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* Set Operations :-

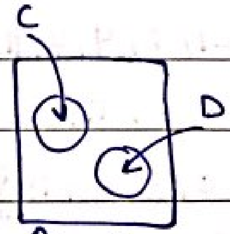
1. Set equality. ($A = B$: when A and B share same elements)

$A \subseteq B$ and $B \subseteq A$

ex:- $A = \{4, 6, 10, 11\}$, $B = \{6, 10, 4, 11\}$, $C = \{4, 6, 10, 3\}$

$A = B$, $A \neq C$

2. Difference

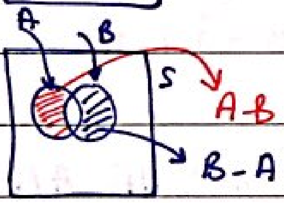


$C - D = C$

$D - C = D$

$A - B$: All elements in A but not in B

$B - A$: All elements in B but not in A



Ex: $A = \{3, 4, 11, 12, 14\}$

$B = \{-1, 2, 4, 12, 16\}$

$A - B = \{3, 11, 14\}$

$B - A = \{-1, 2, 16\}$ Note:- $A - B \neq B - A$.

* Mathematical Model of expirements :-

1. Sample space: The set of all possible outcomes of the exp.

Ex: ⓐ flip a coin → $S = \{H \text{ ① } T\}$ (① → or)

ⓑ Roll a die → $S = \{1, 2, 3, 4, 5, 6\}$

2. Events: event A is a subset of S.

Ex: Roll a die :- A = the appeared number is even. $\{2, 4, 6\}$

B = the appeared number is integer. $\{1, 2, 3, 4, 5, 6\} = S$

C = the appeared number is negative. $\emptyset$

D = the appeared number is $4 < D < 5$. $\emptyset$

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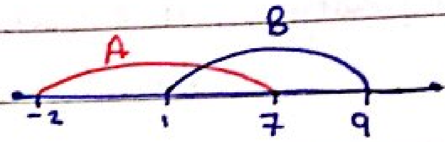
Ex:- $A = \{-2 < a < 7\}$, $B = \{1 \leq b < 9\}$

$A - B = \{-2 < d < 1\}$

$A \cap B = \{1 < 2 < 7\}$

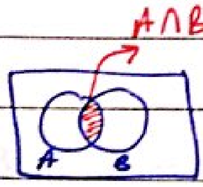


3 Intersection ($\cap$)

$A = \{-1, 2, 4, 7, 11\}$

$B = \{2, 6, 11, 13\}$

$C = \{0, 3, 6, 15\}$



"$\cap$" = and = together

$A \cap B = \{2, 11\}$ $\quad$ $A \cap C = \{\ \}$

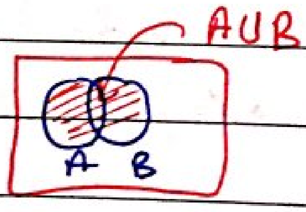
Note: IF $A \cap B = \phi$, then we say A and B are disjoint or mutually execlusive

Note: $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \bigcap_{i=1}^{N} A_i$

4 Union ($\cup$)

$A \cup B$: All elements in A and B

Ex: $B \cup A = \{-1, 2, 4, 7, 11, 6, 13\}$



Note: "$\cup$" means OR (either A or B)

$\bigcup_{i=1}^{N} A_i = A_1 \cup A_2 \cup A_3 \cdots \cup A_4$

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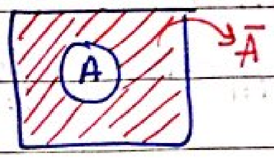
5 Set complement.

$\bar{A}$: All elements Not in A



* $\bar{A} = S - A$ * $A \cup \bar{A} = S$
* $A \cap \bar{A} = \phi$ * $\bar{S} = \phi$ * $\bar{\phi} = S$

* Assign Probability :-

P(A) : The probability of the occurance of A

$$0 \leq P(A) \leq 1$$

$P(\phi) = 0$ $P(S) = 1$

{ } impossible event S : Certain event

* Algebra of sets :-

1 Commutative law

$A \cap B = B \cap A$

$A \cup B = B \cup A$

2 Distributive law

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3 Associative Law

$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$

$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$

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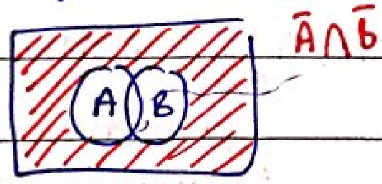
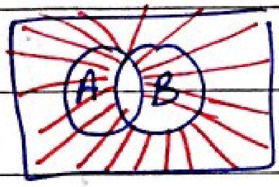
⊛ Demorgan's Law:

① $\overline{A \cup B} = \bar{A} \cap \bar{B}$ → Proof by venn's Diagram

↳ Neither A nor B



② $\overline{A \cap B} = \bar{A} \cup \bar{B}$



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* If $A_1\ A_2\ \cdots\ A_N$ are disjoint

$$\rightarrow P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} P(A_i)$$

Ex: Roll two fair Die and record the appeared number

① find S

② Define the events: A = "the sum = 7"

B = "8 < sum ≤ 11"

C = "10 < sum"

Find P(A), P(B), P(C)

P(A∩B), P(B∩C), P(A∪B), P(B∪C)

* S = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)

(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)

(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

A B C

$P(A) = \frac{6}{36}$

$P(B) = \frac{9}{36}$

$P(C) = \frac{3}{36}$

$P(A \cap B) = \phi \longrightarrow P(A \cap B) = P(\phi) = \phi$

$P(B \cap C) = P(\{5,6\}, \{6,5\}) = 2/36$

$P(A \cup B) = 15/36$ OR $P(A \cup B) = P(A) + P(B) = \frac{6+9}{36}$

$P(B \cup C) = P(B) + P(C) - P(B \cap C)$

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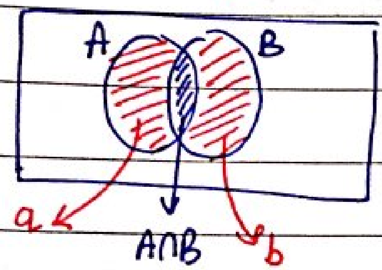
* Joint probability:

$P(A \cap B)$: the probability of the occurance of ~~the~~ A and B together.

$$\underbrace{P(A \cap B)}_{LHS} = \underbrace{P(A) + P(B) - P(A \cup B)}_{RHS}$$

LHS: left hand side — RHS: Right hand side

Proof:



RHS: $P(A) + P(B) - P(A \cup B)$

$= \cancel{P(a)} + P(A \cap B) + \cancel{P(b)} + \cancel{P(A \cap B)} - [\cancel{P(a)} + \cancel{P(A \cap B)} + \cancel{P(b)}]$

$= P(A \cap B)$ LHS #

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$P(A \cup B) \leq P(A) + P(B)$ → أكبر قيمة للاتحاد

لما تكون B,A ← disjoint

$P(A \cap B) = 0$

* Conditional probability:

$P(A/B)$: the probability of A given that B has occured

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Fair event : احتمالية حدوث كل الـ events متساوية

Unfair event: احتمالية حدوث events غير متساوية

ولكن مجموعهم 100%

Date. 30 May No.

*** Mathematical Model of expirements :-**

Sample space S

Assign probability $\rightarrow$ $0 \le P(A) \le 1$ (0% … 100%)

$P(\phi) = \phi$ $\quad$ $P(S) = 1$

Ex: Roll a die $\rightarrow$ $S = \{1, 2, 3, 4, 5, 6\}$

$P(\{1\}) = 1/6$

$P(\{2\}) = 1/6$

$P(\{6\}) \vdots = 1/6$

Event A: the appeared number is even $\quad P(A) = P(\{2, 4, 6\}) = 3/6$

Event B: the appeared number is integer $\quad P(B) = 1 = 100\%$

Ex: flip a fair coin $\quad S = \{H, T\}$

$P(\{H\}) = P(\{T\}) = 1/2$

* If A and B are two disjoint $\rightarrow A \cap B = \phi$

then $\rightarrow P(A \cup B) = P(A) + P(B)$

proof :

$P(A) = \dfrac{aA}{aS} \qquad P(B) = \dfrac{aB}{aS}$

$P(A \cup B) = \dfrac{aA + aB}{aS} = \dfrac{aA}{aS} + \dfrac{aB}{aS} = P(A) + P(B)$

* $P(\{2, 4, 6\}) = P(\{2\} \cup \{4\} \cup \{6\})$ (A1, A2, A3 — Disjoint) $\leftarrow$ زي ده unfair events

$= P(\{2\}) + P(\{4\}) + P(\{6\})$ $\leftarrow$ ممكن تكون احتمالاتهم مختلفة

$1/6 + 1/6 + 1/6 = 3/6$

= 1/2 مش لازم fair event لو unfair مابيطلع 1/2

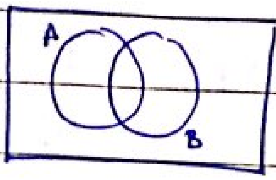
Date. 4. June No.

* Conditional Probability :-

$P(A|B)$: the prob. of A given that event B has occured

(↑ "|" = given that, "conditioned on")

$$P(A|B) = \frac{a_{A\cap B}}{a_B}$$ (a = area)



$$= \frac{a_{A\cap B} / a_s}{a_B / a_s} = \frac{P(A\cap B)}{P(B)}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$P(B|A) = \frac{P(B\cap A)}{P(A)}$$

as a consequence :

→ $P(A\cap B) = P(A|B)\, P(B)$

$P(A\cap B) = P(B|A)\, P(A)$

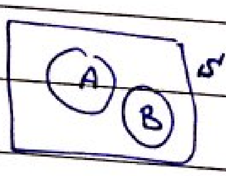
يعني اذا احسب conditional, joint probability

حسب المعطى.

* as a special case :

if $A\cap B = \emptyset$ (disjoint)

$P(A|B) = \text{zero}$



Ex:- A box of 100 resistors

value \ tolerance	5%	10%	Total
22Ω	10	14	24
47Ω	28	16	44
100Ω	24	8	32
Total	62	38	100

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Date. 4. June No.

* Draw out one resistor. Sample space has 100 elements.

Define the events :-

A: the resistor is 47 Ω

B: the resistor is with 5% tolerance.

C: the resistor is 100 Ω

Find:-

$P(A) = 44/100$ $\quad P(B) = 62/100$ $\quad P(C) = 32/100$

$P(A/B) = 28/62$ OR $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{28/100}{62/100} = \frac{28}{62}$

$P(A/C) =$ Zero OR $= \frac{P(A \cap C)}{P(C)} = \frac{P(\phi)}{P(C)} = \frac{0}{32/100} = 0$

$P(B/C) =$

↳ $\frac{24}{32}$ OR $\frac{P(B \cap C)}{P(C)} = \frac{24/100}{32/100} = \frac{24}{32}$

Ex A box of 80 resistors.

10Ω	15Ω	20	30
18	12	33	17

Exp1 :- Draw out one resistor

P(the resistor is 10 Ω) = 18/80 ← event A

P(the resistor is 15 Ω) = 12/80 event B

P(" " is 20 Ω) = 33/80 event C

P(" " is 30 Ω) = 17/80 event D

$P(A \cap B) = 0$

$P(B \cap C) = 0$

$P(B/C) = 0 = \frac{P(B \cap C)}{P(C)} = 0$

Date. 8. June No.

Exp 2 :- Draw out two resistors without replacement.

Find:

$P(\underbrace{\text{2nd is } 30\Omega}_{\text{event } B} \cap \underbrace{\text{1st is } 10\Omega}_{\text{event } A})$

في مسائل بيجي

$P(B \cap A) = P(B/A)\, P(A)$ ✓ — حساب الـ conditional

OR $P(B \cap A) = P(A/B)\, P(B)$ — أسهل من حساب الـ Joint

using Total prob. and Bayes rule

$P(A) = \frac{18}{80}$ $\quad P(B/A) = \frac{17}{79}$

$P(B \cap A) = \frac{17}{79} \times \frac{18}{80}$

* Independent events:-

A and B are said to be independent if

$P(A/B) = P(A)$ — الاحتمالين ما بأثروا ببعض

$P(B/A) = P(B)$

as a consequence :

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Longrightarrow \boxed{P(A \cap B) = P(A) \cdot P(B)}$$

only for independent A and B

Date. 4. June No.

for the previous example :-

Exp #3 :- Draw out two resistors with replacement.

find:-

$P(2^{nd} \text{ is } 30\Omega \cap 1^{st} \text{ is } 10\Omega)$

$= P(2^{nd}\ 30\Omega / 1^{st}\ 10\Omega) \cdot P(1^{st}\ 10\Omega)$

$= \frac{17}{80} \cdot \frac{18}{80}$

$= P(2^{nd}\ 30\Omega) \cdot P(1^{st}\ 10\Omega)$

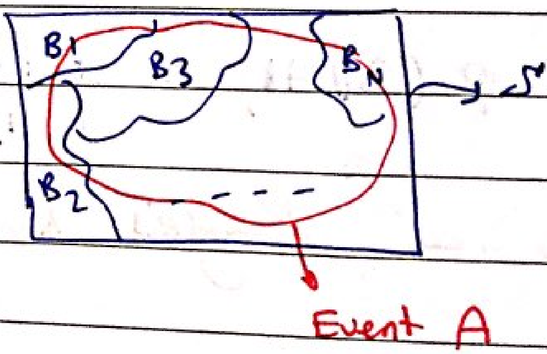
⊛ Total Probability :-

* If you have events $B_1, B_2, ---- B_n$

1) they are disjoint, i.e $B_i \cap B_j = \emptyset \quad \forall i, j$ → for all

2) $\bigcup_{i=1}^{N} B_i = S$

↳ $P(\bigcup_{i=1}^{N} B_i) = 1$



→ $P(B_1) + P(B_2) + ---- P(B_N) = 1$

$$P(A) = \sum_{i=1}^{N} P(A|B_i)\, P(B_i)$$

total probability Law

Date. 5. June No.

Total Probability law Proof :-

proof :-

$$P(A) = P(A \cap S) = P\left(A \cap \bigcup_{i=1}^{N} B_i\right)$$

$$= P(A \cap [B_1 \cup B_2 \cup ---- \cup B_N])$$

$$= P((A \cap B_1) \cup (A \cap B_2) \cup ---- \cup (A \cap B_N))$$ ← Disjoint

$$= P(A \cap B_1) + P(A \cap B_2) + ---- P(A \cap B_N)$$

$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + ---- + P(A|B_N)P(B_N)$$

$$= \sum_{i=1}^{N} P(A|B_i)\, P(B_i)$$

$$P(B_i/A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A/B_i)\, P(B_i)}{P(A)}$$

↗ Baye's Rule.

Date. 5. June No.

Ex:- Box of 80 resistors

10Ω	15Ω	20Ω	30Ω
18	12	33	17

Exp: Draw out 2 resistors without replacement.

Find :-

⊛ $P(2^{nd}\ 15 \cap 1^{st}\ 20\Omega)$

$$= P(2^{nd}\ 15 \mid 1^{st}\ 20)\ P(1^{st}\ 20)$$

$$= \frac{12}{79} \cdot \frac{33}{80}$$

* بستخدم قانون total probability
لما يكون ال event بيعتمد على اكتر من اسـ (B)

⊛ $P(2^{nd}\ 30\Omega)$

$$= P\big((2^{nd}\ 30 \cap 1st\ 10) \cup (2^{nd}\ 30 \cap 1^{st}\ 15) \cup$$

$$(2^{nd}\ 30\Omega \cap 1^{st}\ 20) \cup (2^{nd}\ 30 \cap 1^{st}\ 30)\big)$$

(OR → +)

$$= P(2^{nd}30 / 1^{st}10)\,P(1^{st}10) + P(2^{nd}\,30 / 1^{st}\,15)\,P(1^{st}\,15)$$

$$+ P(2^{nd}\,30 / 1^{st}20)\,P(1^{st}20) + P(2^{nd}\,30 / 1^{st}\,30)\,P(1^{st}\,30)$$

$$= \frac{17}{79} \cdot \frac{18}{80} + \frac{17}{79} \cdot \frac{12}{80} + \frac{17}{79} \cdot \frac{33}{80} + \frac{16}{79} \cdot \frac{17}{80}$$

So, $B_1 = 1^{st}\ 10\Omega$ OR $B_2 = 1^{st}\ 15\Omega$ OR $B_3 = 1^{st}\ 20\Omega$, $B_4 = 1^{st}\ 30\Omega$ → A: $2^{nd}\ 30\Omega$ (depends on)

$$P(B_1) + P(B_2) + P(B_3) + P(B_4) = 1$$

$$P(A) = \sum_{i=1}^{4} P(A/B_i)\,P(B_i)$$

Date. No.

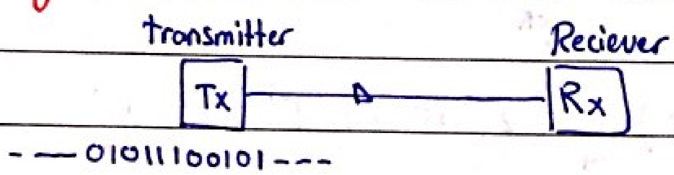
⊛ $P(1^{st}\ 15\Omega \,/\, 2^{nd}\ 20\Omega)$ → $P(B_i / A)$ احتمالية

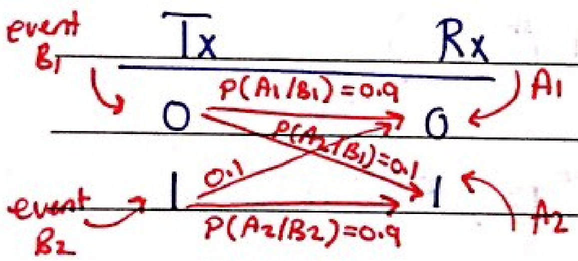
$$= \frac{P(1^{st}\ 15 \cap 2^{nd}\ 20)}{P(2^{nd}\ 20)}$$

$$= \frac{P(2^{nd}\ 20 \,/\, 1st\ 15)\, P(1^{st}\ 15)}{P(2^{nd}\ 20)}$$ ← using Baye's Rule

$$= \frac{\frac{33}{79} \cdot \frac{12}{80}}{\frac{33}{79} \cdot \frac{18}{80} + \frac{33}{79} \cdot \frac{12}{80} + \frac{32}{79} \cdot \frac{33}{80} + \frac{33}{79} \cdot \frac{17}{80}}$$ ← total probability.

* Binary Communication Channel (BCC)





B_1: the transmitted bit is 0

B_2: the txed bit is 1

$P(B_1) = 0.6$

$P(B_2) = 0.4$

A_1 = the Rxed bit is 0

A_2 : The Rxed bit is 1

→

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Date. 5. June No.

Find :-

ذا لاسم

$P(A_1) = P((A_1 \cap B_1) \cup (A_1 \cap B_2))$

~~P(A_2)~~ $= P(A_1 \cap B_1) + P(A_1 \cap B_2)$

$= P(A_1 / B_1) P(B_1) + P(A_1 / B_2) P(B_2)$ ← Total Prob Law

$= (0.9)(0.6) + (0.1)(0.4)$

$= 0.42$

$P(A_2) = 1 - P(A_1) = 1 - 0.42 = 0.58$

OR

$P(A_2) = P(A_2 / B_1) P(B_1) + P(A_2 / B_2) P(B_2)$

$= (0.1)(0.6) + (0.9)(0.4) = 0.58$

$P(B_1 / A_1) = \frac{P(A_1 / B_1) P(B_1)}{P(A_1)} = \frac{(0.9)(0.6)}{0.42} = --$ ← Bayes Rule

الشرط على نفس أ event

يعني B1 متممه B2

$P(B_2 / A_1) = 1 - P(B_1 / A_1)$

$P(B_1 / A_2) = \frac{P(A_2 / B_1) P(B_1)}{P(A_2)}$ ← Bayes Rule

$P(B_2 / A_2) = 1 - P(B_1 / A_2)$ ← Bayes Rule عكس كلها

Date. 6.June No.

⊛ Combined Expirement :-

↳ Consists of multiple sub-expirements (mainly independent)

Ex: Expirement flip a coin (sub-exp 1) and roll a die (sub-exp 2).

$S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)$
$(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

exp
- → sub-exp 1 : $S_1 = \{H, T\}$
- → sub-exp 2 : $S_2 = \{1, 2, 3, 4, 5, 6\}$

$S = S_1 \times S_2$ ← all possible combinations ("and")

Define event C : "the coin shows H (A_{S_1}), and the appear number is odd (A_{S_2})"

$C = \{(H,1), (H,3), (H,5)\}$

$P(C) = 3/12$ → (fair بشرط انهم يكونوا)

$C = A_{S_1} \times A_{S_2}$ → $A_{S_1} = \{H\}$, $A_{S_2} = \{1, 3, 5\}$

$\therefore P(C) = P(A_{S_1}) \times P(A_{S_2})$ (× ضرب) → independent لانهم

$\frac{1}{2} \cdot \frac{3}{6} = \frac{3}{12}$ #

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Date. 6·June No.

Ex: Flip a coin 4 times, find P(HTHH)

S = { (HHHH), (HHHT), ----- (TTTT) }

P(HTHH) = 1/16

OR

$P(\{HTHH\}) = P(H_{s_1}) \cdot P(T_{s_2}) \cdot P(H_{s_3}) \cdot P(H_{s_4})$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$

* Permutations :-

All possible sequences of ordering r elements (order is important) taken from n elements without replacement.

\# of permutations = P^n_r

Ex: 4 cards (n) [A] [B] [C] [D], order 2 cards (r)

$P^4_2 = 12 = 4 \times 3$

Position 1 (4) — 4 possible cards for this position	Position 2 (3)
A	B
A	C
A	D
B	A
B	C
B	D
C	A
C	B
C	D
D	A
D	B
D	C

$P^4_3 = 4 \times 3 \times 2$ $P^6_3 = 6 \times 5 \times 4$

$P^n_r = n(n-1)(n-2) \cdots (n-r+1)$

$$P^n_r = \frac{n!}{(n-r)!}$$

Date. 6. June No.

⊛ Combinations :-

Same def. as permutations but the order is not important. C_r^n

For the same previous example :-

[A] [B] [C] [D]

$C_2^4 = 6 = \frac{12}{2}$

P_1	P_2
A	B
A	C
A	D
B	C
B	D
C	D

6 elements.

$$C_r^n = \frac{P_r^n}{r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

$C_r^n = \binom{n}{r}$ ← n choose r

$\binom{n}{0} = \frac{n!}{n!\,0!} = 1$ $\qquad \binom{n}{1} = \frac{n!}{(n-1)!\,1!} = \frac{n(n-1)!}{(n-1)!\,1!} = n$

$\binom{n}{n} = 1$

Ex :- 7 students, How many 4-member teams we can perform from them ?! combination (order is not imp)

$C_4^7 = \frac{7!}{4! \cdot 3!} = \frac{7 \times 6 \times 5}{3 \times 2} = 35$

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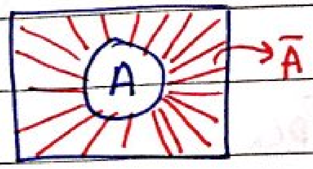
⊛ Bernoulli Trial :-

It's an expirement of two Possible outcomes

outcomes : $A, \bar{A}$

"success" ↗ ↖ "fail"



$P(A) = p$

$P(\bar{A}) = 1 - p$

Ex: flip a coin, 10. June. 2018

$S = \{H, T\}$

$p(H) = p$

$p(F) = 1 - p$

* If we repeat the bernouli trials N times :-

\# of success $K = 0, 1, 2, \ldots, N$

Find $P(K = k)$

ex : Flip a coin (unfair coin) 3-times

Find the prob. that the "H" appears 2-times (success), $p(H) = p$

sol :

$N = 3$, $P(K=2)$

$S = \{$ H, H, H → $K=3$

HHT, HTH, THH → $K=2$

HTT, THT, TTH → $K=1$

TTT $\}$ → $K=0$

Define success "H"

$P(K=2) = P(\{HHT, HTH, THH\})$

$= P(HHT) + P(HTH) + p(THH)$

$= P(H)_{s_1} \cdot pH(s_2) \cdot p(T)_{s_3} + P(H)_{s_1} \cdot P(T)_{s_2} \cdot P(H)_{s_3}$

$+ P(T)_{s_1} \cdot P(H)_{s_2} \cdot P(H)_{s_3}$

→

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$P(K=2) = 3p^2(1-p) = \binom{3}{2} p^2 (1-p)^1$ (N = 3, K = 2, exponent of $(1-p)$ is $N-K$)

In General :-

$$P(K=k) = \binom{N}{k} p^k (1-p)^{N-k}$$

$P(K=1) = \binom{3}{1} p^1 (1-p)^2$

$= 3p(1-p)^2 = P(HTT, THT, TTH)$

$P(K=3) = \binom{3}{3} p^3 (1-p)^0 = p^3 = P(HHH)$

$P(K=0) = \binom{3}{0} p^0 (1-p)^3 = (1-p)^3 = P(TTT)$

Ex:- Flip a coin 100 times, $P(H) = 0.2$

Find the probability that the tail "T" appears at most 2-times.

("tail "T" appears" → success)

Sol: $N = 100$ $P(H) = 0.2$ ($1-p$) $P(T) = 1 - 0.2 = 0.8$ (p)

$K = 0, 1, 2, 3, \ldots, 100$

find $P(K \leq 2) = P(K=0) + P(K=1) + P(K=2)$

$\binom{100}{0}(0.8)^0(0.2)^{100} + \binom{100}{1}(0.8)^1(0.2)^{99} + \binom{100}{2}(0.8)^2(0.2)^{98}$

($\binom{100}{0}(0.8)^0(0.2)^{100} = 0.2^{100}$)

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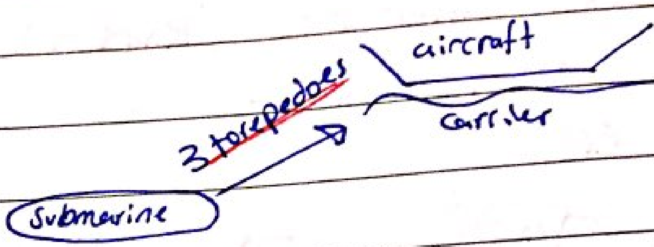
Example 1.7 -1 :-



3- torpedoes

↳ "hit" = A , $P(A) = 0.4$

↳ "doesn't hit" = $\bar{A}$, $P(\bar{A}) = 0.6$

$k = 0, 1, 2, 3$

* Find the prob. that the aircraft carrier will be sunk

$= P(K \geq 2)$

$= P(k=2) + P(k=3)$

$= \binom{3}{2}(0.4)^2(0.6) + \binom{3}{3}(0.4) \cdots$

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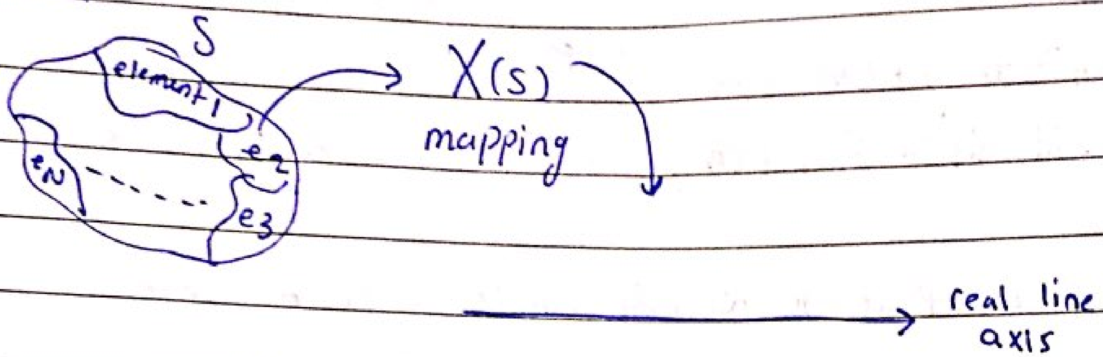
* Chapter two (Random Variables RV)

X, Y, Z, A, - - - - -

exp:



RV → Discrete → (certain values) $S_X = \{x_1, x_2, x_3, ---- x_N\}$

RV → Continuous → interval $X = \{ z_1 < a < z_2 \}$

RV → upper case.

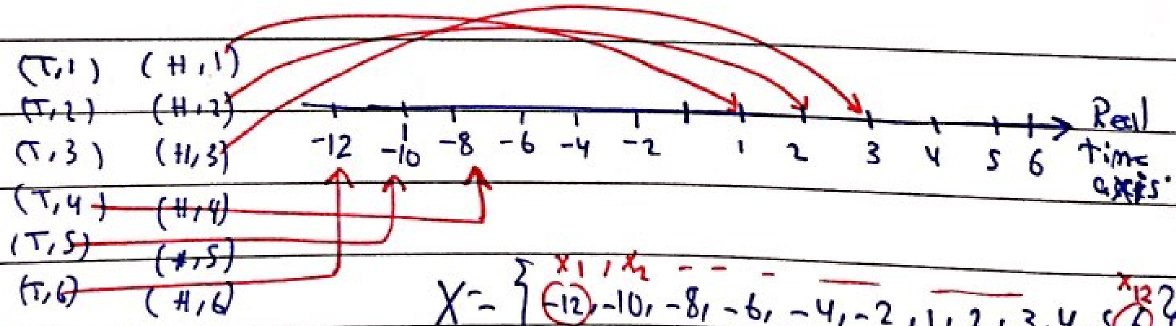
specific element → lower case.

Ex: Exp flip a coin and Roll a die

Define $X = \begin{cases} \text{the * of the die if coin = "H"} \\ (-2)\cdot(\text{* on the die}) \text{ if coin = "T"} \end{cases}$

Find S, X

S =



event

$X = \{ -12, -10, -8, -6, -4, -2, 1, 2, 3, 4, 5, 6 \}$ ($x_1, x_2, ---, x_{12}$)

$P(X = -6) = P(T,3) = 1/12$

$P(X = 15) = P(\emptyset) = 0$

$P(X \leq 14) = P(S) = 1$

$P(X \leq -\infty) = 0$

$P(X \leq -20) = P(\emptyset) = 0$

$P(X \leq \infty) = 1$

for any X

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⊛ Random Variables:

Ex: $X = \{-12, -10, \dots, 1, 2, \dots, 6\}$

$P(1 < X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

event →

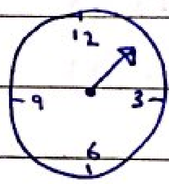
$= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{4}{12}$

$P(4 \leq X \leq 8) = P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + 0 + 0$

Ex:- Random Variable

wheel of chance →



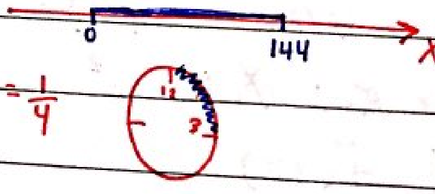
(1) Find S

(2) Define R.V $X = S^2$

(3) find $P(0 < X \leq 9)$ (4) $P(X=7)$ (5) $P(0 < X < 3)$

(1) $S = \{0 < s \leq 12\}$ ← all real numbers between 0 and 12

(2) $X = S^2 = \{0 < x \leq 144\}$ ↵ continuous random variable.

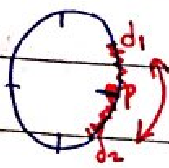


(3) $P(0 < x \leq 9) = P(0 < s \leq 3) = \frac{3-0}{12-0} = \frac{1}{4}$

(4) $P(x=7) = 0$

$P(s=p) = \lim_{\Delta d \to 0} \frac{d_2 - d_1}{12} = 0$

↑ specific point



$d_2 - d_1 = \Delta d$

(5) $P(0 < X < 9) = P(0 < X \leq 9) = \frac{1}{4}$

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Note :-

for Discrete R.V $P(X = x)$ exist.

for continuous R.V $P(X = x) = 0$

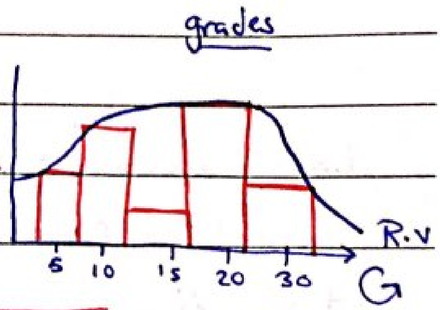
$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2)$

$= P(x_1 < X < x_2)$

✱ Distribution and Density function :-

① CDF $F_X(x)$ Commulative distrivutive func

② PDF $f_X(x)$



$$CDF = \int_{-\infty}^{x} f_X(x)\, dx$$

$$f_X(x) = \frac{d F_X(x)}{dx}$$

[1] CDF : $X \rightarrow F_X(x) = P(X \le x)$, for all $x \in (-\infty, \infty)$

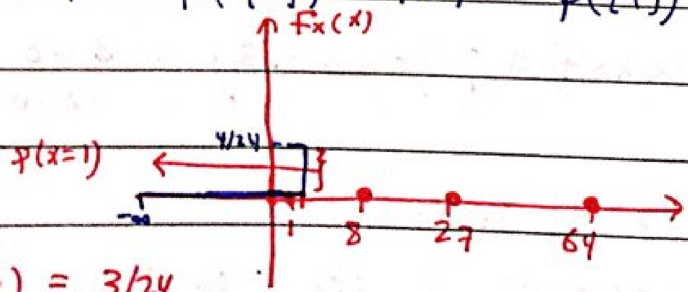
CDF for Discrete R.V :-

→ Ex: expirement : $S = \{1, 2, 3, 4\}$

$P(\{1\}) = 4/24$ $P(2) = 3/24$ $P(\{3\}) = 7/24$ $P(\{4\}) = 10/24$

✱ Define R.V $X = s^3$

✱ Find $F_X(x)$



$X = \{1, 8, 27, 64\}$.

$P(x=1) = 4/24$ $P(x=8) = 3/24$

$P(x=27) = 7/24$ $P(x=64) = 10/24$

قبل ١ بشوي

$F_X(-\infty) = P(X \le -\infty) = 0$ $F_X(1^-) = P(X < 1) = 0$

$F_X(0) = P(X \le 0) = 0$ $F_X(1) = P(X \le 1) = P(X=1) =$

$= 4/24$

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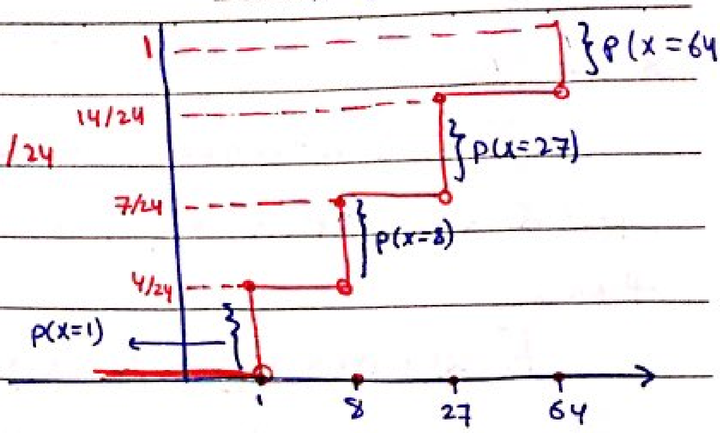
$F_x(6) = P(x < 6) = p(x=1) = 4/24$

$F_x(8^-) = 4/24$

$F_x(8) = P(x \leq 8) =$

$P(x=1) + P(x=8) =$

$4/24 + 3/24 = 7/24$



$F_x(\infty) = P(x < \infty) = 1$

① Non-Decreasing

② starts from 0 and ends with 1

③ stairs.

mathematically :-

$$F_x(x) = \begin{cases} 0 & , x=1 \\ 4/24 & , 1 \leq x < 8 \\ 7/24 & , 8 \leq x < 27 \\ 14/24 & , 27 \leq x < 64 \\ 1 & , 64 \leq x \end{cases}$$

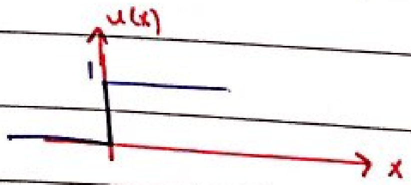
Note $F_x(8) = 7/24$

$P(x=8) = 3/24$

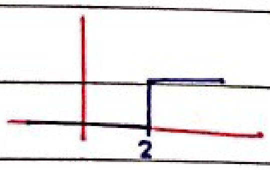
* express $F_x(x)$ in terms of the unit step function :-

preview:

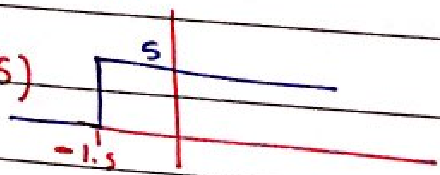
$$u(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$



$u(x-2) \rightarrow$



$5u(x+1.5)$



$$F_x(x) = \frac{4}{24}u(x-1) + \frac{3}{24}u(x-8) + \frac{7}{24}u(x-27) + \frac{10}{24}u(x-64)$$

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* In General :-

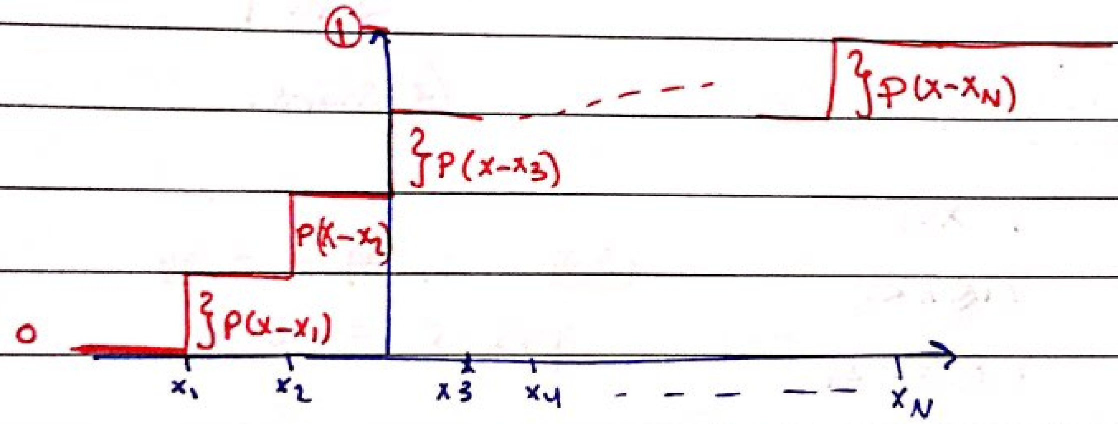
$$X = \{x_1, x_2, x_3, \ldots x_N\}$$

$P(X=x_1)$, $P(X=x_2)$, ... $P(X=x_N)$ are given then,

$$F_X(x) = P(X=x_1)\,u(x-x_1) + P(X=x_2)\,u(x-x_2) + \ldots + \ldots + P(X=x_i)\,u(x-x_i)$$

So,

$$F_X(x) = \sum_{i=1}^{N} P(x=x_i)\,u(x-x_i)$$



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Ex: R.V $X = \{-5, -2, 0, 1, 3\}$ (with x_1, x_2, x_3, x_4, x_5 written above the values)

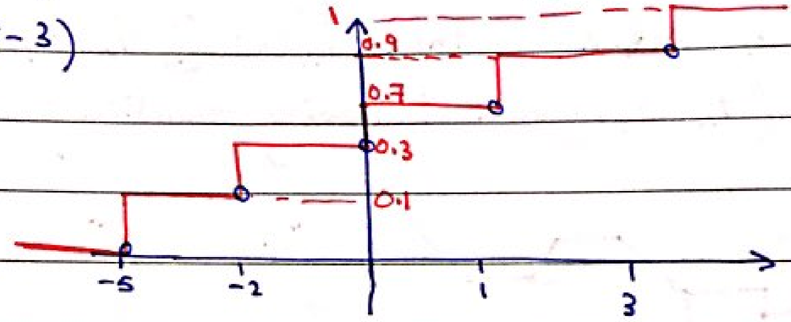
$P(X=-5) = 0.1$ $\quad P(X=0) = 0.4$ $\quad P(X=3) = 0.1$

$P(X=-2) = 0.2$ $\quad P(X=1) = 0.2$

Find and plot $F_X(x)$

Sol: $F_X(x) = \sum_{i=1}^{5} P(X=x_i)\, u(x-x_i)$

$= 0.1\,u(x+5) + 0.2\,u(x+2) + 0.4\,u(x) + 0.2\,u(x-1) + 0.1\,u(x-3)$



Ex:- CDF for continuous R.V

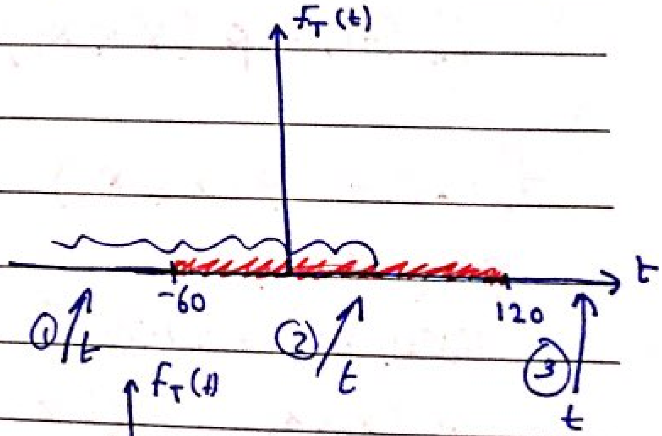
assume R.V $T = \{-60 \le t \le 120\}$, let T a R.V that represents the tempreture in F° for certain location.

Find $f_T(t)$:

$F_T(t) = P(T \le t)$

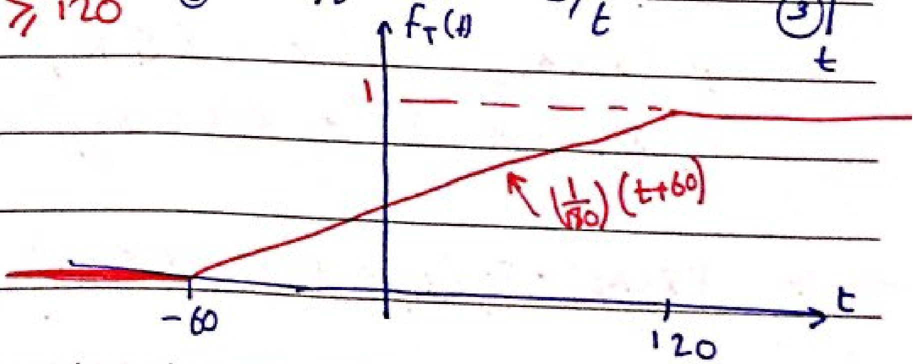
$= \begin{cases} 0, & t < -60 \quad (1) \\ \dfrac{t+60}{180}, & -60 \le t \le 120 \quad (2) \\ 1, & t \ge 120 \quad (3) \end{cases}$

$120+60$



slope $= \dfrac{1-0}{120--60} = \dfrac{1}{180}$

$(y-0) = \dfrac{1}{180}(x--60)$



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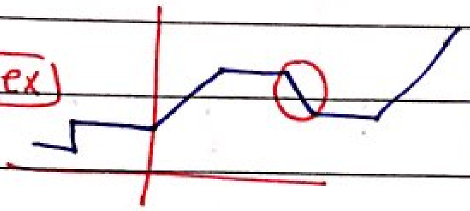
⊛ CDF Properties :-

1. $F_X(-\infty) = 0$
2. $F_X(\infty) = 1$
3. $0 \leq F_X(x) \leq 1$
4. $F_X(x) = F_X(x^+)$ ⟵ $F_X(x)$ is continuous from the right
5. If $x_1 \leq x_2$, then $F_X(x_1) \leq F_X(x_2)$
   ($F_X(x)$ is non-decreasing function)

ex

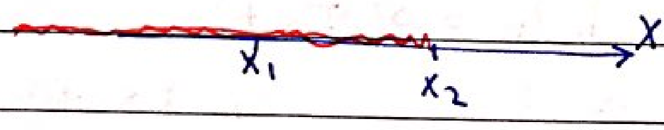


⟵ "Not CDF"

6. $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

Proof:-

$F_X(x_2) = P(X \leq x_2)$



$F_X(x_2) = P(X \leq x_2) =$

$P(X \leq x_1 \cup x_1 < X \leq x_2)$

ما بنفع أحط مساواة بالكاتين.

$= P(X \leq x_1) + P(x_1 < X \leq x_2)$

$= F_X(x_1) + P(x_1 < X \leq x_2)$

⟶ $F_X(x_2) - F_X(x_1) = P(x_1 < X \leq x_2)$

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Ex: $T = \{-60 \le t \le 120^\circ\}$, $f_T(t)$ ✓

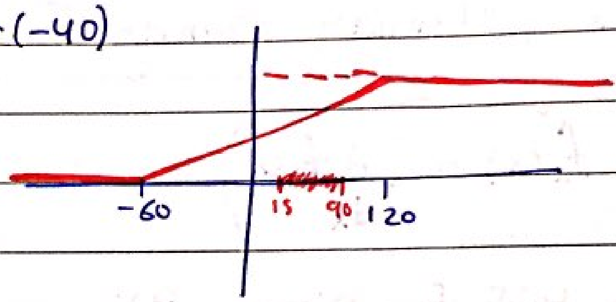
Find ① $P(15 \le T \le 90)$

$= F_T(90) - F_T(15)$

$= \frac{150}{180} - \frac{75}{180} = \frac{75}{180}$

② $P(T < -40) = F_T(-40)$

$= \frac{-40+60}{180}$



③ $P(T < 130) = 1$

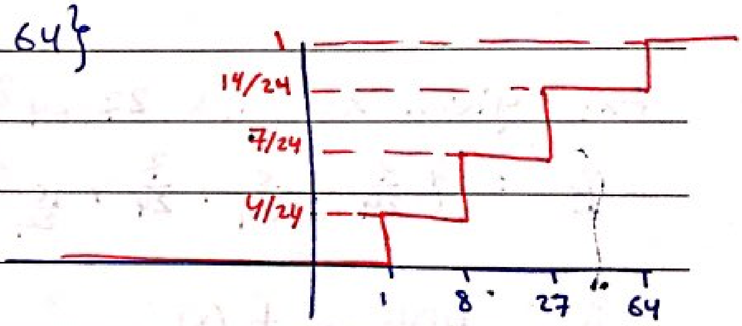
Ex: given $X = \{1, 8, 27, 64\}$

Find:-

① $P(2 < X \le 28)$

$= F_X(28) - F_X(2)$

$= \frac{14}{24} - \frac{4}{24} = 10/24$



OR $P(2 < X \le 28) = P(X=8) + P(X=27)$

$= 3/24 + 7/24 = 10/24$

② $P(3 < X < 27)$

$= P(3 < X \le 27^-) = F_X(27^-) - F_X(3)$

$\frac{7}{24} - \frac{4}{24} = \frac{3}{24}$

or $= P(X=8) = 3/24$

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③ $P(8 \le X \le 64)$

$= P(X=8) \cup 8 < X \le 64)$

$= P(X=8) + P(8 < X \le 64)$

اضم نهاية → $= P(X=8) + F_X(64) - F_X(8)$

$= \frac{3}{24} + 1 - \frac{7}{24} = \frac{3+24-7}{24} = \frac{20}{24}$

* Probability Density Function (PDF)

مشتقة CDF

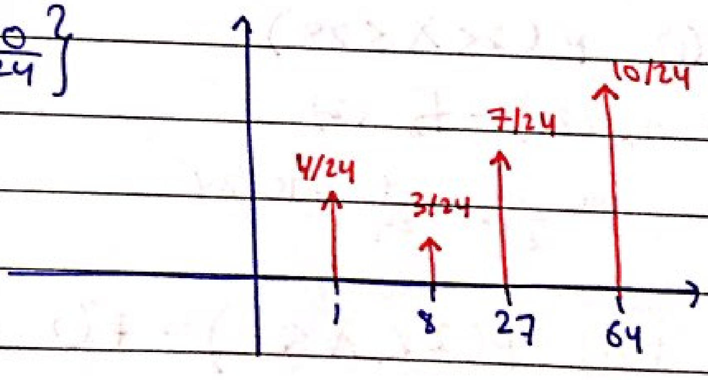
$$f_X(x) = \frac{d\,F_X(x)}{dx}$$

* PDF for Discrete R.V

Ex: given $X = \{1, 8, 27, 64\}$

$P = \{\frac{4}{24}, \frac{3}{24}, \frac{7}{24}, \frac{10}{24}\}$

find PDF → $f_X(x)$



sol:- $f_X(x) = \frac{d\,F_X(x)}{dx}$

$F_X(x) = P(X \le x) =$

$\frac{4}{24} u(x-1) + \frac{3}{24} u(x-8) + \frac{7}{24} u(x-27) + \frac{10}{24} u(x-64)$

$f_X(x) = \frac{4}{24}\delta(x-1) + \frac{3}{24}\delta(x-8) + \frac{7}{24}\delta(x-27) + \frac{10}{24}\delta(x-64)$

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Date. 12. June No.

In General :-

Discrete R.V $X = \{x_1, x_2, x_3, --- x_N\}$

$$f_X(x) = P(X = x_1)\,\delta(x - x_1) + P(X = x_2)\,\delta(x - x_2) + --- + P(X = x_N)\,\delta(x - x_N)$$

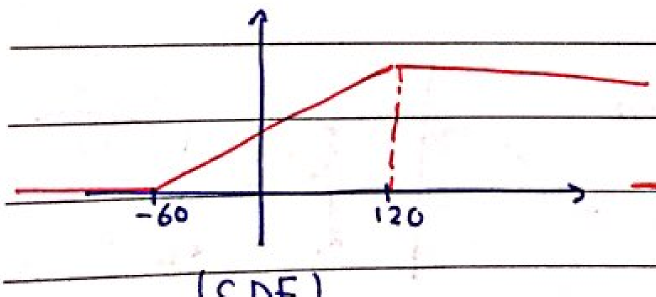
$$\Rightarrow \boxed{f_X(x) = \sum_{i=1}^{N} P(X = x_i)\,\delta(x - x_i)}$$

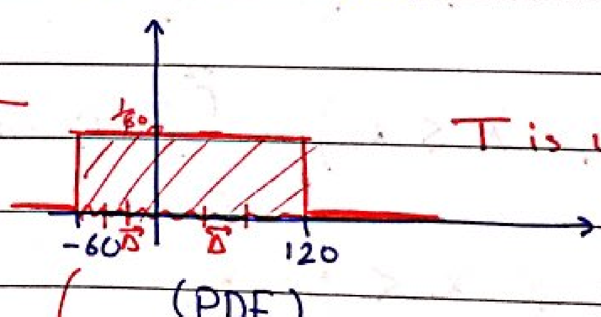
Ex R.V $T = \{-60^\circ \leq t \leq 120^\circ\}$

$$F_T(t) = \begin{cases} 0 & , \; t < -60^\circ \\ \frac{t+60}{180} & , \; -60^\circ \leq t \leq 120 \\ 1 & , \; 120 \leq t \end{cases}$$

Find $f_T(t)$:-

$$f_T(t) = \frac{d F_T(t)}{dt} = \begin{cases} 0 & , \; t < -60 \\ \frac{1}{180} & , \; -60 \leq t \leq 120 \\ 0 & , \; 120 \leq t \end{cases}$$





T is uniform.

$$\text{Area} = \frac{1}{180} \times 180 = 1$$

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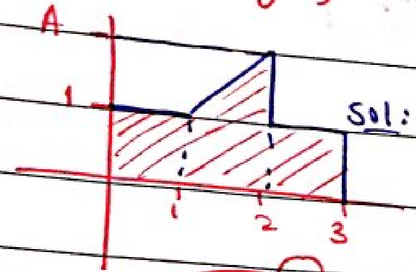
⊛ PDF Properties :-

① $f_X(x) \geq 0$ → لأنه مشتقة ال CDF
ال CDF لازم يتزايد increasing

② $\int_{-\infty}^{\infty} f_X(x)\, dx \triangleq P(S) = 1$

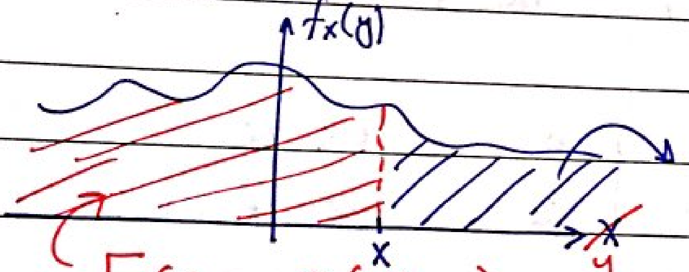
Ex If $g(x)$ is density function, find A



Sol: Area = 1

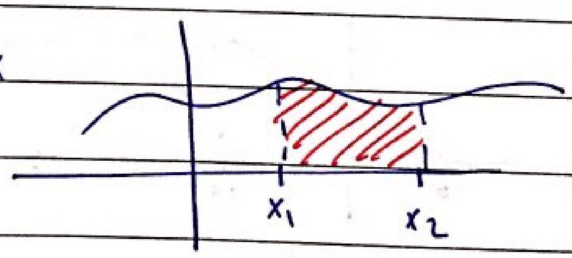
$\int_{-\infty}^{\infty} g(x)\, dx = 1$

③ $F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy = P(X \leq x)$



area = $1 - P(X \leq x)$
$= P(X > x)$

④ $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\, dx$

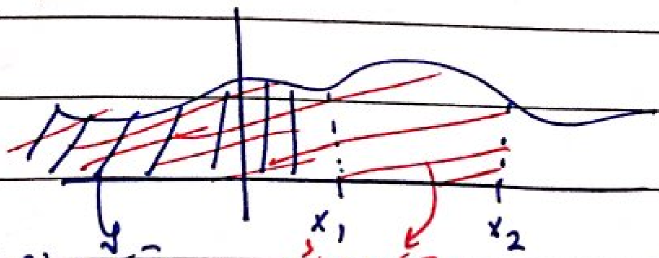


Proof: $P(x_1 < X \leq x_2)$

$= F_X(x_2) - F_X(x_1)$

$= \int_{-\infty}^{x_2} f_X(x)\, dx - \int_{-\infty}^{x_1} f_X(x)\, dx$

$= \int_{z_1}^{z_2} f_X(z)\, dz$

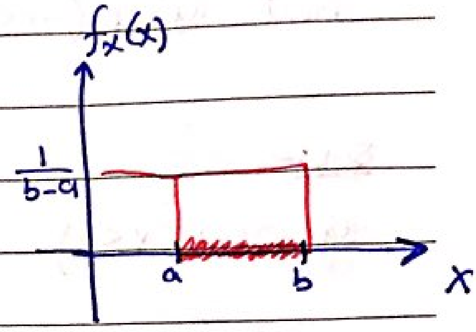


⊛ Common R.V types:-

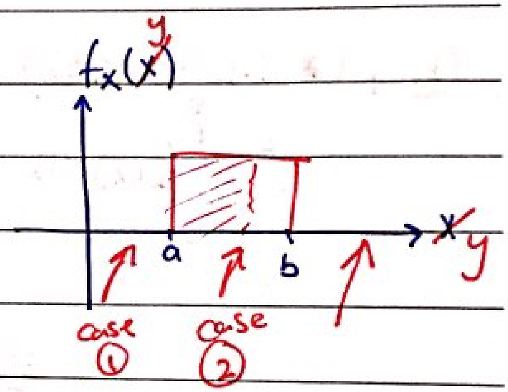
[1] Uniform R.V (continuous R.V)

$X \sim U(a,b)$

- $b > a$
- $a, b \in (-\infty, \infty)$
- $X = \{ a \leq X \leq b \}$

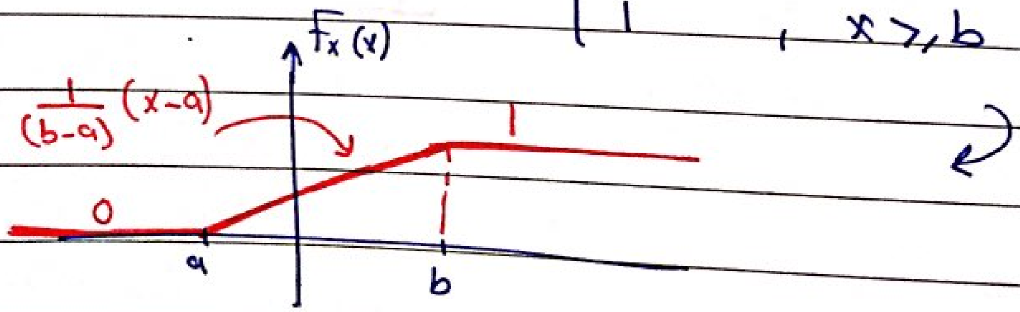


$$f_X(x) = \begin{cases} \frac{1}{b-a} & , \ a \leq x \leq b \\ 0 & , \ \text{otherwise} \end{cases}$$



$$* \ F_X(x) = \int_{-\infty}^{x} f_X(y) \, dy$$

$$= \begin{cases} 0 & , \ x < a \\ \int_a^x \frac{1}{b-a} dz & , \ a \leq x < b \\ 1 & , \ x > b \end{cases} = \begin{cases} 0 & , \ x < a \\ \frac{x-a}{b-a} & , \ a \leq x < b \\ 1 & , \ x \geq b \end{cases}$$

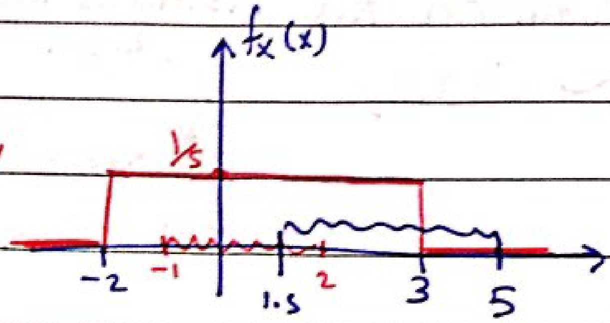


Date. No.

Ex given $X \sim U(-2,3)$

Find (a) $P(-1 < X < 2)$ (b) $P(1.5 < X < 5)$

Sol:-



(a) $P(-1<X<2) = \int_{-1}^{2} f_X(x)\,dx$

$= \int_{-1}^{2} \frac{1}{5}\,dx = \boxed{\frac{3}{5}}$

(b) $P(1.5 < X < 5)$

$= \int_{1.5}^{5} f_X(x)\,dx = \int_{1.5}^{3} f_X(x)\,dx + \int_{3}^{5} f_X(x)\,dx$ (second integral crossed out $= 0$)

$= \frac{1.5}{5} + 0 = \boxed{\frac{1.5}{5}}$

OR $P(1.5 < X < 5) = P(1.5 < X < 3) + P(3 < X < 5)$ (second term crossed out $= 0$)

$= \boxed{\frac{1.5}{5}}$

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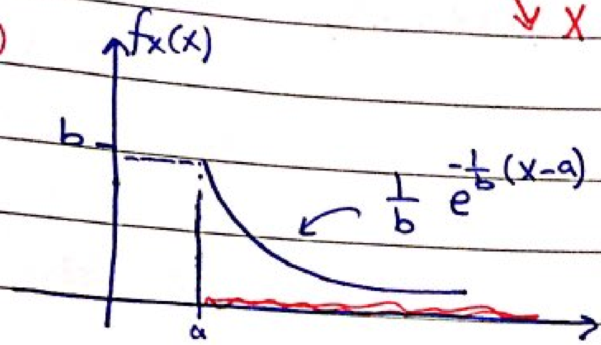
[2] Exponential R.V

① $X \sim \exp(a,b)$ → $b>0$, $a \in (-\infty, \infty)$, $X = \{x > a\}$

②



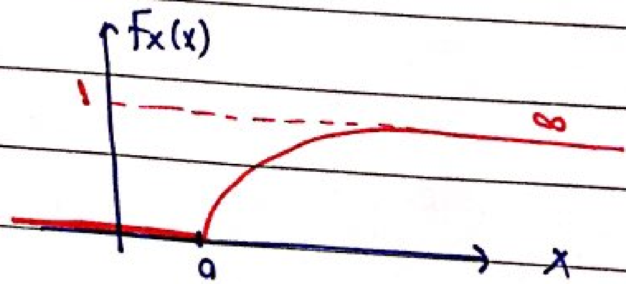
③ $f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{1}{b}(x-a)} , & x \geq a \\ 0 , & x < a \end{cases}$

④ $F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy = \begin{cases} 0 , & x < a \\ \int_{a}^{x} \frac{1}{b} e^{-\frac{1}{b}(y-a)}\, dy , & a \leq x \end{cases}$

$= \frac{1}{b} \frac{e^{-\frac{1}{b}(y-a)}}{-\frac{1}{b}} \Big|_{a}^{x}$

$= \frac{\frac{1}{b} - \frac{1}{b} e^{-\frac{1}{b}(x-a)}}{\frac{1}{b}}$

$F_X(x) = \begin{cases} 0 , & x < a \\ 1 - e^{-\frac{1}{b}(x-a)} , & x \geq a \end{cases}$



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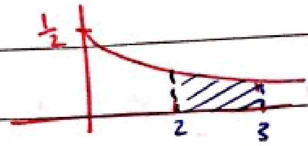
Ex:- $X \sim \exp(\overset{a}{0}, \overset{b}{2})$

Find:- (a) $P(2<X<3)$ (b) $P(2>X \cap X>3)$

(c) $P(2>X \cup X>3)$

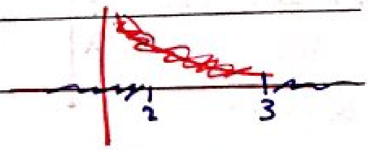
Sol :- $f_X(x) = \frac{1}{2} e^{-\frac{1}{2}x}$, $x>0$



(a) $P(2<X<3) = \int_2^3 \frac{1}{2} e^{-\frac{1}{2}x} \, dx$

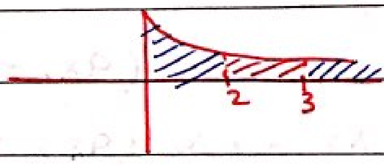
$= \frac{1}{2} \left. \frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}} \right]_2^3 = e^{-1} - e^{-1.5} = 0.147 = 14.7\%$

(b) $P(2>X \cap X>3) = P(\phi) = 0$



(c) $P(2>X \cup X>3)$

$= 1 - 0.147 = \ldots$



OR

$= \int_0^2 \frac{1}{2} e^{-\frac{1}{2}x} \, dx + \int_3^\infty \frac{1}{2} e^{-\frac{1}{2}x} \, dx$

[3] Gaussian (Normal) R.V (contineous)

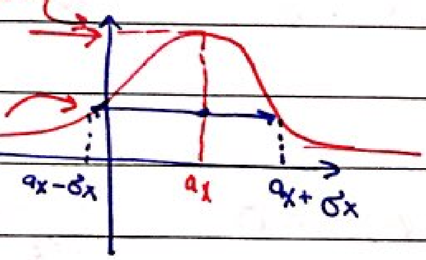
$X \sim N(a_x, \sigma_x^2)$

- a_x → any real number → mean value of X
- $\sigma_x > 0$ → standard deviation

* $X = \{-\infty < x < \infty\}$

$$* \; f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \quad , -\infty < x < \infty$$

Peak

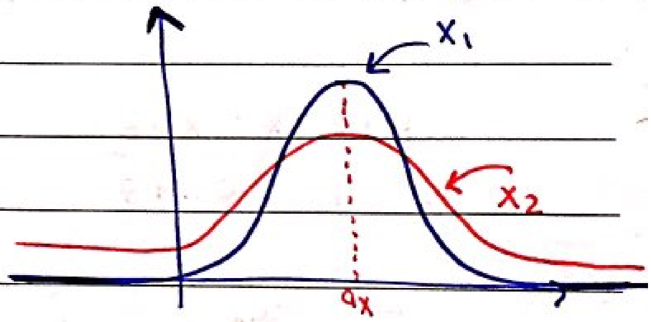


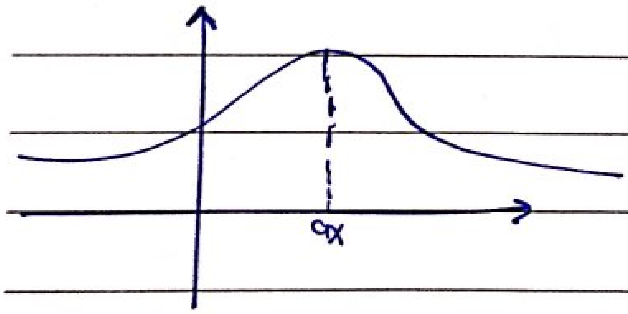
$$\frac{e^{-\frac{1}{2}}}{\sqrt{2\pi\sigma_x^2}} \equiv e^{-\frac{1}{2}} * \text{peak}$$

Note $X_1 \sim N(a_x, \sigma_{x_1}^2)$

$X_2 \sim N(a_x, \sigma_{x_2}^2)$

$\sigma_{x_2} > \sigma_{x_1}$





$$\rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx = 1$$

$$\int_{-\infty}^{a_x} N(a_x, \sigma_x^2) \, dx = 0.5$$

$$\int_{a_x}^{\infty} N(a_x, \sigma_x^2) \, dx = 0.5$$

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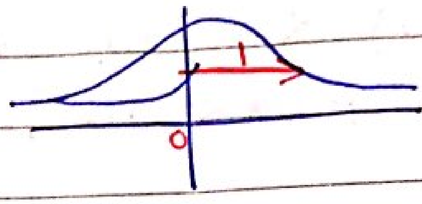
Special case: standard Gaussian

$X \sim N(0,1)$



* CDF for Gaussian R.V :-

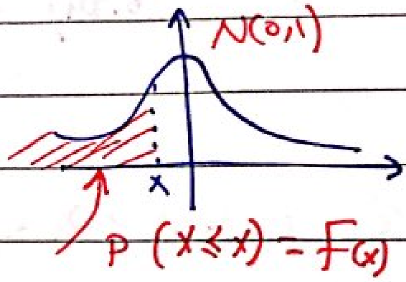
$X \sim N(0,1) \rightarrow F(x)$ → standard

$X \sim N(a_x, \sigma_x^2) \rightarrow F_X(x)$

Find:- $F(x)$ → (standard)

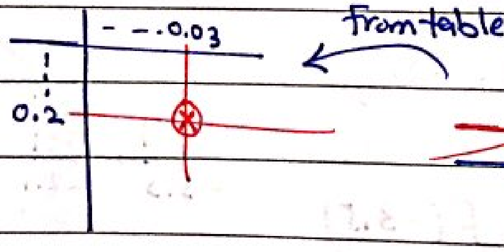
$F(x) = P(X \le x)$

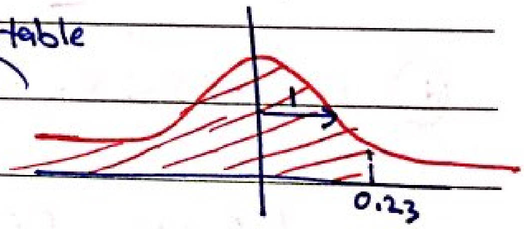
$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy$$



Ex $X \sim N(0,1)$ Find: $F(0.23) = P(X \le 0.23)$

$= 0.591$
(from table)





$P(X > 0.23) = 1 - 0.591$

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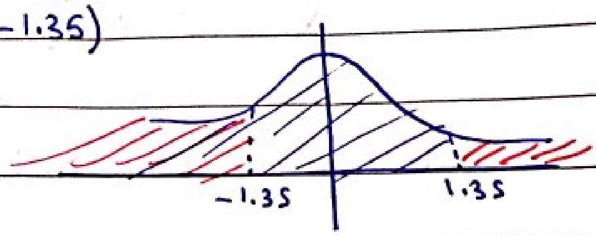
Ex $X \sim N(0,1)$ Find :-

القيم السالبة مش موجودة بالجدول

ⓐ $P(X < -1.35) = F(-1.35)$



from table

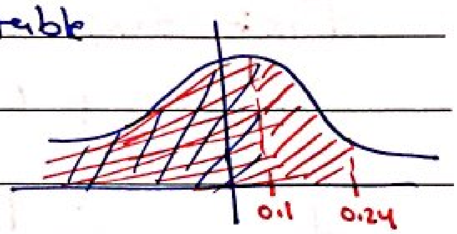
$F(-1.35) = 1 - F(1.35)$

$= 1 - 0.9115$

$$F(-x) = 1 - F(x)$$

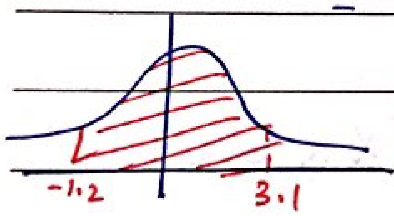
ⓑ $P(0.1 < X < 0.24)$

$= F(0.24) - F(0.1)$ → from table

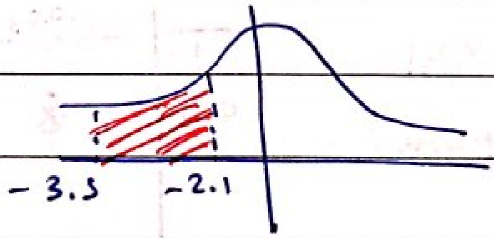


ⓒ $P(-1.2 < X < 3.1)$

$= F(3.1) - 1 + F(1.2)$



ⓓ $P(-3.5 < X < -2.1)$



$= F(-2.1) - F(-3.5)$

$= 1 - F(2.1) - 1 + F(3.5)$

$F(3.5) - F(2.1)$ → from table

Date. 19. June No.

* Find $F_X(x)$ for $X \sim N(a_x, \sigma_x^2)$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(y-a_x)^2}{2\sigma_x^2}} dy$$

by substitution :-

let $u = \frac{y - a_x}{\sigma_x}$

$y = -\infty \Rightarrow u = -\infty$, $y = x \Rightarrow u = \frac{x - a_x}{\sigma_x}$

$\frac{du}{dy} = \frac{1}{\sigma_x} \Rightarrow dy = \sigma_x \, du$

$$F_X(x) = \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-u^2}{2}} \sigma_x \, du$$

$\sim N(0,1)$

$$= \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} du \equiv F\left(\frac{x - a_x}{\sigma_x}\right)$$

$X \sim N(a_x, \sigma_x^2)$

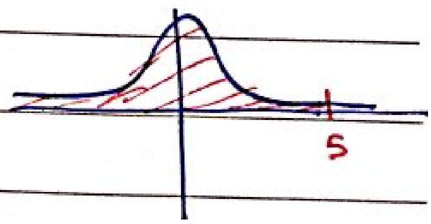
$$F_X(x) = F\left(\frac{x - a_x}{\sigma_x}\right) \rightarrow (*)$$

Ex:- $Y \sim N(10, 25)$ → Not standard)

find : $P(20 < Y < 35)$

$= F_X(35) - F_X(20)$

$= F\left(\frac{35-10}{5}\right) - F\left(\frac{20-10}{5}\right)$



$= F(5) - F(2)$

$\approx 1 - 0.9772$ → from table

عدل أي رقم أكبر من 4 ≈ 1

Date. 20 June No.

4 Bernouli R.V (Discrete)

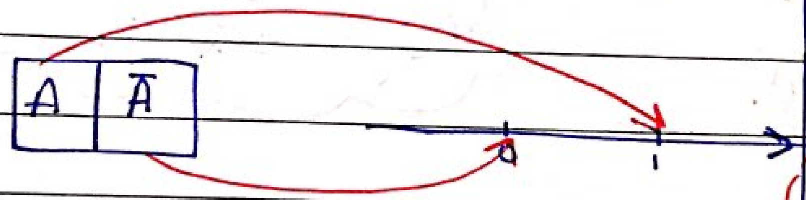
$X \sim B(P)$

Bernouli Trial

A "Success" — $\bar{A}$ "fail"

$P(A) = P \qquad P(\bar{A}) = 1-P$

$$X = \begin{cases} 1 & , \text{ If } A \text{ occurs} \\ 0 & , \text{ If } \bar{A} \text{ occurs} \end{cases}$$



$X = \{0, 1\}$ ($x_1 = 0$, $x_2 = 1$)

$P(X=1) = P$

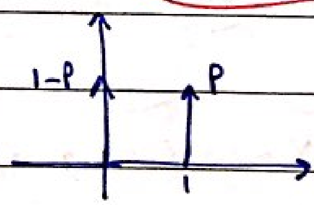
$P(X=0) = 1-P$

$$f_X(x) = \sum_{i=1}^{N} P(X=x_i)\,\delta(x-x_i)$$

$$f_X(x) = P(X=1)\,\delta(x-1) + P(X=0)\,\delta(x-0)$$

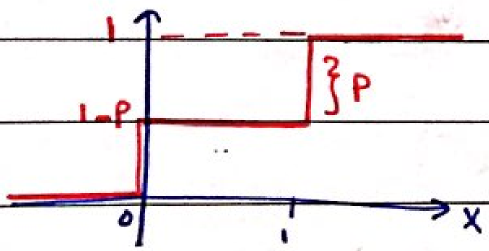
*** PDF**

$$f_X(x) = (1-P)\,\delta(x) + P\,\delta(x-1)$$



*** CDF :**

$$F_X(x) = (1-P)\,u(x) + P\,u(x-1)$$



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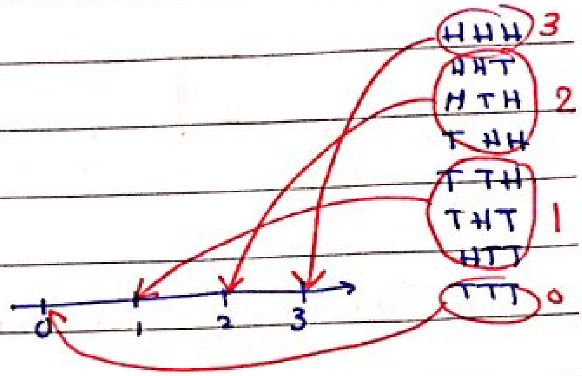
[5] Binomial R.V :- (Discrete) $X \sim b(p, N)$

X : the number of successes in repeated Bernoulli trial (with $P(A) = p$) N-times.

$X = \{ 0, 1, 2, \ldots, N \}$ ($x_1, x_2, x_3, \ldots, x_{N+1}$)

$P(X=k) = \binom{N}{k} p^k (1-p)^{N-k}$



PDF

$f_X(x) = \sum_{k=0}^{N} P(X=k)\, \delta(x-k)$

$= \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \, \delta(x-k)$

$\sum_{k=0}^{N} P(X=k) =$ (*)

$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} = 1 \quad 0 \le p \le 1$

~~CDF~~

Ex $Z \sim b(0.3, 3)$ Find $f_Z(z)$ and $F_Z(z)$

(p → 0.3, N → 3: # of trials)

sol: $Z = \{0, 1, 2, 3\}$

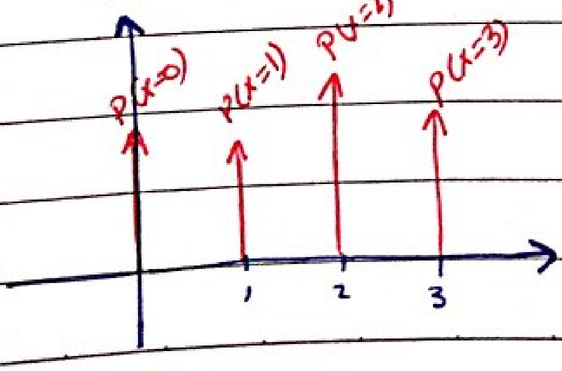
$f_X(x) = P(X=0)\,\delta(x) + P(X=1)\,\delta(x-1) + P(X=2)\,\delta(x-2) + P(X=3)\,\delta(x-3)$

$P(X=0) = \binom{3}{0} p^0 \, 0.7^3 = 0.7^3$

$P(X=1) = \binom{3}{1} (0.3)(0.7)^2 =$

$P(X=2) = \binom{3}{2} 0.3^2 (0.7)^1 = ---$

$P(X=3) = \binom{3}{3} (0.3)^3 (0.7)^0 = (0.3)^3$



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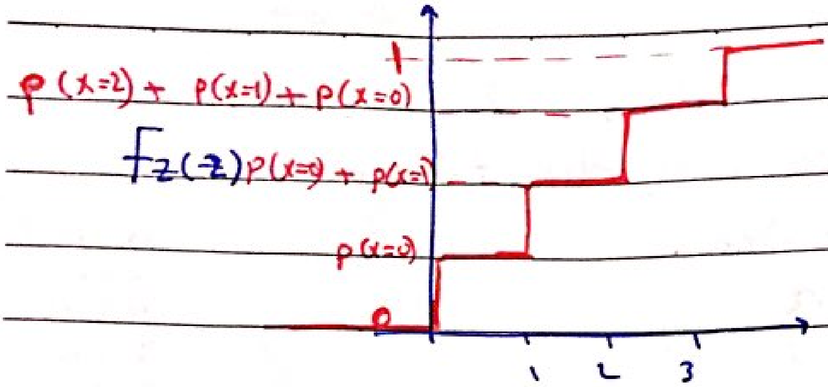


⊛ Conditional CDF and PDF :-

R.V X → $F_x(x) = P(\underbrace{X \leq x}_{\text{event } A})$

→ $f_x(x) = \frac{d F_x(x)}{dx}$

→ Conditional CDF : $F_x(x/B) = P(\underbrace{X \leq x}_{\text{event } A} /B) = \frac{P(X \leq x \cap B)}{P(B)}$

→ Conditional PDF : $f_x(x/B) = \frac{d F_x(x/B)}{dx}$

(ex) $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

⊛ $P(x_1 < X \leq x_2 /C) = F_X(x_2/C) - F_X(x/C)$

Date. 20, June No.

Ex: Given two boxes of balls:-

R	G	B
5	35	60

Box 1

(100 ball)

R	G	B
80	60	10

Box 2

(150 ball)

Exp: Randomly select a Box, and then draw out one ball from the selected box.

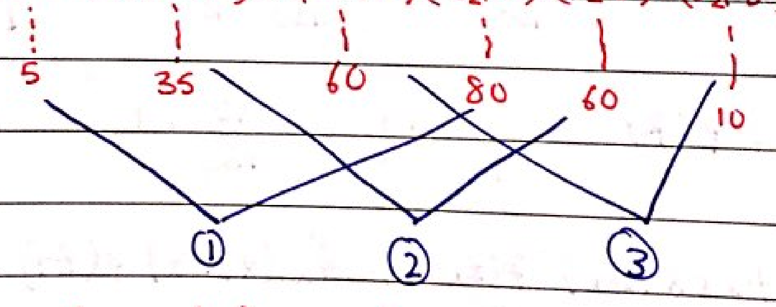
Define R.V $X = \begin{cases} 1, & \text{the ball is } R \\ 2, & \text{the ball is } G \\ 3, & \text{the ball is } B \end{cases}$

Define event B_1: "the selected box is Box #1"

event B_2: "the selected box is Box #2"

Find: (a) $F_X(x/B_1)$ (b) $F_X(x/B_2)$ (c) $F_X(x)$.

$S = \{ (B_1, R)\ (B_1, G), (B_1, B) (B_2, R) (B_2, G)\ (B_2, B) \}$



a) $F(x/B_1) = p(x=1/B_1)\,u(x-1) + p(x=2/B_1)\,u(x-2) + p(x=3/B_1)\,u(x-3)$

$= p(x=R/B_1)\,u(x-1) + p(G/B_1)\,u(x-2) + p(B/B_1)\,u(x-3)$

$= \frac{5}{100}\,u(x-1) + \frac{35}{100}\,u(x-2) + \frac{60}{100}\,u(x-3)$.

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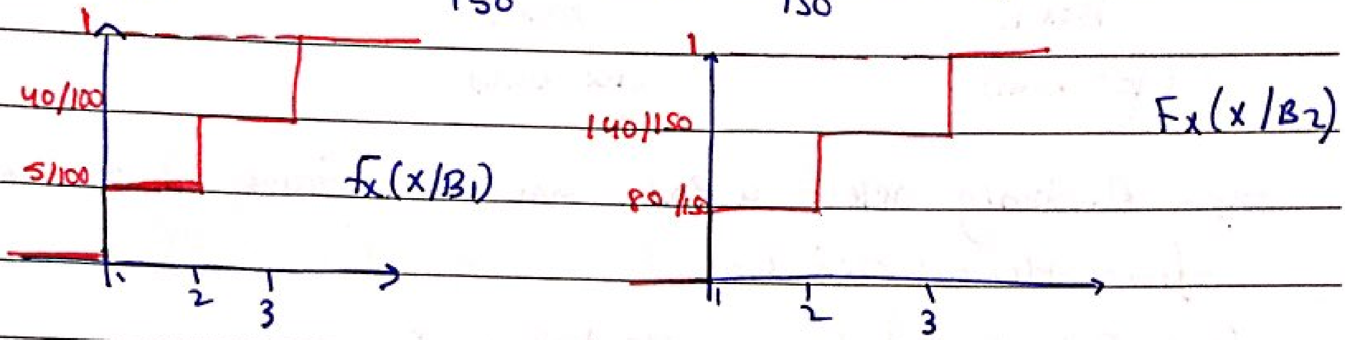
Date. 20.June No.

(b) $F_X(x/B_2) =$

$$\frac{80}{150}u(x-1) + \frac{60}{150}u(x-2) + \frac{10}{150}u(x-3)$$



(c) $F_X(x) =$

$$P(X=1)\,u(x-1) + P(X=2)\,u(x-2) + P(X=3)\,u(x-3)$$

$$P(X=1) = P(R) = P(R/B_1)\,P(B_1) + P(R/B_2)\,P(B_2)$$

Total prob law

$$= \frac{5}{100}\cdot\frac{1}{2} + \frac{80}{150}\cdot\frac{1}{2}$$

$$P(X=2) = P(G) = \frac{35}{100}\cdot\frac{1}{2} + \frac{60}{150}\cdot\frac{1}{2}$$

OR

$P(B_1) = 100/250$

$P(B_2) = 150/250$

$$P(X=3) = P(B) = \frac{60}{100}\cdot\frac{1}{2} + \frac{10}{150}\cdot\frac{1}{2}$$

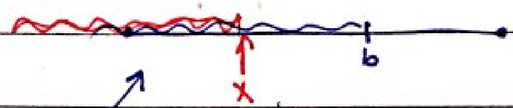
$$F_X(x) = F_X(x/B_1)\,P(B_1) + F_X(x/B_2)\,P(B_2)$$

Date. 24. June No.

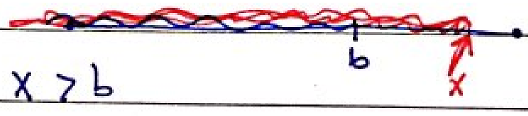
Ex:- Given a R.V X with CDF $F_X(x)$
Determine $F_X(x/B)$ where $B = \{X \leq b\}$ ← constant

Sol: $F_X(x/B) = P(X \leq x \,/\, X \leq b)$ (A = $X \leq x$, B = $X \leq b$)

$$= \frac{P(X \leq x \cap X \leq b)}{P(X \leq b)}$$



$$= \begin{cases} \dfrac{P(X \leq x)}{P(X \leq b)} = \dfrac{F_X(x)}{F_X(b)} & , x < b \\[2ex] \dfrac{P(X \leq b)}{P(X \leq b)} = 1 & , x > b \end{cases}$$



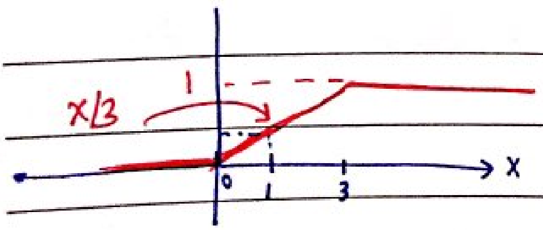
$$F_X(x/X \leq b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & , x < b \\ 1 & , x > b \end{cases}$$

← يمكن نستعملها مباشرة

↳ $f_X(x/\leq b)$ ← بنشتق بنطلع x

Ex: Given $X \sim u(0,3)$, find $F_X(x/X \leq 1)$ → $b = 1$

Sol: $F_X(x)$



→ $$F_X(x/X \leq 1) = \begin{cases} \dfrac{F_X(x)}{F_X(1)} & , x < 1 \\ 1 & , x > 1 \end{cases}$$

$$= \begin{cases} \dfrac{x/3}{1/3} = x & , 0 < x < 1 \\ 0 & , x < 0 \end{cases}$$

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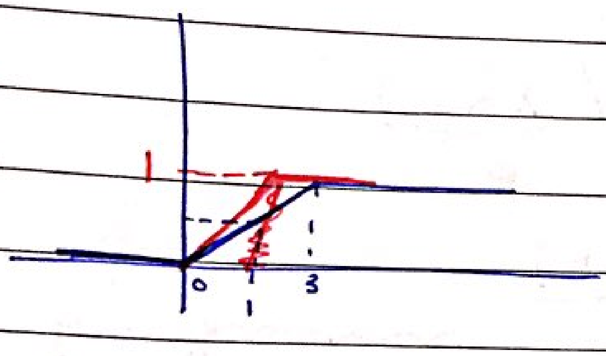
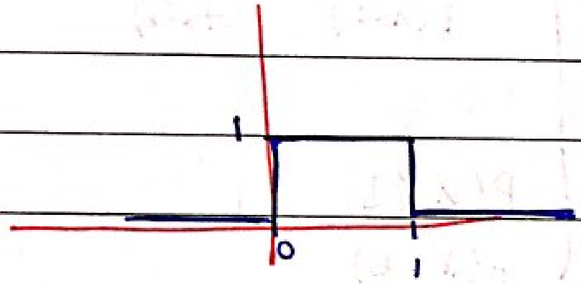
$$F_X(x / X \le 1) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 1, & 1 < x \end{cases}$$



$$f_X(x / X \le 1) = \frac{d F_X(x / X \le 1)}{dx}$$

$$= \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 1 \\ 0, & 1 < x \end{cases}$$

↓



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Chapter 3

* Operations on one Random Variable.

[1] Expectation

Expectation →
- DC value
- Mean value
- Average value

Symbol: X : $E[X]$, $\overline{X}$

* Expectation for Discrete R.V

eg:- 20 students → Find the average grade:

# of students	grade /50
4	36
3	17
3	25
6	45
4	11

(45 →) أكثر مرة تكرر — avg بطلع عنده — weight أخذ

$$\overline{G} = \frac{4*36 + 3*17 + 3*25 + 6*45 + 4*11}{20}$$

$$\overline{G} = 36 \cdot \frac{4}{20} + 17 \cdot \frac{3}{20} + 25 \cdot \frac{3}{20} + 45 \cdot \frac{6}{20} + 11 \cdot \frac{4}{20}$$

let $G = \{\overset{g_1}{36}, \overset{g_2}{17}, \overset{g_3}{25}, \overset{g_4}{45}, \overset{g_5}{11}\}$ as a Discrete R.V

$$P(G = \overset{36}{g_1}) = 4/20, \text{ ------ } P(G=11) = 4/20$$

So,

$$\overline{G} = g_1 P(G=g_1) + g_2 P(G=g_2) + \text{------} \; g_5 P(G=g_5)$$

for any discrete R.V $X = \{x_1, x_2, \text{---}, x_N\}$

with pdf →

$$f_X(x) = \sum_{i=1}^{N} P(X=x_i)\,\delta(x - x_i)$$

$$E[X] = \overline{X} = \sum_{i=1}^{N} x_i P(X = x_i)$$

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Ex: Given an expiment with $S = \{1,2,3,4\}$

$P(\{1\}) = 0.2$ $\quad$ $P(\{3\}) = 0.1$

$P(\{2\}) = 0.4$ $\quad$ $P(\{4\}) = 0.3$

If R.V $Z = S^2 - 1$ find $\bar{Z}$.

Sol: $Z = \{0, 3, 8, 15\}$

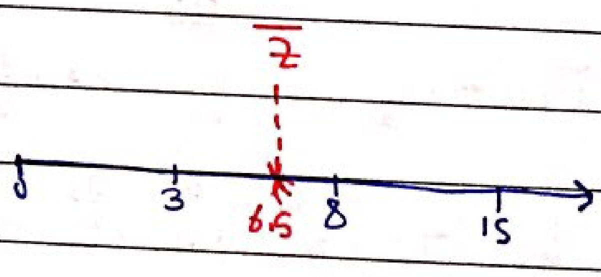
$P(Z=0) = P(\{1\}) = 0.2$ - - - -

$$\bar{Z} = \sum_{i=1}^{4} z_i P(Z = z_i) =$$

$$(0)(0.2) + (3)(0.4) + (8)(0.1) + (0.3)(15)$$

$$= \boxed{6.5}$$

Statistical average.



~~Ex~~ ~~Ex~~

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⊛ Expectation for continuous R.V :

$X \sim f_X(x)$

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

→ as a special case, if X is discrete

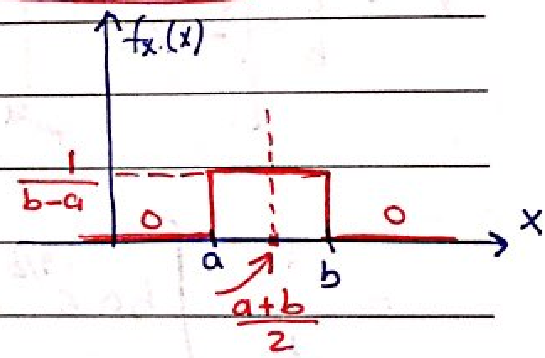
$$E[X] = \int_{-\infty}^{\infty} x \left(\sum_{i=1}^{N} P(X = x_i) \, \delta(x - x_i) \right) dx$$

$$= \sum_{i=1}^{N} P(X = x_i) \int_{-\infty}^{\infty} x \, (\delta(x - x_i) \, dx$$

$$= \sum_{i=1}^{N} x_i \, P(X = x_i)$$

→ Discrete R.V

Ex:- If $X \sim U(a,b)$ show that $\overline{X} = \frac{a+b}{2}$



Sol : $\overline{X} = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$

$$= \int_a^b \frac{1}{b-a} \cdot x \, dx$$

$$= \frac{x^2}{2(b-a)} \Bigg]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

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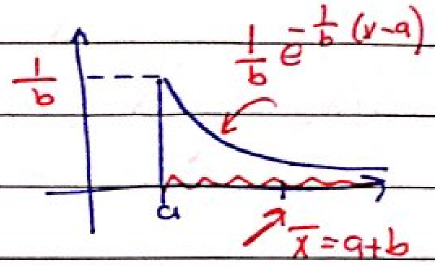
Ex: If $X \sim exp(a,b)$, show that $\bar{X} = a+b$

Solution:



$$\bar{X} = \int_{-\infty}^{\infty} x f_x(x)\, dx =$$

$$\int_{a}^{\infty} x \cdot \frac{1}{b} e^{-\frac{1}{b}(x-a)}\, dx$$

$$\frac{e^{a/b}}{b} \int_{a}^{\infty} x \cdot e^{-x/b}\, dx \quad \text{by parts}$$

$u = x \rightarrow du = dx$

$dv = e^{-x/b} \rightarrow v = \frac{e^{-x/b}}{-1/b}$

$v = -b e^{-x/b}$

$$\bar{X} = \frac{e^{a/b}}{b} \left[-b x e^{-x/b} \right]_{a}^{\infty} + \int_{a}^{\infty} b e^{-x/b}\, dx$$

$$= \frac{e^{a/b}}{b} \left[b a e^{-a/b} - \frac{\infty}{\infty} \right.$$ $=0$ using H'lopital's rule

undetermine

$$= \frac{e^{a/b}}{b} \left[b a e^{-a/b} + \left[\frac{b e^{-x/b}}{-1/b} \right]_{a}^{\infty} \right]$$

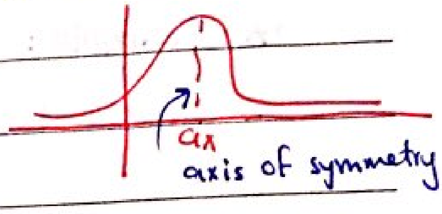
$$= \frac{e^{a/b}}{b} \left[ab e^{-a/b} + b^2 \left(e^{-a/b} - 0 \right) \right]$$

$$= a + b$$

Date. 25. 6 No.

*** Example:** Given $X \sim N(a_x, \sigma_x^2)$, show that

$$\boxed{\overline{X} = a_x}$$



Sol: $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{(2\sigma_x^2)}}$$

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx$$

let $u = \frac{x-a_x}{\sigma_x} \rightarrow du = \frac{dx}{\sigma_x} \rightarrow x = u\sigma_x + a_x$

$u : -\infty \rightarrow \infty$

$$E[X] = \int_{-\infty}^{\infty} \frac{\sigma_x u + a_x}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-u^2/2} \sigma_x \cdot du$$

$$= \frac{\sigma_x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u\, e^{-u^2/2} du + a_x \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

odd x even = odd $\sim N(0,1)$

$$= 0 + a_x(1) = a_x$$

*** Example:** $X \sim$ Bernouli (p), find $\overline{X}$

Sol: $X = \{0, 1\}$ ($x_1 = 0$, $x_2 = 1$)

$P(X=0) = 1-p \qquad P(X=1) = p$

$$\overline{X} = \sum_{i=1}^{2} x_i\, P(X=x_i) = 0(1-p) + 1(p) = p$$

$$\boxed{\overline{X} = p}$$

Date. 25. 6 No.

*** Example:** $X \sim binomial(P, N)$

show that $\bar{X} = N.P$

Sol: $X = \{0, 1, 2, ---, N\}$

$$P(x=i) = \binom{N}{i} P^i (1-P)^{N-i}$$

$$E[X] = \sum_{j=1}^{N} x_j P(x = x_j) = \sum_{i=0}^{N} i \, p(x=i)$$

$$= \sum_{i=1}^{N} i \, p(x=i)$$

$$= \sum_{i=1}^{N} i \binom{N}{i} P^i (1-P)^{N-i} = \sum_{i=1}^{N} i \frac{N!}{i!(N-i)!} P^i (1-P)^{N-i}$$

$(i-1)!$

$$= Np \sum_{i=1}^{N} i \frac{(N-1)!}{(i-1)!(N-n)!} P^{i-1} (1-P)$$

let $k = i-1$, $i = k+1$

$k: 0 \rightarrow N-1$

$$= Np \sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-1-k)!} P^k (1-P)^{N-1-k}$$

let $N-1 = M$

$$Np \sum_{k=0}^{m} \binom{M}{k} P^k (1-P)^{M-K}$$

$\bar{X} = Np$ for binomial.

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** Expectation of function of R.V

Let $g(x)$ is function of R.V x with PDF $f_X(x)$

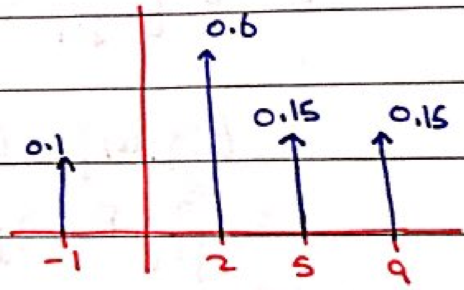
$$E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) \, dx$$

- For discrete RV x :

$$E[g(x)] = \sum_{i=1}^{N} g(x_i) \cdot P(x = x_i)$$

Example :- $X = \{-1, 2, 5, 9\}$, let $g(x) = x^2 - 1$

Find $\bar{y}$?!



Sol: $g(x) = \{0, 3, 24, 80\}$

$$E[g(x)] = \sum_{i=1}^{4} g(x_i) \, P(X = x_i)$$

$$\bar{y} = 0(0.1) + 3(0.6) + 24(0.15) + 80(0.15)$$

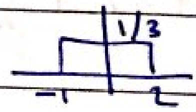
$$\bar{y} = ---$$

Example :- $X \sim U(-1, 2)$ Find ⓐ $E[x]$

ⓑ $E[g(x)]$, where $g(x) = x^2 - 2$

Sol: ⓐ $\bar{X} = \frac{2 + (-1)}{2} = \frac{1}{2}$

ⓑ $E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) \, dx$



$$= \int_{-1}^{2} (x^2 - 2) \frac{1}{3} \, dx = \frac{1}{3} \left(\frac{x^3}{3} - 2x \right) \Big]_{-1}^{3}$$

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* Example: show that $E[ag_1(x) + b g_2(x)] = aE[g_1(x)] + bE[g_2(x)]$

proof:

$$E[g(x)] = \int g(x) f_x(x)\, dx$$

$$= \int (a g_1(x) + b g_2(x))\, f_x(x)\, dx$$

$$= a\int g_1(x) f_x(x)\, dx + b\int g_2(x) f_x(x)\, dx$$

$$= a\, E[g_1(x)] + b E[g_2(x)]$$

$$= E\left[\sum_{j=1}^{J} g_j(x)\right] = E[g_1(x) + g_2(x) + \cdots g_N(x)]$$

$$= E[g_1(x)] + E[g_2(x)] + \cdots E[g_N(x)$$

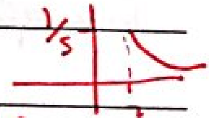
$$= \sum_{j=1}^{N} E[g_j(x)]$$

constant

(*) $E[g] = \int a\, f_x(x)\, dx = a\int f_x(x)\, dx = a$

Note $E[a] = a$ $E[ax] = a\, E[x]$

$E[a g(x)] = aE[g(x)]$



Example: $X \sim \exp(\overset{a}{2}, \overset{b}{5})$, Find $E[-x + 23 + 1/2\, x^2]$

$$= -E[x] + 23 + \frac{1}{2} E[x^2] = 7 + 23 + \int x^2 f_x(x)\, dx$$

$$= 30 + \int_2^{\infty} x^2 \cdot \frac{1}{5} e^{-\frac{1}{5}(x-2)} dx = 30 + \frac{e^{2/5}}{5}\int_2^{\infty} x^2 \cdot e^{-x/5}\, dx$$

→ by parts twice

Date. 27.Jun No.

* Moments about the origin (moments) :-

Let X RV with PDF $f_X(x)$

$$m_n = E[X^n] \qquad n = 0, 1, \ldots$$

$$E[x^n] = \int x^n f_X(x)\, dx$$

* Zeroth order : $n=0 \rightarrow m_0 = E[x^0] = E[1] = 1$
* 1^{st} order : $n=1 \rightarrow m_1 = E[x]$ → mean → average value → DC value
* DC average power $\rightarrow m^2 = \bar{x}^2 = P_{DC}$
* 2^{nd} order moments : $n=2 \rightarrow m = E[x^2]$ = total avg power = P_{tot}

* $P_{AC} = P_{tot} - P_{DC} = E[x^2] - \bar{x}^2$
  ↳ AC avg power in RV X

* $P_{AC} = m_2 - m_1^2 = Var(x)$ → Variance of X
  (m_2 ↳ P_{DC}, m_1^2 ↳ P_{tot})

Example: $X \sim exp(a, b)$, find: (a) m_1 (b) m_2 (c) P_{AC}

$m_1 = E[x] = a+b$

$$m_2 = E[x^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_a^{\infty} x^2 \cdot \frac{1}{b} e^{-\frac{1}{b}(x-a)}$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} x^2 e^{-x/b} \cdot dx$$

→ can be solved using parts integration techniques.

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(c) $P_{DC} = \bar{x}^2 = (a+b)^2$

$P_{AC} = m_2 - P_{DC} = b^2 = Var(x)$ (aka)

Example: $X \sim U(a,b)$

[A*] $m_1 = E[x] = \frac{a+b}{2}$

[B*] $m_2 = E[x^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$

$= \frac{x^3}{3(b-a)}\Big]_a^b = \frac{b^3 - a^3}{3(b-a)}$

$= \frac{(b-a)\cdot(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3} = P_{tot}$

[C*] $P_{AC} = m_2 - m_1^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$

$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$

$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$

$= \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$

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* Moments about the mean (central moments):

$$\mu_n = E[(x-\bar{x})^n] = \int_{-\infty}^{\infty} (x-\bar{x}) f_x(x)\, dx$$

- $\mu_0 = 1$
- $\mu_1 = E[x-\bar{x}] = E[x] - E[\bar{x}] = \bar{x} - \bar{x} = 0$
- $\mu_2 = E[(x-\bar{x})^2]$ → PAC → Variance of X
- $\mu_2 = E[(x-\bar{x})^2] = E[(x^2 - 2\bar{x}x + \bar{x}^2)] =$

$$E[x^2] - 2\bar{x}E[x] + E[\bar{x}^2] - E[x^2] - \bar{x}^2$$

$$\longrightarrow \mu_2 = m_2 - m_1^2 = PAC = Var(x) = \sigma_x^2$$

(*) $X \sim exp(a,b) \longrightarrow Var(x) = b^2$
(*) $X \sim U(a,b) \longrightarrow var(x) = \frac{(b-a)^2}{2}$
(*) $X \sim N(a, \sigma_x^2) \longrightarrow var(x) = \sigma_x^2$

Ex X is a RV with $f_x(x) = 0.1\,\delta(x) + 0.1\,\delta(x-1) + 0.2\,\delta(x-3) + 0.6\,\delta(x-5)$

Find var(x) ?!

$$var(X) = E[x^2] - \bar{x}^2$$

$$\bar{x} = \sum_{i=1}^{4} x_i\, p(x=x_i) = (0.1)(1) + (3)(0.2) + (5)(0.6) = 3.7$$

$$E[x^2] = \sum x_i^2\, p(x=x_i) = (1)^2(0.1) + (3)^2(0.2) + (5)^2(0.6) = 16.9$$

$$var(x) = 16.9 - (3.7)^2 = 3.2$$

* Characteristic function:- $\phi_x(\omega)$

X with $f_x(x)$

$$\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx$$

$\hookrightarrow g(x)$

$$f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega x} \phi_x(\omega) \cdot d\omega$$

$f_x(x) \longrightarrow \phi_x(\omega)$

If X is Discrete : $\phi_x(\omega) = E[e^{j\omega x}] = \sum_{i=1}^{N} e^{j\omega x_i} \cdot p(x = x_i)$

$$* \; m_n = \int x^n \cdot f_x(x)\, dx = (-j)^n \frac{d^n \phi_x(\omega)}{d\omega^n}\Big|_{\omega=0}$$

Ex) $X \sim \exp(a, b)$

Find:- (a) $\phi_x(\omega)$ (b) m_1 and m_2 using $\phi_x(\omega)$

$$\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} \cdot \frac{1}{b} e^{-\frac{1}{b}(x-a)}\, dx$$

$$= \frac{e^{a/b}}{b} \int_{-\infty}^{\infty} e^{-(1/b - j\omega)x} \cdot dx = \frac{e^{a/b}}{b} \left[\frac{e^{-(\frac{1}{b} - j\omega)x}}{-(\frac{1}{b} - j\omega)} \right]_a^{\infty}$$

$$= \frac{e^{a/b}}{b} \left[\frac{e^{-(\frac{1}{b} - j\omega)a}}{(\frac{1}{b} - j\omega)} - 0 \right]$$

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$$= \frac{e^{\frac{a}{b}} \cdot e^{-\frac{a}{b}} \cdot e^{jwa}}{1 - Jwb} = \frac{e^{jwa}}{1 - jwb}$$

$$m_1 = (-j)^1 \left. \frac{d\phi x(w)}{dw} \right|_{w=0}$$

$$= -j \left(\left. \frac{(1 - jwb)(ja\, e^{jwa}) - e^{jwa}(-jb)}{(1 - jwb)^2} \right|_{w=0} \right)$$

$$= -j \left(\frac{ja + jb}{1} \right) = a + b \;\#$$

$$m_2 = E[X^2] = (-j)^2 \left. \frac{d^2 \phi x(w)}{dw^2} \right|_{w=0} = (a+b)^2 + b^2$$

Example $X \sim u(a,b)$ Find $\phi x(w)$

$$\phi x(w) = \int_a^b e^{jwx} \frac{1}{b-a} \, dv$$

$$= \frac{1}{b-a} \left. \frac{e^{jwx}}{jw} \right]_a^b = \frac{e^{jwb} - e^{jwa}}{jw(b-a)}$$

we know. $\bar{x} = m_1 = \frac{a+b}{2}$

OR

$$m_1 = -j \frac{d\phi x(w)}{dw} \; --- = \frac{a+b}{2}$$

using lopital.

Example $X = \{-1, 0, 2, 3\}$

$P = \{0.1, 0.3, 0.2, 0.4\}$

(a) $\bar{x} = -1 \times 0.1 + 0 \times 0.3 + 2 \times 0.2 + 3 \times 0.4$

(b) $\phi x(w) = E\left[e^{jwx}\right] = \sum e^{jwx_i} p(x = x_i)$

$$= 0.1\, e^{-jw} + 0.3\, e^{0} + 0.2\, e^{j2w} + 0.4\, e^{j3w}$$

$$\bar{x} = (-j) \cdot \left. \frac{d\phi x(w)}{dw} \right|_{w=0} ---$$

$(0.1)(-1) + (0) + ---$ ← (a) نفس حل

* Moment generated function $M_X(v)$

$$M_X(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_X(x)\, dx$$

$$m_n = \frac{d M_X(v)}{d v^n}\Big|_{v=0}$$

* Moment for exponential.

$X \sim \exp(a, b)$

$$M_X(v) = E[e^{vX}] = \int_a^{\infty} e^{vX} \cdot \frac{1}{b} e^{-1/b(x-a)} dx$$

$$= \frac{e^{va}}{1-va}, \quad m_1 = \frac{(1-vb)a e^{va} - e^{va}(-b)}{(1-vb)^2}\Bigg|_{v=0}$$

$$= a+b$$

* Transformation of one R.V :-

X R.V (input) → [T(·)] (system, transformation) → Y = T(X) (output)

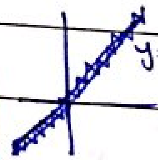
$f_X(x)$, $F_X(x)$ → Determine $f_Y(y)$

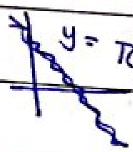
$X \rightarrow [\sqrt{\cdot}] \rightarrow Y = \sqrt{X}$

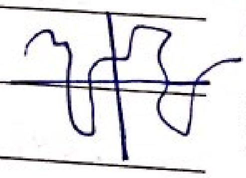
$X \rightarrow [a e^{(\cdot)}] \rightarrow Y = a e^{X}$

$X \rightarrow [|X|] \rightarrow Y = |X|$

Transformation
- monotonic
  - only increasing
  - only decreasing
- non-monotonic

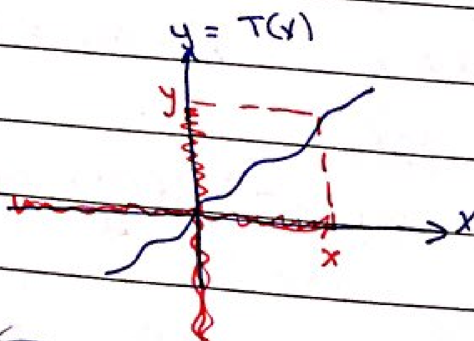






Case 1 Monotonic increasing :

$X \rightarrow [T(\cdot)] \rightarrow Y = T(X)$



$\rightarrow F_Y(y) = P(Y \leq y) = F_X(x)$

$\rightarrow F_Y(y) = F_X(T^{-1}(x))$

$y = T(x) \rightarrow x = T^{-1}(y)$

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* $f_Y(y) = \frac{d F_Y(y)}{dy} = \frac{d F_X(T^{-1}(y))}{dy}$

Using Chain Rule :

$= \frac{d F_X(T^{-1}(y))}{d T^{-1}} \cdot \frac{d T^{-1}(y)}{dy}$

$f_Y(y) = f_X(T^{-1}(y)) \cdot \frac{d T^{-1}(y)}{dy}$ always positive

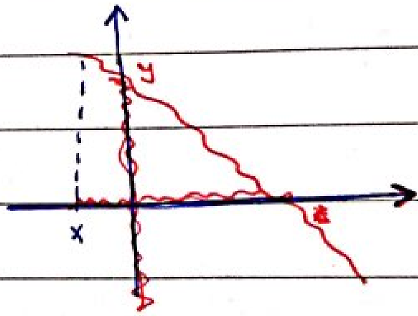
*** Case 2 :-** monotonic decreasing

$X \longrightarrow \boxed{T(\cdot)} \longrightarrow Y = T(X)$

$F_Y(y) = P(Y \leq y)$

$= P(X \geq x)$

$= 1 - F_X(x) = 1 - F_X(T^{-1}(y))$



$F_Y(y) = 1 - F_X(T^{-1}(y))$

$f_Y(y) = \frac{d F_Y(y)}{dy} = \frac{d(1 - F_X(T^{-1}(y)))}{dy}$

$= -\frac{d F_X(T^{-1}(y))}{dy} = -f_X(T^{-1}(y)) \frac{d T^{-1}(y)}{dy}$ always Negative

* For monotonic transformation :

$f_Y(y) = f_X(T^{-1}(y)) \left| \frac{d T^{-1}(y)}{dy} \right|$

Ex $X \longrightarrow \boxed{a(\cdot)+b} \longrightarrow Y = T(X) = aX + b$, a and b are real constants

$G \sim N(a_X, \sigma_X^2)$

Find $f_Y(y)$?!

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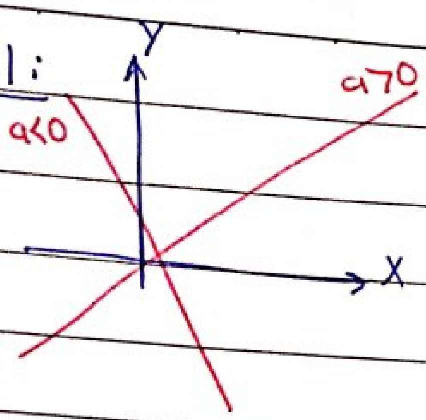
Date. No.

Sol:



* we have monotonic transformation

$$f_Y(y) = f_X(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-a_x)^2/2\sigma_x^2}$$

$$y = ax+b \Rightarrow x = \frac{y-b}{a} = T^{-1}(y).$$

$$d\,T^{-1}(y) = \frac{1}{a}$$

$$* \quad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{\left(\frac{y-b}{a} - a_x\right)^2 \cdot \frac{a}{a}}{2\sigma_x^2}} \cdot \frac{1}{|a|}$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2 a^2}} e^{-\frac{(y-b-a\,a_x)^2}{(2\sigma_x)}}$$

$$= \frac{1}{\sqrt{2\pi a^2 \sigma_x^2}} e^{-\frac{(y-(a\,a_x+b))^2}{2a^2\sigma_x^2}}$$

(red annotations: $a\,a_x + b$ circled and marked a_y; $a^2\sigma_x^2$ circled and marked σ_y^2)

$$\sim N(a_y, \sigma_y^2) = N(a\,a_x+b, a^2\sigma_x^2)$$

∴ linear transformation for gaussian is gaussian.

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* monotonic Transformation :-

$X \longrightarrow \boxed{T(\cdot)} \longrightarrow Y = T(X)$

$$f_Y(y) = f_X\left(T^{-1}(y)\right) \left| \frac{d\, T^{-1}(y)}{dy} \right|$$

$$F_Y(y) = \begin{cases} F_X(T^{-1}(y)) & , \text{monotonic increasing} \\ 1 - F_X(T^{-1}(y)) & , \text{monotonic decreasing.} \end{cases}$$

$X \sim N(a_x, \sigma_x^2)$

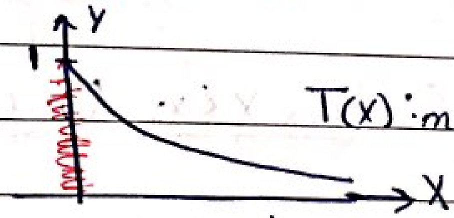
$Y = aX + b \sim N(\underbrace{a\,a_x + b}_{a_y}, \underbrace{a^2 \sigma_x^2}_{\sigma_y^2})$

Ex $X \longrightarrow \boxed{e^{-(\cdot)}} \longrightarrow Y = T(x) = e^{-x}$ (سؤال اقوابة)

$x \sim \exp(0,1)$

Find $f_Y(y)$?!

sol:



$0 < Y < 1$

T(X): monotonic Decreasing

$$f_Y(y) = f_X(T^{-1}(y)) \cdot \left| \frac{d\, T^{-1}(y)}{dy} \right|$$

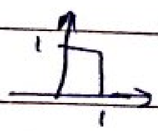
$\Rightarrow T^{-1}(y) = -\ln(y)$, $\frac{d\, T^{-1}(y)}{dy} = -\frac{1}{y}$

substitution

$f_X(x) = e^{-x}$, $x > 0$

$f_Y(y) = \frac{e^{\ln(y)}}{y}$, $\boxed{0 < y < 1}$ ⟶ لازم نحدد الفترة

$= y/y = 1 \quad 0 < y < 1 \longrightarrow$ Uniform



$y \sim U(0,1)$

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Ex: show that $Var(aX) = a^2 Var(X)$

sol: $Var(aX) = E[(aX)^2] - E^2[aX]$ (aX ← g(x))

| $Var(X) = E[X^2] - E[X]^2$

$= E[a^2 X^2] - (a\bar{X})^2$

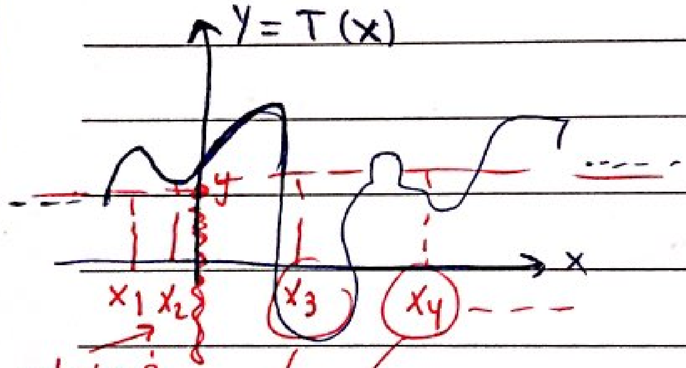
$= a^2 E[X^2] - a^2 \bar{X}^2$

دائماً ال constant

$= a^2 (E[X^2] - \bar{X}^2) = a^2 Var(X)$ ← لـ var$()^2$ بيطلع برا

$Var(b) = 0$ (b → constant)

Case (3): Non-monotonic Transformation.



عشان أكثر من قيمة y بتساوي

كل هاي النقاط من حل المعادلة $y - T(x) = 0$

$F_Y(y) = P(Y \le y)$

must be disjoint

$= P(\cdots x_1 < X < x_2 \cup x_3 < X < x_4 \cdots)$

$= \cdots + P(x_1 < X < x_2) + P(x_3 < X < x_4) + \cdots$

$= \cdots + F_X(x_2) - F_X(x_1) + F_X(x_4) - F_X(x_3) + \cdots$

$f_Y(y) = \dfrac{d\, F_Y(y)}{dy}$

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* In General :

For non-monotonic transformation :

$$f_Y(y) = \sum_{n=1}^{N} \frac{f_X(x_n)}{\left| \frac{dT(x)}{dx}\Big|_{x=x_n} \right|}$$ ← absolute value

where $x_1, x_2, \ldots, x_N$ are the roots for

$$T(x) - y = 0$$

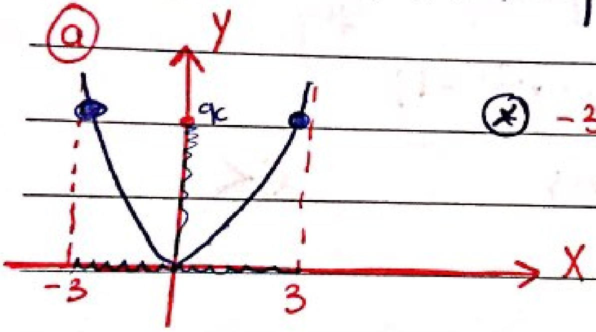
Example :- $X \longrightarrow \boxed{c(\cdot)^2} \longrightarrow Y = T(x) = cX^2$

$c > 0$

(a) $X \sim u(-3,3)$
(b) $X \sim \exp(1,2)$

} Find $f_Y(y)$ for case (a) and (b)

(a)



⊛ $-3 < X < 3 \longrightarrow 0 < y < 9c$

⊛ $T(x) - y = 0$

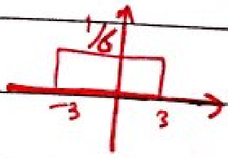
$cx^2 - y = 0 \longrightarrow x^2 = y/c$

$x = \pm\sqrt{y/c}$

⊛ $x_1 = \sqrt{y/c}$, $x_2 = -\sqrt{y/c}$

⊛ $\frac{d\,T(x)}{dx} = 2cX$

⊛ $f(y) = \frac{f_x(x_1)}{|2cx_1|} + \frac{f_x(x_2)}{|2cx_2|}$



$$f_Y(y) = \frac{f_X(\sqrt{y/c})}{|2c\sqrt{y/c}|} + \frac{f_X(-\sqrt{y/c})}{|-2c\sqrt{y/c}|}$$

$$= \frac{1/6}{2\sqrt{cy}} + \frac{1/6}{2\sqrt{cy}} = \boxed{\frac{1}{6\sqrt{cy}}} \quad 0 < y < 9c$$

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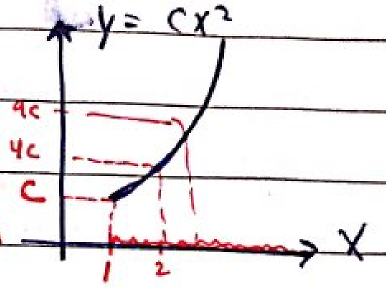
(b) $X \sim \exp(1, 2)$

$f_X(x) = \frac{1}{2} e^{-1/2(x-1)}, \quad x > 1$



$c < y$ monotonic transformation

$$f_Y(y) = f_X(T^{-1}(y)) \cdot \left| \frac{dT^{-1}(y)}{dy} \right|$$

$y = cx^2 \rightarrow X = \pm \sqrt{y/c}$

$f_Y(y) = f_X(\sqrt{y/c})$

$x_1 = \sqrt{y/c} = \frac{1}{\sqrt{c}} y^{1/2}$

$\frac{dx}{dy} = \frac{1}{2\sqrt{cy}}$

$x_2 = -\sqrt{y/c}$

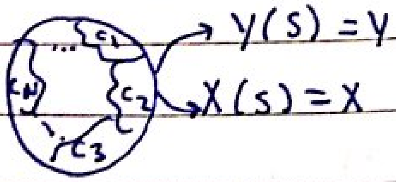
$\frac{dx_2}{dy} = \frac{-1}{2\sqrt{cy}}$

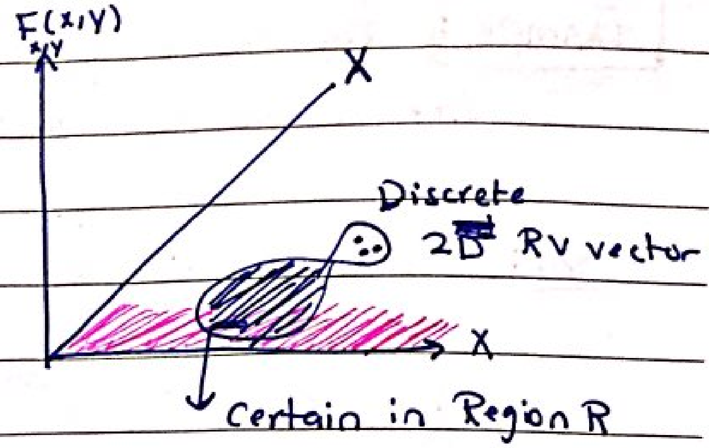
$$f_Y(y) = \frac{f_X(\sqrt{y/c})}{2\sqrt{cy}} + \frac{f_X(-\sqrt{y/c})}{2\sqrt{cy}}$$

$$= \frac{\frac{1}{2} e^{-1/2(\sqrt{y/c} - 1)}}{2\sqrt{cy}} + \frac{\frac{1}{2} e^{-\frac{1}{2}(-\sqrt{y/c} - 1)}}{2\sqrt{cy}}$$

$c < y$

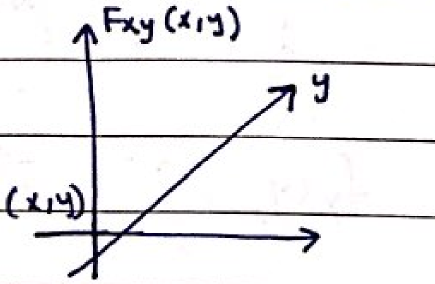
* Chapter 4 :-





(X, Y) : 2^{nd} R.Vector
∩

⊛ Joint CDF for (X,Y) R.Vector



$F_{xy}(x,y) = P(\underbrace{X \le x}_{A} \cap \underbrace{Y \le y}_{B})$ — $x \to (-\infty,\infty)$, $y \to (-\infty,\infty)$

⊛ Joint CDF properties :-

[1] $F_{x,y}(-\infty,\infty) = P(X \le -\infty, Y \le -\infty)$
$= P(\phi \cap \phi) = P(\phi) = 0$

$F_{xy}(-\infty, y) = P(X \le -\infty, \underbrace{Y \le y}_{B}) = P(\phi \cap B) = P(\phi) = 0$

$F_{xy}(x, -\infty) =$ ---- $= P(A \cap \phi) = P(\phi) = 0$

[2] $F_{x,y}(\infty,\infty) = P(X \le \infty, Y \le \infty) = P(S \cap S) = P(S) = 1$

[3] $0 \le F_{x,y}(x,y) \le 1$

[4] $F_{x,y}(x,y)$ is non-decreasing function.

[5] $F_{x,y}(x,\infty) = P(X < x, Y \le \infty) = P(X \le x \cap S) = P(X \le x) = F_X(x)$

marginals cdf's

$F_{xy}(\infty, y) = P(X \le \infty, Y \le y) = P(Y \le y \cap S) = P(Y \le y)$
$= F_X(y)$

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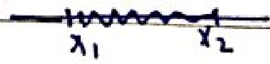
Recall :-

$P(x_1 < X < x_2) = f_X(x_2) - F_X(x_1)$



[6] $P(x_1 < X \leq x_2 \,,\, y_1 < Y < y_2) = P((x,y) \in R)$

$= f_{x,y}(x_2, y_2) + f_{x,y}(x_1, y_1)$

$= F_{x,y}(x_1, y_2) - F_{x,y}(x_2 - y_1)$

[Ex] Joint CDF for discrete 2D R.vector $(X,Y) = \{(1,1)(2,1)(3,3)\}$

Given:

$P\{(1,1)\} = 0.2 \quad P\{(2,1)\} = 0.3 \quad P\{(3,3)\} = 0.5$

Determine $f_{x,y}(x,y)$

Note : $P(1,1) = P(X=1, Y=1)$

$= P(X=1 \cap Y=1)$

<u>solution</u>: $F_{x,y}(-\infty, \infty) = 0$

$F_{x,y}(1,1) = P(X \leq 1, Y \leq 1) = P(1,1) = 0.2$

$F_{x,y}(x,y) = 0.2\, u(x-1)\, u(y-1) + 0.3\, u(x-2)\, u(y-3)$

$+ 0.5\, u(x-3)\, u(y-3)$

$F_X(x) = F_{x,y}(x, \infty) = 0.2\, u(x-1) + 0.3\, u(x-2) + 0.5\, u(x-3)$

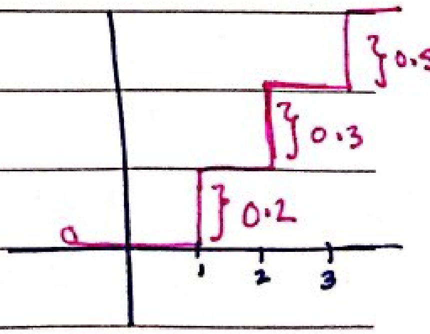
$X = \{1,2,3\}$, $P(X=1) = 0.2 \quad P(X=2) = 0.3 \quad P(X=3) = 0.5$

$F_Y(y) = F_{x,y}(\infty, y) = 0.2\, u(y-1) + 0.3\, u(y-1)$

$+ 0.5\, u(y-3)$

$= 0.5\, u(y-1) + 0.5\, u(y-3)$

$y = \{1, 3\}$



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* Multiple Random variables :-

Joint CDF ← → Joint PDF

2 D R.vector (X, Y)

$$F_{X,Y}(x,y) = P(\underbrace{X \le x}_{A} \cap \underbrace{Y \le y}_{B})$$

$$f_{X,Y}(x,y) = \frac{d\, F_{X,Y}(x,y)}{dx\,dy}$$

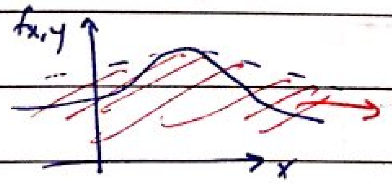
ND R.vector: $(X_1, X_2, \ldots, X_N)$

$F_{X_1 X_2 \ldots X_N}(x_1, x_2, \ldots, x_N)$ ← → $f_{X_1 X_2 \ldots X_N}(x_1, x_2, \ldots, x_N)$

⊛ Joint Density function properties :- (PDF)

[1] $f_{X,Y}(x,y) \ge 0$

[2] $\iint_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$ → volume = 1



[3] $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(z_1, z_2)\, dz_1\, dz_2$

$F_{X,Y}(\infty, \infty) = 1$

[4] $F_X(x) = F_{X,Y}(x, \infty) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(z_1, z_2)\, dz_2\, dz_1$

$F_Y(y) = F_{X,Y}(\infty, y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(z_1, z_2)\, dz_1\, dz_2$

[5] $f_X(x) = \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

↳ marginal density for x

y ل joint لـ

[6] $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$

$f_X(x)$, $f_Y(y)$ } → called marginals PDF's

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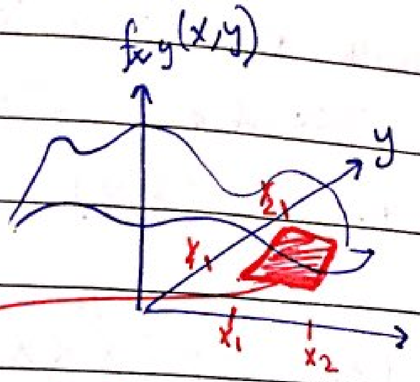
[7] $P(x_1 < X \le x_2 , y_1 < Y \le y_2)$

$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{x,y}(x,y)\, dx\, dy .$$



$= P((X,Y) \in R)$

Example given a two dimentional R.V X, Y with joint PDF

$$f_{x,y}(x,y) = \begin{cases} b e^{-x} \cos(y) & , 0<x<2 , 0<y<\pi/2 \\ 0 & , \text{otherwise}. \end{cases}$$

ⓐ find b ?! $\rightarrow \int_{-\infty}^{\infty}\int f_{x,y}(x,y)\, dx\, dy = 1$

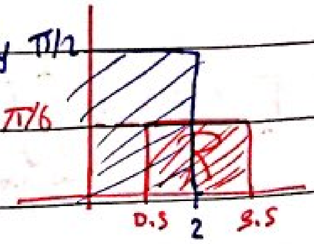
$$\rightarrow \int_0^{\pi/2}\int_0^{2} b e^{-x} \cos(y)\, dx\, dy = b\int_0^{\pi/2} \cos(y) \left(\int_0^2 e^{-x} dx\right) dy$$

$\hookrightarrow = -e^{-x}\Big]_0^2 = 1 - e^{-2}$

$$= b(1-e^{-2}) \int_0^{\pi/2} \cos(y)\, dy = b(1-e^{-2}) \left(\sin(y) \Big]_0^{\pi/2} \right)$$

$$b(1-e^{-2}) = 1 \rightarrow \boxed{b = \frac{1}{1-e^{-2}}}$$

ⓑ find $P((x,y) \in R)$



$P((x,y) \in R) = P\left(\frac{1}{2} < X < 3.5 , 0 < Y < \pi/6\right)$

RV ← عوض بس من 0 → 2

$$= \int_0^{\pi/6}\int_{1/2}^{2} b\, e^{-x} \cos(y)\, dx\, dy$$

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Example :- given $f_{x,y}(x,y) = x\, e^{-x(y+1)}\, u(x)\, u(y)$
find the marginal PDF's $f_X(x)$ and $f_Y(y)$.

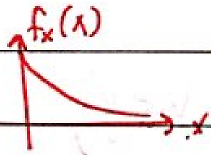
$$f_{x,y}(x,y) = \begin{cases} x\, e^{-x(y+1)} & , x>0 \text{ and } y>0 \\ 0 & , \text{otherwise} \end{cases}$$

Sol: $f_X(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy = \int_{0}^{\infty} x\, e^{-x(y+1)}\, u(x)\, u(y)\, dy$

$$= x\, u(x)\, e^{-x} \int_0^{\infty} e^{-xy}\, dy = x u(x)\, e^{-x} \cdot \left[\frac{e^{-xy}}{-x} \right]_0^{\infty}$$

$$= u(x)\, e^{-x}[1-0] = e^{-x} u(x) = \begin{cases} e^{-x} & , x>0 \\ 0 & , ow \end{cases}$$

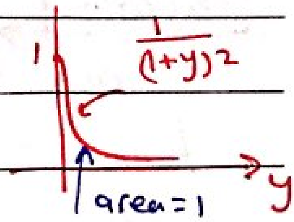
i.e ; $X \sim \exp(0,1)$



$$f_Y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dx = \int_0^{\infty} x\, e^{-x(y+1)}\, u(x)\, u(y)\, dx$$

by parts.

$$= \frac{u(y)}{(1+y)^2} = \begin{cases} \frac{1}{(y+1)^2} & , y>0 \\ 0 & , \text{otherwise} \end{cases}$$



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Independent Random variables:

Recall: two events A and B

$P(A/B) = P(A)$, $P(B/A) = P(B)$

as a consequence → $P(A \cap B) = P(A) \cdot P(B)$

The two RV X and Y with Joint CDF $F_{X,Y}(x,y)$ and joint PDF $f_{X,Y}(x,y)$, are independent if :-

→ $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$

as a result : $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$F_{X,Y}(x,y) = P(X \le x , Y \le y)$

IF X and Y are independent :

$P(X \le x , Y \le y) = P(X \le x) \cdot P(Y \le y)$ (with "," marked as $\cap$)

$= F_X(x) \cdot F_Y(y)$

Example given X and Y with $f_{X,Y}(x,y) = x e^{-x(y+1)} u(x) u(y)$

Are X and Y independent ?!

$f_X(x) = e^{-x} u(x)$

$f_Y(y) = \frac{u(y)}{(1+y)^2}$

$F_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$

So, X and Y are not independent.

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Example: $f_{X,Y}(x,y) = \frac{1}{12} e^{-x/4 - y/3} u(x) u(y)$

Are x and y independent ?!

sol: $f_X(x) = \int_0^\infty \frac{1}{12} e^{-\frac{x}{4} - \frac{y}{3}} u(x) u(y)\, dy$

$= \frac{1}{12} u(x) e^{-x/4} \left[\left[\frac{e^{-y/3}}{-1/3} \right]_0^\infty \right]$

$= \frac{1}{4} u(x) e^{-x/4}$

$f_Y(y) = \int_0^\infty \frac{1}{12} e^{-\frac{x}{4} - \frac{y}{3}} u(x) u(y)\, dx$

$= \frac{1}{12} e^{-y/3} u(y) \left[\left. \frac{e^{-x/4}}{-1/4} \right|_0^\infty \right]$

$= \frac{1}{3} u(y) e^{-y/3}$

$f_X(x) \cdot f_Y(y) = \frac{1}{4} u(x) e^{-x/4} \cdot \frac{1}{3} u(y) e^{-y/3}$

$= \frac{1}{12} e^{-x/4 - y/3} u(x) u(y)$

$= f_{X,Y}(x,y)$

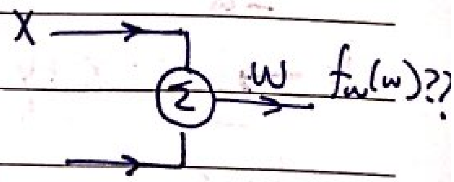
So,

X and Y are independent R.V's

* Sum of two independent R.V's :-

Suppose we have (X,Y) with joint PDF $f_{X,Y}(x,y)$, where X and Y are independent.

Let $W = X+Y$

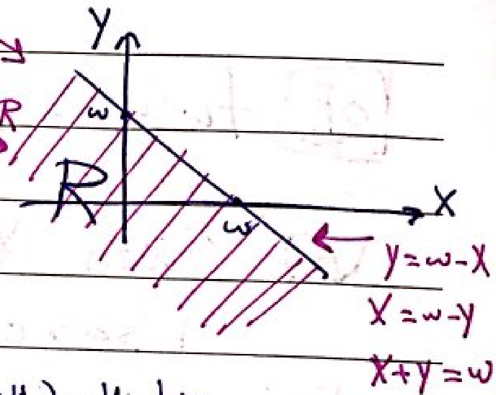


$$F_W(w) = P(W \le w) = P(X+Y \le w)$$

$$X+Y = w \rightarrow \boxed{Y = w - X}$$

$$P(X+Y \le w) = P((X,Y) \in R)$$

Region R



$$= \iint_R f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_{X,Y}(x,y)\,dx\,dy \quad \text{OR} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{w-x} f_{X,Y}(x,y)\,dy\,dx$$

Because X and Y are independent ;

$$* \quad F_W(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_X(x)\cdot f_Y(y)\,dx\,dy$$

constant ($f_Y(y)$)

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)\,dx\,dy$$

$$* \quad f_W(w) = \frac{d\,F_W(w)}{dw} = \int_{-\infty}^{\infty} f_Y(y)\,\frac{d\left[\int_{-\infty}^{w-y} f_X(x)\,dx\right]}{dw}\,dy$$

$$\longrightarrow f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\left[f_X(w-y)\,\frac{d(w-y)}{dw} - f_X(-\infty)\,\frac{d(-\infty)}{dw} + \int_{-\infty}^{x=w} \frac{d\,f_X(x)}{dw}\right]dy$$

(the last two terms are crossed out as 0)

بعوض ال upper limit و بشتقه ، و بعوض ال lower limit و بشتقه بعدين بشتق ال function الي بكامل عليها

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$$= \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy = f_Y(y) * f_X(x)$$ (→ convolution)

(*) → $\boxed{w = x + y}$ x, y independent.

$f_W(w) = f_Y(y) * f_X(x)$ (→ convolution.) $= \int f_Y(y) \cdot f_X(w-y)\, dy$

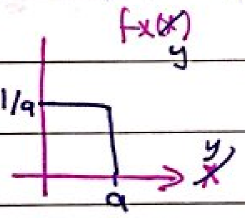
OR $f_W(w) = f_X(x) * f_Y(y) = \int f_X(x) \cdot f_Y(w-x)\, dx$

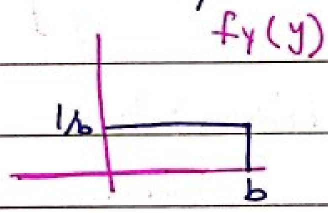
Ex Given $X \sim u(0,a)$, $Y \sim u(0,b)$

$b > a > 0$, X and Y are independent.

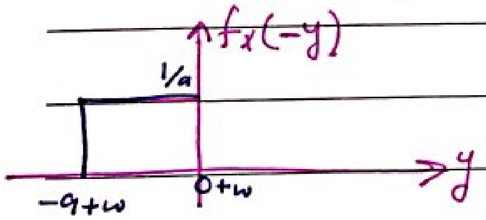
Find the pdf for $W = X + Y$

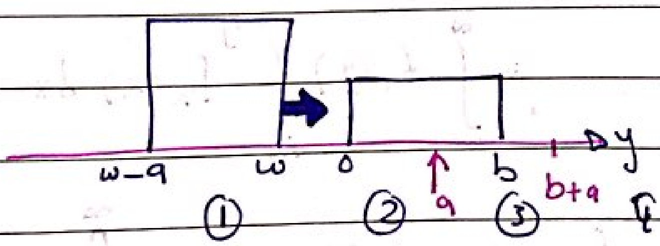
Solution :-





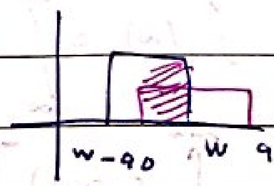
$$f_W(w) = \int f_Y(y) \cdot f_X(w-y)\, dy$$





① $w < 0$: $f_W(w) = \int (0)\, dy = \boxed{0}$

② $0 < w < a$: $f_W(w) = \int_0^w \frac{1}{ab}\, dy = \boxed{\frac{w}{ab}}$



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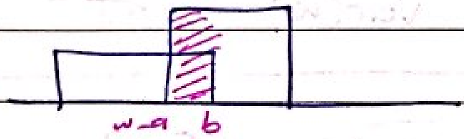
③ $a < w < b$: $f_W(w) = \int_{w-a}^{w} \frac{1}{ab}\, dy = \frac{y}{ab}\Big]_{w-a}^{w}$

$= \frac{w - w + a}{ab} = \boxed{\frac{1}{b}}$

④ $b < w < a+b$: $f_W(w) = \int_{w-a}^{b} \frac{1}{ab}\, dy = \frac{b - w + a}{ab}$

$= \boxed{\frac{(a+b) - w}{ab}}$

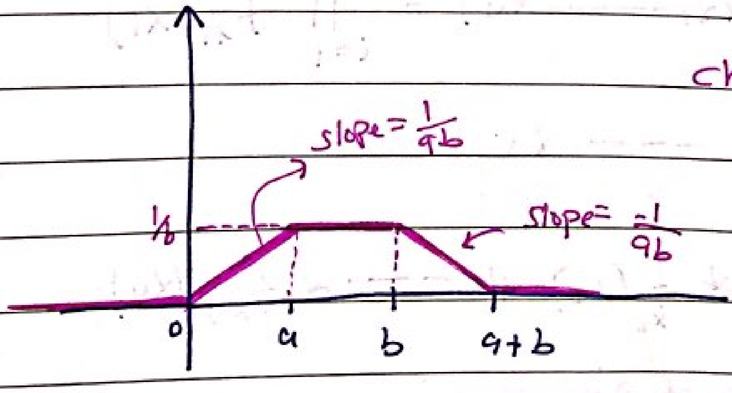


⑤ $a+b < w$: $f_W(w) = \int (0)\, dy = \boxed{0}$

so; $f_W(w) = \begin{cases} 0 & , \; w < 0 \\ \frac{w}{ab} & , \; 0 < w < a \\ \frac{1}{ab} & , \; a < w < b \\ \frac{(a+b)-w}{ab} & , \; b < w < a+b \\ 0 & , \; a+b < w \end{cases}$

~~$b < w < a+b$~~

check $\int_{-\infty}^{\infty} f_W(w)\, dw = 1$?



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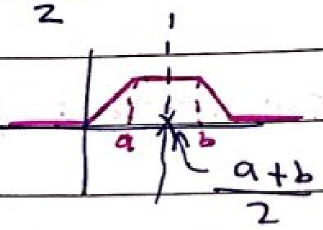
Find $\overline{w}$?!

method ① $\overline{w} = E[w] = E[x+y]$

$= E[x] + E[y] = \frac{a}{2} + \frac{b}{2} = \frac{a+b}{2}$

② $\overline{w} = \int_{-\infty}^{\infty} w f_w(w)\, dw$ ------ $= \frac{a+b}{2}$

③ $\frac{a+b}{2}$ (axis of symmetry)



Find $Var(w) = \sigma_w^2$?!

① $Var(w) = E[w^2] - \overline{w}^2$

AC part

$E[w^2] = \int w^2 f_w(w)\, dw = $ ------

OR $var(w) = Var(x+y) = var(x) + var(y)$ ← ch5

$= \frac{a^2}{12} + \frac{b^2}{12} = \frac{a^2+b^2}{12}$

In General :-

Let $X_1, X_2, ---- X_N$ are independent R.V's

i.e, $f_x(x_1, x_2, ---- x_N) = \prod_{i=1}^{N} f_{x_i}(x_i)$

$W = X_1 + X_2 + --- X_N$

$f_w(w) = f_{x_1}(x_1) * f_{x_2}(x_2) * ---- * f_{x_N}(x_N)$

↑ Exact pdf for W

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* Central Limit theorem (CLT)

If $W = X_1 + X_2 + X_3 + \dots + X_N$, where $N \to \infty$
and $X_1, X_2, \dots, X_N$ are independent,
then the density function of W can be approximated as:

$$W \sim N(a_w, \sigma_w^2)$$

where $a_w = E[w] = \overline{X_1} + \overline{X_2} + \dots \overline{X_N}$

$$\sigma_w^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_N}^2$$

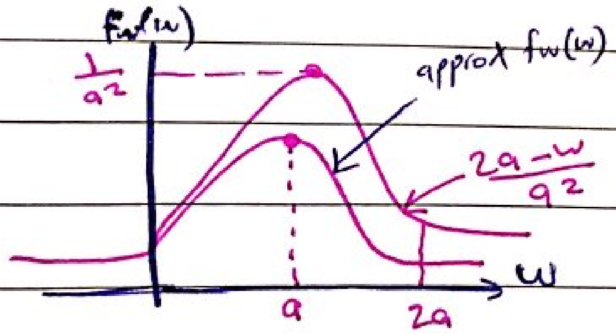
$$f_w(w) \simeq \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(w-a_w)^2}{(2\sigma_w^2)}}, \quad -\infty < w < \infty$$

Ex:- $X \sim u(0,a)$, $Y \sim u(0,a)$, let $w = X+Y$

① Find the exact pdf for w

② find the approximated pdf for w

(a) $f_w = f_x(x) * f_y(y)$



$\frac{\sqrt{3/\pi}}{a} e^{-(w-a^2)/(a^2/3)}$

(b) approximate using CLT

$W \sim N(a_w, \sigma_w^2)$

$a_w = \overline{X} + \overline{Y} = \frac{a}{2} + \frac{a}{2} = a$

$\sigma_w^2 = \sigma_x^2 + \sigma_y^2 = \frac{a^2}{12} + \frac{a^2}{12} = \frac{a^2}{6}$

$f_w(w) \approx \frac{1}{\sqrt{2\pi \frac{a^2}{6}}} e^{-\frac{(w-a)^2}{2a^2/6}}$

$= \frac{1}{\sqrt{\pi \frac{a^2}{3}}} e^{-\frac{(w-a)^2}{a^2/3}}$ $\quad -\infty < w < \infty$

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* Chapter 5 *

Operation on Multi R.V's

* Expectation of function of two R.V's

Let x,y are two joint R.V's with joint pdf $f_{x,y}(x,y)$

IF $g(x,y)$

then: $E[g(x,y)] = \int_{-\infty}^{\infty}\int g(x,y) \cdot f_{x,y}(x,y)\, dx\, dy$

IF $g(x,y) = g_1(x)$

$E[g_1(x)] = \int_{-\infty}^{\infty}\int g_1(x)\, f_{x,y}(x,y)\, dx\, dy$ (dy dx)

$= \int_{-\infty}^{\infty} g_1(x) \underbrace{\int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy}_{f_x(x)}\, dx = \int_{-\infty}^{\infty} g_1(x)\, f_x(x)\, dx$

IF $g(x,y) = g_2(y) \rightarrow E[g(x,y)] = E[g_2(y)]$

$= \int_{-\infty}^{\infty} g_2(y)\, f_y(y)\, dy$

where : $f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dx$

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[Ex] Given $f_{x,y}(x,y) = \{ x e^{-x(y+1)} , x>0 \; y>0 \}$

Find :-

[a] $E\left[\frac{e^{-2x}}{y+1}\right]$ [b] $E[y^3]$ [c] $E[2y^2-3]$

[a] $E\left[\frac{e^{-2x}}{y+1}\right] = \int_0^\infty\int_0^\infty \frac{e^{-2x}}{y+1} \cdot x e^{-x(y+1)} \, dx\, dy$ - - -

[b] $E[x^3]$ = step 1: $f_x(x) = \int_0^\infty f_{x,y}(x,y)\, dy \to E[x^3] = \int_0^\infty x^3 f_x(x)\, dx$

[c] $E[2y^2-3]$ step1: find $f_y(y) \to 2E[y^2]-3 = \int y^2 f_y(y)\, dy$

* let x_1, x_2 are joint R.V's with joint pdf $f_{x_1, x_2}(x_1,x_2)$

show that :

$$E[\alpha_1 x_1 + \alpha_2 x_2] = \alpha_1 E[x_1] + \alpha_2 E[x_2]$$
$$= \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 .$$

[proof] $E[\alpha_1 x_1 + \alpha_2 x_2] = \iint (\alpha x_1 + \alpha_2 x_2) f_{x_1,x_2}(x_1,x_2)\, dx_1 dx_2$

$$= \iint \alpha_1 x_1 f_{x_1 x_2}(x, x_2)\, dx_1 dx_2 + \iint \alpha_2 x_2 f_{x_1 x_2}(x_1,x_2)\, dx_1 dx_2$$

$dx_2 dx_1$

$$= \alpha_1 \int x_1 \left(\int f_{x_1,x_2}(x_1,x_2)\, dx_2\right) dx_1 + \alpha_2 \int x_2 \left(\int f_{x_1 x_2}(x_1,x_2)\, dx_1\right) dx_2$$

$$= \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 \; \#$$

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In General :- $x_1, x_2, \ldots x_N$ with $f(x_1, x_2 \ldots x_N)$

- $E\left[\sum_{i=1}^{Q} \alpha_i x_i\right] = \sum_{i=1}^{N} \alpha_i E[x_i]$
- $E\left[\sum_{i=1}^{N} g_i(x_i)\right] = \sum_{i=1}^{N} E[g(x_i)]$

Ex $E\left[x_1^2 + \cos(x_2) + \frac{1}{x^3}\right] = E[x_1^2] + [\cos(x_2)] + E[1/x^3]$

Ex Given

	$E[\cdot]$	$\sigma^2(\cdot)$
x_1	-2	2
x_2	1/4	1
x_3	0	5

IF $X = \frac{-x_1}{4} + 3x_2^2 + \frac{1}{2}x_3^2 - 4$

find: $E[X]$

$E[X] = \left[\frac{-x_1}{4} + 3x_2^2 + \frac{1}{2}x_3^2 - 4\right]$

$= \frac{-1}{4}\bar{x}_1 + 3E[x_2^2] + \frac{1}{2}E[x_3^2] - 4$

$= (\frac{-1}{4})(-2) + 3(1 + (1/4)^2) + \frac{1}{2}(5 + 0^2)$

(*) Joint Moments :-

- $m_{nk} = E[x^n y^k]$, $n = 0, 1, \ldots$, $k = 0, 1, \ldots$

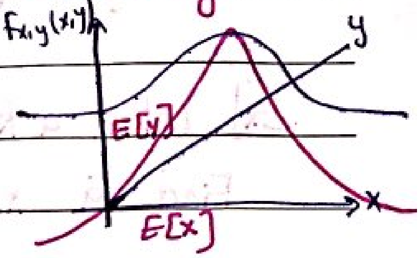
$= \iint_{-\infty}^{+\infty} x^n y^k f_{x,y}(x,y)\, dx\, dy$

* $n+k$ order

- Zeroth order : $m_{00} = E[x^0 y^0] = 1$

1st order: $m_{10} = E[x^1 y^0] = E[x]$ } center of Gravity.

$m_{01} = E[x^0 y^1] = E[Y]$



2nd order: $m_{20} : E[x^2]$

$m_{02} : E[Y^2]$

$m_{11} : E[xy] = R_{xy}$

* $R_{xy} = E[xy]$, the correlation

If $R_{xy} = 0 \rightarrow$ x and Y are orthogonal.

↳ If $R_{xy} = E[xy] = E[x]E[y]$ (x and Y are uncorrelated)

↳ generated from different Sources.

If x&y → Independent: (must be uncorrelated)

→ uncorrelated: (dependent or independent) & يوجد شبه بين x,y

proof

Independent: $f_{x,y}(x,y) = f_x(x) \cdot f_y(y)$

$$R_{xy} = E[xy] = \iint_{-\infty}^{\infty} x\, y\, f_{x,y}(x,y)\, dx\, dy$$

$$= \int\left(\int x \cdot y\, f_x(x) \cdot f_y\, dx\right) dy = \int y\, f_y \int x f_x(x)\, dx$$

$= \bar{x}\bar{y}$ → uncorrelated

If Independent $\xrightarrow{\text{always}}$ uncorrelated

$\not\leftarrow$ valid only for gaussian R.V's

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* $E[g(x) \cdot g(y)]$ if $g(x), (y)$ are Independent
→ $\overline{g(x)} \cdot \overline{g(y)}$

Ex Let x a R.V with $\bar{x} = 3$, $\sigma_x^2 = \sqrt{2}$, If $y = -6x + 22$
Find (a) R_{xy} (b) Are x, y uncorrelated ?!

(a) $R_{xy} = E[xy] = E[x(-6x+22)] = E[-6x^2 + 22x]$
$= -6\,E[x^2] + 22\,E[x]$
$= -6(2+9) + (22)(3) = 0$

(b) Check $E[xy] = \bar{x}\bar{y}$
RHS = 0
LHS = $(3)((-6 \cdot 3) + 22) = 12$
RHS ≠ LHS → (x, y are not uncorrelated)

⊛ Joint central moment :-

$M_{nm} = E[(x-\bar{x})^n, (y-\bar{y})^m]$ $\quad n = 0, 1, \ldots$ $\quad m = 0, 1, \ldots$

$$= \iint_{-\infty}^{\infty} (x-\bar{x})^n (y-\bar{y})^m \cdot f_{xy}(x,y)\, dx\, dy$$

order $n+m$.

1st order → $M_{00} = E[x^0 y^0] = 1$ $\quad M_{01} = E[(y-\bar{y})] = 0$
$M_{10} = E[(x-\bar{x})] = 0$

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2nd order :- $M_{20} = E[(x-\bar{x})^2] = \sigma_x^2 = var(x)$

$M_{02} = E[(y-\bar{y})^2] = \sigma_y^2 = var(y)$

$M_{11} = E[(x-\bar{x})(y-\bar{y})]$ → Covariance C_{xy}

Note: $C_{xx} = \sigma_x^2 = var(x)$

$C_{yy} = \sigma_y^2 = var(y)$

→ Covariance between R.V and itself is the R.V variance.

$C_{x(-x)} = E[(x-\bar{x})(-x-(-\bar{x}))]$

$= E[(x-\bar{x})(x-\bar{x})] = -\sigma_x^2$

$C_{xy} = E[(x-\bar{x})(y-\bar{y})]$

$= E[xy - \bar{y}x - \bar{x}y + \bar{x}\bar{y}]$.

$= E[xy] - \bar{y}E[x] - \bar{x}E[y] + \bar{x}\bar{y}$ (last two terms crossed out)

$= E[xy] - \bar{y}E[x]$

→ $C_{xy} = R_{xy} - \bar{x}\bar{y}$

If x and y are orthogonal → $C_{xy} = -\bar{x}\bar{y}$

If x and y are uncorrelated → $C_{xy} = 0$

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* Correlation parameter :-

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} \quad , \quad -1 \leq \rho_{xy} \leq 1$$

ρ_{xy}
$-1 \quad 1$

If x and y are uncorrelated $\rightarrow C_{xy} = 0 \rightarrow \rho_{xy} = 0$

If $x = y \rightarrow C_{xy} = \sigma_x^2 \rightarrow \rho_{xy} = \frac{\sigma_x^2}{\sigma^2_x} = 1$

If $x = -y \rightarrow$ ------ $\Rightarrow \rho_{xy} = -1$

$$0 \leq |\rho_{xy}| \leq 1$$

$\boxed{\text{Example}}$ Find the variance of $x = \alpha_1 x_1 + \alpha_2 x_2$, where x_1, x_2 constants

$var(x) = E[(x-\bar{x})^2]$

$x - \bar{x} = (\alpha_1 x_1 + \alpha_2 x_2) - (\alpha_1 \bar{x} + \alpha_2 \bar{x}_2)$

$= \alpha_1 (x_1 - \bar{x}_1) + \alpha_2 (x_2 - \bar{x}_2)$

$Var(x) = E[(\alpha_1(x_1 - \bar{x}_1) - \alpha_2(x_2 - \bar{x}_2))^2]$

$= E[\alpha_1^2 (x_1 - \bar{x}_1)^2 + \alpha_2^2 (x - \bar{x}_2)^2 + 2\alpha_1\alpha_2 (x_1 - \bar{x})(x_2 - \bar{x}_2)]$

$= \alpha_1^2 \sigma_{x_1}^2 + \alpha_2^2 \sigma_{x_2}^2 + 2\alpha_1\alpha_2 C_{x_1 x_2}$

$Var(\alpha_1 x_1 + \alpha_2 x_2)$

$= \alpha_1^2 \sigma_{x_1}^2 + \alpha_1 \alpha_2 C_{x_1 x_2} + \alpha_2^2 \sigma_{x_2}^2 + \alpha_2 \alpha_1 C_{x_2 x_1}$

$2\alpha_1\alpha_2 C_{x_1 x_2}$

$\boxed{\text{In General :-}}$

$$Var\left(\sum_{i=1}^{N} \alpha_i x_i\right) = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j C_{x_i x_j}$$

If $x_1, x_2 \text{---} x_N$ are independent

$\Rightarrow$ zero

* $C_{xy} = E[(x-\bar{x})(y-\bar{y})] = R_{xy} - \bar{x}\bar{y}$ → Covariance

* $\rho_{xy} = \dfrac{C_{xy}}{\sigma_x \sigma_y}$ → correlation parameter (coeffecient)

* IF $R_{xy} = \bar{x}\bar{y}$ "x and y are uncorrelated"

  OR

  $C_{xy} = 0$ OR $\rho_{xy} = 0$

* IF x and y are mutually independent → uncorrelated (uncorrelated ↛ independent)

  $f_{x,y}(x,y) = f_x(x) \cdot f_y(y)$

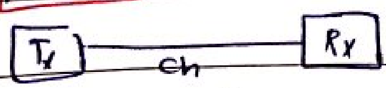
  $\rho_{xy} = 0$, $C_{xy} = 0$

* Not Independent → Uncorrelated OR correlated.

* Uncorrelated → Independent OR Not Independent.

* Correlated → Not Independent.

AWGN

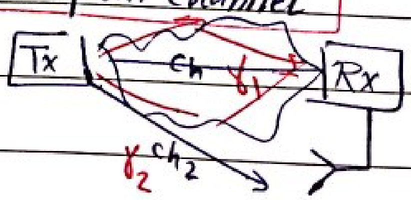


$y = \tilde{x} + n$

$\gamma = \dfrac{P(\tilde{x})}{N_0}$

↳ Not R.V

multipath channel



→ $\gamma = \dfrac{h\, P(\tilde{x})}{N_0}$

↳ R.V

* Joint Gaussian Random Variables.

The two Gaussian R.V's $X_1 \sim N(\bar{X}_1, \sigma_{X_1}^2)$, $X_2 \sim N(\bar{X}_2, \sigma_{X_2}^2)$ are said to be joint Gaussian if their joint pdf is given by:

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_{X_1}^2} - \frac{2\rho(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_{X_1}\sigma_{X_2}} + \frac{(x_2-\bar{x}_2)^2}{\sigma_{X_2}^2}\right]}$$

$$\rho = \frac{C_{X_1X_2}}{\sigma_{X_1}\sigma_{X_2}}$$

* as a special case :- If x_1 and x_2 are uncorrelated, then

$\rho = 0$

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}} e^{-\frac{1}{2}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_{X_1}^2} + \frac{(x_2-\bar{x}_2)^2}{\sigma_{X_2}^2}\right]}$$

$$= \frac{1}{\sqrt{2\pi\sigma_{X_1}^2}} e^{-\frac{1}{2}\frac{(x_1-\bar{x}_1)^2}{\sigma_{X_1}^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_{X_2}^2}} e^{-\frac{1}{2}\frac{(x_2-\bar{x}_2)^2}{\sigma_{X_2}^2}}$$

$= f_{X_1}(x_1) \cdot f_{X_2}(x_2) \longrightarrow x_1$ and x_2 are Independent

Independent $\longrightarrow$ uncorrelated

$\longleftarrow$ ($\rho = 0$, $C_{XY} = 0$)

for joint Gaussian R.V's

⊛ N-Joint Gaussian R.V's :-

$x_1, x_2, \ldots, x_N$ are said to be joint Gaussian If their joint (pdf) is given by :

$$f_{x_1,x_2,\ldots x_N}(x_1, x_2, \ldots x_N) = \frac{|[C_x]^{-1}|^{1/2}}{(2\pi)^{N/2}} \cdot e^{-\frac{[x-\bar{x}]^T [C_x]^{-1} [x-\bar{x}]}{2}}$$

where :-

$[C_x] \rightarrow$ covariance matrix $(N \times N) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$

the variances (the diagonal); symmetry about the Diagonal.

⊛ $C_{11} = C_{x_1 x_1} = \sigma_{x_1}^2$

⊛ $[x-\bar{x}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N - \bar{x}_N \end{bmatrix}$ (OR) $\rightarrow [x] - [\bar{x}] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} - \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_N \end{bmatrix}$

$[x-\bar{x}]^T = [x_1 - \bar{x}_1, \ x_2 - \bar{x}_2, \ x_3 - \bar{x}_3 \ \cdots \ x_N - \bar{x}_N]$ $(1 \times N)$

$(1 \times N)(N \times N)(N \times 1) = (1 \times 1) \rightarrow$ only one element

For two R.V's x_1, x_2 Find $f_{x_1,x_2}(x_1, x_2)$

$$[C_x] = \begin{bmatrix} C_{x_1x_1} & C_{x_1x_2} \\ C_{x_2x_1} & C_{x_2x_2} \end{bmatrix} = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1}\sigma_{x_2}\rho \\ \sigma_{x_1}\sigma_{x_2}\rho & \sigma_{x_2}^2 \end{bmatrix}$$

$$\rho_{x_1x_2} = \frac{C_{x_1x_2}}{\sigma_{x_1}\sigma_{x_2}}$$

$$C[x]^{-1} = \frac{1}{\sigma_{x_1}^2\sigma_{x_2}^2(1-\rho^2)} \begin{bmatrix} \sigma_{x_2}^2 & -\sigma_{x_1}\sigma_{x_2}\rho \\ -\sigma_{x_1}\sigma_{x_2}\rho & \sigma_{x_1}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sigma_{x_1}^2(1-\rho^2)} & \frac{-\rho}{\sigma_{x_1}\sigma_{x_2}(1-\rho^2)} \\ \frac{-\rho}{\sigma_{x_1}\sigma_{x_2}(1-\rho^2)} & \frac{1}{\sigma_{x_2}^2(1-\rho^2)} \end{bmatrix}$$

$$|[C_x]^{-1}| = \frac{1}{\sigma_{x_1}^2\sigma_{x_2}^2(1-\rho^2)^2} - \frac{\rho^2}{\sigma_{x_1}^2\sigma_{x_2}^2(1-\rho^2)^2} = \frac{1}{\sigma_{x_1}^2\sigma_{x_2}^2(1-\rho^2)}$$

$$f_{x_1,x_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2}} \cdot e^{- - - - - - -}$$

$* \; [X - \bar{X}]^T [C_x]^{-1} [X - \bar{X}]$

$$= \begin{bmatrix} x_1 - \bar{x}_1 & x_2 - \bar{x}_2 \end{bmatrix} \overset{\text{ضرب}}{*} \quad \overset{\text{ضرب}}{*} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}$$

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as a special case: If $x_1, x_2, \dots x_N$ are uncorrelated to each other:-

$$[C_X] = \begin{bmatrix} \sigma_{x_1}^2 & 0 & 0 \\ 0 & \sigma_{x_2}^2 & 0 \\ 0 & 0 & \sigma_{x_N}^2 \end{bmatrix}$$

$$f_{x_1 x_2 \dots x_N}(x_1, x_2, \dots x_N) = f_{x_1}(x_1) \cdot f_{x_2}(x_2) \cdot \dots f_{x_N}(x_N)$$

$$= \prod_{i=1}^{N} f_{x_i}(x_i) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi \sigma_{x_i}^2}} e^{-\frac{(x_i - \bar{x}_i)^2}{2\sigma_{x_i}^2}}$$

$$= \frac{1}{(2\pi)^{N/2} \prod_{i=1}^{N} \sigma_{x_i}} e^{-\sum_{i=1}^{N} \frac{(x_i - \bar{x}_i)^2}{2\sigma_{x_i}^2}}$$

$\hookrightarrow \sigma_{x_1} \sigma_{x_2} \dots \sigma_{x_N}$

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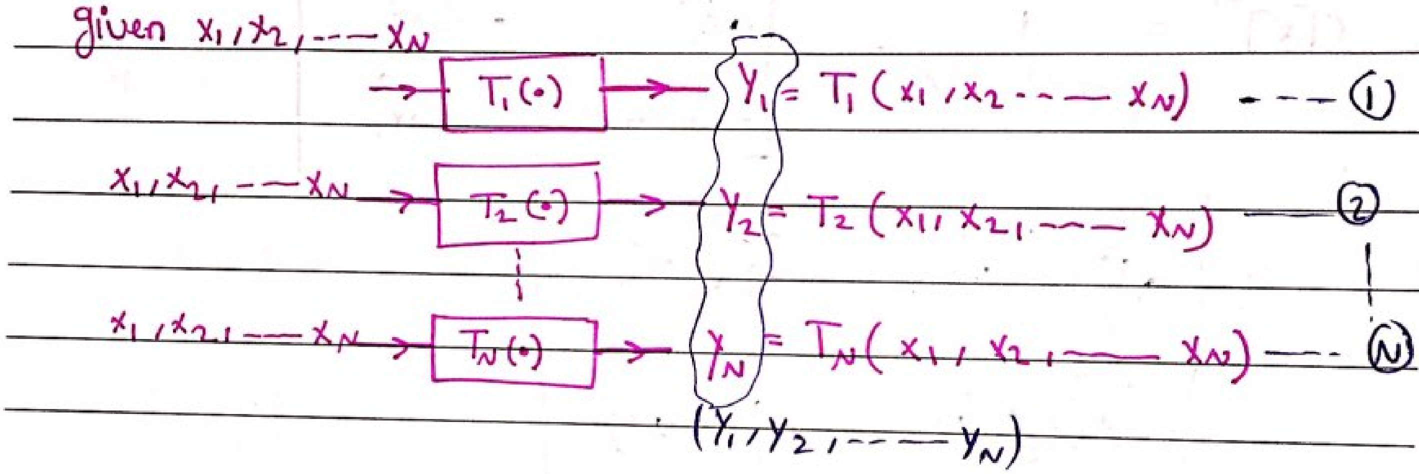
* Transformation of Multiple R.V's.

- Multiple functions :

suppose we have N R.V's $X_1, X_2, \dots, X_N$

defined on set $A = \{ (x_1, x_2, \dots x_N) : f_{X_1, X_2, \dots X_N}(x_1, x_2, \dots x_N) > 0 \}$

given $x_1, x_2, \dots x_N$



* Conditions on Ti's :-

[1] All T_i's are single-valued.

[2] All T_i's are continuous functions.

[3] All T_i's have partial derivatives everywhere.

[4] All T_i's define one-to-one transformation.

→ The solution for the equations (1), (2), --- (N) for $X_1, X_2, \dots X_N$ exist:

$$X_1 = V_1(Y_1, Y_2, \dots Y_N)$$

$$X_2 = V_2(Y_1, Y_2, \dots Y_N)$$

$$\vdots$$

$$X_N = V_N(Y_1, Y_2, \dots Y_N)$$

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$$f_{Y_1,Y_2,\dots,Y_N}(y_1,y_2,\dots,y_N) = f_{X_1,X_2,\dots,X_N}\big(v_1(y_1,y_2,\dots,y_N),\ v_2(y_1,y_2,\dots,y_N)\ \dots\ v_N(y_1,\dots,y_N)\big) \cdot |J|$$

$|J|$ ← absolute value

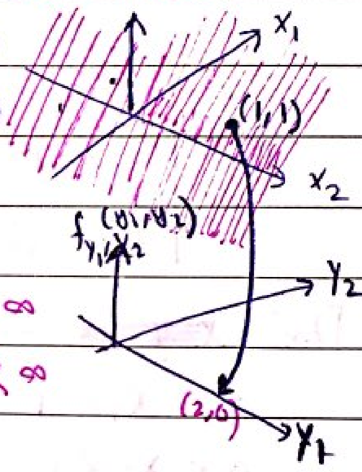
$$J = \left|\begin{bmatrix} \frac{dv_1}{dy_1} & \frac{dv_1}{dy_2} & \cdots & \frac{dv_1}{dy_N} \\ \vdots & & & \\ \frac{dv_N}{dy_N} & \frac{dv_N}{dy_2} & \cdots & \frac{dv_N}{dy_N} \end{bmatrix}\right| \leftarrow \text{determinant}$$

Ex Let x_1 and x_2 are Independent joint R.V.s with

$$f_{x_1,x_2}(x_1,x_2) = \frac{1}{2\pi} e^{-x_1^2/2} \cdot e^{-x_2^2/2} \qquad -\infty < x_1 < \infty, \quad -\infty < x_2 < \infty$$

Let $Y_1 = X_1 + X_2 = T_1(x_1,x_2)$ --(1)

$Y_2 = X_1 - X_2 = T_2(x_1,x_2)$ --(2)



Solve (1) and (2) for x_1, x_2 :-

$-\infty < Y_1 < \infty$

$-\infty < Y_2 < \infty$

$x_1 + x_2 = Y_1$ — (1)

$x_1 - x_2 = Y_2$ — (2)

فبالكزنر

$2x_1 = Y_1 + Y_2 \longrightarrow \boxed{x_1 = \frac{Y_1 + Y_2}{2}} = V_1(Y_1, Y_2)$

$x_2 = x_1 - Y_2 = \frac{Y_1 + Y_2}{2} - Y_2 \longrightarrow \boxed{x_2 = \frac{Y_1 - Y_2}{2}} = V_2(Y_1, Y_2)$

$$J = \left|\begin{bmatrix} \frac{dv_1}{dy_1} & \frac{dv_1}{dy_2} \\ \frac{dv_2}{dy_1} & \frac{dv_2}{dy_2} \end{bmatrix}\right| = \left|\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\right| = \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) - \left(\tfrac{1}{4}\right) = \boxed{\frac{-1}{2}}$$

⟶

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$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right) \cdot \frac{1}{2} \quad (|J|)$$

$$= \frac{1}{4\pi} e^{-\frac{1}{2}\left(\frac{y_1+y_2}{2}\right)^2} \cdot e^{-\frac{1}{2}\left(\frac{y_1-y_2}{2}\right)^2}, \quad -\infty < y_1 < \infty, \; -\infty < y_2 < \infty$$

Ex $Y_1 = aX_1 + bX_2 \rightarrow (1)$, $ad - bc \neq 0 \rightarrow$ حتى أضمن انه $|J| \neq \infty$

$Y_2 = cX_1 + dX_2 \rightarrow (2)$

$$X_1 = V_1(Y_1, Y_2) = \frac{dY_1 - bY_2}{ad-bc}$$

$$X_2 = V_2(Y_1, Y_2) = \frac{aY_2 - cY_1}{ad-bc}$$

$$J = \left|\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}\right| = \frac{1}{ad-bc}$$

Ex two R.v's X_1, X_2 with $f_{X_1,X_2}(x_1,x_2)$ ---

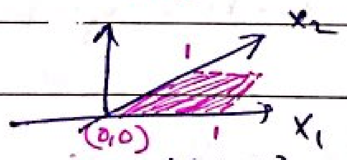
$0 < X_1 < 1$

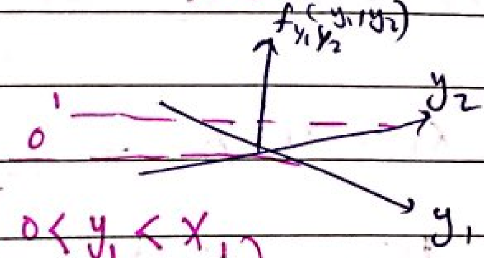
$0 < X_2 < 1$

let $Y_1 = X_1 X_2$

$Y_2 = X_1$

Find J ?!

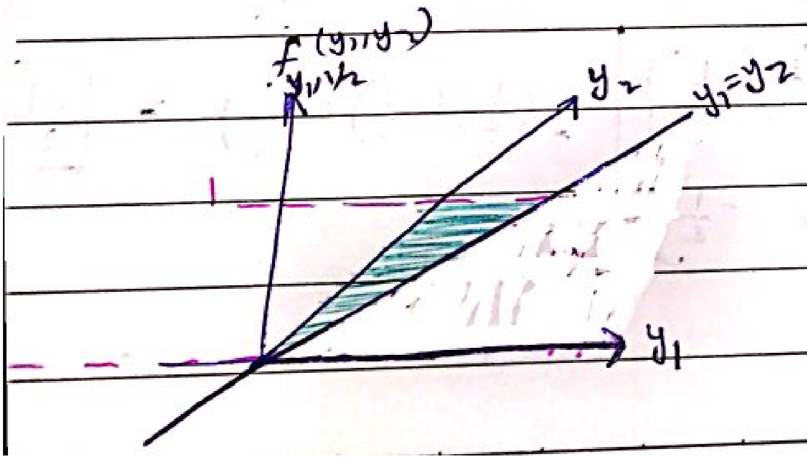




$0 < y_1 < x_1$

$0 < y_1 < y_2$ $\quad$ $0 < y_2 < 1$

$0 < y_1 < y_2 < 1$



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$$x_1 = v_1(y_1, y_2) = y_2$$
$$x_2 = v_2(y_1, y_2) = y_1 / y_2$$

$$|J| = \left| \begin{bmatrix} 0 & 1 \\ \frac{y_1}{y_2} & \frac{-y_1}{y_2} \end{bmatrix} \right| = \left| -\frac{1}{y_2} \right|$$

* Linear Transformation of gaussian R.V's

let $x_1, x_2, ---- x_N$ are Joint gaussian R.V's with joint pdf :

$$f(x_1, x_2, --- x_N) = \frac{|[C_X]^{-1}|^{1/2}}{2\pi^{N/2}} e^{-\frac{[x-\bar{x}]^T [C_X]^{-1} [x-\bar{x}]}{2}}$$

linear transformation :

$$Y_1 = a_{11} X_1 + a_{12} X_2 + a_{13} X_3 + --- a_{1N} X_N$$
$$Y_2 = a_{21} X_1 + a_{22} X_2 + ----- a_{2N} X_N$$
$$\vdots$$
$$Y_N = a_{N1} X_1 + a_{N2} X_2 + ---- a_{NN} X_N$$

a_{ij} are real numbers.

The joint pdf of $Y_1, Y_2, --- Y_N$ is given by :

$$f_{Y_1, Y_2 - Y_N}(y_1, y_2, --- y_N) = \frac{|[C_Y]^{-1}|^{1/2}}{(2\pi)^{N/2}} e^{-\frac{[y-\bar{y}]^T [C_Y]^{-1} [y-\bar{y}]}{2}}$$

$$[C_Y] = [T][C_X][T]^T$$ → transpose

where $[T] = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1N} \\ a_{21} & a_{22} & -- & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & --- & a_{NN} \end{bmatrix}$ ← should be invertable

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~~$[\bar{Y}] = \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \\ \vdots \\ \bar{Y}_N \end{bmatrix}$~~

$$[Y-\bar{Y}] = \begin{bmatrix} y_1 - \bar{Y}_1 \\ y_2 - \bar{Y}_2 \\ \vdots \\ y_N - \bar{Y}_N \end{bmatrix} \qquad \bar{Y}_1 = \sum_{i=1}^{N} a_{1i}\, E[X_i]$$

In General :-

$$\bar{Y}_j = \sum_{i=1}^{N} a_{ji}\, E[X_i] \quad , \quad j = 1, 2, \ldots, N$$

Ex: Two gaussian R.V's X_1 and X_2 have zero means and variances $\sigma_{x_1}^2 = 4$ and $\sigma_{x_2}^2 = 9$

Their covariance $C_{x_1 x_2} = 3$

(*) determine the joint density function of the new R.V's :

$Y_1 = X_1 - 2X_2$.

$Y_2 = 3X_1 - 4X_2$. $\qquad f_{Y_1,Y_2}(y_1, y_2) = ?!$

(*) Find $[C_Y]$

Sol:- $[C_Y] = [T][C_X][T]^T$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 256 \end{bmatrix}$$

(Variances: 4, 9 on the diagonal)

$$[\bar{Y}] = \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} = \begin{bmatrix} \bar{X}_1 - 2\bar{X}_2 \\ 3\bar{X}_1 + 4\bar{X}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$$

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$$\rho = \frac{C_{Y_1,Y_2}}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{-66}{\sqrt{28}\sqrt{256}} = ---$$

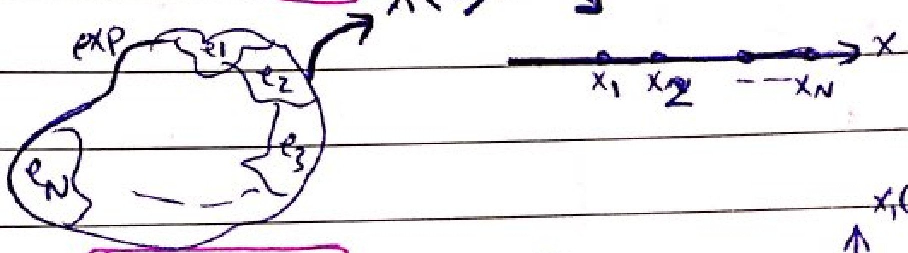
$Y_1 \sim N(0,28)$

$Y_2 \sim N(0,256$

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho^2}}$$

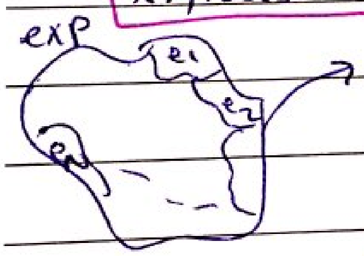
* Chapter 6:- Random (stochastic) Processes (R.P)

Recall: R.V X(s)

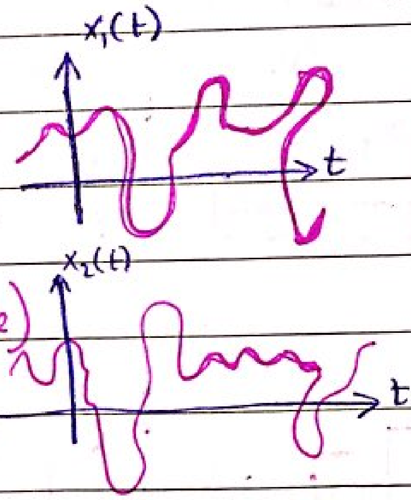


R. process:

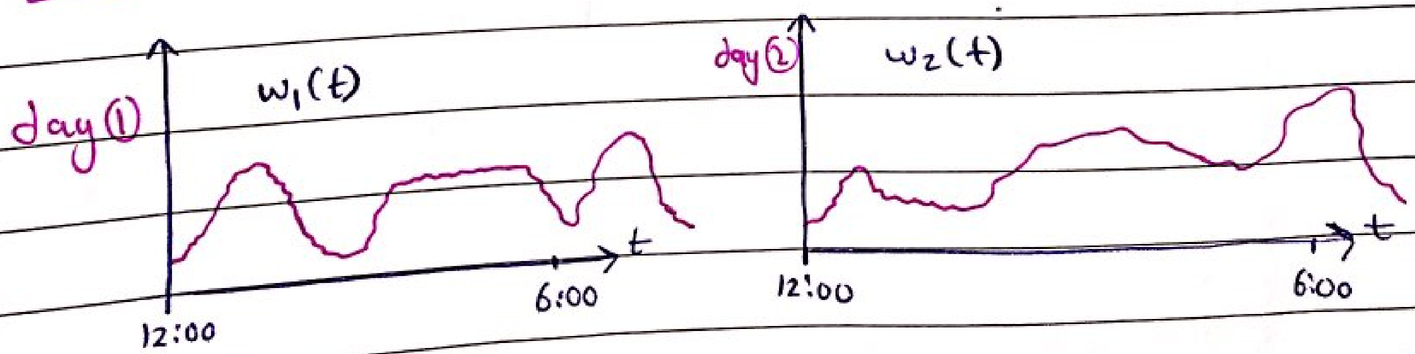
$X(s,t) = x(t)$ (time)



family (or ensemble) of time-waveforms.



Ex Record wind-speed from 12:00 am ⟶ 6:00 am



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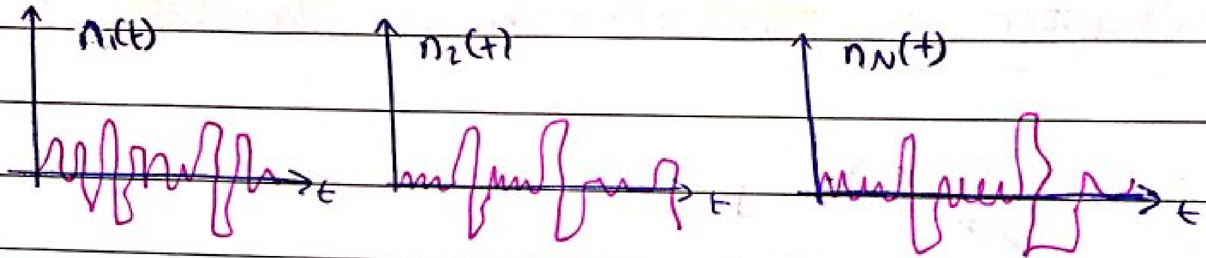
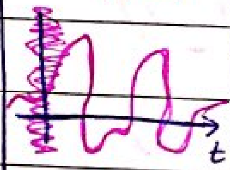
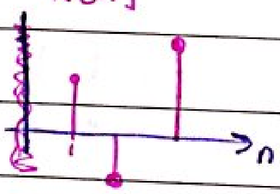
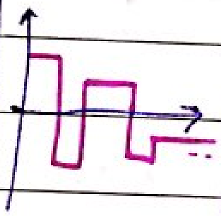
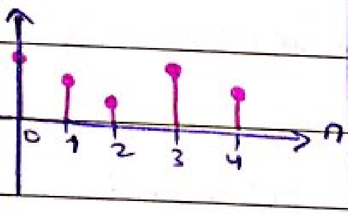
Ex White Noise Process :- $N(t)$



* R.P classification :-

Amplitude \ time	Continuous	Discrete
Continuous	$X(t)$ 	$X[n]$ 
Discrete		

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Random Process

Deterministic

can be discribed mathematically

e.g.: $X(t) = A\cos(w_0 t + \theta)$

at least one of the parameters (A, w_0, θ) should be R.V

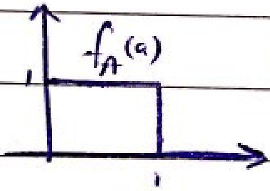
Undeterministic

cannot be described mathematically, future values cannot be determined from the past value.

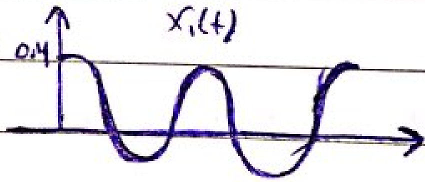
e.g :- noise process

Ex $X(t) = A\cos(w_0 t + \theta)$, $A \sim U(0,1)$

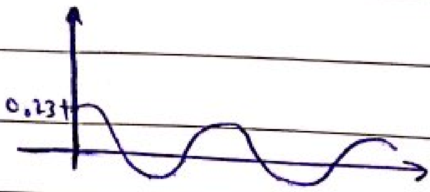
w_0 and θ are constants.



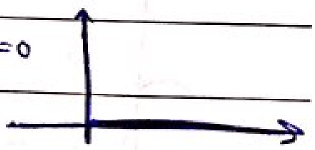
A=0.4 →



A = 0.231



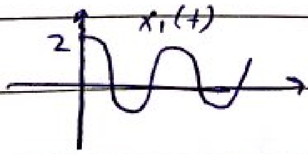
A=0



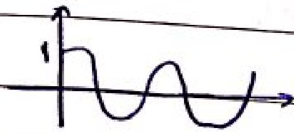
Let $A \sim B(0.1, 3)$

$A = \{0,1,2,3\}$ # of successes

A=2



A=1



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* R. process

family (ensemble) of different time -waveforms ← called sample function or realization

R.P → Continuous time $x(t)$
↳ Discrete time $x[n]$

* R.process at specific time $t = t_i$

$x(t_i) = \{ x_1(t_i), x_2(t_i), ----- x_N(t_i) \}$
↳ is R. variables.

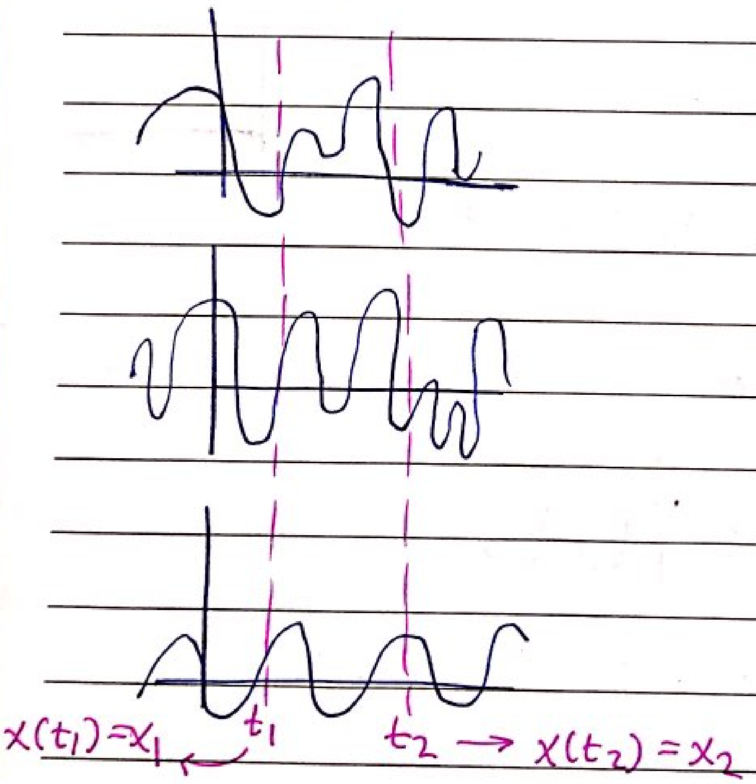
which has statistical properties : pdf, cdf, mean, variance, ----

→ first order distribution.

$x(t_i) = x_i \sim f_x(x_i, t_i)$

$f(x_i, t_i)$

* R.P density function : $x(t) \sim f_x(x, t)$



2^{nd} order distribution:
$(x_1, x_2) \sim f(x_1, x_2, t_1, t_2)$

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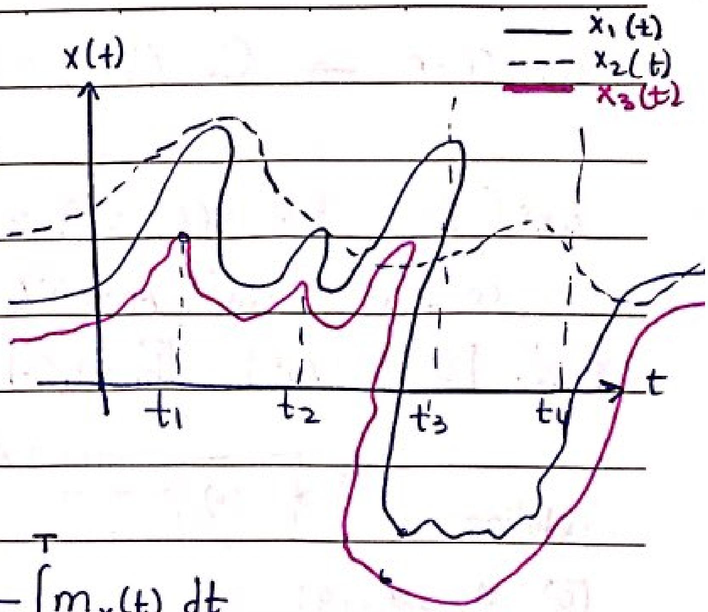
* R.P mean :-

$E[X(t)] = m_x(t)$, R.P mean is in general function of time.



$$m_x(t) = \int x \cdot f_x(x;t)\, dx$$

R.P DC value :

$$DC = A[m_x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} m_x(t)\, dt$$

* R.P variance : $V(x(t)) = \sigma_x^2(t) = E[x^2(t)] - m_x^2(t)$

↳ In general varies with time

* R.P average AC Power $= A[\sigma_x^2(t)]$

* R.P auto-correlation function :

$R_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)]$

$R_{xx}(t, \tau + t) = E[x(t)\, x(t+\tau)]$

$R_{xx}(t, \tau+t) = \iint x_1 x_2 f_{x_1 x_2}(x_1, x_2, t, t+\tau)\, dx_1 dx_2$

For $\tau = 0 \longrightarrow R_{xx}(t,t) = E[x^2(t)]$

Total average power $= A[E[x^2(t)]]$

* R.P auto-covariance function.

$C_{xx}(t, t+\tau) = E[x(t) - m_x(t)][X(t+\tau) - m_x(t+\tau)]$

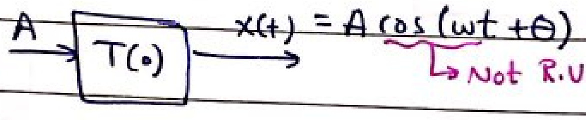
$$C_{xx}(t, t+\tau) = R_{xx}(t, t+\tau) - m_x(t)\, m_x(t+\tau)$$ ⊗

For $\tau = 0 \rightarrow C_{xx}(t,t) = E[x^2(t)] - m_x^2(t) = \text{variance} = \sigma_x^2(t)$

Ex: Given R.P $x(t) = A\cos(\omega_o t + \theta)$, where ω_o and θ are constants and $A \sim N(2,9)$

Find: [a] $f_x(x,t)$ [b] $m_x(t)$ [c] $\sigma_x^2(t)$.

Solution:



[a] $A \sim N(2,9)$

$x(t) \sim N(2\cos(\omega_o t + \theta),\ 9\cos^2(\omega_o t + \theta))$

$$f_x(x,t) = \frac{1}{\sqrt{2\pi(9\cos^2(\omega t + \theta))}} e^{-\frac{(x - 2\cos(\omega t + \theta))^2}{2(9\cos^2(\omega t + \theta))}}$$

[b] $m_x(t) = E[x(t)] = \int x f_x(x,t)\, dx = ----$

OR

$= 2\cos(\omega t + \theta)$

[c] $\sigma_x^2(t) = \text{var}(x(t)) = 9\cos^2(\omega t + \theta)$

[or] $m_x(t) = E[x(t)] = E[A\cos(\omega t + \theta)]$

$= \cos(\omega_o t + \theta) E[A]$ → distribution ⟶

$= 2\cos(\omega_o t + \theta)$

$\sigma_x^2(t) = E[x^2(t)] - m_x^2(t)$

$E[x^2(t)] = E[A^2\cos^2(\omega_o t + \theta)] = 13\cos^2(\omega_o t + \theta)$

$\sigma_x^2(t) = 13\cos^2(\omega_o t + \theta) - 4\cos^2(\omega_o t + \theta)$

$= 9\cos^2(\omega_o t + \theta)$

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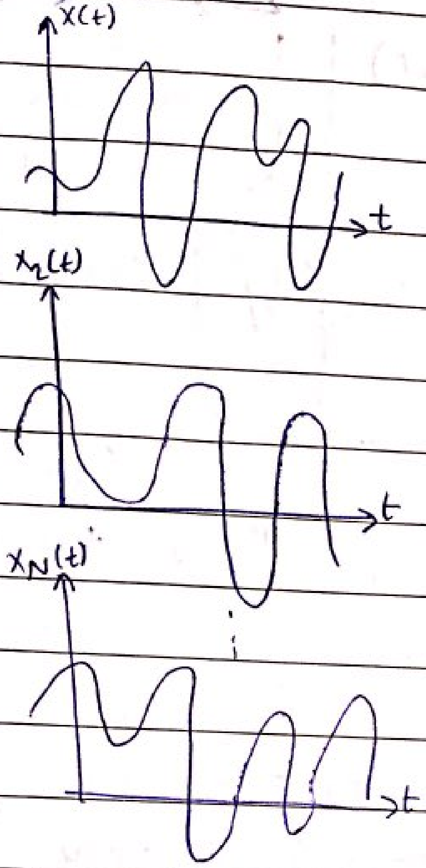
* Ro Process :-



$$x(t_i) = x_i$$

$$\hookrightarrow \bar{x}_i \, , \, \sigma_i^2 \, , \, f_x(x,t)$$

$$x(t) \sim f_x(x,t)$$

$$m_x(t) = E[x(t)]$$

$$= \int x \, f_x(x,t) \, dx$$

$$R_{xx}(t_1, t_1+\tau) = E[x(t) \, x(t+\tau)]$$

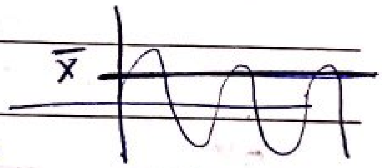
* Stationrity :- In General a R.P $X(t)$ is to be stationrity If it does not change its statistical properties with time.

(*) First order stationrity :-

$$f_x(x, t_i) = f_x(x, t_j) \quad \text{for all } t_i \text{ and } t_j$$

i.e : $f_x(x, t) = f_x(x)$

As a result: $m_x(t) = \int_{-\infty}^{\infty} x \, f_x(x) \, dx$



$m_x(t) = \bar{x} \rightarrow$ It's not a function of time (constant)

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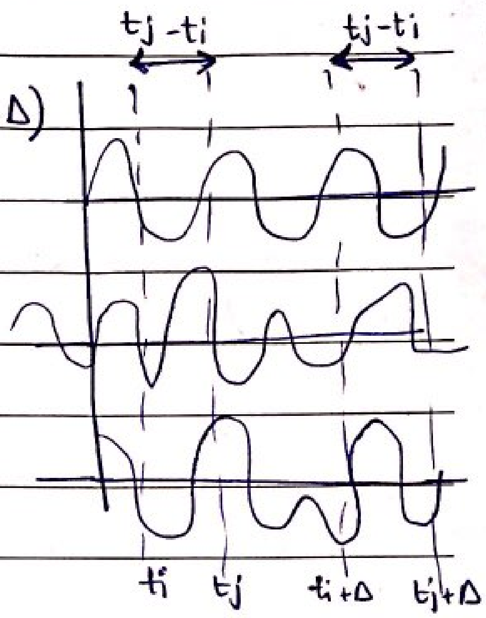
* second order stationarity :-

$f_x(x_1, x_2, t_i, t_j) = f_x(x_1, x_2, t_i+\Delta, t_j+\Delta)$

for any t_i, t_j and Δ

$\rightarrow f_x(x_1, x_2, t_i, t_j) = f_x(x_1, x_2, t_j - t_i)$



* As a result :-

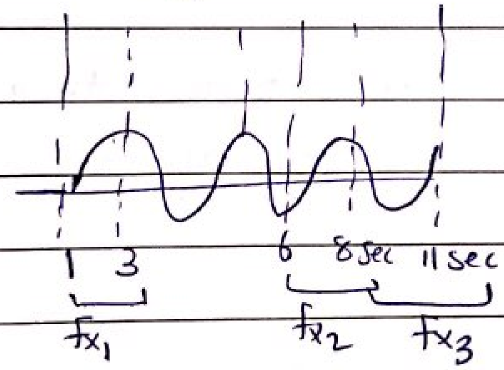
R.P auto correlation function

$R_{xx}(t, t+\tau) = E[x(t), x(t+\tau)]$

$= \iint_{-\infty}^{\infty} x_1 x_2 \; f_x(x_1, x_2, t, t+\tau) \, dx_1 \, dx_2$

$= \iint_{-\infty}^{\infty} x_1 x_2 \; f_x(x_1, x_2, \tau) \, dx_1 \, dx_2 \quad = R_x(\tau)$

function of τ only



$f_{x_1} = f_{x_2} \neq f_{x_3}$

eg: $R_x(2\,sec, 6\,sec) = R_x(12\,sec, 16\,sec)$

$\tau = 4$ $\quad$ $\tau = 4$

2^{nd} order stationarity $\longrightarrow$ 1^{st} order stationarity

2^{nd} order stationarity $\not\longleftarrow$ 1^{st} order stationarity

↳ $R_{xx}(t, \tau+t) = R_{xx}(\tau)$

$m_x(t) = \bar{x}$

↳ $\mu_x(t) = \bar{x}$

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* Wide-Sense-Stationarity (WSS)

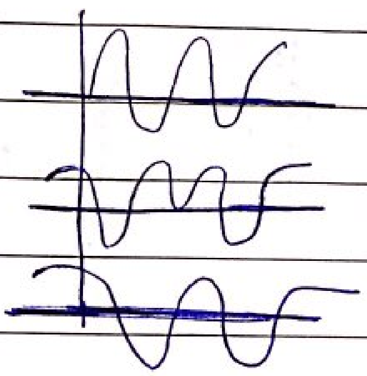
Def: If R.P $x(t)$ is said to be WSS iff:

① $m_x(t) = E[x(t)] = \bar{x}$

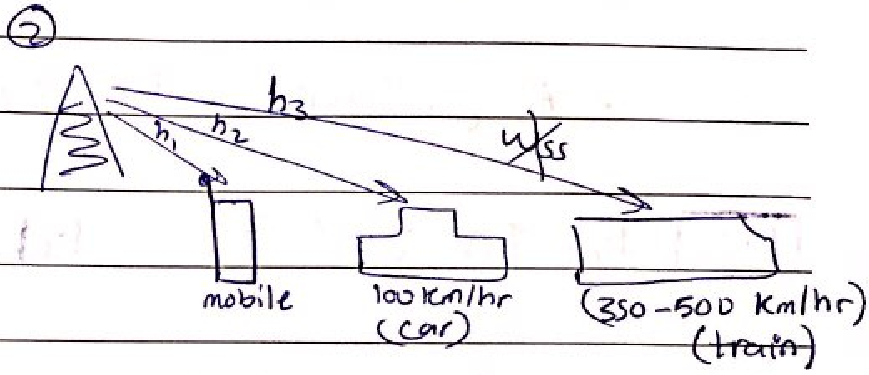
② $R_{xx}(t, t+\tau) = R_{xx}(\tau)$

If $x(t)$ is 2^{nd} order stationarity $\longrightarrow$ WSS (reverse direction crossed out: ✗)

example: noise Process: $N(t)$ in communication systems is WSS



②



Ex: Given a R.P $X(t) = A\cos(\omega t + \theta)$ where A and ω are constant and $\theta \sim U(0, 2\pi)$

check if $x(t)$ is WSS

Sol: $m_x(t) = E[x(t)] = E[\underbrace{A\cos(\omega_o t + \overset{R.V}{\theta})}_{g(\theta)}]$

$= \int A\cos(\omega_o t + \theta)\, f_\theta(\theta)\, d\theta$

$= \int_0^{2\pi} A\cos(\omega_o t + \theta)\, \frac{1}{2\pi}\, d\theta$

$= \frac{1}{2\pi}(0) = \text{zero} = \bar{x}$

$\longrightarrow$

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$R_x(t, t+\tau) = E[x(t), x(t+\tau)]$

$= E[A^2 \cos(\omega_0 t + \theta) \cdot \cos(\omega_0 t + \omega_0 \tau + \theta)]$

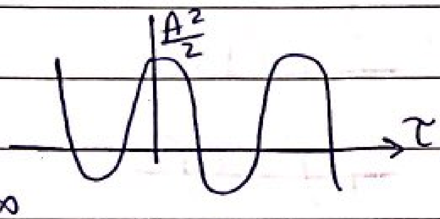
↳ $g(\theta)$

$= E[\frac{A^2}{2}\cos(\omega_0 \tau) + \frac{A^2}{2}\cos(2\omega_0 t + \omega_0 \tau + 2\theta)]$

$= E[\frac{A^2}{2}\cos(\omega_0 \tau) + \frac{A^2}{2}\cos(2\omega_0 t + \omega_0 \tau + 2\theta)]$

$= \frac{A^2}{2}\cos(\omega_0 \tau) + \int_0^{2\pi} \frac{A^2}{2}\cos(2\omega_0 t + \omega_0 \tau + 2\theta) \cdot \frac{1}{2\pi} d\theta$ (integral = 0)

$= \frac{A^2}{2}\cos(\omega_0 \tau) = R_{xx}(\tau)$



Note $\lim_{|\tau| \to \infty} R_{xx}(\tau) \neq \bar{X}^2$

⊛ for WSS : $R_{xx}(\tau)$ Properties:

① $|R_{xx}(\tau)| \leq R_{xx}(0)$

$R_{xx}(-\tau) = R_{xx}(\tau)$ [$R_{xx}(1,4)$ or $R_{xx}(4,1)$ are same] ($\tau=3$, $\tau=-3$)

even function.

② $R_{xx}(0) = E[x^2(t)]$ (R.P Power)

↳ not function of t.

③ $\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{X}^2$ } conditions:

The R.P has no Periodic components

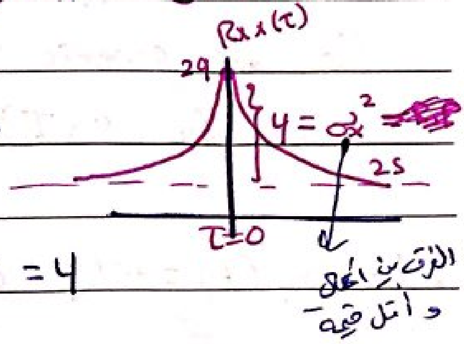
The R.P is ergodic.

Ex let $x(t)$ a WSS R.P with no periodic components with:

$R_{xx}(\tau) = 25 + \dfrac{4}{1+6\tau^2}$, Find: (a) $E[x^2(t)]$ (b) $\bar{X}$ (c) σ_x^2

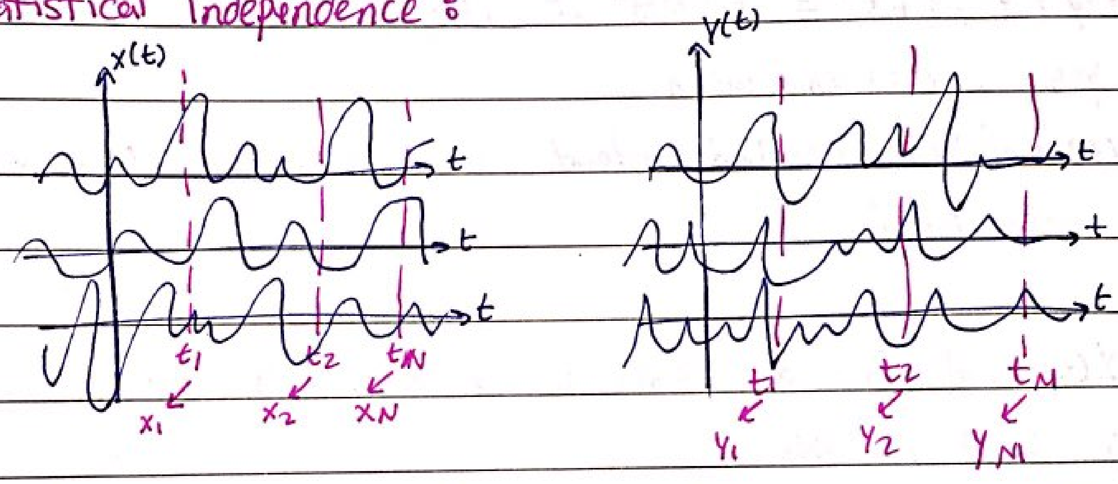
Sol: (a) $E[x^2(t)] = R_{xx}(0) = 29$

(b) $\bar{X} = \sqrt{\lim_{|\tau|\to\infty} R_{xx}(\tau)} = \sqrt{25} = \pm 5$



(c) $\sigma_x^2 = E[x^2(t)] - \bar{X}^2 = 29 - 25 = 4$

* Statistical Independence:



$f_x(x_1, x_N, t_1, \ldots t_N)$ $\qquad$ $f_y(y_1, y_N, t_1', \ldots t_M')$

* Joint density function:-

$f_{xy}(x_1, \ldots x_N, y_1, \ldots y_N, t_1, \ldots t_N, t_1', \ldots t_M')$

$f_x(x_1, \ldots x_N, t_1, \ldots t_N) \cdot f_y(y_1, \ldots y_M, t_1', \ldots t_M')$

For any N and M the $x(t)$ and $y(t)$ are Independent.

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* Cross Correlation :

→ Given two R.P's $x(t)$ and $y(t)$

$$R_{xy}(t_1, t_2) = E[x(t_1) \cdot y(t_2)]$$

To make generic :

$$R_{xy}(t, t+\tau) = E[x(t) \cdot y(t+\tau)]$$

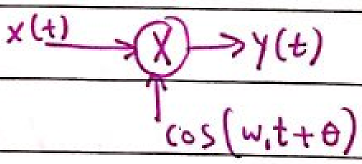
- If $R_{xy}(t, t+\tau) = 0$, then $x(t)$ and $y(t)$ are orthogonal.
- If $R_{xy}(t, t+\tau) = E[x(t)] \cdot E[y(t+\tau)]$, then $x(t)$ and $y(t)$ are uncorrelated.
- If $x(t)$ and $y(t)$ are Independent, then $x(t)$ and $y(t)$ are uncorrelated.

→ $x(t)$ and $y(t)$ are said to be joint WSS if :

1) $x(t)$ is WSS
2) $y(t)$ is WSS
3) $R_{xy}(t, t+\tau) = R_{xy}(\tau)$

Ex:- let $x(t)$ as WSS R.P with no Periodic Compunts and $R_{xx}(\tau) = e^{-a|\tau|}$, $a > 0$

where $\theta \sim U(-\pi, \pi)$ and Independent $x(t)$



ⓐ Find $E[y(t)]$:-

$$E[y(t)] = m_y(t) = E[x(t) \cdot \cos(w_0 t + \theta)]$$

$$= E[x(t)] \cdot E[\cos(w_0 t + \theta)] = (0)(0) = 0$$

↳ Integration on one-period.

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(b) $R_{YY}(\tau)$?

$R_{YY}(\tau) = E[Y(t) \cdot Y(t+\tau)]$

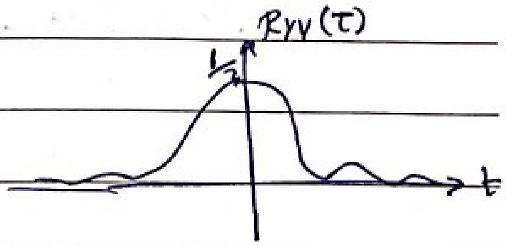
$= E[X(t) \cdot \cos(\omega_0 t + \theta) \cdot X(t+\tau) \cdot \cos(\omega_0 t + \omega_0 \tau + \theta)]$

$= E[X(t) \cdot X(t+\tau)] \cdot E[\cos(\omega_0 t + \theta) \cdot \cos(\omega_0 t + \omega_0 \tau + \theta)]$

$= R_{XX}(\tau) \cdot E\left[\frac{\cos(\omega_0 t)}{2} + \frac{1}{2}\cos(2\omega_0 t + \omega_0 \tau + 2\theta)\right]$

$\rightarrow R_{XX}(\tau) \cdot \left[\frac{1}{2}\cos(\omega_0 \tau) + 0\right]$

$= \frac{1}{2} e^{-a|\tau|} \cos(\omega_0 \tau)$



(c) $R_{XY}(t, t+\tau)$?!

[Ex] (6.3 - 3) text Book

$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$

where ω_0 is a constant, A and B uncorrelated R.V

R.V's with mean zero and same variance

Check If $X(t)$ and $Y(t)$ are joint WSS.

1) $X(t)$ is WSS
2) $Y(t)$ is WSS
} since they are uncorrelated

3) $R_{XY}(t, t+\tau) \stackrel{?}{=} R_{XY}(t)$

→

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Sol:

$R_{xy}(t, t+\tau) = E[x(t) \cdot y(t+\tau)]$

$= E[(A\cos(w_0 t) + B\sin(w_0 t)) \cdot (B\cos(w_0 t + w_0\tau) - A\sin(w_0 t + w_0\tau))]$

$\rightarrow E[AB\cos(w_0 t)\cdot\cos(w_0 t + w_0\tau)]$

$- E[A^2\cos(w_0 t)\sin(w_0 t + w_0\tau)]$

$+ E[B^2\sin(w_0 t)\cdot\cos(w_0 t + w_0\tau)] - E[AB\sin(w_0 t)\cdot\sin(w_0 t + w_0\tau)]$

all cos terms will go out (constants)

$\rightarrow E[AB]\cos(2w_0 t + w_0\tau) +$

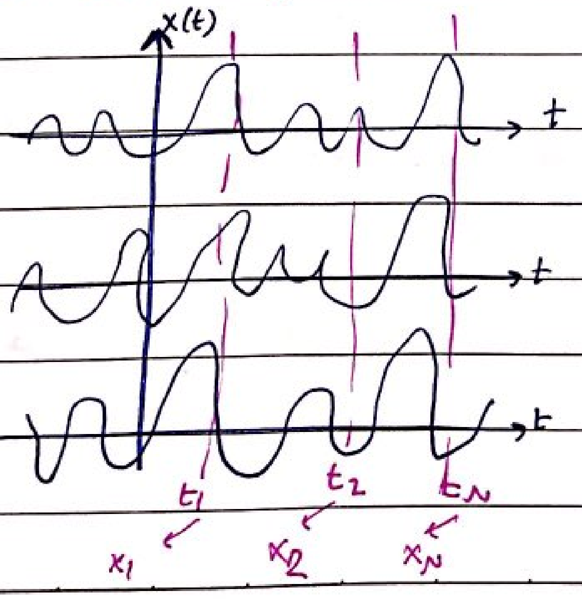
$E[B^2]\sin(w_0 t)\cdot\cos(w_0 t + w_0\tau)$

$- E[A^2]\cos(w_0 t)\sin(w_0 t + w_0\tau)$

$= -\sigma^2\sin(w_0\tau)$

So, $x(t)$ and $y(t)$ are joint WSS.

* Gaussian Process :-



→ If $f_X(x_1, \ldots x_N, t_1, \ldots t_N)$ is gaussian then $x(t)$ is said to be gaussian.

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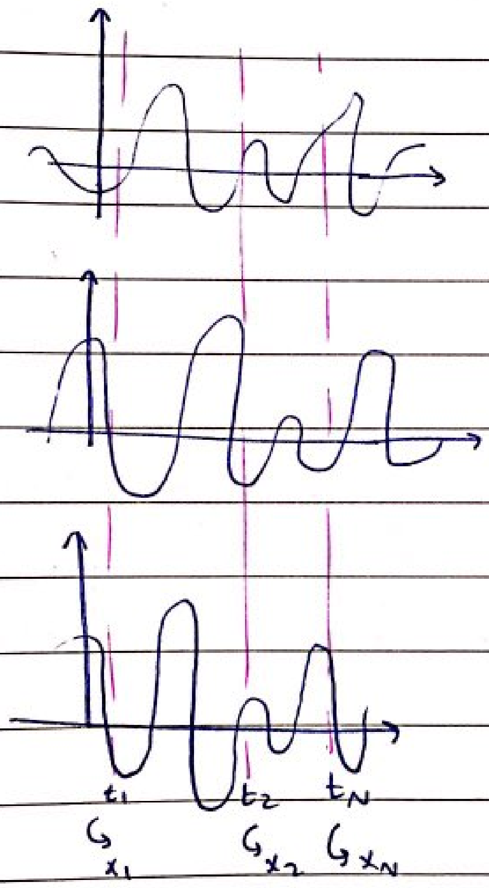
* Gaussian R.P :-

Def: a R.P $x(t)$ is said to be gaussian if the N R.V's $x(t_1)$, $x(t_2)$, ---- $x(t_N)$ for any N are joint gaussian with:

$$f_X(x_1, x_2, --- x_N ; t_1, t_2, --- t_N) = \frac{|[C_X]^{-1}|^{1/2}}{(2\pi)^{N/2}} e^{-\frac{[x-\bar{x}]^T [C_X]^{-1} [x-\bar{x}]}{2}}$$



- $$[C_X] = \begin{bmatrix} C_{11} & C_{12} & ---- & C_{1N} \\ C_{21} & C_{22} & ---- & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & --- & C_{NN} \end{bmatrix}$$

where $C_{ij} = C_{XX}(t_i, t_j)$

$= R_{XX}(t_i, t_j) - m_x(t_i) . m_x$

- $$[X - \bar{X}] = \begin{bmatrix} X - m_x(t) \\ x_2 - m_x(t) \\ \vdots \\ x_N - m_x(t_N) \end{bmatrix}$$

Note :- as a special case if the R.p $x(t)$ is WSS

$C_{ij}(t_i, t_j) = C_{ij}(t_j, t_i) = R_{xx}(t_1, -t_i) - \bar{x}^2$

$$[X - \bar{X}] = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_N - \bar{y} \end{bmatrix}$$

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Example: Given a WSS R.P $x(t)$ with mean $\bar{X} = 4$, $R_{XX}(\tau) = 25e^{-3|\tau|}$

Determine the joint pdf for 3 R.V's $x(t_i)$, $i = 1, 2, 3, -$

Define at time $t_i = t_0 + \frac{i-1}{2}$, with t_0 constant.

Solution: $[X - \bar{X}] = \begin{bmatrix} x_1 - 4 \\ x_2 - 4 \\ \vdots \\ x_3 - 4 \end{bmatrix}$ $\quad [C_X]_{3\times3} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$C_{ij} = C_{XX}(t_j - t_i) - R_{XX}(t_j - t_i) - 4^2$

$$\begin{cases} C_{11} = R_{XX}(t_1 - t_1) - 16 = 25 - 16 = 9 \\ C_{22} = R_{XX}(t_2 - t_2) - 16 = 25 - 16 = 9 \\ C_{33} = 9 \end{cases}$$

$\rightarrow C_{12} = R_{XX}(t_2 - t_1) - 16 = R_{XX}(\frac{1}{2}) - 16 = 25e^{-3/2} - 16 = C_{21}$

$\rightarrow C_{13} = R_{XX}(t_3 - t_1) - 16 = R_{XX}(1) - 16 = 25e^{-3} - 16 = C_{31}$

$\rightarrow C_{23} = R_{XX}(t_3 - t_2) - 16 = R_{XX}(1/2) - 16 = 25e^{-3/2} - 16 = C_{32}$

$t_1 = t_0$

$t_2 = t_0 + \frac{1}{2}$

$t_3 = t_0 + 1$

$$[C_X] = \begin{bmatrix} 9 & 25e^{-3/2} - 16 & 25e^{-3} - 16 \\ 25e^{-3/2} - 16 & 9 & 25e^{-3/2} - 16 \\ 25e^{-3} - 16 & 25e^{-3/2} - 16 & 9 \end{bmatrix}$$

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** Time Average and Ergodicity.

WSS R.P $x(t)$ → $m_x(t) = E[x(t)] = \int x f_x(x) t \, dx$ (exact mean)

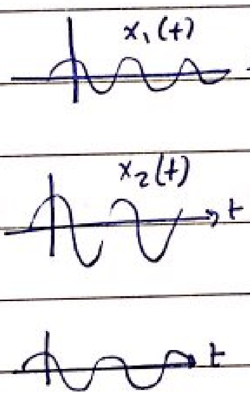
→ $R_{xx}(\tau) = E[x(t) . x(t+\tau)] = \iint x_1 x_2 f_x(x_1, x_2, t_1, t_2) dx$
(exact auto-correlation)

practically: estimated mean : $\hat{m}_x(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t)$

estimated $R_{xx}(\tau)$: $\hat{R}_{xx}(\tau) = \frac{1}{N} \sum_{i=1}^{N} x_i(t) . x_i(t+\tau)$

→ Recall : Time-Average for signal $g(t) = A[g(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t) \, dt$

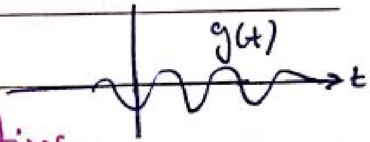
(Give a WSS R.P $x(t)$)



$\overline{x_1} = A[x_1(t)]$ → $R_{xx}(\tau) = A[x_1(t) x(t+\tau)]$ time auto-correlation function

$\overline{x_2}$, $R_{x_2x_2}(\tau)$ → sample functions.

$\overline{x_3}$, $R_{x_3x_3}(\tau)$



$\overline{x} = \{\overline{x_1}, \overline{x_2}, ----\overline{x_N}\}$, $R_{xx} = \{R_{x_1x_1}(\tau), R_{x_2x_2}(\tau), --- R_{x_Nx_N}(\tau)\}$

$E[\overline{x}] = E[A[x(t)]] = E\left[\lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \, dt\right]$

$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[x(t)] \, dt = \overline{x}$ → statistical mean

(circled: $E[x(t)]$ = $\overline{x}$)

Ergodic in mean

time, avg for sample function = statistical mean $\qquad \bar{X} = \bar{X}$

$\mathcal{R}_{XX}(\tau) = R_{XX}(\tau)$

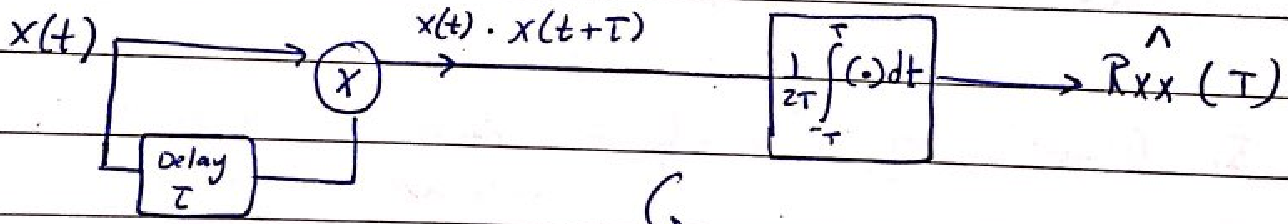
↑ time, avg auto correlation

↑ Statistical auto-correlation function.

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\cdot x(t+\tau)\,dt = E[x(t)\cdot x(t+\tau)]$$

↳ any sample function

To measure $R_{XX}(\tau)$



↳ take T as large as possible.

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Recall :-

WSS R.P is ergodic if :

1) $\bar{X} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$ ← Avg sample function

statistical avg

2) $R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt$

statis. auto-cor

← time auto-correlation

Ex: given R.P $x(t) = A\cos(\omega_0 t + \theta)$, $\theta \sim U(0, 2\pi)$

Find (a) $R_{xx}(\tau)$ (b) $\hat{R}_{xx}(2T)$

sol: (a) $R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)] = --- = \frac{A^2}{2}\cos(\omega_0 \tau)$

(b) $\tilde{R}_{xx}(2T) = \frac{1}{2T}\int_{-T}^{T} x(t) \cdot x(t+\tau)\, dt$

$x(t) = A\cos(\omega_0 t + \theta)$ "one sample function"

$\therefore \hat{R}_{xx}(2T) = \frac{1}{2T}\int_{-T}^{T} A^2 \cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0 \tau + \theta)\, dt$

$\rightarrow \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\cos(\omega_0 \tau)\, dt + \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\cos(2\omega_0 t + \omega_0 \tau + 2\theta)\, d\theta$

$= \underbrace{\frac{A^2}{2}\cos(\omega_0 \tau)}_{\text{exact}} + \underbrace{\frac{A^2}{2}\cos(\omega_0 \tau + 2\theta)\frac{\sin(2\omega_0 T)}{2\omega_0 T}}_{\text{error since 'T' not} \to \infty}$

$= R_{xx}(\tau) + \mathcal{E}(\tau)$

Note:- $\lim_{T\to\infty} \hat{R_{xx}}(2T) = R_{xx}(\tau) + 0$

→ To measure the mean:-

$$x(t) \longrightarrow \boxed{\frac{1}{2T}\int_{-T}^{T}(\cdot)} \longrightarrow \hat{\bar{x}}(t)$$

Ex:- given two WSS R.P's $x_1(t)$ and $x_2(t)$ with:

$$R_{x_1x_1}(\tau) = A\,e^{-|\tau|}$$

$$R_{x_2x_2}(\tau) = A\,e^{-3|\tau|}$$

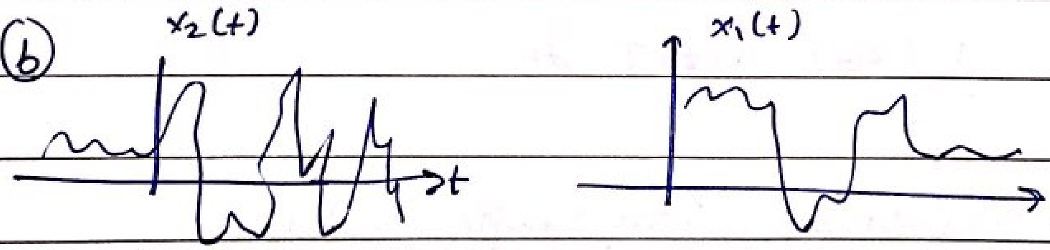
ⓐ find the total power in each process ?!

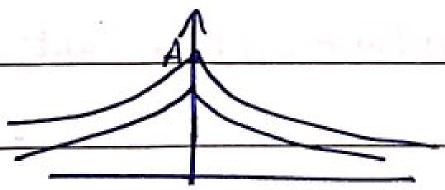
ⓑ Determine which process experiences faster variation in time?

Sol: ⓐ $P_{x_1x_1} = E[x_1^2(t)] = R_{x_1x_1}(0) = A$

$P_{x_2x_2} = E[x_2^2(t)] = R_{x_2x_2}(0) = A$

ⓑ





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Chapter 7 Random Processes – Spectral characteristics:-

R.P
$x(t)$ ⟶ $\mathcal{S}_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$

$P_{XX} = A[E[x^2(t)]]$ $X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$

$$* \; P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\, d\omega.$$

Ex:- given R.P $x(t) = A\cos(\omega_0 t + \theta)$, $\theta \sim U(0, \pi/2)$
($A\cos(\omega_0$: constant.)

find: (a) P_{XX} in time domain
(b) $\mathcal{S}_{XX}(\omega)$
(c) use b to find P_{XX}.

Solution:

(a) $P_{XX} = A[E[x^2(t)]]$

$E[x^2(t)] = E[A^2\cos^2(\omega_0 t + \theta)]$

$= \frac{A^2}{2} E[\frac{A^2}{2}\cos^2(\omega_0 t + 2\theta)]$

$= \frac{A^2}{2} + \frac{A^2}{2}\int_0^{\pi/2} \frac{2}{\pi}\cos(2\omega_0 t + 2\theta)\, d\theta$

$= \frac{A^2}{2} - \frac{A^2}{2}\sin(2\omega_0 t)$, Note → $x(t)$ is not wss

$P_{XX} = A[\frac{A^2}{2} - \frac{A^2}{2}\sin(2\omega_0 t)]$

$= \frac{A^2}{2} - \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\sin(2\omega_0 t)\, dt$

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(b) $S_{xx}(\omega) = \lim_{T \to \infty} \frac{E\left[|X_T(\omega)^2|\right]}{2T}$

$$X_T(\omega) = \int_{-T}^{T} A\cos(\omega_0 t + \theta)\, e^{-j\omega t}\, dt = \int_{-T}^{T}\left(\frac{A}{2} e^{j(\omega_0 t + \theta)} + \frac{A}{2} e^{-j(\omega_0 t + \theta)}\right) e^{-j\omega t}$$

$$= \frac{A}{2} e^{j\theta} \int_{-T}^{T} e^{-j(\omega - \omega_0)t}\, dt + \frac{A}{2} e^{-j\theta} \int_{-T}^{T} e^{-j(\omega + \omega_0)}\, dt$$

$$= \frac{A}{2} e^{j\theta} \left(\frac{e^{-j(\omega - \omega_0)t}}{-j(\omega - \omega_0)} \right]_{-T}^{T} + \frac{A}{2} e^{-j\theta} \left(\frac{e^{-j(\omega + \omega_0)t}}{-j(\omega + \omega_0)} \right]_{-T}^{T} \Bigg)$$

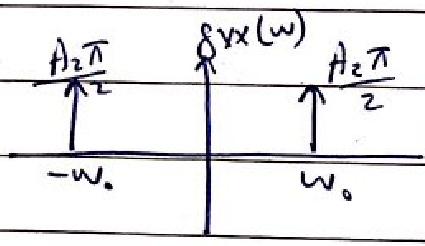
$$X_T(\omega) = A T e^{j\theta} \frac{\sin((\omega - \omega_0)T)}{(\omega - \omega_0)T} + A T e^{-j\theta} \frac{\sin((\omega + \omega_0)T)}{(\omega + \omega_0)T}$$

$$|X_T(\omega)|^2 = X_T(\omega) \cdot X_T^*(\omega) = ----$$

$$\frac{E[|X_T(\omega)|^2]}{2T} \simeq \frac{A_0^2 \pi}{2} \left[\frac{T}{\pi} \frac{\sin(\omega - \omega_0)T}{[(\omega - \omega_0)T]^2} + \frac{T}{\pi} \frac{\sin^2(\omega + \omega_0)T}{\pi((\omega + \omega_0)T)^2} \right.$$

$$S_{xx}(\omega) = \lim_{T \to \infty} (\ )$$

$$S_{xx}(\omega) = \frac{A_0^2 \pi}{2} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]$$



(*) Note:-

$$\lim_{T \to \infty} \frac{T}{\pi} \left[\frac{\sin(\alpha T)}{\alpha T} \right]^2 = \delta(\alpha)$$

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(c) $P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(w)\,dw =$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A^2\pi}{2}\delta(w-w_0)\,dw + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A^2\pi}{2}\delta(w+w_0)\,dw$$

$$= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

(*) $S_{xx}(w)$ properties :-

[1] $S_{xx}(w) \geq 0$

[2] $S_{xx}(-w) = S_{xx}(w)$, $x(t)$ real
even function

[3] $S_{xx}(w)$ is real

[4] $\int_{-\infty}^{\infty} S_{xx}(w)\,dw = 2\pi P_{xx}$

[5] $x(t) \rightarrow S_{xx}(w)$

$\frac{dx(t)}{dt} \rightarrow w^2 S_{xx}(w)$

[6] $S_{xx}(w) = \int_{-\infty}^{\infty} A[R_{xx}(t, t+\tau)]\, e^{-jw\tau}\, d\tau$

$$A[R_{xx}(t, t+\tau)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(w)\, e^{jw\tau}\, dw$$

$$A[R_{xx}(t, t+\tau)] \xleftrightarrow{F.T} S_{xx}(w)$$

As a special case → If $x(t)$ is WSS : $R_{xx}(t, t+\tau) = R_{xx}(\tau)$

(*) $A[R_{xx}(\tau)] = R_{xx}(\tau)$

$R_{xx}(\tau) \xleftrightarrow{F.T} S_{xx}(w)$

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Ex :- $x(t) = A\cos(\omega t + \theta)$, $\theta \sim U(0, 2\pi)$ Find $S_{xx}(\omega)$

Sol: $R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)] = - - - = \frac{A^2}{2}\cos(\omega_0 \tau)$

$S_{xx}(\omega) = FT\left\{\frac{A^2}{2}\cos(\omega_0 \tau)\right\} = \frac{A^2\pi}{2}\delta(\omega - \omega_0) + \frac{A^2\pi}{2}\delta(\omega + \omega_0)$

Ex: Given $x(t)$ a WSS R.P with no periodic components

$$R_{xx}(\tau) = \begin{cases} A_0\left(1 - \frac{|\tau|}{T}\right) & , -T < \tau < T \\ 0 & , o.w \end{cases}$$

Find: (a) P_{xx}

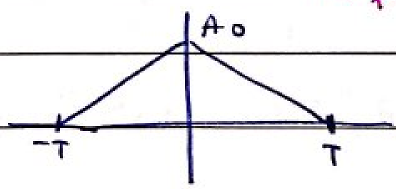
(b) R.P DC value

(c) $S_{xx}(\omega)$

(a) $P_{xx} = R_{xx}(0) = A_0$

(b) $\bar{X}^2 = \lim_{\tau \to \infty} R_{xx}(\tau) = 0 \quad (\bar{X} = 0)$

$R_{xx}(\tau) = A_0\, tri\left(\frac{\tau}{T}\right)$ ↳ FT sinc



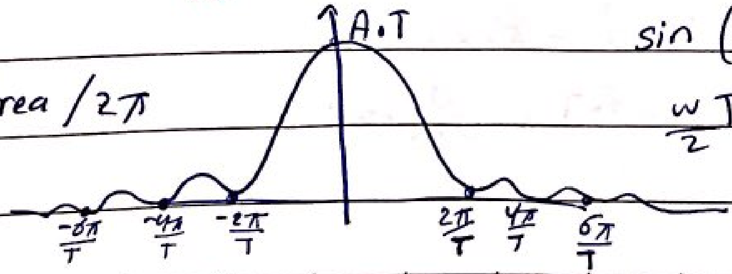
(c) $S_{xx}(\omega) = FT\{R_{xx}(\tau)\}$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-T}^{0} A_0\left(1 + \frac{\tau}{T}\right) e^{-j\omega\tau}\, d\tau + \int_{0}^{T} A_0\left(1 - \frac{\tau}{T}\right) e^{-j\omega\tau}\, d\tau$$

$R_{xx}(\tau) = A_0\, tri\left(\frac{\tau}{T}\right) \xleftrightarrow{FT} S_{xx}(\omega) = A_0 T\, \text{sinc}^2\left(\frac{\omega T}{2}\right)$

$P_{xx} = \text{Area} / 2\pi$



$\sin\left(\frac{\omega T}{2}\right) = 0$

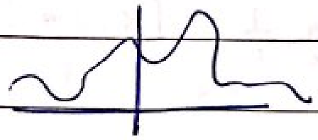
$\frac{\omega T}{2} = \pm n\pi$

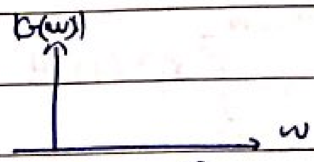
$\omega = \pm \frac{2\pi n}{T}$

Ch.7 Recall F.T :-

$g(t) \rightarrow$ deterministic time-waveform $\xleftrightarrow{F.T} G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$

$= |G(\omega)| \angle \phi(\omega)$





$E_g = \int_{-\infty}^{\infty} g^2(t)\, dt$

Energy of signal.

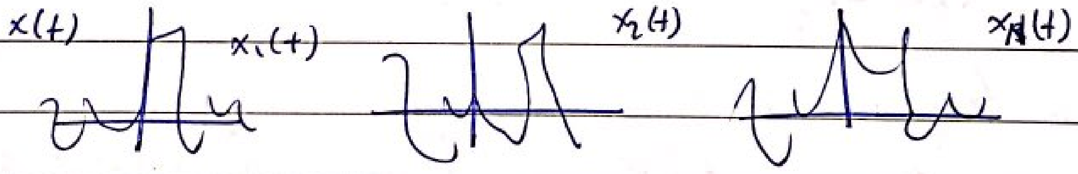
$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega$

$E_g \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2\, d\omega \right.$

- $\int_{-\infty}^{\infty} |g(t)|^2\, dt < \infty$: suffecient but not nessessory for FT existence

* See the table of F.T.s

→ for Random Processes :-



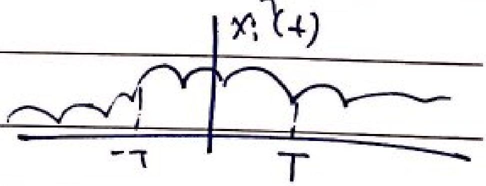
* P.P Power spectral density function :

for $x(t)$ take one sample function.

$x_i(t)$ " assume $x_i(t)$ is time un-limited "

$x_i(t) \longleftrightarrow FT$ X

$x_i^T(t) = \begin{cases} x_i(t) , & -T \le t < T \\ 0 , & o.w \end{cases}$



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$x_i^T(t) \xleftrightarrow{F.T} X^T(\omega) = \int_{-\infty}^{\infty} x_i^T(t)\, e^{-j\omega t}\, dt$

- $P_i^T = \int_{-T}^{T} x_i(t)\, dt$
- $P_i^T = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X^T(\omega)|^2}{2T}\, d\omega$

- $P_i = \lim_{T\to\infty} P_i^T$

$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x_i^2(t)\, dt$

- $P_i = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\, d\omega$

$i: 0, 1, 2, \ldots$ (power required)

- For all sample function:

$P = \{P_1, P_2, \ldots P_N\}$

$\therefore$

R.P avg Power $= P_{xx}$

$P_{xx} = E[P]$

$= E\left[\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)^2\, dt\right]$

$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[x(t)^2]\, dt = A\left[E[x(t)^2]\right]$

In freq domain:-

$P_{xx} = E[P] = E\left[\lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\, d\omega\right]$

$\rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)^2|]}{2T}\, d\omega$

[watt / Hz]

$S_{xx}(\omega)$ = R.P PDS

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$$X(t) \rightarrow \mathcal{P}_{XX}(\omega) = \lim_{T\to\infty} E\left[\frac{|X_T(\omega)|^2}{2T}\right]$$

$$X_T(\omega) = \int_{-T}^{+T} X(t)\, e^{-j\omega t}\, dt.$$

- $A\left[R_{XX}(t, t+\tau)\right] \overset{F.T}{\longleftrightarrow} \mathcal{P}_{XX}(\omega)$

WSS $\longrightarrow R_{XX}(\tau) \overset{F.T}{\longleftrightarrow} \mathcal{P}_{XX}$

Ex. Given two WSS R.P $x_1(t)$ and $x_2(t)$ with:

$$R_{x_1x_1}(\tau) = \sigma_{x_1}^2\, e^{-\beta_1|\tau|}$$

$$R_{x_2x_2}(\tau) = \sigma_{x_2}^2\, e^{-\beta_2|\tau|} \qquad \text{where } \beta_2 > \beta_1$$

Find:- (a) $P_{x_1x_1}$ and $P_{x_2x_2}$

(b) $\mathcal{P}_{x_1x_1}(\omega)$ and $\mathcal{P}_{x_2x_2}(\omega)$

(c) which process has higher frequency components ?!

Sol: (a) $P_{x_1x_1} = R_{x_1x_1}(0) = \sigma^2$

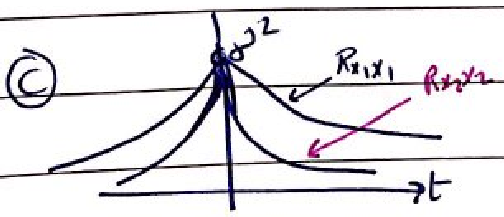
$P_{x_2x_2} = R_{x_2x_2}(0) = \sigma^2$

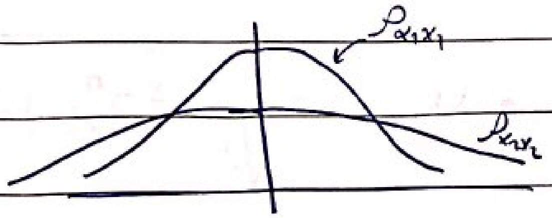
(b) $\mathcal{P}_{x_1x_1}(\omega) = FT\left(\sigma^2 e^{-\beta_1|\tau|}\right) = \int_{-\infty}^{\infty} \sigma^2 e^{-\beta_1} e^{-j\omega\tau}\, d\tau$

using F.T table :

$$\mathcal{P}_{x_1x_1}(\omega) = \frac{2\sigma^2\beta_1}{\beta_1^2+\omega^2} \qquad \mathcal{P}_{x_2x_2}(\omega) = \frac{2\sigma^2\beta_2}{\beta_2^2+\omega^2}$$

(c)





$R_{x_2x_2}(\tau) < R_{x_1x_1}(\tau)$

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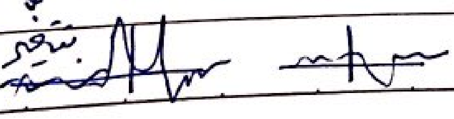
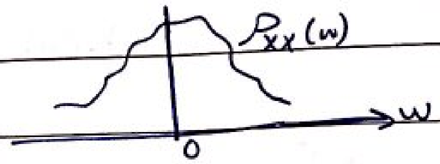
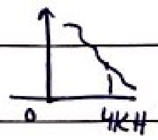


* R.P Bandwidth and classification :-

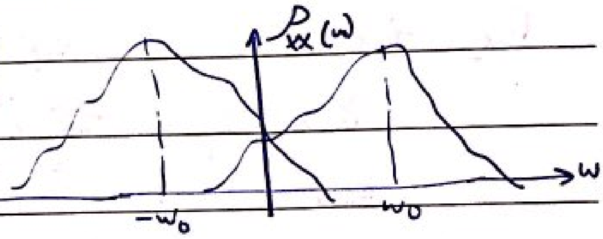
[1] **Baseband R.P :** it's frequency components, are clustered around $\omega = 0$



ex: human speech



[2] **Band-pass R.P :** freq. componets are clustered around certein frequency (ω_0)

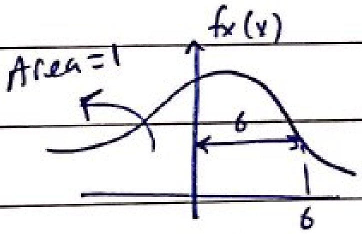


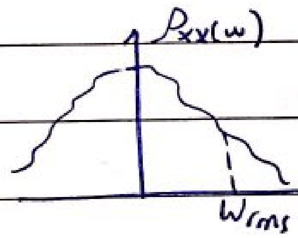
$x(t) \longrightarrow X(\omega)$

$x(t)\cos(\omega_0 t) \longrightarrow X(\omega-\omega_0) + X(\omega+\omega_0)$

* Root-mean-square BW :- W_{rms}

① W_{rms} BW for BaseBand R.P's :





$x \sim N(0, \sigma^2)$

$$\sigma = \sqrt{var(x)} = \sqrt{\int x^2 f_x(x)\,dx}$$

normalized ←

- $$P^{(n)}_{xx}(\omega) = \frac{P_{xx}(\omega)}{\int_{-\infty}^{\infty} P_{xx}(\omega)\,d\omega}$$

- $$W_{rms} = \sqrt{\int \omega^2 P^{n}_{xx}(\omega)\,d\omega}$$

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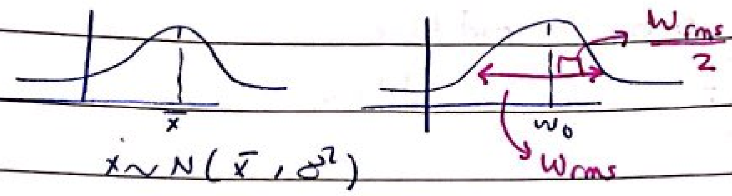
$\therefore \; W_{rms} = \sqrt{\dfrac{\int_{-\infty}^{\infty} w^2 P_{xx}(w)\, dw}{\int_{-\infty}^{\infty} P_{xx}(w)\, dw}}$ for base-band.

② W_{rms} for Bandpass process :-



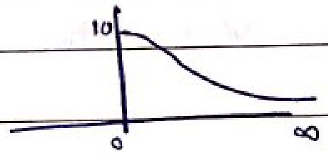
$x \sim N(\bar{x}, \sigma^2)$

$* \; \sigma = \sqrt{\int (x-\bar{x})^2 f_x(x)\, dx}$

$* \; \dfrac{W_{rms}}{2} = 2\sqrt{\dfrac{\int_{-\infty}^{\infty} (w-w_0)^2 P_{xx}(w)\, dw}{\int_{-\infty}^{\infty} P_{xx}(w)\, dw}}$

Example: Given $x(t)$ with $P_{xx}(w) = \dfrac{10}{\left[1+\left(\frac{w}{10}\right)^2\right]^2}$

find W_{rms} :-



Sol: from the figure, its base-band

$\therefore$

$W_{rms} = \sqrt{\dfrac{\int_{-\infty}^{\infty} w^2 P_{xx}(w)\, dw}{\int_{-\infty}^{\infty} P_{xx}(w)\, dw}}$

$\rightarrow \int_{-\infty}^{\infty} P_{xx}(w)\, dw = \int_{-\infty}^{\infty} \dfrac{10}{\left[1+\left(\frac{w}{10}\right)^2\right]^2} dw \cdot \dfrac{10^4}{10^4}$

go to table of Integrals.

$= \int_{-\infty}^{\infty} \dfrac{10^5}{[100+w^2]^2} dw = 50\pi \text{ rad/s}$ → from table Appendix C-28.

$$\rightarrow \int_{-\infty}^{\infty} w^2 \, \mathcal{P}_{xx}(w)\, dw = \int_{-\infty}^{\infty} \frac{10 w^2}{\left[1+\left(\frac{w}{10}\right)^2\right]^2} dw = \int_{-\infty}^{\infty} \frac{10^5 w^2}{(100+w^2)^2} dw$$

$$= 5000\pi \text{ rad/sec}$$

$$* \; W_{rms}^{bb} = \sqrt{\frac{5000\pi}{50\pi}} = 10 \text{ rad/sec}$$

1st/Aug/2018.

* Cross-power and cross-PDS :-

$$x(t) = P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} E \frac{[|X_T(w)|^2]}{2T}}_{\mathcal{P}_{xx}(w)} dw$$

$$y(t) = P_{yy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} E \frac{[|Y_T(w)|^2]}{2T}}_{\mathcal{P}_{YY}(w)} dw$$

→ Cross power :-

$$\rightarrow P_{xy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} E \frac{[X_T^*(w)\, Y_T(w)]}{2T}}_{\mathcal{P}_{xy}(w)} dw$$

$\mathcal{P}_{xy}(w)$ → cross PDS.

$$\rightarrow P_{yx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} E \left[\frac{Y_T^*(w)\, X_T(w)}{2T}\right]}_{\mathcal{P}_{yx}(w)} dw$$

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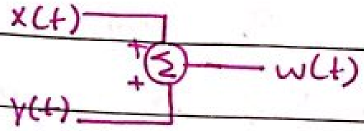
- $S_{xy}(w)$, $S_{yx}(w)$ are complex functions
- $S_{xy}^{*}(w) = S_{yx}(w)$
- $P_{xy}^{*} = P_{yx}$



Given $x(t)$ and $y(t)$

Let $w(t) = x(t) + y(t)$

what are: (a) $R_{ww}(t, t+\tau)$ (b) $S_{ww}(w)$ (c) P_{ww}

(a) $R_{ww}(t,t+\tau) = E[w(t) \cdot w(t+\tau)]$

$= E[(x(t)+y(t))(x(t+\tau)+y(t+\tau))]$

$= E[x(t) \cdot x(t+\tau)] + E[x(t) \cdot x(t+\tau)] + E[x(t) \cdot x(t+\tau)] + E[y(t) \cdot y(t+\tau)]$

$= R_{xx}(t,t+\tau) + R_{xy}(t,t+\tau) + R_{yx}(t,t+\tau) + R_{yy}(t,t+\tau)$

cross-correlation functions

autocorrelation function

→ IF $x(t)$ and $y(t)$ are orthogonal:

$R_{xy}(t,t+\tau) = R_{yx}(t,t+\tau) = 0$

$R_{ww}(t,t+\tau) = R_{xx}(t,t+\tau) + R_{yy}(t,t+\tau)$.

(b) $S_{ww}(w) = FT\{A[R_{ww}(t,t+\tau)]\}$

$= FT\{A[R_{xx}(t,t+\tau) + R_{xy}(t,t+y)$

$+ R_{yx}(t,t+\tau) + R_{yy}(t,t+\tau)\}$

توزيع

$= FT\{A[R_{xx}(t,t+\tau)]\} + FT[R_{xy}(t,t+\tau)]$

$+ ----$

$S_{xx}(w) + S_{yy}(w) + S_{xy}(w) + S_{yx}(w)$

PSD's — cross PSD's

IF $x(t)$ and $y(t)$ are orthogonal:

$S_{xy}(w) = 0$, $S_{yx}(w) = 0$

$S_{ww}(w) = S_{xx}(w) + S_{yy}(w)$

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ⓒ $P_{ww} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{ww}(\omega)\, d\omega = P_{xx} + P_{yy} + P_{xy} + P_{yx}$

If $x(t)$ and $y(t)$ are orthogonal, $P_{xy} = P_{yx} = 0$

$$P_{ww} = P_{xx} + P_{yy}$$

Ex If $x(t)$ and $y(t)$ are uncorrelated R.p's and have constant means $\bar{x}$ and $\bar{y}$, show that:

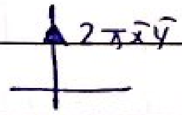
$$\mathcal{S}_{xy} = 2\pi\, \bar{x}\bar{y}\, \delta(\omega)$$

Sol: $\mathcal{S}_{xy} = \int_{-\infty}^{\infty} A\left[E\left[x(t)\,y(t+\tau)\right]\right] e^{-j\omega t}\, dt$

$= \int_{-\infty}^{\infty} \bar{x}.\bar{y}\, e^{-j\omega\tau}\, d\tau = FT[\bar{x}\bar{y}]$

$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$

$\bar{x}\bar{y} \xleftrightarrow{FT} 2\pi\bar{x}\bar{y}\,\delta(\omega)$



Ex 7.3-1 Cross Power Spectrum:

$$\mathcal{S}_{xy}(\omega) = \begin{cases} a + j\frac{b\omega}{W}, & -W < \omega < W \\ 0, & o.w \end{cases}$$

$W > 0$, a, b are real constants, Find $R_{xy}(\tau)$?!

Sol:

$R_{xy}(\tau) = FT^{-1}\{\mathcal{S}_{xy}(\omega)\} = \frac{1}{2\pi}\int_{-W}^{W} \mathcal{S}_{xy}(\omega)\, e^{j\omega\tau}\, d\omega$

$= \frac{1}{2\pi}\int_{-W}^{W} \left(a + \frac{jb\omega}{W}\right) e^{j\omega\tau}\, d\omega$

$R_{xy}(\tau) = \frac{1}{\pi W\tau^2}\left[(aW - \tau)\sin(W\tau) + bW\tau\cos(W\tau)\right]$

Ex 7.3-2 $\mathcal{S}_{xy}(\omega) = \frac{8}{(\alpha + j\omega)^2}$

$u(\tau)\,\tau^2 e^{-\alpha\tau} \longleftrightarrow \frac{2}{(\alpha + j\omega)^3}$

$R_{xy}(\tau) = FT^{-1}\left\{\frac{8}{(\alpha + j\omega)^3}\right\}$

$R_{xy}(\tau) = 4\,\tau\, e^{-5\tau}\, u(\tau)$

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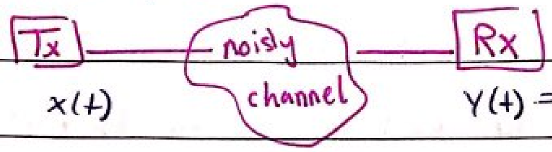
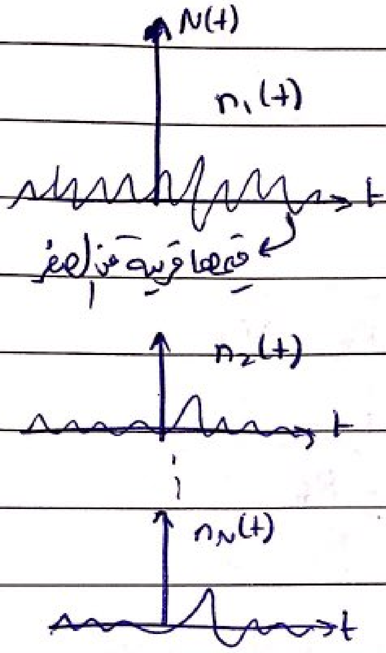
* Noise process :-



$Y(t) = x(t) + N(t)$

additional



$N(t)$ → Gaussian Process (G)

→ Addition "A"

→ white "W"

$N(t)$ → AWGN

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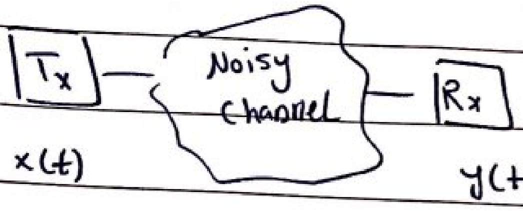
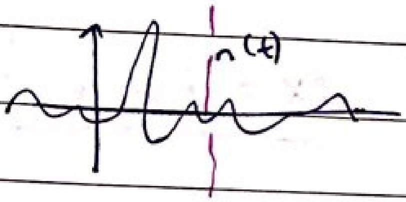
* White Noise :-

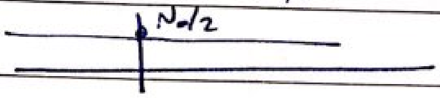


$y(t) = x(t) + N(t)$ → Noise process



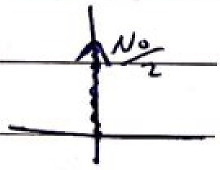
AWGN → Noise
additive, white, gaussian

$N(t) \rightarrow \mathcal{S}_{NN}(\omega)$



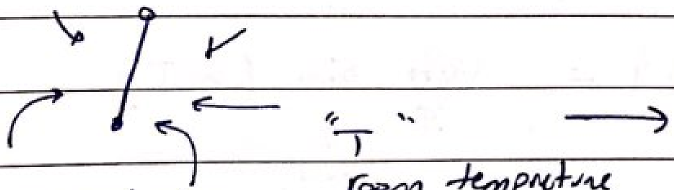
- WSS
- Ergodic

$$R_{NN}(\tau) = F^{-1}\{\mathcal{S}_{NN}(\omega)\} = \frac{N_0}{2}\,\delta(\tau)$$

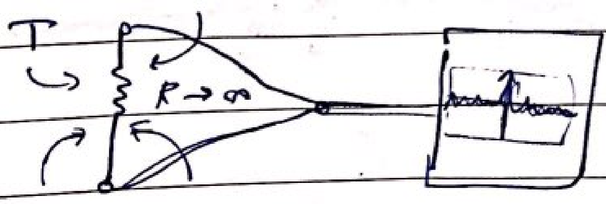


$$P_{NN} = \int_{-\infty}^{\infty} \frac{\mathcal{S}_{NN}(\omega)}{2\pi}\, d\omega = \infty$$

* Thermal Noise (حراري)



Random motion of electrons
↓
Random current $i_n(t)$
↓
Noise voltage $V_n(t) = N(t)$

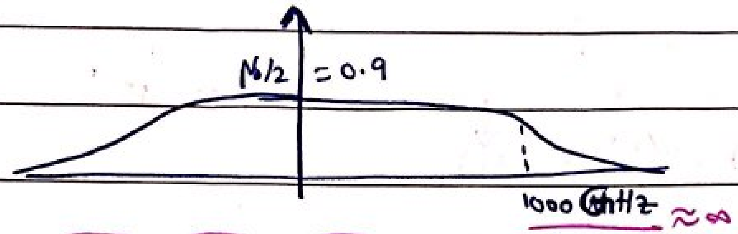


High Gain oscope

T ↑ , P_{NN} ↑
علاقة طردية

Date. 5-Aug No.

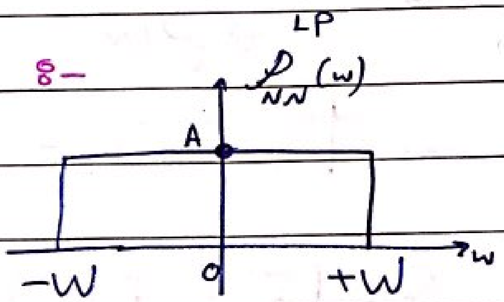
« Thermal Noise »



$$\rho_{NN}^{th}(\omega) = \frac{(N_0/2)(\alpha|\omega|/T)}{e^{\alpha|\omega|/T} - 1}$$

$\alpha = 7.64 * 10^{-12}$

* Band limited lowpass Noise :-



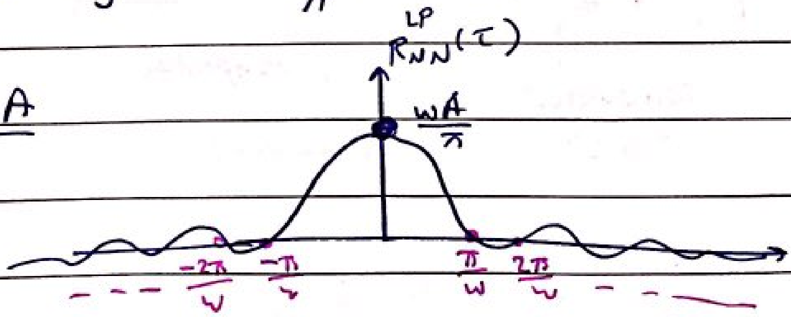
$$\rho_{NN}^{LP}(\omega) = \begin{cases} A, & -W < \omega < +W \\ 0, & \text{otherwise.} \end{cases}$$

$$P_{NN}^{LP} = \frac{\text{Area}}{2\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_{NN}^{LP}(\omega)\, d\omega = \frac{1}{2\pi} \cdot 2WA$$

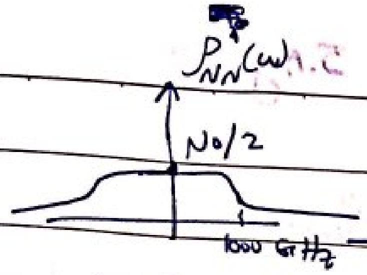
$$P_{NN}^{LP} = \frac{AW}{\pi}$$

$$R_{NN}^{LP}(\tau) = F_{\tau}^{-1}\{\rho_{NN}^{LP}(\omega)\} = \frac{WA}{\pi}\,\text{sinc}(W\tau)$$

$$P_{NN} = R_{NN}^{LP}(0) = \frac{WA}{\pi}$$



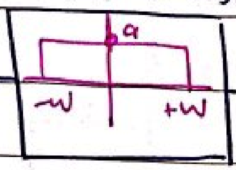
Date. 5 Aug No.



input Noise

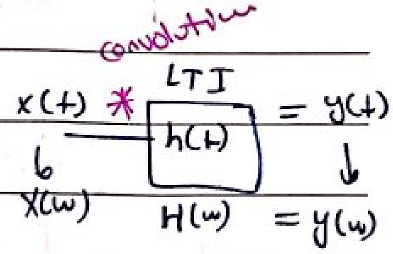
$N^{i}(t)$

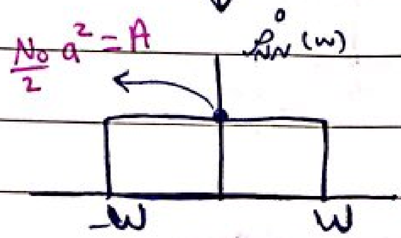
LPF (low pass filter)



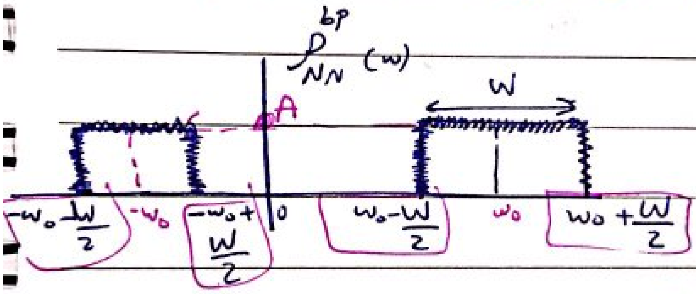
$|H(\omega)|$

$$P^{o}_{NN}(\omega) = P^{i}_{NN}(\omega)\,|H(\omega)|^2$$





Band limited Band Pass Noise :



$$P^{bP}_{NN} = \frac{1}{2\pi}\int_{-\infty}^{\infty} P^{bP}_{NN}(\omega)\,d\omega$$

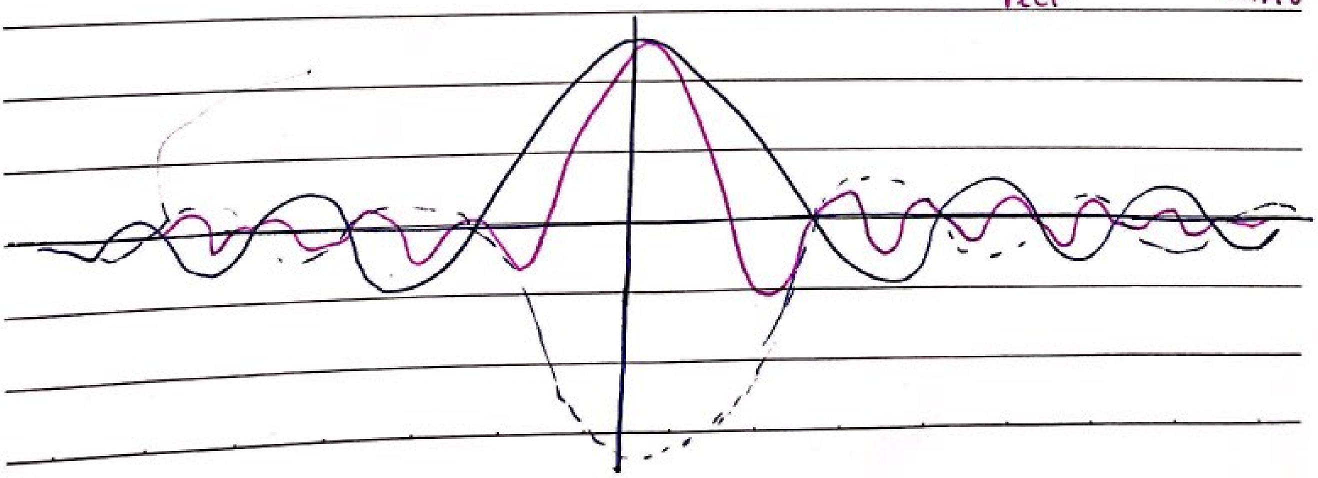
$$= \frac{1}{2\pi}\cdot\left[A \cdot W \cdot 2\right]$$

(W: width)

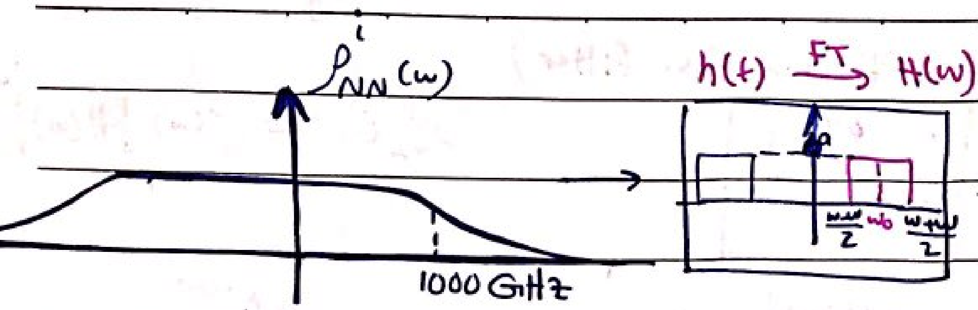
$$P^{bP}_{NN} = \frac{A\,W}{\pi}$$

$$R^{bP}_{NN}(\tau) = F_{\tau}^{-1}\left\{P^{bP}_{NN}(\omega)\right\} = \frac{WA}{2}\,\operatorname{sinc}\left(\frac{W\tau}{2}\right)\cos(\omega_0\tau)$$

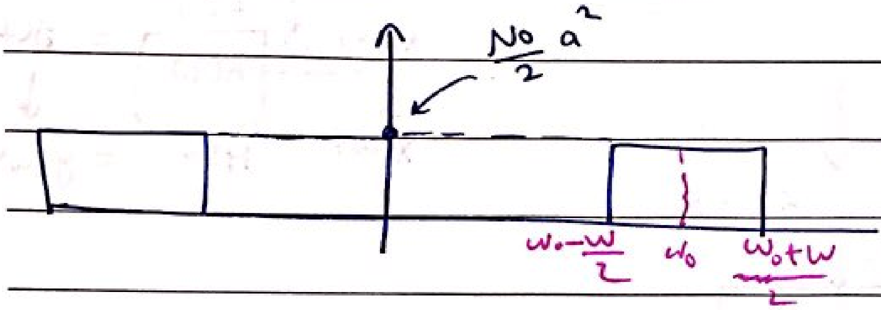
(sinc: rect; cos: shift)

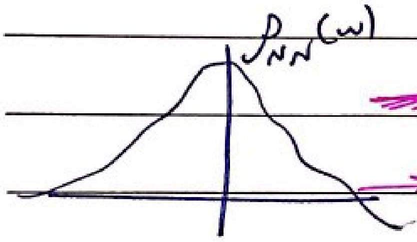


Date. 5.Aug No.



$$\rho^o_{NN}(\omega) = \rho^i_{NN}(\omega)\,|H(\omega)|$$





→ Power spectral density for colored Noise