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Signals Notebook

Signals, Systems + transformers
Charles Phillips, John Parr 4th edition.

SI System of units :-

Derived quantities	Name	Symbol	Expression in terms of SI units
plane angle	Radian	rad	—
Solid angle	Steradian	sr	—
Frequency	Hertz	Hz	— → $\frac{1}{s}$ Base unit
Force	Newton	N	— N/m^2
Pressure, Stress	Pascal	Pa	N/m
Energy, heat	Joule	J	J/s
Power, radiant flux	Watt	W	—
Electric charge	coulombs	C	—
Electric potential difference	Volt	V	W/A
Capacitance	Farads	F	C/V
Electric resistance	ohm	Ω	V/A
Electric conductance	Siemens	S	A/V
magnetic flux	weber	Wb	V - s
magnetic flux density	Tesla	T	Wb/m^2
Inductance	Henry	H	Wb/A
celc. Temperature	Degrees celc.	C°	—
Luminous flux	lumen	lm	—
Illuminance	Lux	lx	—
Activity (nuclear)	Baccquerel	Bq	—
Absorbed dose/specific energy	Gray	Gy	J/kg
Dose equivalent	Sievert	Sv	J/kg
Catalytic Activity	katal	kat	

Factor	Name	Symbol	Factor	Name	Symbol
10^{24}	Yotta	Y	10^6	Mega	M
10^{21}	Zetta	Z	10^3	Kilo	K
10^{18}	Exa	E	10^{-3}	Milli	m
10^{15}	Peta	P	10^{-6}	Micro	μ
10^{12}	Tera	T	10^{-9}	nano	n
10^9	Giga	G	10^{-12}	pico	p
			10^{-15}	[illegible]	[illegible]

ckt

* Basic Circuit elements :-

A) Passive elements :-

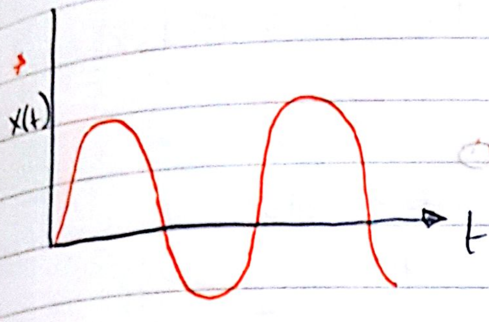
1. resistances (R) ohms
2. Electric inductance Henry — μ perimiability
3. Capacitances Farads — ϵ permitivity

B) Active elements :-

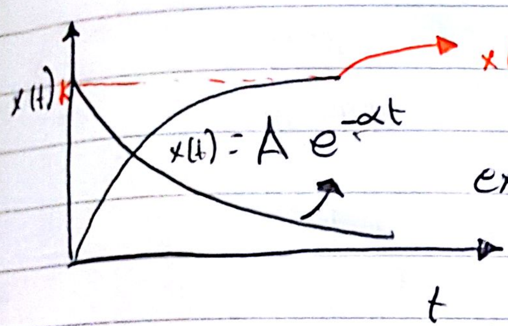
1. Diode
2. BJT transistor
3. FET transitor
4. OP-Amp.
5. transformers
6. 3 phase Systems

needs energy to be active.

* Basic mathematical represintation of continuous time Signals :-



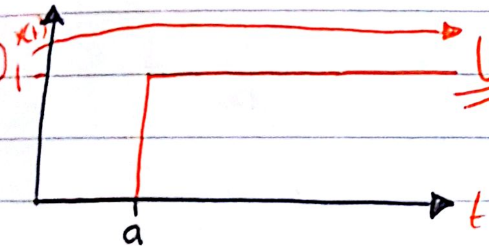
$x(t) = A \sin \omega t$



$x(t) = k(1 - e^{-\alpha t})$ exponentially growing

$x(t) = A e^{-\alpha t}$ exponentially decaying.

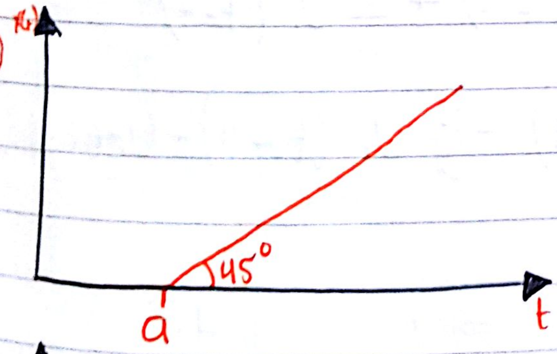
* Basic mathematical functions :-

① 

Unit Step function
(at the moment of closing the circuit).

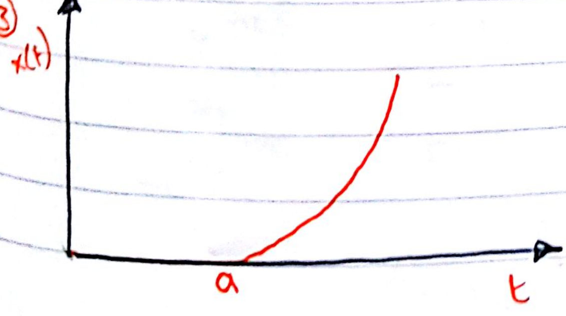
$$u_s(t-a) = \begin{cases} 1, & t \geq a \\ 0, & \text{else} \end{cases}$$

Functions with singularities.

② 

gradient =
Unit ramp function

$$x(t-a) = U_r(t) = \begin{cases} (t-a), & (t-a) \geq 0 \\ 0 & \text{else} \end{cases}$$

③ 

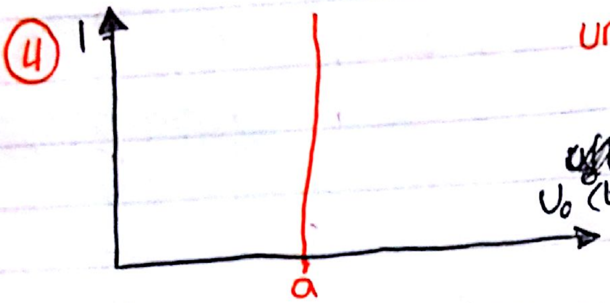
Unit Parabolic function

$$u_p(t-a) = \begin{cases} \frac{1}{2}(t-a)^2 & t \geq a \\ 0, & \text{else} \end{cases}$$

③

* Assignment :-

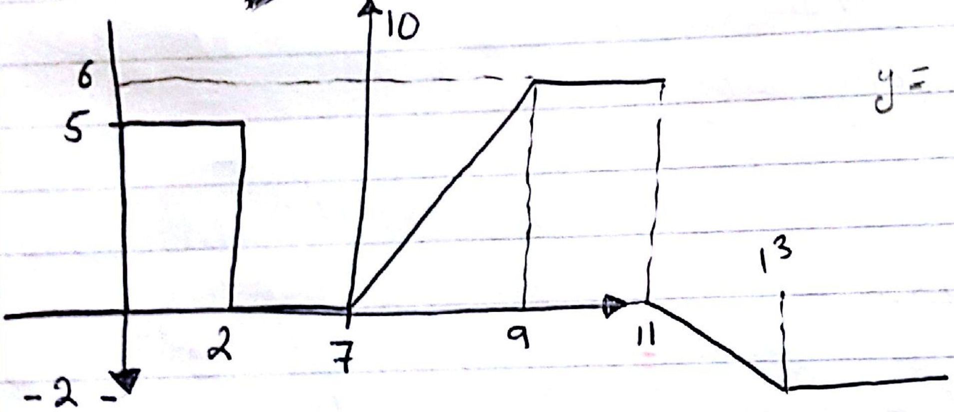
(4)



unit delta function

$$U_0(t-a) = \begin{cases} 1 , & t = 0 \\ 0 , & \text{else} \end{cases}$$

H.W :- write the function (expression)



$y = 3(x-7)$

$y = -(x-11)$

$$u(t) = \begin{cases} 5 & 0 \le t < 2 \\ 0 & 2 < t < 7 \quad , 10 \quad t = 7 \\ 3(x-7) & 7 < t < 9 \\ 6 & 9 < t < 11 \\ -(x-11) & 11 < t < 13 \end{cases}$$

$$X(t) = 5U_s(t) - 5U_s(t-2) + 10\,U_0(t-7) + \frac{6}{2}U_r(t-7)$$

$$- \frac{6}{2}U_r(t-9) - 6U_s(t-11) - \frac{2}{2}U_r(t-11) + U_r(t-13)$$

-ve to cancel the effect of the +ve slope.

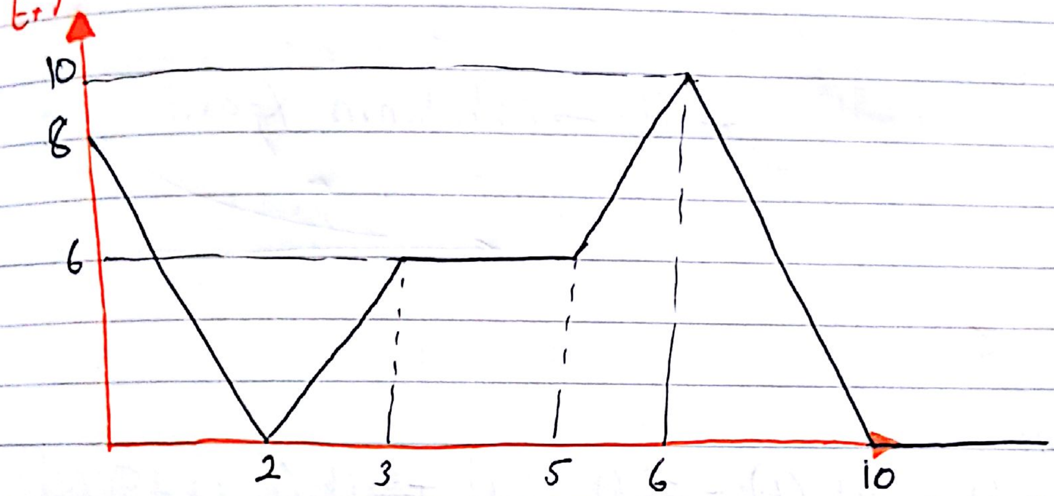
Ex) Find $x(8)$

$x(8) = 5 - 5 + 3(8-7)$ ignore the unit delta function.

it returned to zero.

only exist at one point

Ex)



$x(t) = \underline{\underline{8u_s(t)}} - \frac{8}{2}u_r(t) + \frac{8}{2}u_r(t-2)$

attention → function can't start at 8 by itself.

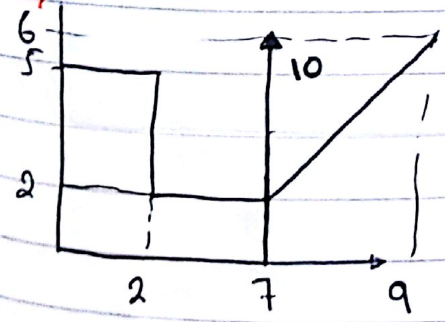
$+ 6U_r(t-2) - 6U_r(t-3) + 4U_r(t-5) - 4U_r(t-5)$

$- \frac{10}{4}U_r(t-6) + \frac{10}{4}U_r(t-6)$

these canceled so now where on the straight horiz line.

$x(4.5) = 8 - \frac{8}{2}(4.5) + \frac{8}{2}(4.5-2) + 6(4.5-2) - 6(4.5-3)$

Ex)



$x(t) = 5U_s(t) - 3U_s(t-2) + 2U_r(t-7)$

$+ 10U_0(t-7)$

$x(7) = 2 + 10_0(t-7)$

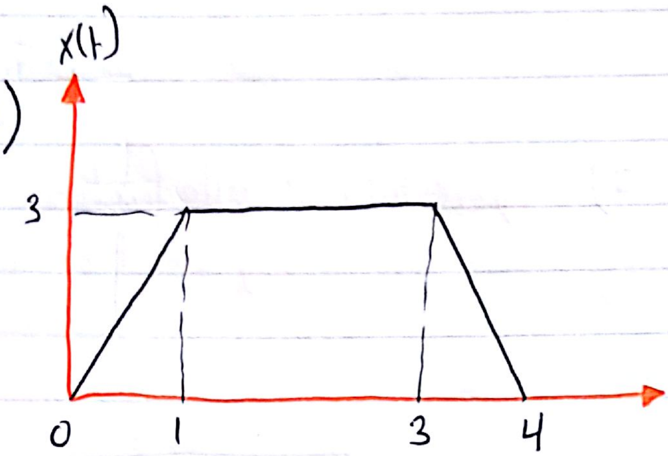
or $x(7) = 2 + 10\delta(7)$

delta.

pulse always starts from zero

5

Let x(t)



plot $x(-t)$

$-x(t) \rightarrow$ reflection in X-axis

* Plot the Function :-

Homework

$$x(t) = u_\delta(t) + 7u_s(t) - \frac{2}{3}u_r(t-4) + \frac{4}{3}u_r(t-6) - 5u_s(t-9)$$
$$- 2u_r(t-11)$$

Find $x(3)$, $x(7)$, $x(10)$, $x(20)$

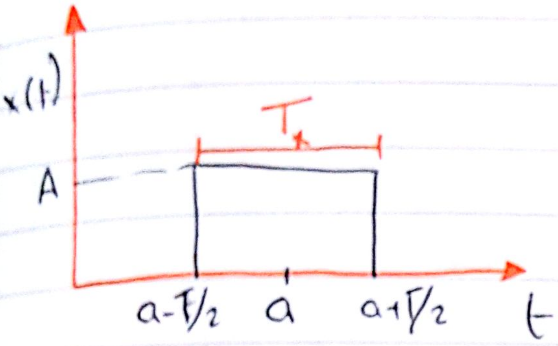
external

on the paper

$$= \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$$

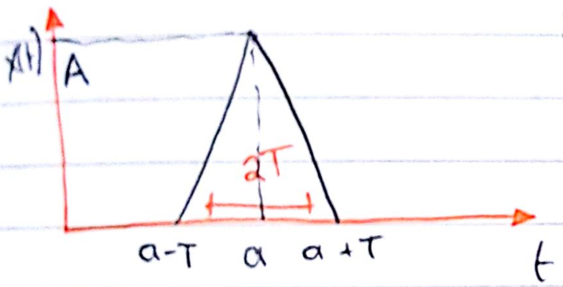
$$= \frac{1}{|a|} \int_{-\infty}^{t} \delta\left(t - \frac{t_0}{2}\right) dt$$ time Scaling property

* Rectangular Function :



$$x(t) = A\,\text{rect}\left(\frac{t-a}{T_1}\right) = A\,\Pi\left(\frac{t-a}{T_x}\right)$$

* Triangular function



$$x(t) = A\,\text{tri}\left(\frac{t-a}{T}\right) = A\,\Lambda\left(\frac{t-a}{T}\right)$$

* properities of the unit delta function:-

① $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$ sifting property

② $\int_{-\infty}^{\infty} f(t-t_0)\,\delta(t)\,dt = f(-t_0)$ $\delta \equiv u\delta$

③ $f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)$

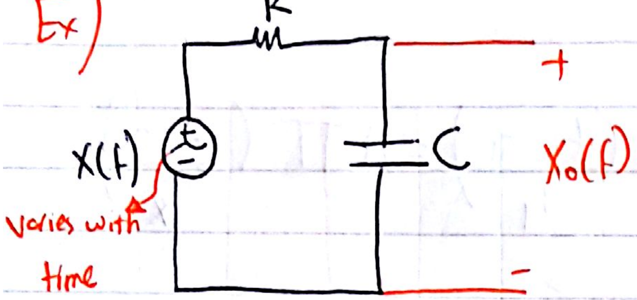
④ $u\delta(t-t_0) = \dfrac{d\,U_s(t-t_0)}{dt}$

⑤ $u(t-t_0) = \int_{-\infty}^{t} \delta(\tau-t_0)\,d\tau = \begin{cases} 1, & t>t_0 \\ 0, & t<t_0 \end{cases}$

⑥ $\int_{-\infty}^{\infty} \delta(at-t_0)\,dt = \dfrac{1}{|a|}\int_{-\infty}^{t} \delta\left(t-\frac{t_0}{a}\right)dt$ time scaling property

⑦ $\delta(t) = \delta(-t)$

Ex)



$$i(t) = C\frac{dx_o}{dt}$$

$x(t) =$ ________

$$x(t) = Ri(t) + \frac{1}{c}\int i(t)\,dt$$

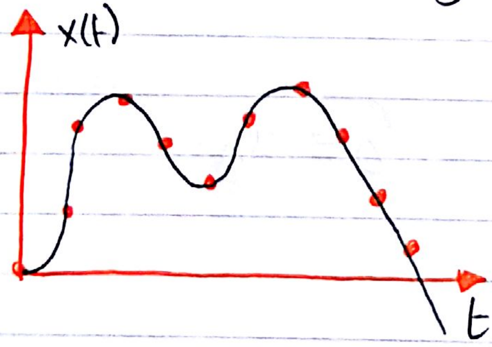
[redacted]

$$x(t) = RC\frac{dx_o(t)}{dt} + \frac{1}{\cancel{c}}\cancel{c}\,x_o(t)$$

$$RC\frac{dx_o(t)}{dt} + x_o(t) = x(t)$$

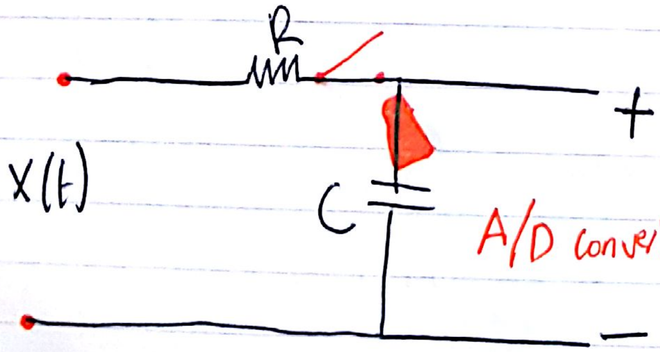
if $d\,x(t)$ assume $x(t)$ & derive the final solution.

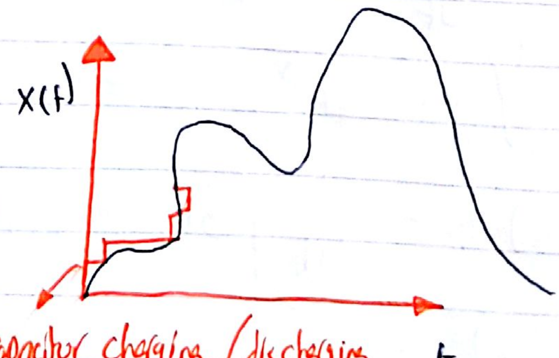
* Continuous time signals:-



* Samples (sample time signal)

$f_s \geq 2W$ largest signal in the sample.





capacitor charging / discharging

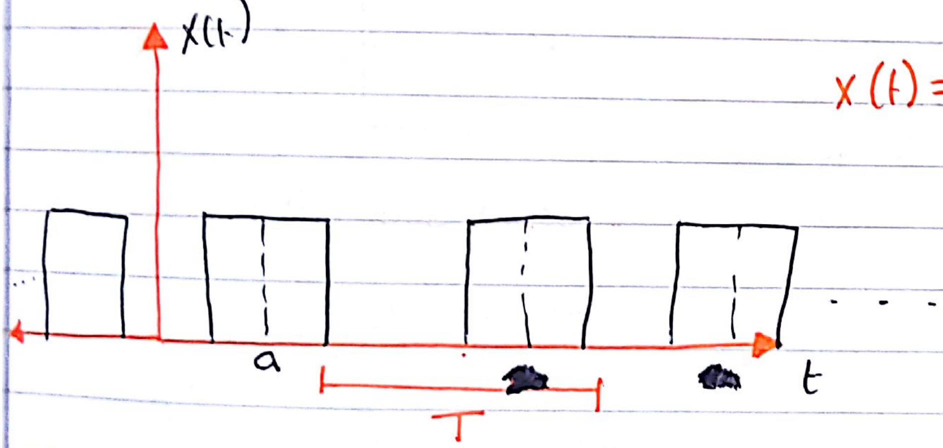
A/D Converter

Sampling — Converts to dots (Samples)

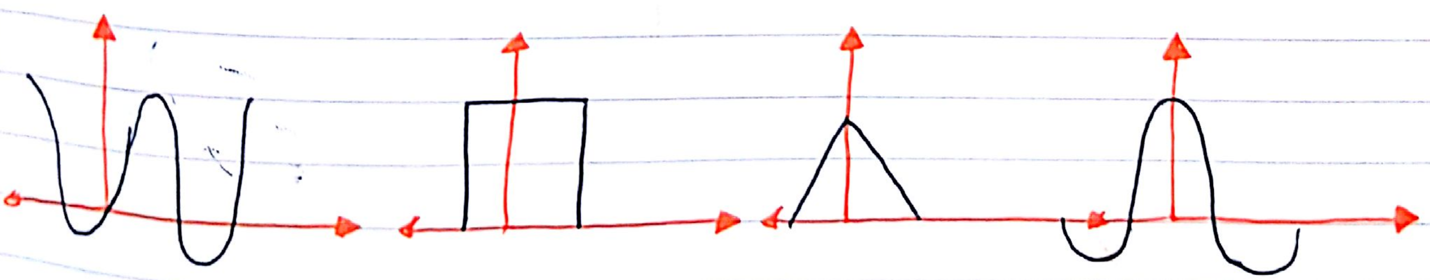
Quantization → Discretization

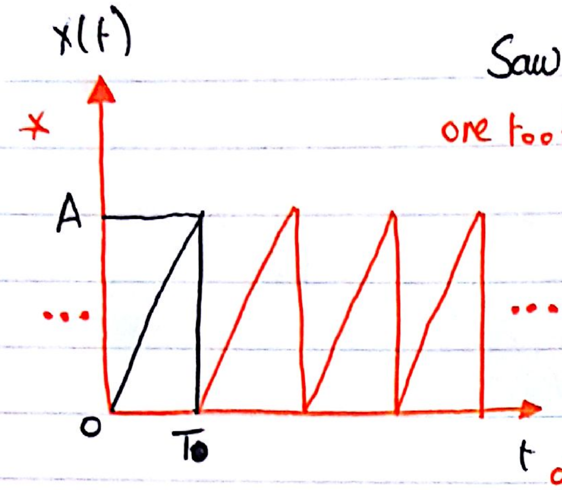
* A periodic function is one for which $x(t) = x(t \pm nT)$ where T is the period of the base function & n is an integer.

$$x(t) = A \sum_{i=-\infty}^{\infty} rect\left(\frac{t - iT}{T_x}\right)$$



* A function $x(t)$ is said to be odd if → $x(t) = -x(-t)$ for all t.
* A function $x(t)$ is said to be even if → $x(t) = x(-t)$ for all t





Saw tooth

one tooth:- $X_{swb}(t) = \frac{A}{T} U_r(t) - \frac{A}{T} U_r(t-T) - A U_s(t-T)$

all together $\rightarrow$ $X_{sw}(t) = A \Sigma X_{swb}(t - iT)$

* <u>Linear time invariant Systems:-</u>

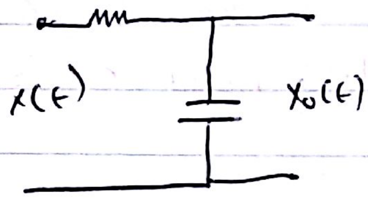
* properties of linear Systems:-

System

input $\rightarrow$ $T[\cdot]$ $\rightarrow$ response of the System to the input. output $y(t)$

(Continuous time signal) $x(t)$

$$y(t) = T[x(t)]$$



$$x_o(t) = \frac{1}{c}\int_{-\infty}^{\infty} i(t)\,dt$$

$$T[\cdot] = \frac{1}{c}\int_{-t}^{t} \cdot \, dt$$

$$T[\cdot] = \frac{1}{c}\int_{-\infty}^{t_0} (\cdot)\,dt + \frac{1}{c}\int_{t_0}^{t} (\cdot)\,dt$$

initial condition

|x| is not linear

1- Linearity Property :-

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b).$$

- Homogenous $\longrightarrow$ $f(\alpha a) = \alpha f(a)$
- Superposition $\longrightarrow$ $f(a+b) = f(a) + f(b)$

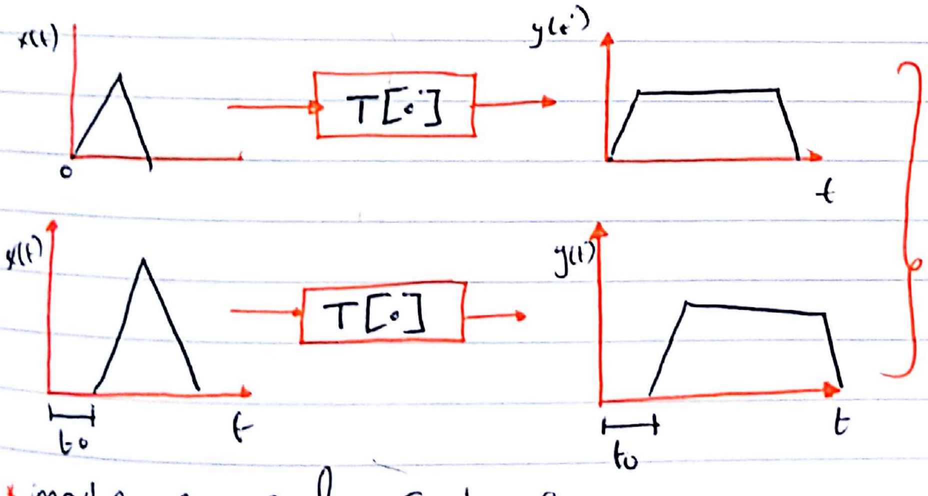
2- Causality :-

Based on the principle of Cause + effect. a causal System would not function before an excitation is applied; a Causal System Cannot rely on future excitation for proper operation.

3- Stability (Bounded input Bounded output (BIBO)) :-

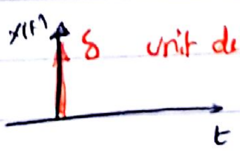
if For Some input $|x(t)| < M$ (Amplitude) the resulting output $|y(t)| < N$ (Amplitude) then we Call the underlying System a Stable one.

4- Time / shift invariant :-

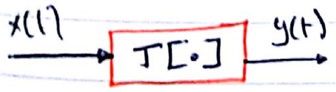


Example of a time invariant Signal.

* impulse response of a System :-



unit delta function (impulse)



$$y(t) = T[\delta(t)]$$ ← impulse response of the System

① $\int_{-\infty}^{t_1} f(t-t_0)\,\delta(t)\,dt = f(-t_0)$

② $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$ sifting property

Using ① inserting a delta function to the System will show us the function inside the System. ⑪

$$y(t) = T[\delta(t)] = f(\cdot)$$

$$T[\cdot] = h(t)$$

$$\delta(t-t_0) \longrightarrow \boxed{h(t)} \longrightarrow y(t, t_0)$$

if LTI

$$\delta(t-t_0) \longrightarrow \boxed{h(t)} \longrightarrow y(t-t_0)$$

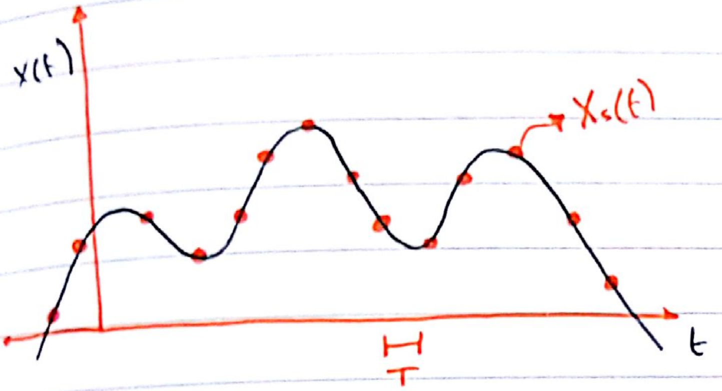
$$\int_{-\infty}^{\infty} \delta(t,\tau)\, T[\cdot]\, dt = y(t) \longrightarrow \int_{-\infty}^{\infty} h(t,\tau)\, \delta(\tau)\, d\tau = y(t,\tau)$$

if the system is shift invariant.

$$\int_{-\infty}^{\infty} h(t-\tau)\, \delta(\tau)\, d\tau = y(t-\tau)$$

* $\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau$

L this function reveals what the System is.



$$X_s(t) = \sum_{i=-\infty}^{\infty} x(t)\,\delta(t-iT)$$

$$X_s(t) \longrightarrow \boxed{T[\cdot]} \longrightarrow y_s(t)$$

$$y_s(t) = T[x_s(t)] = T\Big[\sum_{i=-\infty}^{\infty} x(t)\,\delta(t-i\underset{\text{Period}}{T})\Big]$$

$$\overset{\text{linear}}{=} \sum_{i=-\infty}^{\infty} x(t)\,T[\delta(t-iT)]$$

* Define $h(t,\tau) \triangleq T[\delta(t-iT)]$ where $\tau = iT$

$$\Rightarrow \sum_{i=-\infty}^{\infty} x(t)\,h(t,\tau) \overset{\tau \xrightarrow{\text{limit}} 0}{=} \sum_{i=-\infty}^{\infty} x(\tau)\,h(t,\tau) \overset{\lim \tau \to 0}{=} \int_{-\infty}^{\infty} x(\tau)\,h(t,\tau)\,d\tau$$

$h(t,\tau) \triangleq T[\delta(t-iT)]$ impulse response of a System

Assuming that the System is time inv.

$$x(t) \longrightarrow \boxed{h(t,\tau)} \longrightarrow y(t,\tau)$$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

linear & shift invariant

System response

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Convolution integral.

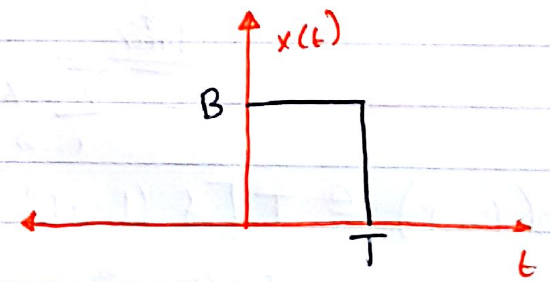
$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

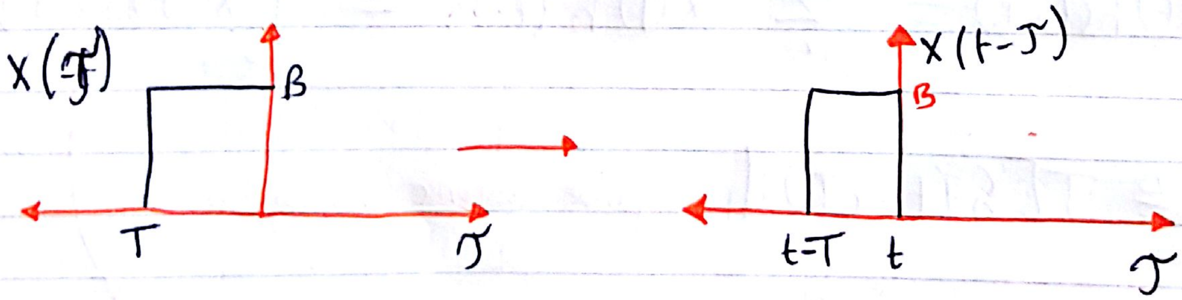
impulse response.

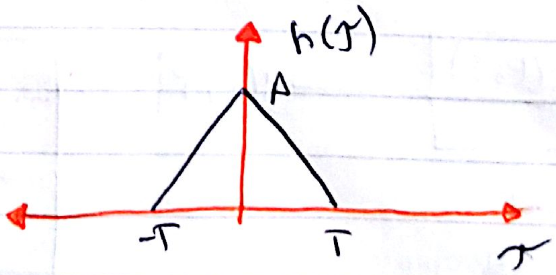
$$\equiv \quad \boxed{y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau}$$

* Convolution Computation:-

let $h(t)$ be given by & $x(t)$ be given by







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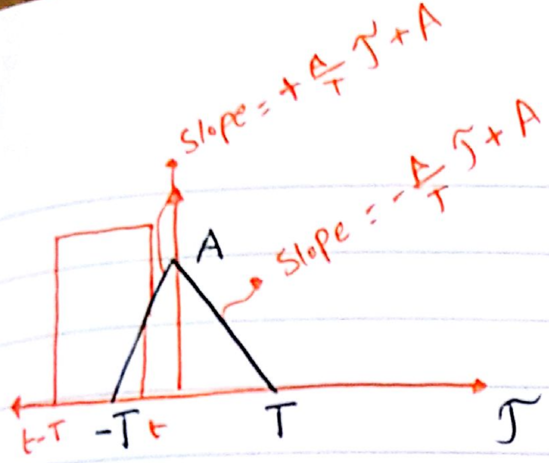


$$x(t-\tau)\,h(\tau) =$$

① For $t < -T$

$$x(t-\tau)\,h(\tau) = 0$$

② For $-T \leq t \leq 0$

$$x(t-\tau)\,h(\tau) = \int_{-T}^{t} \left(\frac{A}{T}\tau + A\right)\cdot B$$

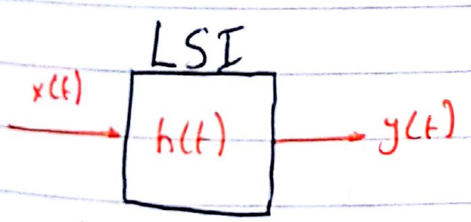
③ For $0 \leq t \leq T$

$$\int_{0}^{t} x(t-\tau)\,h(\tau) = \int_{0}^{t} \left(\frac{-A}{T}\tau + A\right)\cdot B$$

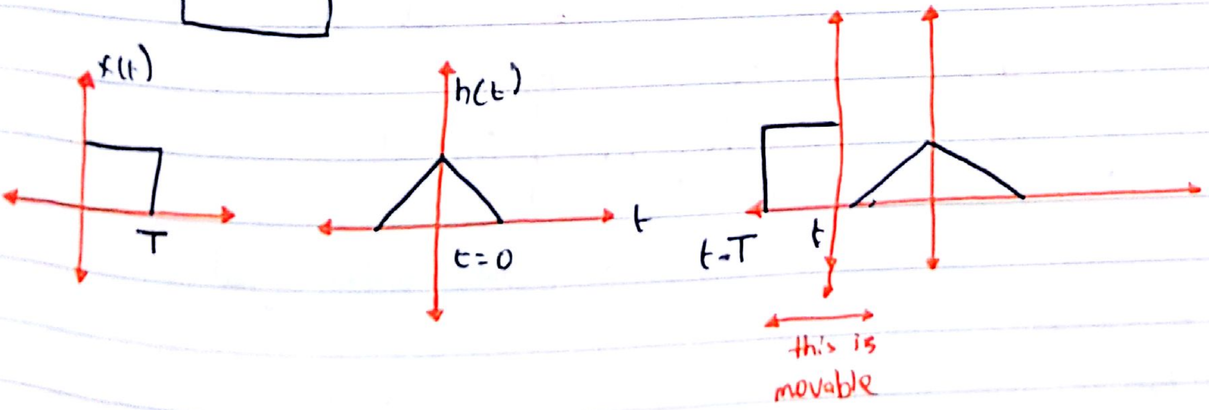
④ For $t \geq T$

zero.

*

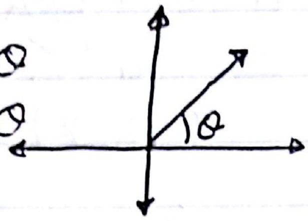


Convolution :- how the System reacts with the input.



* Mathematical equations and functions :-

Euler's eq.
$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$



$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

* Even and odd functions :-

* $f(x)$ is even if $f(x) = f(-x)$
* $f(x)$ is odd if $f(x) = -f(-x)$

* $X_e(t) = x_e(-t)$
* $X_o(t) = -X_o(-t)$
* Any Signal can be expressed as a Sum of an even function + an odd function.

* $x(t) = X_e(t) + X_o(t)$

* $X_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

* $X_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$

* Average value for a signal :-

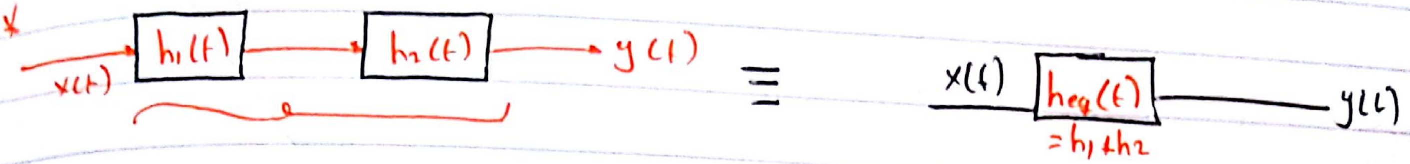
$$A_x = \lim_{t\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\, dt$$

* RMS Value of ~~the~~ function :-

$$\sqrt{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)\, dt}$$

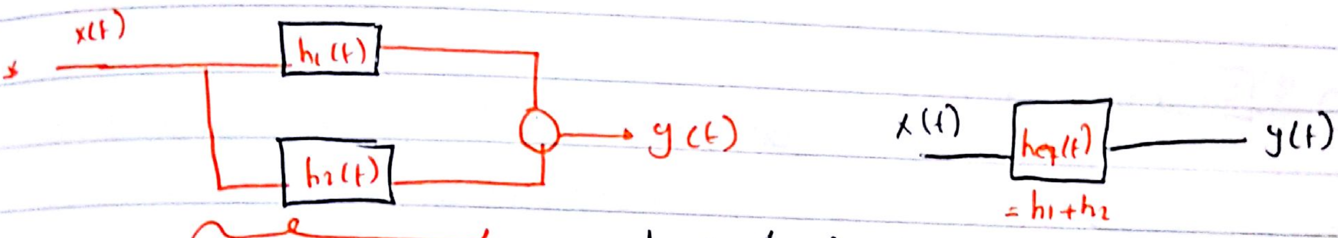
effective value of the Signal.

*



we can replace these 2 with 1 Block $h(t) = h_1(t) * h_2(t)$.

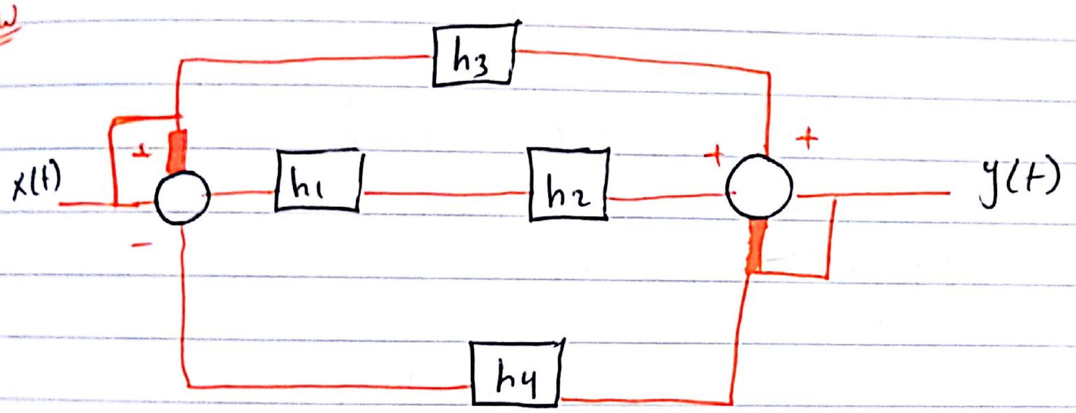
*



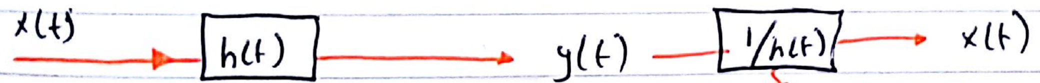
we can replace these two with $h(t) = h_1(t) + h_2(t)$.

H.W

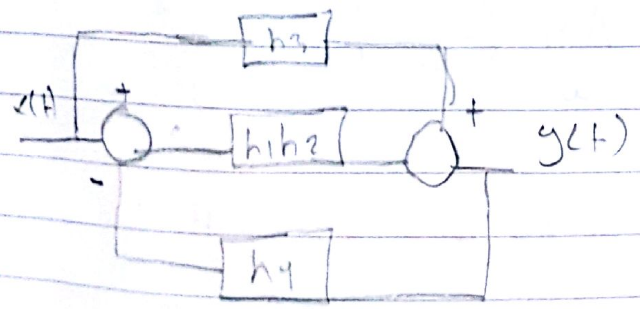
*



* Inverse of a System :-



inverse of a function (to get the input again) provided that the System has an inverse



$$e(t) = x(t) - h_4\, y(t)$$

$$y(t) = h_3\, x(t) + h_1 h_2\, e(t)$$

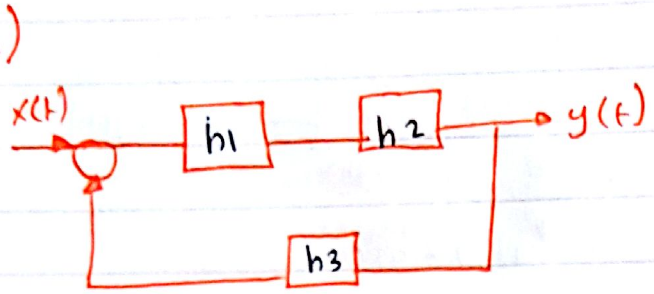
$$y(t) = h_3\, x(t) + h_1 h_2\, x(t) - h_1 h_2 h_4\, y(t)$$

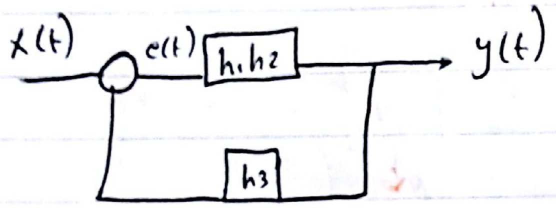
$$y(t) + h_1 h_2 h_4\, y(t) = h_3\, x(t) + h_1 h_2\, x(t)$$

$$y(t)[1 + h_1 h_2 h_4] = x(t)[h_3 + h_1 h_2]$$

$$\frac{y(t)}{x(t)} = \frac{[h_3 + h_1 h_2]}{1 + h_1 h_2 h_4}$$

Ex)





$$e(t) = x(t) - h_3\, y(t)$$

$$y(t) = e(t)\, h_1 h_2$$

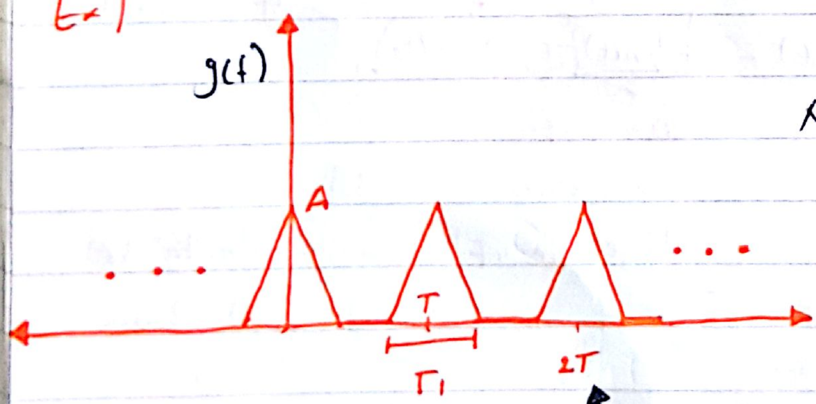
$$\frac{y(t)}{h_1 h_2} = e(t)$$

$$\frac{y(t)}{h_1 h_2} + h_3\, y(t) = x(t)$$

$$\left(\frac{h_3 + 1}{h_1 h_2}\right) y(t) = x(t)$$

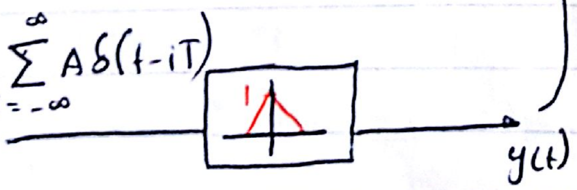
$$y(t) = \frac{h_1 h_2}{1 + h_1 h_2 h_3}\, x(t)$$

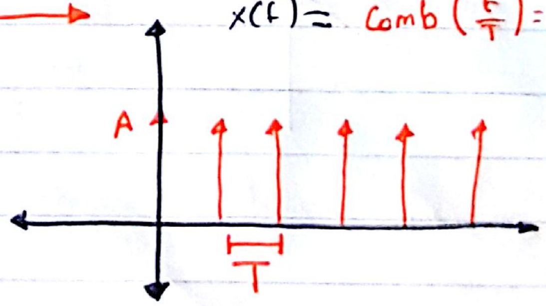
Ex)



$$x(t) = \sum_{i=-\infty}^{\infty} A\, tri\left(\frac{t - iT}{T_1}\right)$$

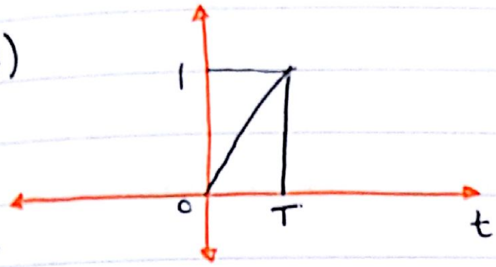
$$x(t) = comb\left(\frac{t}{T}\right) = \sum_{i=-\infty}^{\infty} A\delta(t - iT)$$



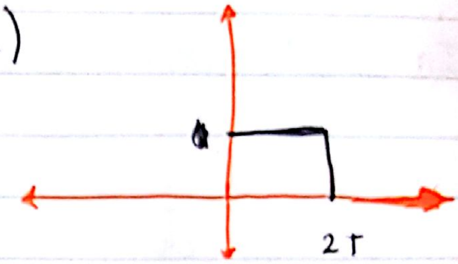


Example on Convolution :-

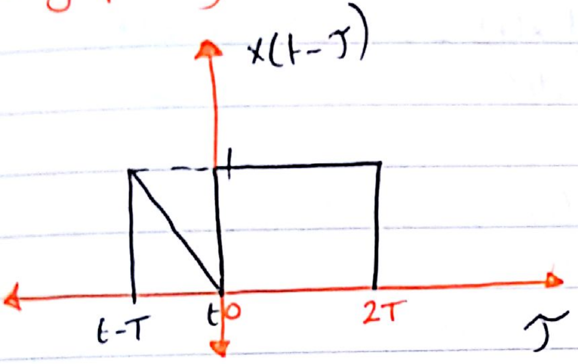
* Let $x(t)$



* $h(t)$



* Doing it graphically



So, for ① $t<0$

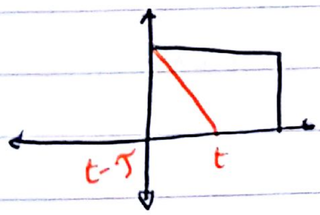
* $\int h(\tau)\,x(t-\tau)\,d\tau = 0$ no overlap.

② for $t>0$ & $t-T\leq 0$

$$\int_0^T h(\tau)\,x(t-\tau)\,d\tau$$

$$= \int_0^t -\frac{1}{T}\tau + 1 \;d\tau$$

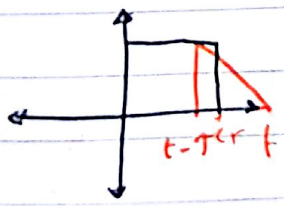
for $0\leq t\leq T$ $= -\frac{1}{2T}\tau^2\Big|_0^t = \boxed{-\frac{t^2}{2T}} = -\frac{t^2}{2T}$



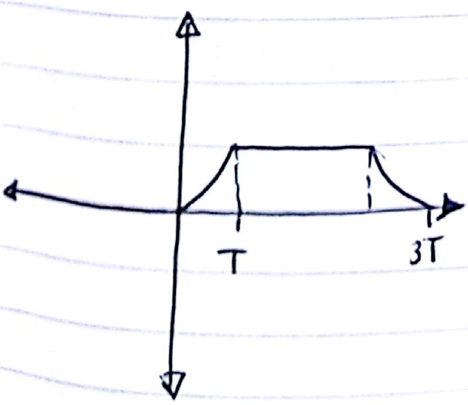
③ for $t-T>0$ & $t\leq 2T$

$t>T \rightarrow \boxed{T<t\leq 2T}$

$$\int_{t-T}^{t} x(t-\tau)\,h(\tau)\,d\tau = \frac{1}{2}T$$



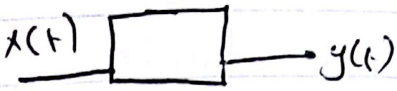
④ $$\int_{t-T}^{2T} x(t-\tau) * 1 \;d\tau =$$



⑤ $t-T>2T$

$$\int = 0$$

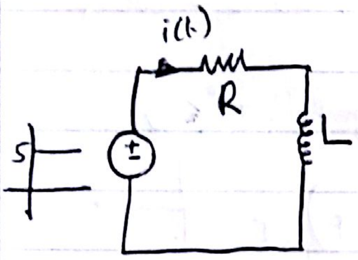
* Differential Equation System models :-



* In general, System responses Can be modeled as

$$a_m \frac{d^m y(t)}{dt^m} + a_{m-1} \frac{d^{m-1} y(t)}{dt^{m-1}} --- + a_0 y(t)$$

$$= b_0 x(t) + b_1 \frac{d x(t)}{dt} + b_2 \frac{d^2 x(t)}{dt^2} + \dots + b_n \frac{d^n x(t)}{dt^n}$$



$$R i(t) + L \frac{di(t)}{dt} = 5 u_s(t)$$

$$i(t) = i_{\substack{\text{steady} \\ \text{state}}}(t) + i_{\substack{\text{natural} \\ \text{response} \\ \text{(transient)}}}(t)$$

* To find the transient Solution :-

$$L \frac{di}{dt} + R i(t) = 0$$

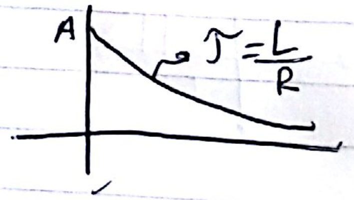
Assume that the form of the Solution is a Complex exponential.
is $i(t) = Ae^{st}$

now Substitute :-

$$L \cancel{A} s e^{st} + R \cancel{A} e^{st} = 0$$

$$Ls + R = 0$$

$$\hookrightarrow s = \frac{-R}{L} \longrightarrow \boxed{i(t) = Ae^{-\frac{R}{L}t}}$$



if we work with the fact the the excitation signal x(t) is Constant for $t \geq 0$

$$L \cancel{\frac{di}{dt}}^{\,0} + R i(t) = 5 \qquad \text{assuming that } i(t) \text{ is Constant}$$

$$RK = 5$$

$$\boxed{K = \frac{5}{R}}$$

$$i(t) = \left(A e^{-\frac{R}{L}t} + \frac{5}{R}\right) ~~u_s(t)~~$$

* Note that the multiplying Constant is closely coupled with the initial Conditions of the System.

* Compute it after determining the Complete Solution. i.e : $i(t)$ = i_{ss} + i_t

Ex) $\frac{dy(t)}{dt} + 2y(t) = 2 \quad , t \geq 0 \qquad y(0) = 4$.

$$\frac{dy_t}{dt} + 2y_t(t) = 0$$

$$\frac{dy_t(t)}{dt} + 2y_t(t) = 0$$

$\hookrightarrow (s+2)Ce^{st} = 0 \rightarrow s = -2$

$y_t(t) = Ce^{-2t}$ where C is to be determined from the initial condition.

if we assume that the input is Constant.

$$y_{ss}(t) = P$$

$$\frac{dP}{dt} + 2P = 0 + 2P = 2$$

$$\boxed{P = 1}$$

$$y(t) = y_t(t) + y_{ss}(t) = Ce^{-2t} + 1$$

$$\rightarrow y(0) = 4 = Ce^{0} + 1 \rightarrow \boxed{C = 3}$$

$$\boxed{y(t) = 1 + 3e^{-2t} \quad , t \geq 0}$$

Ex) Lets Consider a System excited by a Sinusoidal input

P126 3.17

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 10\, x(t)$$

$$x(t) = 5\cos(2t + 40^\circ)$$

if we assume that the Solution again is in the form Ae^{st}

$$As^2 e^{st} + 3As\, e^{st} + 2Ae^{st} = s^2 + 3s + 2 = 0$$

$$[s^2 + 3s + 2]\, Y(s) = [\quad]\, X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{10}{s^2 + 3s + 2}$$

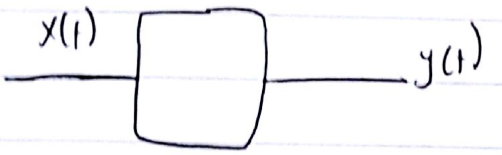
convolution only applies on LSC

* Transfer Function :-

For an LTI System that can be represented by the n^{th} order differential equation :-

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_0 x(t) + b_1 \frac{d x(t)}{dt^2} + \cdots + b_m \frac{d^m x(t)}{dt^m}$$



Complex variable

$x(t) = A e^{st}$

$y(t) = B e^{st}$

$$a_n s^n B \cancel{e^{st}} + a_{n-1} s^{n-1} B \cancel{e^{st}} + \cdots a_1 s B \cancel{e^{st}} + a_0 Y \cancel{e^{st}} =$$

$$b_m A \cancel{e^{st}} + b_1 s A \cancel{e^{st}} + b_2 s^2 A \cancel{e^{st}} + \cdots b_m s^3 A \cancel{e^{st}} + b_m s^m A \cancel{e^{st}}$$

$$B \left[a_n s^n + a_{n-1} s^{n-1} + \cdots a_n s + a_0 \right] = A \left[b_0 + b_1 s + b_2 s^2 + \cdots b_m s^n \right]$$

$$\frac{B}{A} = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n} = H(s)$$ transfer function

Ex) 3.15

$$\frac{d^2 y(t)}{dt^2} + 1.25 \frac{dy(t)}{dt} + 0.375\, y(t) = x(t)$$

characteristic equation:- $s^2 + 1.25 s + 0.375 = 0$

$$(s + 0.75)(s + 0.5) = 0$$

$\longrightarrow s_1 = -0.75 \quad s_2 = -0.5$

$$y_c(t) = C_1 e^{-0.75t} + C_2 e^{-0.5t}$$

natural output

$t \longrightarrow \infty \longrightarrow 0$ (stable) order of DE → no. of elements that store energy

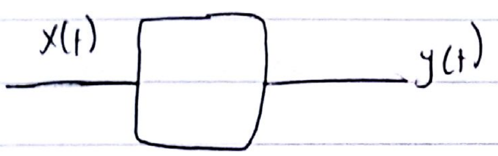
Convolution only applies on LTI

Transfer function :-

For an LTI System that can be represented by the n^{th} order differential equation :-

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_0 x(t) + b_1 \frac{dx(t)}{dt^2} + \cdots + b_m \frac{d^m x(t)}{dt^m}$$



Complex variable

$x(t) = A e^{st}$

$y(t) = B e^{st}$

$$a_n s^n B \cancel{e^{st}} + a_{n-1} s^{-1} B \cancel{e^{st}} + \cdots a_1 s B \cancel{e^{st}} + a_0 Y \cancel{e^{st}} =$$
$$b_m A \cancel{e^{st}} + b_1 s A \cancel{e^{st}} + b_2 s^2 A \cancel{e^{st}} + \cdots b_m s^3 A \cancel{e^{st}} + b_m s^m A \cancel{e^{st}}$$

$$B\left[a_n s^n + a_{n-1} s^{n-1} + \cdots a_n s + a_0\right] = A\left[b_0 + b_1 s + b_2 s^2 + \cdots b_m s^m\right]$$

$$\frac{B}{A} = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n} = H(s)$$ transfer function

Ex) 3.15

$$\frac{d^2 y(t)}{dt^2} + 1.25 \frac{dy(t)}{dt} + 0.375\, y(t) = x(t)$$

characteristic equation:- $s^2 + 1.25 s + 0.375 = 0$

$$(s + 0.75)(s + 0.5) = 0$$

$\longrightarrow s_1 = -0.75 \qquad s_2 = -0.5$

$$y_c(t) = c_1 e^{-0.75t} + C_2 e^{-0.5t}$$

natural response

$t \to \infty \longrightarrow 0$ (stable)

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$5 - j6 = \sqrt{25+36} \angle \tan^{-1} \frac{-6}{5}$

Ex)

$$\frac{d^2y}{dt^2} + 0.25 \frac{dy(t)}{dt} - 0.375\, y(t) = x(t)$$

$$s^2 + 0.25s - 0.375$$

$$(s + 0.75)(s - 0.5) = 0$$

$$s_1 = -0.75 \quad s_2 = 0.5$$

$$y_c(t) = C_1 e^{-0.75t} + C_2 e^{0.5t}$$

$t \longrightarrow \infty \quad \longrightarrow \infty$ (unstable)

Example:- Consider the system equation

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 10x(t) \quad , \; x(t) = 5\cos(2t + 40^\circ)$$

$$H_s = \frac{10}{s^2+3s+2}$$

$$Y = H(s)\Big|_{j2} X = \frac{10}{s^2+3s+2}\Bigg| 5\angle 40^\circ = \frac{50 \angle 40}{-2 + j6}$$

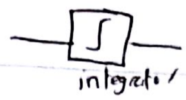
$$= \frac{50\angle 40^\circ}{6.325\angle 108.4} = 7.905 \angle -68.4^\circ$$

$$= 7.905\, e^{-j68.4}$$

$$y_c(t) = 7.905 \cos(2t - 68.4)$$

to implement a System, it must be Causal.

a block diagram with
$h(t) \rightarrow$ Convolution.

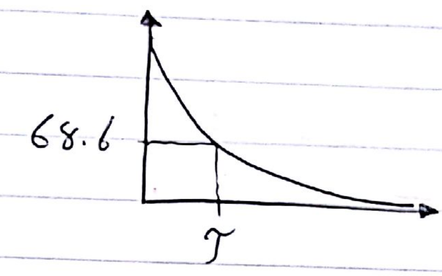


* $Y = X(H_s)$

$$\frac{Y}{X} \triangleq H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

* τ for an RC circuit
$\tau = RC$
* τ for an RL ckt
$\tau = \frac{L}{R}$

$x(t) = Xe^{st}$
$Y(t) = Ye^{st}$



In Cases where the input Signal is a Sinusoid :-

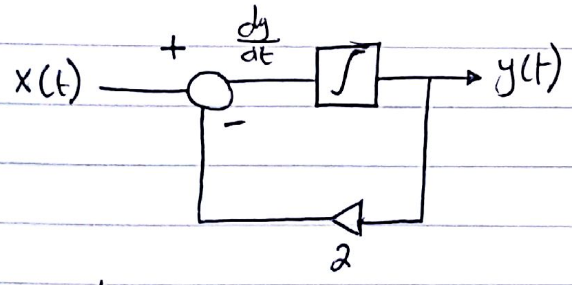
$$A \cos(\omega_1 t + \phi)$$

$$y_{ss}(t) = |X||H(\omega_1)| \cos(\omega_1 t + \phi - \theta) \rightarrow \text{delay time of a System}$$

* **Time domain System implementation :-**

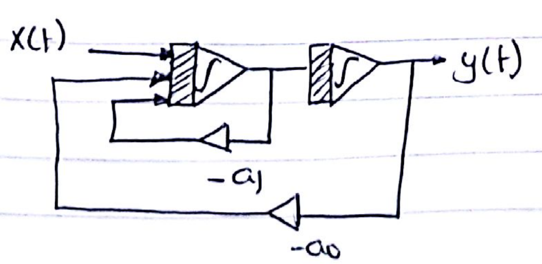
$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

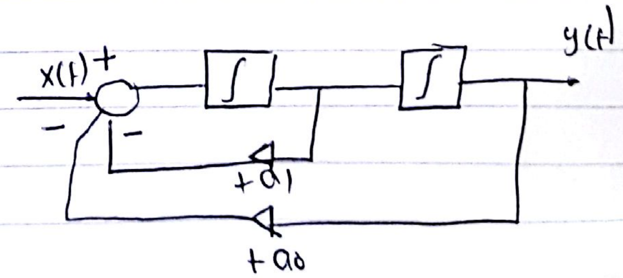
$$\frac{dy(t)}{dt} = -2y(t) + x(t)$$



* $$\frac{d^2y(t)}{dt^2} + a_1 \frac{dy}{dt}(t) + a_0 y(t) = x(t)$$

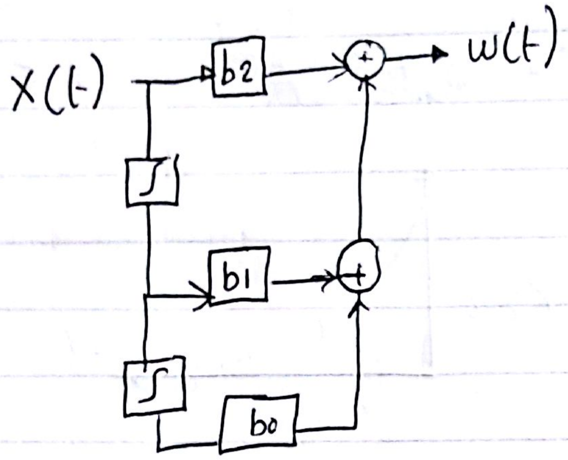
$$\frac{d^2y(t)}{dt^2} = -a_1 \frac{dy(t)}{dt} - a_0 y(t) + x(t)$$





$$* \sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = \sum_{j=0}^{m} b_j \frac{d^j x(t)}{dt^j}$$

2nd order

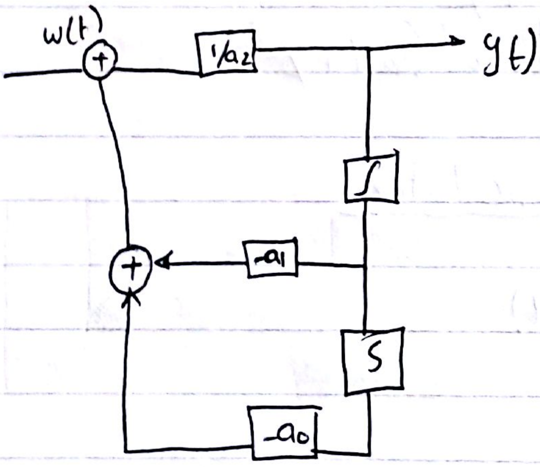


Direct form I

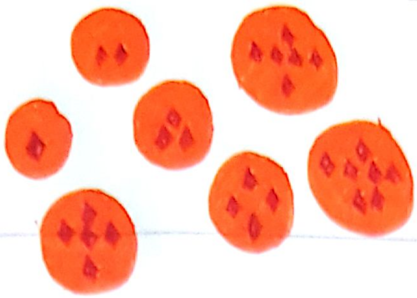
4

Direct form II

check the book (last pages of chapter 3)



Tiësto



The Dragon Balls.

Time domain representation of Continivos time Signals

Introduction to Systems & their properities

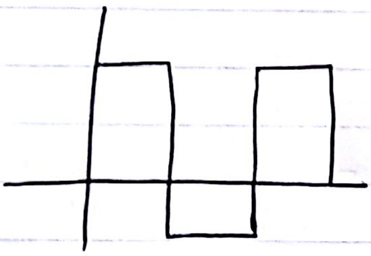
* Introduction to Fourier Series :-

Recall, for any function $f(x)$ that is well behaved & continuous with partial derivatives everywhere $f(x)$ can be expanded / appx-ed about some point a as:-

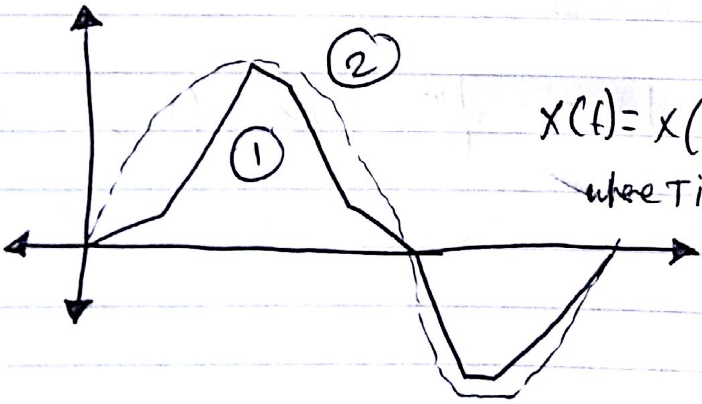
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \underbrace{R_n}_{\text{remainder}}$$

Special Case when $a=0$.

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \dots \frac{f^{n}(0)}{n!}x^n + R_n$$. Power series expansion (McLaurin).



→ can't use Fourier Series because some points are defined by more than one value.



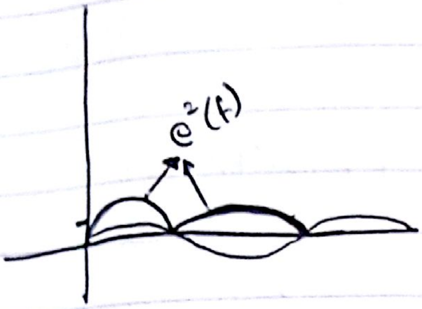
$x(t) = x(t+nT)$ where T is the period

we can use Fourier Series on ② but not ①.

$X_{appx} = B_1 \sin\omega_0 t$

$x(t) - x_{appx.}(t) = \underbrace{e(t)}_{\text{error}}$ → Periodic.

$e(t) = x(t) - B_1 \sin\omega_0 t$ (we can only control this)



$$\frac{1}{T_0}\int_0^{T_0} e(t)\,dt = 0$$ ← average value
↳ m/s reading!

$$\frac{1}{T_0}\int_0^{T_0} e^2(t)\,dt = J[e^2(t)]$$ → mean Square error.

Cost function

$$= \frac{1}{T_0}\int_0^{T_0}\left[x(t) - B_1 \sin w_0 t\right]^2 dt$$

$$\left.\frac{\partial J[e^2(t)]}{\partial B}\right|_{B_1 = B_{1(optimum)}} = \frac{1}{T_0}\int_0^{T_0}\frac{\partial}{\partial B_1}\left[x(t) - B_1 \sin w_0 t\right]^2 = 0\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} -2\left[x(t) - B_1 \sin w_0 t\right]\sin w_0 t\, dt$$

$$\left.\frac{1}{T_0}\int_0^{T_0} -2\left[x(t) - B_1 \sin w_0 t\right]\sin w_0 t\, dt\right|_{B_1 = B_1^*(optimum)} = 0$$

$$\longrightarrow \frac{-2}{T_0}\int_0^{T_0} x(t)\sin w_0 t + \frac{2}{T_0}\int_0^{T_0} B_1 \sin^2 w_0 t\, dt = 0$$

$$\frac{1}{T_0}\int_0^{T_0} x(t)\sin w_0 t = \frac{1}{T_0}\int_0^{T_0}\frac{B_1}{2}\left[1 - \cos 2w_0 t\right] dt.$$ ← average of sin/cos over a period = 0

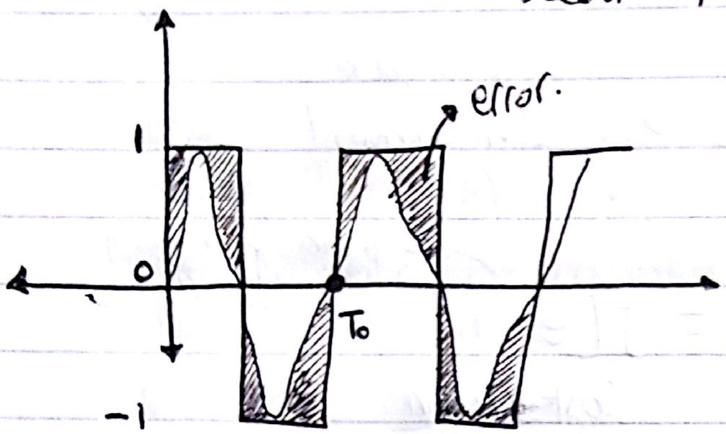
$$= \frac{1}{T_0}\int_0^{T_0}\frac{B_1^*}{2}\, dt = \frac{B_1^*}{2T_0}\left[t\right]_0^{T_0} = \frac{B_1^*}{2\cancel{T_0}}\cancel{T_0}$$

$$= \frac{B_1^*}{2}$$

$$\rightarrow B_1^* = \frac{2}{T_0}\int_0^{T_0} x(t)\sin w_0 t\, dt.$$

we can't use Cos

because the error would be huge.



$$x(t) \begin{cases} 1, & 0<t<T_0/2 \\ -1, & T_0/2<t<T_0 \end{cases}$$

$$B_1^* = \frac{2}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t \, dt.$$

* $\sin\omega_0 t$ has the same period as the original function.

$$B_1^* = \frac{2}{T_0}\int_0^{T_0/2} 1\cdot\sin\omega_0 (t) + \frac{2}{T_0}\int_{T_0/2}^{T_0} -\sin\omega_0 t$$

$$B_1^* = \frac{2}{T_0}\left[\frac{-\cos\omega_0 t}{\omega_0}\Big|_0^{T_0/2} + \frac{\cos\omega_0 t}{\omega_0}\Big|_{T_0/2}^{T_0} \right]$$

$$\omega_0 t\Big|_{T_0/2} = \frac{2\pi}{T_0}\left(\frac{T_0}{2}\right) = \pi \qquad = \frac{1}{\pi}\left[-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi\right]$$

$$B_1^* = \boxed{\frac{4}{\pi}}$$

$$e(t) = x(t) - \frac{4}{\pi}\sin\omega_0 t = \begin{cases} 1-\frac{4}{\pi}\sin\omega_0 t & 0<t<T_0/2 \\ -1-\frac{4}{\pi}\sin\omega_0 t, & \frac{T_0}{2}<t<T_0 \end{cases}$$

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$$\frac{2}{j} \times \frac{-j}{-j} = -2j$$

$$\tan^{-1}\left(\frac{-2}{1}\right) = \boxed{\pi/2}$$

Example :-

Consider the function

$$x(t) = 10 + 3\cos \omega_0 t + 5\cos(2\omega_0 t + 30^\circ) + 4\sin 3\omega_0 t$$

using euler's identity

$$= 10 + \frac{3}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{5}{2}\left[e^{j(2\omega_0 t + 30^\circ)} + e^{-j(2\omega_0 t + 30^\circ)}\right]$$

$$+ \frac{4}{2j}\left[e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right]$$

$$x(t) = 2e^{j\pi/2} e^{-j3\omega_0 t} + 2.5 e^{-j\pi/6} e^{-j2\omega_0 t} + 1.5 e^{-j\omega_0 t} + 10$$
$$+ 1.5 e^{j\omega_0 t} + (2.5 e^{j\pi/6}) e^{j2\omega_0 t} + 2e^{-j\pi/2} e^{j3\omega_0 t}$$

$$x(t) = C_{-3} e^{-j3\omega_0 t} + C_{-2} e^{-j2\omega_0 t} + C_{-1} e^{-j\omega_0 t} + C_0$$

$$+ C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + C_3 e^{j3\omega_0 t}.$$

$$= \sum_{k=-3}^{k=3} C_k e^{jk\omega_0 t}.$$

Complex Conjugate

k	C_k	C_{-k}
0	10	—
1	1.5	1.5
2	$2.5\angle 30^\circ$	$2.5\angle -30^\circ$
3	$2\angle -90^\circ$	$2\angle 90^\circ$

$$(a + jb)^* \longrightarrow a - jb$$

$$(C\angle\theta_k)^* \longrightarrow C\angle -\theta_k$$

Fourier Series :-

Suppose that x(t) is a periodic time function with a base harmonic frequency ω_0, the Fourier series is defined as:-

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} , \quad C_k = C^*_{-k}$$

$$C_k = |C_k| e^{j\theta_k}$$

$$C_{-k} e^{-jk\omega_0 t} + C_k e^{jk\omega_0 t} = |C_k| e^{-j\theta_k} e^{-jk\omega_0 t} + |C_k| e^{j\theta} e^{jk\omega_0 t}$$

$$= |C_k| \left[e^{-j[k\omega_0 t + \theta_k]} + e^{j[k\omega_0 t + \theta_k]} \right]$$

$$= 2|C_k| \cos(k\omega_0 t + \theta_k)$$

the original expression becomes:- $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k)$$

$$= C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos\theta_k \cos \omega_0 t k - 2|C_k| \sin\theta_k \sin k\omega_0 t]$$

$$2 C_k = 2|C_k| e^{j\theta k}$$

$$= 2|C_k| \cos \theta_k + j 2|C_k| \sin\theta_k = A_k - jB_x$$

$$\Rightarrow x(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos k\omega_0 t + B_k \sin k\omega_0 t]$$

$$2C_k = A_k - jB_k \; ; \; C_k = |C_k| e^{j\theta_k}, \quad A_0 = C_0$$

* Any two functions, $g(t)$ & $x(t)$, are said to be orthogonal if $\int g(t)\, x(t)\, dt = 0$

$\int \sin w_0 t \cdot \cos w_0 t \, dt$ Zero always

$\int \sin i w_0 t \, \sin k w_0 t \, dt = 0, \forall\, i \neq k$

$\int \cos i w_0 t \, \cos k w_0 t \, dt = 0, \forall\, i \neq k$

$$B_k^* = \frac{2}{T_0}\int_0^{T_0} x(t) \sin k w_0 t \, dt$$

$$A_k = \frac{2}{T_0}\int_0^{T_0} x(t) \cos k w_0 t \, dt$$ → normalize.

$$A_0 = \frac{1 \; 2}{T_0}\int_0^{T_0} x(t)\, dt.$$

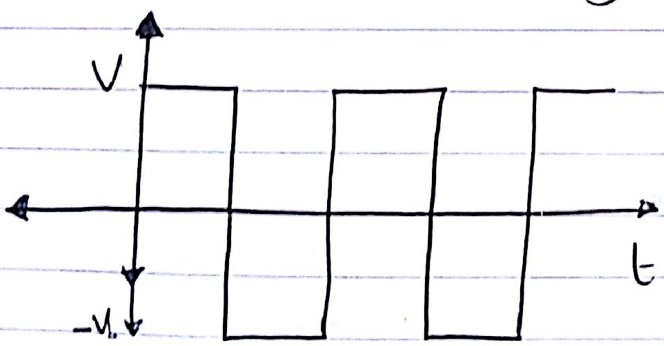
Average value from the graph

$\sin^2 w_0 t = \frac{1}{2}\left[1 - \cos 2 w_0 t\right]$

→ average value = $\boxed{\frac{1}{2}}$

zero average value

* Fourier series and the Frequency spectrum:-



taylor series

$$x(t) = C_0 + \sum_{k=-\infty}^{\infty} C_k e^{-jkw_0 t}$$

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jkw_0 t}\, dt$$

$$x(t) = \begin{cases} V & 0 < t < \frac{T_0}{2} \\ -V & T_0/2 < t < T_0 \end{cases}$$

$$= \frac{V}{T_0}\int_0^{T_0/2} e^{-jkw_0 t}\, dt - \frac{V_0}{T_0}\int_{T_0/2}^{T_0} e^{-jkw_0 t}\, dt$$

$$= \frac{V}{T_0[-jkw_0]}\left[e^{-jkw_0 t}\Big|_0^{T_0/2} - e^{-jkw_0 t}\Big|_{T_0/2}^{T_0} \right]$$

$$w_0 t \Big|_{t=\frac{T_0}{2}} = \frac{2\pi}{\cancel{T_0}} \times \frac{\cancel{T_0}}{2} = \pi$$

$$w_0 T_0 = 2\pi$$

$$C_k = \frac{jV}{2\pi k}\left(e^{-jk\pi} - e^{-j(0)} - e^{-jk2\pi} + e^{-jk\pi}\right)$$

$$C_k = \begin{cases} \frac{-2jV}{k\pi} = \frac{2V}{k\pi} \angle -90^\circ & , k\ \text{odd} \\ \text{zero} & , k\ \text{even} \end{cases}$$?

because the original function is purely odd.

$$C_0 = \frac{2}{T_0} \int_0^{T_0} x(t)\, dt.$$ → or Sometimes just Substitute k in the equation

$$x(t) = \sum_{\substack{k=-\infty \\ k\ \text{odd}}}^{\infty} \frac{2V}{k\pi} e^{-j\pi/2} e^{jk\omega_0 t}$$

90°

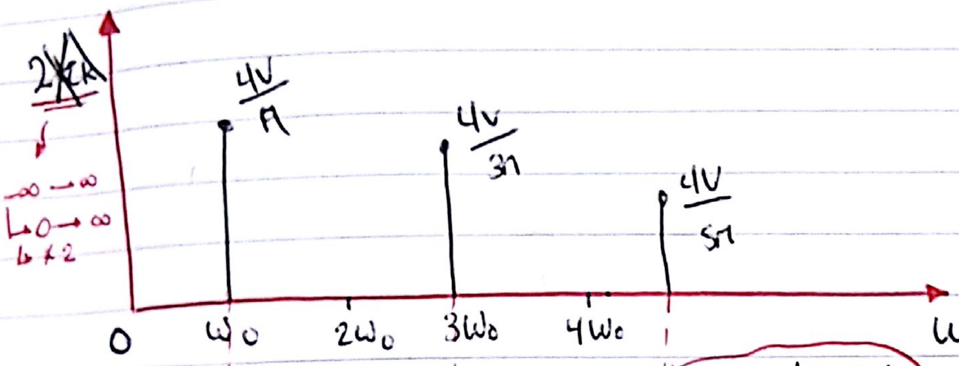
First harmonic frequency. (Base Frequency)

$$= \sum_{\substack{k=1 \\ k\ \text{odd}}}^{\infty} \frac{4V}{k\pi} \cos(k\omega_0 t - 90^\circ)$$

$$= \sum_{k=1}^{\infty} \frac{4V}{k\pi} \sin(k\omega_0 t)$$

* For −ve → phase becomes +90°

* Fourier Series & the Frequency [redacted] Spectrum :-



$$X(t) = \sum_{\substack{k=-\infty \\ k=odd}} \frac{4V}{k\pi} e^{-j\pi/2} e^{jk\omega_0 t}$$

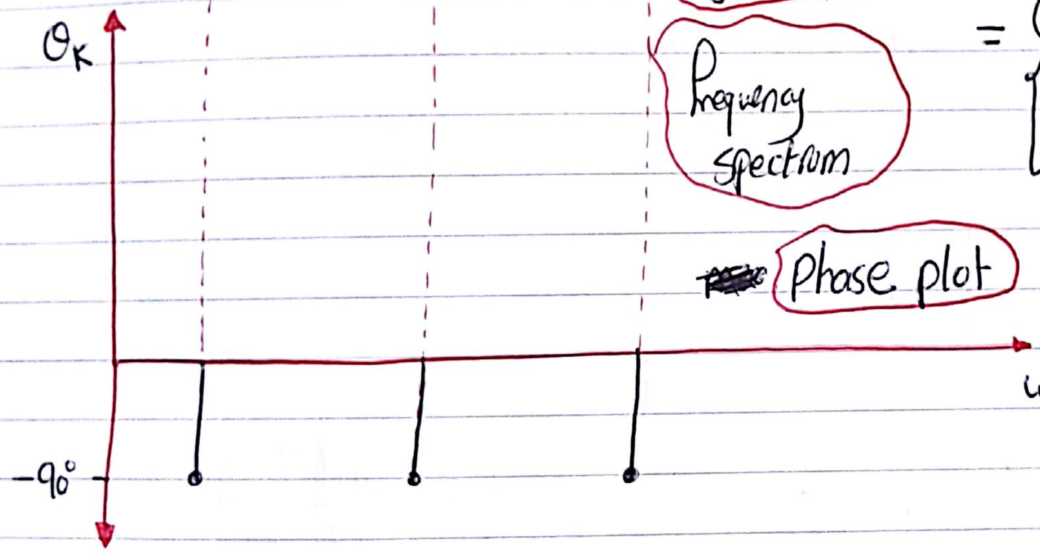
$$C_k = \frac{jV}{2\pi k}\left(e^{-jk\pi} - e^{-j0} - e^{-jk2\pi} + e^{-jk\pi}\right)$$

$$= \begin{cases} \frac{-2jV}{k\pi} \Rightarrow \frac{2V}{k\pi}\angle -90^\circ & k\,odd \\ 0, & k\,even \end{cases}$$

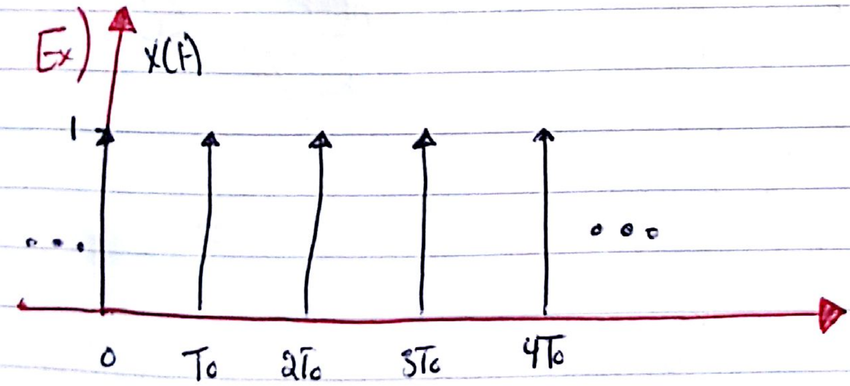
magnitude plot

Frequency spectrum

[redacted] Phase plot



Ex)



* Find the Fourier Series

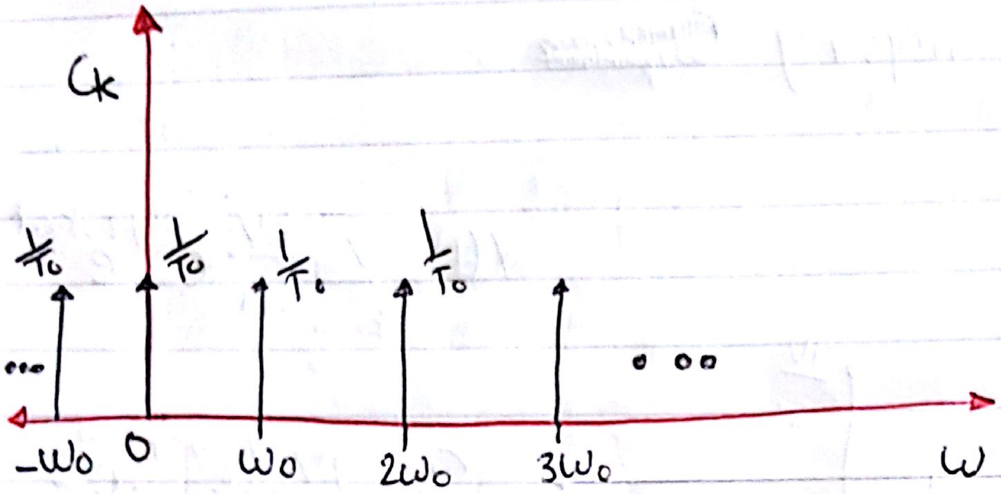
To find C_k's for this functions:-

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

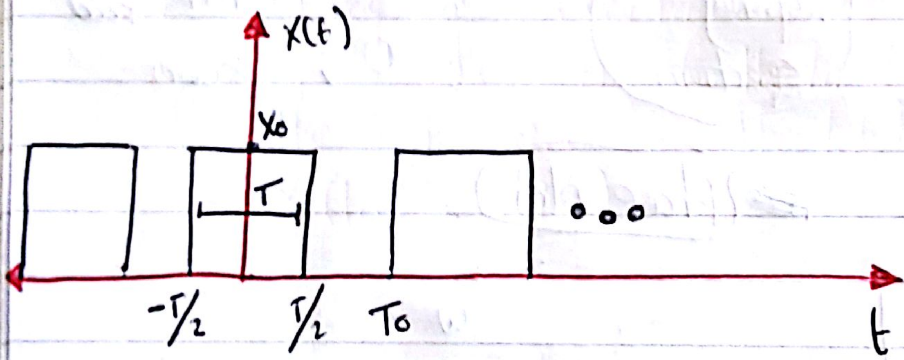
$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0} e^{-jk\omega_0 t}\Big|_{t=0} = \frac{1}{T_0}$$

Sifting property $= e^{-jk\omega_0 t}\Big|_{t=0}$

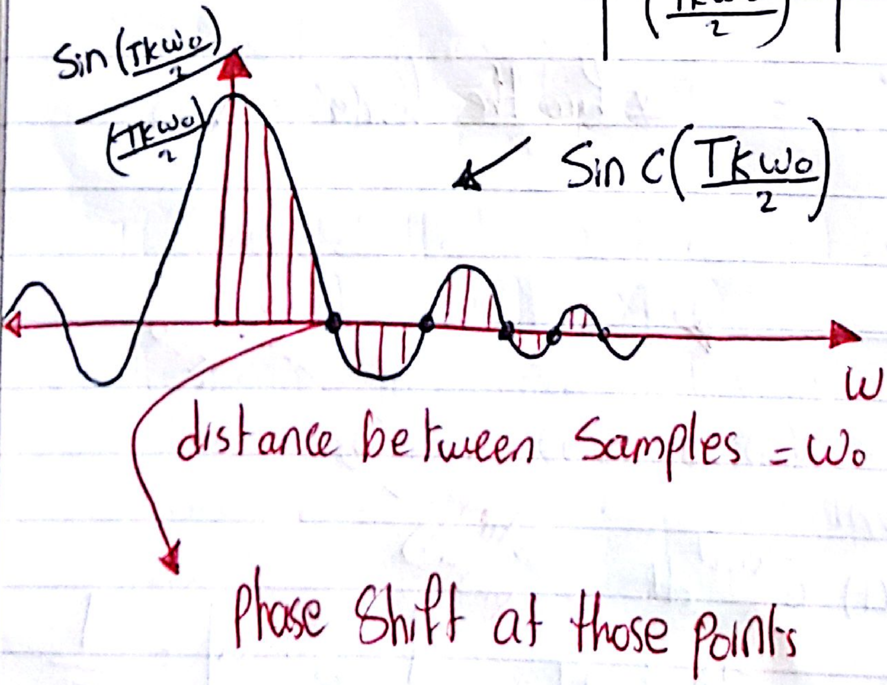
or $\boxed{x(t) = \frac{1}{T_0} + \sum_{k=1}^{\infty} \frac{2}{T_0}\cos k\omega_0 t}$ $\boxed{X(t) = \sum_{-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}}$



* Finding the Fourier Series for a rectangular wave :-



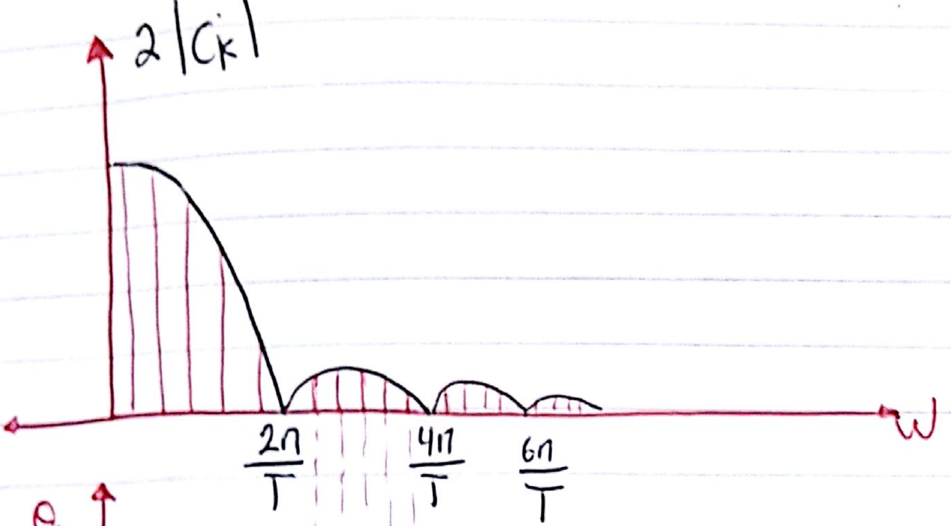
$$2|C_k| = \frac{2T X_0}{T_0}\left|\frac{\sin\left(\frac{Tk\omega_0}{2}\right)}{\left(\frac{Tk\omega_0}{2}\right)}\right| , \quad \theta_k = \begin{cases} 0, & \frac{\sin\left(\frac{Tk\omega_0}{2}\right)}{Tk\omega_0/2} > 0 \\ 180^\circ & \frac{\sin\left(\frac{Tk\omega_0}{2}\right)}{Tk\omega_0/2} < 0 \end{cases}$$

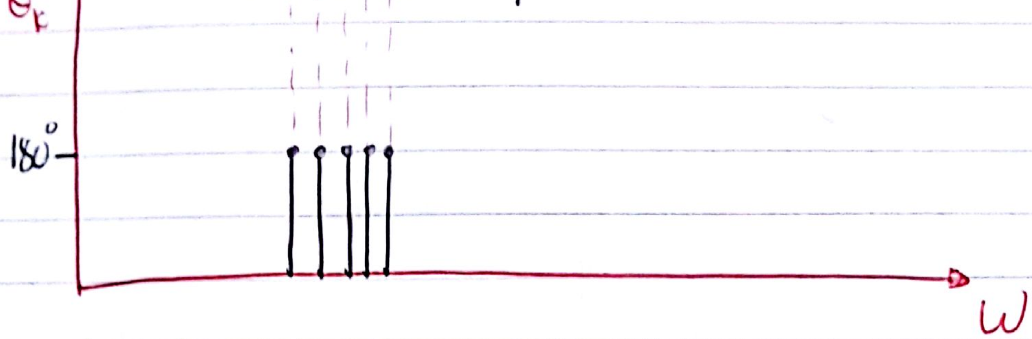


$|\text{Sinc}| \longrightarrow 180^\circ$

$$\theta_k = \begin{cases} 0, & \text{Sinc}\left(\frac{Tk\omega_0}{2}\right) > 0 \\ 180^\circ, & \text{Sinc}\left(\frac{Tk\omega_0}{2}\right) < 0 \end{cases}$$

even in the -ve x-axis





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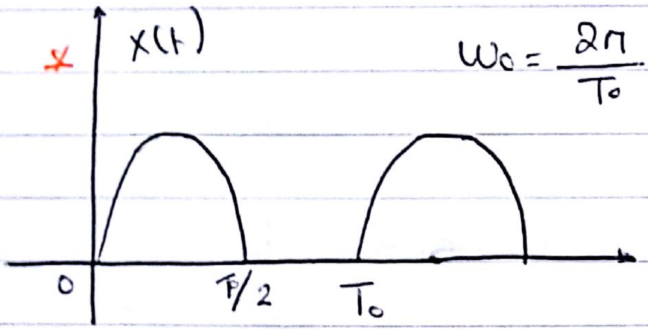


$\omega_0 = \frac{2\pi}{T_0}$

$$x(t) = \begin{cases} \sin \omega_0 t, & 0 < t < \frac{T_0}{2} \\ 0, & \frac{T_0}{2} < t < T_0 \end{cases}$$

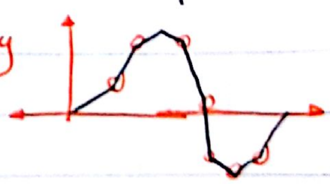
* Properties of the Fourier Series :-

(*) Any Single valued periodic function (one to one function) $x(t)$ that Satisfies the Dirichlet Conditions can be expanded into a Fourier series.

① $x(t)$ has at most a finite number of discontinuities in one period

② $x(t)$ has at most a finite number of maxima and minima in one period

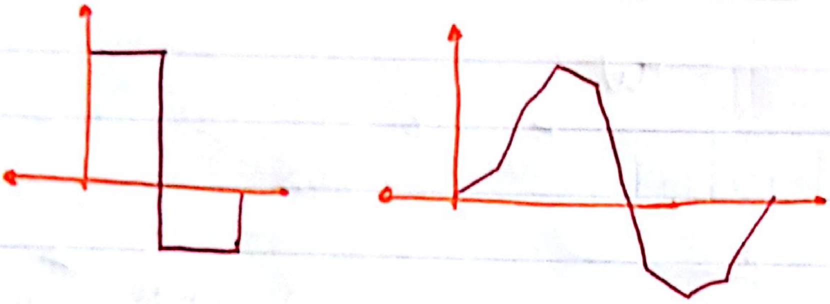
③ $x(t)$ is bounded. i.e: $\int_{T_0} |x(t)|\, dt < \infty$ → Finite energy



* function keeps on changing → more terms of Fourier series needed.

* Properties:-

① the Fourier Series Converges to the value of $x(t)$ at every point of continuity where $x(t)$ has a right-hand and left-hand derivatives whether these derivatives are the Same or not



② if $x(t)$ has a discontinuity at a point, the Fourier Series Converges to the mean of the limits approached by $x(t)$ from the right and from the left; that is $\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t_a}$ discont. point.

$$= \frac{x(t_a^-) + x(t_a^+)}{2}$$

③ almost any Continuous function $x(t)$ of period T_0 Can be uniformly approximated by a truncated (cropped) Fourier Series with any pre-assigned degree of accuracy where the Series is given by:- N → no. of terms.

$$X_N(t) = \sum_{k=-N}^{N} C_k e^{jk\omega_0 t} = C_0 + \sum_{k=1} 2|C_k| \cos(k\omega_0 t + \theta_k)$$

→ If we did that an error results;

$$e(t) = x(t) - X_N(t).$$

④ The mean Square error that is optimized by virtue of the Fourier Series Coefficients (C_k, A_k, B_k) is Computed/optimized as

$$\text{mean Square-error} = \frac{1}{T_0}\int_{T_0} e^2(t)\, dt.$$

→ Always gives the best coeff.

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odd / even

sin / cos

⑤ A sum of trigonometric functions of w_0t that is periodic is its own fourier series

Ex) $y(t) = 3\sin 2w_0t - 5\cos(3w_0t + \theta_1) + 7\sin(5w_0t - \theta_2) + 10$

↳ fourier series gives the same function + same number of terms.

⑥ The fourier coefficient of the k_{th} harmonic (kw_0) for $x(t)$ decreases in magnitude at least as fast as $\frac{1}{k}$ for sufficiently large k. ($\delta(t)$ is an exception). if $x(t)$ has one or more discontinuities in a period, the coefficients can decrease no faster than this. * at first point of discontinuity → fourier series approach zero.

↳ discontinuous functions → we need a large number of terms

↳ 〃 〃 → max speed $= 1/k$.

* if the n^{th} derivative of $x(t)$ is the first derivative that contains a discontinuity, and if all derivatives through the n_{th} (including the nth) satisfy the dirichlet conditions, then the fourier coefficients approach zero as $\frac{1}{k^{n+1}}$, for sufficiently large k.

⑦ The fourier series of a periodic sum of periodic functions is equal to the sum of the fourier series of the functions.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

↳ input → fourier series

↳ output → fourier series.

↳ $H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$

$X_1 e^{s_1t} \longrightarrow X H(s_1) e^{s_1t}$ $\qquad x(t) \longrightarrow y(t)$

↳ $X = |X| e^{j\phi}$; x can have a phase.

$|X| \cos(w_1t + \phi) \longrightarrow |X|\,|H(jw_1)| \cos(w_1t + \phi + \angle H(jw_1))$

$x(t) = \sum C_{kx} e^{jkw_0t} \longrightarrow y_{ss}(t) = \sum_{k=-\infty}^{\infty} H(jkw_0)\, C_{kx}\, e^{jkw_0t}$

↳ $= \sum^{\infty} C_{ky}\, e^{jkw_0t}$

$$+\; x(t) = C_{0x} + \sum_{k=1}^{\infty} 2|C_{kx}| \cos(k\omega_0 t + \theta_{kx})$$

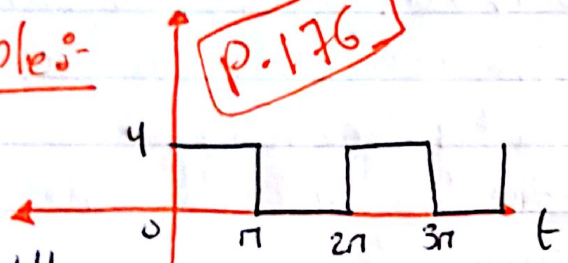
$$|X|\cos(\omega_1 t + \phi) \longrightarrow |X|\,|H(j\omega_1)| \cos(\omega_1 t + \phi + \angle H(j\omega_1))$$

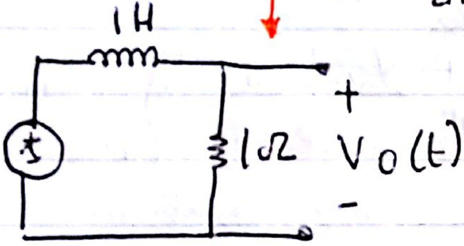
$$y_{ss}(t) = C_{0y} + \sum_{k=1}^{\infty} 2|C_{ky}| \cos(k\omega_0 t + \theta_{ky})$$

$$\Downarrow$$

$$C_{ky} = |C_{ky}|\angle\theta_{ky} = H(jk\omega_0)\, C_{kx}$$

Example:- P.176





$$x(t) = C_{0x} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_{kx} e^{jk\omega_0 t}$$

from table

$$= 2 + \sum \frac{4}{k\pi} e^{-j\pi/2} e^{jkt}$$

$$h(t) = e^{-t}u(t)$$

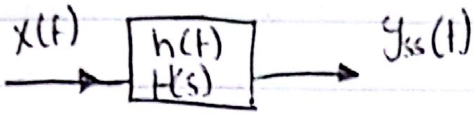
$$H(s) = \frac{1}{s+1}$$

OH MY EAGLE EYES

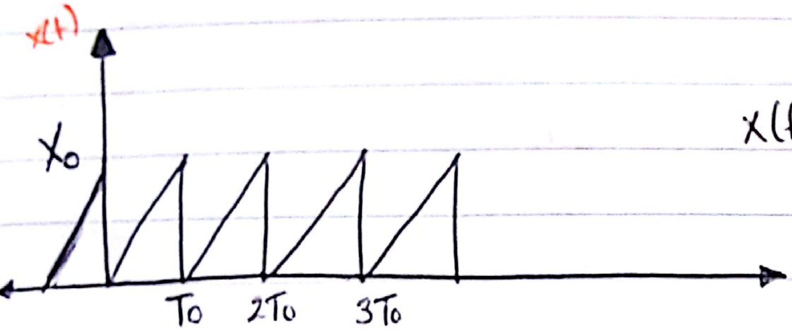
$\boxed{\text{Ex. } 4.28}$

* Fourier Series transformations:-

* Amplitude transformation

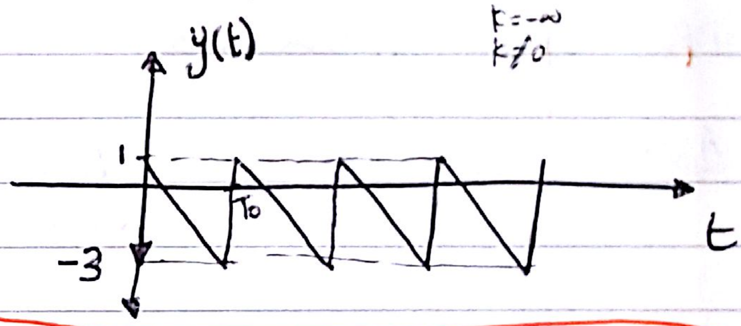


* time transformation



$$x(t) = \frac{X_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{X_0}{2\pi k} e^{j\pi/2} e^{jk\omega_0 t}$$

$$x(t) = C_{0x} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_{kx} e^{jk\omega_0 t}$$



* Amplitude transformation:-

$$y(t) = Ax(t) + B$$

$$= \frac{-4}{X_0} x(t) + 1$$

turn ~~to make~~ the amplitude into unity.

$$C_{0y} = A C_{0x} + B = \left(\frac{-4}{X_0}\right)\left(\frac{X_0}{2}\right) + 1$$

$$C_{ky} = A_{kx} = \left(\frac{-4}{X_0}\right) \frac{X_0}{2\pi k} e^{j\pi/2}$$

$$= \frac{2}{\pi k} e^{-j\pi/2}, \quad k \neq 0$$

$$y(t) = C_{0y} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_{ky} e^{jk\omega_0 t}$$

* Time transformation :-

$$\tau = at + b$$

$$x(t) \longrightarrow y(t) = x(at+b)$$

* Summary of Chapter 4:-

* Periodic functions → what warrants the periodicity of a Sum of periodic components:-

$x(t) = x(t \pm nT)$, n - integers ; T - period.

Calculated Fourier series Coefficients.

All of these minimize the cost function (error).

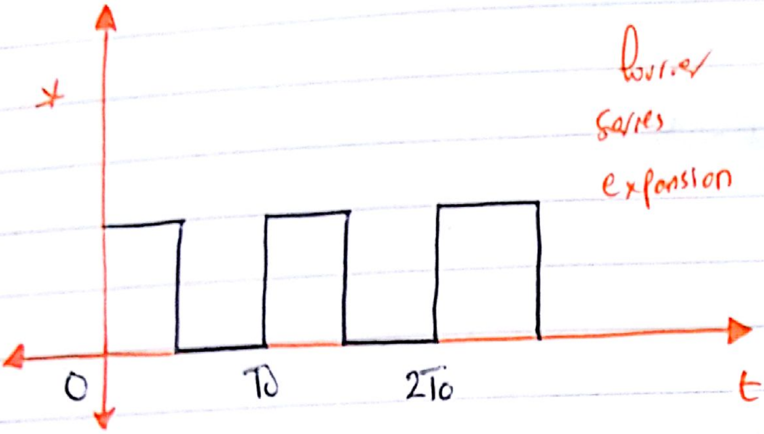
* properties of Fourier Coefficients 4 the Series.

* input - System interactions.

* Amplitude transformation 4 time transformation.

END OF CHAPTER 4

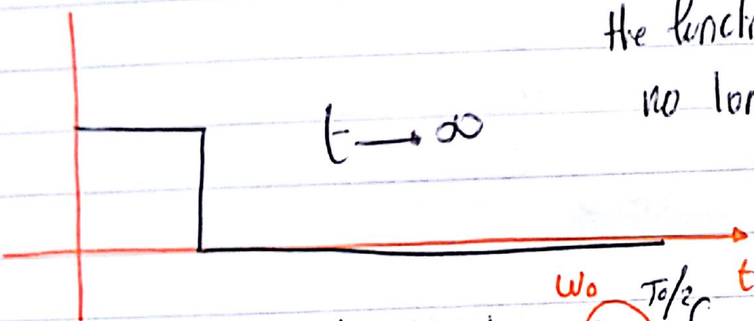
Chapter 5



Fourier series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \, e^{jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{jk\omega_0 t}\, dt$$



$t \to \infty$

the functions no longer periodic

as $T_0 \to \infty$

$\Delta\omega \to 0$

$$C_{k(\infty)} = \lim_{T_0 \to \infty} \frac{1}{2\pi} \overbrace{\frac{2\pi}{T_0}}^{\omega_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-(jk2\pi/T_0)t}\, dt$$

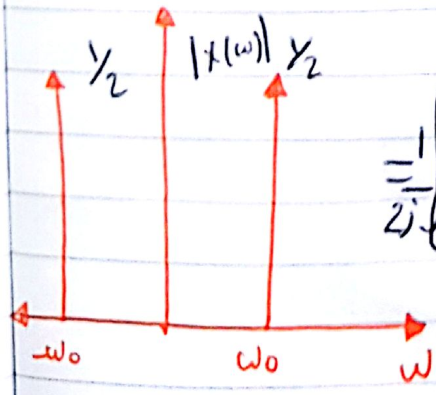
$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right] d\omega$$

$$\mathcal{F}\{x(t)\} \triangleq X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier transform operator.

* Consider $x(t) = \sin(\omega_0 t) = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty}\left(\frac{e^{j\omega_0 t}}{2j}\, e^{-j\omega t}\, dt\right) - \int_{-\infty}^{\infty} \frac{e^{-j\omega_0 t}\, e^{-j\omega t}}{2j}$$



$$= \frac{1}{2j}\left[\int e^{-j(\omega-\omega_0)t}\, dt - \int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t}\, dt\right]$$

$$\int_{-\infty}^{\infty} g(x,y)\,dy = f(x) \to \int 4\,dy$$

↳ we'll get rid of y

+ $x(t) = \sin \omega_0 t$

$$\langle x(t) \rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \sin \omega_0 t \, dt = 0$$

+ $x(t) = \sin^2 \omega_0 t$

$$\langle x(t) \rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \sin^2 \omega_0 t \, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{2}\left(1 - \cos 2\omega_0 t\right) dt$$

$$= \frac{1}{2T_0} \int 1 \, dt - \frac{1}{2T_0} \int \cancel{\cos 2\omega_0 t}^{\,0} \, dt$$

$$= \frac{1}{2T_0}\, t \Big|_{-T_0/2}^{T_0/2} = \boxed{\frac{1}{2}}$$

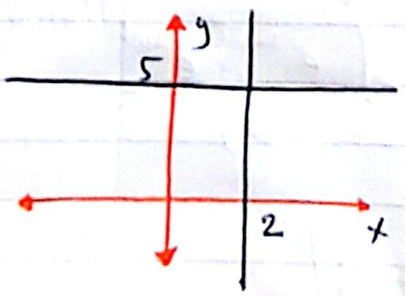
only half the power → the other half is lost in the -ve frequency.

+ $$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} X(\omega)\, d\omega \, e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega)\, e^{jk\omega_0 t} \, d\omega$$

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} \, d\omega} = \text{Fourier transfer inverse}$$

$$\mathcal{F}^{-1}\{X(\omega)\}$$



$y = 5 \;\; \forall x$

$x = 2 \;\; \forall y$

$$\boxed{\begin{aligned} * \; x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \\ * \; X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \end{aligned}}$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

* recall that for a function $f(t)$, the fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

and Conversly;

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Ex) $x(t) = \sin \omega_0 t$

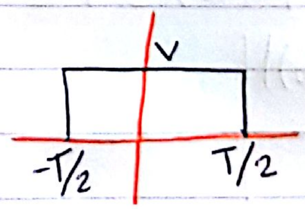
$$F(\omega) = \int_{-\infty}^{\infty} \sin\omega_0 t \; e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-j\omega t}\, dt$$

$$= \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt = \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j(\omega\pm\omega_0)t} = \delta(\omega\pm\omega_0)$$

* $f(t) = V \,\mathrm{rect}\left(\frac{t}{T}\right)$



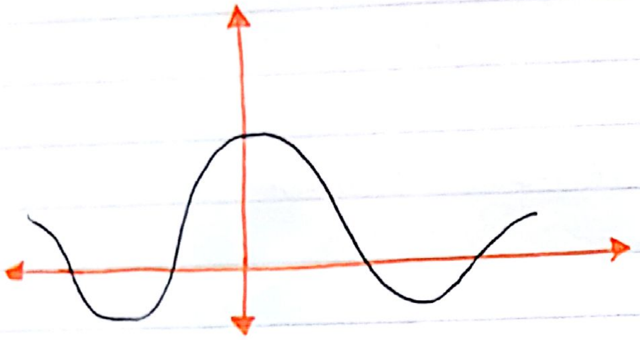
$$= V u_s\left(t+\frac{T}{2}\right) - V u_s\left(t - T/2\right)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} V e^{-j\omega t}\, dt$$

$$= \frac{-V}{j\omega}\, e^{-j\omega t}\Big|_{-T/2}^{T/2} = -\frac{V}{j\omega}\, e^{j\omega t}\Big|_{-T/2}^{T/2} = \frac{V}{j\omega}\left[-e^{-j\omega T/2} + e^{j\omega T/2}\right]$$

$$= \frac{2V}{\omega}\left(\frac{e^{j\omega T/2} - e^{j\omega T/2}}{2j}\right)$$

$$= \frac{2V}{\omega}\sin\frac{\omega T}{2} = VT\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} = VT\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)$$



$$\omega = \frac{n\pi}{T}$$

$$\sin\left(\frac{n\pi}{T}\right) = 0$$

* The dirichlet Conditions that apply to the fourier transformare:

1- on any finite interval:-

a- $f(t)$ is bounded

b- $f(t)$ has a finite no. of maxima and minima

c- $f(t)$ has a finite no. of dis Continuities

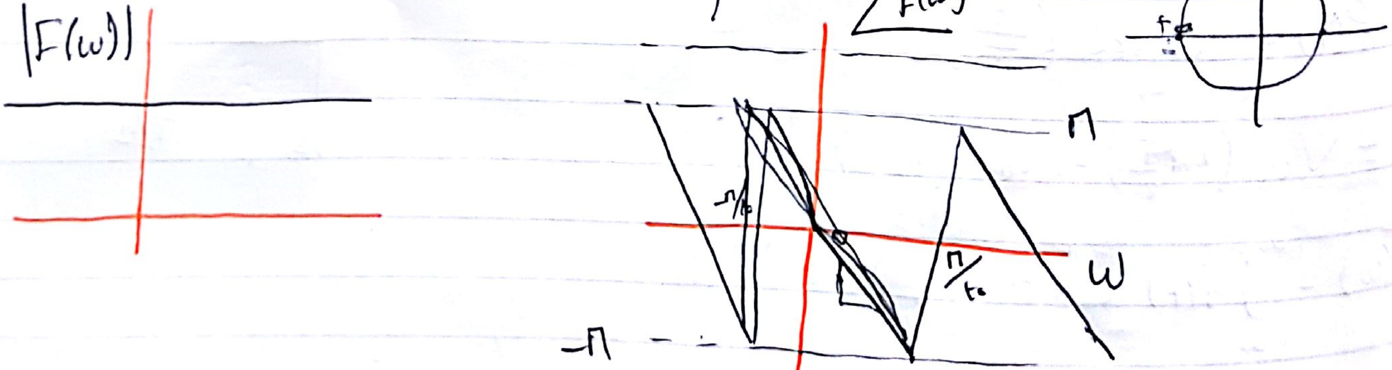
2- $f(t)$ is absolutely integrable :-

$$\int_{-\infty}^{\infty} f(t)\,dt < \infty$$

Ex) $f(t) = A\delta(t-t_0)$

$$F(\omega) = \int_{-\infty}^{\infty} A\,\delta(t-t_0)\,e^{-j\omega t}\,dt = A e^{-j\omega t_0}$$ sifting property

$$F(\omega) = Ae^{-j\omega t_0} = A\angle -t_0\omega$$



$$\frac{2\pi}{T_0} = \omega_0 \qquad T$$

$$T_0 = \frac{2\pi}{\omega_0}$$

$$F(\omega) \triangleq \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt.$$

* Properties of the fourier transform :-

① The fourier trans form is linear :-

$$f(t) = B\cos\omega t = \frac{B}{2} e^{j\omega_0 t} + \frac{B}{2} e^{-j\omega_0 t}$$

$$\mathcal{F}\left\{ \frac{B}{2} e^{j\omega_0 t} + \frac{B}{2} e^{-j\omega_0 t} \right\} = \int_{-\infty}^{\infty} \left(\frac{B}{2} e^{j\omega_0 t} + \frac{B}{2} e^{-j\omega_0 t} \right) e^{-j\omega_0 t}\, dt.$$

$$= \int_{-\infty}^{\infty} \frac{B}{2} e^{j\omega_0 t} e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \frac{B}{2} e^{-j\omega_0 t} e^{-j\omega t}\, dt$$

$$= \mathcal{F}\left\{ \frac{B}{2} e^{j\omega_0 t} \right\} + \mathcal{F}\left\{ \frac{B}{2} e^{-j\omega_0 t} \right\}$$

② Time Scaling :-

$$f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$$

Then for some Constant a,

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\, e^{j\omega t}\, dt$$

Let $\xi = at \longrightarrow t = \frac{\xi}{a}$, $dt = \frac{1}{a}\, d\xi$

$$\int_{-\infty}^{\infty} f(\xi)\, e^{-j\omega \frac{\xi}{a}}\, \frac{1}{a}\, d\xi$$

$$f(at) \xrightarrow{\mathcal{F}} \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

③ Time shift property

$$f(t) \xrightarrow{\mathcal{F}} F(\omega)$$

$$\int_{-\infty}^{\infty} f(t-t_a)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} f(\xi)\, e^{-j\omega(\xi + t_a)}\, d\xi$$

$\xi = t - t_a$

$\boxed{t = \xi + t_a}$

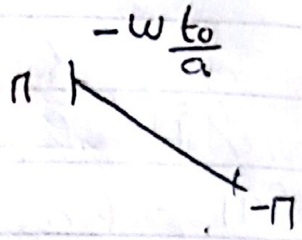
$dt = d\xi$

$$= \underbrace{\int_{-\infty}^{\infty} f(\xi)\, e^{-j\omega\xi}\, d\xi}_{F(\omega)}\; e^{-j\omega t_a}$$

$$F(\omega)\, e^{-j\omega t_a} \quad \text{phase.}$$

$$* \; F(\omega-\omega_1) \xleftrightarrow{\mathcal{F}^{-1}} e^{-j\omega_1 t} f(t)$$

$$* \; f(t-t_0) \longleftrightarrow F(\omega)\, e^{-j\omega t_0}$$

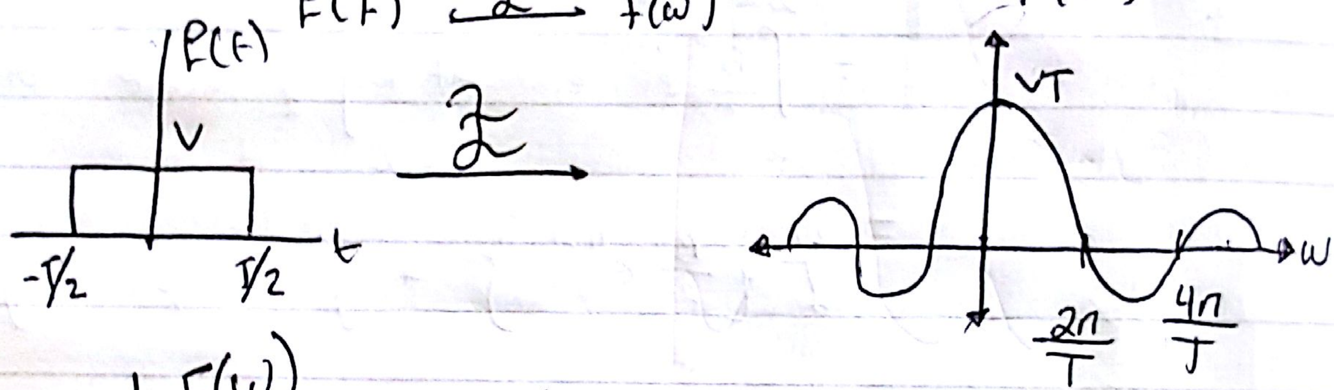


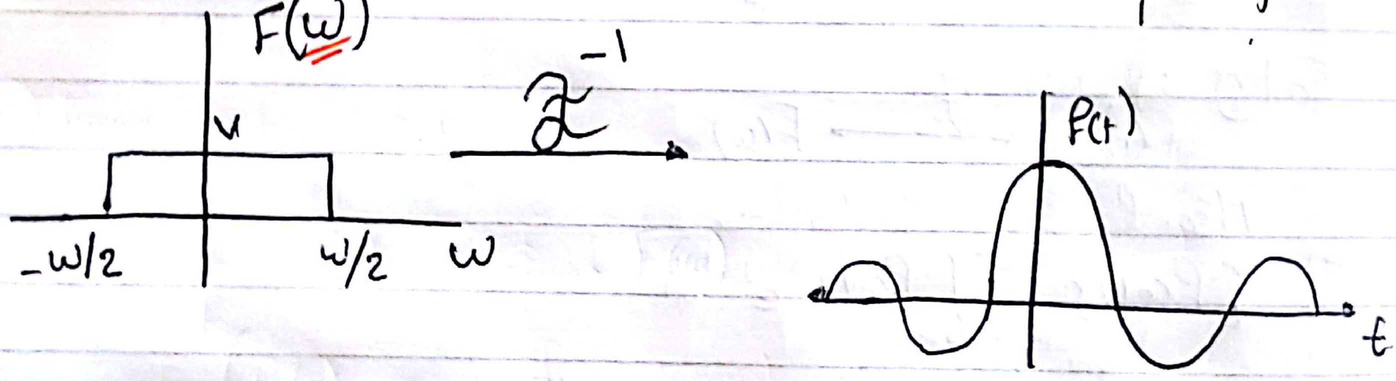
$$* \text{Ex)} \; f(at-t_0) \longleftrightarrow \frac{1}{a} F\left(\frac{\omega}{a}\right) e^{-jt_0\left(\frac{\omega}{a}\right)}$$

* properties of the fourier transform .. continued.

$$f(t) \xrightarrow{\mathcal{F}} F(\omega)$$

4





5 Convolution Property :-

$$f_1(t) \xleftrightarrow{\mathcal{F}} F_1(\omega)$$

$$f_2(t) \xleftrightarrow{\mathcal{F}} F_2(\omega)$$

$$\underbrace{f_1(t) * f_2(t)}_{\text{Convolution}} = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

$$\mathcal{F}\{f_1(t) * f_2(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau\right] e^{-j\omega t}\, dt.$$

$$\xi = t-\tau$$

* Conversly ;

$$f_1(t) \xleftrightarrow{\mathcal{F}} F_1(\omega)$$
$$f_2(t) \xleftrightarrow{\mathcal{F}} F_2(\omega)$$

* Quiz material.

$$F_1(\omega) * F_2(\omega) = \int_{-\infty}^{\infty} F_1(\tau)\, F_2(\omega - \tau)\, d\tau$$

$$\mathcal{F}^{-1}\{F_1(\omega) * F_2(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F_1(\tau) F_2(\omega - \tau)\, d\tau\right] e^{j\omega t}\, d\omega$$

6 Time differentiation

$$f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$$

not by parts

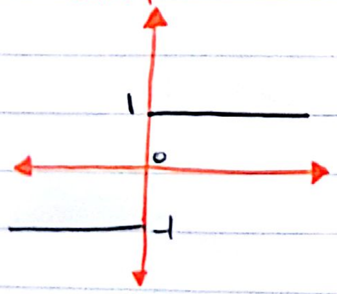
$$\frac{df(t)}{dt} \xleftrightarrow{\mathcal{F}} \boxed{j\omega F(\omega)} \quad ; \quad \frac{d^n f(t)}{dt} \xleftrightarrow{\mathcal{F}} (j\omega)^n F(\omega).$$

Proof .

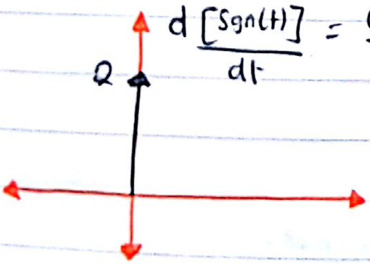
$$\int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{-j\omega t}\, dt = $$

Example :- $f(t) = Sgn(t)$

$$Sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$



$$\int_{-\infty}^{\infty} Sgn(t) e^{-j\omega t}\, dt$$

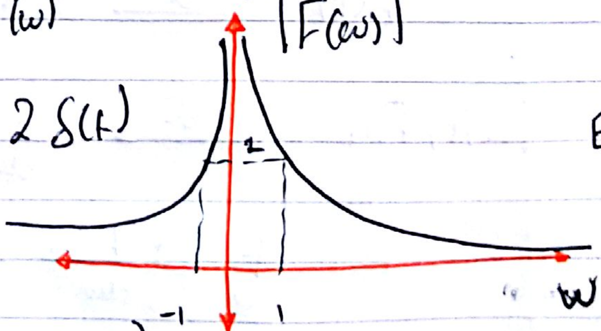


$$\frac{d[Sgn(t)]}{dt} = \frac{df(t)}{dt} = 2\delta(t)(\omega)$$

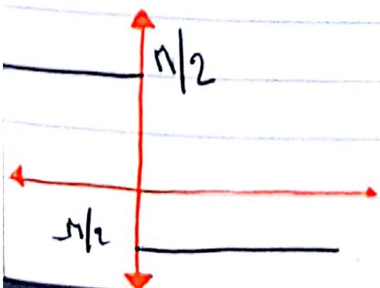
$$\frac{df(t)}{dt} = 2\delta(t)$$

$$j\omega F(\omega) = 2$$

$$F(\omega) = \frac{2}{j\omega} + k\,\delta(\omega)$$



Ex) $\boxed{5.11}$



$$Sgn(t) \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega}$$

* Conversly ;

$$f_1(t) \xleftrightarrow{\mathcal{F}} F_1(\omega)$$
$$f_2(t) \xleftrightarrow{\mathcal{F}} F_2(\omega)$$

* Quiz material.

$$F_1(\omega) * F_2(\omega) = \int_{-\infty}^{\infty} F_1(\tau)\, F_2(\omega - \tau)\, d\tau$$

$$\mathcal{F}^{-1}\{F_1(\omega) * F_2(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F_1(\tau) F_2(\omega-\tau)\, d\tau\right] e^{j\omega t}\, d\omega$$

(6) Time differentiation

$$f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$$

not by parts

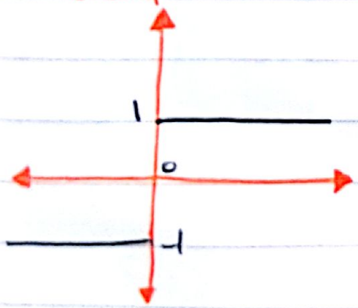
$$\frac{df(t)}{dt} \xleftrightarrow{\mathcal{F}} \boxed{j\omega F(\omega)} \quad ; \quad \frac{d^n f(t)}{dt^n} \xleftrightarrow{\mathcal{F}} (j\omega)^n F(\omega).$$

Proof .

$$\int_{-\infty}^{\infty} \frac{df(t)}{dt}\, e^{-j\omega t}\, dt =$$

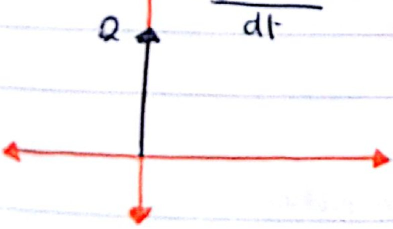
Example :- $f(t) = \text{Sgn}(t)$

$$\text{Sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

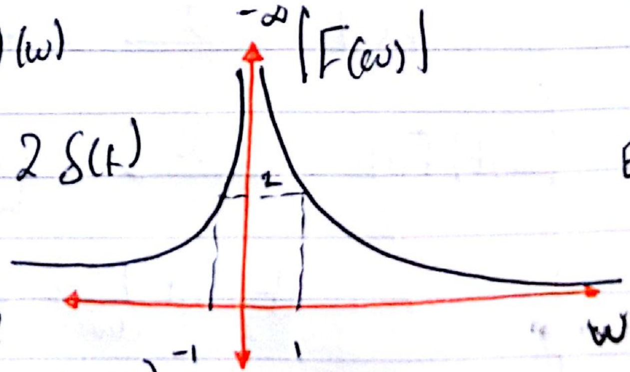


$$\int_{-\infty}^{\infty} \text{Sgn}(t)\, e^{-j\omega t}\, dt$$

$$\frac{d\,[\text{Sgn}(t)]}{dt} = \frac{df(t)}{dt} = 2\delta(t)(\omega)$$

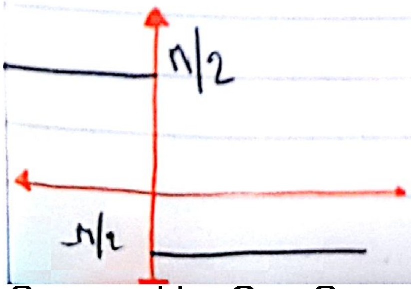


$$\frac{df(t)}{dt} = 2\delta(t)$$
$$\downarrow$$
$$j\omega F(\omega) = 2$$
$$F(\omega) = \frac{2}{j\omega} + k\,\delta(\omega)$$

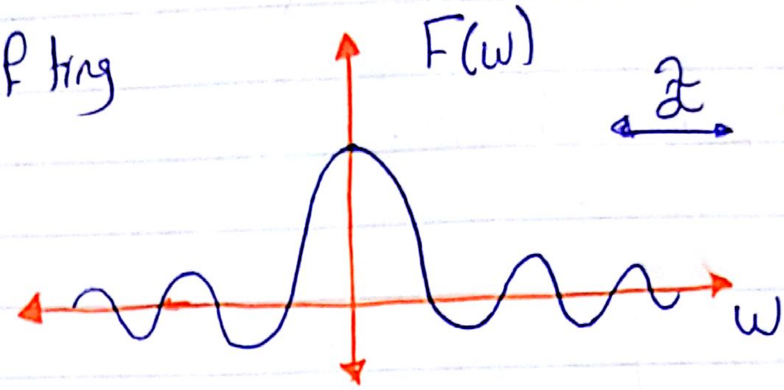


Ex) $\boxed{5.11}$

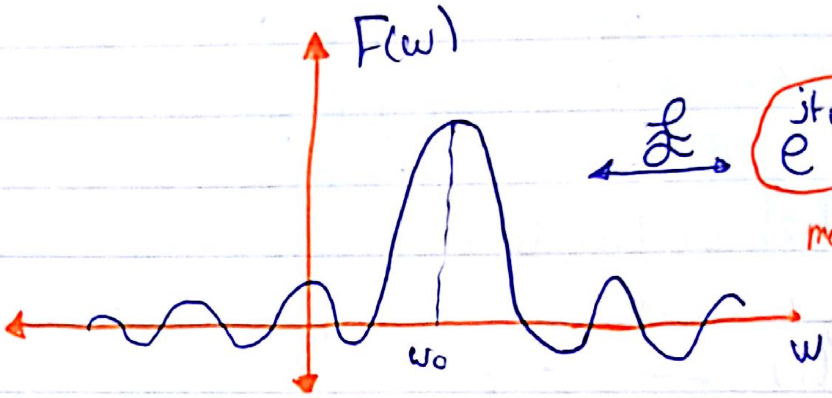
$$\text{Sgn}(t) \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega}$$



(7) Frequency Shifting



$\xleftrightarrow{\mathcal{F}} f(t)$



$\xleftrightarrow{\mathcal{F}} e^{jtw_0} f(t)$

modulation

(8) Time integration

$\boxed{\text{Ex } 5.12}$

$$f(t) \xleftrightarrow{\mathcal{F}} F(w)$$

$$\int_{-\infty}^{t} f(\tau)\, d\tau \xleftrightarrow{\mathcal{F}} \frac{F(w)}{jw} + \pi F(0)\, \delta(w)$$

Initial Condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{t} f(\tau)\, d\tau\, e^{-jwt}\, dt$$

prove this

(9) Frequency differentiation

$$f(t) \xleftrightarrow{\mathcal{F}} F(w)$$

$$(-jt)\, f(t) \xleftrightarrow{\mathcal{F}} \frac{d\, F(w)}{dw}$$

Prove.

$$4 \quad \frac{d^n F(w)}{dw} \xleftrightarrow{\mathcal{F}} (-jt)^n f(t).$$

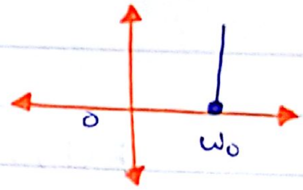
Ex) Applications :-

* DC level :-

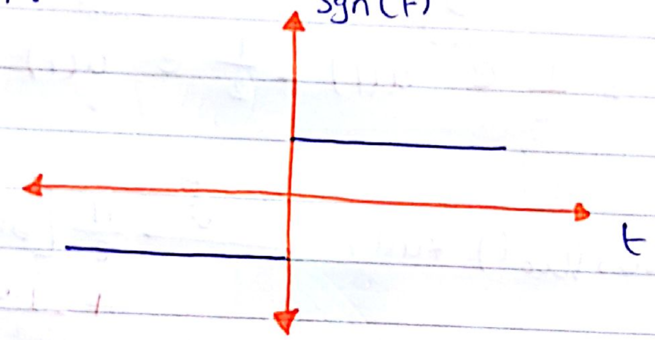
from cos subst : $\frac{e^{j\omega_0 t}+e^{-j\omega t}}{2}$

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\omega-\omega_0)$$

$$\omega_0 = 0 \quad e^{0} = 1 \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\omega)$$



* Unit step function :-



$$\text{Sgn}(t) = \begin{cases} 1 & t>0 \\ 0 & t=0 \\ -1 & t<0 \end{cases}$$

$$\frac{1+\text{Sgn}(t)}{\cancel{2}} = \frac{\cancel{2}u_s(t)}{\cancel{2}}$$

$$u_s(t) = \frac{1}{2}\left[1+\text{Sgn}(t)\right]$$

normally:- $$\int_0^{\infty} 1 \; e^{-j\omega t}\,dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^{\infty} = 0 - \left(-\frac{1}{j\omega}\right) = \boxed{\frac{1}{j\omega}}$$

Using the Sgn function :- $$\text{Sgn} \longleftrightarrow \frac{2}{j\omega}$$

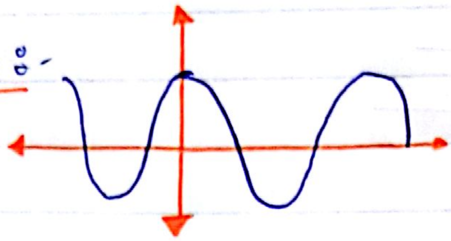
$$u(t) = \frac{1}{2}\left[1+\text{Sgn}(t)\right] \xleftrightarrow{\mathcal{F}} \boxed{\pi\,\delta(\omega) + \frac{1}{j\omega}}$$

a more general term that takes the DC initial Conditions into Consideration.

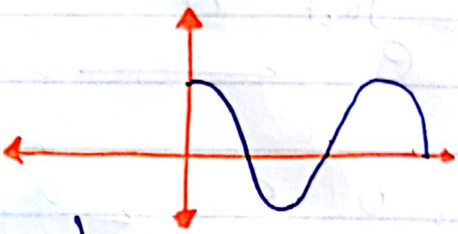
Convolution in time ⟶ multiplication in frequency

*** Switched Cosine:-**

$f(t) = \cos(\omega_0 t)$



$$f(t) = \cos(\omega_0 t)\, u_s(t)$$



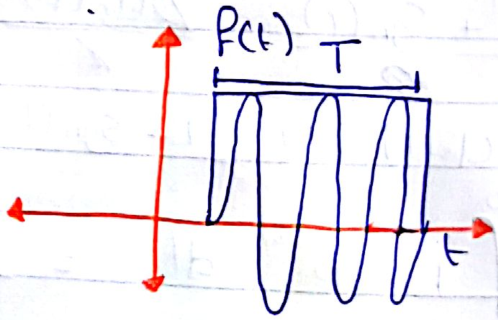
$$f(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\, u_s(t)$$

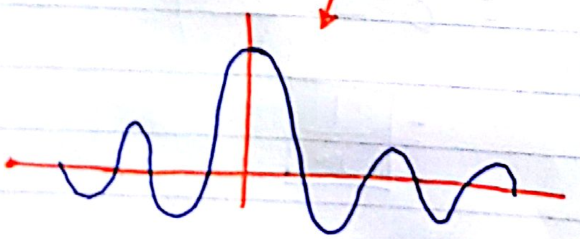
$$= \frac{1}{2} e^{j\omega_0 t} u(t) + \frac{1}{2} e^{-j\omega_0 t} u(t)$$ from $0 \to \infty$

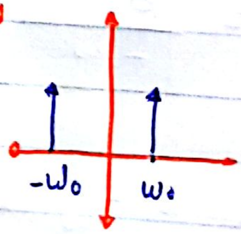
$$\cos(\omega_0 t) * u(t) \xleftrightarrow{\mathcal{F}} \frac{\pi}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] + \frac{j\omega}{\omega_0^2 - \omega^2}$$

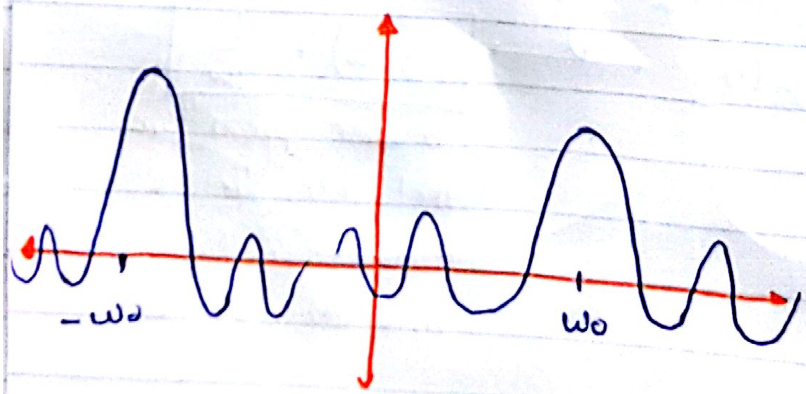
*** Pulsed Cosine:-**

$$f(t) = \text{rect}\left(\frac{t}{T}\right) \cos(\omega_0 t)$$









* two Sinc functions, one at ω_0, one at $-\omega_0$.

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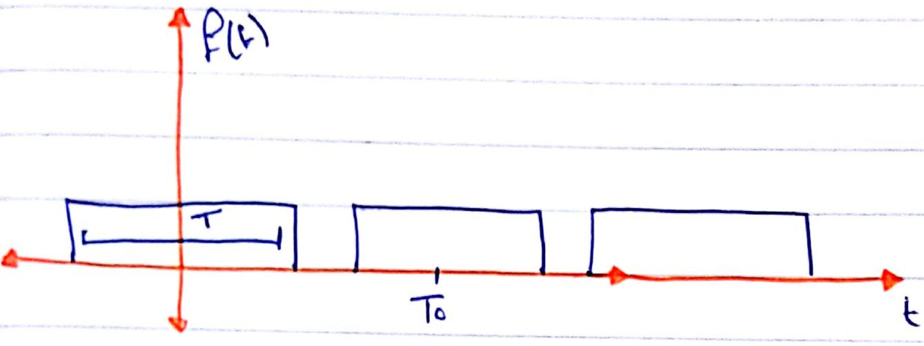
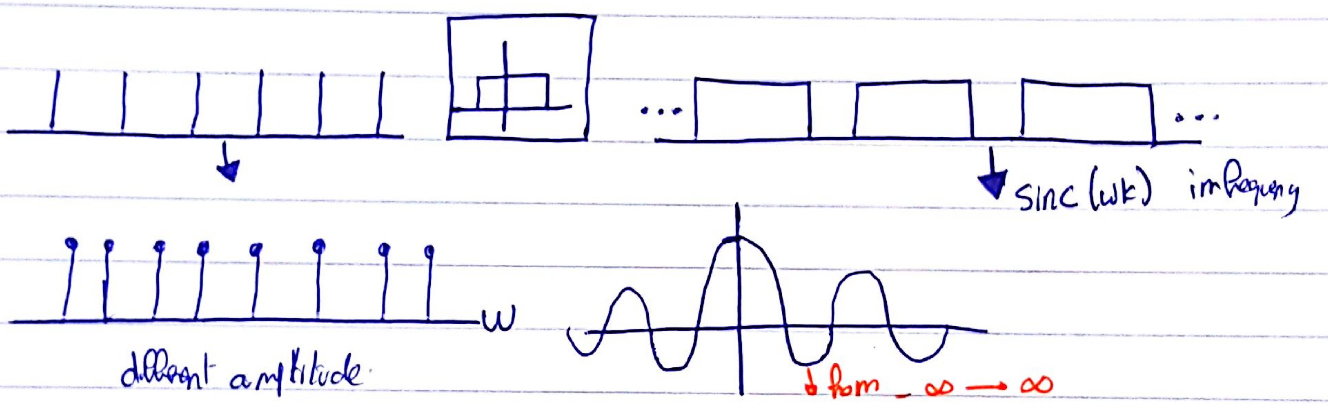
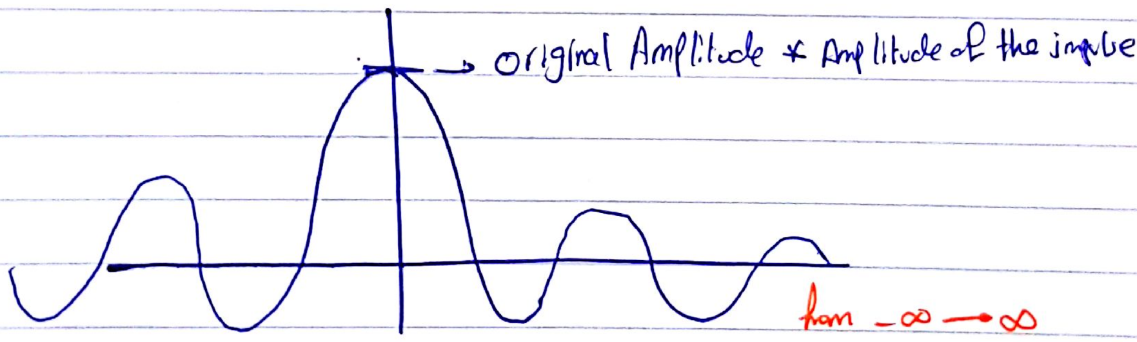
* $f(t) = e^{-at} u(t)$ → Homework.







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