_n}{2}$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t)\cos(n\omega_0 t)\,dt.$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t)\sin(n\omega_0 t)\,dt.$$

Example: find trigonometric form of fourier series of $x(t)$:



$T_0 = 2, \omega_0 = \pi$

$$A_0 = \frac{1}{2}\int_{2} x(t)dt = \frac{1}{2}\left[\int_{-0.5}^{0.5} 2t\,dt + \int_{0.5}^{1.5} -2(t-1)dt\right]$$

$= Zero$ (odd signal)

Both triangles will cancel eachother.

$$a_n = \frac{2}{2}\int_{2} x(t)\cos(n\pi t)dt$$

$$= \int_{-0.5}^{0.5} \underbrace{2t\cos(n\pi t)}_{odd} dt + \int_{0.5}^{1.5} -2(t-1)\cos(n\pi t)dt = 0$$

since it is an odd signal so just the fourier series contain b_n.

$$b_n = \int_{-0.5}^{0.5} 2t\sin(n\pi t)dt + \int_{0.5}^{1.5} -2(t-1)\sin(n\pi t)dt$$

do the integration:

$$\Rightarrow b_n = \begin{cases} 0 & , n \text{ even} \\ \frac{8}{(n\pi)^2} & , n = 1, 5, 9, 13, \ldots \\ \frac{-8}{(n\pi)^2} & , n = 3, 7, 11, \ldots \end{cases}$$

** $x(t) = \frac{8}{\pi^2}\sin(\pi t) - \frac{8}{9\pi^2}\sin(3\pi t) + \frac{8}{25\pi^2}\sin(5\pi t) - \frac{8}{49\pi^2}\sin(7\pi t) + \cdots$

if we want $x(t)$ in compact form:

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \begin{cases} -\frac{\pi}{2} & , b_n \text{ +ve} \\ \frac{\pi}{2} & , b_n \text{ -ve} \end{cases}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \frac{8}{(n\pi)^2}$$

OR: remember: $\cos(\theta \mp \frac{\pi}{2}) = \pm \sin\theta$

$$x(t) = \frac{8}{\pi^2}\cos(\pi t - \frac{\pi}{2}) + \frac{8}{9\pi^2}\cos(3\pi t + \frac{\pi}{2}) + \frac{8}{25\pi^2}\cos(5\pi t - \frac{\pi}{2}) + \cdots$$

as a notation: $x(t) = \frac{8}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}\cos\left[(2m+1)\pi t + (-1)^{m+1}\frac{\pi}{2}\right]$

* Existence of fourier Series:

- * for any power series ⇒ fourier series exist.
- * for any absolutly integrable signal ⇒ fourier series exist

* Proparties of Fourier Series:

let $x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t}$, $y(t) = \sum \beta_n e^{jn\omega_0 t}$, $Z(t) = \sum \gamma_n e^{jn\omega_0 t}$

$\alpha_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$ $\quad$ $\beta_n = \frac{1}{T_0}\int_{T_0} y(t) e^{-jn\omega_0 t} dt$ $\quad$ $\gamma_n = \frac{1}{T_0}\int_{T_0} Z(t) e^{-jn\omega_0 t} dt$

Amplitude Transformation:

Coefficients:

$x(t) \Longrightarrow \alpha_n$ $\quad$ we need the relation between $\underline{\underline{\beta_n}} \propto \underline{\underline{\alpha_n}}$

* $y(t) = x(t) + B$

The only affected part: α_0

$\Rightarrow \quad \underline{\underline{\alpha_0 + B = \beta_0}}$, $\alpha_n = \beta_n , n \neq 0$

the proof: $\sum \beta_n e^{jn\omega_0 t} = B + \sum \alpha_n e^{jn\omega_0 t}$

$\underline{\underline{\beta_0}} + \sum \underline{\beta_n} e^{jn\omega_0 t} = \underline{\underline{B + \alpha_0}} + \sum \underline{\alpha_n} e^{jn\omega_0 t}$

---

* $y(t) = A x(t)$

$\underline{\underline{\boxed{\beta_n = A\alpha_n}}}$ $\quad$ prove it: $\sum_{n=-\infty}^{\infty} \underline{\beta_n} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \underline{\underline{A\alpha_n}} e^{jn\omega_0 t}$

---

* $y(t) = A x(t) + B$

$\boxed{\beta_0 = A\alpha_0 + B}$ $\quad$ $\boxed{\beta_n = A\alpha_n}$, $n \neq 0$

Time Transformation:

$$x(t) \Longrightarrow \alpha_n$$

* $y(t) = x(t-t_0)$

$$\underline{\underline{\beta_n = \alpha_n e^{-jn\omega_0 t_0}}}$$

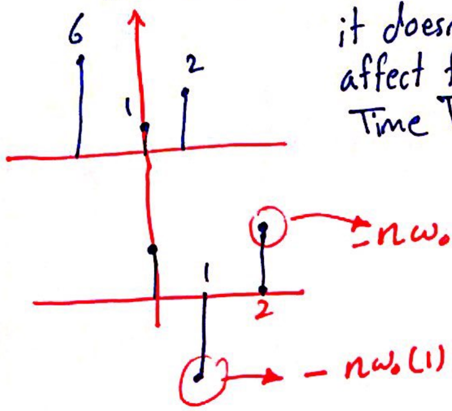
$$|\beta_n| e^{j\measuredangle\beta_n} = |\alpha_n| e^{j\measuredangle\alpha_n} e^{-jn\omega_0 t_0}$$

$$= |\alpha_n| e^{j(\measuredangle\alpha_n - n\omega_0 t_0)}$$

$$\Rightarrow |\beta_n| = |\alpha_n|$$

$$\measuredangle\beta_n = \measuredangle\alpha_n - n\omega_0 t_0$$

$|\beta_n| = |\alpha_n|$



it doesn't affect from Time Transformation.

---

* $y(t) = x(-t)$

$$\beta_n = \alpha_{-n} = \alpha_n^* \Rightarrow |\beta_n| e^{\underline{\underline{j\measuredangle\beta_n}}} = |\alpha_n| e^{\underline{\underline{-j\measuredangle\alpha_n}}}$$

we just reflect the angle in the x-axis.

proof:

$$\sum_{n=-\infty}^{\infty} \beta_n \underline{\underline{e^{jn\omega_0 t}}} = \sum_{n=-\infty}^{\infty} \alpha_n e^{-jn\omega_0 t}$$

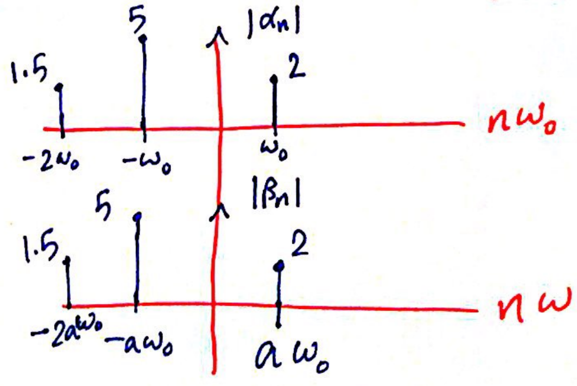
replace (n) by (−n).

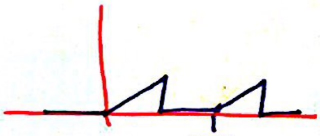
$$= \sum_{n=-\infty}^{\infty} \alpha_{-n} \underline{\underline{e^{jn\omega_0 t}}}$$

so $\underline{\underline{\beta_n = \alpha_{-n}}}$.

---

* $y(t) = x(at) \Rightarrow \underline{\underline{\beta_n = \alpha_n}}$



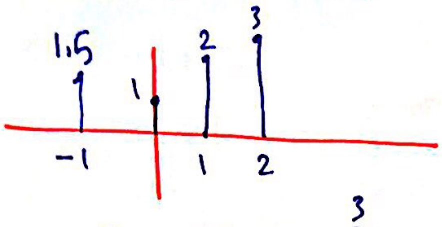


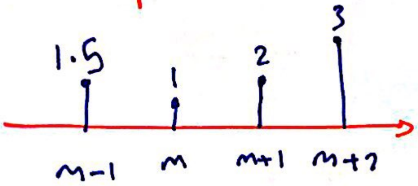
$y(t) = x(at)$

$T = \frac{T_0}{a}$

$\omega = a\omega_0$

* Frequancy shift:





$\alpha_n = \alpha[n]$

$\beta_n = \alpha_{n-m} = \alpha[n-m]$

$y(t) = x(t) \cdot e^{jm\omega_0 t}.$

*prove:

$$\beta_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{jm\omega_0 t}\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j(n-m)\omega_0 t} = \alpha_{n-m}$$

* Linearity:

*scaling & additivity: $AX(t) \Rightarrow A\alpha_n$

$$\begin{array}{cccc} Z(t) = & K_1 x(t) & + K_2\, y(t) & + \dots \\ \downarrow & \downarrow & \downarrow & \\ \gamma_n = & K_1\alpha_n & + K_2\,\beta_n & + \dots \end{array}$$

* Multiplication:

$$\underset{\gamma_n}{Z(t)} = \underset{\alpha_n}{x(t)} \cdot \underset{\beta_n}{y(t)} \Rightarrow \gamma_n = \sum_{m=-\infty}^{\infty} \alpha_m \beta_{n-m} = \alpha_n \star \beta_n$$

$$= \sum_{K=-\infty}^{\infty} \beta_K \alpha_{n-K} = \beta_n \star \alpha_n$$

*periodic convolution:

$$\underset{\downarrow\, T_0}{x(t)} \star \underset{\downarrow\, T_0}{y(t)} = \underset{\hookrightarrow \text{also } T_0}{Z(t)} = \frac{1}{T_0}\int_{T_0} x(\tau)\, y(t-\tau)\, d\tau$$

$\gamma_n = \alpha_n \cdot \beta_n$ $\quad Z(t) = \sum_{n=-\infty}^{\infty} \gamma_n e^{jn\omega_0 t}$ "see proof in slides".

*Differentiation:

$x(t) \longrightarrow \alpha_n$

$y(t) = \dfrac{dx(t)}{dt}$

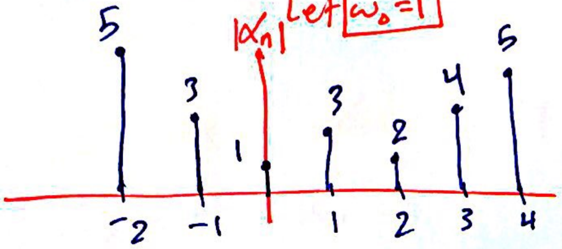
$\beta_n = jn\omega_0 \alpha_n$

$\boxed{\beta_0 = 0}$

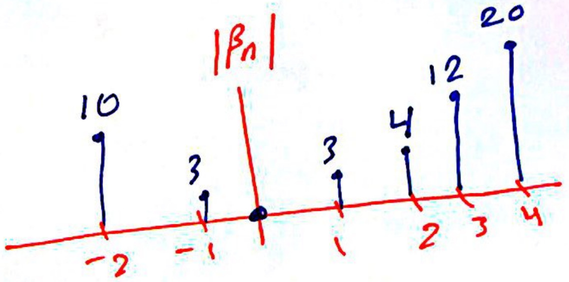
$|\beta_n| e^{j\measuredangle\beta_n} = |n|\omega_0 e^{j\frac{\pi}{2}} \cdot |\alpha_n| e^{j\measuredangle\alpha_n}$

$$\boxed{\begin{aligned} |\beta_n| &= |n|\omega_0|\alpha_n| \\ \measuredangle\beta_n &= \measuredangle\alpha_n \pm 90^\circ \end{aligned}}$$

Amplification.



$\Longrightarrow$



"see proof in slides"

*Integration:

Attenuation.

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

if y(t) is periodic, Then:

$\beta_n = \dfrac{\alpha_n}{jn\omega_0}$

$$\boxed{\begin{aligned} |\beta_n| &= \frac{|\alpha_n|}{|n|\omega_0} \\ \measuredangle\beta_n &= \measuredangle\alpha_n \mp 90^\circ \end{aligned}}$$

"see proof in slides"

Condition:

$\Rightarrow$ if $\underline{\alpha_0 = 0}$ $\Rightarrow$ y(t) is periodic.

if $\alpha_0 \neq 0$ $\Rightarrow$ y(t) isn't periodic $\Rightarrow$ fourier series can't be found.

$\boxed{\beta_0 = 0}$.

* Effect of symmetry:

$$x(t) = A_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$A_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$\begin{matrix}a_n\\ b_n\end{matrix} = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \begin{matrix}\cos\\ \sin\end{matrix}(n\omega_0 t)\, dt$$

if $x(t)$ even:

$$a_0 = \frac{2}{T_0}\int_{T_0/2} x(t)dt$$

$$a_n = \frac{4}{T_0}\int_{T_0/2} x(t)\cos(n\omega_0 t)dt.$$

$b_n = 0$

$$\alpha_n = \frac{a_n}{2} \Rightarrow \boxed{|\alpha_n| = \frac{|a_n|}{2}}$$

$$\boxed{\measuredangle \alpha_n = 0, \pm\pi}$$

if $x(t)$ odd:

$A_0 = 0, \ a_n = 0$

$$b_n = \frac{4}{T_0}\int_{T_0/2} x(t)\sin(n\omega_0 t)dt$$

$$\alpha_n = -j\frac{b_n}{2} \Rightarrow \boxed{|\alpha_n| = \frac{|b_n|}{2}}$$

$$\boxed{\measuredangle \alpha_n = X, \pm 90^\circ}$$

* Half wave symmetry:

$x(t) = -x(-t), \ \forall t$

$x(t \pm \frac{T_0}{2}) = -x(t)$

always $A_0 = 0$

$$\begin{matrix}a_n\\ b_n\end{matrix} = \{0, \ n \text{ even}\}$$

Half-wave & even:

$b_n = 0 \ \forall n$

$$a_n = \begin{cases}\frac{4}{T_0}\int_{T_0/2} x(t)\cos(n\omega_0 t)\, dt & , n \text{ odd.}\\ 0 & , n \text{ even.}\end{cases}$$

$A_0 = 0$

Half-wave & odd:

$A_0 = 0$, $a_n = 0 \ \forall n$

$$b_n = \begin{cases} \frac{4}{T_0}\int_{T_0/2} x(t)\sin(n\omega_0 t)dt & , n \text{ odd.} \\ 0 & , n \text{ even.} \end{cases}$$

proof:

$x(t \pm \frac{T_0}{2}) = -x(t)$

$\Rightarrow y(t) = z(t)$

$\beta_n = \gamma_n$

$\Rightarrow^{*} y(t) = x(t \pm \frac{T_0}{2})$

$\Rightarrow \beta_n = \alpha_n e^{\mp j n \omega_0 \frac{T_0}{2}}$

$* \ z(t) = -x(t) \Rrightarrow \gamma_n = -\alpha_n$

$$\beta_n = \alpha_n \cdot e^{\mp j n \frac{2\pi}{T_0} \cdot \frac{T_0}{2}} = \alpha_n e^{\mp j n \pi}$$

$$= \alpha_n \cdot \begin{cases} +1 & , n \text{ even.} \\ -1 & , n \text{ odd.} \end{cases}$$

so $\beta_n = \begin{cases} +\alpha_n & , n \text{ even} \\ -\alpha_n & , n \text{ odd} \end{cases}$

$$\begin{Bmatrix} +\alpha_n & , n \text{ even} \\ -\alpha_n & , n \text{ odd} \end{Bmatrix} = -\alpha_n$$

$+\alpha_n = -\alpha_n$ just in case of $\alpha_n = 0$

so $\alpha_n = 0$ n even.

---

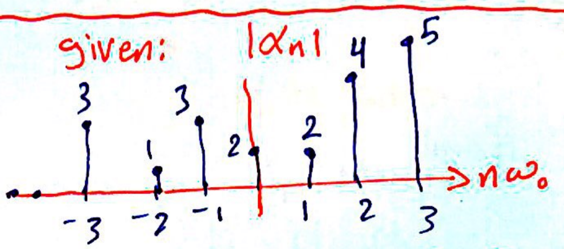
* Parseval's Theorem:

$$\underset{\overset{\downarrow}{T_0}}{x(t)} = \sum_{n=-\infty}^{\infty} \alpha_n e^{j n \omega_0 t}$$

$$P_T = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt$$

$$= \sum_{n=-\infty}^{\infty} |\alpha_n|^2$$

given:



$\Downarrow$ find P_T ?

$P_T = 9 + 1 + 9 + 4 + 4 + 16 + 25$

find P for first harmonic?

$P = 9 + 4 = 13 \ W$

$\Rightarrow$

For previous figure $|\alpha_n|$:

find P up to first harmonic?

$$P = 2^2+2^2+3^2 = 17\, W.$$

find P up to second harmonic?

$$P = 1+9+4+4+16 = 34\, W.$$

** for $\underbrace{x(t)}_{\text{Real } x(t)=x^*(t)} = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t+\theta_n)$

$$\boxed{P_T = A_0^2 + \sum_{n=1}^{\infty} \frac{A_n^2}{2}}$$

Proof: $x(t)= \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t}$

$$P_T = \frac{1}{T_0}\int_{T_0} x(t)\, x^*(t)dt = \frac{1}{T_0}\int_{T_0}\left(\sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t}\right)\cdot\left(\sum_{m=-\infty}^{\infty} \alpha_m^* e^{-jm\omega_0 t}\right)dt$$

$$= \frac{1}{T_0}\int \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \alpha_n \alpha_m^* e^{jn\omega_0 t}\cdot e^{-jm\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \underbrace{\alpha_n \alpha_m^*}_{=|\alpha_n|^2}\left(\frac{1}{T_0}\underbrace{\int e^{jn\omega_0 t}\cdot e^{-jm\omega_0 t}\, dt}_{\frac{1}{T_0}\langle \phi_n(t), \phi_m(t)\rangle}\right)$$

$\phi_n(t) = e^{jn\omega_0 t}$

*remember:

$$\sum_{n=1}^{K} a_n * \sum_{m=1}^{K} b_n = \sum_{n=1}^{K}\sum_{m=1}^{K} a_n\cdot b_n$$

$\Downarrow$

$$\sum_{n=-\infty}^{\infty} |\alpha_n|^2$$

#.

Now proof: $x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$

$$P_T = \frac{1}{T_0}\int_{T_0} x^2(t)\,dt = \frac{1}{T_0}\int_{T_0}\left(A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta)\right)^2$$

$$= \frac{1}{T_0}\int_{T_0}\left(A_0^2 + 2A_0\sum_{n=1}^{\infty} A_n \cos + \left(\sum A_n \cos\right)^2\right)dt$$

$$= \underbrace{\frac{1}{T_0}\int_{T_0} A_0^2\,dt}_{A_0^2} + 0 + \frac{1}{T_0}\int_{T_0}\left(\sum A_n \cos\right)^2 dt$$

$$\rightarrow \frac{1}{T_0}\int_{T_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n A_m \cos(n\omega_0 t + \theta_n) * \cos(m\omega_0 t + \theta_m)\,dt$$

$$\frac{A_n A_m}{2}\left[\cos((n-m)\omega_0 t + \theta_n - \theta_m) + \cos((n+m)\omega_0 t + \theta_n + \theta_m)\right]$$

you can reach that:

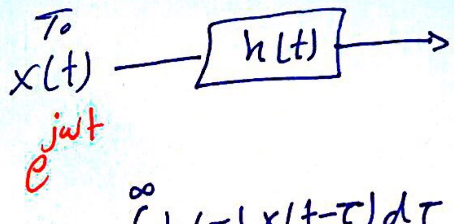
$$= \sum_{n=1}^{\infty} \frac{A_n^2}{2}$$

$$\boxed{\text{So } P_T = A_0^2 + \sum_{n=1}^{\infty} \frac{A_n^2}{2}}$$

* * *

end of second material.

* System Response to periodic Inputs:



$y(t) = x(t) * h(t)$

$y_e(t) = e^{j\omega t} \cdot H(\omega)$

proof: $y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,e^{j\omega(t-\tau)}\,d\tau = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)\,e^{-j\omega\tau}\,d\tau$

$= e^{j\omega t}\int_{-\infty}^{\infty} h(t)\,e^{-j\omega t}\,dt$

so $H(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{j\omega t}\,dt$

$H(\omega) = \mathcal{F}\{h(t)\}$

Signals Analysis and Systems.

Second Semester (2017)

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* * *

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* * *

Now proof:

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

$$P_T = \frac{1}{T_0}\int_{T_0} x^2(t)\,dt = \frac{1}{T_0}\int_{T_0}\left(A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta)\right)^2$$

$$= \frac{1}{T_0}\int_{T_0}\left(A_0^2 + 2A_0\sum_{n=1}^{\infty} A_n \cos + \left(\sum A_n \cos\right)^2\right)dt$$

$$= \underbrace{\frac{1}{T_0}\int_{T_0} A_0^2\,dt}_{A_0^2} + 0 + \frac{1}{T_0}\int_{T_0}\left(\sum A_n \cos\right)^2 dt$$

$$\rightarrow \frac{1}{T_0}\int_{T_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n A_m \cos(n\omega_0 t + \theta_n) * \cos(m\omega_0 t + \theta_m)\,dt$$

$$\frac{A_n A_m}{2}\left[\cos((n-m)\omega_0 t + \theta_n - \theta_m) + \cos((n+m)\omega_0 t + \theta_n + \theta_m)\right]$$

you can reach that:

$$= \sum_{n=1}^{\infty}\frac{A_n^2}{2}$$

$$\text{So } P_T = A_0^2 + \sum_{n=1}^{\infty}\frac{A_n^2}{2}$$

* * * end of second material.

* System Response to periodic Inputs:

$$\underset{e^{j\omega t}}{\overset{T_0}{x(t)}} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$y_e(t) = e^{j\omega t}\cdot H(\omega)$$

$$\text{proof: } y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,e^{j\omega(t-\tau)}\,d\tau = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} h(t)\,e^{-j\omega t}\,dt$$

$$\text{so } H(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{j\omega t}\,dt$$

$$H(\omega) = \mathcal{F}\left\{h(t)\right\}$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \longrightarrow y(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 n t} \cdot H(n\omega_0)$$

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n) \longrightarrow A_0 H(0) + \sum_{n=1}^{\infty} A_n |H(n\omega_0)| \cdot \cos(n\omega_0 t + \theta_n + \measuredangle H(n\omega_0))$$

$$A\cos(\omega t+\theta) \longrightarrow \overset{LTI}{\boxed{}} \rightarrow B\cos(\omega t+\phi)$$

$$A\measuredangle\theta \qquad\qquad B\measuredangle\phi$$

Ex. in slides:

$$x(t) = 4\cos(t) - 2\cos(2t)$$

$$\text{KVL:} \quad x(t) = L\frac{di(t)}{dt} + \underline{\underline{R\,i(t)}} \rightarrow y(t) = R\,i(t)$$

$$\frac{dy(t)}{dt} = R\frac{di(t)}{dt}$$

$$= L \cdot \frac{1}{R} \frac{dy(t)}{dt} + y(t)$$

$$x(t) = \frac{L}{R}\frac{dy(t)}{dt} + y(t)$$

$$\Rightarrow \quad y(t) = e^{j\omega t} H(\omega)$$

$$y'(t) = H(\omega)\, j\omega\, e^{j\omega t}$$

$$x(t) = \frac{dy(t)}{dt} + y(t)$$

$$\cancel{e^{j\omega t}} = H(\omega)\, j\omega\, \cancel{e^{j\omega t}} + \cancel{e^{j\omega t}} H(\omega)$$

$$\Rightarrow \boxed{H(\omega) = \frac{1}{1+j\omega}}$$

$$\Rightarrow H(n\omega_0) = \frac{1}{1+jn\omega_0}$$

$$\underline{\underline{|H| = \frac{1}{\sqrt{1+(n\omega_0)^2}}}}, \quad \underline{\underline{\measuredangle H = -\tan^{-1}(n\omega_0)}}$$

*in the example:

$$x(t) = 4\cos \underset{\omega_0}{\underset{\downarrow}{t}} - 2\cos \underset{2\omega_0}{\underset{\downarrow}{2t}} \Rightarrow \boxed{\omega_0 = 1}$$

$$\text{so} \quad H(n) = \frac{1}{1+jn}, \quad |H| = \frac{1}{\sqrt{1+n^2}}, \quad \measuredangle H = -\tan^{-1}(n)$$

$$\Longrightarrow$$

$$\Rightarrow C_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$x(t) = 4\left(\frac{1}{2}e^{jt}+\frac{1}{2}e^{-jt}\right) - 2\left(\frac{1}{2}e^{j2t}+\frac{1}{2}e^{-j2t}\right)$$

$$= 2e^{jt} + 2e^{-jt} - e^{j2t} - e^{-j2t}$$

$C_1=2$ $\quad C_{-1}=2 \quad C_2=-1 \quad C_{-2}=-1$

(No DC in this ex.)

so $y(t) = 2H(1)\, e^{jt} + 2H(-1)\, e^{-jt} + -1\, H(2)\, e^{j2t} + -1\, H(-2)\, e^{-j2t}$

where:

$H(1) = \frac{1}{j+1}$, $H(-1) = \frac{1}{-j+1}$, $H(2) = \frac{1}{j2+1}$, $H(-2) = \frac{1}{-j2+1}$

** Note: if we want to use compact form:

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

$\Rightarrow A_0 = 0$, $A_1 = 4$, $A_2 = 2$

$\theta_1 = 0$ $\quad \theta_2 = \pm 180°$

$$y(t) = A_1|H(1)|\cos(t + \angle H(1)) + A_2|H(2)|\cos(2t + 180° + \angle H(2))$$

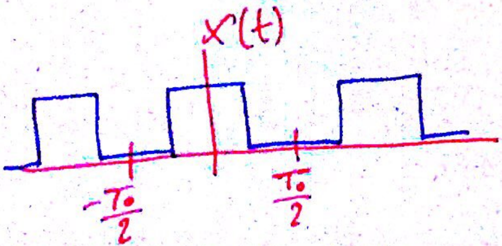
$$= \frac{4}{\sqrt{2}}\cos(t - \tan^{-1}(1)) + \frac{2}{\sqrt{5}}\cos(2t + 180° - \tan^{-1}(2))$$

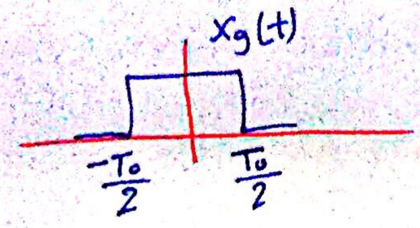
* * *

* Fourier Transform:

$x(t)$ periodic with T_0 $\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$; $C_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$

$\downarrow$ $C[n\omega_0]$





$$x(t) = \sum_{n=-\infty}^{\infty} x_g(t - nT_0)$$

$$x_g(t) = \lim_{T_0 \to \infty} x(t)$$

$\hookrightarrow \omega_0 \to 0$

$\Rightarrow$

$$x_g(t) = \lim_{\substack{T_0 \to \infty \\ \omega_0 \to 0 \\ \Delta\omega \to 0}} x(t)$$

$$\downarrow$$

$$\lim_{\substack{T_0 \to \infty \\ \omega_0 \to 0}} \Delta w = \underline{\underline{dw}}.$$

$$C[n\omega_0] = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$X[n\omega_0] = T_0 \; C[n\omega_0] = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-jn\omega_0 t} dt$$

$$C[n\omega_0] = \frac{1}{T_0} X[n\omega_0]$$

$$\boxed{C[n\omega_0] = \frac{\omega_0}{2\pi} X[n\omega_0]}$$

$$x_g(t) = \lim_{\Delta\omega \to 0} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \lim_{\Delta\omega \to 0} \frac{\Delta\omega}{2\pi} X[n\Delta\omega] e^{jn\Delta\omega t} = \frac{1}{2\pi} \lim_{\Delta\omega \to 0} \sum_{n=-\infty}^{\infty} X[n\Delta\omega] e^{jn\Delta\omega t} \Delta\omega$$

$$\Rightarrow x_g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{\Delta\omega \to 0} X[n\Delta\omega] e^{j\omega t} dw$$

$$\lim_{T_0 \to \infty} X[n\Delta\omega] = \lim_{T_0 \to \infty} \int_{-\frac{T_0}{2}}^{T_0/2} x(t) e^{-jn\Delta\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x_g(t) e^{-j\omega t} dt$$

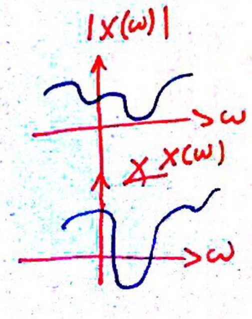
$$\Rightarrow \boxed{x_g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dw}$$

$$x(t) \xleftrightarrow{\text{Forier Transform} \; f} X(\omega)$$

Time Domain. Frequancy Domain.

$$\omega = 2\pi f$$

ω → rad/s. f → HZ.



$$\left.\begin{aligned} X(\omega) &= f\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \\ x(t) &= f^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dw \end{aligned}\right\} \text{F.T pair.}$$

$$y(t) \xrightarrow{f} Y(\omega)$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$y(t) = f^{-1}\{Y(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} dw$$

* Laplace Transform: general Transform

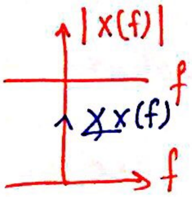
$$x(t) \xrightarrow{\mathcal{L}} X(s) \quad s \to \sigma + j\omega$$

when $\sigma = 0$

⇒ Fourier Transform.

* Note:

$$x(t) \xrightarrow{\mathcal{F}} X(\underline{f})$$



$\omega = 2\pi f$
$d\omega = 2\pi df$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt.$$

$$H(j\omega) \equiv H(\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

Ex. find fourier transform for $x(t) = \mathrm{rect}\left(\frac{t}{2T}\right)$:

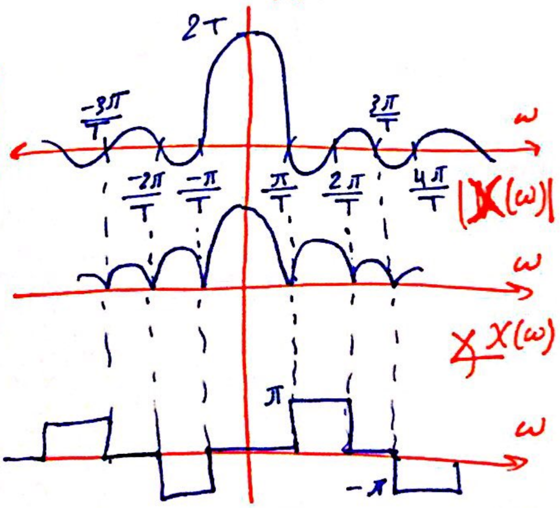
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} (1)\, e^{-j\omega t}\, dt = \frac{1}{-j\omega}\, e^{-j\omega t}\Big|_{-T}^{T}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega T} - e^{j\omega T}\right] = \frac{2}{\omega}\left[\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right] = \frac{2}{\omega}\sin(\omega T)$$

$$\mathrm{sinc}\,x = \frac{\sin(\pi x)}{\pi x}$$

$$= \frac{2T}{\pi \underset{x}{\left(\frac{\omega T}{\pi}\right)}} \sin\left(\overset{x}{\left(\frac{\omega T}{\pi}\right)} \pi\right) = 2T\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right) = X(\omega)$$

$$\Rightarrow \boxed{\mathrm{rect}\left(\frac{t}{2T}\right) \xrightarrow{\mathcal{F}} 2T\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)}$$



*** Time Limited $\xrightarrow{\quad f \quad}$ Time unlimited.

Time unlimited $\xrightarrow{\quad f \quad}$ Time Limited.

⇒ special case:

Time unlimited $\xrightarrow{\quad f \quad}$ Time unlimited.

※ Existence of Fourier Transform:

* every Energy signal → you can find fourier transform for it.

* if $\int_{-\infty}^{\infty} |x(t)| < \infty \longrightarrow$

* if fourier transform exist, Then $X(\omega)$ bounded.

$$\Rightarrow \quad |X(\omega)| \le \int_{-\infty}^{\infty} |x(t)|\,dt$$

proof:

$$|X(\omega)| = \left| \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right| \Rightarrow \le \int_{-\infty}^{\infty} |x(t)|\, \cancelto{1}{|e^{-j\omega t}|}\, dt$$

$$\le \underline{\underline{\infty}}.$$

* $x(t) = x^*(t)$

$|X(\omega)| = |X(-\omega)| \rightarrow$ even

$\measuredangle X(\omega) = -\measuredangle X(-\omega) \rightarrow$ odd

$X(\omega) = X^*(-\omega)$

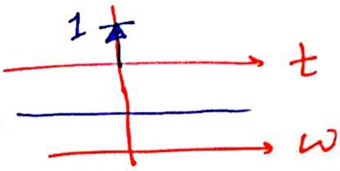
proof:

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{+j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{+j\omega t}\, dt = X(-\omega)$$

$$|X(\omega)| -\measuredangle X(\omega) = |X(-\omega)| \measuredangle X(-\omega)$$

$\delta(t) \xrightarrow{f} f\{\delta(t)\}$



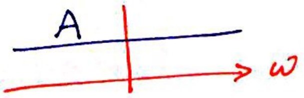
$f\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$ "sifting"

$= e^{-j\omega(0)} = e^{0} = \underline{\underline{1}}$

$\boxed{A\delta(t) \xrightarrow{f} A}$

$A\delta(t) \xrightarrow{f} A\, f\{\delta(t)\}$

$f\{\delta(t)\} = \underline{\underline{A}}$



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* $A\delta(t-t_0) \xrightarrow{f}$??

$f\{A\delta(t-t_0)\} = A\int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\omega t}\, dt = A e^{-j\omega t_0}$

$\boxed{A\delta(t-t_0) \xrightarrow{f} A e^{-j\omega t_0}}$

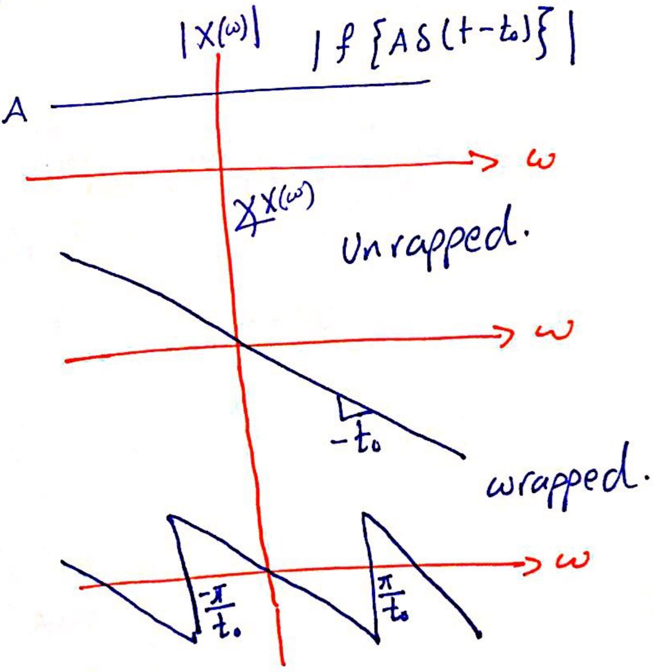
$x(t) \xrightarrow{f} X(\omega)$

$x(t-t_0) \xrightarrow{f} X(\omega)\, e^{-j\omega t_0}$

t_0 +ve

$-\omega t_0 = -\pi$

$\omega = \frac{\pi}{t_0}$



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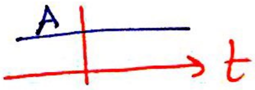
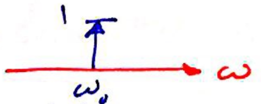
* $? \xrightarrow{f} \delta(\omega)$ $f^{-1}\{\delta(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j(0)t} = \frac{1}{2\pi}$

$2\pi A \times \frac{1}{2\pi} \xrightarrow{f} \delta(\omega) \times 2\pi A$



$\Rightarrow \boxed{A \xrightarrow{f} 2\pi A\delta(\omega)}$ $\boxed{1 \xrightarrow{f} \delta(f)}$

* $? \xrightarrow{f} \delta(\omega-\omega_0)$



$$f^{-1}\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

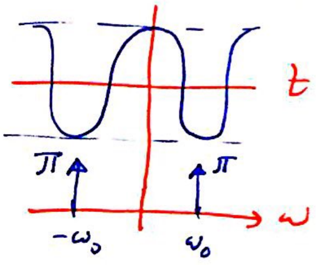
$$\boxed{a e^{j\omega_0 t} \xrightarrow{f} a 2\pi\delta(\omega-\omega_0)}$$ ①

$$\boxed{b e^{-j\omega_0 t} \rightarrow 2b\pi\delta(\omega+\omega_0)}$$ ②

① + ② : [take a=b=1]

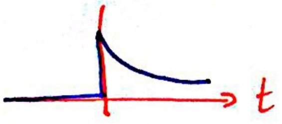
$$\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \xrightarrow{f} \frac{2\pi}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

$$\Rightarrow \boxed{\cos(\omega_0 t) \xrightarrow{f} \pi\delta(\omega-\omega_0)+\pi\delta(\omega-\omega_0)}$$



Ex. find fourier transform:

$e^{-at}u(t), a>0 \xrightarrow{f} ?$



$$f\{e^{-at}u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t) . e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$

$$\boxed{e^{-at}u(t) \xrightarrow{f} \frac{1}{a+j\omega}}, \underline{\underline{a>0}}$$

* Proporties of forier Transform:

* Linearity:

$K_1\, x(t) \xrightarrow{f} K_1\, X(\omega)$

$+\ K_2\, y(t) \xrightarrow{f} K_2\, Y(\omega)$

$f\{z(t) = K_1 x(t) + K_2 y(t)\}$

$f\{z(t)\} = f\{K_1 x(t)\} + f\{K_2 y(t)\}$

$Z(\omega) = K_1\, X(\omega) = K_2\, Y(\omega)$

In general:

$x(t) = \sum_i a_i \cdot x_i(t)$

$f\{x(t)\} = X(\omega) = f\{\sum_i a_i \cdot x_i(t)\}$

$= \sum_i a_i \cdot f\{x_i(t)\}$

$= \sum_i a_i \cdot X_i(\omega)$

* For compact form:

$X(\omega) = f\{A_0\} + \sum_{n=1}^{\infty} A_n f\{\cos(n\omega_0 t + \theta_n)\}$ $\Longrightarrow$

$$\Rightarrow X(\omega) = A_0\, 2\pi\delta(\omega) + \sum_{n=1}^{\infty} \pi A_n \left(e^{j\theta\omega}\delta(\omega - n\omega_0) + e^{+j\theta\omega}\delta(\omega + n\omega_0)\right)$$

$\cos(\omega_0 t + \theta)$

$= \frac{1}{2} e^{j\theta} e^{j\omega_0 t} + \frac{1}{2} e^{-j\theta} e^{-j\omega_0 t}$

* Time Transformation: & Amplitude Transformation:

$$AX(t) + B \xrightarrow{f} A\,X(\omega) + 2\pi B\,\delta(\omega)$$

*shift:

$$x(t) \xrightarrow{f} X(\omega)$$

$$y(t) = x(t-t_0) \xrightarrow{f} Y(\omega) = e^{-j\omega t_0} X(\omega)$$

$|Y(\omega)| = |X(\omega)|$

$\measuredangle Y(\omega) = \measuredangle X(\omega) + -\omega t_0$

$$\boxed{\begin{array}{l} \delta(t) \xrightarrow{f} 1 \\ \delta(t-t_0) \xrightarrow{f} e^{-j\omega t_0} \end{array}}$$

* scaling:

$$x(t) \xrightarrow{f} X(\omega)$$

$$\boxed{\underbrace{x(at)}_{y(t)} \xrightarrow{f} \underbrace{\frac{1}{|a|} X\left(\frac{\omega}{a}\right)}_{Y(\omega)}}$$

proof → see slides.

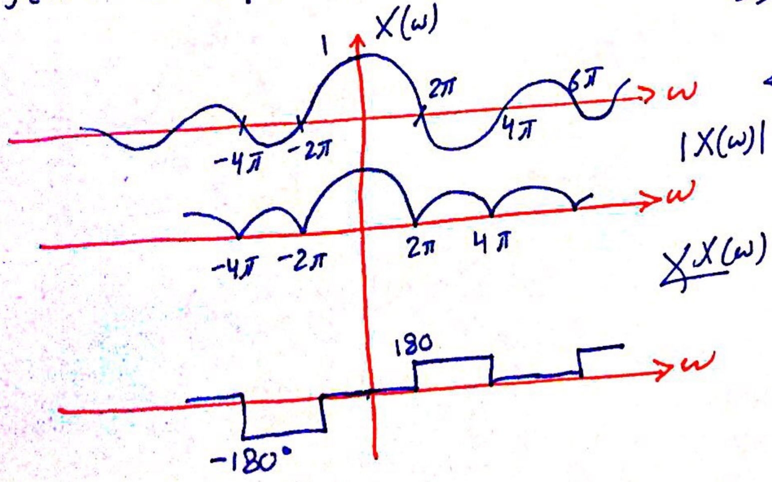
Ex. find $f\{rect(\frac{t}{T})\}$?

we found $x(t) = rect\left(\frac{t}{2T}\right) \xrightarrow{f} 2T\, sinc\left(\frac{\omega T}{\pi}\right) = X(\omega)$

Take $x(2t) = rect\left(\frac{\cancel{2}t}{\cancel{2}T}\right) \xrightarrow{f} \frac{1}{2}\, 2T\, sinc\left(\frac{\omega T}{2\pi}\right)$

$\Rightarrow y(t) = rect\left(\frac{t}{T}\right) \xrightarrow{f} Y(\omega) = T\, sinc\left(\frac{\omega T}{2\pi}\right)$

$\Rightarrow \boxed{rect(t) \xrightarrow{f} sinc\left(\frac{\omega}{2\pi}\right)}$



* $x(t) \xleftrightarrow{f} X(\omega)$

$y(t) = x(at - t_0) \xleftrightarrow{f} \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \cdot e^{-j\omega\left(\frac{t_0}{a}\right)}$

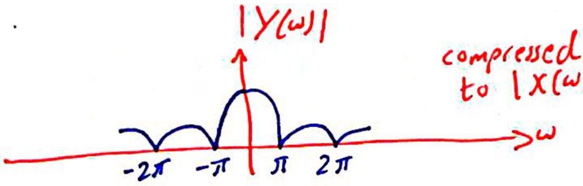
$x(at - t_0)$
$= x\left(a\left(t - \underline{\underline{\frac{t_0}{a}}}\right)\right)$

Ex. $g(t) = \text{rect}\left(\frac{1}{2}t - 1\right) \xleftrightarrow{f}$?

$y(t) \xrightarrow{f} 2X(2\omega) \cdot e^{-j2\omega}$ where $X(\omega) = \text{sinc}\left(\frac{\omega}{2\pi}\right)$

$Y(\omega) = 2\,\text{sinc}\left(\frac{\omega}{\pi}\right) \cdot e^{-j2\omega}$ ⇒ $|Y(\omega)| = 2\left|\text{sinc}\left(\frac{\omega}{\pi}\right)\right|$
$= 2\,|X(2\omega)|$

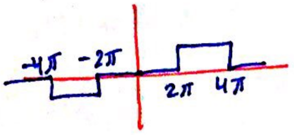
⇒ continue to previous figure .
$X(\omega)$, $|X(\omega)|$, $\measuredangle X(\omega)$:



compressed to $|X(\omega)|$

for the phase:

$\measuredangle Y(\omega) = \measuredangle X(\omega) + (-2\omega)$



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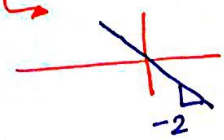
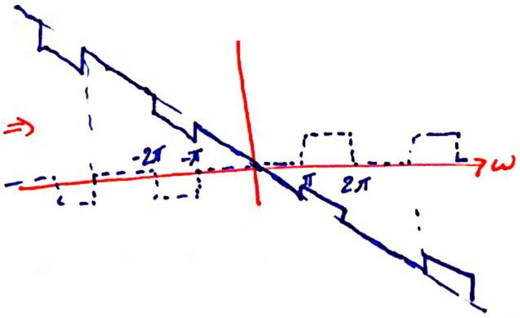
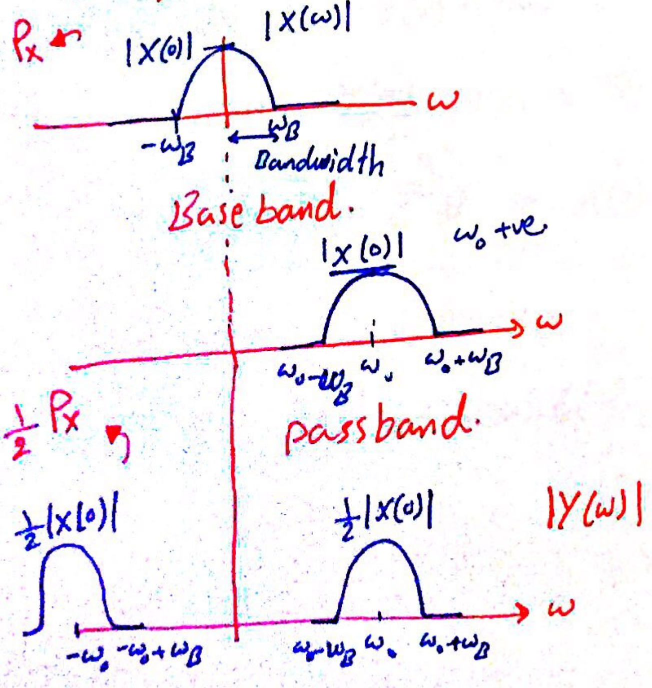


⇒



*Freq. shifting:



$x(t) \xrightarrow{f} X(\omega)$

$x(t - t_0) \xrightarrow{f} X(\omega)\, e^{-j\omega t_0}$

$x(t)\, e^{j\omega_0 t} \xrightarrow{f} X(\omega - \omega_0)$ +

\+ $x(t)\, e^{-j\omega_0 t} \xrightarrow{f} X(\omega + \omega_0)$

$x(t)\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) \xrightarrow{f} \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0)$

$\Rightarrow y(t) = x(t)\cos(\omega_0 t) \xleftrightarrow{f} \frac{1}{2}X(\omega - \omega_0)$
$+ \frac{1}{2}X(\omega + \omega_0)$
$= Y(\omega)$

Ex. $x(t) = \begin{cases} e^{j10t} & , |t| < \pi \\ 0 & , \text{o.w.} \end{cases}$ find $X(\omega)$?

$$= e^{j10t}\,\mathrm{rect}\left(\frac{t}{2\pi}\right)$$

we Know: $\mathrm{rect}\left(\frac{t}{\underset{\to 2\pi}{\cancel{T}}}\right) \xrightarrow{f} \overset{2\pi}{\cancel{T}}\,\mathrm{sinc}\left(\frac{\omega \overset{2\pi}{\cancel{T}}}{2\pi}\right) \Rightarrow \mathrm{rect}\left(\frac{t}{2\pi}\right) \xrightarrow{f} 2\pi\,\mathrm{sinc}(\omega)$

$$e^{j10t} \cdot \mathrm{rect}\left(\frac{t}{2\pi}\right) \xrightarrow{f} 2\pi\ \mathrm{sinc}(\omega - 10)$$

---

* freq. scaling:

$$\frac{1}{|a|}x\left(\frac{t}{a}\right) \xrightarrow{f} X(a\omega) \qquad \frac{1}{|a|}x\left(\frac{t}{a}\right) \xrightarrow{f} \frac{|a|}{|a|}X(a\,\omega)$$

$$x(bt) \xrightarrow{f} \frac{1}{|b|}X\left(\frac{\omega}{b}\right)$$

---

* Time differentiation:

$$x(t) \xleftrightarrow{f} X(\omega)$$

$$y(t) = \frac{dx}{dt} \xleftrightarrow{f} j\omega X(\omega) = Y(\omega)$$

see proof in slides.

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{f} (j\omega)^n X(\omega)$$

** $x(t) \xleftrightarrow{f} X(\omega)$ → UnKnown.

$y(t) = \frac{dx}{dt} \xleftrightarrow{f} j\omega X(\omega) = Y(\omega)$ → Known

$\Rightarrow X(\omega) = \frac{Y(\omega)}{j\omega}$, This is True under condition if $X(0) = 0$

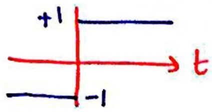
$X(\overset{0}{\cancel{\omega}}) = \int_{-\infty}^{\infty} x(t)\, e^{-j\overset{0}{\cancel{\omega}}t}\,dt \Rightarrow X(0) = \int_{-\infty}^{\infty} x(t)\,dt$ = Average. $= 0$

so Average must equal zero.

*if $X(0) \equiv$ has a DC value :

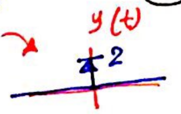
$X(\omega) = \frac{Y(\omega)}{j\omega} + \text{Constant}.\delta(\omega)$ ⇒ explained later.

Ex. find $f\{sgn(t)\}$?



$x(t) = sgn(t) \xleftrightarrow{f} X(\omega) = ?$

let $y(t) = \frac{d}{dt} sgn(t)$



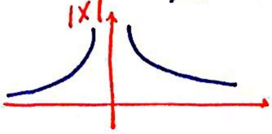
$\Rightarrow \; y(t) = 2\delta(t) \xleftrightarrow{f} 2 \equiv Y(\omega)$

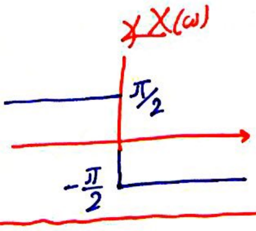
since the average value of $x(t) = \underline{\underline{0}}$ $\quad X(0) = 0$

so $X(\omega) = \frac{Y(\omega)}{j\omega} \Rightarrow X(\omega) = \frac{2}{j\omega}$

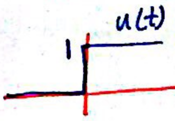
$$\boxed{sgn(t) \xleftrightarrow{f} \frac{2}{j\omega}}$$

$|X(\omega)| = \frac{2}{|\omega|}$, $\measuredangle X(\omega) = \mp \frac{\pi}{2}$





Ex. find $f\{u(t)\}$?



Av. $\neq 0$

here we use the $f\{sgn(t)\}$.

since $u(t) = \frac{1}{2}(sgn(t) + 1) = \frac{1}{2} sgn(t) + \frac{1}{2}$

so $f\{u(t)\} \Leftrightarrow f\{\frac{1}{2} sgn(t) + \frac{1}{2}\}$

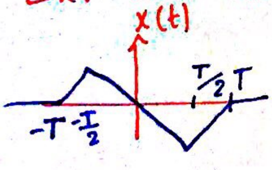
$f\{u(t)\} = \cancel{\frac{1}{2}}(\frac{\cancel{2}}{j\omega}) + \cancel{\frac{1}{2}} \cancel{2}\pi\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

similer to:

$X(\omega) = \frac{Y(\omega)}{j\omega} + K\delta(\omega)$

$$\boxed{u(t) \xrightarrow{f} \frac{1}{j\omega} + \pi\delta(\omega)}$$

Ex. see the slides for the full solution:



$\Rightarrow \; \ldots \quad Z(\omega) = \frac{2}{T} e^{j\omega T} - \frac{4}{T} e^{j\omega \frac{T}{2}} + \frac{4}{T} e^{-j\omega \frac{T}{2}} - \frac{2}{T} e^{-j\omega T}$

$= 2j \frac{2}{T} \left(\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right) - \frac{4}{T} 2j \left(\frac{e^{j\omega \frac{T}{2}} - e^{-j\omega \frac{T}{2}}}{2j}\right)$

$\Rightarrow \; \cancel{X}(\omega) = \frac{4j}{-T\omega^2} \sin(\omega T) - \frac{4j}{-\frac{T}{2}\omega 2} \sin(\omega \frac{T}{2})$

* Time Integration:

$y(t) = \int_{-\infty}^{t} x(\tau) d\tau \xleftrightarrow{f} Y(\omega) = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

proof:

$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left(\underline{\underline{\int_{-\infty}^{t} x(\tau) d\tau}} \right) e^{-j\omega t} dt = \int_{t=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} x(\tau) u(t-\tau) d\tau \right] e^{-j\omega t} dt = \int_{\tau=-\infty}^{\infty} x(\tau) \left\{ \int_{t=-\infty}^{\infty} u(t-\tau) e^{-j\omega t} dt \right\} d\tau$

$f\{u(t-\tau)\}$

$= \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau$

$= \int_{-\infty}^{\infty} x(\tau) \cdot \left[\frac{1}{j\omega} + \pi\delta(\omega) \right] e^{-j\omega\tau} d\tau \quad \Longrightarrow$

function of ω.

$$\Rightarrow Y(\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) \underline{\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau} \longrightarrow X(\omega)$$

$$\Rightarrow Y(\omega) = \frac{X(\omega)}{j\omega} + \pi\delta(\omega) X(\omega) = \frac{X(\omega)}{j\omega} + \pi\delta(\omega) X(0)$$

* Freq. diff. :

$$x(t) \xrightarrow{f} X(\omega)$$

$$-jt\, x(t) \xrightarrow{f} \frac{dX(\omega)}{d\omega} \Rightarrow \boxed{t\, x(t) \xrightarrow{f} j\frac{dX(\omega)}{d\omega}}$$

$$(-jt)^n x(t) \xrightarrow{f} \frac{d^n X(\omega)}{d\omega^n} \Rightarrow \boxed{t^n x(t) \xrightarrow{f} j^n \frac{d^n X(\omega)}{d\omega^n}}$$

see proof in slides.

Ex. if $y = t e^{-at} u(t)$, $a>0$, find $Y(\omega)$?

$$x(t) = e^{-at} u(t) \xrightarrow{f} \frac{1}{a+j\omega} = X(\omega), \quad a>0$$

$$y(t) = t e^{-at} u(t) \xrightarrow{f} Y(\omega) = j\frac{dX(\omega)}{d\omega} = \frac{1}{(a+j\omega)^2} = j\frac{-j}{(a+j\omega)^2}$$

$$\boxed{Y(\omega) = \frac{1}{(a+j\omega)^2}}$$

---

* Time Convelution:

$$x(t) \xrightarrow{f} X(\omega)$$
$$y(t) \xrightarrow{f} Y(\omega)$$

$$\Rightarrow f\{z(t)\} = f\{x(t) * y(t)\} \Rightarrow Z(\omega) = X(\omega) \cdot Y(\omega).$$
$$= f\{x(t)\} \cdot f\{y(t)\}$$

توزيع ال fourier على ال conv.
بيتحول لضرب .

$$x(t) \longrightarrow \overset{LTI}{\boxed{h(t)}} \longrightarrow y(t) = x(t) * h(t) \Longrightarrow X(\omega) \longrightarrow \overset{LTI}{\boxed{H(\omega)}} \longrightarrow Y(\omega) = X(\omega) \cdot H(\omega)$$

$$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$$
$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

Ex. if $y(t) = x(t) * u(t)$, find $Y(\omega)$?

$$x(t) \longrightarrow \boxed{u(t)} \longrightarrow y(t) = x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$= \begin{cases} 1, & \tau < t \\ 0, & \tau > t \end{cases}$$

$$\Rightarrow Y(\omega) = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

$$f\{y(t)\} = f\{x(t)\} \cdot f\{u(t)\}$$

* freq. Conv.:

$z(t) = x(t) \cdot y(t)$

$$\boxed{Z(\omega) = \frac{1}{2\pi} X(\omega) \star Y(\omega)} \rightarrow Z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\, Y(\omega - \lambda)\, d\lambda$$

if we deal with f:

$z(f) = X(f) \star Y(f)$

remember:
$\cos(\omega_0 t) \xrightarrow{\mathcal{F}} \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$

Ex. $z(t) = x(t) \cdot \cos(\omega_0 t)$, find $Z(\omega)$?

$$Z(\omega) = \frac{1}{2\pi} X(\omega) \star \mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{2}\left[X(\omega) \star \delta(\omega - \omega_0) + X(\omega) \star \delta(\omega + \omega_0)\right]$$

$$= \underline{\underline{\frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]}}$$

---

* Duality:

see proof is slides.

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \Rightarrow X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

Ex. find $\mathcal{F}^{-1}\{\text{sgn}(\omega)\}$?

remember: $\text{sgn}(t) = x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{j\omega} = X(\omega)$

$$\Rightarrow \frac{2}{jt} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, \text{sgn}(-\omega)$$

↳ sgn function is an odd function $f(x) = -f(-x)$

$$\frac{2}{jt} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi(-\text{sgn}(\omega)) \Rightarrow \boxed{\mathcal{F}^{-1}\{\text{sgn}(\omega)\} = \frac{j}{\pi t}}$$

Ex. find $\mathcal{F}^{-1}\{\text{rect}(\frac{\omega}{2\beta})\}$?

$$x(t) = \text{rect}\left(\frac{t}{2\beta}\right) \xrightarrow{\mathcal{F}} 2\beta\, \text{sinc}\left(\frac{\omega\beta}{\pi}\right) = X(\omega)$$

$$\Rightarrow 2\beta\, \text{sinc}\left(\frac{t\beta}{\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, \text{rect}\left(\frac{-\omega}{2\beta}\right)$$

→ even function so $= \text{rect}(\frac{\omega}{2\beta})$

$$\Rightarrow \boxed{\mathcal{F}^{-1}\{\text{rect}(\frac{\omega}{2\beta})\} = \frac{\beta}{\pi} \text{sinc}\left(\frac{\beta t}{\pi}\right)}$$

* Real & imag. signals:

$x(t)$ real.

$\Rightarrow X^*(\omega) = X(-\omega)$

see proof. & Ex. in slides.

$x(t)$ imag.

$-x(t) = x^*(t)$

$\Rightarrow X^*(\omega) = -X(-\omega)$

see proof in slides.

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	$x(t)$	$X(\omega)$
①	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
②	$e^{at}u(t)$	$1/(-a+j\omega)$
③	$\delta(t)$, $\delta(t-t_0)$	1 , $e^{-j\omega t_0}$
④	$\text{rect}(t)$, $\text{rect}(t/T)$	$\text{sinc}\left(\frac{\omega}{2\pi}\right)$, $T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right)$
⑤	a (constant) , $a\,e^{-jat}$	$2\pi a\,\delta(\omega)$, $2\pi a\,\delta(\omega+a)$
⑥	$e^{-a\|t\|}$	$2a/(a^2+\omega^2)$
⑦	$\cos(at)$, $\sin(at)$	$\pi[\delta(\omega-a)+\delta(\omega+a)]$, $j\pi[\delta(\omega+a)-\delta(\omega-a)]$
⑧	$u(t)$, $\text{sgn}(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$, $\frac{2}{j\omega}$
⑨	impulse train $\delta_T(t)$.	$\sum_{n=-\infty}^{\infty} 2\pi C_n\,\delta(\omega-n\omega_0)$; $C_n=\frac{1}{T}$
⑩	$y(t)=\frac{dx(t)}{dt}$ & Averag $=0$	$X(\omega)=\frac{Y(\omega)}{j\omega}$
⑪	$Z(t)=t^n\cdot x(t)$ freq. diff.	$Z(\omega)=j^n\frac{d^nX(\omega)}{d\omega^n}$
⑫	$y(t)=\int_{-\infty}^{t}x(\tau)d\tau$ Time Integration	$Y(\omega)=\frac{X(\omega)}{j\omega}+\pi X(0)\,\delta(\omega)$
⑬	$y(t)=x(t)*h(t)$	$Y(\omega)=X(\omega)\cdot H(\omega)$
⑭	$y(t)=x(t)\cdot h(t)$	$Y(\omega)=\frac{1}{2\pi}X(\omega)*H(\omega)$
⑮	$x(t)\xrightarrow{f}X(\omega)$	$\Longrightarrow X(t)\xrightarrow{f}2\pi x(-\omega)$
⑯	$\text{Tri}(t/T)$	$T\,\text{sinc}^2\left(\frac{\omega T}{2\pi}\right)$

* Energy spectral Density:

$\psi(\omega)=|X(\omega)|^2$

$$E=\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2d\omega$$

even

$$=\frac{1}{\pi}\int_{0}^{\infty}|X(\omega)|^2d\omega$$

* power spectral Density:

$$S(\omega)=\lim_{T\to\infty}\frac{|X_T(\omega)|^2}{2T}$$

$$P=\frac{1}{T}\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)d\omega$$

↳ power spec. Den.

※ Even & odd:

$x(t)=x^*(t)$ Real. $\xleftrightarrow{f}$ $X(\omega)=X^*(\omega)$ real.

$x(t)=x(-t)$ even $\qquad$ $X(\omega)=X(-\omega)$ even.

proof:

$$X(\omega)=\int_{-\infty}^{\infty}x(t)\,e^{-j\omega t}\,dt \Rightarrow X^*(\omega)=\int_{-\infty}^{\infty}\underbrace{x^*(t)}_{=x(t)}\,e^{j\omega t}\,dt \qquad x(t)=x(-t)$$

$$X^*(\omega)=\int_{-\infty}^{\infty}x(-t)\,e^{j\omega t}\,dt$$ put $-t=\lambda$, $dt=-d\lambda$

$$=\int_{-\infty}^{\infty}x(\lambda)\,e^{-j\omega\lambda}\,d\lambda = X(\omega)$$

so $\underline{X^*(\omega)=X(\omega)}$ #.

$$\rightarrow\ x^*(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)\,e^{-j\omega t}\,d\omega = x(t) \Rightarrow x(t)=\int_{-\infty}^{\infty}\frac{X(\omega)}{2\pi}\,e^{-j\omega t}\,d\omega$$ put $\tau=-\omega$

$$\Rightarrow\ x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{X(-\tau)}_{=X(\tau)}\,e^{j\tau t}\,d\tau \Rightarrow \underline{\underline{X(\omega)=X(-\omega)}}\ \#$$

prove the other three cases.

※ Parseval's Theorem:

$$E_T=\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2d\omega=\int_{-\infty}^{\infty}|X(f)|^2df$$

see proof in slides.

Ex. evaluate: $F=\int_{-\infty}^{\infty}\frac{2}{|j\omega+2|^2}\,d\omega = 2\int_{-\infty}^{\infty}\left|\frac{1}{j\omega+2}\right|^2d\omega$ let $X(\omega)=\frac{1}{j\omega+2}$

$F=2\,(2\pi)\int_{-\infty}^{\infty}|x(t)|^2dt$ remember: $e^{-at}u(t)\xrightarrow{f}\frac{1}{a+j\omega}$

so $x(t)=e^{-2t}u(t) \Rightarrow F=4\pi\int_{-\infty}^{\infty}e^{-2t}u(t)\,dt$

$\Rightarrow F=4\pi\int_{0}^{\infty}e^{-2t}dt=\boxed{\pi}$.

$$E_T=\int_{-\infty}^{\infty}\frac{|X(\omega)|^2}{2\pi}\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{E}(\omega)\,d\omega$$

↓

Joule.

$\mathcal{E}(\omega)=|X(\omega)|^2$

ESD

Energy spectral Density.

Joule/(rad/s)

or Joule/HZ if $\mathcal{E}(f)=|X(f)|^2$

since $x(t)$ real $\Rightarrow |X(\omega)|^2$ even so $\mathcal{E}(\omega)$ even.

$$E_T = \frac{1}{\pi}\int_0^{\infty} \mathcal{E}(\omega)\,d\omega$$

*Always Desity function must be over a band $[\omega_1, \omega_2]$, if it was at a value $\Rightarrow$ Density $= 0$

Ex. find E_T for $x(t) = e^{-t}u(t)$ & E for $-4<\omega<4$?

$$E_T = \int_{-\infty}^{\infty} |e^{-t}u(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = \boxed{\frac{1}{2}} \text{ Joule.}$$

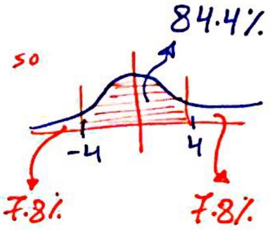
$$X(\omega) = \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1+j\omega} \rightarrow \mathcal{E}(\omega) = |X(\omega)|^2 = \frac{1}{\omega^2+1}$$

so $E = \frac{1}{\pi}\int_0^4 \frac{1}{1+\omega^2} d\omega = \frac{1}{\pi} \tan^{-1}(\omega)\Big|_0^4 = \boxed{0.422}$ Joule

0.422 as a percentage from $\frac{1}{2} \Rightarrow \frac{0.422}{0.5} * 100\% = 84.4\%$

$\Rightarrow$ most of the energy concentrated around Zero.

so



* Power Spectral Density:

(PSD). $\Rightarrow S(\omega)$ watt/(rad/s)
$S(f)$ watt/HZ.

$$\boxed{P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega}$$

$$x_T(t) = \begin{cases} x(t) & ; -T<t<T \\ 0 & , \text{o.w.} \end{cases}$$

$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x_T(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

$$\text{so } P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(\underbrace{\lim_{T\to\infty} \frac{|X_T(\omega)|^2}{2T}}_{S(\omega)}\Big) d\omega \Rightarrow S(\omega) = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{2T}$$

it is Not easly to find but we will Learn Autocorrelation, and use it to find $S(\omega)$.

$$X(\omega) \longrightarrow \overset{\text{LTI}}{\boxed{H(\omega)}} \longrightarrow Y(\omega) = X(\omega)\cdot H(\omega) \Rightarrow Y^*(\omega)\,Y(\omega) = X(\omega)\cdot X^*(\omega)\cdot H(\omega)\cdot H^*(\omega)$$

$\mathcal{E}_x(\omega)$ $\qquad \mathcal{E}_y(\omega) = ?$

$S_x(\omega)$ $\qquad S_y(\omega) = ?$

$$|Y(\omega)|^2 = |X(\omega)|^2 \cdot |H(\omega)|^2 \text{ --- ①}$$

$$\Rightarrow \boxed{\mathcal{E}_y(\omega) = \mathcal{E}_x(\omega)\cdot|H(\omega)|^2}$$

$\Rightarrow$ from ①:

$$\lim_{T\to\infty} \frac{|Y_T(\omega)|^2}{2T} = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{2T} \cdot |H(\omega)|^2 \Rightarrow \boxed{S_y(\omega) = S_x(\omega)\cdot|H(\omega)|^2}$$

Energy & power Transfer functions

* Correlation :

$R_{xx}(t) \xrightarrow{f} E(\omega) = |X(\omega)|^2$ $\qquad$ $R_{xx}(t) \xrightarrow{f} \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{2T}$

* Correlation : it is a function of time measure how much two functions are close to each other.

* Correlation Coefficient:

$-100\% \leqslant \rho \leqslant +100\%$

* Correlation between two signals called Cross Correlation.

* Correlation between same signal called Auto Correlation.

* Cross Correlation:

$$R_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)\, y^*(\tau-t)\, d\tau = \int_{-\infty}^{\infty} x(\tau+t)\, y^*(\tau)\, d\tau$$

↳ it could be written as: $R_{xy}(\tau) = \int_{-\infty} x(t)\, y^*(t-\tau)\, dt$

* the relation with Convelution:

$$\boxed{R_{xy}(t) = x(t) \star y(-t)}$$

$$\Rightarrow f\{R_{xy}(t)\} = X(\omega)\cdot Y(-\omega) = X(\omega)\cdot Y^*(\omega)$$

* Auto Correlation:

$$R_{xx}(t) = \int_{-\infty}^{\infty} x(\tau)\, x^*(\tau-t)\, d\tau = \int_{-\infty}^{\infty} x(\tau-t)\, x^*(\tau)\, d\tau$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(\tau)|^2 d\tau = E_x$$

* Auto Correlation is an Even function. $R_{xx}(t) = R_{xx}(-t)$

$f\{R_{xx}(t)\} = E(\omega)$ ⇒ see proof in slides.

$$R_{xx}(t) = \lim_{T\to\infty} \frac{1}{T}\int_T x(\tau)\, x^*(\tau-t)\, d\tau \Longrightarrow R_{xx}(0) = P_x = \lim_{T\to\infty}\frac{1}{T}\int_T |x(t)|^2 dt$$

$$f^{-1}\{S(\omega)\} = R_{xx}(t)$$

* Fourier Transform for periodic signals:

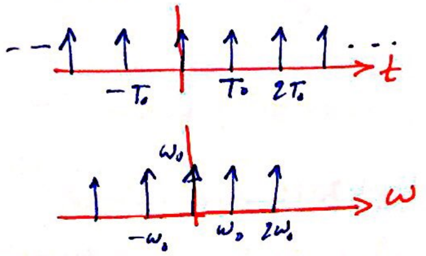
$$f\left\{ x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \right\} \Rightarrow X(\omega) = \sum_{n=-\infty}^{\infty} C_n \, f\left\{ e^{jn\omega_0 t} \right\}$$

$$\Rightarrow \boxed{X(\omega) = \sum_{n=-\infty}^{\infty} 2\pi C_n \, \delta(\omega - n\omega_0)}$$

remember:

$e^{jn\omega_0 t} \xrightarrow{f} 2\pi\delta(\omega - n\omega_0)$

Ex. find $f\{\delta_T(t)\}$?



$$\Rightarrow \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{jn\omega_0 t}$$

$$\text{so } f\{\delta_T(t)\} = \sum_{n=-\infty}^{\infty} \overbrace{2\pi \frac{1}{T_0}}^{\omega_0} \delta(\omega - n\omega_0)$$

$$= \sum_{n=-\infty}^{\infty} \omega_0 \, \delta(\omega - n\omega_0)$$

Ex. $g(t) = \begin{cases} x(t), & \frac{-T_0}{2} < t < \frac{T_0}{2} \\ 0, & o.w \end{cases}$

$f\{g(t)\} = G(\omega)$

Need $X(\omega) \propto G(\omega)$

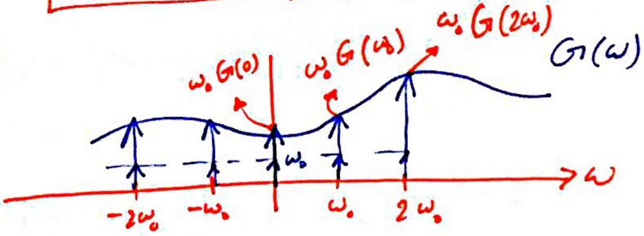
$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0) \longrightarrow = g(t) * \delta(t - nT_0)$$

$$f\left\{ x(t) = g(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \right\}$$

$$\Rightarrow X(\omega) = \underline{\underline{G(\omega)}} \cdot \sum_{n=-\infty}^{\infty} \omega_0 \, \underline{\underline{\delta(\omega - n\omega_0)}}$$

sampling property.

$$\Rightarrow X(\omega) = \sum_{n=-\infty}^{\infty} \omega_0 G(n\omega_0) \, \delta(\omega - n\omega_0)$$



$$\cancel{2\pi} C_n = \frac{\cancel{2\pi}}{T_0} G(n\omega_0)$$

$$\Rightarrow T_0 C_n = G(n\omega_0)$$

$$R_{xx}(t) \xrightarrow{f} S(\omega) = \lim_{T \to \infty} \frac{|X_T(\omega)|^2}{2T} \Rightarrow S(\omega) = \sum_{n=-\infty}^{\infty} 2\pi |C_n|^2 \delta(\omega - n\omega_0)$$

$$P_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega \overset{?}{=} \sum_{n=-\infty}^{\infty} |C_n|^2$$

$$= \cancel{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \cancel{2\pi} |C_n|^2 \delta(\omega - n\omega_0) \, d\omega = \sum_{n=-\infty}^{\infty} |C_n|^2 \overbrace{\int_{-\infty}^{\infty} \delta(\omega - n\omega_0) d\omega}^{1.}$$

#

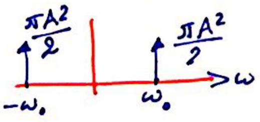
Ex. find $S(\omega)$ for $X(t) = A\cos(\omega_o t+\theta)$ for $n=-1,1$

$$x(t) = \frac{A}{2} e^{j\theta} \cdot e^{j\omega_o t} + \frac{A}{2} e^{-j\theta} \cdot e^{-j\omega_o t} \Rightarrow \begin{array}{l} C_1 = \frac{A}{2} e^{j\theta} \Rightarrow |C_1| = \frac{A}{2} \\ C_{-1} = \frac{A}{2} e^{-j\theta} \Rightarrow |C_{-1}| = \frac{A}{2} \end{array}$$

$|C_1|^2 = A^2/4$ & $|C_{-1}|^2 = A^2/4$

so $$S(\omega) = \sum_{n=-\infty}^{\infty} 2\pi |C_n|^2 \delta(\omega - n\omega_o) = 2\pi \frac{A^2}{4} \delta(\omega+\omega_o) + 2\pi \frac{A^2}{4} \delta(\omega-\omega_o)$$

P_x for $x(t) = \frac{A^2}{2}$



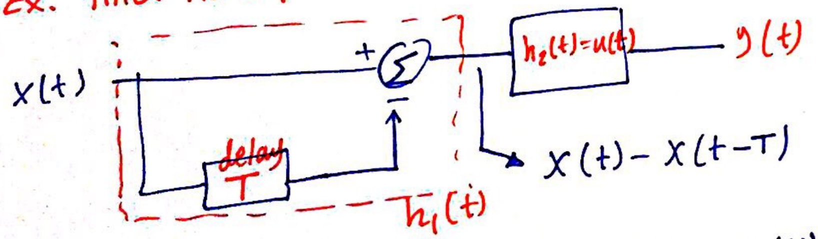
※ Frequancy Response (Transfer Function) $H(\omega)$:

$h_{eq}(t) = h_1(t) * h_2(t) * \cdots$ } in cascaded form.

$H_{eq}(\omega) = H_1(\omega) \cdot H_2(\omega) \cdot \cdots\cdots$

$= \prod_{i=1}^{M} H_i(\omega)$ $\Rightarrow H_{eq}(\omega) = \sum_{i=1}^{M} H_i(\omega) \Rightarrow$ in parallel form.

Ex. find $H(\omega)$?



By sampling.

$H_1(\omega) = (1 - e^{-j\omega T})$ $\Rightarrow H(\omega) = H_1(\omega) \cdot H_2(\omega)$

$H_2(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$ $= (1 - e^{-j\omega T}) \frac{1}{j\omega} + \cancel{\pi\delta(\omega)(1-e^{-j\omega T})}$

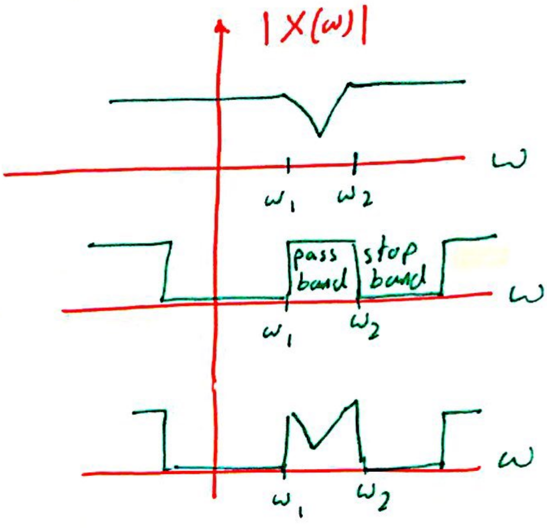
$$\Rightarrow H(\omega) = \frac{1}{j\omega}\left(e^{j\omega T/2} e^{-j\omega T/2} - e^{-j\omega T/2} \cdot e^{-j\omega T/2}\right) = \frac{2e^{-j\omega T/2}}{\omega}\left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right)$$

$$= \frac{2}{\omega} e^{-j\omega T/2} \sin\left(\frac{\omega T}{2}\right) = T \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) e^{-j\omega T/2}$$

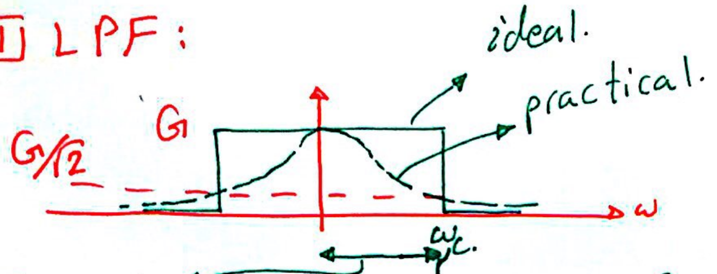
※ Signal Filtering:

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega) \implies |Y(\omega)| = |H(\omega)| \cdot |X(\omega)|$$

* Main Types:



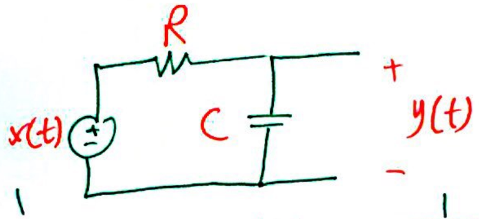
1 LPF:



Bandwidth (BW.

cutoff freq., Corner freq.
3dB freq., half power freq.

*practical example:
passive filter.

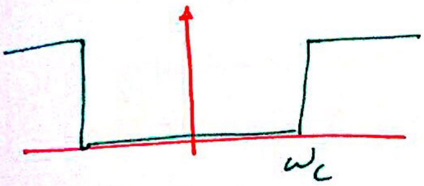


$$H(\omega) = \frac{1/j\omega c}{1/j\omega c + R} = \frac{1}{1+j\omega c R} \implies |H| = \frac{1}{\sqrt{1+(\omega c R)^2}}$$

for ω_c:

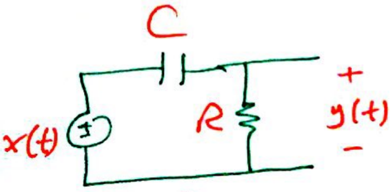
$$|H(\omega_c)| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+(\omega c R)^2}} \quad \text{solve it} \implies \boxed{\omega_c = \frac{1}{RC}}$$

2 HPF:



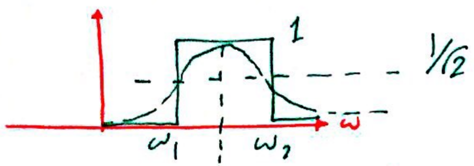
*practical ex.:

$$H(\omega) = \frac{R}{R+\frac{1}{j\omega c}} = \frac{j\omega c R}{1+j\omega c R}$$



$$\boxed{\omega_c = \frac{1}{RC}}$$

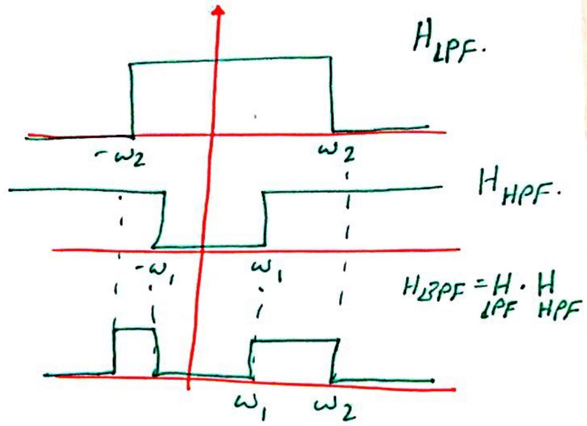
3 BPF :



$\beta = \omega_2 - \omega_1$
Bandwidth

ω_c : center freq.
"Resonance freq."

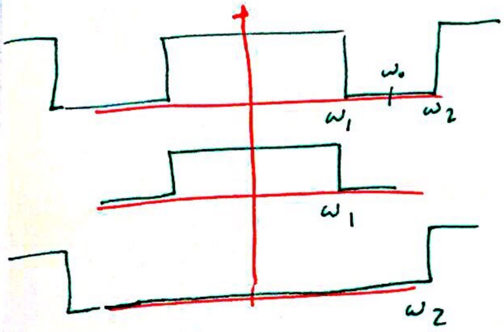
ω_1: lower cutoff freq.
ω_2: Higher cutoff freq



This is true under condition:
$\omega_2 > \omega_1$
$\Rightarrow$ cutoff of LPF > cutoff HPF

Type of connection:
Cascaded.

4 BSF :

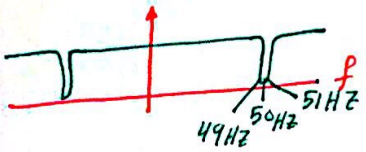


H_{BSF}
$\downarrow$
$=$
$\downarrow$
H_{LPF}
$+$
H_{HPF}

$\Rightarrow$ Type of connection:
Parallel.

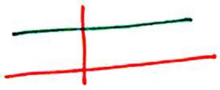
condition:
cutoff LPF < cutoff HPF

* Special Type of BSF:



This BSF called
Notch Filter.

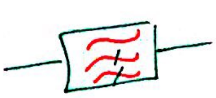
* Another special type filters:
All pass filter.



passing all frequancies.

Filters as Blocks:

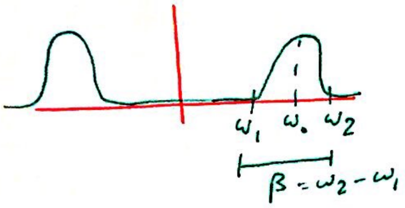
HPF.



LPF.

BSF.

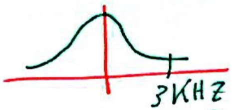
BPF.



"even"

$$\frac{2}{2\pi}\,|X(\omega_0)|^2\beta = \frac{2}{2\pi}\int_0^\infty |X(\omega)|^2 d\omega$$

※Amplitude Modulation (AM):



$\lambda = \frac{c}{f} = \frac{3*10^8}{3*10^3} = 100Km$

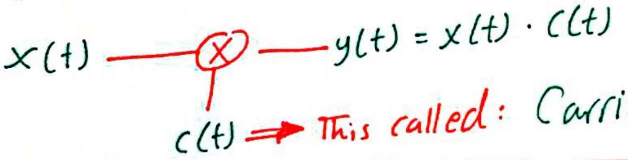
$L = \frac{\lambda}{10} = 10Km$

↳ length of Antenna
Not-practical.

$\lambda = \frac{3*10^8}{3*10^8} = 1m$.

$L = \frac{1}{10} = 0.1 = \underline{10\ Cm}$.
practical.

$y(t) = x(t)\ \cos(\omega_c t)$



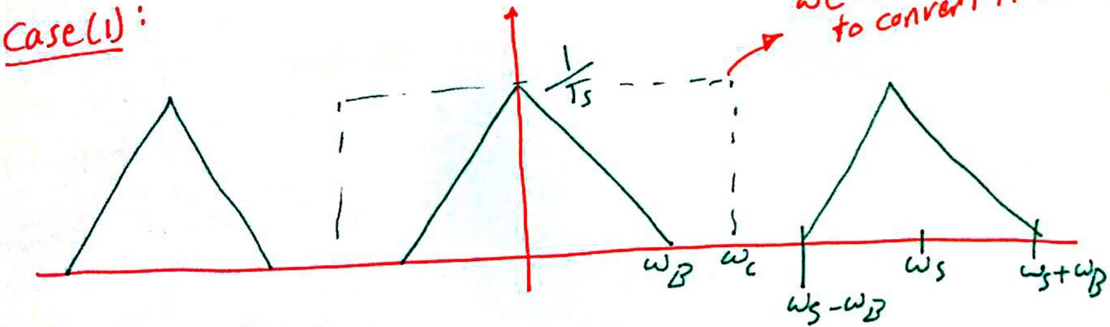
⇒ This called: Carrier.

$x_s(t) = \frac{1}{2\pi}x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$ Take $\mathcal{F}$ for both sides.

$$\Rightarrow X_s(\omega) = \frac{1}{2\pi}X(\omega) * \sum \omega_s\delta(\omega-n\omega_s)$$

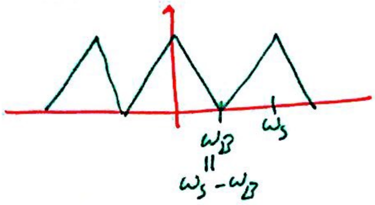
$$= \frac{1}{2\pi}\sum_{-\infty}^{\infty}\omega_s X(\omega-n\omega_s) = \sum_{n=-\infty}^{\infty} T_s\ X(\omega-n\omega_s)$$

Case(1):



$\omega_s - \omega_B > \omega_B$ $\Rightarrow \boxed{\omega_s > 2\omega_B}$

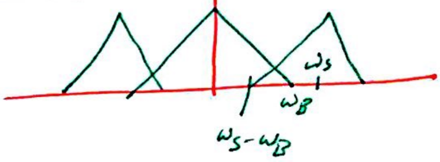
case(2):



$\omega_s-\omega_B=\omega_B$

so $\boxed{\omega_s=2\omega_B}$

case(3):



$\Rightarrow \omega_s-\omega_B<\omega_B$

$\Rightarrow \boxed{\omega_s<2\omega_B}$

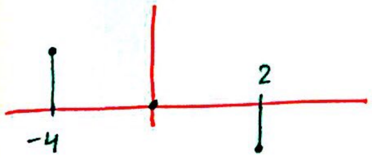
in case (3): we never ever can find $x_s(t)$ for it.

so just $\omega_s \geqslant 2\omega_B \Rightarrow$ This called: Nyquist Rate.

$\omega_s = 2~~\pi~~\omega_B \Rightarrow \frac{2\pi}{T_s}=2(2\pi f_B) \Rightarrow \boxed{T_s=\frac{1}{2f_B}}$

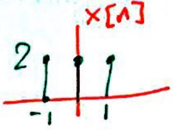
$T_s \leq \frac{1}{2f_B}$ if $T_s>\frac{1}{2f_B}$ this called aliasing.

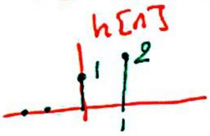
※Discrete Time:



length $=7$.

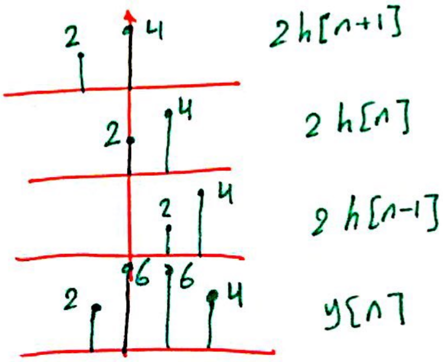
*find $y[n]=x[n]*h[n]$





$x[n]=2\delta[n+1]+2\delta[n]+2\delta[n-1]$

$y[n]=2h[n+1]+2h[n]+2h[n-1]$



length $x[n]=3$

length $h[n]=2$

Always length $y[n]=$ length $x[n]+$ length $h[n]-1$

so length $y[n]=3+2-1=4$

$\boxed{\text{starting of } y[n]=\text{start}(x)+\text{start}(h).}$

$y[n]$ start $=0-1=-1$

$\boxed{\text{end of } y[n]=\text{end}(x)+\text{end}(h)}$ $\Rightarrow$ end $y[n]=1+1=2$

* * * End of Material * * *