PARTIAL

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partial

$F : \mathbb{R}^2 \longrightarrow \underline{\underline{\mathbb{R}}}$ Scalar.

$F(x,y) = x^2 + y^2 - 2$ (Scalar field) " Scalar function "

$F(1,2) = \underline{\underline{3}}$

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$F(x,y) = 2x\,\underline{i} + y^2\,\underline{j}$ → the result is vector.

$F(1,1) = 2\underline{i} + \underline{j} \rightarrow \underline{\underline{\langle 2,1 \rangle}}$

* vector field = function with vector result " vector rang "

\# let $\underline{F}$ be avector field →

$$\begin{tikzcd} \underline{F} \arrow[r] \arrow[dr] & \text{div } F \\ & \underline{\text{curl}}\ F \end{tikzcd}$$

just for avector.

$v(t) = 2t\,\underline{i} + \underline{j} - t^2\,\underline{k}$ $\mathbb{R} \longrightarrow \mathbb{R}^3$ " vector field " ✓

$\langle 2t, 1, -t^2 \rangle$

$v(t) = (2+)\,\underline{i} + x\,\underline{j}$ → Scalar + Vector → Not vector.

Scalar * Vector → Vector.

$F(x,y,z) = f(x,y,z)\,\underline{i} + g(x,y,z)\,\underline{j} + h(x,y,z)\,\underline{k}$

$\text{div } F = f_x + g_y + h_z = \underline{\nabla} \cdot \underline{F}$

let $\underline{F} = 2x^2y\,\underline{i} + x^2y^2\,\underline{j} + x^2z\,\underline{k}$

$\text{div } \underline{F} = 4xy + 2x^2y + x^2$ → Scalar. (!!! div $\underline{F}$ is Scalar fie

* $F = f\,\underline{i} + g\,\underline{j} + h\,\underline{k}$

$\nabla = \frac{d}{dx}\,\underline{i} + \frac{d}{dy}\,\underline{j} + \frac{d}{dz}\,\underline{k}$

$\underset{\text{vector}}{\underline{\nabla}} \overset{\text{dot product}}{\cdot} \underset{\text{vector}}{\underline{F}} = \text{div } F = f_x + g_y + h_z$

\# let $F = f\,\underline{i} + g\,\underline{j} + h\,\underline{k}$

the curl $\underline{F} = \underline{\nabla} \times \underline{F} \rightsquigarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f & g & h \end{vmatrix}$

$= \left(\frac{dg}{dx} - \frac{d\theta}{dy}\right)\underline{i} - \left(\frac{dh}{dx} - \frac{df}{dz}\right)\underline{j} + \left(\frac{dg}{dx} - \frac{df}{dy}\right)\underline{k}$ " vector field "

$F(x,y,z) = xy\,\underline{i} + xz\,\underline{j} + x^2z\,\underline{k}$

$\underline{\boxed{1}}$ find div F $\underline{\boxed{2}}$ curl F

$\text{div } F = y + x^2$

$\text{curl } F = -x\,\underline{i} + (2xz - 0)\,\underline{j} + (z - x)\,\underline{k} = -x\,\underline{i} - 2xz\,\underline{j} + (z-x)\,\underline{k}$

$\langle -x, -2xz, (z-x)\rangle$

$\boxed{\text{* Remark}}$

$F(x,y,z) = x^2y^2 + z^2$ "Scalar"

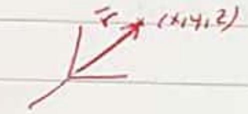
vector. $\underline{\nabla F} = \frac{dF}{dx}\,\underline{i} + \frac{dF}{dy}\,\underline{j} + \frac{dF}{dz}\,\underline{k} = 2xy^2\,\underline{i} + 2x^2y\,\underline{j} + 2z\,\underline{k}$

gradient. scalar

$\langle 2xy^2, 2x^2y, 2z\rangle$

vector.

=



$\underline{r}$ (distance) $= x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$

according to origin.

$\langle x,y,z\rangle$

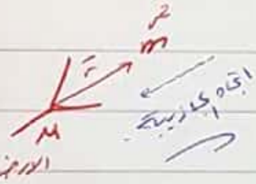
r or $\|r\| = \sqrt{x^2+y^2+z^2}$

distance

"length of vector"

r : magnitude. $= \|r\|$

$\underline{r}$: vector.



$F = \frac{GmM}{r^2} * (-\hat{a}_r) = \frac{GmM}{r^2} * \frac{\underline{r}}{r} = \frac{\boxed{GmM}\,\underline{r}}{r^3}$

$= \frac{c\,\underline{r}}{r^3}$ $\underline{r} \rightarrow \langle x,y,z\rangle$

$r^3 \swarrow \sqrt{x^2+y^2+z^2}$

"Gravitational field"

(III) let F, G be a vector field ; φ : scalar field

then :

$\text{div}(\underline{F} + \underline{G}) = \text{div}\,\underline{F} + \text{div}\,\underline{G}$

$\text{div}(K\underline{F}) = K\,\text{div}\,\underline{F}$

$\text{div}(\varphi \underline{F}) = \varphi\,\text{div}\,\underline{F} + \nabla\varphi \, . \, \underline{F}$ ↳ dot not cross " because div is scalar

$\text{div}(\text{curl}\,\underline{F}) =$ zero

↳ zero scalar

$\text{curl}(\underline{F} + \underline{G}) = \text{curl}\,\underline{F} + \text{curl}\,\underline{G}$

$\text{curl}(K\underline{F}) = K\,\text{curl}(\underline{F})$

$\text{curl}(\varphi \underline{F}) = \varphi\,\text{curl}\,\underline{F} + \nabla\varphi \times \underline{F}$

$\text{curl}(\nabla . \varphi) =$ zero $\underline{0}$

gradinant ↳ zero vector.

*** Example.**

let $F(x,y,z) = f(x,y,z)\,\underline{i} + g(x,y,z)\,\underline{j} + h(x,y,z)\,\underline{k}$

φ : scalar function

show that :-

$$\text{div}(\varphi F) = \varphi\,\text{div}\,\underline{F} + \nabla . \varphi . F$$

→ $\varphi F = \varphi f(x,y,z)\,\underline{i} + \varphi g(x,y,z)\,\underline{j} + \varphi h(x,y,z)\,\underline{k}$

$$\text{div}(\varphi F) = \frac{d\varphi f}{dt} + \frac{d\varphi g}{dy} + \frac{d\varphi h}{dz}$$

$$\varphi f_x + f\varphi_x + \varphi g_y + g\varphi_y + \varphi h_z + h\varphi_z$$

$(\nabla . \varphi . \underline{F})$

→ $\nabla . \varphi = \varphi_x \underline{i} + \varphi_y \underline{j} + \varphi_z \underline{k}$

$\nabla . \varphi . \underline{F} = \varphi_x f + \varphi_y g + \varphi_z h$

$\# \ \ \varphi(f_x + g_y + h_z) + \nabla . \varphi . F$

$\varphi\,\text{div}\,\underline{F} + \nabla . \varphi . \underline{F}$ $\#$ Dow.

9/

* **Example:-** Does there exsit a vector field $\underline{G} \rightarrow \text{curl}\,\underline{G} = \underline{F}$

let $F = 2x i + y j + z k$

→ Suppose that there is G

$\text{curl } G = F$

$\text{div}(\text{curl } G) = \text{div}\underline{F}$

$\text{zero} = 2 + 1 + 1$

$\text{zer} \neq 4$, so there is no G that curl of it $= \underline{F}$.

$\circledast$

$r = x i + y j + z k$ | $\frac{dr}{dx} = \frac{x}{r}$; $\frac{dr}{dy} = \frac{y}{r}$; $\frac{dr}{dz} = \frac{z}{r}$

$\|r\| = \sqrt{x^2+y^2+z^2}$

$\boxed{1}$ $\frac{d(r^2)}{dx} = 2rr' = 2r\frac{x}{r} = \underline{\underline{2x}}$

$\boxed{2}$ $\frac{d\sqrt{r}}{dy} = \frac{r'}{2\sqrt{r}} = \frac{y}{2r\sqrt{r}}$

$\boxed{3}$ let $\underline{F} = r^3\underline{r}$ Find $\text{div}\,\underline{F}$

$\underline{F} = r^3 \langle x,y,z\rangle = xr^3\underline{i} + yr^3\underline{j} + zr^3\underline{k}$

$\text{div}\,\underline{F} = \frac{d(xr^3)}{dx} + \frac{d(yr^3)}{dy} + \frac{d(zr^3)}{dz}$

$y.3r^2.r' + r^3$ $\quad z.3r^2.r' + r^3$

$x.3r^2.r' + r^3$ $\quad 3y^2r + r^3$ $\quad \underline{3z^2r + r^3}$

$x.3r^2.\frac{x}{r} + r^3$

$\text{div}\,F = 3x^2r + 3y^2r + 3z^2r + 3r^3 = 3r\underbrace{(x^2+y^2+z^2)}_{r^2} + 3r^3 = \underline{\underline{6r^3}}$

H.w

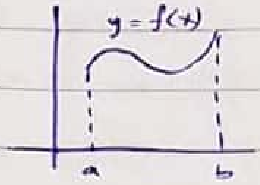
$\overline{F} = e^r.\underline{r}$ find div F

$\text{div}(e^r.\underline{r}) = \text{div}(xe^r\underline{i} + ye^r\underline{j} + ze^r\underline{k})$

$xe^r.r' + e^r + ye^r.r' + e^r + ze^r.r' + e^r$

$3e^r + e^r\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right) = 3e^r + \frac{e^r}{r}(x^2+y^2+z^2) = 3e^r + e^r.r$

* Arc Length



$$L = \int_a^b \underbrace{ds}_{\text{وحدة طول}} \quad ; \text{where} \quad ds = \sqrt{1 + f'(x)^2}\, dx$$

* Parametric curves.

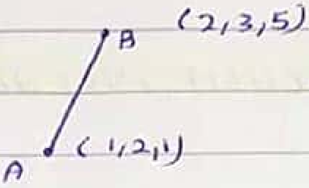
line.

$\underset{\text{position vector}}{A} + \underset{\text{tangent}}{(B-A)}t$

$\underset{\sim\sim\sim}{0 \leq t \leq 1}$

اذا طلبنا عكس طريقه

لازم ال t بس نغير الاتجاه.



$\rightarrow \langle 1,2,1 \rangle + \langle 1,1,4 \rangle t$

$\langle 1+t,\ 2+t,\ 1+4t \rangle$

&

$x(t) = 1+t$

$y(t) = 2+t$

$z(t) = 1+4t$

$0 \leq t \leq 1$

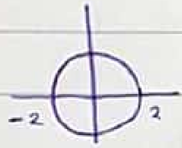
"parametrization"

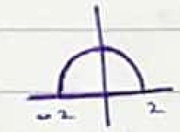
كتابة كل المتغيرات بدلالة

متغير واحد.

circle.

$x^2+y^2=4$





$x(t) = 2\cos t$

$y(t) = 2\sin t$

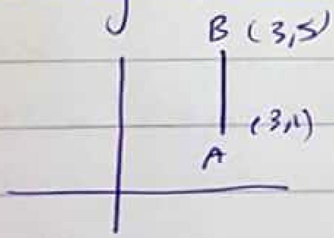
$0 \leq t \leq 2\pi$

$x(t) = 2\cos t$

$y(t) = 2\sin t$

$0 \leq t \leq \pi$

Another way for line.



$x(t) = 3$

$y(t) = 1+4t$

$0 \leq t \leq 1$

الطريقة العادية.

$x(t) = 3$

$y(t) = t$

$1 \leq t \leq 5$

* ch.10

Vector integral , sec. 10.1 → line integral

$\int_a^b f(x)\,dx$, $r(x) = \langle x(t), y(t), z(t) \rangle$

$r(x) = x(t)\,i + y(t)\,j + z(t)\,k$

$$\int_{C_{curve}} F(r)\,dr = \int_a^b F(r(t)) . r'(t)\,dt$$

~~does~~ closed curve → $\oint$

Example.

$F(r) = \langle -y, -xy \rangle$

$r(t) : \langle \cos t, \sin t \rangle \quad 0 \leq t \leq 2\pi$

→ $F(r(t)) = \langle -\sin t, -\cos t \sin t \rangle$

→ $r'(t) = \langle -\sin t, \cos t \rangle$

$$\int_0^{2\pi} \langle -\sin t, -\cos t \sin t \rangle . \langle -\sin t, \cos t \rangle\,dt$$

dot product.

$$= \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)\Big]_0^{2\pi} = 0$$

* $\int KF\,dr = k\int F\,dr$.
* $\int F + G\,dr = \int F\,dr + \int G\,dr$.
* $\int F\,dr = \int_{\underline{\underline{c_1}}} F\,dr + \int_{\underline{\underline{c_2}}} F\,dr$

تقسيم ال curve.

Example.

$F \langle x+y, z+y, z+x \rangle$

$c : r = \langle 4\cos t, \sin t, 0 \rangle$

$0 \leq t \leq \pi$

→ $F(r(t)) = \langle 4\cos t + \sin t, \sin t, 4\cos t \rangle$

$r'(t) = \langle -4\sin t, \cos t, 0 \rangle$

$$\int_C F(r(t))\,dt = \int_0^{\pi} F(r(t)) . r'(t)\,dt .$$

$F = \langle y^2, -x^2 \rangle$

$c : y = 4x^2$

$(0,0) \longrightarrow (1,4)$

→ solution

$x = \cos t$? لا نستخدم

$y = \sin t$

$r(t) = \langle \sin t , 4\cos^2 t \rangle$

$1 = \cos t \rightarrow t = 2\pi$

$0 \leq t \leq 2\pi$

$$\int_0^{2\pi} \langle \sin^2 t , -\cos^2 t \rangle . \langle \cos t , -8\cos t \sin t \rangle .$$

example.

$F = \langle x^2, z^2, x^2$

$r = \langle 3\cos t, 3\sin t, 2t \rangle \qquad \longrightarrow F(r(t))$

$0 \leq t \leq 4\pi \qquad \longrightarrow \int F(r(t)) . r'(t)\, dt .$

نظرية

- * Th direction - proper preserving parametric Transformatio
- * any representation of curve that give you the same value of the line

Example.

$F(t) = \langle xy , yz , z \rangle$

$r(t) = \langle \cos t , \sin t \rangle$

$0 \leq t \leq 2\pi$.

$F(r(t)) = \langle \cos t + \sin t , 3t \sin t , 3t \rangle$

$r'(t) = \langle -\sin t , \cos t , 3 \rangle$

$$\int_c F(r)\, dr = \int_0^{2\pi} \langle \cos t + \sin t , 3t \sin t , 3t \rangle . \langle -\sin t , \cos t , 3 \rangle\, dt .$$

D.r : Banan.

* Def : The arc length of curve from a to b is:

$$L = \int_a^b \sqrt{r'(t) . r'(t)} \; . dt.$$

$r'(t) = x'(t) i + y'(t) j + z(t) k$

$r'(t) . r'(t) = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$

Ex: find the Arc Length, where : $r(t) = [t, \cosh t]$
from $t=0$ to $t=1$

Solution.

$r'(t) = \langle 1, \sinh t \rangle$

$r'(t). r'(t) = 1 + \sinh^2 t = \cosh^2 t$ $\quad [\cosh^2 t - \sinh^2 t = 1]$

$$L = \int_0^1 \sqrt{\cosh^2 t} \, dt = \sinh t \Big|_0^1 = \sinh 1 - \overset{=0}{\cancel{\sinh 0}} = \underline{\underline{\sinh 1}}$$

Ex: find Arc length of the circular helix with parametric representation $r(t) = \cos t \; i + \sin t \; j + t k$
from points $(1,0,0) \rightarrow (1,0,2\pi)$

Solution. $r(t) = \langle \cos t, \sin t, t \rangle$

$r'(t) = -\sin t \; i + \cos t \; j + 1 \; k = \langle -\sin t, \cos t, 1 \rangle$

$r'(t) \cdot r'(t) = \sin^2 t + \cos^2 t + 1 = \boxed{2}$

$\langle \cos t, \sin t, t \rangle$

points $\langle 1, 0, 0 \rangle \rightarrow \boxed{t=0}$ $\quad \langle 1, 0, 2\pi \rangle \rightarrow t = 2\pi$

$\cos t = 1$ $\quad \sin t = 0$

$0 \qquad 0, \pi, 2\pi ----$

π

2π

$\vdots$

$$\underline{\underline{\text{So}}} \quad L = \int_0^{2\pi} \sqrt{2} \, dt = \underline{\underline{2\sqrt{2}\pi}}$$

* Def: The Arc length function of a curve from a :-

no end point

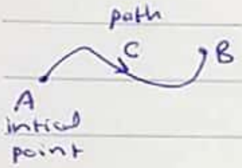
$$s(t) = \int_a^t \sqrt{r'(u) . r'(u)} \; du .$$

" chapter 10 "

" Line Integration "

Def: let $F(t) = F_1(t) i + F_2(t) j + F_3(t) k$ then :-

$$\int_a^b F(t) \, dt = \int_a^b F_1(t) \, dt + \int_a^b F_2(t) \, dt + \int_c^b F_3(t) \, dt .$$



" terminal point '

$$r(t) = \underbrace{x(t) i + y(t) j + z(t) k}_{\text{parametric representation}} \quad , a \le t \le b$$

Def: let $F(r) = F_1(r) i + F_2(r) j + F_3(r) \underline{k}$; be a vector faild.
$F(t) = F(r(t)) = F(x, y, z)$, then the line integral is given

$$\boxed{1} \int_C F(r) \, \underline{\underline{dr}} = \int_{t_0}^{t_1} F(r(t) . \underline{r'(t) . dt} .$$

منحنى ← C
curve.

$$\rightarrow r'(t) = \frac{dr}{dt}$$

$$\boxed{2} \int_C F(r) . dr = \int F_1(x,y,z) \, dx + \int F_2(x,y,z) \, dy + \int F_3(x,y,z) \, dz$$

$\boxed{3}$ where : $dx = x'(t) . dt$, $dy = y'(t) . dt$, $dz = z'(t) . dt$.

$$\int_C F(r) . dr = \int F_1 \, x'(t) . dt + \int F_2 \, y'(t) \, dt + \int F_3 \, z'(t) . dt .$$
$$= \int [F_1 \, x'(t) + F_2 \, y'(t) + F_3 \, z'(t)] . dt .$$

P: Closed bath :

Note: if the bath of integration C is closed curve then we write.

$$\int_C F \cdot dr = \oint F \cdot dr.$$

* Proporities of line integration :-

→ $\int K F \cdot dr = K \int F \cdot dr$

→ $\int F \pm G \; dr = \int F \cdot dr \pm \int G \cdot dr$

→ $\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$

→ $\int_{-C} F \cdot dr = -\int_C F \cdot dr$

→ page lost.

* $r(t) = \cos t \, i + \sin t \, j$

$0 \leq t \leq 2\pi$. "closed path"

$r(0) = r(2\pi)$ so it is closed path.

· we are looking for point zero & 2π not the limit of t

we are care of images not domain "

* find $\int_C F \cdot dr$ where.

$F(r) = [y^2, x]$

ⓐ C_1 : line segment from $\underline{\underline{(-5,-3)}}$ to $\underline{\underline{(0,2)}}$

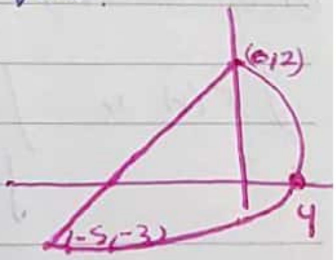
$\searrow \; \swarrow$ $\underline{\underline{r(t)}}$

· I have to write it in parametric form".

$r(t) = r_0 + (r_1 - r_0)t$

$< -5, 3 > + (<0,2> - <-5,-3>)t$ $\quad 0 \leq t \leq 1$

$< -5 + 5t , -3 + 5t >$ ....



ⓑ C_2: arc ~~Form~~ of the parabola $x = 4 - y^2$ from $(-5,3)$ to $(0,2)$

ⓑ C_2 :

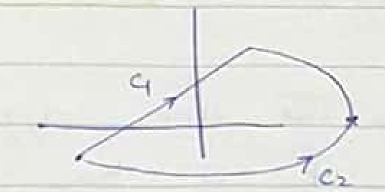
$x = 4 - y^2$

$(-5, -3) \rightarrow (0, 2)$

$\rightarrow y = t$

$x = 4 - t^2 \quad -3 \leq t \leq 2$

$\int_c F. dr = \int_{t_0}^{t_1} F. r' dt.$



$r(t) = x(t) i + y(t) j$

$(4-t^2) i + t j \quad ; -3 \leq t \leq 2$

$F(r(t)) = t^2 i + 4-t^2 j \quad ; \quad r'(t) = -2t i + j.$

$$\int_c F. dr = \int_{-3}^{2} (-2t^3 + 4 - t^2)\, dt = \frac{245}{6}$$

P.S

line integral depends on → F
→ r
→ path. (path dependents)

H.W

Find $\int_c y\, dx + z\, dy + dz$

when c consider of the line segment from (2,0,0) to (3,4,5) followed by the vertical line segment c_2 from (3,4,5) to (3,4,0)

10.2 path independance of line integral.

*. The line integral $\int F\cdot dr$ is path independant if it has same value of all path in a domain.

* Theorems.

Thm. [1] line integral $\int_C F\cdot dr$ where F_1, F_2, F_3 are continuous is path independant if and only if F is the gradiant of some function (f)

$$\boxed{F = \nabla\cdot f} \rightarrow \text{potential}$$

Conservative field

then. $\int_A^B F\cdot dr = f(B) - f(A)$ for any path. (c).

Ex.

show that: $\int_C F\cdot dr = \int 3x^2\,dx + 2yz\,dy + y^2\,dz$ is path independance.

that find $\int_{(0,1,2)}^{(1,-1,7)} F\cdot dr$.

p.s

$F = F_1 i + F_2 j + F_3 k$

$\nabla f = \frac{df}{dx} i + \frac{\partial f}{\partial y} j + \frac{df}{\partial z} k$

→ solution.

$F = 3x^2 i + 2yz j + y^2 k = \nabla f$.

$3x^2 = \frac{df}{\partial x}$ $\quad 2yz = \frac{df}{\partial y}$ $\quad y^2 = \frac{df}{\partial z}$.

$\int 3x^2\,dx = \int \partial f$

$$\boxed{x^3 + g(y,z) = f(x,y,z)}$$

$\to x^3 + g(y,z)$

$\frac{df}{dy} = 2yz$.

$0 + \frac{dg(y,z)}{dy} = 2yz \to \int dg = \int 2yz\, dy$

$g(y,z) = y^2 z + h(z)$

So

$f(x,y,z) = x^3 + y^2 z + \underline{h(z)}$

$\frac{df}{dz} = y^2$

$0 + \cancel{y^2} + h'(z) = \cancel{y^2}$

$h'(z) = 0$

$\int dh(z) \quad \int 0\, dz$

$h(z) = \text{constant} = \text{Zero}$

$f(x,y,z) = x^3 + y^2 z$. So, the line integral is independant

لانه طلع عندي f

So

$$\int_{(0,1,2)}^{(1,-1,7)} F \cdot dr = f(1,-1,7) - f(0,1,2)$$

← بغض النظر عن ال path

P.S → IF $\int F \cdot dr$ is path independent & $A = B$

then $\int_A^B F \cdot dr = \text{Zero}$.

+ ~~Exa~~

Ex: Find $\int_C 3x^2\, dx + 2yz\, dy + y^2\, dz$.

$C: r(t) = [\cos t, \sin t, 0] \quad 0 \le t \le 2\pi$

$r(0) = (1,0,0) \quad r(2\pi) = (1,0,0)$

from the previoos exp.

F is line independent & $A = B$ so $\int_C F \cdot dr = \underline{\text{Zero}}$.

Thm 2 The line integral $\int F.dr$ is path independent if and only if $\oint_c F.dr$ = zero for any closed path of c.

→ Note:

let $F = F_1 i + F_2 j + F_3 k = \nabla f$.

$F.dr = F_1 dx + F_2 dy + F_3 . dz$.

$$df = \frac{df}{dx} . dx + \frac{df}{dy} . dy + \frac{df}{\partial z} . \partial z$$

this form is Called exact differential form if $F.dr = df$.

→ The line integral is path independant if and only if the differential form has condiuns coefficent F_1, F_2, F_3 is exact in D.

Thm 3

let F_1, F_2, F_3 be Continuous and have cont. first partial derivative in D.

$F = [F_1, F_2, F_3]$, then

if $F.dr$ is exact in D ($\int F.dr$ path independent)

then:

$$\boxed{Curl \vec{F} = \vec{0}}$$

* الطريقة غير صحيحة بالدكاترين

إذا مباشرة :-

* $Curl \vec{F} = \vec{0}$

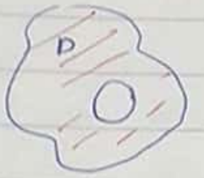
* and domain D is simply connected

→ then

$\int F.dr$ is $\underset{\Leftrightarrow}{\text{exact}}$ ($\int F.dr$ is path indep.)

Domain.

→ simply connected.



→ multiple connected

→ not connected

~~Method $\boxed{I}$~~

~~" find f "~~

Ex.

Show that the differential form under the line integral.

$$I = \int_c 2xyz^2\, dx + (x^2 z^2 + z\cos(yz))\, dy + 2x^2yz + y\cos(yx)\, dz.$$

is exact, then find

$$I = \int_{(0,0,1)}^{(1,\pi/4,2)}$$

Solution.

Method 1

" find f "

$$\frac{df}{\partial x} = 2xyz^2 \longrightarrow \frac{df}{\partial y} = x^2 \cancel{y} z^2 + z\cos(yz)$$

$$f = x^2yz^2 + g(y,z) \qquad \cancel{x^2 z^2} + \frac{dg}{\partial y} = \cancel{x^2 z^2} + z\cos(yz)$$

$$g(y,z) = \sin(yz) + h(z)$$

$f = x^2 y z^2 + \sin(yz) + h(z)$

$\frac{\partial f}{\partial z} = 2\cancel{x^3 y}z + \cancel{y\cos}(yz) + h'(z) = 2x^2\cancel{yz} + y\cancel{\cos}(yz)$

$h'(z) = 0$

$h(z) = 0$

$f(x,y,z) = 2x^2 y z + y\cos(yz)$

F. dr is exact ~~[illegible]~~. so, $\int$ F. dr is path indep.

Method 2

Since the domain ~~[illegible]~~ apply connected.

then find curl $\underline{F}$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z. \\ F_1 & F_2 & F_3 \end{vmatrix} = \underline{\underline{0}}$$

zero بدم تطلع

بس ما نحسب

$\underline{\underline{\text{then}}}$. $f(1, \frac{\pi}{2}, 2) - f(0,0,1) = \underline{\underline{\pi + 1}}$

Ex.

let $F = [F_1, F_2, F_3]$

$F_1 = \frac{-y}{x^2+y^2}$; $F_2 = \frac{x}{x^2+y^2}$, $F_3 = 0$.

let the domain (D) = $\frac{1}{2} < \sqrt{x^2+y^2} < \frac{3}{2}$

Find $\int_c F. dr$; c: $x^2+y^2=1$

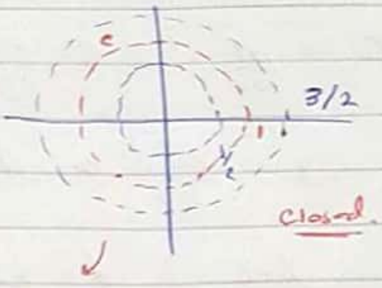
closed

$$\frac{dF_1}{dy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{dF_2}{dx} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ F_1 & F_2 & 0 \end{vmatrix}$$

$$\text{curl } F = \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) = \underline{\underline{\text{Zero}}}$$



Closed.

not simply
Connected.

← لو طلعت ال curl F ←
Zero لا يعني معي
دكتنا غلط

* ~~Bdk~~

* But since D is not simply Connect we can't say that $\oint F.\,dr = \text{Zero}$.

→ So, we must calculate $\int_c F.\,dr$.

Solution.

$r = \cos t\ i + \sin t\ j$

$r' = -\sin t\ i + \cos t\ j$

$$\int_{2\pi} \frac{-y}{x^2+y^2}\,i + \frac{x}{x^2+y^2}\,j$$

$$= \int_{2\pi} \left(\frac{-\sin t}{\sin^2 t + \cos^2 t}(-\sin t) + \frac{\cos t}{\sin^2 t + \cos^2 t}\cdot \cos t\right) dt.$$

Ex. find

$$\int_c y\,dx + (x+z)\,dy + y\,dz.$$

when C:

$$r(t) = \frac{t^2+1}{t^2-1}\,i + \cos(\pi t)\,j + 2t\sin(\pi t)\,k \quad , \quad 0 \leqslant t \leqslant \frac{1}{2}$$

بما انه C مغلقة ودي احلها بالتكامل

بيصح لـ $F = \nabla . f.$

$F_1 = \frac{\partial f}{\partial x}$, $F_2 = \frac{\partial f}{\partial y}$, $F_3 = \frac{\partial f}{\partial z}$

(1) $y = \frac{\partial f}{\partial x} \Rightarrow f(x,y,z) = yx + g(y,z)$

(2) $\cancel{x} + z = \cancel{x} + \frac{\partial g}{\partial y}$

$g(y,z) = zy + h(z)$

(3) $\cancel{y} = 0 + \cancel{y} + h'(z) \rightarrow h'(z) = 0 \rightarrow h(z) = 0$

$f(x,y,z) = xy + yz$

so $\rightarrow \int_C y\,dx + (x+z)dy + y\,dz = f(B) - f(A)$

$t \rightarrow 0 \rightarrow A\ (-1,1,0)$

$t \rightarrow 1/2 \rightarrow B\ (-\frac{5}{3}, 0, 1)$

$= f(-\frac{5}{3}, 0, 1) - f(-1,1,0) = 0 - -1 = \boxed{1}$

HOMEWORK

$\int_C F \cdot dr$ where $F = \left[\frac{x}{1+x^2+y^2+z^2} , \frac{y}{1+x^2+y^2+z^2} , \frac{z}{1+x^2+y^2+z^2} \right]$

where $C: r(t) = [t, t^2, t^4]$ $0 \leq t \leq 1$

solution

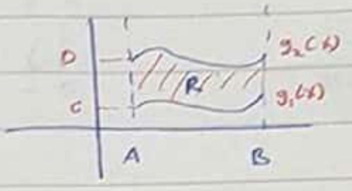
$f = \frac{1}{2}\ln(x^2+y^2+z^2+1)$

10.4 Double integral

$$\iint_R f(x,y)\ dA.$$

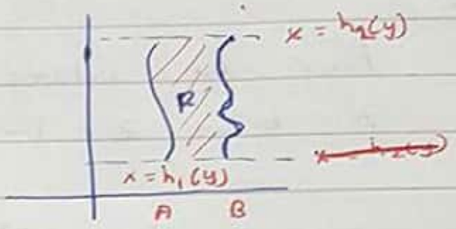
#Type I

$$\iint_R f(x,y) . dA = \int_C^D \int_{g_1(x)}^{g_2(x)} f(x,y)\ dx\ dy$$



Type II

$$\iint_R f(x,y)\ dA = \int_A^B \int_{h_1(y)}^{h_2(y)} f(x,y)\ dy . dx$$



* Polar Coordinate :-

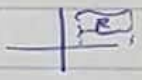
$x = r\cos\theta$, $y = r\sin\theta$

$r^2 = x^2 + y^2$

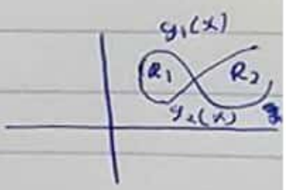
$$\iint_R f(x,y) . dA = \int_{\theta_1}^{\theta_2} \int_{\alpha(\theta)}^{\beta(\theta)} f(r\cos\theta, r\sin\theta) . r . dr . d\theta .$$

+ Applications:

[1] $\iint_R$ --- . dA "Area of region R"

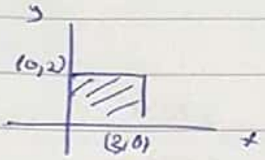


[2] $\iint_R f(x,y) . dA$ = "Volume of $z = f(x,y) > 0$ & above region R in the xy-plane."



Ex. 9. p. 432.

Find the volume of the region $z = 4x^2 + 9y^2$ and above the ractangle with vertices $(0,0)$, $(3,0)$, $(3,2)$, $(0,2)$ in the xy-plane.



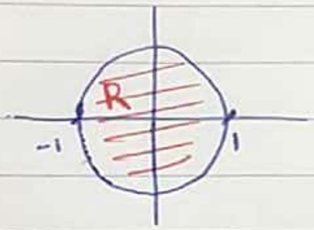
$$\int_0^2\int_0^3 (4x^2 + 9y^2)\, dx \cdot dy \quad = \underline{\underline{144}}.$$

Ex.

Find the volume of the region above the xy-plane & belwo $z = 1 - (x^2 + y^2)$ $\longrightarrow$ So $z = 0$

لأنه بده التقاطع
مع ال xy-plane

$$0 = 1 - (x^2 + y^2)$$



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 - (x^2 + y^2)\ dy \cdot dx.$$

$$y^2 = 1 - x^2$$
$$y = \pm\sqrt{1 - x^2}$$

$\downarrow$ OR in <u>polar</u>

$$\iint_R 1 - (x^2 + y^2)\ dx \cdot dy.$$

$$\int_0^{2\pi}\int_0^1 (1 - r^2) \cdot r \cdot dr \cdot d\theta \,. \qquad \#$$

Sec. 10.4.

* Green's Thm :-

(transphormation between double integral & line integral).

Thm :-

let R be a closed bounded region in the xy-plane boundary c. ~~Const~~ consist of finite many smoth Curves, let $F_1(x,y)$, $F_2(x,y)$ be functions that are continuous & have continuous partial derevative $\frac{dF_1}{dy}$, $\frac{dF_2}{dx}$ everywhere in domain containing R, then :-

$$\iint_R \left(\frac{dF_2}{dy} - \frac{dF_1}{dx} \right) dx\, dy = \oint F_1\, dx + F_2\, dy.$$

Ex.

$F_1 = y^2 + 7x$, $F_2 = 2xy + x$.

$C: x^2 + y^2 = 1$, $D: [0, 2\pi]$

Solu :-

$x = \cos t$, $y = \sin t$.

$r(t) = \cos t\ i + \sin t\ j$

$r'(t) = -\sin t\ i + \cos t\ j$

$$\oint (F_1 x' + F_2 y').\, dt.$$

$$= \int_0^{2\pi} [(\sin^2 t - 7\sin t).\sin t + (2\cos t.\sin t + \cos t).\cos t].\, dt.$$

$$= \int_0^{2\pi} -\sin^3 t - 7\sin t + 2\cos^2 t \sin t + \cos^2 t\ .\, dt = \underline{\underline{9\pi}}$$

Solu 2 :

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy, \quad \frac{dF_2}{dx} = 2y + 1, \quad \frac{dF_1}{dy} = 2y - 7.$$

$$\iint_R (2y+1) - (2y-7).\ dx\, dy = \underline{\underline{9\pi}}$$

Ex. $\oint xy \cdot dx + x^2y^3 \cdot dy$ $\rightarrow$ $F_1 = xy$, $F_2 = x^2y3$

R (0,0) , (1,0) , (1,2)

$\frac{dF_1}{dy} = x$, $\frac{dF_2}{dx} = 2xy^3$

$$\int_0^{2x}\int_0^{1} \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) \cdot dx\,dy = \frac{(2x)^4}{4} - \frac{(2x)^2}{4}$$

Ex.

$F = [6y^2 , 2x - 2y^4]$

R : square with vertices ± (2,2) , ∓ (2,-2)

Solu :-

$-2 \leqslant x \leqslant 2$ | $F_1 = 6y^2 \longrightarrow dF_1/dy = 12y$

$-2 \leqslant y \leqslant 2$ | $F_2 = 2x - 2y^2 \rightarrow dF_2/dx = 2$

$$\rightarrow \int_{-2}^{2}\int_{-2}^{2} 2 - 12y \cdot dx \cdot dy = \int_{-2} 2x - 12yx \Big]_{-2}^{2} \cdot dy = \int_{-2}^{2} 8 - 48y \cdot dx$$

$$= 8y - \frac{48}{2}y^2 \Big|_{-2}^{2}$$ #

* Some app. of Green's Thm

(III) Area of a plane as line integral on the boundary.

$F_1 = 0$, $F_2 = x$

$$\oint F(r(t)) \cdot dr = \iint_R 1 \cdot dx \cdot dy$$

$A = \frac{1}{2}\oint x \cdot dy - y\,dx$, $F_1 = -y$; $F_2 = 0$

$$\iint_R - dx\,dy$$

$x = a\,\cos t$, $y = b\,\sin t$, $x' = -a\sin t$, $y' = b\cos t$

~~$A = \frac{1}{2}\int_0^{2\pi}$ acost - bsint (~~ $A = \frac{1}{2}\int_0^{2\pi} a\cos t * b\cos t - b\sin t^{*}(-a\sin t) \cdot dt$

$$A = \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t \cdot dt = \frac{1}{2}\int_0^{2\pi} ab \cdot dt = ab\pi.$$

(III) Area of plane ~~Rea~~ Region on polar coordinate.

$x = r\cos\theta \longrightarrow x' = \cos\theta . dr - r\sin\theta . d\theta .$

$y = r\sin\theta \longrightarrow y' = \sin\theta . dr + r\cos\theta . d\theta .$

$A = \frac{1}{2} \oint r\cos\theta \left(\sin\theta . dr + r\cos\theta . d\theta \right) - r\sin\theta \left(\cos\theta . dr - r\sin\theta . d\theta \right)$

$A = \frac{1}{2} \oint \cancel{r\cos\theta\sin\theta . dr} + r^2\cos^2\theta . d\theta - \cancel{r\sin\theta\cos\theta . dr} + r^2\sin^2\theta . d\theta .$

$A = \frac{1}{2} \oint r^2\cos^2\theta \, d\theta + r^2\sin^2\theta \, d\theta$

$= \frac{1}{2} \oint r^2 \left(\cos^2 t + \sin^2 t \right) . d\theta$

$A = \frac{1}{2} \oint r^2 . d\theta$

in polar

*

$r = a(1 - \cos\theta) \quad , \quad 0 \leqslant \theta \leqslant 2\pi$

$A = \frac{1}{2} \int_0^{2\pi} a^2 \left(1 - 2\cos\theta + \cos^2\theta \right) . d\theta$

$= \frac{1}{2} \int_0^{2\pi} a^2 (1 - 2\cos\theta + \frac{1}{2} (1 + \cos 2\theta) . d\theta$

(III) transphormation od double integral of laplacian of a function into a line integral of its normal derivative.

$w(x,y) \Rightarrow \quad F_1 = dw/dy \quad , \quad F_2 = dw/dx \quad , \quad$ cont. on R.

$\frac{dF_1}{dy} = \frac{d^2w}{dy^2} \quad , \quad \frac{dF_2}{dx} = \frac{d^2w}{dx^2}$

$\oint F_1 \, dx + F_2 \, dy$

$\oint \left(F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds} \right) . ds = \oint \left(-\frac{dw}{dy} . \frac{dx}{ds} + \frac{dw}{dx} . \frac{dy}{ds} \right) . ds$

$\text{grad}(w) . \underset{\swarrow}{n} = \left[\frac{dw}{dx} , \frac{dw}{dy} \right] . \left[\frac{dy}{ds} , \frac{-dx}{ds} \right] = \frac{dw}{dx} . \frac{dy}{ds} - \frac{dw}{dy} . \frac{dx}{ds}$

normal

محاضرة الإرشاد

→

* Green's thm.

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) . \underbrace{dx . dy}_{dA} = \oint F_1\, dx + F_2\, dy.$$

OR:

$$\iint_R \text{curl } F . K \; dx . dy = \oint F . dr \quad \text{where :}$$

$$F = F_1 i + F_2 j + 0 K.$$

Ex.

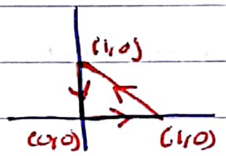
Find $\int_C x^4 . dx + xy . dy.$

where:

C: triangle a consisting line segments from (0,0) to (1,0) Line 1

& from (1,0) to (0,1) → L2

& from (0,1) to (0,0) → L3



رجعنا لنفس النقطة

ولكن لا يوجد

f that F = ∇.f.

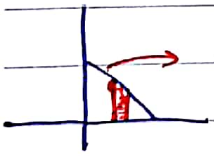
↑ لذلك لا أطبق الـ path. indep.

Solu:-

عايز تكامل على L1 و L2 و L3

ولكن يوجد طريقة أسهل بدل ما

أعمل ٣ تكاملات



$y = 1 - x \;, (x = 1 - y)$

$$\oint_C \underset{F_1}{\boxed{x^4}} . dx + \underset{F_2}{\boxed{xy}} . dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \int_0^1 \int_0^{1-y} (y - 0) . dx . dy = \boxed{\frac{1}{6}}$$

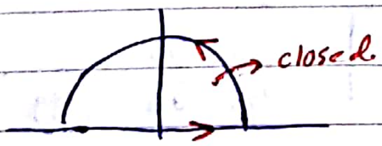
Ex.

ⓐ let $F = [-e^{-x} \cos y \,, -e^{-x} \sin y]$

R : is semidisk $x^2 + y^2 \leq 16$, $x \geq 0$

find :

$\int_C F . dr$. counter clock wise around C.



$F = -e^{-x}\cos y\, i - e^{-x}\sin y\, j$

then

we can find f that $\nabla \cdot f = F$ $(f = e^{-x}\cos y)$

so $\int F \cdot dr$ is path independent

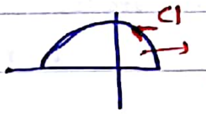
So, $\int F \cdot dr$ = Zero.

ⓑ

Find $\int_{C1} F \cdot dr$ where

C1 : is the upper half of $x^2 + y^2 = 16$

Solu:



not closed.

$\int F \cdot dr.$

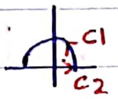
$r = 4\cos$ - - - -

التكاملات رح تكون طويلة

لو ← نفكر بال Green thm

بس ال green لازم تكون

ال C ← closed →



so → خليها انا

closed.

$$\boxed{\int_C F \cdot dr} = \int_{C1} F \cdot dr + \int_{C2} F \cdot dr$$

صرلفنع الثابت ↘ $0 = \int_{C1} F \cdot dr + \int_{C2} F \cdot dr$

$$\int_{C1} F \cdot dr = -\underline{\int_{C2} F \cdot dr}$$

أسهل التكامل

ع ال

line. من التكامل صعبه

$r(t) = r_0 + (r_1 - r_0)t.$

$[-4,0] + [8,0]t$

$r(t) = [-4+8t, 0]$, $0 \leq t \leq 1$

$r'(t) = [8, 0]$

$$= \int_0^1 -e^{-(-4+8t)} \cdot \cos(0)\, i - e^{-(-4+8t)} \sin(0)\, j$$

$= e^4 - e^{-4}$ #

integral

10.5 Surfaces for surface integral

" Surface in xyz − space are
$z = f(x,y)$ or $g(x,y,z) = 0$ ".

* parametric representation of a surface S is
$r(u,v) = [x(u,v), y(u,v), z(u,v)]$

* Tangent plane & surface normal.

→ normal vector of a surface (s) at point (p) is

$$N = \underline{\underline{r_u}} \times \underline{\underline{r_v}} \neq 0$$

$r_u = \frac{dr}{du}$ $r_v = \frac{dr}{dv}$.

unit normal : $n = \frac{N}{|N|}$

اذا ما قدرت اوجد
parametric representation form
IF S represented by
$g(x,y,z) = 0$
$$n = \frac{\nabla g}{|\nabla g|}$$

→ to write plane equation I need
$\boxed{1}$ normal vector $\boxed{2}$ point unit.

vector equation of the plane $= \underline{n}\,(r - \underline{r_0}) = 0$

$r = (x,y,z)$

r_0 → $[x_0, y_0, z_0]$
$P_0(x_0, y_0, z_0)$
(point on plane.)

$n = [a,b,c]$
↓
is normal
(orthogonal)
vector to
the plane.

→ + اذا كان معطيني ٣ نقاط عل ال plane وطلب معادلته
make $\underline{\underline{2}}$ vectors from the 3 points then by
cross product find the normal.

Ex.

find the tangent plane to the surface S with parametric eq.

$x = u^2$, $y = v^2$, $z = u + 2v$; at the point $(1,1,3)$

$r_u = 2u\,i + 0j + k$
$r_v = 0i + 2v\,j + 2k$
} cross product.

ال دة surface التي [illegible] parametric eq represents by $\underline{\underline{2}}$ parameters.

$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,i + 4u\,j + 4uv\,k$$

at point $(1,1,3)$

$x=1 \quad y=1 \quad z=3$

So :

$x = u^2 = 1 \longrightarrow u = \pm 1$

$y = v^2 = 1 \longrightarrow v = \pm 1$

$z = u + 2v$

$u=1$	$v=1$	$z=3$	✓
$u=-1$	$v=-1$	$z\neq 3$	✗
$u=-1$	$v=1$	$z\neq 3$	✗
$u=1$	$v=-1$	$z\neq 3$	✗

So, Normal vector

~~$N = (2i, 4j$~~

$N = -2i - 4j + 4k$.

* tangent plane eq is:

$\langle -2,-4,4\rangle \cdot \langle x-1, y-1, z-3\rangle = 0$

$-2x - 4y + 4z - 2 = 0$.

parametric eq of plane is:-

~~$0 = a(x - x_0) + b(y - y_0) + c(z - z_0)$~~

~~$ax + by + cz + d$~~ $= 0$

$r(u,v)$

$r(u,v) = [u, v, -\frac{1}{c}(au + bv + d)]$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$ax+by+cz+d=0$

$z = \text{function}(x,y)$

$z = \frac{(-ax-by-d)}{c}$, let $x=u$, $y=v$

$z = \frac{-au-bv-d}{c}$, so parametric eq of plane is.

so $[u, v, \frac{-1}{c}(au+bv+d)]$

~~$r(u,v) = E\, a\cos$~~ V

Parametric eq of surface :-

$x^2+y^2+z^2=a^2$

$r(u,v) = [a\cos V\cos u, a\cos V\sin u, a\sin V]$

→ find unit normal vector of the plane

$g(x,y,z) = x^2+y^2+z^2-a^2$

First way $r_u \times r_v = N$

then $n = \frac{N}{|N|}$

Second way.

$\nabla g = 2xi+2yj+2zk$

$|\nabla g| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{a^2} = 2a$

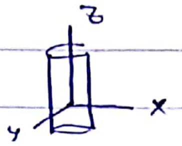
$n = \frac{\nabla g}{|\nabla g|} = \frac{2xi+2yj+2zk}{2a} = \frac{1}{a}(xi+yj+zk)$

[center of (0,0)]

H.W

find parametric representation $z = 4x^2 + y^2$ then find unit normal vector.

* parametric eq of Circular cylinder



$x^2+y^2 = a^2$, $-1 \leq z \leq 1$

$r(u,v) = [a\cos u , a\sin u , v]$ $\quad 0 \leq u \leq 2\pi$

$-1 \leq v \leq 1$

* Find parametric representation of the cylinder

$\frac{x^2}{9} + \frac{z^2}{4} = 1$, $0 \leq y \leq 1$

$x = 3\cos u$

$z = 2\sin u$

$y = v$

$r(u,v) = [3\cos u , 2\sin u , v]$

$0 \leq u \leq 2\pi$

$0 \leq v \leq 1$

* parametric representation of circular cone

$z^2 = x^2 + y^2$

$-a \leq z \leq a$

$r(u,v) = [a\cos v , a\sin v , u]$

Ex → Find unit normal of circular cone

Solu →

$g = x^2+y^2 - z^2$

$\nabla g = 2xi + 2yj - 2zk$

$|\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{z^2+z^2} = 2\sqrt{2}z$

$n = \frac{1}{\sqrt{2}}\left(\frac{x}{2}i + \frac{y}{2}j + \frac{z}{2}k\right)$