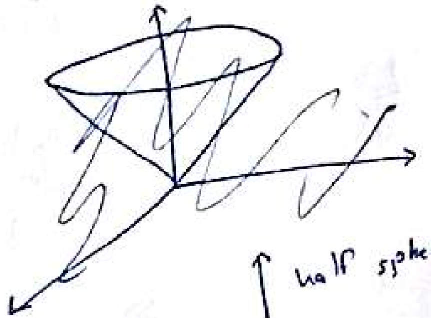
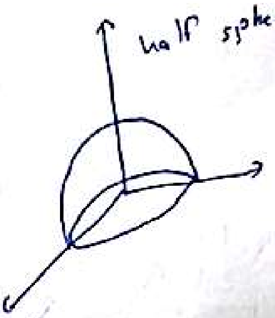
**Q1. (4 points) Use Stokes' theorem to evaluate** $\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, dA$ ,

**where** $\vec{F}(x,y,z) = [3y, 5-2x, z^2-2]$,

**S is the surface of** $z = \sqrt{4-x^2-y^2}$ .






$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix}$$

$[0, 0, -2-3]$

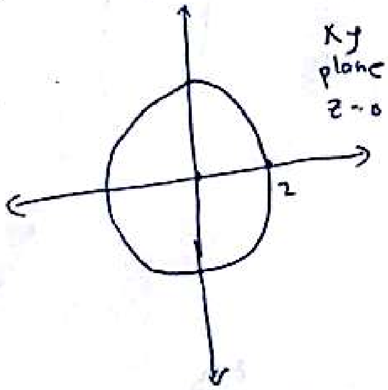
$[0, 0, -5]$

$z^2 + x^2 + y^2 = 4 \qquad z \geq 0$

$\nabla S = [2x, 2y, 2z] = \hat{n}$

$\frac{1}{\sqrt{4(\underbrace{x^2+y^2+z^2}_{4})}} [2x, 2y, 2z] = \hat{n} = [\frac{x}{2}, \frac{y}{2}, \frac{z}{2}]$

proj on xy plane

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$\iint [0,0,-5][\frac{x}{2}, \frac{y}{2}, \frac{z}{2}] \cdot \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$\iint \frac{-5}{2} z \cdot \frac{dx\,dy}{z/2} = \iint -5\, dx\,dy$

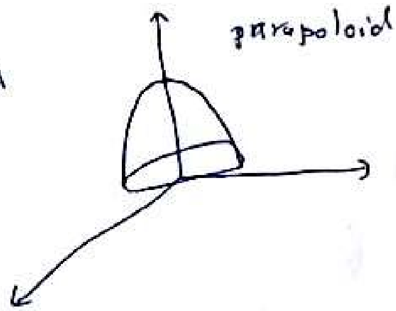
$-5\pi r^2 = -20\pi$

**Q2. (6 points)** Use the divergence theorem to evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dA$,

where $\vec{F}(x,y,z) = [7y^3z^2, -8x^2z^5, -4xy^4]$ and

S is the surface of the paraboloid $z = 9 - x^2 - y^2$, that lies above the plane $z = 1$.

$$\iiint \nabla \cdot (\nabla \times \vec{F}) \cdot dv = \iint_{S_1+S_2} (\nabla \times \vec{F}) \cdot \hat{n}\, dA$$




since $\nabla \cdot (\nabla \times \vec{F}) = 0$

$$0 = \iint_{S_1} \nabla \times \vec{F} \cdot \hat{n}\, dA + \iint_{S_2} \nabla \times \hat{F} \cdot \hat{n}\, dA$$

$$\iint_{S_1} \nabla \times \hat{F} \cdot \hat{n}\, dA = -\iint_{S_2} \nabla \times \vec{F} \cdot \hat{n}\, dA$$

→ now: $\iint_{S_2} \nabla \times \hat{F} \cdot \hat{n}\, dA$

$S_2$ : $\cancel{x^2}$

$1 = 9 - x^2 - y^2$

$x^2 + y^2 = 8$

$\boxed{\hat{n} = -\hat{k}}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 7y^3z^2 & -8x^2z^5 & -4xy^4 \end{vmatrix}$$

$$(-16xy^3 + 8 \times 5\, x^2 z^5)\,\hat{i} - (-4y^4 - 14y^3 z)\,\hat{j} + (-16xz^5 - 21y^2z^2)\,\hat{k}$$

$$\iint \nabla \times \hat{F} \cdot \hat{n}\, dA =$$

$$\iint +16x + 21y^2\, dx\,dy = \int_0^{2\pi}\int_0^{\sqrt{8}} 16r^2\cos\theta + 21r^3\sin^2\theta\, dr\, d\theta$$

$\cancel{16r^2\sin\theta + \frac{21}{2}}$

$$\int_0^{2\pi}\int_0^{\sqrt{8}} 16r^2\cos\theta + 21r^3\sin^2\theta \, dr\, d\theta$$

$$\int_0^{\sqrt{8}}\int_0^{2\pi} \underbrace{\cancel{16r^2\cos\theta}}_{\text{zero}} + \frac{21}{2}r^3\left(1 - \underbrace{\cancel{\cos 2\theta}}_{\text{zero}}\right) d\theta\, dr$$

$$\int_0^{\sqrt{8}}\int_0^{2\pi} \frac{21}{2} r^3 \, d\theta\, dr$$

$$\int_0^{\sqrt{8}} 2\pi \cdot \frac{21}{2} r^3 \, dr$$

$$21\pi \left.\frac{r^4}{4}\right]_0^{\sqrt{8}} = 21\pi \cdot \frac{64}{4}$$

$$16 \times 21 \times \pi$$

$$236\pi$$

$$\begin{array}{r} 16 \\ 21 \times \\ \hline 16 \\ 20 \end{array}$$

**Q3.** (4 points) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ is any constant vector, show that

a) $\iint_S (\vec{a}\times\vec{r})\cdot\hat{n}\,dA = 0$, for any closed surface S,

b) $\oint_C (\vec{a}\times\vec{r})\cdot d\vec{r} = 2\iint_S \vec{a}\cdot\hat{n}\,dA$, where S is any open surface with boundary C.

a) $\iint_S (\vec{a}\times\vec{r})\cdot\hat{n}\,dA = \iiint \nabla\cdot(\vec{a}\times\vec{r})\,dV$

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ x & y & z \end{vmatrix}$$

$[(a_y z - a_z y)\hat{i},\ (-a_x z + a_z x)\hat{j},\ a_x y - a_y x]$

so $\nabla\cdot(\vec{a}\times\vec{r}) = \text{zero}$

$\longrightarrow \iiint 0\, dV = \text{zero}$ #

b) $\oint \vec{a}\times\vec{r}\cdot d\vec{r} = \iint \nabla\times(\vec{a}\times\vec{r})\cdot\hat{n}\,dA$

$$\nabla\times(\vec{a}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_y)z-(a_z)y & (a_z)x-(a_x)z & (a_x)y-(a_y)x \end{vmatrix}$$

$(ax + ax)\hat{i} - (-a\hat{y} - a\hat{y})\hat{j} + (az - a\hat{z})\hat{k}$

$= 2\vec{a}$

$\iint \nabla\times(\vec{a}\times\vec{r})\cdot\hat{n}\,dA$

$= 2\iint \vec{a}\cdot\hat{n}\,dA$ #

4

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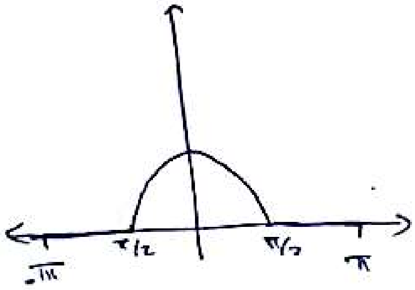
**Q4.(6 points)** Let $f(x)=\begin{cases}\cos x, & 0<x<\pi/2,\\ 0, & \pi/2<x<\pi.\end{cases}$

a) Show that the Fourier cosine series for $f(x)$ is

$$f(x)\approx\frac{1}{\pi}+\frac{1}{2}\cos x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos 2nx,$$

b) Show that $\sum\limits_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}$.




الحل على
الصفحة ←
المقابلة

Cosine series $\rightarrow$ $b_n=0$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}$$

1) $L=\pi$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\cos x\,dx$$

$$\frac{2}{\pi}\left[+\sin x\right]_0^{\pi/2}=\frac{2}{\pi}\sin\pi/2-0$$

$$\boxed{\frac{2}{\pi}}$$

$\frac{2}{\pi}\sin\frac{n\pi}{2}=\begin{cases}0 & n=\text{even}\\ -1/1 & n\text{ odd}\end{cases}$

4

$$a_n=\frac{2}{\pi}\int_0^{\pi/2}\cos x\cos\frac{n\pi x}{L}\,dx$$

$$\frac{2}{\pi}\int_0^{\pi/2}\cos x\cos nx\,dx$$

$$\frac{1}{\pi}\int_0^{\pi/2}\cos(1-n)x+\cos(1+n)x\,dx$$

$$\frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n}+\frac{\sin(1+n)x}{1+n}\right]_0^{\pi/2}$$

$$\frac{1}{\pi}\left(\frac{\sin(1-n)\pi/2}{1-n}+\frac{\sin(1+n)\pi/2}{1+n}\right)$$

$$\frac{1}{\pi}\left(\frac{\sin\pi/2\cos n\pi/2+\cos\pi/2\underbrace{\sin(-n\pi/2)}_{\text{zero}}}{1-n}+\right.$$

$+\sin\pi/2\cos n\pi/2$
$+$

$\sin 2x=2\sin x\cos x$
$\sin\pi/2\cos n\pi/2$
$+\cos\pi/2\cdot\sin -n\pi/2$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{L}$$

cosine series

$b_n = 0$

$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

$a_0 = \frac{2}{\pi}\int_0^{\pi/2} \cos x\,dx$

$\frac{2}{\pi}\Big[+\sin x\Big]_0^{\pi/2}$

$\frac{2}{\pi}(\sin\pi/2 - \sin 0) = \frac{2}{\pi}$

$a_n = \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos\frac{n\pi x}{\pi}\,dx$

$\frac{1}{\pi}\int_0^{\pi/2} \cos(1-n)x + \cos(1+n)x\,dx$

for $n \neq 1$

$\frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n} + \frac{\sin(1+n)x}{1+n}\right]_0^{\pi/2}$

$\frac{1}{\pi}\left[\frac{\sin(1-n)\pi/2}{1-n} + \frac{\sin(1+n)\pi/2}{1+n}\right]$

$\rightarrow \frac{1}{\pi}\left[\frac{\overset{1}{\sin\pi/2}\cos n\pi/2 - \sin n\pi/2\,\overset{zero}{\cancel{\cos\pi/2}}}{1-n} + \frac{\overset{1}{\sin\pi/2}\cos n\pi/2 + \sin n\pi/2\,\overset{zero}{\cancel{\cos\pi/2}}}{1+n}\right]$

$\frac{1}{\pi}\left[\frac{\cos n\pi/2}{1-n} + \frac{\cos n\pi/2}{1+n}\right]$

$\frac{1}{\pi}\left[\frac{\cos n\pi/2 + n\cancel{\cos}\,n\pi/2 + \cos n\pi/2 - n\cancel{\sin}\,n\pi/2}{1-n^2}\right]$

$\frac{2}{\pi}\left[\frac{2\cos n\pi/2}{1-n^2}\right] = \frac{2}{\pi}\begin{cases} 0 & n \text{ odd} \\ \left(\frac{2}{1-n^2}\right) / \frac{-2}{1-n^2} & n \text{ even}\end{cases}$

$\boxed{f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1-(2n)^2}\cos 2nx}$

$f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n^2-1)}\cos 2nx$

$n = 1$

$\int_0^{\pi/2} \cos^2 x\,dx$

$\int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2x\,dx$

$\left[\frac{\pi}{2} + \frac{1}{2}\sin 2x\right]_0^{\pi/2}$

$+\left[\frac{1}{2\pi}(\cos\pi - 1)\right]$

$= \frac{-2}{2\pi}$

$\boxed{\frac{1}{\pi} + \frac{1}{\pi}}$

for $n = 1$

$\frac{2}{\pi}\int_0^{\pi/2} \cos x \cos x$

$\int_0^{\pi/2} \cos^2 x\,dx$

$\int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2x$

$\frac{\pi}{4} + \frac{1}{4}\sin 2x$

$\frac{2}{\pi}\left(\frac{\pi}{4}\right)$

$\frac{\pi}{4}$

Q4.(6

حل فرع b آخر صفحة

**You can use formulas**

$$\sin A\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

Q4 b)

since $f(x) = \frac{1}{\pi} + \frac{1}{2}\cos x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2+1}\cos 2nx$

let $x = \pi/2$

$$\frac{f(\pi/2^+) + f(\pi/2^-)}{2} = 0$$

$$0 = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot(-1)^n}{4n^2-1}$$

$(-1)^{n+1}\cdot(-1)^n = -1$

$$-\frac{1}{\cancel{\pi}} = -\frac{2}{\cancel{\pi}}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

$$\boxed{\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2+1}}$$ #

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Q5.(5 points) Let $f(x)=\begin{cases}0, & x<0,\\ 1, & 0<x<2,\\ 0, & x>2.\end{cases}$

a) Show that $f(x)\approx\frac{1}{\pi}\int_0^\infty \frac{\sin[w(2-x)]+\sin wx}{w}\,dw$

b) Show that $\int_0^\infty \frac{\sin w}{w}\,dw=\frac{\pi}{2}$.

$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx$

$\frac{1}{\pi}\int_0^2 \cos\omega x\,dx = \frac{1}{\omega\pi}\sin\omega x\Big]_0^2$

$\frac{1}{\pi\omega}\sin 2\omega - 0$

$B(\omega)=\frac{1}{\pi}\int_0^2 \sin\omega x\,dx = \frac{-1}{\pi\omega}\cos\omega x\Big]_0^2$

$\frac{-1}{\pi\omega}(\cos 2\omega - 1)$

$f(x)=\int_0^\infty (A(\omega)\cos\omega x + B(\omega)\sin\omega x)\,d\omega$

$=\frac{1}{\pi}\int_0^\infty \frac{\sin 2\omega\cos\omega x - (\cos 2\omega + 1)\sin\omega x}{\omega}\,d\omega$

$\frac{1}{\pi}\int_0^\infty \left(\frac{\sin 2\omega\cos\omega x - \cos 2\omega\sin\omega x + \sin\omega x}{\omega}\right)d\omega$

$\frac{1}{\pi}\int_0^\infty \frac{\sin(2-x)\omega + \sin\omega x}{\omega}\,d\omega$ #

b) $\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \pi/2$

let $x=2 \rightarrow \frac{f(2^+)+f(2^-)}{2} = \frac{1+0}{2}=\frac{1}{2}$

$\frac{1}{2}=\frac{1}{\pi}\int_0^\infty \frac{\cancel{\sin 0} + \sin 2\omega}{2\omega}\,d\omega$

let $2\omega = z$, $2d\omega = dz$

$\frac{1}{2}=\frac{1}{\pi}\int_0^\infty \frac{\sin z}{z}\,dz$ #

5 so $\boxed{\frac{\pi}{2}=\int_0^\infty \frac{\sin z}{z}\,dz}$