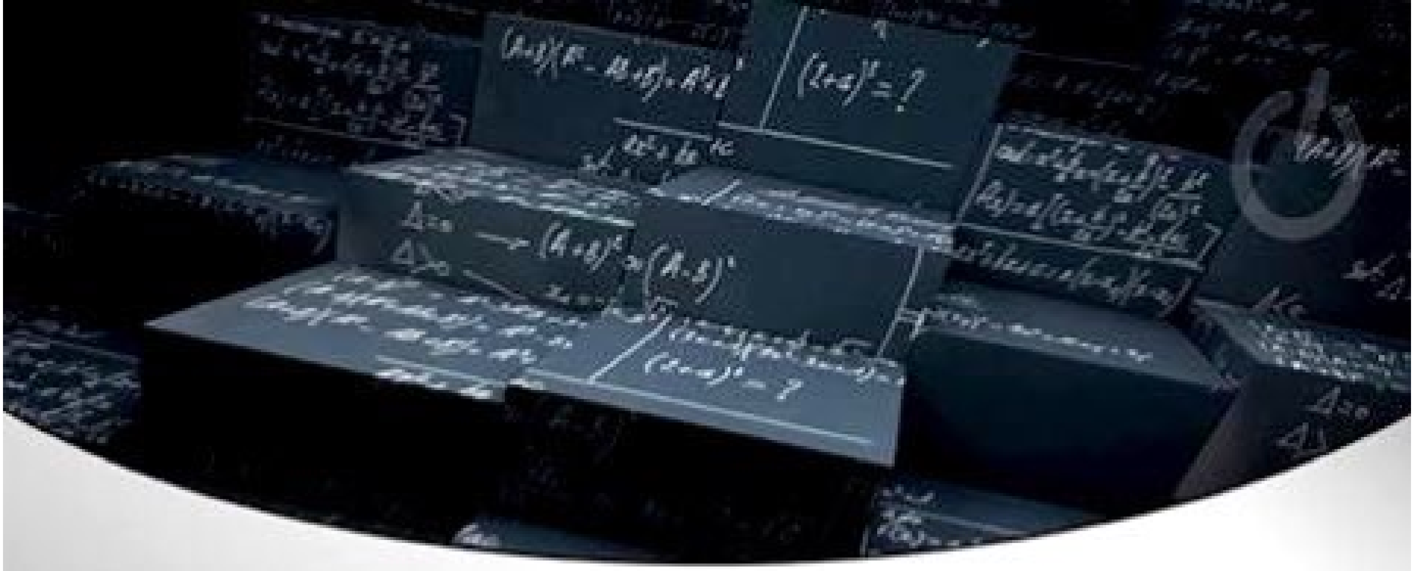

Probabiliy

Fall017

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probability

20/9/2017 Dr. Yazid khattabi

① set definition :-

Set :- is a collection of objects (called elements)

e.g :- set $A = \{a, b, d, c, f\}$ $d \in A$, $g \notin A$

"set upper case" "lower case"

e.g set $B = \{10, 12, 13\}$, $12 \in B$, $1 \notin B$

$a \in A$: a is an element in set A.

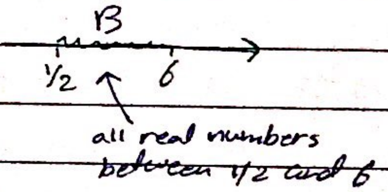
$a \notin A$: a is not an element in set A.

② Two methods to specify a set :-

① Tabular method :-

$A = \{1, 3, 5\}$

$B = \{\frac{1}{2} \le a \le 6\}$ → rule method not tabular



all real numbers between 1/2 and 6

② Rule method :

A = { all integer odd numbers between 1 and 5 ; 1 & 5 are included }

B = { all real numbers between 1/2 and 6 } .

③ Sets can be ~~sp~~ classified as :-

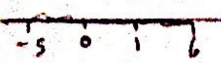
① Countable set :-

$A = \{\underset{\textcircled{1}}{2}, \underset{\textcircled{2}}{4}, \underset{\textcircled{3}}{6}, \underset{\textcircled{4}}{8}\}$

$B = \{\underset{\textcircled{1}}{1}, \underset{\textcircled{2}}{3}, \underset{\textcircled{3}}{7}, \underset{\textcircled{4}}{9}, \underset{\textcircled{5}}{11}, \ldots\ldots \underset{\textcircled{\infty}}{}\}$

② un Countable set :-

$C = \{-5 \le b \le 6\}$.



- sets can be classified as :-

① finite :-

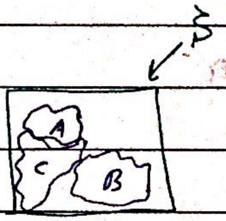
$A = \{3, 6, 9, 12\}$

② infinite :-

$B = \{1, 2, 3, 4, 5, \dots\}$

$C = \{-1 \leq b < 7.5\}$

- Venn diagram :-



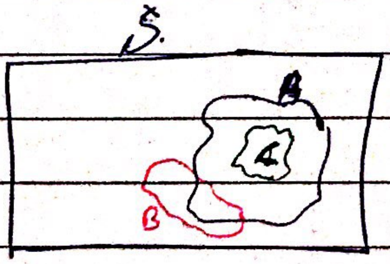
- $A \subseteq B$: set A is contained in B (all elements in A are in B)

  - A is subset of B

e.g $A = \{3, 5, 7, 9, 12\}$

$B = \{5, 9, 11\} \rightarrow B \not\subseteq A$

$C = \{7, 5, 9\} \rightarrow C \subseteq A$



- empty set :- $\{\}$, ϕ

↳ the set that does not contained any element

- universal set :-

The set that contains all other sets (in certain situation)

- if we have set of N elements;
How many subsets we can get? $\boxed{2^N}$ subsets

$\boxed{\text{e.g}}$ $A = \{a, b, c\}$; $N = 3$

Solution :- all subsets

$\{a\}, \{b\}, \{c\}$
$\{a, b\}, \{b, c\}, \{a, c\}$
$\{\ \}, \{a, b, c\}$

$8 = 2^3$ subsets.

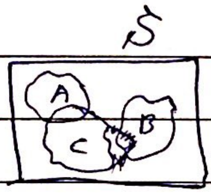
for this situation; the set A is the universal set "S"

- Sets A & B are said to be disjoint (mutully exclusive) if they do not have common elements.

$\boxed{\text{e.g}}$ $A = \{5, 10, 13, 15\}$
$B = \{1, 2, 3\}$
$C = \{5, 20, 30\}$

A & B are disjoint

A & C } not
~~B & C~~ } are disjoint



B & C are disjoint

- Set Equality :-

$A = B$, when $A \subseteq B$ & $B \subseteq A$

- Set difference :-

$A - B$: all elements in A but not in B

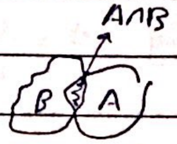
$\boxed{\text{e.g}}$ $A = \{-1 \leq a < 3\}$
$B = \{0.5 < a \leq 7\}$

$A - B = \{-1 \leq a \leq 0.5\}$ $B - A = \{3 \leq a \leq 7\}$

① Intersection :-

$A \cap B$: set of all common elements between A & B

" and "
" together"



e.g $A \cap B = \{0.5 < a < 3\}$

Note : $A \cap B = \phi \quad \therefore \Rightarrow$ A and B are disjoint.

① $A_1, A_2, \ldots, A_n$

$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$

① union :-

$A \cup B$:- all elements in A and B.

OR

e.g : $A = \{1, 3, 5, 7, 8\}$

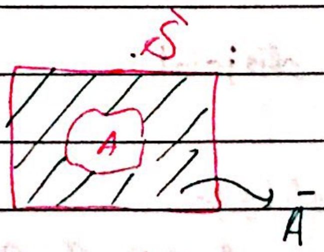
$B = \{1, 8, 12, 13\}$

$A \cap B = \{1, 8\}$

$A \cup B = \{1, 3, 5, 7, 8, 12, 13\}$

① Complement :-

$\bar{A}$: all element not in A



$\bar{A} = S - A$

$A = S - \bar{A}$

$A \cap \bar{A} = \phi$

$A \cup \bar{A} = S$

$\bar{S} = \phi$

$\bar{\phi} = S$

- Commutative law :-

$A \cap B = B \cap A$

$A \cup B = B \cup A$

- Distributive law :-

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- Associative law :-

$A \cap B \cap C - (A \cap B) \cap C - A \cap (B \cap C)$

$A \cup B \cup C = (A \cup B) \cup C$

$A \cup (B \cup C)$

- DeMorgans Rule :-

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$ → using venn diagram

25/7/2017

⓪ Mathematical Models of Expirements :-

1 Sample space "S" :-

The set of all possible outcomes.

e.g : Roll a die and record (observe) the appeared number

$S = \{1, 2, 3, 4, 5, 6\}$

e.g Flipp a Coin

$S = \{H, T\}$

2 Events :-

event : Subset of the sample space

e.g : Roll a die

Define event A "the appeared number is odd"
B " " " " is negative"
C " " " " " integer"

Sol: $A = \{1, 3, 5\}$

$B = \{\ \} = \phi$ "impossible event" 0%.

$C = \{1, 2, 3, 4, 5, 6\} = S$ "certain event" 100%

3 Assign probabilities :-

$p(A)$: The probability of the occurrence of event A.

probability axioms :-

① $0 \le p(A) \le 1$

impossible event

certain event

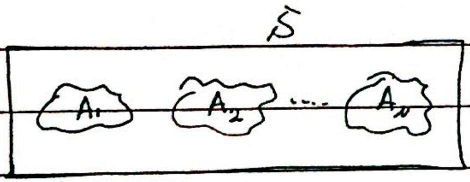
$p(\phi) = 0$ (0%)

$p(\tilde{S}) = 1$ (100%)

② let $A_1, A_2, A_3, \dots, A_N$ are disjoint events, then :-

$$p(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} p(A_i$$

$$p(A_1) \cup p(A_2) \cup p(A_3) \cup \dots \cup A_N = p(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_N)$$



$$p(\bigcup_{i=1}^{N} A_i) = \frac{area A_1 + area A_2 + area A_3 + \dots + area A_N}{area \tilde{S}}$$

$$= p(A_1) + p(A_2) + p(A_3) + \dots + p(A_N)$$

e.g. Roll two dic and observe the appeared numbers.

ⓐ find $\tilde{S}$ ⓑ Define events $A = \{ sum = 7 \}$

$B = \{ 8 < sum \le 11 \}$

$C = \{ sum > 10 \}$

ⓒ find $p(A), p(B), p(c)$

ⓓ find $p(A \cup B), p(B \cup C), p(A \cap B), p(B \cap c)$

Sol :-

"A"

$\tilde{S} = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$

$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$ "B"

$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$

$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$ "C"

$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$p(A) = \frac{6}{36} = 1/6$

$p(B) - \frac{9}{36} = 1/4$

$p(c) = 3/36 = 1/12$

ⓓ $p(A \underset{\text{"OR"}}{\cup} B) = \frac{15}{36}$

OR : A and B are disjoint

$p(A \cup B) = p(A) + p(B)$

$\frac{1}{6} + \frac{1}{4} = 15/36$

$p(B \cup C) = \frac{10}{36} \neq p(B) + p(c)$

$p(A \underset{\text{and}}{\cap} B) = p\{\emptyset\} = 0$

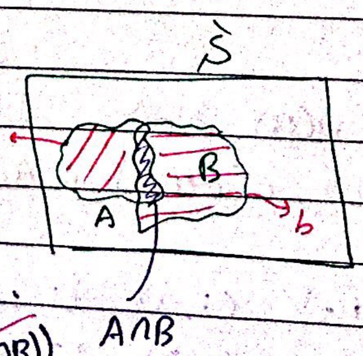
$p(B \cap C\} = \frac{2}{36}$

ⓔ Joint probability :-

$p(A \cap B)$: the probability of the occurrence of A and B together.

$$\boxed{p(A \cap B) = p(A) + p(B) - p(A \cup B)}$$

proof :-



left hand side ← LHS :

$p(A) + p(B) - p(A \cup B) = \cancel{p(a)} + \cancel{p(A \cap B)} + \cancel{p(b)}$

$+ p(A \cap B)$

$- (\cancel{p(a)} + \cancel{p(b)} + \cancel{p(A \cap B)})$

$= p(A \cap B)$ RHS

$$\boxed{p(A\cup B) = p(A) + p(B) - p(A\cap B)}$$

as special case ; if A and B are disjoint

$$p(A\cup B) = p(A) + p(B) - p\{\phi\} = p(A) + p(B)$$

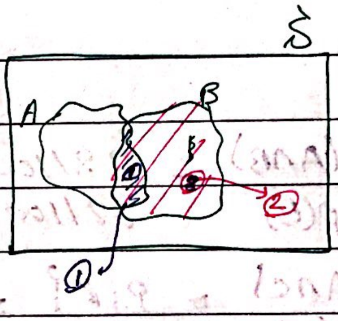
note : $p(A\cup B) = p(A) + p(B)$

"$A\cap B = \phi$"

Conditional probability :-

$p(A/B)$: the probability of A given that B has occurred.

$$\boxed{p(A/B) = \frac{p(A\cap B)}{p(B)}}$$



$$p(A/B) = \frac{①}{②} = \frac{①/\text{area } \grave{S}}{②/\text{area } \grave{S}}$$

$$= \frac{p(A\cap B)}{p(B)}$$

$$p(A\cap B) = p(A/B)\,p(B)$$
$$= p(B/A)\,p(A)$$

Ex :- a box with 100 resistors with resistance and tolerance as shown in the table .

	5%	10%	total
22 Ω	10	14	24
47 Ω	28	16	44
100 Ω	24	8	32
total	62	38	100

experiment :- Dawe out one resistor and ~~record~~ observe the (Ω, tolerance)

Define events :

A : " the resistor is 47Ω "

B : " the resistor is 5% tolerance "

C : " the resistor is 100Ω "

find :- ⓐ $p(A), p(B), p(c)$

ⓑ $p(A/B), p(A/c), p(B/c)$

Sol :- ⓐ $p(A) = \frac{44}{100}$

$p(B) = \frac{62}{100}$

$p(c) = \frac{32}{100}$

ⓑ $p(A/B) = \frac{p(A \cap B)}{p(B)} = \frac{28/100}{62/100} = \frac{28}{62}$

$p(A/c) = \frac{p(A \cap c)}{p(c)} = \frac{p\{\emptyset\}}{32/100} = \text{zero}$

$p(B/c) = \frac{p(B \cap c)}{p(c)} = \frac{24/100}{32/100} = \frac{24}{32}$

Ex :-

10Ω	22Ω	27Ω	47Ω
18	12	33	17

80 resistors

exp①: Dawe out one resistor

A : " the resistor is 10Ω "

B : " " " " 22Ω "

C : " " " " 27Ω "

D : " " " " 47Ω "

$p(A) = \frac{18}{80}$

$p(B) = \frac{12}{80}$

$p(c) = \frac{33}{80}$

$p(D) = \frac{17}{80}$

exp ② : Dave out two resistors without replacement

A : the 1^{st} resistor is 22Ω
B : the 2^{nd} resistor is ~~22~~ 27Ω

p(A) =
p(B/A) =

Sol :-
p(A) = 12/80
p(B/A) = P(~~B~~ 2^{nd} is 27 / 1st is 22Ω)

$$= \underline{\underline{\frac{33}{79}}}$$

$$p(A \cap B) = p(B/A) \times p(A) = \frac{33}{79} \times \frac{12}{80}$$

27/9/2017

⓪ Independent events :-

A and B are said to be dependent if :

① $p(A/B) = p(A)$ ← A is not affected by B

② $p(B/A) = p(B)$ ← B is not affected by A

Consequence of independence :

$$p(A/B) = \frac{P(A \cap B)}{P(B)} = p(A)$$

$$\hookrightarrow \boxed{p(A \cap B) = p(A) . p(B)}$$

- If A_1 and A_2 are independent ; then

① A_1 & $\bar{A}_2$ are independent.

② $\bar{A}_1$ & A_2 " "

③ $\bar{A}_1$ & $\bar{A}_2$ " " .

Ex : Two events A_1 and A_2 are independent

$p(A_1) = 0.6$, $p(A_2) = 0.4$

Find ⓐ $p(A_1 \cap \bar{A}_2)$

ⓑ $p(\bar{A}_2 / \bar{A}_1)$

ⓐ $p(A_1) \cdot p(\bar{A}_2) = 0.6(1-0.4) = 0.6^2$

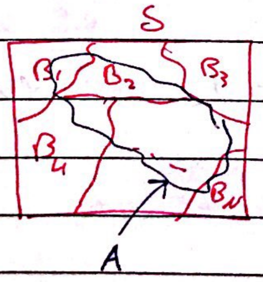
ⓑ $p(\bar{A}_2 / \bar{A}_1) = p(\bar{A}_2) = 1 - 0.4 = 0.6$

- Total probability :-

Let $B_1, B_2, \ldots, B_N$ are disjoint events $(B_i \cap B_j = \phi)$ and
$i \neq j$

$\underbrace{\bigcup_{i=1}^{N} B_i = S}$

↳ $p(B_1) + p(B_2) + \ldots + p(B_N) = 1$



$$\boxed{p(A) = \sum_{i=1}^{N} p(A/B_i)\, p(B_i)}$$

proof :- $p(A) = p(A \cap S)$

$$= p\Big(A \cap (B_1 \cup B_2 \cup \dots \cup B_N)\Big)$$

$$= p\Big(\underline{\underline{(A \cap B_1)}} \cup \underline{\underline{(A \cap B_2)}} \cup \dots \cup \underline{\underline{(A \cap B_N)}}\Big)$$

$$= p(A \cap B_1) + p(A \cap B_2) + \dots + p(A \cap B_N)$$

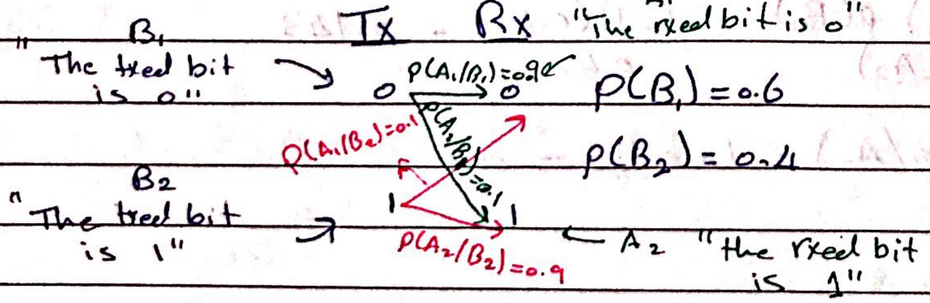
$$= \sum_{i=1}^{N} p(A \cap B_i)$$

$$= \sum_{i=1}^{N} \underline{p(A/B_i)}\,\underline{p(B_i)}$$

$$\Rightarrow p(B_i/A) = \frac{p(B_i \cap A)}{p(A)} = \frac{p(A/B_i)\,p(B_i)}{p(A)}$$

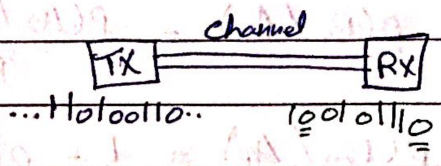
$$\boxed{p(B_i/A) = \frac{p(A/B_i)\,p(B_i)}{\underbrace{\sum_{j=1}^{N} p(A/B_j)\,p(B_j)}_{p(A)}}} \Longrightarrow \text{Bayes rule}$$

Ex :- Binary communication channel (BCC)



$p(B_1) = 0.6$

$p(B_2) = 0.4$



Find ⓐ $p(A_1)$, $p(A_2)$

ⓑ $p(B_1/A_1)$, $p(B_2/A_2)$

ⓒ $p(B_1/A_2)$, $p(B_2/A_2)$

Sol:-

ⓐ $p(A_1) = p\big((A_1 \cap B_1) \cup (A_1 \cap B_2)\big)$

"OR"

"and"

$= p(A_1 \cap B_1) + p(A_1 \cap B_2)$

$= \underbrace{p(A_1/B_1)p(B_1) + p(A_1/B_2)p(B_2)}$ (Total probability)

$= (0.9)(0.6) + (0.1)(0.4) = 0.58$

- $p(A_2) = 1 - p(A_1) = 1 - 0.58 = 0.42$

- $p(A_2) = p(A_2/B_1)p(B_1) + p(A_2/B_2)p(B_2)$
  $= 0.42$

ⓑ $p(B_1/A_1) = \frac{P(B_1 \cap A_1)}{p(A_1)} = \frac{p(A_1/B_1)p(B_1)}{p(A_1)} = \frac{(0.9)(0.6)}{(0.58)} = 0.931$

$p(B_2/A_1) = 1 - p(B_1/A_1) = 1 - 0.931 = 0.069$

oR $p(B_2/A) = \frac{P(B_2 \cap A_1)}{P(A_1)} = \frac{P(A_1/B_2)\,P(B_2)}{P(A_1)} = \frac{(0.1)(0.4)}{0.58} = 0.069$

ⓒ $p(B_1/A_2) = \frac{P(A_2/B_1)\,p(B_1)}{P(A_2)} = \frac{(0.1)(0.6)}{0.42} = 0.143$

$p(B_2/A_2) = 1 - p(B_1/A_2) = 1 - 0.143 = 0.857$.

Combined Experiments :-

Experiments formed by more than one sub-expirement.

Ex :- Flip a coin & Roll a die

ⓐ Sample space :-

$S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)$
$(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

exp "two sub-exp"

→ sub-exp1 Flip a Coin : $S_1 = \{H, T\}$

→ sub-exp2 : $S_2 = \{\cancel{H}1, 2, 3, 4, 5, 6\}$ Roll a die

$$S = S_1 \underset{\uparrow}{\times} S_2 = \left\{ \begin{array}{l} (H,1), (H,2), ---, (H,6) \\ (T,1), (T,2), ---, (T,6) \end{array} \right\}$$

~~means~~ Combination

ⓑ define event C : "The Coin is T and the number on die is even"

$C = \{(T,2), (T,4), (T,6)\}$

OR : event C

→ event A $= \{T\}$ (S_1)

→ event B $= \{2, 4, 6\}$ (S_2)

$C = A \times B$

Canpination ↑

ⓒ $p(c) = \frac{3}{12}$

OR $p(c) = P(A \underset{\substack{\uparrow \\ \text{"and"} \\ \cap}}{\times} B) = p(A) \cdot p(B) = \frac{1}{2} \times \frac{3}{6} = \frac{3}{12}$

Ex : Flip a Coin 3 times. find p(all are head)

$S = \{(HHH), (HTH), (HHT), (THH), (TTH), (THT), (HTT), (TTT)\}$

$p(\text{all are head}) = p(\{HHH\}) = 1/8$

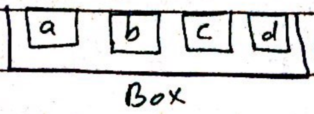
OR : $p(\{\underset{s_1}{H}\,\underset{s_2}{H}\,\underset{s_3}{H}\}) = p(\{H\}) \cdot p(\{H\}) \cdot p(\{H\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

① permutations :

all possible sequences of ordering r elements (the order is important) taken from n elements without replacement.

P^n_r = # of permutations

Ex :



⇒

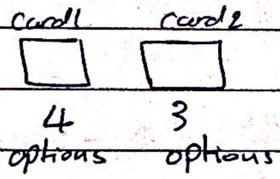
place 1	place 2
a	b
a	c
a	d
b	a
b	c
b	d
c	a
c	b
c	d
d	a
d	b
d	c

order 2 cards

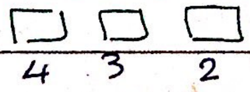
$n=4$

$r=2$

$P^4_2 = 12 = 4 \times 3$



$r=3$



$P^4_3 = 4 \times 3 \times 2 = 24$

$P^n_r = n(n-1)(n-2) - (n-r+1)$

$= \frac{n!}{(n-r)!}$

Ex : $n=50$ ⇒ $P^{50}_{15} = \frac{50!}{35!}$

$r=15$

2/10/2017

① Combinations:-

Same as permulations but the order is not important. $P^4_2 = 12$

Ex: 4 Cards. $\boxed{a}\ \boxed{b}\ \boxed{c}\ \boxed{d}$

n = 4

r = 2

place 1	place 2
a	b
a	c
a	d
b	c
b	d
c	d

$\Rightarrow$ In general: $C^n_r = \frac{P^n_r}{r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$

# of Combinations = $C^4_2 = 6 = \frac{P^4_2}{2}$

Ex:- 5 students. How many 3-member teams can be performed?

# of teams $= C^5_3 = \binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3!\,2!} = \ldots$

note $0! = 1$

$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{n} = 1$

@ Bernouli Trail

G is trail (experiment) with two outcomes

$A \rightsquigarrow$ success , $p(A) = p$

$\bar{A} \rightsquigarrow$ fail , $p(\bar{A}) = 1-p$

* if we repeat the Bernouli trail N times, @How many times the outcome A will appear ?!

(OR) what is the number of successes?

number of successes $\triangleq k = 0, 1, 2, 3, \dots, N$

ⓑ $p(k=i) = ??$ (what is the probability that the # of successes is i ?!

$i = 0, 1, 2, 3, \dots, N$

$$\overbrace{\underbrace{|\,s\,|\,s\,|\,\cdots\,|\,s\,|}_{i}\underbrace{\,f\,|\,f\,|\,\cdots\,|\,f\,|}_{N-i}}^{N \text{ times}}$$

$$\boxed{p(k=i) = p^i \binom{N}{i} \cdot (1-p)^{N-i}}$$

$i = 0, 1, \dots, N$

Ex 8- Flip a coin 3 times

$N = 3$

Success : $A = \{H\}$

fail : $\bar{A} = \{T\}$

$k = 0, 1, 2, 3.$

$$S = \{\overbrace{HHH}^{k=3}, \overbrace{HHT, HTH, THH}^{k=2}, TTH,$$
$$\underbrace{THT, HTT}_{k=1}, \underbrace{TTT}_{k=0}\}$$

$$p(k=2) = p(\{HHT, HTH, THH\}) = \overset{p(H)\cdot p(H)\cdot p(T)}{p(HHT)} + p(HTH) + p(THH) = 3p^2(1-p)$$
$$= \binom{3}{2} p^2 (1-p)$$

$p(k=0) = p(TTT) = (1-p)^3$

OR $p(k=0) = \binom{3}{0} p^0 (1-p)^{3-0} = (1-p)^3$

$p(k=1) = \binom{3}{1} p^1 (1-p)^{3-1} = 3p(1-p)^2$

Ex:- flip a coin 100 times, what is the probability that the tail will appear at most 2 times ?- $p(H) = 0.4$ in each flip

Sol:- $N = 100$

Success $= \{T\} \rightsquigarrow p(T) = 0.6 \leftarrow p$

Fail $= \{H\} \rightsquigarrow p(H) = 1 - 0.6 = 0.4$

$k = 0, 1, 2, \ldots, 100$

p(the tail appears at most 2 times) $= p(k \leq 2) = p(k=0) + p(k=1) + p(k=2)$

$$= \binom{100}{0} 0.6^0 (0.4)^{100} + \underset{\underset{100}{\downarrow}}{\binom{100}{1}} 0.6^1\, 0.4^{99} + \underset{\underset{\frac{100!}{2!\,98!}}{\downarrow}}{\binom{100}{2}} 0.6^2\, 0.4^{98}$$

$$= 0.4^{100} +$$

$$= 1.48 \times 10^{-19}$$

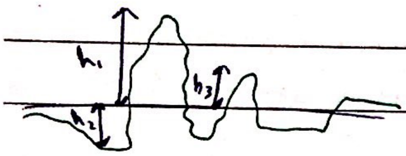
$$\boxed{\sum_{i=0}^{N} p(k=i) = 1}$$

$$\sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} = 1$$

1.71 ⟶ examples in the book (4^{th} edition) we should Solve
1.72
1.73

4/10/2017

Chapter 2 :- Random variable



h : is a random variable

⟹ a Random variable X(s) is function of the sample space S

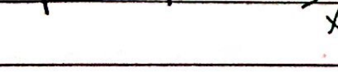
exp 1 : $S = \{e_1, e_2, \dots, e_{10}\}$

set of all possible outcomes

$X(s) \triangleq X$

Discrete R.V

· Contineous R.V

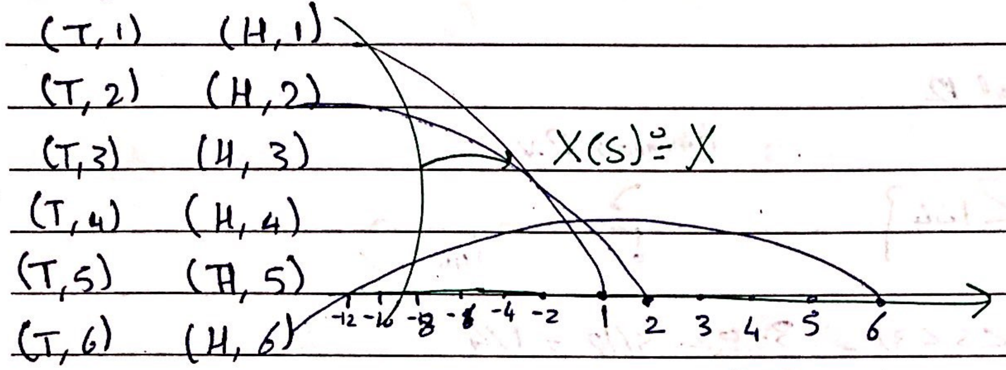


Ex :- flip a Coin and roll a die

ⓐ $\tilde{S}$ ⓑ $X(\tilde{S})$: H appears $X \doteq$ ※ on the die

: T appears $X \triangleq -2$ (※ on the die)

ⓐ $\underset{=}{\tilde{S}}$



ⓑ $X = \{-12,-10,-8,-6,-4,-2,1,2,3,4,5,6\}$ ← Discrete R.V

$p(\overbrace{X=-10}^{\text{event}}) = p(\{T,5\}) = 1/12$

$p(x=20) = p(\phi) = 0$

$p(-2<x\leq 3) = p(x=1)+p(x=2)+p(x=3) = \frac{1}{12} \cdot 3 = 14$

$p(x \leq 12) = p(\tilde{S}) = 1$

$p(x < \infty) = p(\tilde{S}) = 1$

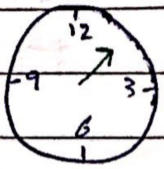
$p(x < -\infty) = p(\phi) = 0$

$p(x < -14) = 0$

Note :- for discrete R.V $p(X=x)$ exists

Ex 2 :- Contineous R.V : wheel of chance

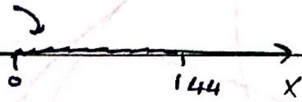
ⓐ $\tilde{S}$ ⓑ $X(\tilde{S}) = \tilde{S}^2$



ⓐ $\tilde{S} = \{0 < s \le 12\}$

all real numbers greater than zero, less than or equal 12

Contineous R.V



ⓑ $X = \{ 0 < x \le 144 \}$

$p(0 < x \le 9) = p(0 < s \le 3) = \frac{3-0}{12-0} = 3/12 = 1/4$

$p(X = 25) - p(\{5\}) = \frac{1}{\infty} = 0$

$p(d_1 < s < d_2) = \frac{d_2 - d_1}{12}$

$p(d_1 < s < d_2) = \lim_{d_2 - d_1 \to 0} \frac{d_2 - d_1}{12} = 0$

$\lim_{d_2 - d_1 \to 0}$ they are one point in this case

Note :- for Contineous R.V $p(X = x) = 0$

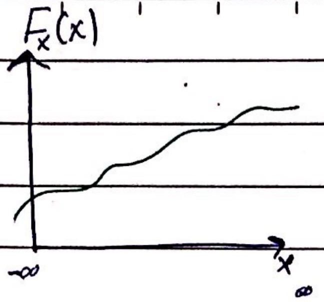
Any R.V X has two functions :-

① Distribution function (CDF) $F_X(x)$

② Density function (PDF) $f_X(x)$

⑪ Distribution function :-

$X \sim F_X(x)$
$(-\infty, \infty)$



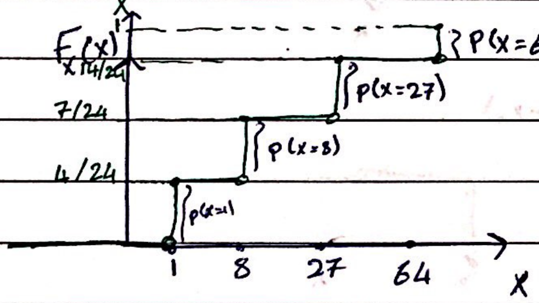
Def : $F_X(x) = p(X \le x)$

back to Ex 2 :- $F_X(9) = p(x \le 9) = p(s \le 3) = \frac{3}{12} = \frac{1}{4}$

Ex : given discrete R.v $x = \{1, 8, 27, 64\}$

$p(x=1) = 4/24$ $\quad p(x=27) = 7/24$
$p(x=8) = 3/24$ $\quad p(x=64) = 10/24$

Find $F_X(x)$



$$\Rightarrow \quad F_X(x) = \begin{cases} 0 , & x < 1 \\ 4/24 , & 1 \le x < 8 \\ 7/24 , & 8 \le x < 27 \\ 14/24 ; & 27 \le x < 64 \\ 1 , & 64 \le x \end{cases}$$

$F_X(x) = p(\cancel{X} \le \boxed{x})$ $(-\infty, \infty)$

$F_X(5) = p(x \le 5) = p(x=1) = 4/24$

$F_X(-\infty) = p(x \le -\infty) = 0$

$F_X(8) = p(x \le 8) = p(x=1) + p(x=8)$
$= 4/24 + 3/24 = 7/24$

$F_X(0) = p(X \le 0) = 0$

$F_X(8^-) = 4/24$

$F_X(0.999) = 0 \Rightarrow F_X(1^-) = 0$

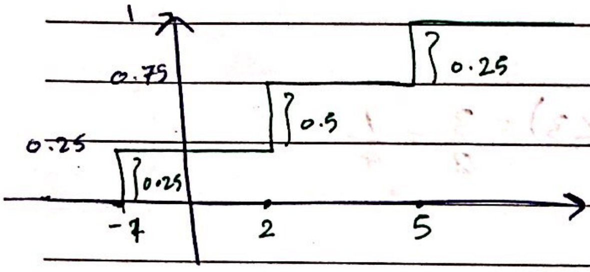
$F_X(27^-) = 7/24$

$F_X(1) = 4/24 = p(X \le 1) = p(x=1)$

$F_X(27) = p(x \le 27)$
$= p(x=1) + p(x=8) + p(x=27)$
$= 7/24 + 7/24$

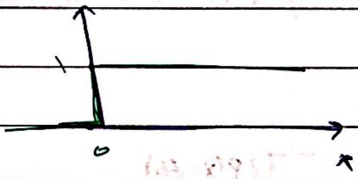
$Y = \{-1, 2, 5\}$

$p(y=-1) = 0.25$
$p(y=2) = 0.5$
$p(y=5) = 0.25$

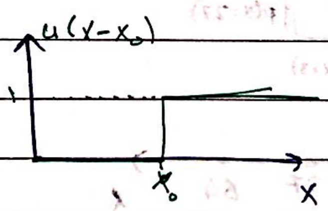


unit-step-function

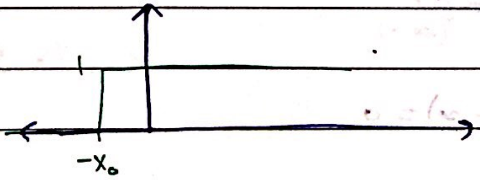
$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



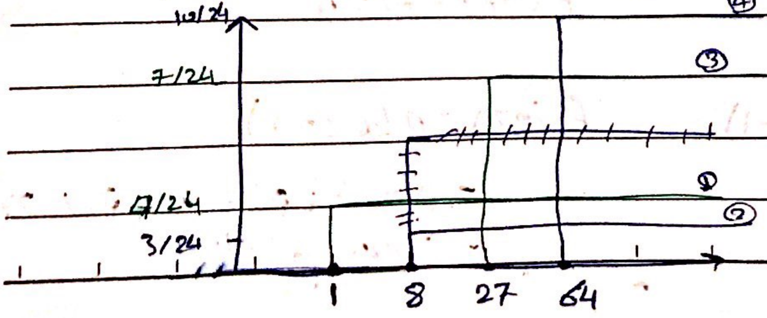
$$u(x-x_0) = \begin{cases} 1, & x \geq x_0 \\ 0, & x < x_0 \end{cases}$$

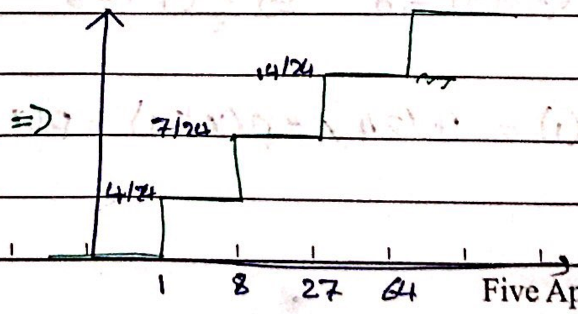


$$u(x+x_0) = \begin{cases} 1, & x \geq -x_0 \\ 0, & x < -x_0 \end{cases}$$



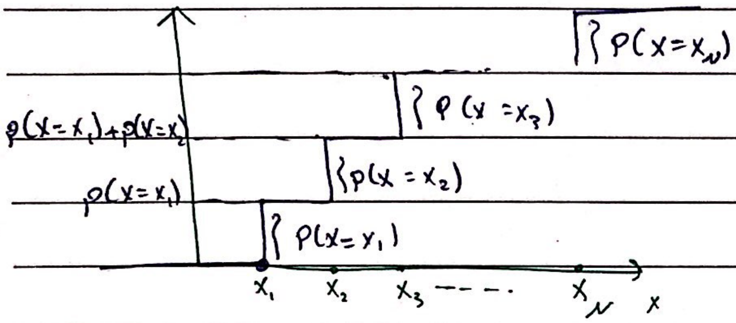
$$F_x(x) = \overbrace{\frac{4}{24} u(x-1)}^{①} + \overbrace{\frac{3}{24} u(x-8)}^{②} + \overbrace{\frac{7}{24} u(x-27)}^{③} + \overbrace{\frac{10}{24} u(x-64)}^{④}$$





● for discrete R.V $X = \{x_1, x_2, x_3, \dots, x_N\}$

$$F_X(x) = \sum_{i=1}^{N} P(X = x_i)\, u(x - x_i)$$



Note: $F_X(-\infty) = 0$

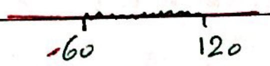
$F_X(\infty) = 1$

$F_X(x^+) = F_X(x)$

9/10/2017

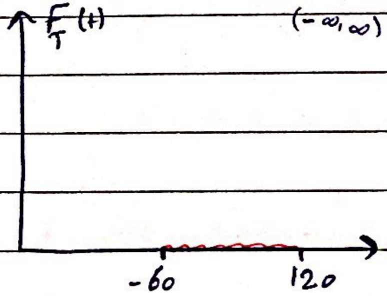
Ex: CDF for continuous R.V

T: R.V that models the temperature for certain location

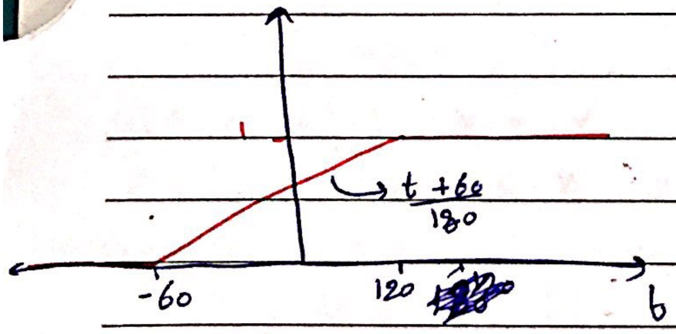


$T = \{-60 \le t \le 120\}$ F°, find $F_T(t)$?

Sol: $F_T(t) = p(T \le \underset{\overset{\uparrow}{(-\infty,\infty)}}{t})$



$$F_T(t) = \begin{cases} 0 & , t < -60 \\ \frac{t+60}{180} & , -60 \le t \le 120 \\ 1 & , 120 \le t \end{cases}$$



$F_T(-\infty) = 0$

$F_T(\infty) = 1$

(*) preparties for (CDF)

$\boxed{1}$ $F_x(-\infty) = 0$

$\boxed{2}$ $F_x(\infty) = 1$

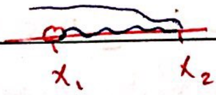
$\boxed{3}$ $0 \le F_x(x) \le 1$

$\boxed{4}$ $F_x(\overset{+}{x}) = F_x(x)$ Continuous from the right

$\boxed{5}$ if $x_1 \le x_2$, $F_x(x_1) \le F_x(x_2)$

$\therefore$ CDF is non decreasing function

$\boxed{6}$ $p(x_1 < X \le x_2) = F_x(x_2) - F_x(x_1)$



proof:

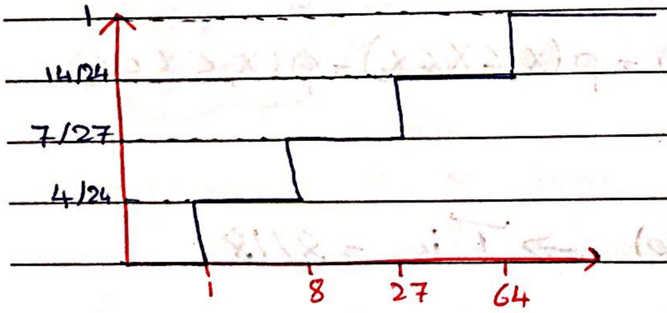
$$p(X \le x_2) = p(X \le x_1 \cup x_1 < X \le x_2)$$

Disjoint

$$p(X \le x_1) + p(x_1 < X \le x_2)$$

$$\Rightarrow p(x_1 < X \le x_2) = p(X \le x_2) - p(X \le x_1) = F_X(x_2) - F_x(x_1)$$

Ex:- $X = \{1, 8, 27, 64\}$



Find a) $p(1 < x \le 27)$

b) $p(3 < x < 64)$

c) $p(8 \le x < 64)$

Sol:-

a) $p(1 < x \leq 27) = F_x(27) - F_x(1)$

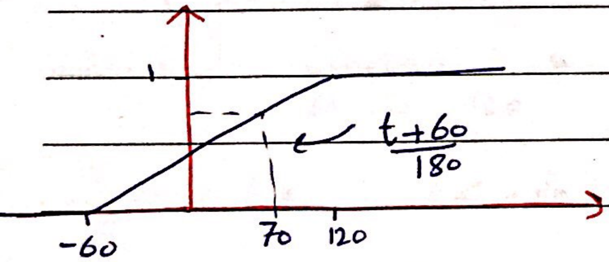
$= \frac{14}{24} - \frac{4}{24} = \frac{10}{24}$

oR $p(1 < x \leq 27) = p(x = 8) + p(x = 27) = \frac{3}{24} + \frac{7}{24} = \frac{10}{24}$

b) $p(3 < x < 64) = p(3 < x \leq 64^-) = F_x(64^-) - F_x(3) = \frac{14}{24} - \frac{4}{24} = \frac{10}{24}$

c) $p(8 \leq x < 64) = p(x=8) + p(8 < x \leq 64^-) = \frac{3}{24} + \frac{7}{24} = \frac{10}{24}$

$\boxed{\text{Ex}}$:- ⓐ $T = \{-60 \leq t \leq 120\}$



$p(-10 < T \leq 70) = F_T(70) - F_T(-10) = \frac{130}{180} - \frac{50}{180} = \frac{80}{180} = 8/18$

$p(-10 < T < 70) = 8/18$

Note :- $p(x_1 < X < x_2) = p(x_1 \leq X < x_2) = p(x_1 < X \leq x_2) = p(x_1 \leq X \leq x_2)$

ⓘⓕ X is a Continous R.V

ⓑ $p(20 < T < 100) = F_T(100) - F_T(20)$, $\Rightarrow$ T is $= 8/18$

$\uparrow \quad \uparrow$

uniform random variable

probability Density function $f_x(x)$

Def : $\boxed{f_x(x) = \frac{d F_x(x)}{dx}}$

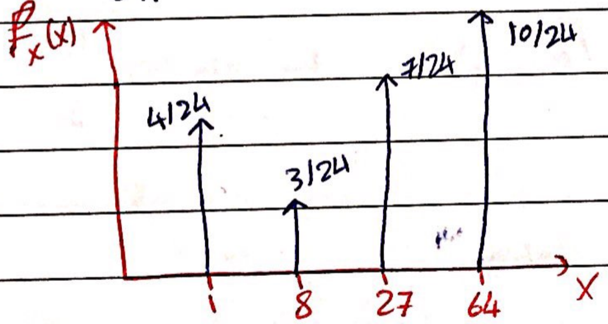
Ex :- Density function for Discrete R.V

$X = \{1, 8, 27, 64\}$

$p(x=1) = 4/24$ $p(x=27) = 7/24$
$p(x=8) = 3/24$ $p(x=64) = 10/24$

Find $F_x(x) = \frac{4}{24} u(x-1) + \frac{3}{24} u(\cancel{x}-8) + \frac{7}{24} u(x-27) + \frac{10}{24} u(x-64)$

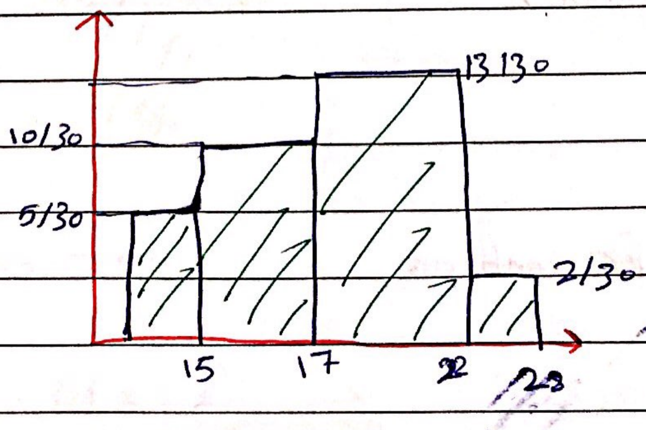
$$f_x(x) = \frac{\partial F_x(x)}{\partial x} = \frac{4}{24}\delta(x-1) + \frac{3}{24}\delta(x-1) + \frac{7}{24}\delta(x-27) + \frac{10}{24}\delta(x-64)$$



p.m.f ⇒ = probability mass function

Histogram :- 30 Students

15 → 5
17 → 10
22 → 13
28 → 2



$G = \{15, 17, 22, 28\}$

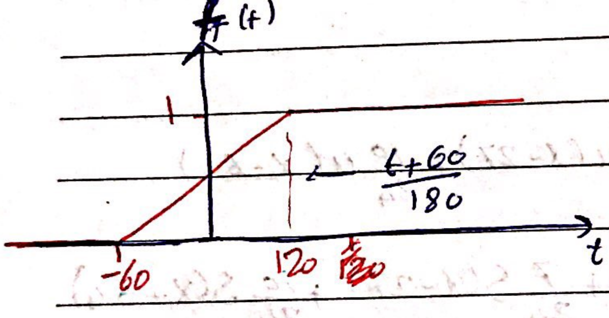
~~Ex~~

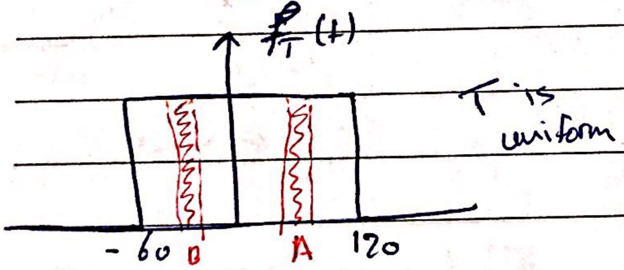
Note :- for discrete R.V $X = \{x_1, x_2, \dots, x_n\}$

$$f_X(x) = \sum_{i=1}^{n} P(X = x_1)\,\delta(x - x_i)$$

Ex :- $T = \{-60 \leq t \leq 120\}$ find $f_T(t) = ?$

Sol $f_T(t) = \dfrac{dF_T(t)}{dt}$





$$= \begin{cases} 1/180 , & -60 \leq t \leq 120 \\ 0 , & \text{o.w} \end{cases}$$

① Density function properties

① Density function

$f_X(x) \geq 0$

② $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ for continous for Discrete R.V check

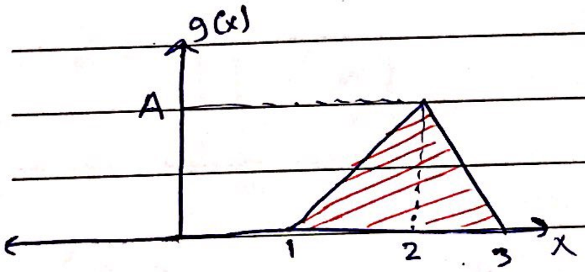
$$f_x(x) = \sum_{i=1}^{N} p(x = x_i)\,\delta(x - x_i)$$

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_{-\infty}^{\infty} \sum_{i=1}^{N} p(x = x_i)\,\delta(x - x_i)\,dx$$

$$= \sum_{i=1}^{N} \int_{-\infty}^{\infty} p(x = x_i)\,\delta(x - x_i)\,dx$$

$$= \sum_{i=1}^{N} p(x = x_i) \int_{-\infty}^{\infty} \delta(x - x_i)\,dx = \boxed{\sum_{i=1}^{N} p(x = x_i) = 1}$$ for discrete

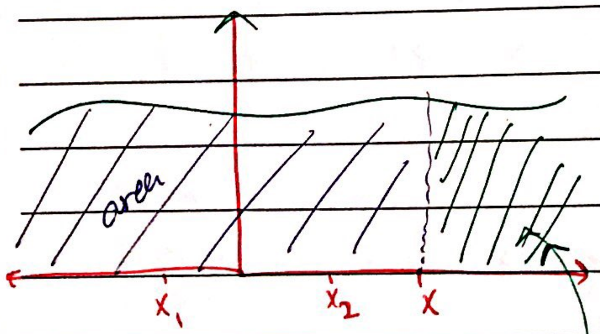
Ex 8- Find the Constant A such that g(x) is density function



Sol :- $\int_{-\infty}^{\infty} g(x)\,dx = 1 \quad = \left(\frac{1}{2}\right)(2)\,A = 1 \quad \Rightarrow \boxed{A = 1}$

③ probability Continous

$$F_x(x) = \int_{-\infty}^{x} f_x(\xi)\,d\xi$$ → Zeta



$$- F_x(x) = p(x \le x)$$

$$p(x > x) = 1 - p(x \le x) \longrightarrow 1 - F_x(x) = p(X > x) = \int_{x}^{\infty} f_x(\xi)\,d\xi$$

11/10/2017

⑫ Density function :-

$$\begin{tikzcd} & F_X(x) = p(X \leq x) \\ X \arrow[ur] \arrow[dr] & \\ & f_X(x) = \end{tikzcd}$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

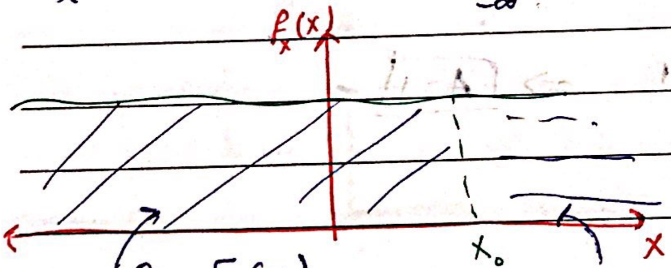
$$f_X(x) = \sum_{i=1}^{N} p(X = x_i)\, \delta(x - x_i)$$

① $f_X(x) \geq 0$ ② $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

③ $F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi$

For point x_0

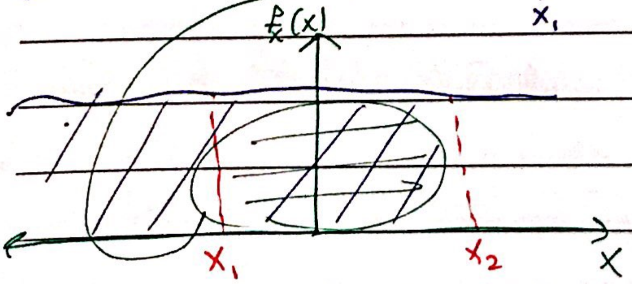
$$F_X(x_0) = p(X \leq x_0) = \int_{-\infty}^{x_0} f_X(x)\, dx$$



area ① $= F_X(x_0)$
$= P(X \leq x_0)$

area ②
$1 - F_X(x_0) = p(X > x_0)$

④ $p(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\, dx$



proof : $p(x_1 < X \leq x_2)$

$= F_X(x_2) - F_X(x_1)$

$$= \int_{-\infty}^{x_2} f_X(x)dx - \int_{-\infty}^{x_1} f_X(x)\, dx = \int_{x_1}^{x_2} f_X(x)\, dx$$

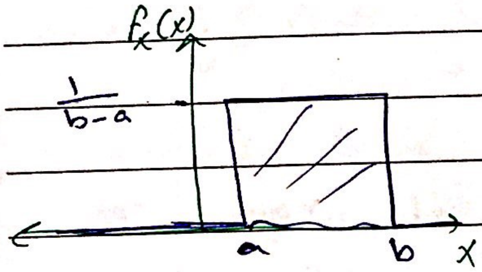
Ø Important R.V's :-

[1] uniform R.V

$X \sim u(a,b)$

- a & $b \in (-\infty, \infty)$
- $b > a$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & , a \leq x \leq b \\ 0 & , \text{o.w} \end{cases}$$



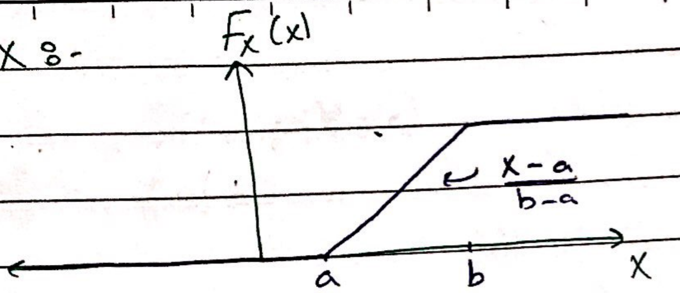
$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = 1$$

$$F_X(x) = \int_{-\infty}^{\boxed{x}} \underline{\underline{f_X(\xi)}}\,d\xi$$

$$= \begin{cases} 0 & , x \leq a \\ \int_a^x \frac{1}{b-a}\,d\xi & , a \leq x < b \\ 1 & , b \geq x \end{cases}$$

$$= \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x < b \\ 1 & , b \leq x \end{cases}$$

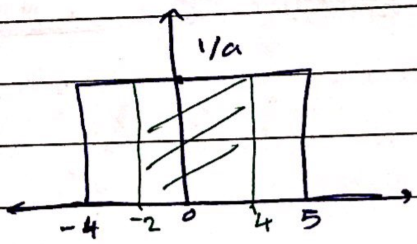
Ex :-



$X \sim u(\overset{a}{-4}, \overset{b}{5})$

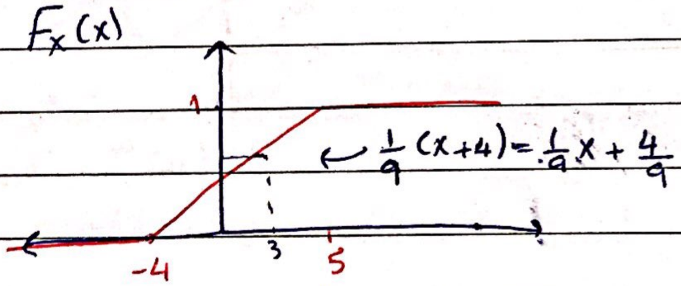
ⓐ $f_X(x)$ ⓑ $F_X(x)$ ⓒ $p(-2 \le x \le 4)$ ⓓ $p(X < 3)$ ⓔ $p(-5 < X < 0)$

Sol :-



$$f_X(x) = \begin{cases} 1/a, & -4 \le x \le 5 \\ 0, & o.w \end{cases}$$

ⓑ $F_X(x)$



ⓒ $p(-2 \le x \le 4) = \int_{-2}^{4} \frac{1}{9} dx = 6/9$

oR $\overset{\circ}{=} F_X(4) - F_X(-2) = \frac{8}{9} - \left(\frac{4}{9} - \frac{2}{9}\right) = 6/9$

ⓓ $p(X < 3) = \int_{-\infty}^{3} f_X(x)\, dx = \int_{-4}^{3} \frac{1}{9} dx = 7/9$

oR $\overset{\circ}{=} F_X(3) = \frac{3+4}{9} = 7/9$

ⓔ $p(-5 < X < 0) = \int_{-5}^{0} f_X(x)\, dx = \int_{-4}^{0} \frac{1}{9} dx = 4/9$

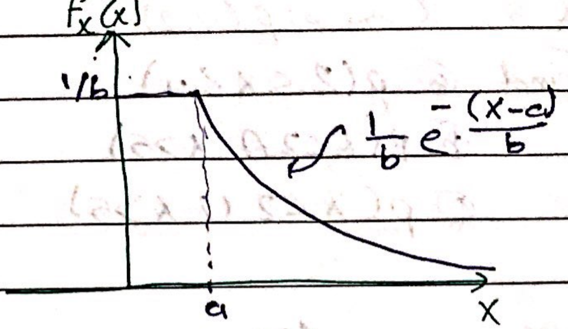
oR $\overset{\circ}{=} F_X(0) - F_X(-5) = \frac{4}{9} - 0 = 4/9$

$\boxed{2}$ exponential R.V :-

$X \sim \exp(a,b)$

$b>0$

$a \in (-\infty,\infty)$



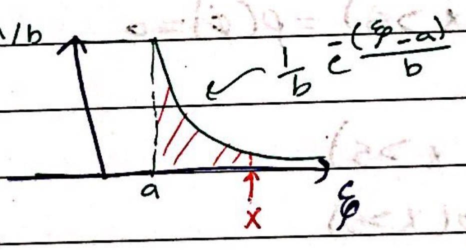
$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} , & x \geq a \\ 0 , & x < a \end{cases}$$

Note : $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{a}^{\infty} \frac{1}{b} e^{-\frac{(x-a)}{b}} dx = \frac{1}{b} \frac{e^{-\frac{(x-a)}{b}}}{-1/b} \Big]_a^{\infty} = 1$

⊙ waiting time for phone calls

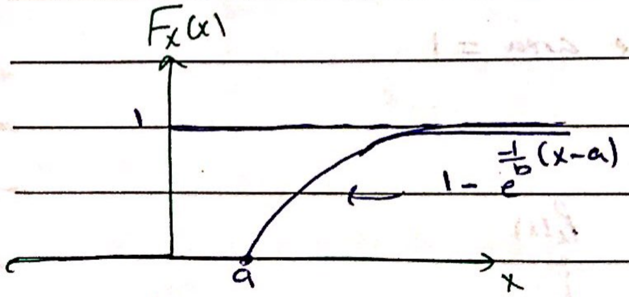
$T \sim \exp(0,\lambda)$

$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi$



- $x<a \to 0$,
- $x>a$: $F_X(x) = \int_{a}^{x} \frac{1}{b} e^{-\frac{(\xi-a)}{b}} d\xi = \frac{1}{b} \frac{e^{-\frac{(\xi-a)}{b}}}{-1/b} \Big]_a^{\infty} = \frac{e^{-\frac{(x-a)}{b}} - 1}{-1}$

$= 1 - e^{-\frac{1}{b}(x-a)}$, $x>a$



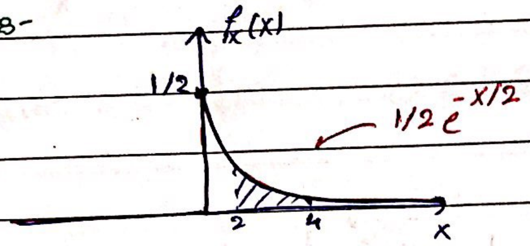
Ex 8 $X \sim exp(0,2)$

Find ⓐ $p(2<x<4)$

ⓑ $p(x<2 \cap x>5)$

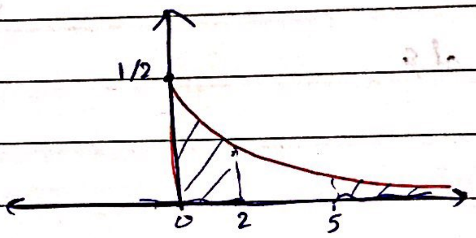
ⓒ $p(x<2 \cup x>5)$

Sol 8-



ⓐ $p(2<x<4) = \int_2^4 \frac{1}{2} e^{-x/2} dx = \frac{1}{2} \left. \frac{e^{-x/2}}{-1/2} \right|_2^4 = e^{-1} - e^{-2} = --$

ⓑ $p(x<2 \cap x>5) = p(\phi) = 0$



ⓒ $p(x<2 \cup x>5)$

$= p(x<2) + p(x>5)$

$= \int_0^2 \frac{1}{2} e^{-x/2} dx + \int_5^\infty \frac{1}{2} e^{-x/2} dx$

OR $\triangleq 1 - p(2<x<5)$ → because the area = 1

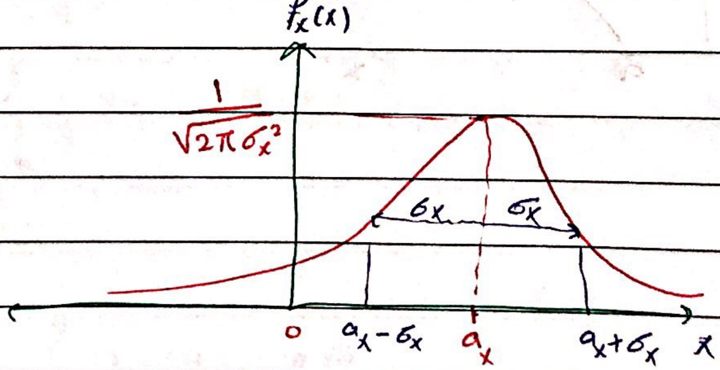
3 Gaussian (Normal) R.V

$X \sim N(a_x, \sigma_x^2)$

$a_x \in (-\infty, \infty)$

$\sigma_x > 0$

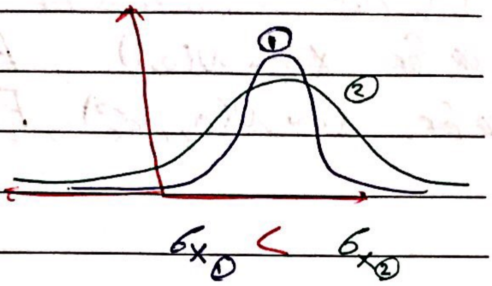
$\mathcal{X} = \{-\infty < x < \infty\}$



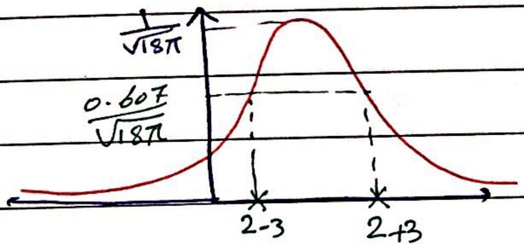
a_x : average (mean) value for X.

σ_x : standard deviation.

σ_x^2 : variance.

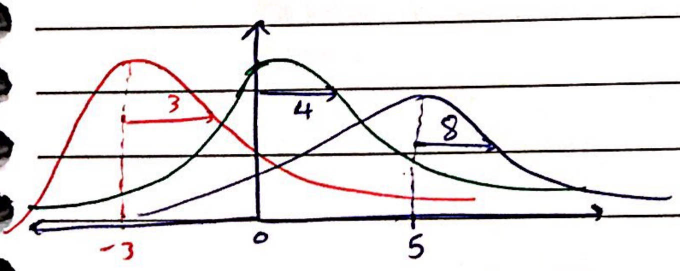


$$\boxed{f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}}$$ $-\infty < x < \infty$



$N(2, 9)$

Ex: plot
- $* \; X \sim N(0, 16)$
- $* \; X \sim N(-3, 9)$
- $* \; X \sim N(5, 64)$



● special case :-

$X \sim N(0,1)$; $f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

↗

called standard normal R.V.

16/10/2017

Distribution function for the gaussian

- for $X \dot{\sim} N(0,1)$, $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} d\xi$
- for $X \sim N(a_x, \sigma_x^2)$, $F_x(x) \longrightarrow = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi \sigma_x^2}} e^{-\frac{(\xi - a_x)^2}{2\sigma_x^2}} d\xi$

gaussian (Normal) R.V :-

$X \sim N(a_x, \sigma_x^2)$

$\searrow f_x(x) = \frac{1}{\sqrt{2\pi \sigma_x^2}} e^{\frac{-(x\xi - a_x)^2}{2\sigma_x^2}}$

special case :-

$\underbrace{X \sim N(0,1)}$

"standard normal"

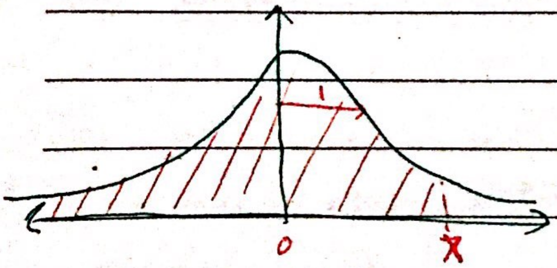
Distrubution Function :-

Case 1 : $X \sim N(0,1) \longrightarrow F(x)$

Case 2 : $X \sim N(a_x, \sigma_x^2) \longrightarrow F_x(x)$

$X \sim N(0,1)$

$F(x) = p(X \le x) = \int_{-\infty}^{x} N(0,1) d\xi$



Ex:- given $X \sim N(0,1)$ Find.

ⓐ $p(X<1.52) = F(1.52) = 0.9357$ → From table

ⓑ $p(1.2<X<3.1) = F(3.1) - F(1.2) = 0.999 - 0.8849 = ...$

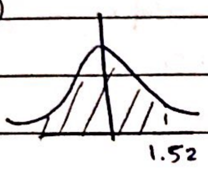
ⓒ $p(X \le -2.31) = 1 - F(2.31) = 1 - 0.9896$

ⓓ $p(-2<X<2.4) = F(2.4) - (1 - F(2)) = ...$

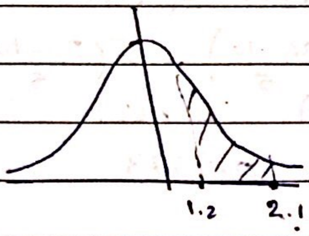
ⓔ $p(-3.2<X<-0.6) = F(-0.6) - F(-3.2) = 1 - F(0.6) - (1 - F(3.2))$

$= F(3.2) - F(0.6)$

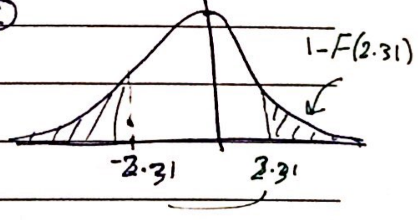
ⓐ



ⓑ



ⓒ



same area

$\boxed{F(-x) = 1 - F(x)}$

ⓔ ⇒ $p(-x_1 < \overset{N(0,1)}{X} < -x_2) = F(x_1) - F(x_2)$

Case ※2 :- $X \sim N(a_x, \sigma_x^2)$

$$F_X(x) = p(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(\xi - a_x)^2}{2\sigma_x^2}} d\xi$$

By substitution :- let $\frac{\xi - a_x}{\sigma_x} = u$

$\xi = -\infty$, $u = -\infty$

$\xi = x$, $u = \frac{x - a_x}{\sigma_x}$

$\frac{1}{\sigma_x} d\xi = du$

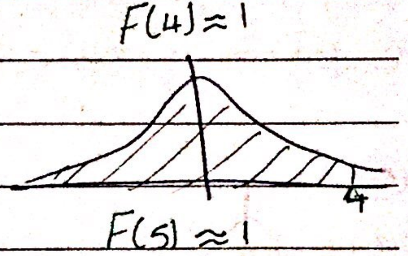
$$F_X(x) = \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-u^2/2} \sigma_x \, du$$

$$= \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-u^2/2}}_{N(0,1)} du = F\left(\frac{x - a_x}{\sigma_x}\right)$$

- $X \sim N(a_x, \sigma_x^2)$

$$F_x(x) = F\left(\frac{x - a_x}{\sigma_x}\right)$$

Ex :- $X \sim N(10, 25)$



ⓐ $p(20 < X \leq 35) = F_x(35) - F_x(20)$

$= F\left(\frac{35-10}{5}\right) - F\left(\frac{20-10}{5}\right)$

$= F(5) - F(2)$

$= 1 - 0.9773$

ⓑ $p(5 < X \leq 15) = F_x(15) - F_x(5) = F\left(\frac{15-10}{5}\right) - F\left(\frac{5-10}{5}\right)$

$= F(1) - F(-1)$

$= 2F(1) - 1$

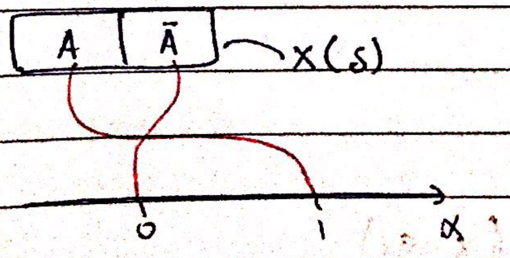
$\boxed{4}$ Bernouli R.V :- $\sim$ Bernouli (p)

Comes from Bernouli Trail : A : success, $p(A) = p$

$\bar{A}$: fail , $p(\bar{A}) = 1 - p$

$S = \{A, \bar{A}\}$

$$X = \begin{cases} 1, & \text{if the output is } A \\ 0, & \text{'' '' '' '' } \bar{A} \end{cases}$$



$X = \{\overset{x_1}{0}, \overset{x_2}{1}\}$

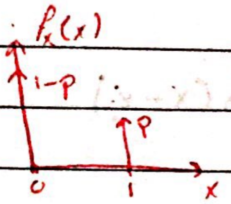
$p(X=0) = 1-p$

$p(X=1) = p$

$$f_X(x) = \sum_{i=1}^{2} p(X=x_i)\,\delta(x-x_i)$$

$$= p(X=0)\,\delta(x) + p(X=1)\,\delta(x-1)$$

$$\boxed{f_X(x) = (1-p)\,\delta(x) + p\,\delta(x-1)}$$



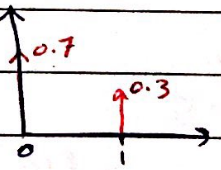
Ex :- $X \sim$ Bernouli (0.3)

Sol: $f_X(x) = 0.7\,\delta(x) + 0.3\,\delta(x-1)$

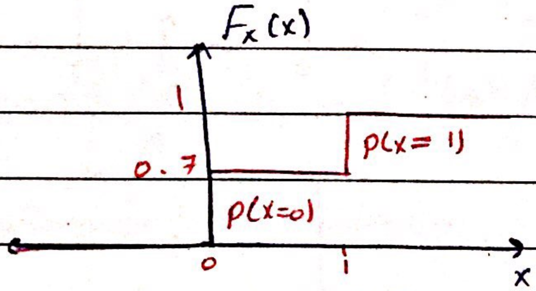
$X = \{0,1\}$

$p(X=1) = 0.3$

$p(X=0) = 0.7$



$F_X(x) = 0.7\,u(x) + 0.3\,u(x-1)$

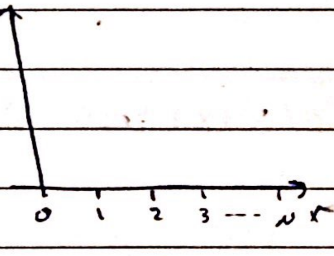


$\boxed{5}$ Binomial R.V :- $\rightarrow$ Binomial (P, N)

$\rightarrow p(\{A\}) = p$

X : "The number of success in N Bernouli trails"

$$X = \{\overset{x_1}{0}, \overset{x_2}{1}, \overset{x_3}{2}, \ldots, \overset{x_{N+1}}{N}\}$$



$$p(X=x_i) = \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i}$$

$$f_X(x) = \sum_{i=1}^{N+1} p(X=x_i)\, \delta(x-x_i)$$

$$= \sum_{i=1}^{N+1} \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i}\, \delta(x-x_i)$$

Ex :- Binomial $(0.6, 3)$ ※H.w

$X = \{0, 1, 2, 3\}$

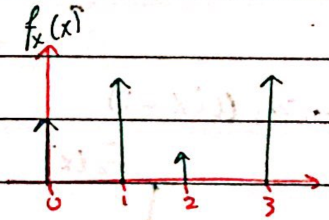
$$p(x=x_i) = \binom{3}{x_i} 0.6^{x_i}\; 0.4^{3-x_i}.$$

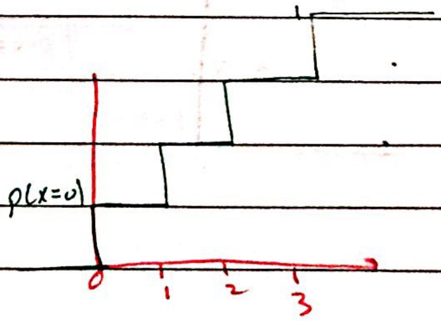
$p(x=0) =$

$p(x=1) =$

$p(x=2) =$

$p(x=3) =$





Ex: Def : $X =$ " # of flipping a coin until we get a head "
↳ geometrical (p)

$X = \{1, 2, 3, 4, ----\}$

$p(x=1) = p(\{H\}) = p$
$p(x=2) = p(\{TH\}) = (1-p).p$
$p(x=3) = (1-p)^2 p$
$p(x=4) = (1-p)^3 p$
$\vdots$
$p(x=x_i) = (1-p)^{x_i-1} p$

$$f_X(x) = \sum_{x_i=1}^{\infty} (1-p)^{x_i-1} p \, \delta(x - x_i)$$

● Conditional Distribution :-

$$\begin{tikzcd} & f_X(x) \\ X \arrow[ur] \arrow[dr] & \\ & F_X(x) = p(\underbrace{X \le x}_{\text{event}}) \end{tikzcd}$$

~~p(X~~ $F_X(x/B) = p(\underbrace{X \le x}_{A} / B) = \dfrac{p(X \le x \cap B)}{p(B)}$

↑ Conditional distribution function

$$f_X(x/B) = \frac{dF_X(x/B)}{dx}$$

Ex:-

R	G	B
5	35	60

Box 1

total = 100

R	G	B
80	60	10

Box 2

total = 150

Experiment :- Randomly select a box, and then draw out a ball from the selected box.

Define R.V $X = \begin{cases} 1, & \text{the ball is R} \\ 2, & \text{the ball is G} \\ 3, & \text{the ball is B} \end{cases}$

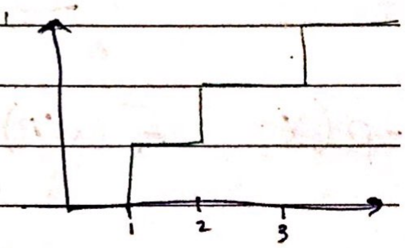
B_1 = the selected box is B_1

B_2 = the Selected box is B_2

Sol:-

ⓐ $F_X(x/B_1) = p(X=1/B_1)\,u(x-1) + p(X=2/B_1)\,u(x-2) + p(x=3/B_1)\,u(x-3)$

$$= \frac{5}{100}u(x-1) + \frac{35}{100}u(x-2) + \frac{60}{100}u(x-3)$$



ⓑ $F_X(x/B_2) = p(X=1/B_2)u(x-1) + p(X=2/B_2)u(x-2) + p(X=3)/B_2)u(x-3)$

$$= \frac{80}{150}u(x-1) + \frac{60}{150}u(x-2) + \frac{10}{150}u(x-3)$$

ⓒ $F_X(x) = p(X=1)\,u(x-1) + p(X=2)\,u(x-2) + p(X=3)\,u(x-3)$

$p(X=1) = p((R\cap B_1) \cup (R\cap B_2)) = p(R\cap B_1) + p(R\cap B_2)$

$= p(R/B_1)p(B_1) + p(R/B_2)p(B_2)$

$= \frac{5}{100} \cdot 0.5 + \frac{80}{150}\, 0.5 = \ldots$

← عشان وجود أكثر من
اتنين [illegible]
total probability

18/10/2017

Conditional Distribution and Density Functions :-

Recall: $X \rightarrow F_X(x) = p(\boxed{X \le x}) \rightarrow$ "event A"

$\quad\;\; \rightarrow f_X(x) = \dfrac{dF_X(x)}{dx}$

- $F_X(x/B) = p(\underbrace{X \le x}_{\text{"A"}} / B)$

  ↑ Conditional Distribution function of X

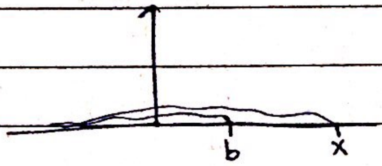
- $f_X(x/B) = \dfrac{dF_X(x/B)}{dx}$

  ↑ Conditional density function

R.V X Determine $F_X(x/B)$ where $B = \{X \le b\}$ ← "constant"

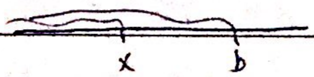
$$F_X(x/B) = p(X \le x / B) = p(X \le x / X \le b) = \frac{p(X \le x \wedge X \le b)}{p(X \le b)}$$

Case [1] :- $x > b$



$$= \begin{cases} \dfrac{p(X \le b)}{p(X \le b)} = 1 , & x \geq b \\ \dfrac{p(X \le x)}{p(X \le b)} , & x < b \end{cases}$$

Case [2] :- $x < b$

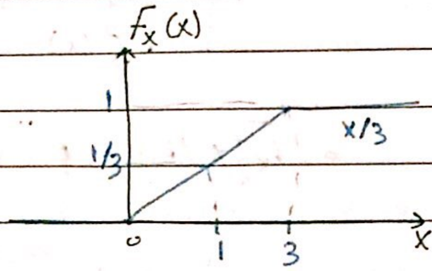


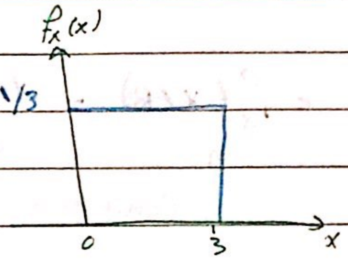
$$F_X(x/B) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} , & x < b \\ 1 , & b < x \end{cases}$$

* for $x < b$: $F_X(x/B) > F_X(x)$

Ex 8- $X \sim u(\overset{a}{0},\overset{b}{3})$ Find $F_x(x / x \leq 1)$

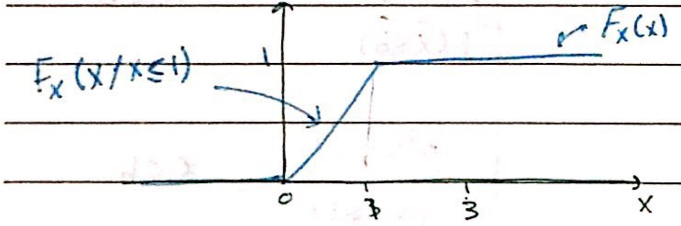
Sol: ~~$F_x(X \leq x)$~~ $F_X(x / x \leq 1) = \begin{cases} \dfrac{F_x(x)}{F_x(b)} & , x \leq 1 \\ 1 & , x > 1 \end{cases}$





$F_x(1) = \frac{1}{3}$

$$F_x(x / x \leq 1) = \begin{cases} 0 & , x < 0 \\ \dfrac{x/3}{1/3} = x & , 0 < x < 1 \\ 1 & , x > 1 \end{cases}$$



cHapter 3

① operations on one R.V :

① expectation (mean value / average value)

$E[X] \doteq \overline{X}$ denotes the mean value of R.V x

e.g 40 students

# of students	grades /50
4	15
10	20
4	31
20	38
2	49

Find the average grade !

$$\overline{G} = \frac{15\times4 + 20\times10 + 31\times4 + 38\times20 + 49\times2}{40}$$

$$= \frac{15\times4}{40} + \frac{20\times10}{40} + \frac{31\times4}{40} + \frac{38\times20}{40} + \frac{49\times2}{40}$$

$p(G=15)$ $p(G=20)$

as Discrete R.V $\curvearrowright$ $G = \{15, 20, 31, 38, 49\}$

$$p(G=15) = \frac{4}{40}$$

① for Discrete R.V $X = \{x_1, x_2, \dots, x_N\}$

$$E[X] = \sum_{i=1}^{N} x_i \, p(X = x_i)$$

Ex: exp : $S = \{1, 2, 3, 4\}$ $p(1) = 0.2$ $p(3) = 0.1$

$p(2) = 0.4$

Define

~~Find~~ R.V ; $Y = S^2 - 1$

Find $E[Y]$?!

$Y = \{0, 3, 8, 15\}$

$p(Y=0) = 0.2$ $p(Y=8) = 0.1$

$p(Y=3) = 0.4$ $p(Y=15) = 0.3$

$$E[y] = \sum_{i=1}^{4} y_i\, p(y = y_i)$$

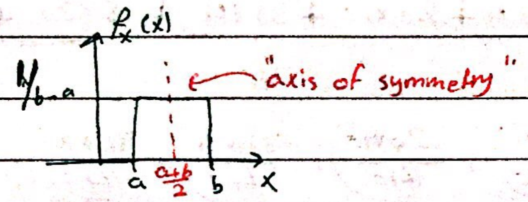
$$= (0)(0.2) + (3)(0.4) + (8)(0.1) + (15)(0.3) = 6.5$$

⑩ Expection for contineous R.V :-

$$\boxed{E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx}$$

Ex1 :- $X \sim u(a,b)$. Find $\bar{X}$?

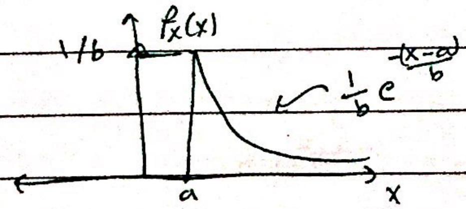
Sol :-



$$E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \frac{x^2}{(b-a)^2}\Big|_a^b .$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \boxed{\frac{b+a}{2}}$$

Ex2 : $X \sim \exp(a,b)$. Find $\bar{x}$?



$$E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = \int_a^{\infty} x\, \frac{1}{b}\, e^{\frac{-(x-a)}{b}}\, dx = \frac{e^{a/b}}{b} \int \underbrace{x}_{u} \underbrace{e^{-x/b}\, dx}_{dv}$$

By parts : let $u = x \longrightarrow du = dx$

$dv = e^{-x/b} \longrightarrow V = \frac{e^{-x/b}}{-1/b}$

$$= \frac{e^{a/b}}{b}\left[\underbrace{x\,\frac{e^{-x/b}}{-1/b}}_{\frac{\infty}{\infty} \Rightarrow L.p}\Bigg|_{\infty}^{\infty} - \int_{\infty}^{\infty} \frac{e^{-x/b}}{-1/b}\, dx \right]$$

$$= \frac{e^{a/b}}{b}\left[0 + ab\,e^{-a/b} - b^2 e^{-x/b}\Big|_{a}^{\infty} \right]$$

$$= \frac{e^{a/b}}{b}\left[ab\,e^{-a/b} + b^2 e^{-a/b} \right]$$

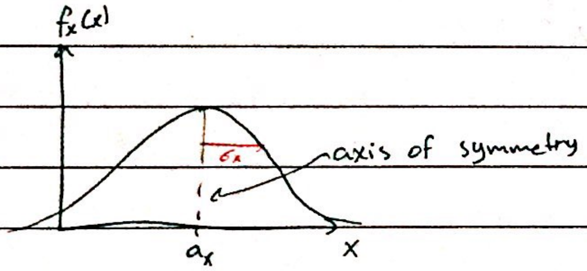
$$= a + b \qquad \Rightarrow \quad X \sim \exp(a, b)$$

$$\overline{X} = a + b$$

Ex : $X_1 \sim U(-1,3) \rightarrow \overline{X} = 1$

$X_2 \sim \exp(1,4) \rightarrow \overline{X} = 5$

Ex: $X \sim N(a_x, \sigma_x^2)$. Find $\overline{X}$?



$$E[x] = a_x$$

@analytically : $E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}\, dx$

let $u = \frac{x - a_x}{\sigma_x}$ $\qquad du = \frac{1}{\sigma_x} dx$

$$E[x] = \int_{-\infty}^{\infty} (\sigma_x u + a_x) \frac{1}{\sqrt{2\pi \cancel{\sigma_x^2}}}\, e^{-u^2/2}\, \cancel{\sigma_x}\, du$$

Five Apple

Scanned by CamScanner

odd × even = odd

$$= \int_{-\infty}^{\infty} a_x \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du + \int_{-\infty}^{\infty} \frac{\sigma_x}{\sqrt{2\pi}}\, u\, e^{-u^2/2}\, du$$

$$= (a_x)(1) + 0 = a_x$$

p(success) = A

Ex : $X \sim$ Bernoulli (p)

Find $\overline{X}$?

Sol :- $X = \{0, 1\}$

$p(X=0) = 1-p$

$p(X=1) = p$

$$\overline{X} = \sum_{i=1}^{2} x_i\, p(x = x_i) = (0)(1-p) + (1)(p) = p$$

Ex : $X \sim$ Binomial (p, N). Find $\overline{X}$?

Sol : $X = \{0, 1, 2, \ldots, N\}$

$$p(X=i) = \binom{N}{i} p^i (1-p)^{N-i}$$

$i = 0, 1, 2, —, N$

$$\overline{X} = \sum_{i=1}^{N} i\, p(X=i) = \sum_{i=1}^{N} i \cdot \binom{N}{i} p^i (1-p)^{N-i}$$

Zero لا

مش موجود

بالSummation

$$= \sum_{i=1}^{N} i \frac{\overset{N(N-1)!}{N!}}{i!\,(N-i)!}\, p\, p^{i-1} (1-p)^{N-i}$$

$$\frac{i}{i!} = \frac{1}{(i-1)!}$$

$$= Np \sum_{i=1}^{N} \frac{(N-1)!}{(i-1)!\,(N-i)!}\, p^{i-1} (1-p)^{N-i}$$

let $k = i-1$, $i = k+1$

$$= Np \sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} p^{k} (1-p)^{N-k-1}$$

let $N-1 = M$

$$= Np \sum_{k=0}^{M} \frac{M!}{k!\,(M-k)!} p^{k} (1-p)^{M-k}$$

$$= Np \sum_{k=0}^{M} \underbrace{\binom{M}{k} p^{k} (1-p)^{M-k}}_{P(\underset{\substack{\uparrow \\ \text{Binomial}(M,p)}}{X} = k)}$$

$$Np(1) = Np$$

Ex: $X \sim \text{geometrical}(p)$ $\boxed{\text{H.W}}$

$X = \{1, 2, 3, \ldots \infty\}$

$P(X=i) = (1-p)^{i-1} p$

$\bar{X} = \sum_{i=1}^{\infty} i\,(1-p)^{i-1}\, p = - - -$

23/10/2017

R.V : $\bar{x} = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$

Contineous →

$\bar{x} = \sum_{i=1}^{N} x_i\, p(X=x_i)$

Discrete →

$X \sim \mathrm{Eu}(a,b) \longrightarrow \bar{x} = \frac{a+b}{2}$

$X \sim \exp(a,b) \longrightarrow \bar{x} = a+b$

$X \sim N(a_x, \sigma_x^2) \longrightarrow \bar{x} = a_x$

$X \sim \text{Bernouli}(p) \longrightarrow \bar{x} = p$

$X \sim \text{Binomial}(p, N) \longrightarrow \bar{x} = Np$

⑩ Expectation of function of R.V :-

X is R.V

g(x) is function of x, then:

- $E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx$
- $E[g(x)] = \sum_{i=1}^{N} g(x_i)\, p(X=x_i)$

Ex :- $X = \{-1, 2, 5, 9\}$, $p(X=-1) = 0.1$, $p(X=2) = 0.6$

$p(X=5) = 0.15$, $p(X=9) = 0.15$

Find the mean of $g(x) = x^2 - 1$

Sol :- $g(x) = \{\underset{\underset{g(-1)}{\nearrow}}{0}, \underset{\underset{g(2)}{\updownarrow}}{3}, \underset{\underset{g(5)}{\uparrow}}{24}, \underset{\underset{g(9)}{\nwarrow}}{80}\}$

$\bar{g} = E[g(x)] = \sum_{i=1}^{4} g(x_i)\, p(X=x_i) = (0)(0.1) + (3)(0.6) + (24)(0.15) + (80)(0.15) = --$

Ex:- let R.V x with $f_x(x)$. Define $g(x) = aX + b$, where a and b are real Constants. Determine $E[g(x)]$

Sol:-

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx = \int_{-\infty}^{\infty} (ax+b)\, f_x(x)\, dx$$

$$= a\int_{-\infty}^{\infty} x\, f_x(x)\, dx + b\int_{-\infty}^{\infty} f_x(x)\, dx$$

$$= a\,E[x] + b$$

$$\boxed{E[ax+b] = a\,E[x] + b}$$

Consequence :-

- $E[ax] = a\,E[x]$
- $E[\underset{\uparrow\ \text{Constant}}{b}] = b$

e.g let x a R.V with mean of -2.

ⓐ $E[4x-1] = ((4)\times(-2)) - 1 = -9$

ⓑ $E[1/2 + 3x] = \frac{1}{2} + (3)(-2) = -5.5$

practice :- $x \sim u(\underset{a}{-1}, \underset{b}{4})$ Find $E[\underbrace{x^2-1}_{g(x)}] = E[x^2] - 1$

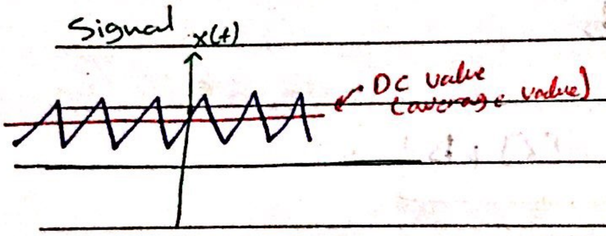
⊛ Moments about the origion (Moments):-

$$m_n = E[\underset{g(x)}{X^n}] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

$n = 0, 1, 2, \dots$

- Zeroth-moment : $m_0 = E[X^0] = 1$
- first-moment : $m_1 = E[X]$

Signal



$m_1 = E[X]$

expectation

average value

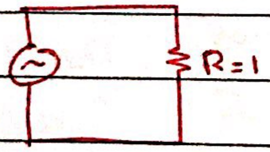
mean value

Dc-value

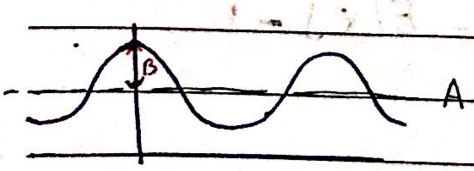
m_1^2 = Dc-avg-power

- Second moment : $\underbrace{m_2 = E[X^2]}$

total-average power in X
= AC power + Dc-power



$v(t) = A + B\cos(w_c t)$



Find total-averge-power

$p(t) = v(t)\,i(t) = \frac{v^2(t)}{R} = v^2(t)$

$\bar{p} = \frac{1}{T}\int p(t)\,dt$

$= \frac{1}{T}\int A^2 + B^2\cos^2(w_c t) + 2AB\cos(w_c t)\,dt$

$$= A^2 + \frac{B^2}{2} + 0 = \underbrace{A^2}_{\text{Dc-avg power}} + \underbrace{\frac{B^2}{2}}_{\text{Ac-average power}}$$

$$\text{Ac-power} = \text{total-avg-power} - \text{Dc-avg-power}$$
$$= E[x^2] - E^2[x] = \boxed{m_2 - m_1^2}$$

Variance of R.V X :-

$$Var(x) \triangleq \sigma_x^2 = m_2 - m_1^2 = E[x^2] - E^2[x] \triangleq \text{Ac-avg-power}$$

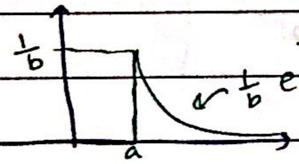
Ex:- $X \sim exp(a,b)$

ⓐ Find m_1 ⓑ Find m_2

ⓒ Find $Var(x)$

ⓐ $m_1 = E[x] = a+b$

ⓑ $m_2 = E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx$



$$m_2 = \int_a^{\infty} x^2 \cdot \frac{1}{b} e^{\frac{-(x-a)}{b}} dx = \frac{1}{b} e^{a/b} \int_a^{\infty} x^2 e^{-x/b} dx = \underset{\text{by parts twice}}{\underline{\qquad\qquad}} = (a+b)^2 + b^2$$

ⓒ $Var(x) = m_2 - m_1^2 = (a+b)^2 + b^2 - (a+b)^2 = b^2$

Ex 3- $X \sim u(a,b)$

ⓐ m_0 ⓑ m_1 ⓒ m_2 ⓓ $Var(x)$

Sol: ⓐ $m_0 = 1$

ⓑ $m_1 = \bar{x} = \frac{a+b}{2}$

ⓒ $m_2 = E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$

$$\int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^3}{3}\Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+2ab+a^2)}{3(b-a)}$$

$$= \boxed{\frac{b^2+ab+a^2}{3}}$$

ⓓ $Var(x) = m_2 - m_1^2 = \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$

$$= \frac{4b^2+4ab+4a^2+3a^2 - 6ab - 3b^2}{12}$$

$$\frac{b^2+a^2-2ab}{12} = \boxed{\frac{(b-a)^2}{12}}$$

① Moments about the mean : (central moments).

$$\mu_n = E[(x-\bar{x})^n] = \int_{-\infty}^{\infty} (x-\bar{x})^n f_x(x)\,dx$$

$n = 0, 1, 2, \dots$

- $\mu_0 = 1$
- $\mu_1 = E[x - \bar{x}] = E[x] - \bar{x} \rightsquigarrow$ constant $= 0$
- $\mu_2 = E[(x-\bar{x})^2] = Var(x)$

$$\mu_2 = E[(x-\bar{x})^2] = E[x^2 - 2\bar{x}x + \bar{x}^2] = E[x^2] - E[\textcircled{2\bar{x}}x] + E[\bar{x}^2]$$

$$= E[x^2] - 2\bar{x}^2 + \bar{x}^2$$

$$= E[x^2] - \bar{x}^2$$

$$= m_2 - m_1^2 = Var(x)$$

$$\boxed{\mu_2 = m_2 - m_1^2 = Var(x)}$$

Ex 3- $X \sim N(a_x, \sigma_x^2)$

① $m_1 = a_x$ ② $m_2 = E[X^2] = \int x^2 f_x(x)\,dx = \sigma_x^2 + a_x^2$.

③ $Var(x) = m_2 - m_1^2 = \sigma_x^2 + a_x^2 - a_x^2 = \sigma_x^2$

⑪ characteristic Function :-

$$\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx$$

$$\boxed{f_x(x) = \frac{1}{2\pi}\int \phi_x(\omega)\, e^{-j\omega x}\, d\omega}$$

$$m_n = (-j)^n \left.\frac{d^n \phi_x(\omega)}{d\omega^n}\right|_{\omega=0}$$

Ex :- $X \sim \exp(a,b)$

ⓐ $\phi_x(\omega)$ ⓑ m_1, m_2 using part ⓐ

sol:- $$\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx = \int_{a}^{\infty} e^{j\omega x}\, \frac{1}{b}\, e^{-\frac{(x-a)}{b}}\, dx$$

$$= \frac{e^{a/b}}{b}\int_a^{\infty} e^{j\omega x}\, e^{-x/b}\, dx$$

$$= \frac{e^{a/b}}{b}\int_a^{\infty} e^{-(\frac{1}{b}-j\omega)x}\, dx$$

$$= \frac{e^{a/b}}{b}\left(\left.\frac{e^{-(\frac{1}{b}-j\omega)x}}{-(\frac{1}{b}-j\omega)}\right|_a^{\infty}\right)$$

$$= \frac{e^{a/b}}{b}\left(\frac{e^{-(\frac{1}{b}-j\omega)a}}{(\frac{1}{b}-j\omega)}\right) = \boxed{\frac{e^{j\omega a}}{1-j\omega b}}$$

$$m_1 = (-j)^1 \frac{d\,\phi_x(\omega)}{d\omega}\Big|_{\omega=0}$$

$$= -j \left[\frac{(1-j\omega b)(ja)e^{j\omega a} - e^{j\omega a}(-jb)}{(1-j\omega b)^2} \right]\Bigg|_{\omega=0}$$

$$= -j \left[\frac{ja + jb}{1} \right] = \boxed{a+b}$$ same as before

ⓒ $$m_2 = (-j)^2 \frac{d^2\phi_x(\omega)}{d\omega^2}\Big|_{\omega=0}$$

~~$m_2 =$~~

$$m_2 = (a+b)^2 + b^2$$

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$$Q_X(\omega) = E\left[e^{j\omega x}\right] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx$$

$$m_n = (-j)^n \frac{d^n Q_x(\omega)}{d\omega^n}$$

◎ Moment Generating Function:

- $M_X(\nu) = E\left[e^{\nu X}\right] = \int_{-\infty}^{\infty} e^{\nu x} f_x(x)\,dx$

- $m_n = \left.\frac{d^n M_x(\nu)}{d\nu^n}\right|_{\nu=0}$

Ex: $X \sim \exp(a,b)$

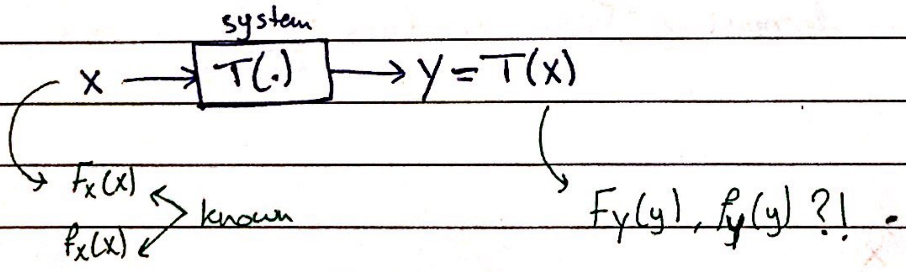
- $Q_x(\omega) = \frac{e^{j\omega a}}{1-j\omega b}$

- $M_x(\nu) = \frac{e^{\nu a}}{1-b\nu}$

$$m_1 = E[X] = \left.\frac{(1-b\nu)\,a\,e^{\nu a} - e^{\nu a}(-b)}{(1-b\nu)^2}\right|_{\nu=0}$$

$$= a + b$$

◎ Transformation of one R.V :-

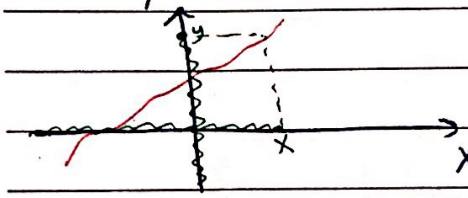


Transformation

Transformation → monotonic → increasing
Transformation → monotonic → decreasing
Transformation → non-monotonic

Case 1 : Monotonic Increasing Transformation

$y = T(x)$



$$F_y(y) = p(Y \le y) = p(X \le \overset{T^{-1}(y)}{x}) = F_x(T^{-1}(y))$$

$$\boxed{F_y(y) = F_x(T^{-1}(y))}$$

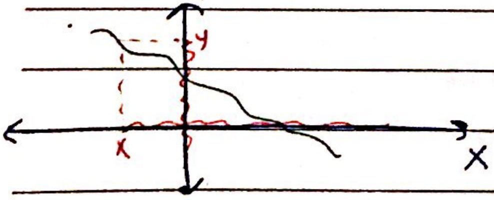
$$f_y(y) = \frac{d\,F_Y(y)}{dy} = \frac{d\,F_x(T^{-1}(y))}{dy} \overset{\text{by chain rule}}{=} \frac{d\,F_x(T^{-1}(y))}{dT^{-1}} \frac{dT^{-1}(y)}{dy}$$

$$= \boxed{f_x (T^{-1}(y) \frac{dT^{-1}(y)}{dy}}$$

$\rightarrow \frac{dT^{-1}(y)}{dy}$ (+ve)

Case 2 :- Monotonic decreasing Transformation :

$Y = T(x)$



$$F_y(y) = p(Y \le y) = p(X \ge \underset{T^{-1}(y)}{x}) = \boxed{1 - F_x(T^{-1}(y))}$$

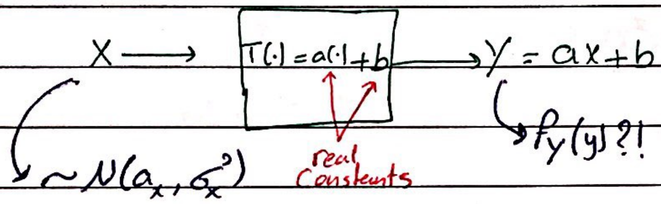
$$f_y(y) = \frac{d F_y(y)}{dy} = \frac{d[1 - F_x(T^{-1}(y))]}{dy}$$

$$= \frac{-d F_x(T^{-1}(y))}{dy} = - f_x(T^{-1}(y)) \frac{dT^{-1}(y)}{dy} \quad -ve$$

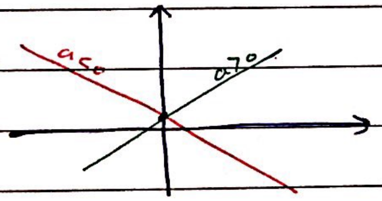
⊗ for monotonic Transformation :-

$$f_y(y) = f_x(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|$$

Ex :-



Sol :-



we have monotonic transformation

- $f_y(y) = f_x(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|$ $\qquad T^{-1}(y) = \frac{y-b}{a}$
- $f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}$

$$\frac{dT^{-1}(y)}{dy} = 1/a$$

$$f_y(y) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(\frac{y-b}{a} - \frac{a}{a}a_x)^2}{2\sigma_x^2}} \cdot \frac{1}{|a|} \boxed{= \frac{1}{\sqrt{2\pi a^2\sigma_x^2}} e^{\frac{-(y-(aa_x+b))^2}{2a^2\sigma_x^2}} = f_y(y)}$$

$$Y \sim N(\underbrace{aa_x+b}_{a_y}, \underbrace{a^2\sigma_x^2}_{\sigma_y^2})$$

$Y = ax + b$

- $E[y] = E[ax+b] = a\bar{x} + b = a a_x + b$
- $Var(x) = Var(ax) = a^2 \, var(x) = a^2 \sigma_x^{\;2}$

The ~~linear~~ Transformation of gaussian is gaussian

$Var(ax) = E[a^2x^2] - E^2[ax]$

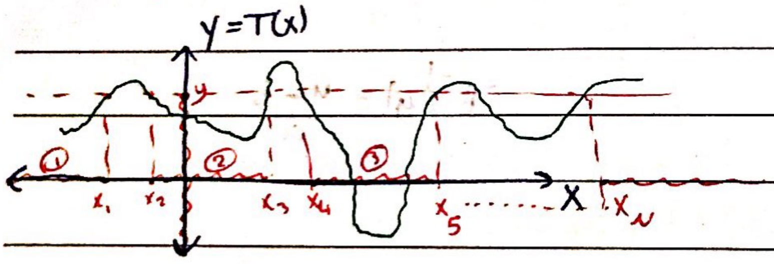
$= a^2 E[x^2] - a^2 E^2[x]$

$= a^2 \, var(x)$

$var(b) = E[b^2] - E^2[b]$

$= b^2 - b^2 = 0$

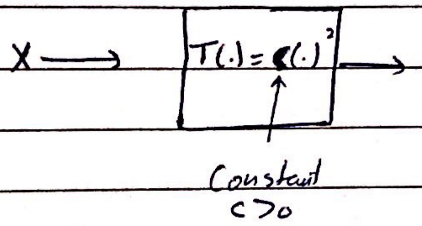
Case 3 :- Non-monotonic Transformation



$$\boxed{f_Y(y) = \sum_n \frac{f_X(x_n)}{\left| \frac{dT(x)}{dx} \Big|_{X=x_n} \right|}}$$

$x_1, x_2, \ldots x_N$ are the roots for $Y - T(x) = 0$

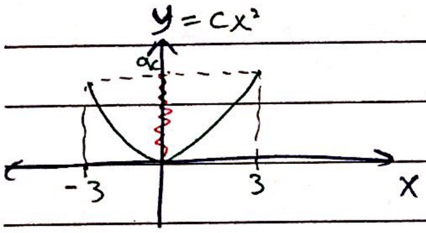
Ex : $X \sim u(-3,3)$



$Y = T(x) = cX^2$

Find $f_Y(y)$?!

Sol :-



$-3 < x < 3\,,\quad 0 < y < 9c$

step 1 :- Find the roots for $Y - T(x) = 0$

~~$\Rightarrow y = cx$~~

$\Rightarrow y - cx^2 = 0 \quad \Rightarrow y = cx^2$

$\Rightarrow x^2 = \frac{y}{c}$

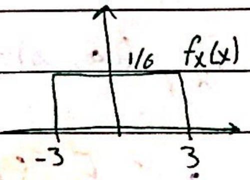
$\Rightarrow x = \pm\sqrt{y/c}$

$x_1 = \sqrt{y/c}$

$x_2 = -\sqrt{y/c}$

step 2 :-

$$\frac{dT(x)}{dx} = 2cx$$



$$f_y(y) = \frac{f_x(x_1)}{\left|\frac{dT(x)}{dx}\Big|_{x_1}\right|} + \frac{f_x(x_2)}{\left|\frac{dT(x)}{dx}\Big|_{x_2}\right|} = \frac{1/6}{|2c\sqrt{y/c}|} + \frac{1/6}{|-2c\sqrt{y/c}|}$$

$$= \frac{1/6 + 1/6}{2c\sqrt{y/c}} = \frac{1}{6c\sqrt{y/c}} = \frac{1}{6\sqrt{cy}}$$

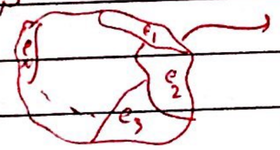
$0 < y < 9c$

Chapter 4

1/11/2017

⓪ Multiple R.V's :-

exp ξ



$X(s) = X$

$Y(s) = Y$

(X,Y) : 2D Random vector

ND - Random vector

$(X_1, X_2, \ldots, X_N)$

⓪ Joint Distribution Function :-

$$F_{X,Y}(x,y) = p(\underbrace{X\le x}_{\substack{\text{event}\\ A}} \underset{\text{and}, \text{"}\cap\text{"}}{,} \underbrace{Y\le y}_{\substack{\text{event}\\ B}})$$

properties :-

① $F_{X,Y}(-\infty,-\infty) = p(X\le -\infty, Y\le -\infty)$
$= p(\phi\cap\phi) = 0 = p(\phi)$

② $F_{X,Y}(-\infty,y) = p(X\le \infty , Y\le y) = p(\phi\cap B) = p(\phi) = 0$

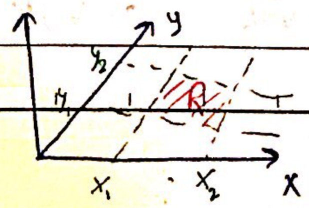
③ $F_{X,Y}(x,-\infty) = \ldots\ldots = 0$

④ $F_{X,Y}(\infty,\infty) = p(X\le\infty, Y\le\infty) = p(S\cap S) = p(S) = 1$

⑤ $0 \le F_{X,Y}(x,y) \le 1$

⑥ $F_{X,Y}(x,y)$ is non decreasing 2D- function.

⑦ $p(x_1 < X \le x_2 , y_1 < Y \le y_2) = p((X,Y)\in R)$



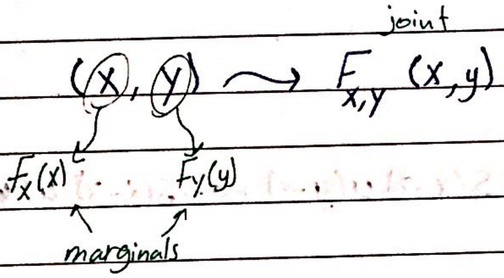
$= F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)$

marginal distribution functions :-

$F_{X,Y}(x, \infty) = p(X \leq x \ , Y \leq \infty) = p(X \leq x \cap S) = p(X \leq x) = F_X(x)$

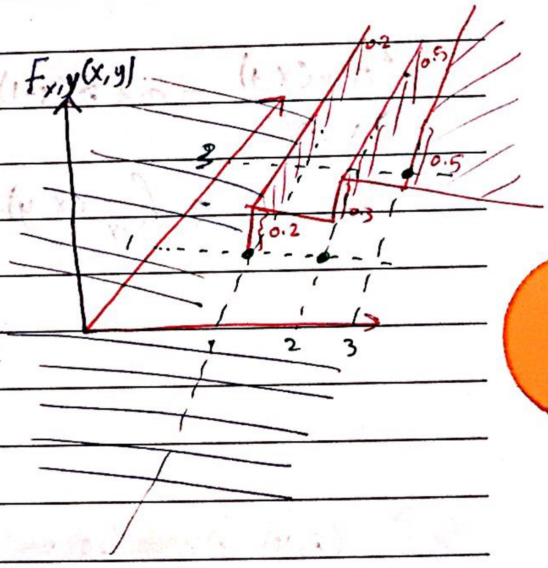
$\boxed{F_{X,Y}(x, \infty) = F_X(x)}$ ← morginal distribution function of X

$\boxed{F_{X,Y}(\infty, y) = F_Y(y)}$ → " " " " Y



Ex: 2D - Discrete R. vector

$(X,Y) = \{(1,1), (2,1), (3,3)\}$



$p((X,Y) = (1,1)) = 0.2$ ① $F_{X,Y}(x,y)$

$p((2,1)) = 0.3$ ② $F_X(x)$

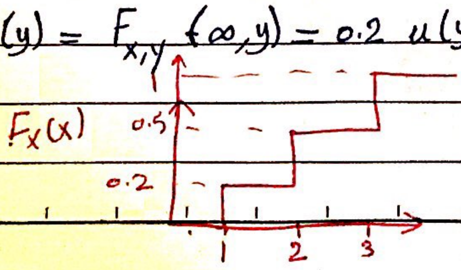
$p((3,3)) = 0.5$ ③ $F_Y(y)$

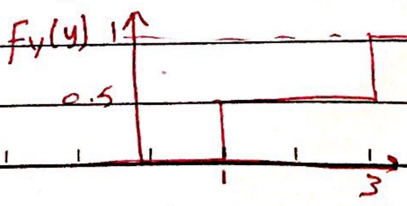
Sol :- $F_{X,Y}(x,y) = p(X \leq x \ , Y \leq y)$

① $F_{X,Y}(x,y) = 0.2\ u(x-1)\, u(y-1) + 0.3\ u(x-2)\, u(y-1) + 0.5\ u(x-3)\, u(y-3)$

② $F_X(x) = 0.2\, u(x-1) + 0.3\, u(x-2) + 0.5\ u(x-3)$

③ $F_Y(y) = F_{X,Y}(\infty, y) = 0.2\ u(y-1) + 0.3\ u(y-1) + 0.5\ u(y-3) = 0.5\, u(y-1) + 0.5\ u(y-3)$





$X = \{1, 2, 3\}$ $Y = \{1, 3\}$

① Joint Density function :-

$$(X\ ,\ Y) \longrightarrow F_{X,Y}(x,y)$$

$$\downarrow$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

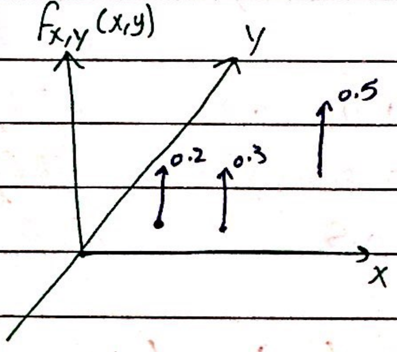
Ex :- for the previous example, Find $f_{X,Y}(x,y)$?

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

$$\frac{\partial F_{X,Y}(x,y)}{\partial x} = 0.2\,\delta(x-1)\,u(y-1) + 0.3\,\delta(x-2)\,u(y-1) + 0.5\,\delta(x-3)\,u(y-3)$$

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y} = 0.2\,\delta(x-1)\,\delta(y-1) + 0.3\,\delta(x-2)\,\delta(y-1) + 0.5\,\delta(x-3)\,\delta(y-3)$$

$$= f_{X,Y}(x,y)$$

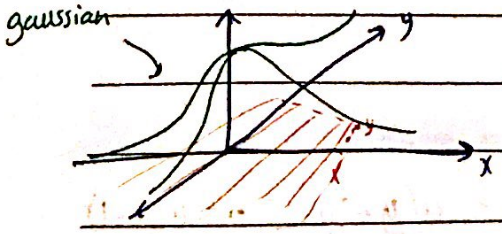


② $f_{X,Y}(x,y)$ properties :-

① $f_{X,Y}(x,y) \geq 0$

② $$\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy \overset{\circ}{=} 1}_{}$$

volume under $f_{X,Y}(x,y) = 1$



③ $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2$.

④ $F_X(x) = F_{X,Y}(x,\infty) = \frac{d}{dx}\int_{-\infty}^{\cancel{x}}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)\, d\underline{\underline{\xi_2}}\, d\xi_1$ it will be a function of ξ_1

⑤ $F_Y(y) = F_{X,Y}(\infty,y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)\, d\underline{\underline{\xi_1}}\, d\xi_2$ it will be a function of ξ_2

⑥ $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

⑦ $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$

marginal Density function

Ex: given $g(x,y) = \begin{cases} b e^{-x}\cos(y), & 0 \le x \le 2 \\ & 0 \le y \le \cancel{2}\,\pi/2 \\ \text{Zero}, & \text{o.w} \end{cases}$

Find the value of b such that g(x,y) is joint density function:

Sol: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy = 1 \Rightarrow \int_{0}^{\pi/2}\left(\int_{0}^{2} b e^{-x}\cos(y)\,dx\right) dy$

$$= \int_{0}^{\pi/2} \cos(y)\,\Big(\underbrace{\int_{0}^{2} b e^{-x}\,dx}_{\triangleq I}\Big)\,dy$$

$$I = \int_{0}^{2} b e^{-x}\,dx = b\,\frac{e^{-x}}{-1}\Big|_{0}^{2} = b(1 - e^{-2})$$

$$= \int_0^{\pi/2} \cos(y)\, b(1-e^{-2})\, dy \quad = b(1-e^{-2})\, \sin(y)\Big|_0^{\pi/2}$$

$$= b(1-e^{-2})$$

$$\Rightarrow b(1-e^{-2}) = 1 \quad \Rightarrow \boxed{b = \frac{1}{1-e^{-2}}}$$

Ex : $f_{X,Y}(x,y) = x\, e^{-x(y+1)}\, u(x)\, u(y)$ find ① $f_X(x)$ ② $f_Y(y)$

Sol : ① $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{-\infty}^{\infty} \underbrace{x\, e^{-x(y+1)}\, u(x)\, u(y)}\, dy$

$$= x\, u(x)\, e^{-x} \int_0^{\infty} e^{-xy}\, dy$$

$$= x e^{-x} u(x) \left[\frac{e^{-xy}}{-x} \Big|_0^{\infty} \right] = \boxed{e^{-x}\, u(x)}$$

② $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$

$$= u(y) \int_0^{\infty} x\, e^{-\overset{\text{Constant}}{(y+1)}x}\, dx = \underset{\substack{\text{by} \\ \text{parts} \\ \text{(practice)}}}{\cdots\cdots} = \frac{u(y)}{1+y^2}$$

6/11/2017

① statistical Independency :-

$$\bullet\ (X,Y) \begin{cases} \longrightarrow F_{X,Y}(x,y) \\ \longrightarrow f_{X,Y}(x,y) \end{cases}$$

$$F_{X,Y}(x,y) = p(\underbrace{X\le x}_{\text{event } A} \underset{\substack{\uparrow \\ \cap}}{,} \underbrace{Y\le y}_{\text{event } B})$$

If X and Y are independent ;

$$F_{X,Y}(x,y) = p(X\le x) \,.\, p(Y\le y)$$

$$\boxed{F_{X,Y}(x,y) = F_X(x) \,.\, F_Y(y)}$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y} = \frac{\partial^2 F_X(x).\, F_Y(y)}{\partial x\, \partial y} = \frac{\partial F_X(x)}{\partial x} \,.\, \frac{\partial F_Y(y)}{\partial y}$$

$$\boxed{f_{X,Y}(x,y) = f_X(x) .\, f_Y(y)}$$

• In general ; $X_1, X_2, \dots X_N$ are statistically independent, then

$$f(x_1, x_2, \dots, x_N) = f_{X_1}(x_1) \,.\, f_{X_2}(x_2) \dots f_{X_N}(x_N) = \prod_{i=1}^{N} f_{X_i}(x_i)$$

Ex : $f_{X,Y}(x,y) = x\, e^{-x(y+1)}\, u(x)\, u(y)$ are X and Y independent ?

first find $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = e^{-x} u(x)$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \frac{u(y)}{1+y^2}$$

$$f_Y(y) \neq f_{X,Y}(x,y)$$

So ; X and Y are not independent

Ex:- $f_{X,Y}(x,y) = \frac{1}{12} e^{-x/4 - y/3} u(x)\, u(y)$, are X and Y independent?

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{12} e^{-x/4 - y/3} u(x) u(y)\, dy$$

$$= \frac{1}{12} e^{-x/4} u(x) \int_0^{\infty} e^{-y/3} dy = \frac{1}{12} e^{-x/4} u(x) \underbrace{\left(\frac{e^{-y/3}}{-1/3} \Big|_0^{\infty} \right)}_{\to 3} = \boxed{\frac{1}{4} e^{-x/4} u(x)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{12} e^{-x/4 - y/3} u(x)\, u(y)\, dx = \frac{1}{12} e^{-y/3} u(y) \int_0^{\infty} e^{-x/4} dx = \frac{1}{3} e^{-y/3} u(y)$$

$$f_X(x) . f_Y(y) = \frac{1}{12} e^{-y/3} e^{-x/4} u(x)\, u(y) = f_{X,Y}(x,y)$$

∴ X and y are independent

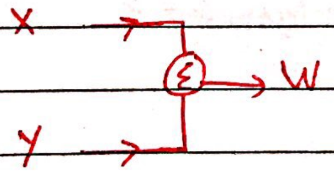
⑪ Density function for sum of two independent R.V's :-

$$\begin{tikzcd} & F_{X,Y}(x,y) \\ (X,Y) \arrow[ur] \arrow[dr] & \\ & f_{X,Y}(x,y) \end{tikzcd}$$

are independent

$f_X(x) . f_Y(y)$

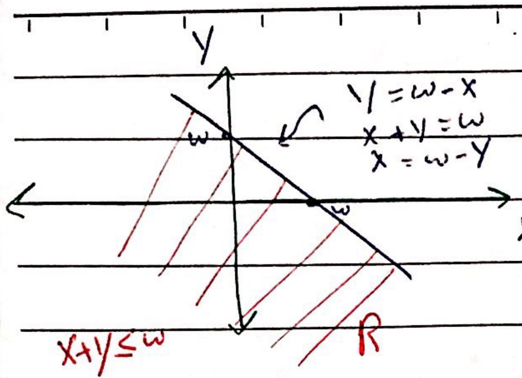
$W = X + Y$, find $f_W(w)$?!



$$F_W(w) = P(W \le w) = P(X + Y \le w)$$

$$X + Y = w \rightarrow Y = w - X$$

$$\rightarrow X = w - Y$$



$$p(X+Y \le w) = p((X,Y) \in R)$$

$$= \iint_R f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{-\infty}\int_{-\infty}^{w-y} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_X(x) \cdot f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)\,dx\,dy$$

$$f_W(w) = \frac{d F_W(w)}{dw}$$

$$= \frac{d\left(\int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)\,dx\,dy\right)}{dw}$$

$$= \int_{-\infty}^{\infty} f_Y(y)\,\frac{d\left(\int_{-\infty}^{w-y} f_X(x)\,dx\right)}{dw}\,dy$$

$$= f_X(w-y)\,\frac{d(w-y)}{dw} - f_X(-\infty)\,\frac{d(-\infty)}{dw} + \int_{-\infty}^{w-y} \frac{d f_X(x)}{dw}\,dx$$

$$= f_X(w-y) - 0 + 0 = f_X(w-y)$$

using Appendix G $\dfrac{d\int_{f_1}^{f_2(w)} g(x)\,dx}{dw}$

$$\boxed{\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\,dy \\ f_W(w) &= f_Y(y) * f_X(x) \end{aligned}}$$

Convolution

oR:

$$f_W(w) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dX$$

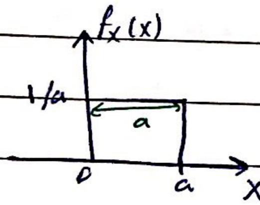
Ex: given $X \sim u(0,a)$

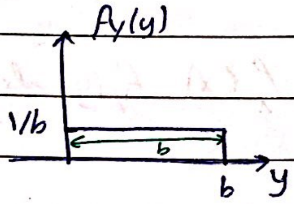
$Y \sim u(0,b)$

$b > a > 0$

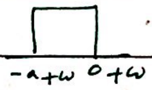
$W = X+Y$, find exact density function of W; $f_W(w)$

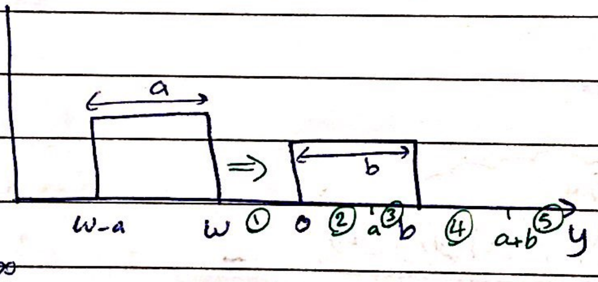
$$f_W(w) = f_X(x) * f_Y(y)$$



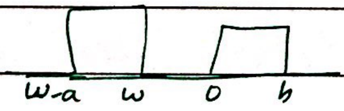


$$f_W(w) = f_Y(y) * f_X(x) = \int_{-\infty}^{\infty} f_Y(y)\cdot f_X(w-y)\, dy$$

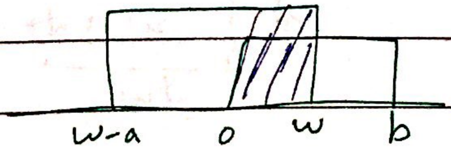




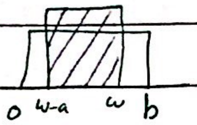
① $w<0$, $f_W(w) = \int_{-\infty}^{\infty} 0\, dy = \boxed{0}$



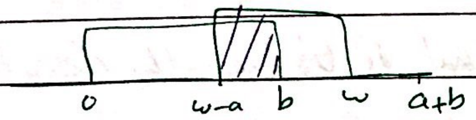
② $0<w<a$ $f_W(w) = \int_{0}^{w} \frac{1}{ab}\, dy = \boxed{\frac{w}{ab}}$



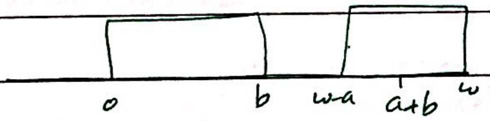
③ $a < w < b$ $\quad f_W(w) = \int_{w-a}^{w} 1/ab \, dy = \boxed{1/b}$



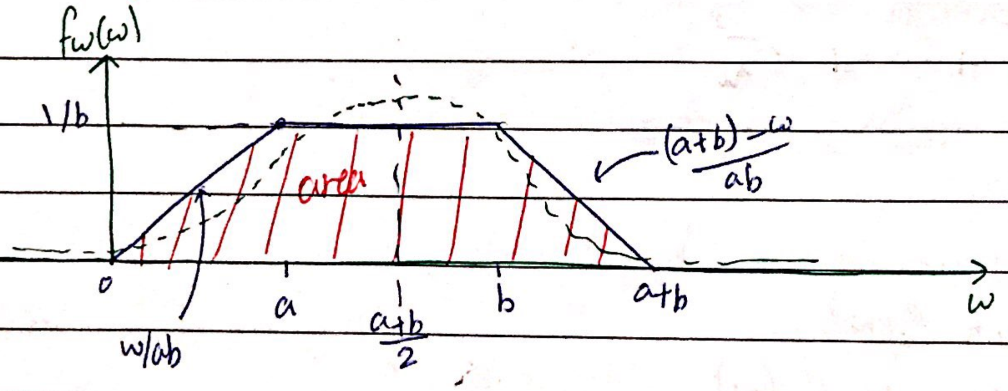
④ $b < w < a+b$ $\quad f_W(w) = \int_{w-a}^{b} 1/ab \, dy = \boxed{\frac{(a+b)-w}{ab}}$



⑤ $a+b < w$ $\quad f_W(w) = \int 0 \, dy = \boxed{0}$



$$f_W(w) = \begin{cases} 0 & , \ w < 0 \\ w/ab & , \ 0 < w < a \\ 1/b & , \ a < w < b \\ \frac{(a+b)-w}{ab} & , \ b < w < a+b \\ 0 & , \ w > a+b \end{cases}$$



$\int_{-\infty}^{\infty} f_W(w) \, dw = \text{area} = 1$

$E[w] = \int_{-\infty}^{\infty} w f_w(w)\, dw = \cdots = \boxed{\frac{a+b}{2}}$

or

$E[w] = \frac{a+b}{2}$ (axis of symmetry)

or

$E[w] = E[x+y] = E[x] + E[y] = a/2 + b/2 = \frac{a+b}{2}$

⊚ In general , if $x_1, x_2, \ldots, x_N$ are independent R.V's then the density function for

$W = x_1 + x_2 + \cdots + x_N$

is given as

$f_w(w) = f_{x_1}(x_1) * f_{x_2}(x_2) * \cdots * f_{x_N}(x_N)$

⊚ Central limit theorem (CLT) :

If $x_1, x_2, \ldots, x_N$ are independent R.V's, then the density function for

$W = X_1 + X_2 + \cdots + X_N$ can be approximated as :

$W \sim N(a_w, \sigma_w^2)$

$a_w = E[w] = \bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_N$

$\sigma_w^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \cdots + \sigma_{x_N}^2$

Ex 3- $X \sim U(0,a)$

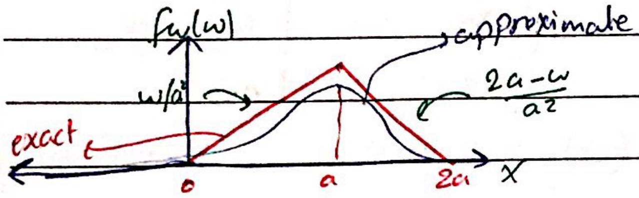
$Y \sim U(0,a)$

$W = X + Y$ ⓐ exact $f_w(w)$

ⓑ approximate $f_w(w)$

ⓐ exact

$$f_\omega(\omega) = f_x(x) * f_y(y)$$



ⓑ approximate :-

$$W \sim N(a_\omega, \sigma_\omega^{\,2})$$

$$a_\omega = \bar{X} + \bar{Y} = \frac{a}{2} + \frac{a}{2} = \frac{a+a}{2} = \frac{2a}{2} = a$$

$$\sigma_\omega^{\,2} = \sigma_x^{\,2} + \sigma_y^{\,2} = \frac{a^2}{12} + \frac{a^2}{12} = \frac{a^2}{6}$$

$$\omega \sim N\left(a, \frac{a^2}{6}\right)$$

$$f_\omega(\omega) = \frac{1}{\sqrt{2\pi . a^2/6}} \, e^{\frac{-(\omega-a)^2}{2(a^2/6)}}$$

$$= \frac{1}{\frac{\sqrt{2\pi}}{\sqrt{6}}\, a} = \frac{1}{\sqrt{\pi/3}\; a}$$

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8/11/2017

Ex 8- X_1, X_2 and X_3 are independent R.V's

$X = X_1 + X_2 + X_3$

Find the approximate density function for X.

	mean	variance
X_1	-1	2
X_2	0.6	1.5
X_3	1.8	0.8

ng T $\rightarrow$ $X \sim N(a_x, \sigma_x^2)$

$a_X = \bar{X}_1 + \bar{X}_2 + \bar{X}_3 = 1.4$

$\sigma_x^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_3}^2 = 4.3$

$$f_X(x) \simeq \frac{1}{\sqrt{2\pi(4.3)}} e^{\frac{-(x-1.4)^2}{(2)(4.3)}}$$

CHAPTER 5 : operations on Multiple R.V's.

① Expectation (mean) for function of R.V.

- let X and Y are joint R.V's with joint density function $f_{X,Y}(x,y)$ and $g(x,y)$ is function of the R.V's X and Y, then

$$\underbrace{E[g(x,y)]}_{\bar{g}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{x,y}(x,y)\, dx\, dy$$

- if $g(x,y) = g(x)$ ← function of X only

$$E[g(x,y)] = \iint g(x)\, f_{x,y}(x,y)\, dx\, dy$$

$$= \iint g(x)\, f_{x,y}(x,y)\, dy\, dx$$

$$= \int g(x) \left(\int f_{x,y}(x,y)\, dy\right) dx = \int g(x)\, f_x(x)\, dx$$

• If $g(x,y) = g(y)$ ← function of y only

$$E[g(y)] = \int g(y)\, f_Y(y)\, dy$$

practice 3- $(x,y) \rightarrow f_{X,Y}(x,y) = u(x)\, u(y)\, e^{-x(y+1)}$

$g_1(x,y) = x e^{-2y}$ $\qquad$ $g_2(x,y) = 2x-1$

↓ $\qquad$ ↓

Find $\bar{g}_1$?! $\qquad$ Find $\bar{g}_2$?!

$$\bar{g}_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} x e^{-2y} \cdot e^{-x(y+1)}\, u(x)\, u(y)\, dx\, dy$$

$$\bar{g}_2 \Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \dots$$

Then $\bar{g}_2 = \int_{-\infty}^{\infty} (2x-1)\, f_X(x)\, dx$

Ex: X_1 and X_2 are joint R.V's with $f_{X_1,X_2}(x_1,x_2)$

let $X = \alpha_1 X_1 + \alpha_2 X_2$. Show that

$\bar{X} = \alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2$

$$E[X] = E[\underbrace{\alpha_1 X_1 + \alpha_2 X_2}_{g(x_1,x_2)}] = \iint (\underline{\underline{\alpha_1 x_1}} + \underline{\underline{\alpha_2 x_2}})\, f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \iint \alpha_1 x_1 f_{X_1,X_2}(x_1,x_2)\, dx_2\, dx_1 + \iint \alpha_2 x_2 f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \alpha_1 \int x_1 \underbrace{\left(\int f_{X_1,X_2}(x_1,x_2)\, dx_2\right)}_{f_{X_1}(x_1)} dx_1 + \alpha_2 \int x_2 \underbrace{\left\{\int f_{X_1,X_2}(x_1,x_2)\, dx_1\right\}}_{f_{X_2}(x_2)} dx_2$$

$= \alpha_1 \int x_1 f_{X_1}(x_1)dx_1 + \alpha_2 \int x_2 f_{X_2}(x_2)\, dx_2 \quad = \alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2$

- In general

$X_1, X_2, \ldots, X_N \rightsquigarrow f_{X_1, X_2, \ldots X_N}(x_1, x_2, \ldots, x_N)$

$X = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_N X_N$, then

$\bar{X} = \alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2 + \cdots + \alpha_N \bar{X}_N$

$\bar{X} = \sum_{i=1}^{N} \alpha_i \cancel{X}_i$

$E[X] = \sum_{i=1}^{N} \alpha_i \, E[X_i]$

- $E\left[\sum_{i=1}^{N} \alpha_i X_i\right] = \sum_{i=1}^{N} \alpha_i E[X_i]$

Ex :- given X, Y, Z with $\bar{X} = -1$, $\bar{Y} = 2$, $\bar{Z} = 3$

let $W = 2X - Y + \frac{1}{2} Z$

Find Dc value for W.

$E[W] = E[2X - Y + \frac{1}{2} Z] = 2\bar{X} - \bar{Y} + \frac{1}{2}\bar{Z} = -2.5$

- $X = g(X_1) + g(X_2)$

then $E[X] = E[g(X_1)] + E[g(X_2)]$

e.g :- $Z = 2X + Y^2$

$\bar{Z} = E[2X + Y^2]$

$= 2\bar{X} + E[Y^2]$

- Re call moment

$X \rightsquigarrow f_X(x)$

$m_n = E[X^n] = \int x^n f_X(x)\, dx$

joint Moments :

$$m_{nk} = E[x^n y^k] = \iint x^n y^k f_{x,y}(x,y)\,dx\,dy$$

order is n+k :

- Zeroth order $n+k=0$

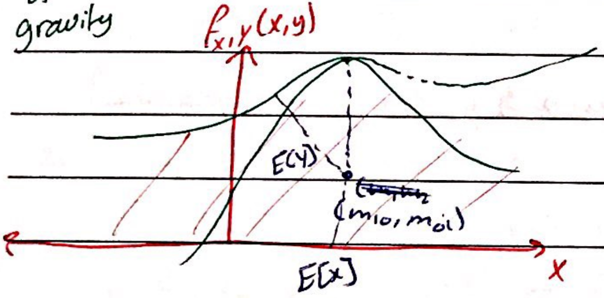
$$m_{00} = E[x^0 y^0] = 1$$

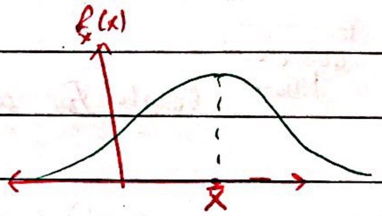
- 1'st order $n+k=1$

$$m_{10} = E[x] = m_1^x$$

$$m_{01} = E[y] = m_1^y$$

Center of gravity

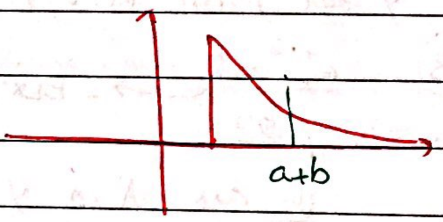




- 2nd order : $n+k=2$

$$m_{20} = E[x^2] = m_2^x$$

$$m_{02} = E[y^2] = m_2^y$$



$$m_{11} = E[xy] = R_{xy}$$

Correction between x and y

$$x(t), y(t) \Rightarrow \int x(t)\, y(t+\tau)\, d\tau$$

- $R_{xy} = E[xy] = m_{11}$

* If $R_{xy} = 0$, then x and y are orthogonal

* If $R_{xy} = E[x]\,E[y]$: x & y are uncorrelated

$$\downarrow$$

$$E[x,y] = \bar{x}\,\bar{y}$$

- If X and Y are independent, then they must be uncorrelated

proof :-

$$R_{XY} = E[XY] = \iint xy\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \iint x\, y\, f_X(x) \cdot f_Y(y)\, dx\, dy$$

↑ because X and Y are independent

$$= \int y\, f_Y(y) \underbrace{\int x\, f_X(x)\, dx}_{=\bar{x}}\, dy$$

$= \bar{X}\bar{Y}$, so X & Y are uncorrelated. Center of g

independent $\xrightarrow{\text{must}}$ uncorrelated

$\xleftarrow{\text{we don't know}}$ (only for one case when x & y are joint gaussian)

Ex :- X & Y are joint R.V's :-

$\bar{X} = 3$, $\sigma_X^2 = 2 \longrightarrow = E[X^2] - E^2[X]$

$Y = -6X + 22$

Find ⓐ R_{XY} ⓑ are X & Y orthogonal

ⓒ are X & Y uncorrelated

Sol: ⓐ $R_{XY} = E[XY] = E[X(-6X+22)] = E[-6X^2 + 22X] = -6E[X^2] + 22\,E[X]$

$\cancel{= -6E[}$ $= -6(\sigma_x^2 + \bar{X}) + 22\bar{X} = -6(2+9) + (22)(3) = 0$

ⓑ yes

ⓒ check $R_{XY} = \bar{X}\bar{Y} \longrightarrow \bar{X}\bar{Y} \overset{?}{=} 0$

$\bar{X}\bar{Y} = 3((-6)(3) + 22) \neq 0$

X and Y are not correlated

⑩ joint central moments :-

$$\mu_{nk} = E[(x-\bar{x})^n (y-\bar{y})^k] = \iint (\,) f_{x,y}(x,y)\,dx\,dy$$

The order is n+k :

- Zeroth order : $\mu_{00} = 1$

- 1'st order :

$$\mu_{10} = E[(x-\bar{x})] = 0$$

$$\mu_{01} = E[(y-\bar{y})] = 0$$

- 2nd order :-

$$\mu_{20} = E[(x-\bar{x})^2] = \sigma_x^2$$

$$\mu_{02} = E[(y-\bar{y})^2] = \sigma_y^2$$

$$\mu_{11} = E[(x-\bar{x})(y-\bar{y})] = C_{xy}$$ ← covariance

Note :- $C_{xx} = \sigma_x^2$

$C_{yy} = \sigma_y^2$ The covariance between R.V and itself is it's variance.

- $C_{xy} = E[(x-\bar{x})(y-\bar{y})] = E[xy - \bar{y}x - \bar{x}y + \bar{x}\bar{y}]$

$$= E[xy] - \bar{x}\bar{y} - \bar{x}\bar{y} + \bar{x}\bar{y}$$

$$\boxed{C_{xy} = R_{xy} - \bar{x}\bar{y}}$$

- If x & y are orthogonal then $R_{xy} = 0$, $C_{xy} = -\bar{x}\bar{y}$
- If x & y are uncorrelated ; $R_{xy} = \bar{x}\bar{y}$ $C_{xx} = 0$

note : C_{xy} → +ve, 0 (uncorrelated), −ve

13/11/2017

Joint moments :

$m_{11} = R_{xy} = E[xy]$

if $R_{xy} = \bar{x}\,\bar{y}$ (uncorrelated)

Joint central moments:

$\mu_{11} = E[(x-\bar{x})(y-\bar{y})] = C_{xy}$

$\rightarrow C_{xy} = R_{xy} - \bar{x}\,\bar{y}$

if x & y are uncorrelated:

$C_{xy} = 0$

- Correlation parameter : ρ_{xy}

$$\rho_{xy} = \frac{\mu_{11}}{\sqrt{\underbrace{\mu_{20}}_{\sigma_x^2}\,\underbrace{\mu_{02}}_{\sigma_y^2}}} = \frac{C_{xy}}{\sigma_x\,\sigma_y}$$

. if $y = x$:

$$\rho_{xy} = \frac{C_{xx}}{\sigma_x\,\sigma_x} = \frac{\sigma_x^2}{\sigma_x^2} = 1$$

. if y & x are independent $\xrightarrow{\text{which yields}}$ (uncorrelated)

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x\,\sigma_y} = \frac{0}{\sigma_x\,\sigma_y} = 0$$

• if $Y = -X$

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} = \frac{C_{x(-x)}}{\sigma_x \sigma_{-x}}$$

$$* \ \sigma_{-x}^2 = Var(-X) = Var(X) = \sigma_x$$

$$\sigma_{-x} = \sigma_x$$

$$* \ C_{x(-x)} = E\left[(x-\bar{x})(-x-(-\bar{x})\right]$$

$$= -E\left[(x-\bar{x})(x-\bar{x})\right]$$

$$= -\sigma_x^2 \longrightarrow \boxed{\rho_{xy} = \frac{-\sigma_x^2}{\sigma_x \sigma_x} = -1}$$

$$\Rightarrow -1 \le \rho_{xy} \le 1 \Rightarrow \underset{\substack{\uparrow \\ \text{uncorrelated}}}{0} \le |\rho_{xy}| \le \underset{\substack{\uparrow \\ \text{highly Correlated}}}{1}$$

Recall: $X = \alpha_1 X_1 + \alpha_2 X_2$

$$E[X] = \alpha_1 E[X_1] + \alpha_2 E[X_2]$$

Ex : let $X = \alpha_1 X_1 + \alpha_2 X_2$, where α_1 and α_2 are Constants. Determine the variance of X.

$$Var(X) \triangleq \sigma_x^2 = E\left[(X-\bar{X})^2\right] = E\left[\left(\underbrace{(\alpha_1 X_1 + \alpha_2 X_2)}_{} - \underbrace{(\alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2)}_{}\right)^2\right]$$

$$= E\left[\left(\alpha_1 (X_1 - \bar{X}_1) + \alpha_2 (X_2 - \bar{X}_2)\right)^2\right]$$

$$= E\left[\alpha_1^2 (X_1 - \bar{X}_1)^2 + \alpha_2^2 (X_2 - \bar{X}_2)^2 + 2\alpha_1\alpha_2 (X_1 - \bar{X}_1)(X_2 - \bar{X}_2)\right]$$

$$= \alpha_1^2 \sigma_{X_1}^2 + \alpha_2^2 \sigma_{X_2}^2 + \underline{\underline{2\alpha_1\alpha_2 C_{X_1} C_{X_2}}}$$

$$= \alpha_1^2 \sigma_{X_1}^2 + \alpha_2 \sigma_{X_2}^2 + \alpha_1\alpha_2 C_{X_1} C_{X_2} + \alpha_2\alpha_1 C_{X_2} C_{X_1}$$

this term appeared because they are Correlated, if they weren't Correlated this term will equal zero.

In general :-

$$X = \sum_{i=1}^{N} \alpha_i X_i$$

$$\sigma_x^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} \alpha_i \alpha_j C_{x_i x_j}$$

Special case :- when $X_1, X_2, \dots, X_N$ are uncorrelated :

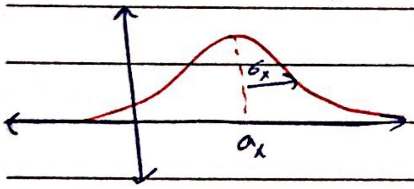
$$\sigma_x^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2$$

$$= \alpha_1 \sigma_{x_1}^2 + \alpha_2 \sigma_{x_2}^2 + \dots + \alpha_N^2 \sigma_{x_N}^2$$

⑪ joint Gaussian R.V's :-

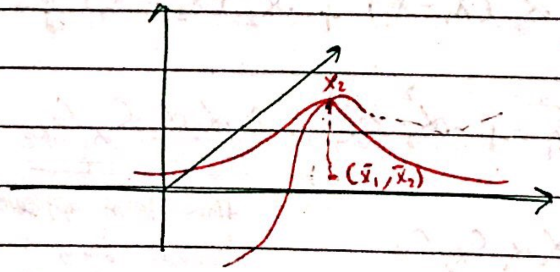
Recall : $X \sim N(a_x, \sigma_x^2)$

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}$$



Def :- X_1 and X_2 are said to be jointly gaussian if ~~there~~ their joint density function is given by :-

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2}} * e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_x^2} - \frac{2\rho(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2-\bar{x}_2)^2}{\sigma_{x_2}^2}\right]}$$



- if X_1 & X_2 are uncorrelated ~~density~~ jointly gaussian:

$\rho = 0$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}} e^{-\frac{1}{2}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_{x_1}^2} + \frac{(x_2-\bar{x}_2)^2}{\sigma_{x_2}^2}\right]}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma_{x_1}^2}} e^{\frac{-(x_1-\bar{x}_1)^2}{2\sigma_x^2}}}_{f_{X_1}(x_1)} \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma_{x_2}^2}} e^{\frac{-(x_2-\bar{x}_2)^2}{2\sigma_{x_2}^2}}}_{f_{X_2}(x_2)} = \prod_{i=1}^{2} f_{x_i}(x_i) \rightarrow N(a_{x_i}, \sigma_{x_i}^2)$$

$\Rightarrow \therefore$ X_1 & X_2 are independent

$$\text{independent} \underset{\text{only for gaussian R.V's}}{\overset{\text{must}}{\rightleftarrows}} \text{uncorrelated}$$

⑩ N jointly Gaussian R.V's

Def :- $X_1, X_2, \dots, X_N$ are said to be jointly gaussian if the joint density function is given by.

$$f_{X_1,X_2,\dots X_N}(x_1,x_2,\dots,x_N) = \frac{\left|[C_X]^{-1}\right|^{1/2}}{(2\pi)^{N/2}} e^{-\frac{[X-\bar{X}]^T [C_X]^{-1} [X-\bar{X}]}{2}}$$

$$[C_X]_{N\times N} = \begin{bmatrix} \overset{\sigma_{x_1}^2}{C_{X_1X_1}} & C_{X_1X_2} & \cdots & C_{X_1X_N} \\ C_{X_2X_1} & \overset{\sigma_{x_2}^2}{C_{X_2X_2}} & \cdots & C_{X_2X_N} \\ \vdots & & & \\ C_{X_NX_1} & C_{X_NX_2} & \cdots & \underset{\sigma_{x_N}^2}{C_{X_NX_N}} \end{bmatrix}$$

↑ Covariance matrix

→ symmetry around diagonal

the same

$$[X-\bar{X}] = \begin{bmatrix} X_1-\bar{X}_1 \\ X_2-\bar{X}_2 \\ \vdots \\ X_N-\bar{X}_N \end{bmatrix}$$

$$[X-\bar{X}]^T = [x_1-\bar{x}_1 \quad x_2-\bar{x}_2 \;\ldots\; x_N-\bar{x}_N]$$

$[\cdot]^{-1}$: matrix inverse

$|[\cdot]|$: determinant

if $x_1, x_2, \ldots, x_N$ are uncorrelated ; then

$$[C_X] = \begin{bmatrix} \sigma_{x_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{x_2}^2 & - & 0 \\ & & & \\ 0 & \cdots 0 & \cdots & \sigma_{x_N}^2 \end{bmatrix}$$

$$f_{x_1,x_2,x_N}(x_1,x_2,\ldots x_N) = f_{x_1}(x_1)\cdot f_{x_2}(x_2)\cdots f_{x_N}(x_N)$$

$$= \prod_{i=1}^{N} f_{x_i}(x_i) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_{x_i}^2}}\, e^{-\frac{(x_i-\bar{x}_i)^2}{2\sigma_{x_i}^2}}$$

$$= \frac{1}{(2\pi)^{N/2}\prod_{i=1}^{N}\sigma_{x_i}}\, e^{-\sum_{i=1}^{N}\frac{(x_i-\bar{x}_i)^2}{2\sigma_{x_i}^2}}$$

Use ⊛ to find the joint density function for x_1, x_2

$$[C_X]_{2\times2} = \begin{bmatrix} \sigma_{x_1}^2 & C_{x_1x_2} \\ C_{x_2x_1} & \sigma_{x_2}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1}\sigma_{x_2}\rho \\ \sigma_{x_1}\sigma_{x_2}\rho & \sigma_{x_2}^2 \end{bmatrix}$$

$$[C_X]^{-1} = \frac{1}{\sigma_{x_1}^2\sigma_{x_2}^2(1-\rho^2)} \begin{bmatrix} \sigma_{x_2}^2 & -\sigma_{x_1}\sigma_{x_2}\rho \\ -\sigma_{x_1}\sigma_{x_2}\rho & \sigma_{x_1}^2 \end{bmatrix}$$

$$[C_X]^{-1} = \begin{bmatrix} \frac{1}{(1-\rho^2)\sigma_{x_1}^2} & -\frac{\rho}{\sigma_{x_1}\sigma_{x_2}(1-\rho^2)} \\ \frac{-\rho}{\sigma_{x_1}\sigma_{x_2}(1-\rho^2)} & \frac{1}{(1-\rho^2)\sigma_{x_2}^2} \end{bmatrix} \ldots ①$$

$$\left|[C_X]^{-1}\right| = \frac{1}{\sigma_{x_1}^2 \, \sigma_{x_2}^2 \, (1-\rho^2)} \quad \dots ②$$

$$\cdot \; [x-\bar{x}]^T = \begin{bmatrix} x_1-\bar{x}_1 & x_2-\bar{x}_2 \end{bmatrix} \quad \dots ③$$

$$\cdot \; [x-\bar{x}] = \begin{bmatrix} x_1-\bar{x}_1 \\ x_2-\bar{x}_2 \end{bmatrix} \quad \dots ④$$

$$f_{x_1 x_2}(x_1, x_2) = \frac{1}{2\pi \, \sigma_{x_1} \sigma_{x_2} \sqrt{1-\rho^2}} \, e^{-\frac{③ \, ① \, ④}{2}}$$

20/11/2017

• Transformation in multiple R.V's :-

Recall :

$$X \longrightarrow \boxed{T(\cdot)} \longrightarrow y$$

$F_x(x)$, $f_x(x)$ $\qquad$ $F_y(y)$?

$f_y(y)$?

$x_1, x_2, \dots, x_N$ are joint R.V's :-

$$(x_1, x_2, \dots, x_N) \longrightarrow \boxed{T_1(\cdot)} \longrightarrow Y_1 = T_1(x_1, x_2, \dots, x_N)$$

$$(x_1, x_2, \dots, x_N) \longrightarrow \boxed{T_2(\cdot)} \longrightarrow Y_2 = T_2(x_1, x_2, \dots, x_N)$$

$$\vdots$$

$$(x_1, x_2, \dots, x_N) \longrightarrow \boxed{T_N(\cdot)} \longrightarrow Y_N = T_N(x_1, x_2, \dots, x_N)$$

Conditions :-

① All $T_1, T_2, \dots, T_N$ are continuous functions.

② The partial derivatives of all $T_1, T_2, \dots, T_N$ exist & contineous everywhere.

③ ~~The functions~~ we can find :

$X_1 = V_1(Y_1, Y_2, \dots, Y_N)$

$X_2 = V_2(Y_1, Y_2, \dots, Y_N)$

$\vdots$

$X_N = V_N(Y_1, Y_2, \dots, Y_N)$

Then the joint density function of $Y_1, Y_2, \dots, Y_N$ can be given as:

$$f_{Y_1,Y_2,\dots,Y_N}(y_1,y_2,\dots,y_N) = f_{X_1,X_2,\dots,X_N}\big(v_1(y_1,y_2,\dots,y_N), v_2(y_1,y_2,\dots,y_N),\dots,v_N(y_1,y_2,\dots,y_N)\big) \cdot |J|$$

absolute value

$$J = \begin{bmatrix} \frac{dv_1}{dy_1} & \frac{dv_1}{dy_2} & \cdots & \frac{dv_1}{dy_N} \\ \frac{dv_2}{dy_1} & \frac{dv_2}{dy_2} & \cdots & \frac{dv_2}{dy_N} \\ \vdots & & & \\ \frac{dv_N}{dy_1} & \frac{dv_N}{dy_2} & \cdots & \frac{dv_N}{dy_N} \end{bmatrix}$$

Ex 3:- given X_1 & X_2 are joint R.V's with joint density function

$$f_{X_1,X_2}(x_1,x_2) = q\, x_1\, e^{-x_2(x_1-1)}\, u(x_1)\, u(x_2)$$

q ← constant

let $Y_1 = \boxed{aX_1 + bX_2}\; T_1(X_1,X_2)$

$Y_2 = \boxed{cX_1 + dX_2}\; T_2(X_1,X_2)$

$X_1, X_2 \longrightarrow \boxed{T_1(\cdot)} \longrightarrow Y_1 = aX_1 + bX_2$

$X_1, X_2 \longrightarrow \boxed{T_2(\cdot)} \longrightarrow Y_2 = cX_1 + dX_2$

where $ad - bc \neq 0$

Determine the $f_{Y_1,Y_2}(y_1,y_2)$?!

Sol 3:- $Y_1 = aX_1 + bX_2$

$Y_2 = (cX_1 + dX_2) * \frac{-b}{d}$

$Y_1 = aX_1 + bX_2$

$-\frac{b}{d} Y_2 = \frac{-bc}{d} X_1 - bX_2$

$\left[Y_1 - \frac{b}{d} Y_2 = \left(a - \frac{bc}{d}\right) X_2 \right] * d$

$X_1 = \boxed{\frac{dY_1 - bY_2}{ad - bc}}$ $v_1(Y_1, Y_2)$

$Y_1 = aX_1 + bX_2$

$Y_2 = (cX_1 + dX_2)\left(\frac{-a}{c}\right)$

$Y_1 = aX_1 + bX_2$

$\frac{-a}{c} Y_2 = -aX_1 - \frac{da}{c} X_2$

$\left[Y_1 - \frac{a}{c} Y_2 = \left(b - \frac{da}{c}\right) X_2\right] * c$

$$X_2 = \boxed{\frac{cY_1 - aY_2}{bc - da}}$$

$v_2(Y_1, Y_2)$

$$J = \left| \begin{bmatrix} \frac{dv_1}{dY_1} & \frac{dv_1}{dY_2} \\ \frac{dv_2}{dY_1} & \frac{dv_2}{dY_2} \end{bmatrix} \right| = \left| \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{bc-da} & \frac{+a}{bc-da} \end{bmatrix} \right|$$

$$\Rightarrow J = \frac{ad}{(ad-bc)^2} - \frac{bc}{(ad-bc)^2} = \frac{1}{ad-bc}$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(v_1(y_1,y_2), v_2(y_1,y_2))\,|J|$$

~~$-dY_1 - bY_2$~~

$$= \frac{q\, dY_1 - bY_2}{t\, ad - bc}\, e^{-\left(\frac{cY_1 - aY_2}{bc-da}\right)\left[\frac{dY_1 - bY_2}{ad-bc} - 1\right]}\, u\left(\frac{dY_1 - bY_2}{ad-bc}\right) \cdot u\left(\frac{cY_1 - aY_2}{bc-da}\right) \cdot \frac{1}{|ad-bc|}$$

⑪ linear Transformation of joint gaussian R.v's :-

$X_1, X_2, \dots, X_N$ are joint gaussian R.v's, i.e :-

$$f_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N) = \frac{\left|[C_X]^{-1}\right|^{1/2}}{(2\pi)^{N/2}} * e^{-\frac{[x-\bar{x}]^T [C_X]^{-1} [x-\bar{x}]}{2}}$$

let
$$\left.\begin{aligned} Y_1 &= a_{11}X_1 + a_{12}X_2 + \dots + a_{1N}X_N \\ Y_2 &= a_{21}X_1 + a_{22}X_2 + \dots + a_{2N}X_N \\ &\vdots \\ Y_N &= a_{N1}X_1 + a_{N2}X_2 + \dots + a_{NN}X_N \end{aligned}\right\} \text{linear Transformation.}$$

a_{ij} are Constant

$i = 1, \dots, N$

$j = 1, \dots, N$

Then ;

$$f_{Y_1,Y_2,\dots,Y_N}(y_1,y_2,\dots,y_N) = \frac{\left|[C_Y]^{-1}\right|^{1/2}}{(2\pi)^{N/2}} e^{-\frac{[y-\bar{y}]^T [C_Y]^{-1} [y-\bar{y}]}{2}}$$

which is jointly gaussian.

$$[C_Y] = [T][C_X][T]^T$$

$$[Y-\bar{Y}] = \begin{bmatrix} y_1-\bar{Y}_1 \\ y_2-\bar{Y}_2 \\ \vdots \\ y_N-\bar{Y}_N \end{bmatrix}$$

$$\bar{Y}_i = \sum_{j=1}^{N} a_{ij}\bar{X}_j \qquad , \qquad T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & & \vdots & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}$$

$i = 1, 2, \dots, N$

Ex :- X_1, X_2 are joint gaussian R.V's :-

$Y_1 = X_1 - 2X_2$

$Y_2 = 3X_1 + 4X_2$, $[C_X] = \begin{bmatrix} \overset{\sigma_{X_1}^2}{4} & 3 \\ 3 & \underset{\sigma_{X_2}^2}{9} \end{bmatrix}$, $\bar{X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \bar{X}_1 \\ \leftarrow \bar{X}_2 \end{matrix}$

ⓐ $\bar{Y}$ ⓑ $[C_Y]$ ⓒ $f_{Y_1,Y_2}(y_1,y_2)$

$$[C_X]_{N\times N} = \begin{bmatrix} \overset{\sigma_{X_1}^2}{C_{X_1,X_1}} & C_{X_1,X_2} \cdots C_{X_1,X_N} \\ \vdots & \\ C_{X_N,X_1} \cdots & \underset{\sigma_{X_N}^2}{C_{X_N,X_N}} \end{bmatrix}$$

ⓐ $\bar{Y} = \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} = \begin{bmatrix} 2-(2)(1) \\ (3)(2)+4(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$

ⓑ $[C_Y] = [T][C_X][T]^T$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \overset{\sigma_{Y_1}^2}{28} & \overset{C_{Y_1,Y_2}}{-66} \\ \underset{C_{Y_2,Y_1}}{-66} & \underset{\sigma_{Y_2}^2}{252} \end{bmatrix}$$

ⓒ $\rho_{Y_1,Y_2} = \dfrac{C_{Y_1,Y_2}}{\sigma_{Y_1}\sigma_{Y_2}}$

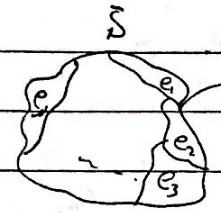
$\vdots$

22/11/2017

Chapter 6 :- Random (Stochastic) Processes - Temporal characteristics.

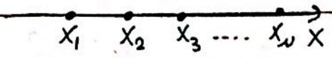
Recall :-

exp :

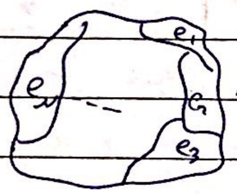


$X(\mathcal{S}) = X$

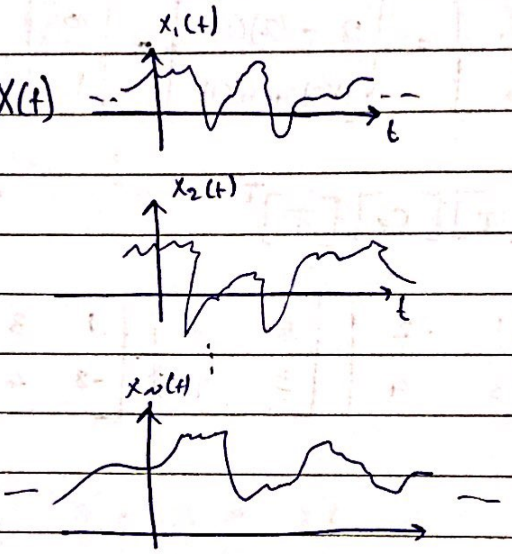
$X = \{x_1, x_2, \dots, x_N\}$



Def : R.P



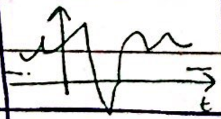
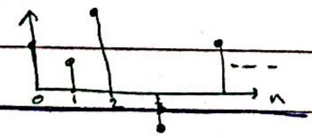
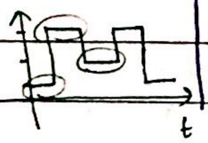
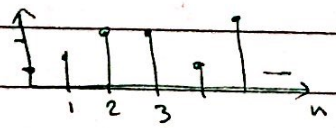
$$X(\mathcal{S}, \overset{\text{time}}{t}) \triangleq X(t)$$



R.P X(t) is family of realizations (sample functions)

--- $x_1(t)$, $x_2(t)$, ... $x_N(t)$, . —

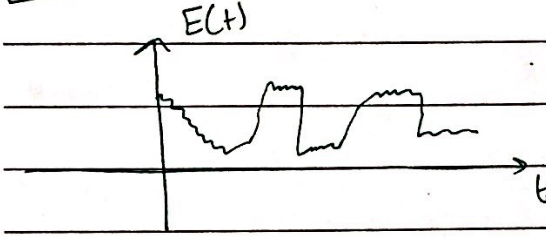
R.P classification :-

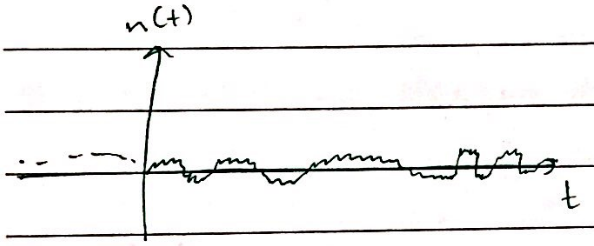
	Contineous time	Discrete Time
Contineous amplitude		
Discrete amplitude		

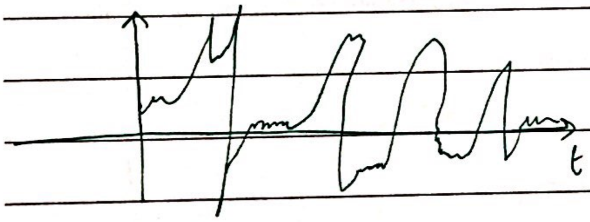
⊗ R.p $\rightarrow$ Deterministic (Th R.p has a mathmatical representation)

$\rightarrow$ non-deterministic (Does not have " "). Future values cannot be predicted).

$\boxed{e.g}$:- non-Deterministic R.p







(human speech)

$\boxed{e.g}$ Deterministic R.p

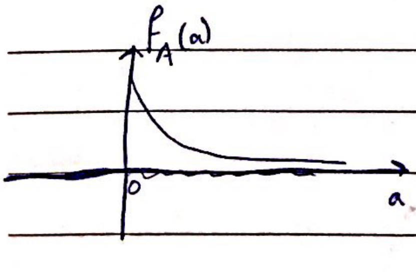
$$X(t) = \underline{\underline{A}} \cos(\underline{\underline{\omega_0}} t + \underline{\underline{\theta}})$$

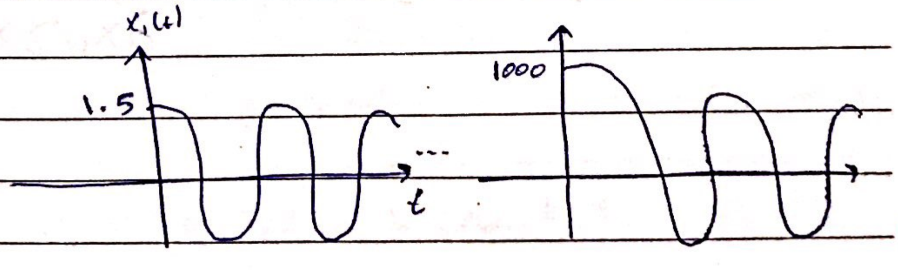
At least one of A, ω_0, θ should be R.v

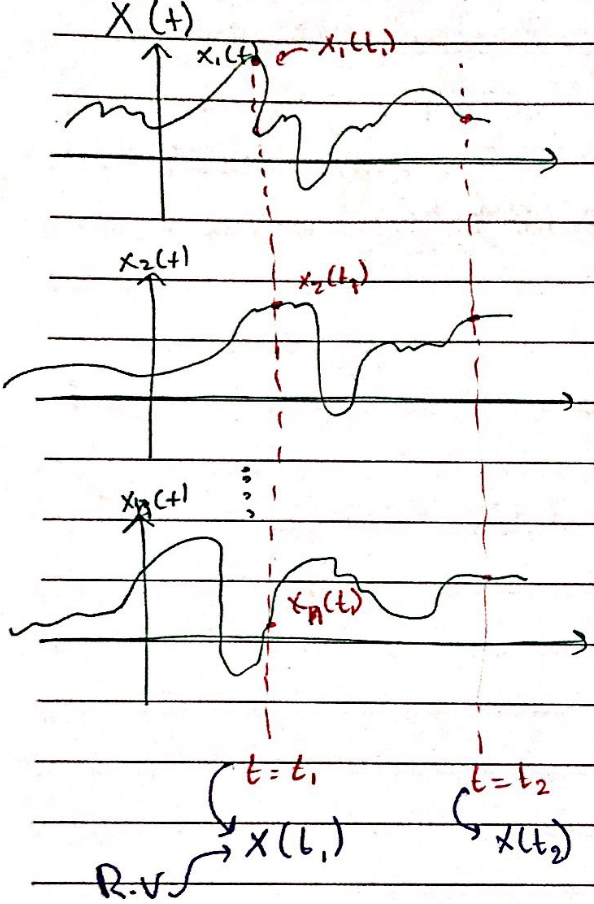
e.g :- $X(t) = A \cos(\omega_0 t + \theta)$, ω_0, θ are constants

, $A \sim \exp(0,1)$

Sol:-







$$\underset{\text{R.V}}{\underline{\underline{X(t_1)}}} = \{\overbrace{x_1(t_1), x_2(t_1), \dots, x_n(t_1)}^{\text{real numbers}}\}$$

- 1st order distribution:

$$X(t_1) \longrightarrow F_X(x;t_1)$$

$$\searrow f_X(x_1;t_1) = \frac{dF_X(x_1;t_1)}{dx_1}$$

- 2nd order distribution

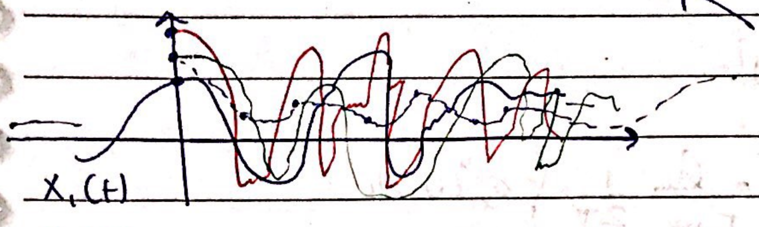
$$\left.\begin{matrix} X(t_1) \\ X(t_2) \end{matrix}\right\} \longrightarrow F_X(x_1, x_2; t_1, t_2)$$

$$\searrow f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \, \partial x_2}$$

R.P mean

$$x(t) \longrightarrow E[x(t)] \overset{\Delta}{=} m_x(t) = \int x\, f_x(x;t)\, dx$$

R.P mean is function of time



$$\text{R.P - DC - value} = A\left[m_x(t)\right]$$

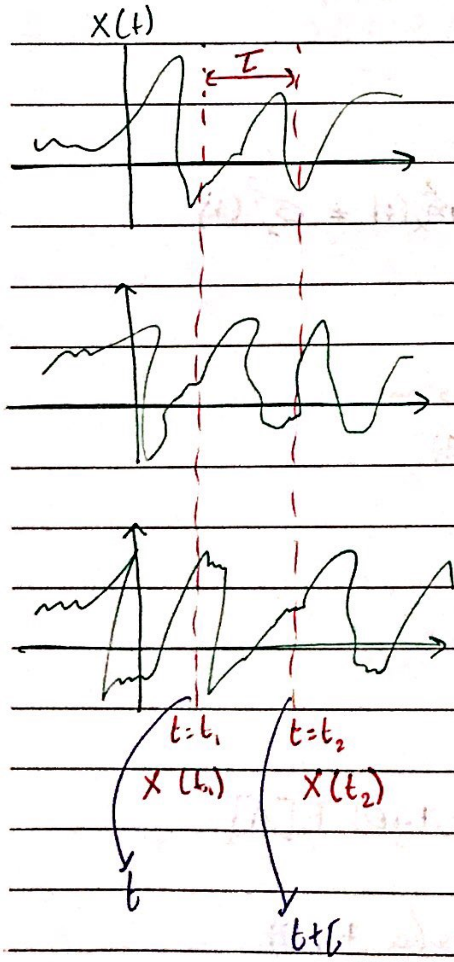
time - average

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} m_x(t)\, dt$$

auto - correlation function :-

X_1 , X_2

$$R_{x_1,x_2} = E[X_1 X_2]$$



$$R_{xx}(t_1, t_2) = E\left[X(t_1) \cdot X(t_2)\right]$$

$$\boxed{R_{xx}(t, t+\tau) = E\left[x(t)\, X(t+\tau)\right]}$$

if $\tau = 0$

$$R_{xx}(t,t) = E\left[x(t)\, x(t)\right] = E\left[X^2(t)\right]$$

R.P - Total - power

$$= A\left[E[x^2(t)]\right]$$

$$= \iint_{-\infty}^{\infty} x_1 x_2\, f_x(x_1, x_2; t, t+\tau)\, dx_1\, dx_2$$

- R.P variance :-

$$Var(X(t)) = E[X^2(t)] - m_x^2(t)$$

$$\Rightarrow \boxed{\sigma_x^2(t) = E[X^2(t)] - m_x^2(t)}$$

$$\text{R.P Ac-power} = A[\sigma_x^2(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma_x^2(t)\, dt$$

- auto-Covariance function :-

$$\left.\begin{matrix} X_1 \\ X_2 \end{matrix}\right\} C_{X_1X_2} = E[(X_1-\bar{x})(X_2-\bar{x})] = R_{X_1X_2} - \bar{X}_1\bar{X}_2$$

$$C_{xx}(t, t+\tau) = E[(X(t) - m_x(t))(X(t+\tau) - m_x(t+\tau))]$$

$$= R_{xx}(t, t+\tau) - m_x(t)\, m_x(t+\tau)$$

if $\tau = 0$

$$\Rightarrow C_{xx}(t,t) = R_{xx}(t,t) - m_x^2(t) = E[X^2(t)] - m_x^2(t) = \sigma_x^2(t)$$

Ex: $X(t) = A\cos(w_0t+\theta)$, w_0, θ are Constant

$A \sim N(2,9)$

Find ⓐ $m_x(t)$ ⓑ $\sigma_x^2(t)$

Sol: ⓐ $m_x(t) = E[X(t)] = \int x \underbrace{f_x(x;t)}_{N(2,9)}\, dx$

$$m_x(t) = E[X(t)] = E[A\,\underbrace{\cos(w_0t+\theta)}_{\text{not R.P}}] = \cos(w_0t+\theta)\, E[A]$$

$$= 2\cos(w_0t+\theta)$$

ⓑ $\sigma_x^2(t) = Var(X(t)) = Var(A\cos(w_0t+\theta)) = \cos^2(w_0t+\theta)\, Var(A)$

$$= 9\cos^2(w_0t+\theta)$$

$$X(t) = \underbrace{\boxed{A}}_{N(2,9)} \underbrace{\cos(\omega_0 t+\theta)}_{\parallel \atop c}$$

$$X = cA$$

$$A \longrightarrow \boxed{T(\cdot)=c(\cdot)} \longrightarrow X = cA$$

$$E[X] = 2c$$

$$\sigma_x^2 = 9c^2$$

$$x(t) \sim N\left(2\cos(\omega_0 t+\theta),\ 9\cos^2(\omega_0 t+\theta)\right)$$

$$f_x(x;t) = \frac{1}{\sqrt{2\pi\ 9\cos^2(\omega_0 t+\theta)}}\, e^{-\frac{(x-2\cos(\omega_0 t+\theta))}{2(9)\cos^2(\omega_0 t+\theta)}}$$

27/11/2017

② Stationarity :-

In general random process is said to be stationary if all of it's Statistics, characteristics do not change with time

• 1'st order Stationarity :-

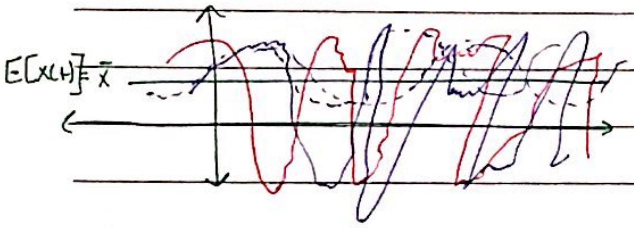
Def : a R.p is said to be 1st order stationary if ; $f_x(x, t_i) = f_x(x ; t_j)$ for all t_i & t_j

⇒ The R.p density function is not function of time.

Result :- $E[x(t)] = m_x(t) = \int_{-\infty}^{\infty} x \boxed{f_x(x, t)} dx$

Because $f_x(x ; t)$ is not function of t ; the R.p mean is Constant

$m(t) = \bar{x}$



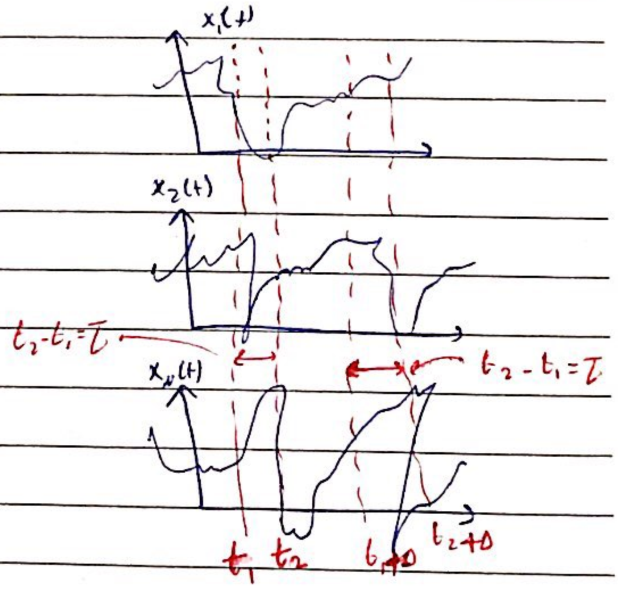
R.p Dc-Value $= A[m_x(t)] = A[\bar{x}] = \bar{x}$

• 2nd order stationarity :-

Def : The R.p is said to be 2nd order stationary, if :

$f_x(x_1, x_2 ; t_1, t_2) = f_x(x_1, x_2 ; t_1+\Delta, t_2+\Delta)$

for all t_1, t_2 & Δ



$R_{xx}(t_1,t_2) = E[X(t_1)X(t_2)] = \iint_{-\infty}^{\infty} x_1 x_2 \, f_x(x_1,x_2;t_1,t_2)\,dx_1\,dx_2$

• $R_{xx}(t_1+\Delta,t_2+\Delta) = E[X(t_1+\Delta)\,X(t_2+\Delta)]$

$= \iint x_1 x_2 \, f_x(x_1,x_2;t_1+\Delta,t_2+\Delta)\,dx_1\,dx_2$

$\therefore \Rightarrow R_{xx}(\underbrace{t_1,t_2}_{t_2-t_1=\tau}) = R_{xx}(\underbrace{t_1+\Delta,t_2+\Delta}_{t_2-t_1=\tau})$

Result : $R_{xx}(t,t+\tau) = R_{xx}(\tau)$

e.g. : $R_{xx}(101,104) = R_{xx}(0,3)$

⑪ Wide sence stedionarity :- (WSS)

Def :- R.P $X(t)$ is said to be WSS if :

① $E[X(t)] \triangleq m_x(t) = \bar{x}$

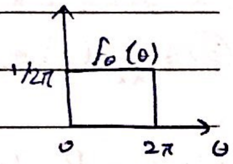
② $R_{xx}(t,t+\tau) = R_{xx}(\tau)$

$\sigma_x^2(t) = \sigma_x^2$

Ex :- $X(t) = A\cos(\omega_0 t+\theta)$; A & ω_0 are constants

$\theta \sim U(0,2\pi)$

Show that $X(t)$ is WSS.



Sol :-

$E[X(t)] = \cancel{\int x f_x(x;t)\,dx} = E[A\cos(\omega_0 t+\theta)]$

$= A\,E[\underbrace{\cos(\omega_0 t+\theta)}_{g(\theta)}] = A\int_{-\infty}^{\infty} g(\theta)\,f_\theta(\theta)\,d\theta$

$= A\int_0^{2\pi}\cos(\theta+\omega_0 t)\cdot 1/2\pi\,d\theta$

$= \frac{A}{2\pi}\int_0^{2\pi}\cancelto{zero}{\cos(\omega_0 t+\theta)}\,d\theta = \text{Zero}$

• $E[X(t)] = \bar{x} = 0$

- Check if $R_{xx}(t, t+\tau) \overset{?}{=} R_{xx}(\tau)$

$$R_{xx}(t, t+\tau) = E\left[x(t), x(t+\tau)\right]$$

$$= E\left[A\cos(\omega_0 t+\theta) \cdot A\cos(\omega_0 t + \omega_0\tau+\theta)\right]$$

$$= A^2 E\left[\underbrace{\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)}_{g(\theta)}\right]$$

$$= \frac{A^2}{2} E\left[\cos(\omega_0 t) + \cos(2\theta+2\omega_0 t+\omega_0\tau)\right]$$

$$= \frac{A^2}{2}\cos(\omega_0\tau) + \frac{A^2}{2}\int_0^{2\pi} \cancelto{\theta_0}{\cos(2\theta+2\omega_0 t+\omega_0\tau)} \cdot \frac{1}{2\pi}\,d\theta$$

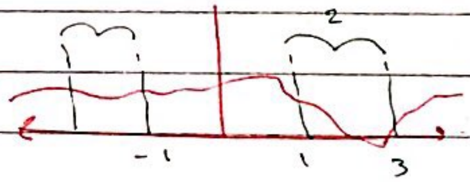
$$\Rightarrow R_{xx}(t, t+\tau) = \frac{A^2}{2}\cos(\omega_0\tau) \overset{\Delta}{=} R_{xx}(\tau)$$

So, $X(t)$ is WSS

- $R_{xx}(\tau)$ properties: $\tau = (-\infty, \infty)$

① $|R_{xx}(\tau)| \leq R_{xx}(0)$ (biggest value of $R_{xx}(\tau)$ is at zero)

② $R_{xx}(-\tau) = R_{xx}(\tau)$
($R_{xx}(\tau)$ is even function of τ)



③ $R_{xx}(0) = E[X^2(t)]$ total power

$t_2 - t_1 < 0$

$R_{xx}(1, 3) = R_{xx}(3, 1)$

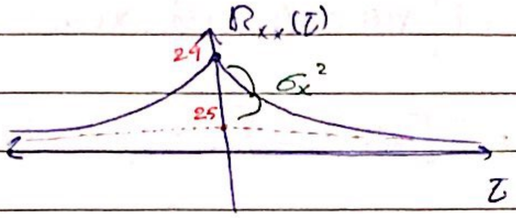
④ $\lim\limits_{\tau \to \infty} R_{xx}(\tau) = \bar{X}^2$

Ex: given a wss R.p with $R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$

Find ⓐ $E[x^2(t)]$

ⓑ The R.p mean

ⓒ The R.p variance



Sol:-

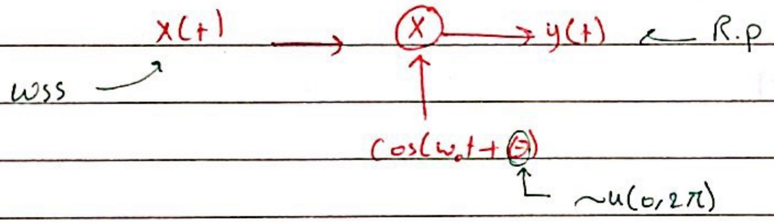
ⓐ $E[x^2(t)] = R_{xx}(0) = 29$

ⓑ $\lim_{\tau \to \infty} R_{xx}(\tau) = 25 = \bar{x}^2 \therefore \bar{x} = \sqrt{25}$

$= +5, -5$

ⓒ $Var(x(t)) = E[x^2(t)] - \bar{x}^2 = 29 - 25 = 4$

Ex: given x(t) a wss R.p with $R_{xx}(\tau) = e^{-a|\tau|}$



is y(t) wss ?!

$y(t) = x(t) \cos(w_0 t + \theta)$

$E[y(t)] = E[x(t) \cos(w_0 t + \theta)]$

x(t) & $\cos(w_0 t + \theta)$ are independent so they're uncorrelated

$= E[x(t)] \, E[\cos(w_0 t + \theta)]$

$= \bar{x} \quad E[\cos(w_0 t + \theta)]$

$\bar{x}^2 = \lim_{\tau \to \infty} R_{xx}(\tau) = 0 \quad \therefore \bar{x} = 0$

$E[\cos(w_0 t + \theta)] = 0$

$\Rightarrow E[y(t)] = 0 \triangleq \bar{y}$ ✓

4/12/2017

$$R_{yy}(t+\tau) \overset{?}{=} R_{yx}(\tau)$$

$$R_{yy}(t, t+\tau) = E\left[y(t)\, y(t+\tau)\right]$$

$$= E\left[x(t)\cos(\omega_0 t+\theta)\, .\, x(t+\tau)\cos(\omega_0 t+\omega_0\tau+\theta)\right]$$

$$= E\left[\underline{x(t)\, x(t+\tau)}\right] .\; E\left[\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)\right]$$

$$= R_{xx}(\tau)\;\; E\left[\frac{1}{2}\cos(\omega_0\tau) + \frac{1}{2}\cos(2\omega_0 t+\omega_0\tau+2\theta)\right]$$

$$= \frac{1}{2} R_{xx}(\tau)\cos(\omega_0\tau) + \frac{1}{2} R_{xx}(\tau)\,(0)$$

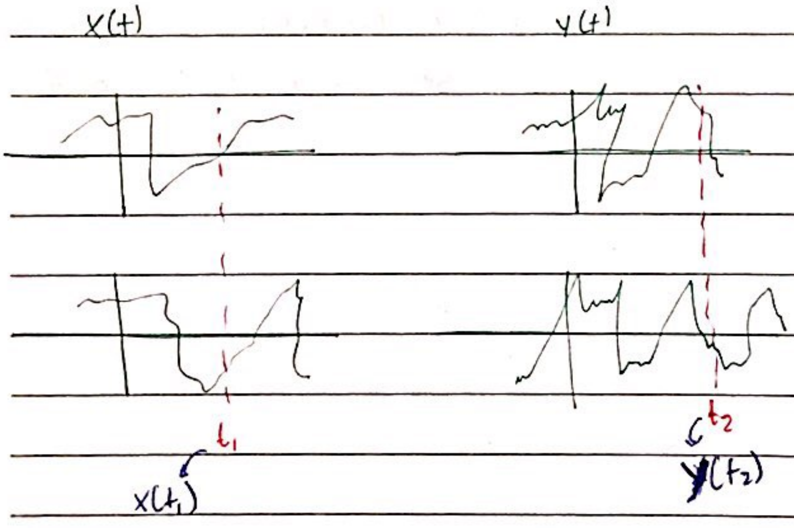
$$\Rightarrow R_{yy}(t, t+\tau) = \frac{1}{2} R_{xx}(\tau)\cos(\omega_0\tau)$$

$$= \frac{1}{2} e^{-a/\tau}\cos(\omega_0\tau)$$

$$\triangleq R_{yy}(\tau)$$

So, y(t) is wss

@ Cross - Correlation function :-



$R_{xy}(t_1, t_2) = E[x(t_1)\, y(t_2)]$

In general: $R_{xy}(t, t+\tau) = E[x(t)\, y(t+\tau)]$

- if $R_{xy}(t, t+\tau) = 0$, for all t, τ then; $x(t)$ & $y(t)$ are orthogonal

- if $R_{xy}(t, t+\tau) = E[x(t)]\, E[y(t+\tau)]$ then; $x(t)$ & $y(t)$ are uncorrelated.
  for any t & τ

- Cross - Covariance function :-

$$C_{xy}(t, t+\tau) = E[(x(t) - m_x(t))(y(t+\tau) - m_y(t+\tau)]$$

$$= R_{xy}(t, t+\tau) - m_x(t)\, m_y(t+\tau)$$

- joint wss :-

Def :- two R.p's $x(t)$ & $y(t)$ are said to be joint wss, if :-

① $x(t)$ is wss $\to m_x(t) = \bar{x}$
$\to R_{xx}(t, t+\tau) = R_{xx}(\tau)$

② $y(t)$ is wss $\to m_y(t) = \bar{y}$
$\to R_{yy}(t, t+\tau) = R_{yy}(\tau)$

③ $R_{xy}(t, t+\tau) = R_{xy}(\tau)$

↳ as a result:

$$C_{xy}(t, t+\tau) = R_{yy}(\tau) - \bar{x}\bar{y} \triangleq C_{xy}(\tau)$$

⑪ Gaussian R. process :

Def : A R.p $x(t)$ is said to be gaussian if the R.v's $X_1, X_2, \dots, X_N$ corresponding to time instants $t_1, t_2, \dots, t_N$ are jointly gaussian, for any N & $t_1, t_2, \dots, t_N$, i.e, their joint density function is given by :-

$x(t_1) = x_1$, $x(t_2) = x_2$, ---

$$f_X(x_1, x_2, \dots, x_N ; t_1, t_2, \dots, t_N) = \frac{\left|[C_x]^{-1}\right|^{1/2}}{(2\pi)^{N/2}} e^{-\frac{[x-\bar{x}]^T [C_x]^{-1} [x-\bar{x}]}{2}}$$

where
$$[X-\bar{X}] = \begin{bmatrix} x_1 - m_x(t_1) \\ X_2 - m_x(t_2) \\ \vdots \\ X_N - m_x(t_N) \end{bmatrix}$$

$$\underset{N \times N}{[C_x]} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1N} \\ C_{21} & C_{22} & \dots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \dots & C_{NN} \end{bmatrix}$$

$$C_{ij} = C_{XX}(t_i, t_j) = R_{xx}(t_i, t_j) - m_x(t_i)\, m_x(t_j)$$

$i = 1, 2, \dots, N$

$j = 1, 2, \dots, N$

as special case :- if $x(t)$ is WSS $\curvearrowright m_x(t) = \bar{X}$

$$[X - \bar{X}] = \begin{bmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_N - \bar{X} \end{bmatrix}$$

$$C_{ij} = C_{XX}\underbrace{(t_j - t_i)}_{\tau} = R_{XX}\underbrace{(t_j - t_i)}_{\tau} - \bar{X}^2$$

Ex3- given x(t) a wss R.P with $\bar{X}=4$ & $R_{XX}(\tau)=25e^{-3|\tau|}+16$

Determine Covariance matrix for the three R.V's $\underset{X_1}{\underline{x(t_1)}}, \underset{X_2}{\underline{x(t_2)}}, \underset{X_3}{\underline{x(t_3)}}$

where $t_j = t_0 + \frac{j-1}{2}$, $j=1,2,3$

$$\underset{3\times3}{[C_X]} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{ij} = C_{XX}(t_j - t_i)$$

$$= R_{XX}(\underbrace{t_j - t_i}) - \bar{X}^2$$

$$= 25e^{-3|t_j - t_i|} + 16 - 4^2$$

$$\boxed{C_{ij} = 25e^{-3|t_j - t_i|}}$$

$t_1 = t_0$

$t_2 = t_0 + \frac{1}{2}$

$t_3 = t_0 + 1$

$C_{11} = 25e^{-3|t_1 - t_1|} = 25 = C_{22} = C_{33} = \sigma_X^2$ (bc it's wss) σ_X^2 is Constant

$C_{12} = 25e^{-3|t_2 - t_1|} = 25e^{-3/2} = C_{21}$

$C_{13} = 25e^{-3|t_3 - t_1|} = 25e^{-3} = C_{31}$

$C_{23} = 25e^{-3|t_3 - t_2|} = 25e^{-3/2} = C_{32}$

$$[C_X] = \begin{bmatrix} 25 & 25e^{-3/2} & 25e^{-3} \\ 25e^{-3/2} & 25 & 25e^{-3/2} \\ 25e^{-3} & 25^{-3/2} & 25 \end{bmatrix} \qquad [X - \bar{X}] = \begin{bmatrix} X_1 - 4 \\ X_2 - 4 \\ X_3 - 4 \end{bmatrix}$$

⊛ Time - average & Ergodicity :-

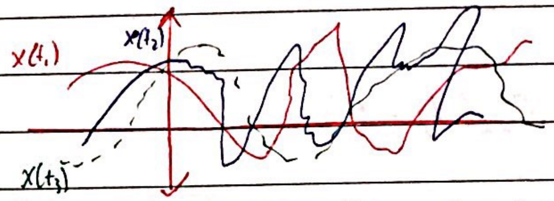
R.p $x(t)$ $\rightarrow$ $m_x(t) = E[x(t)] = \int x\, f_x(x;t)\, dx$

$\rightarrow$ $R_{xx}(t, t+\tau) = E[x(\tau)\; x(t+\tau)]$

$= \iint x_1 x_2\, f_x(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$

practically :- we estimate mean :-

$\hat{m}_x(t) = \frac{1}{k} \sum_{i=1}^{k} x_i(t)$ k is large



$\hat{R}_{xx}(t, t+\tau) = \frac{1}{k} \sum_{i=1}^{k} x_i(t)\, x_i(t+\tau)$

$\boxed{\text{e.g}}$:- we add them then divide them by 3 to get the avg. signal

Ergodicity :-

Def: A wss R.p $x(t)$ is said to be ergodic if

① $\bar{x} \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$

$= m_x(t)$ statistical average

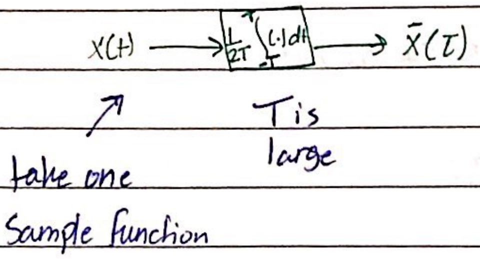
time average for any $x(t)$ → sample function

② $R_{xx}(\tau) \triangleq \boxed{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt}$

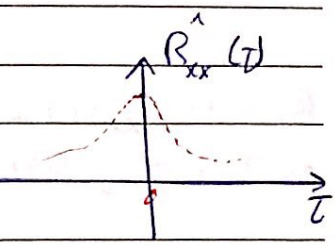
$E[x(t), x(t+\tau)]$ statistical auto-correlation

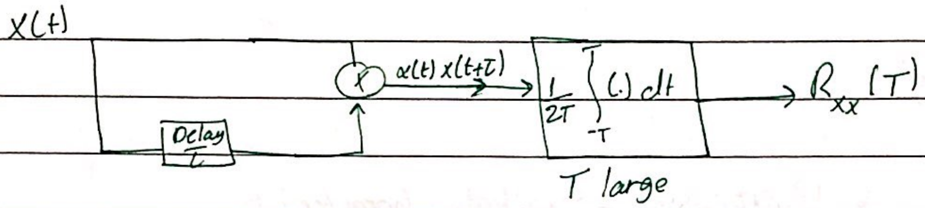
time - average auto Correlation function.

how to measure the $\bar{x}$



how to measure $R_{xx}(\tau)$





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⊛ Ex : given wss R.p $X(t) = \underbrace{A}\cos(\underbrace{\omega_0}t+\theta)$ Find :-

Constants

$\llcorner U(0,2\pi)$

① $R_{xx}(\tau)$

② measure $R_{xx}(\tau)$

① $R_{xx}(\tau) = R_{xx}(t, t+\tau) = E[x(t)\,x(t+\tau)]$.

$= E[A^2\cos(\omega_0t+\theta)\cdot\cos(\omega_0t+\omega_0\tau+\theta)] = \dots = \frac{A^2}{2}\cos(\omega_0\tau)$

② take one sample function $x(t) = A\cos(\omega_0t+\theta)$

↑ it's constant

$R_0(2\cancel{T}) = \frac{1}{T}\int_{-T}^{+T} x(t)\,x(t+\tau)\,dt$

↓ T

$$= \frac{1}{T}\int_{-T}^{T} A^2 \cos(\omega_o t + \theta^c)\cos(\omega_o t + \omega_o \tau + \theta^c)\, dt$$

$$= \frac{1}{T}\int_{-T}^{T} \frac{A^2}{2}\cos(\omega_o \tau)\, dt + \frac{1}{T}\int_{-T}^{T} \frac{A^2}{2}\cos(2\omega_o t + \omega_o \tau + 2\theta^c)\, dt$$

$$= \frac{A^2}{2}\cos(\omega_o \tau) \underbrace{+ \frac{A^2}{2}\cos(\omega_o \tau + 2\theta^c)\,\frac{\sin(2\omega_o T)}{2\omega_o T}}_{\text{error}} \quad E(2T)$$

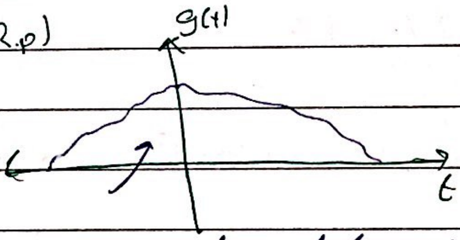
$R_o(2T) = R_{xx}(\tau) + E(T)$

as $T \to \infty$, $E(T) \to 0$

& $R_o(2T) \longrightarrow R_{xx}(\tau)$

⑩ CHAPTER 7 : R.processes – spectral characteristics

Recall :- $g(t) \to$ deterministic signal (not R.p)



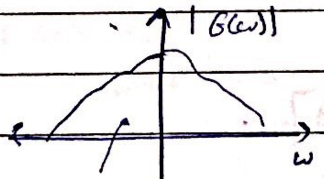
time-limited signal (energy signal)

$$E_{gg} = \int_{-T}^{T} g^2(t)\, dt$$

$g(t) \xleftarrow{\;FT\;}\!\!\xrightarrow{} G(\omega)$

spectrum

$$= \int_{-\infty}^{\infty} g(t) e^{-j\omega t}\, dt$$

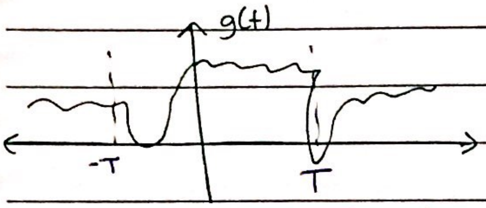


voltage spectrum density

$$E_{gg} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2\, d\omega$$

↗ (parseval's Theorem)

- If the signal $g(t)$ is unbounded (power signal)



$$P_{gg} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g^{2}(t)\, dt$$

To find FT :-

$$g_T(t) = \begin{cases} g(t) , & -T < t < +T \\ 0 , & o.w \end{cases}$$

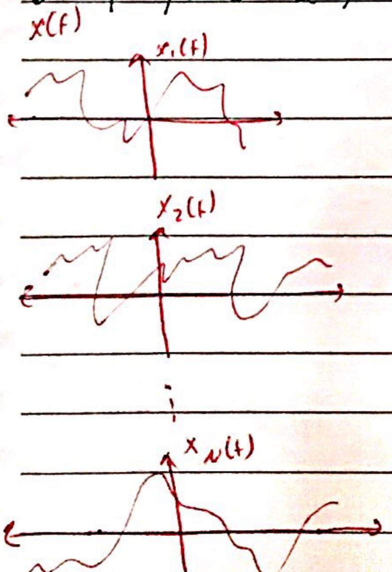
$$g_T(t) \xleftrightarrow{FT} G_T(\omega) = \int_{\cancel{-\infty}\ -T}^{\cancel{\infty}\ T} \overset{g(t)}{g_T(t)}\, e^{-j\omega t}\, dt \quad = \int_{-T}^{T} g(t)\, e^{-j\omega t}\, dt$$

$$g(t) \longleftrightarrow \lim_{T\to\infty} \int_{-T}^{T} g(t)\, e^{-j\omega t}\, dt$$

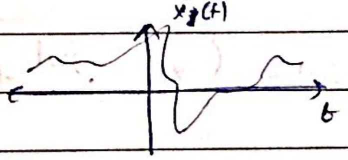
$$= G(\omega)$$

$$\underset{\uparrow \atop \text{watts}}{P_{gg}} = \lim_{T\to\infty} \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{|G(\omega)|^{2}}{2T}\, d\omega}_{Hz \times \frac{\omega}{Hz}} \quad \rightsquigarrow \text{power in frequency domain (PSD)}$$

- R.p power & power Density Spectrum :-



- Pick one sample function : $x^{i}(t)$ ← i th sample function



- take $x_T^{i}(t) = \begin{cases} x(t) , & -T < t < T \\ 0 , & o.w \end{cases}$

$$X_T^i(t) \xleftrightarrow{FT} X_T^i(\omega)$$

$$= \int_{-T}^{T} x^i(t)\, e^{-j\omega t}\, dt$$

$$\boxed{\rho^i = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^{i^2}(t)\, dt} \qquad \rho^i = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T^i(\omega)|^2}{2T}\, d\omega$$

Random processes average power :-

$$P_{XX} = E[\rho^i]$$

$$= E\left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^{i^2}(t)\, dt\right]$$

$$\boxed{P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[x^2(t)]\, dt} \; \cancel{= A}$$

$$= A\left[E[x^2(t)]\right]$$

In frequency domain :-

$$P_{XX} = E[\rho^i] = E\left[\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T} \frac{|X_T^i(\omega)|^2}{2T}\, dt\right]$$

$$\boxed{P_{XX} = \frac{1}{2\pi} \int_{-T}^{T} \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}}\, d\omega} \rightarrow P_{XX}(\omega) = PDS$$

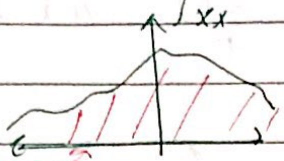
$$\boxed{X_T(\omega) = \int_{-T}^{T} X(t)\, e^{-j\omega t}\, dt}$$

R.p expression

R.P $x(t)$ PDS :-

$$P_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

So; $P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$



Ex : given $x(t) = A_0 \cos(\omega_0 t + \underset{\uparrow U(0,\frac{\pi}{2})}{\theta})$ Find :-

ⓐ P_{xx} (in time domain)

ⓑ $S_{xx}(\omega)$ ⓒ P_{xx} using part ⓑ

a) $P_{xx} = A\left[E[x^2(t)]\right]$

• $E[x^2(t)] = E\left[A_0^2 \cos^2(\omega_0 t + \theta)\right] = E\left[\frac{A_0^2}{2} + \frac{A_0^2}{2}\cos(2\theta + 2\omega_0 t)\right]$

$= \frac{A_0^2}{2} + \frac{A_0^2}{2}\int_0^{\pi/2} \cos(2\theta + 2\omega_0 t)\, \frac{2}{\pi}\, d\theta$

$= \frac{A_0^2}{2} + -\frac{A_0^3}{2}\sin(2\omega_0 t)$

$P_{xx} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{A_0^2}{2} - \frac{A_0^2}{2}\sin(2\omega_0 t)\, dt$

$= \frac{A_0^2}{2} - \cancel{\frac{A_0^2}{4T}\int_{-T} \sin(2\omega_0 t)\, dt} \rightarrow 0$

$$\boxed{P_{x\alpha} = \frac{A_0^2}{2}}$$

ⓑ $S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(\omega)|^2\right]$

• $X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt = \int_{-T} A_0 \underbrace{\cos(\omega_0 t + \theta)}\, e^{-j\omega t})\, dt$

$\hookrightarrow = \frac{e^{j(\omega_0 t + \theta)} + e^{-j(\omega_0 t + \theta)}}{2}$

$= \frac{A_0}{2} e^{j0} \int e^{j(\omega_0 - \omega)t}\, dt + \frac{A_0}{2} e^{-j\theta} \int_{T}^{T} e^{-j(\omega_0 + \omega)t}\, dt$

$$= A_0 T e^{j\theta} \frac{\sin((\omega-\omega_0)T)}{(\omega-\omega_0)T} + A_0 T e^{-j\theta} \frac{\sin((\omega+\omega_0)T)}{(\omega+\omega_0)T}$$

$$|X_T(\omega)|^2 = X_T(\omega)\, X_T^*(\omega)$$

$$\frac{1}{2T} E\left[X_T(\omega)\, X_T^*(\omega)\right]$$

$$= \frac{A_0^2\pi}{2}\left[\frac{T}{\pi}\,\frac{\sin^2(\omega-\omega_0)T}{[(\omega-\omega_0)T]^2} + \frac{T}{\pi}\,\frac{\sin^2((\omega+\omega_0)T)}{((\omega+\omega_0)T)^2}\right] \quad \circledast$$

$$\lim_{T\to\infty} \circledast = \frac{A_0^2\pi}{2}\,\delta(\omega-\omega_0) + \frac{A_0^2\pi}{2}\,\delta(\omega+\omega_0)$$

$$\triangleq \mathcal{S}_{XX}(\omega)$$

ⓒ $$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\,d\omega = \frac{1}{2\pi}\,\frac{A_0^2\pi}{2}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,d\omega + \frac{1}{2\pi}\,\frac{A_0^2\pi}{2}\int_{-\infty}^{\infty}\delta(\omega+\omega_0)\,d\omega$$

$$= A_0^2/2$$

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⑪ R. processes _ spectral characteristics :-

$$X(t) \longrightarrow S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(\omega)|^2\right]$$

"PDS"

$$X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

total average power in x(t) $\downarrow$

$$P_{XX} = A\left[E[X^2(t)]\right] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)]\, dt$$

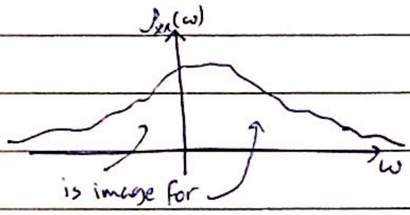
$$P_{XX} = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}_{\text{Area under } S_{XX}(\omega)}$$

PDS properties :-

① $S_{XX}(\omega) \geq 0$

② $S_{XX}(-\omega) = S_{XX}(\omega)$, if $x(t)$ is real

even function



③ $S_{XX}(\omega)$ is real function

④ $X(t) \downarrow S_{XX}(\omega)$ | $\dot{X}(t) = \frac{dX(t)}{dt} \downarrow S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$

⑤ $\underset{\sim\sim}{S_{XX}(\omega)} = \int_{-\infty}^{\infty} A\left[R_{XX}(t, t+\tau)\right] e^{-j\omega\tau} \, d\tau$ $\circledast\circledast$

i.e ; $S_{XX}(\omega) = FT\left\{ A\left[R_{XX}(t, t+\tau)\right] \right\}$

⑥ $A\left[R_{XX}(t, t+\tau)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \, e^{+j\omega\tau} \, d\omega$

i.e ; $A\left[R_{XX}(t, t+\tau)\right] = FT^{-1}\left\{ S_{XX}(\omega) \right\}$

So ; $A\left[R_{XX}(t, t+\tau)\right]$ & $S_{XX}(\omega)$ Form a FT pair

$$A\left[R_{XX}(t, t+\tau)\right] \xleftrightarrow{FT} S_{XX}(\omega)$$

as a special case ; if $X(t)$ is WSS R.P :-

$R_{XX}(t, t+\tau) = R_{XX}(\tau)$, then

$A\left[R_{XX}(\tau)\right] = R_{XX}(\tau)$, then

- $S_{XX}(\omega) = FT\left\{ R_{XX}(\tau) \right\} = \int_{-\infty}^{\infty} R_{XX}(\tau) \, e^{-j\omega\tau} \, d\tau$
- $R_{XX}(\tau) = FT^{-1}\left\{ S_{XX}(\omega) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \, e^{+j\omega\tau} \, d\omega$

$$R_{XX}(\tau) \xleftrightarrow{FT} S_{XX}(\omega)$$

Ex:- given $x(t) = A\cos(\omega_0 t + \underset{\sim U(0,2\pi)}{\theta})$

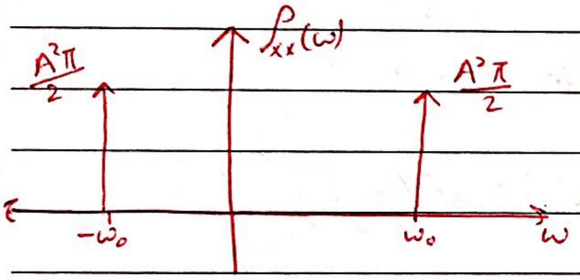
Find $S_{xx}(\omega)$ using ⊛⊛

$$R_{xx}(t, t+\tau) = E[x(t)\,x(t+\tau)] = \cdots = \frac{A^2}{2}\cos(\omega_0\tau) \triangleq R_{xx}(\tau)$$

so ; $S_{xx}(\omega) = FT\{R_{xx}(\tau)\} = FT\{\frac{A^2}{2}\cos(\omega_0\tau)\}$

$$\cos(\omega_0\tau) \xleftrightarrow{FT} \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

$$\underbrace{\frac{A^2}{2}\cos(\omega_0\tau)}_{R_{xx}(\tau)} \xleftrightarrow{FT} \underbrace{\frac{A^2\pi}{2}\delta(\omega-\omega_0) + \frac{A^2\pi}{2}\delta(\omega+\omega_0)}_{S_{xx}(\omega)}$$

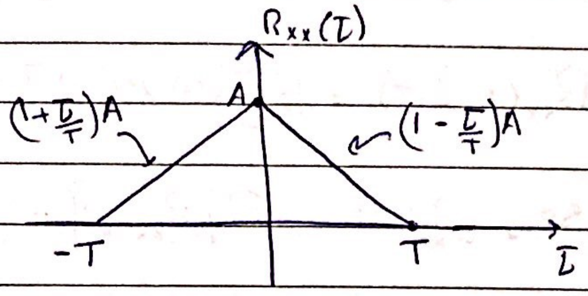


$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega = \left(\frac{1}{2\pi}\int_{0}^{\infty} S_{xx}(\omega)\,d\omega\right)\cdot 2$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{A^2\pi}{2}\,\delta(\omega-\omega_0)\,d\omega \quad = \frac{A^2}{2}\cdot(1) = \frac{A^2}{2}$$

Ex: given WSS R.P $X(t)$ with $R_{xx}(\tau) = \begin{cases} A\left(1-\frac{|\tau|}{T}\right) , & -T<\tau<T \\ 0 , & o.w \end{cases}$

Find $S_{xx}(\omega)$?!



$$R_{xx}(\tau) = A\,\text{Tri}\left(\frac{\tau}{T}\right)$$

$$S_{xx}(\omega) = FT\{R_{xx}(\tau)\} = \int_{-\infty}^{\infty} R_{xx}(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$= \int_{-T}^{0} A\left(1+\frac{\tau}{T}\right)e^{-j\omega\tau}\,d\tau + \int_{0}^{T} A\left(1-\frac{\tau}{T}\right)e^{-j\omega\tau}\,d\tau$$

$$= \int_{-T}^{0} A e^{-j\omega\tau}\,d\tau + \int_{-T}^{0} \frac{A}{T}\tau\, e^{-j\omega\tau}\,d\tau + \int_{0}^{T} A e^{-j\omega\tau}\,d\tau - \int_{0}^{T} \frac{A}{T}\tau\, e^{-j\omega\tau}\,d\tau$$

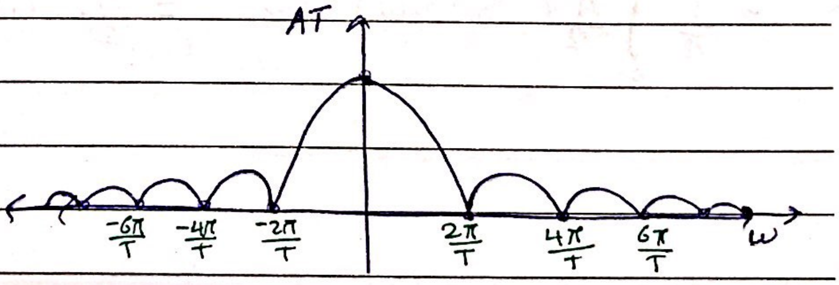
$=$ ⋮

* using the FT pair table :-

$$\text{tri}\left(\frac{\tau}{T}\right) \longleftrightarrow T\,\text{sinc}^2\left(\frac{\omega T}{2}\right)$$

$$= T\,\text{Sinc}^2\left(\frac{\omega T}{2}\right)$$

$$R_{xx}(\tau) = A\,\text{tri}\left(\frac{\tau}{T}\right) \xleftrightarrow{FT} S_{xx}(\omega) = AT\,\text{sinc}^2\left(\frac{\omega T}{2}\right)$$



nulls : $\frac{\omega T}{2} = \pm n\pi$

$$\omega_n = \pm\frac{n2\pi}{T}$$

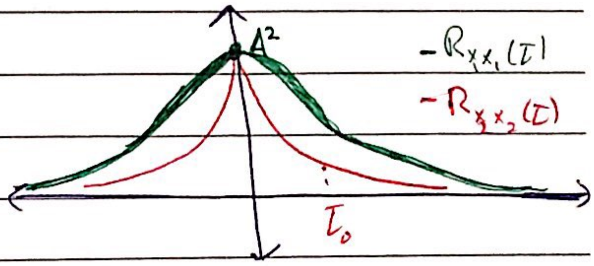
Ex: suppose we have two WSS R.p's

$X_1(t)$: $R_{X_1X_1}(\tau) = \sigma^2 e^{-|\tau|}$

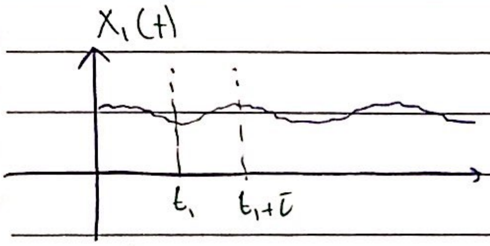
$X_2(t)$: $R_{X_2X_2}(\tau) = \sigma^2 e^{-2|\tau|}$

· $P_{X_1X_1} = R_{X_1X_1}(0) = A^2$

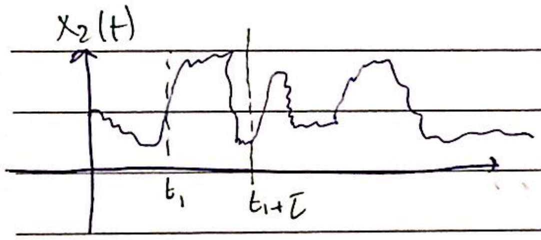
· $P_{X_2X_2} = R_{X_2X_2}(0) = A^2$



$R_{X_2X_2}(\tau_0) < R_{X_1X_1}(\tau_0)$



values are close to each other so $R_{XX}(\tau)$ will be big value

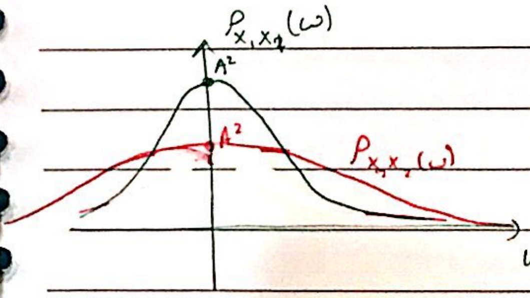


values are far from each other so $R_{XX}(\tau)$ will be small value

* Because $R_{X_2X_2}(\tau) < R_{X_1X_1}(\tau)$: $X_2(t)$ has faster variance.

· $S_{X_1X_1}(\omega) = FT\{\sigma^2 e^{-|\tau|}\} = \frac{2A^2}{\omega^2+1}$

· $S_{X_2X_2}(\omega) = FT\{A^2 e^{-2|\tau|}\} = \frac{4A^2}{\omega^2+4}$



$$P_{X_2X_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{X_2X_2}(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{\infty}\frac{4A^2}{\omega^2+A^2}\,d\omega = A^2$$

$$P_{X_1X_1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{X_1X_2}(\omega)\,d\omega = A^2$$

18/12/2017

⑩ R.P - spectral characteristics :-

$$X(t) \longrightarrow S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\Big[_T |X_T(\omega)|^2\Big]$$

$$X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

$$S_{XX}(\omega) = FT\Big\{A\big[R_{XX}(t, t+\tau)\big]\Big\}$$

$$R_{XX}(t, t+t) \xleftrightarrow{FT} S_{XX}(\omega)$$

WSS:

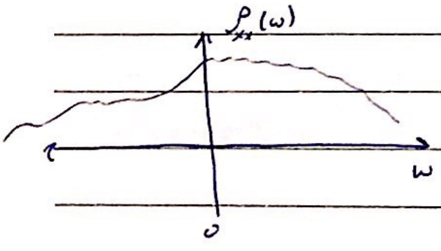
$$R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$$

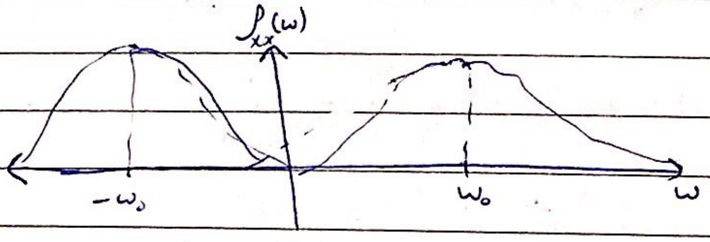
is real and even

⑪ A R.P $x(t)$ Can be Classified as :-

① Baseband
most of the process power is clustered around $W=0$

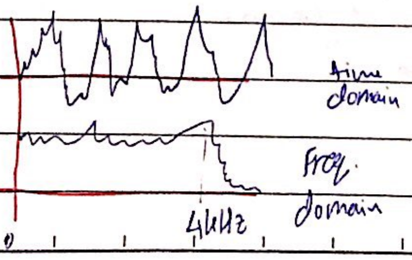
② Bandpass
most of the process power is clustered about a certain freq $w=w_0$





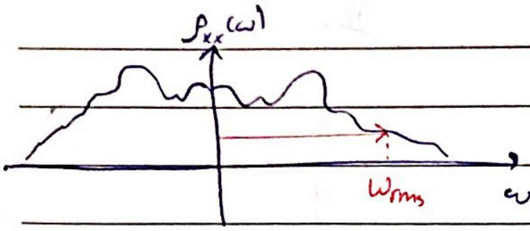
Ex: Information (messages)
e.g:- human speech

Ex:- modulated signals

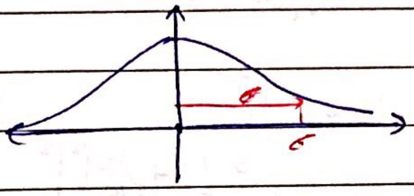


⊛ R.P Bandwidth :- (root-mean-square -BW) "ω_{rms}"

① rms-BW for baseband R.P's :-



$X \sim N(0, \sigma^2)$



$$W_{rms} = \sqrt{\int \omega^2 \overset{norm}{\rho_{xx}(\omega)}\, d\omega}$$

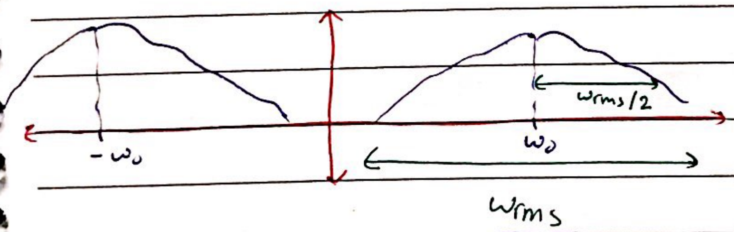
$$\sigma = \sqrt{\sigma^2} = \sqrt{Var(x)}$$
$$= \sqrt{E[x^2] - \cancel{\bar{x}^2}}$$
$$= \sqrt{\int_{-\infty}^{\infty} x^2 f_x(x)\, dx}$$

$$\overset{norm}{\rho}_{xx}(\omega) = \frac{\rho_{xx}(\omega)}{\int_{-\infty}^{\infty} \rho_{xx}(\omega)\, d\omega}$$

has area=1

$$\Rightarrow W_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty} \omega^2 \rho_{xx}(\omega)\, d\omega}{\int_{-\infty}^{\infty} \rho_{xx}(\omega)\, d\omega}}$$

② rms-BW for Bandpass R.P :-

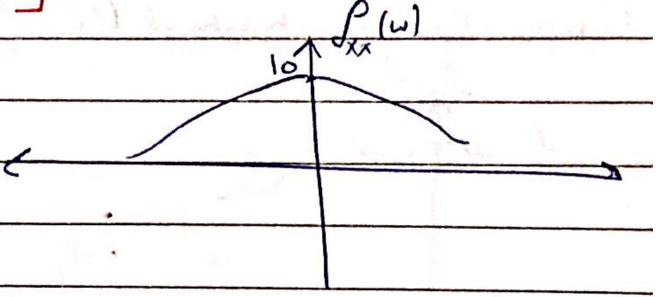


$$\frac{W_{rms}}{\cancel{2}} = 2\sqrt{\frac{\int_{-\infty}^{\infty} (\omega-\omega_0)^2 \rho_{xx}(\omega)\, d\omega}{\int_{-\infty}^{\infty} \rho_{xx}(\omega)\, d\omega}}$$

Ex:- given $x(t)$ with $S_{xx}(\omega) = \dfrac{10}{\left[1+\left(\frac{\omega}{10}\right)^2\right]^2}$, Find ω_{rms} ?

Sol :-

$$\omega_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty} \omega^2 S_{xx}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega}}$$



∴ (Baseband)

$$\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{10}{\left[1+\left(\frac{\omega}{10}\right)^2\right]^2}\, d\omega \times \frac{10^4}{10^4}$$

$$= 10^5 \int \frac{1}{[100+\omega^2]}\, d\omega \underset{\substack{\downarrow \\ \text{using} \\ \text{table of} \\ \text{integral}}}{=} 50\pi$$

$$\int_{-\infty}^{\infty} \omega^2 S_{xx}(\omega)\, d\omega = 10^5 \int_{-\infty}^{\infty} \frac{\omega^2}{[100+\omega^2]^2}\, d\omega \underset{\downarrow}{=} 500\pi$$

$$\omega_{rms} = \sqrt{\frac{500\pi}{50\pi}} = 10 \text{ rad/sec}$$

● Cross power Density spectrum :-

Suppose we have

$x(t)$	$y(t)$
~~$S_{xx}(\omega) =$~~	
$S_{xx}(\omega) = \lim\limits_{T\to\infty} \frac{1}{2T} E\left[\lvert X_T(\omega)\rvert^2\right]$	$S_{YY}(\omega) = \lim\limits_{T\to\infty} \frac{1}{2T} E\left[\lvert Y_T(\omega)\rvert^2\right]$
$X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$	$Y_T(\omega) = \int_{-T}^{T} y(t)\, e^{-j\omega t}\, dt$

Cross – PDS :-

$$S_{XY} = \lim_{T\to\infty} \frac{1}{2T} E\left[X_T(\omega) \cdot Y_T^*(\omega)\right] \qquad S_{XY}^* = S_{YX}$$

$$S_{YX} = \lim_{T\to\infty} \frac{1}{2T} E\left[Y_T(\omega)\, X_T^*(\omega)\right] \qquad S_{YX}^* = S_{XY}$$

In general Complex functions

- Cross. power

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega \qquad P_{XY} = P_{YX}^*$$

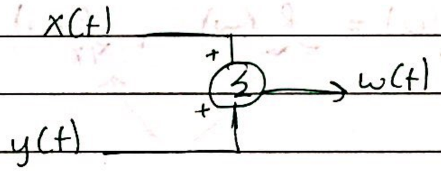
$$P_{YX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)\, d\omega \qquad P_{YX} = P_{XY}^*$$

let $w(t) = X(t) + y(t)$

Determine 1) $R_{ww}(t, t+\tau)$

2) $S_{ww}(\omega)$

3) P_{ww}



so 1) $R_{ww}(t, t+\tau) = E\left[w(t)\, w(t+\tau)\right]$

$$= E\left[(x(t)+y(t))(x(t+\tau)+y(t+\tau))\right]$$

$$= E\left[x(t)\,x(t+\tau) + y(t)\,y(t+\tau) + \underline{x(t)\,y(t+\tau)} + y(t)\,x(t+\tau)\right]$$

$$= R_{XX}(t, t+\tau) + R_{YY}(t, t+\tau) + \cancel{R_{YX}(t, t+\tau)} + R_{XY}(t, t+\tau) + R_{YX}(t, t+\tau)$$

auto correlation function

Cross-~~corelation~~ Correlation function.

As a special case ; when $x(t)$ & $y(t)$ are orthogonal :

then $R_{xy}(t,\ t+\tau) = R_{yx}(t,\ t+\tau) = 0$

then $R_{ww}(t, t+\tau) = R_{xx}(t, t+\tau) + R_{yy}(t, t+\tau)$

if $x(t)$ and $y(t)$ are $\perp$ & wss

$R_{ww}(\tau) = R_{xx}(\tau) + R_{yy}(\tau)$

$$\overline{w} = \overline{x} + \overline{y}$$

2) $S_{ww}(\omega) = \cancel{[} FT\left\{A\left[R_{ww}(t, t+\tau)\right]\right\}$

$$= FT\left\{A\left[R_{xx}(t,t+\tau) + R_{yy}(t,t+\tau) + R_{xy}(t,t+\tau) + R_{yx}(t,t+\tau)\right]\right\}$$

$$S_{ww}(\omega) = \underbrace{S_{xx}(\omega) + S_{yy}(\omega)}_{\text{PDS's}} + \underbrace{S_{xy}(\omega) + S_{yx}(\omega)}_{\text{Cross\_PDS's}} \ldots \circledast$$

as special case : $x(t) \perp y(t) \Longrightarrow S_{ww}(\omega) = S_{xx}(\omega) + S_{yy}(\omega)$

3) $P_{ww} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{ww}(\omega)\, d\omega$

$$= \underbrace{P_{xx} + P_{yy}}_{\text{average powers}} + \underbrace{P_{xy} + P_{yx}}_{\text{cross-average powers}}$$

subt $\circledast$

if $x(t)$ & $y(t)$ are $\perp$

then $P_{ww} = P_{xx} + P_{yy}$

Cross PDS properties :- $S_{xy}(\omega)$ & $S_{yx}(\omega)$

1) $S_{xy}(\omega) = S^{*}_{yx}(\omega)$

$S_{xy}(-\omega) = S^{*}_{yx}(\omega)$

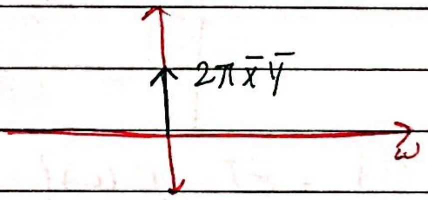
2) $\mathrm{Re}\{S_{xy}(\omega)\}$ & $\mathrm{Re}\{S_{yx}(\omega)\}$ are even.

3) $\mathrm{Im}\{S_{xy}(\omega)\}$ & $\mathrm{Im}\{S_{yx}(\omega)\}$ are odd.

4) If x(t) & y(t) are uncorrelated and have constant means, then

$$S_{xy}(\omega) = S_{yx}(\omega) = 2\pi \bar{X}\bar{Y}\,\delta(\omega).$$

$$R_{xy} = R_{yx} = \bar{X}\bar{Y}$$



Proof :- $S_{xy}(\omega) = FT\{A[R_{xy}(t, t+\tau)]\}$

$$1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$$

$$= FT\{A[E[x(t) \cdot y(t+\tau)]]\}$$

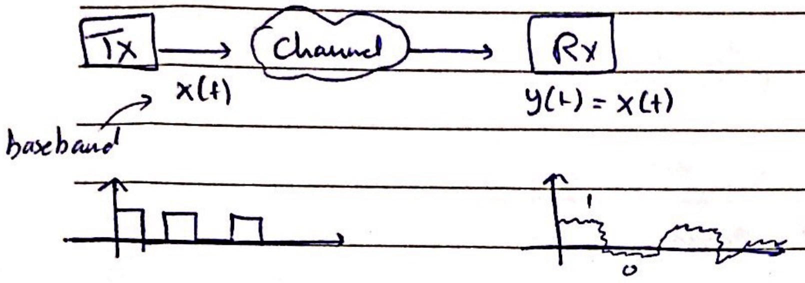
$$= FT\{A[\underset{\bar{X}}{E[x(t)]} \cdot \underset{\bar{Y}}{E[y(t+\tau)]}]\}$$

$$= FT\{\bar{X}\bar{Y}\} = 2\pi\bar{X}\bar{Y}\,\delta(\omega)$$

21/12/2017

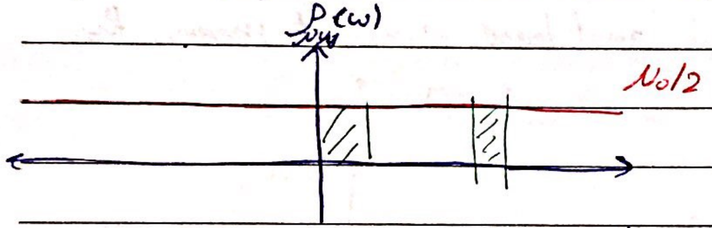
Noise process : $N(t)$ AWGN

In any Communication System

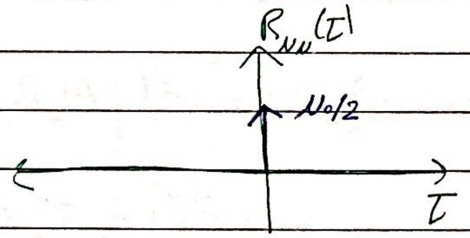


(*) white noise :-

any Noise process $N(t)$ is said to be white. "wss"



$$R_{NN}(\tau) = FT^{-1}\{S_{NN}(\omega)\} = FT^{-1}\{N_0/2\} = \frac{N_0}{2}\delta(\tau)$$

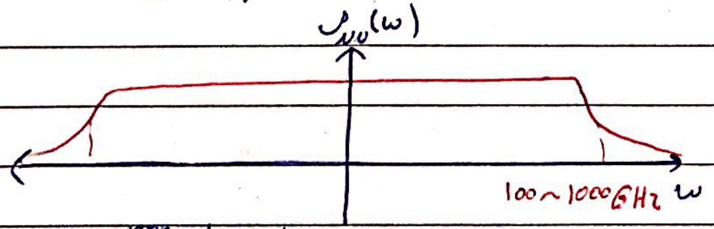


$$SNR = \frac{P_y}{P_N}$$

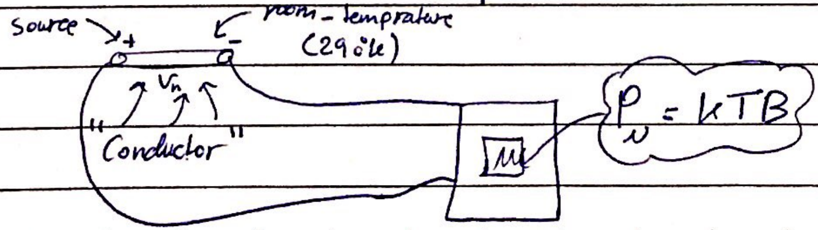
$$P_N = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\,d\omega = \infty$$

Themal Noise :-

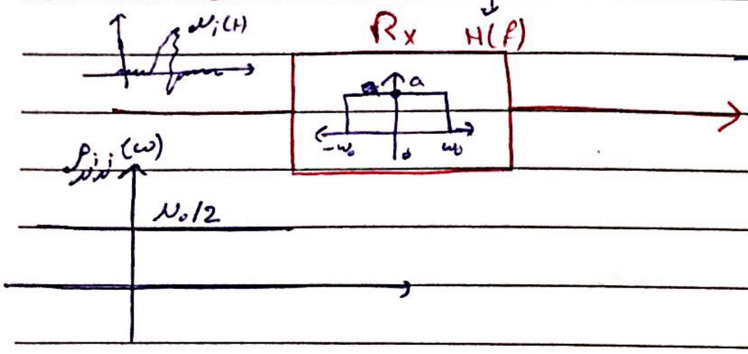
a real-world closely approximates white noise.



Def :-



Bandlimited Noise :-

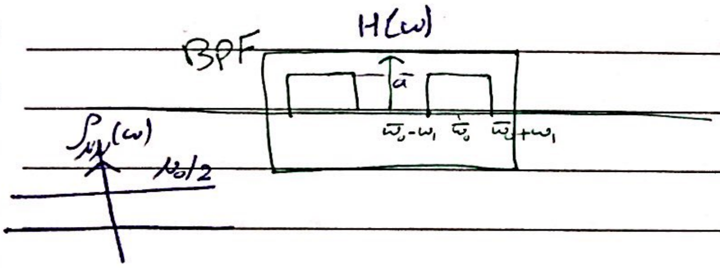


$$S_{n_o n_o}(\omega) = S_{n_i n_i}(\omega)\,|H(\omega)|^2$$

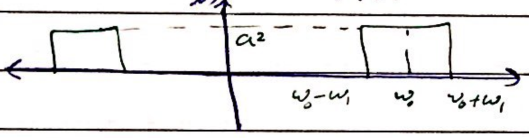
$$P_{N_0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_o n_o}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\,\frac{N_0}{2}\,a^2 \cdot \bar{\omega}_0 \cdot 2$$

$$= \frac{N_0\,a^2\,\bar{\omega}_0}{2\pi} \text{ watt}$$



$$S_{n_o n_o}(\omega) = S_{n_i n_i}(\omega)\,|H(\omega)|^2$$



$$P_{N_0 N_0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{N_0 N_0}(\omega)\,d\omega$$

$$= \frac{1}{2\pi} \cdot 2\, a^2 \cdot 2\omega_1 \cdot \frac{N_0}{2}$$

$$= \frac{a^2\,\omega_1\,N_0}{\pi}$$

Noise → external

Noise → Internal

* white (Thermal) Noise