Probability and Random Variables EE321 | The University of Jordan
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Section 8 – 9

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Question 1 (5 pts)

Given the function: $G_{XY}(x,y) = u(x)\,u(y)\,(1-\exp(-(x+y)))$.

Is this a density or distribution function? distribution

Is this a valid probability function? Yes

$1-e^{-(x+y)} = 1-(e^{-x}e^{-y})$

Why? because it satisfy the conditions (1) & (2) described below and $P(x_a<X\le x_b,\ y_a<Y\le y_b)$ is between 0 & 1 from figure, and it's not density because area under it is not (1)

$G_{XY}(x,y) = 1-e^{-(x+y)}$ for $x>0$ & $y>0$

$\therefore 0 \le G_{XY}(x,y) \le 1$ because $1-e^{-(x+y)} \to 1-e^{0} = 1-1 = 0$ min, $\to 1-e^{-\infty} = 1$ max

condition (1)

$G_{XY}(\infty,y) = 1-e^{-\infty} = 1$
$G_{XY}(x,\infty) = 1$
$G_{XY}(\infty,\infty) = 1$
condition (2)

$P(x_a<X\le x_b,\ y_a<Y\le y_b) = F_{X,Y}(x_b,y_b) + F_{X,Y}(x_a,y_a) - F_{X,Y}(x_b,y_a) - F_{X,Y}(x_a,y_b)$

Question 2 (5 pts)

Find the constant b (in terms of a) for the function: $f_{XY}(x,y)=b\exp(-(x+y))$ for $0<x<a$ and $0<y<\inf$, and zero elsewhere, so that it is a valid density function

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$\int_0^\infty\int_0^a b\,e^{-(x+y)}\,dx\,dy = 1 \to -b\,e^{-(x+y)}\Big|_0^a = -b\left(e^{-(a+y)} - e^{-y}\right)$

$\to \int_0^\infty -b\left(e^{-(a+y)} - e^{-y}\right)dy = -b\left[-e^{-(a+y)} + e^{-y}\right]_0^\infty$

$= -b\left[0+0-\left(-e^{-a}+1\right)\right]$

$= b(1-e^{-a})$, now $b(1-e^{-a}) = 1$

$b = \dfrac{1}{1-e^{-a}}$

b = $\dfrac{1}{1-e^{-a}}$

Find the distribution function

$$F_{X,Y}(x,y) = \begin{cases} 0 & x<0 \text{ or } y<0 \\ b e^{-(x+y)} - b e^{-y} - b e^{-x} + b & 0<x<a \text{ and } 0<y<\infty \\ b e^{-(a+y)} - b e^{-y} - b e^{-a} + b & a<x \text{ and } 0<y<\infty \end{cases}$$

$\int_0^y\int_0^x b\,e^{-(x+y)}\,dx\,dy$

$-b\,e^{-(x+y)}\Big|_0^x = -b e^{-(x+y)} + b e^{-y}$

$\int_0^y -b e^{-(x+y)} + b e^{-y}\,dy$

$= \left(+b e^{-(x+y)} - b e^{-y}\right)\Big|_0^y$

$= b e^{-(x+y)} - b e^{-y} - (b e^{-x} - b)$

$F_{XY}(x,y)$

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Question 3, (5 pts)

Find the marginal density functions of the joint function:

$f_{XY}(x,y) = u(x)\, u(y)\, x \exp\{-x(1+y)\}$

$$f_X(x) = \int_{-\infty}^{\infty} u(x)u(y)\, x\, e^{-x(1+y)}\, dy = \int_0^{\infty} u(x)\, x\, e^{-x(1+y)}\, dy = x\,u(x) \left. \frac{e^{-x(1+y)}}{-x} \right|_0^{\infty}$$

$$= -u(x)\left[0 - e^{-x}\right] = u(x)\, e^{-x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} u(x)u(y)\, x\, e^{-x(1+y)}\, dx = \int_0^{\infty} u(y)\, x\, e^{-x(1+y)}\, dx = \frac{u(y)}{(1+y)^2}$$

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$f_X(x) =$ $u(x)\, e^{-x}$

$f_Y(y) =$ $\dfrac{u(y)}{(1+y)^2}$

Find $P(Y<=2|X=1)$ $= F_Y(2|X=1) = \int^{2} f_Y(y|x=1) = \dfrac{\int_{-\infty}^{y} f_{X,Y}(1,y)\, dy}{f_X(1)}$

$$\int_{-\infty}^{y} u(y)\, e^{-(1+y)}\, dy = \int_0^{y} e^{-(1+y)}\, dy = \left. -e^{-(1+y)} \right|_0^{y} = -e^{-(1+y)} + e^{-1}$$

$f_X(1) = u(1)\, e^{-1} = e^{-1}$ now $P(Y \leq 2|X=1) = \dfrac{e^{-1} - e^{-(1+y)}}{e^{-1}}$

Question 4 (5 pts)

Given $f_{XY}(x,y)$ is a valid joint density function. Find the area under the density function $f_Y(y|x)$

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$\text{Area} = \int_{-\infty}^{\infty} f_Y(y|x)\, dy = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy}{f_X(x)} = \frac{f_X(x)}{f_X(x)} = 1$$

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Area = 1

$1200 - 900 = 300$

Question 5 (6 pts)

A pointer is spun on a fair wheel of chance numbered from 0 to 60 around its circumference

What is the average value of the pointer position

$f_X(x) = \begin{cases} \frac{1}{60-0} & 0 \le x \le 60 \\ 0 & \text{elsewhere} \end{cases}$

$\bar{X} = \int_0^{60} x \frac{1}{60} dx = \frac{x^2}{2(60)}\Big|_0^{60} = \frac{(60)(60)}{2(60)} = 30$

Average position = 30

What deviation from the mean value will the pointer take on average (RMS deviation from average)

$m_2 = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_0^{60} x^2 \frac{1}{60} = \frac{x^3}{3(60)}\Big|_0^{60} = \frac{(60)(60)(60)}{3(60)} = 1200$

$\sigma_X^2 = m_2 - m_1^2 = 1200 - (30)^2 = 1200 - 900 = 300$

$\sqrt{\sigma_X^2} = \sqrt{300}$

RMS deviation = $\sqrt{300}$

Question 6 (5 pts)

A university conducts an entrance exam for the first year students (few thousands). Known is that the exam has 50 questions of different topics (assume the questions are not related). Find the characteristic function and density distribution of the normalized (mean and variance) sum of the total grades. State clearly your evidence.

$f_x = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$

X_1: grade of first student

X_2: " " second "

X_3 ...

Y: sum of total grades

$Y = X_1 + X_2 + X_3 + \dots X_N$

N is very large can be described as $N \to \infty$ and Y is not decreasing and we can describe every X_i that they are independent of each other.

∴ Y has a density function that is similar to the Gaussian density function.

$f_Y(y) = \frac{1}{\sqrt{2\pi(1)}} e^{-y^2/2}$

$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

$\text{Phi}_z(w) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{jwy} dy$

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{jwy} dy$

why zero

$e^{-0} = 1$

Question 7 (5 pts)

Find the 10[th] moment of a Gaussian function of zero mean and unit variance (normalized), ~N(0,1)

$$m_{10} = \int_{-\infty}^{\infty} x^{10} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \; !!$$

$$G_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{2} + j\omega x\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\left(-\frac{x^2}{2} + j\omega x\right)^n}{n!} dx$$

$$m_n = (-j)^n \left.\frac{dG(\omega)}{d\omega}\right|_{\omega=0}$$

x^2 is dominant against x

$$= (-j)^n \frac{1}{\sqrt{2\pi}} \left(e^{-\infty} - e^{-\infty}\right) = 0$$

long

m_{10} = 0 Zero

$$f_X(x) = \frac{x}{s^2} e^{-\left(\frac{x^2}{2s^2}\right)} \quad x \geq 0$$

Question 8 (5 pts)

non-monotonic

The envelope of a radio detection signal has Rayleigh density: $f_X(x) = (x/s^2) \exp(-(x^2/(2 s^2)))$, $x >= 0$, and you are asked to find the power as $Y = c X^2$. Find the density of Y

$$f_Y(y) = \sum \frac{f_X(x_n)}{\left|\frac{dT}{dx}\right|_{x_n}} \quad Y = cx^2 \rightarrow \frac{dT}{dx} = 2cx$$

$$x = T^{-1}(Y) = \pm\sqrt{\frac{y}{c}}$$

$$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{c}}\right)}{\left|2c\sqrt{\frac{y}{c}}\right|} + \frac{f_X\left(-\sqrt{\frac{y}{c}}\right)}{\left|-2c\sqrt{\frac{y}{c}}\right|} = \frac{f_X\left(\sqrt{\frac{y}{c}}\right) + f_X\left(-\sqrt{\frac{y}{c}}\right)}{\left|2c\sqrt{\frac{y}{c}}\right|}$$

$$f_X\left(\sqrt{\tfrac{y}{c}}\right) = \frac{\sqrt{\frac{y}{c}}}{s^2} e^{-\frac{y}{2cs^2}}$$

$$= \frac{\frac{2\sqrt{\frac{y}{c}}}{s^2} e^{-y/2cs^2}}{\left|2c\sqrt{\frac{y}{c}}\right|}$$

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$$f_Y(y) = \frac{2\sqrt{\frac{y}{c}}\; e^{-y/2cs^2}}{\left|2c\sqrt{\frac{y}{c}}\right|}$$

No aid given, received, nor observed ____

I swear to Allah

~~I state, with my word of honor,~~ that I didn't use, or see, any form of cheating, including electronic forms ____ حرام العقم بالشرف