Q1. (4 points) If $f(x,y,z)$ and $g(x,y,z)$ are scalar functions, then prove

$$\nabla\cdot(f\nabla g)=f\nabla^2 g+\nabla f\cdot\nabla g$$

$$\nabla g = [g_x , g_y , g_z] = [\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}]g$$

$$f\nabla g = [f(g_x), f(g_y), f(g_z)]$$

$$\nabla\cdot(f\nabla g) = [\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}]\cdot[f(g_x), f(g_y), f(g_z)]$$

$$= \left[\frac{\partial(fg_x)}{\partial x} + \frac{\partial(fg_y)}{\partial y} + \frac{\partial(fg_z)}{\partial z}\right]$$

$$\boxed{f g_{xx} + f_x g_x + f_y g_y + f g_{yy} + f g_{zz} + g_z f_z}$$

مشتقه ضرب ←

الاول X مشتقة الثاني

+ الثاني X مشتقة الاول

$$\cancel{f\frac{\partial g_x}{\partial x}}$$

$$= \{F(g_{xx}) + F_x g_x + F(g_{yy}) + f_y g_y + f(g_{zz}) + f_z g_z\}$$

$$= \{F(g_{xx}) + F(g_{yy}) + F(g_{zz})\} + \{F_x g_x + F_y g_y + F_z g_z\}$$

$$F\underbrace{\{g_{xx} + g_{yy} + g_{zz}\}}_{\nabla^2 g} + \underbrace{[F_x , F_y , F_z]}_{\nabla f}\cdot\underbrace{[g_x , g_y , g_z]}_{\nabla g}$$

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$$F\nabla^2 g + \nabla f\cdot\nabla g = \nabla\cdot(f\nabla g) \quad \#$$

where

$$\nabla^2 g = (g_{xx} + g_{yy} + g_{zz})$$

$$\nabla f = [f_x , f_y , f_z]$$

$$\nabla g = [g_x , g_y , g_z]$$

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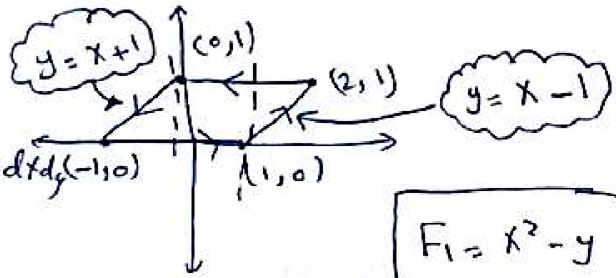
Q2.(5 points) **Evaluate** $\oint_C (x^2 - y)dx + (x + y^3)dy$ **, where C is the parallelogram with vertices (-1,0), (1,0), (2,1), (0,1).**



$$\oint F_1 dx + F_2 dy = \iint \frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\, dxdy$$

$$\boxed{\begin{array}{l} F_1 = x^2 - y \\ F_2 = x + y^3 \end{array}}$$

$$= \iint 1 + -1 \; dxdy$$

~~= zero → path dependent field~~

$f_{1y} = -1$

$f_{2x} = 1$

~~$\phi_x = x^2 - y \rightarrow \phi = x$~~

$$\int_0^1 \int_{y-1}^{y+1} 2\, dx\, dy = \int_0^1 2x \Big]_{y-1}^{y+1} dy$$

$$\int_0^1 2(y+1) - 2(y-1)\, dy$$

$$\int_0^1 \cancel{2y} + 2 \cancel{- 2y} + 2\, dy$$

$$\int_0^1 4\, dy = 4y\Big]_0^1$$

$$= 4$$

5

Q3. (6 points)

a) Show that $\vec{F}(x,y,z) = (2x\sin y + e^{3z})\hat{i} + (x^2\cos y)\hat{j} + (3xe^{3z} + 5)\hat{k}$ is conservative.

b) Find the scalar potential for $\vec{F}(x,y,z)$.

c) Find the work done in moving a particle in the field $\vec{F}(x,y,z)$ from the point (1, 0,0) to the point $(2,\pi,1)$.

فرع ٣ على الصفحة المقابلة

5

a) 1) conseravative field $\rightarrow$ $\nabla \times \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\sin y + e^{3z} & x^2\cos y & 3xe^{3z} + 5 \end{vmatrix}$$

$\cancel{\frac{\partial}{\partial x} x^2\cos y}$ $= \left(\frac{\partial}{\partial y}(3xe^{3z}+5) - \frac{\partial}{\partial z}(x^2\cos y)\right)\hat{i}$

$- \left(\frac{\partial}{\partial x}(3xe^{3z}+5) - \frac{\partial}{\partial z}(2x\sin y + e^{3z})\right)\hat{j}$

$+ \left(\frac{\partial}{\partial x}(x^2\cos y) - \frac{\partial}{\partial y}(2x\sin y + e^{3z})\right)\hat{k}$

$= (0 - 0)\hat{i} - (3e^{3z} - 3e^{3z})\hat{j}$

$+ (2x\cos y - 2x\cos y)\hat{k}$

$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$ #

b) $\phi_x = 2x\sin y + e^{3z} \rightarrow \phi = \frac{2x^2}{2}\sin y + xe^{3z} + g(y,z)$

$\phi_y = x^2\cos y \rightarrow \phi = x^2\sin y + f(x,z)$

$\phi_z = 3xe^{3z} + 5 \rightarrow \phi = 3x\frac{e^{3z}}{3} + 5z + m(x,y)$

$$\boxed{\phi = x^2\sin y + xe^{3z} + 5z}$$

$(0 + 1 + 0) - (0 + 2e^3 + 5)$

$$c)\quad w = \int_{(1,0,0)}^{(2,\pi,1)} f \cdot dr = \phi(1,0,0) - \phi(2,\pi,1)$$

$$= (0 + 1 + 0) - (0, 2e^3, 5)$$

$$1 - 2e^3 - 5$$

$$= \boxed{-2e^3 - 4}$$

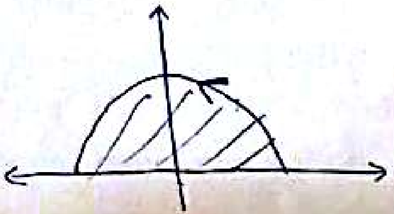
Q4. (4 points) Find the value of $\oint_C \frac{\partial w}{\partial \hat{n}} dt$ taken counterclockwise over the boundary curve C of the region R, where $w(x,y) = x^3 - 3xy^2 + y^4$ and R: $x^2 + y^2 \le 4$, $y \ge 0$.

$$\oint \frac{\partial w}{\partial \hat{n}} dt = \iint \nabla^2 w \, dx\,dy$$

1) $\nabla^2 w = (w_{xx} + w_{yy} + w_{zz})$

$= (3x^2 - 3y^2) + (-6xy + 4y^3) + 0$

$= 3x^2 - 3y^2 - 6xy + 4y^3$



$$\iint \nabla^2 w \, dx\,dy$$

$= \iint (3x^2 - 3y^2 - 6xy + 4y^3)\, dx\,dy$

to polar

$= \iint (3r^2\cos^2\theta - 3r^2\sin^2\theta - 6r^2\cos\theta\sin\theta + 4r^3\sin^3\theta)\, r\,dr\,d\theta$

$\iint 3r^3\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) - 3r^3\left(\frac{1}{2} - \frac{1}{2}\cos\theta\right) - 3r^3\sin 2\theta + 4r^4\sin\theta\,(1-\cos^2\theta) \, .dr\,d\theta$

التكملة في الصفحة المقابلة

$64 \times 2 + 2 = 128$

$16 \times 4 = 64$

$64 \times 3 = 192 - 64$

$192 - 64 = 128$

$= \int_0^{\pi}\int_0 \frac{3r}{2} + \frac{3r}{2}\cos 2\theta - \frac{3r}{2} + \frac{3r}{2}\cos\theta - 3r\sin 2\theta + 4r\sin\theta - 4r\sin\theta\cos^2\theta \; d\theta\,dr$

$\int \left[\frac{3r}{2}\sin 2\theta + \frac{3r}{2}\cos 2\theta + -4r\cos\theta + \frac{4r\cos^3\theta}{3}\right]_0^{\pi} dr$

$\frac{3r}{2}(-1-1) \quad - 4r(-1-1) \quad + \frac{4r(-1-1)}{3}$

$\int 8r + -\frac{8r}{3}\, dr$

$\left[4r^2 - \frac{4r^2}{3}\right]_0^4 = 64 - \frac{64}{3}$

$= \boxed{\frac{128}{3}}$

$$- 3r^3 \sin 2\theta$$

$$\int_0^2\int_0^{\pi} \cancel{\frac{3r^3}{2}} + \frac{3r^3}{2}\cos 2\theta - \cancel{\frac{3r^3}{2}} + \cancel{\frac{3r^3}{2}\cos 2\theta} \; d\theta \, dr$$

$$+ 4r^4 \sin\theta - 4r^4 \sin\theta \cos^2\theta$$

$$\int_0^2 \underbrace{\frac{3r^3}{2}\sin 2\theta}_{\text{zero}} + \underbrace{\frac{3r^3}{2}\cos 2\theta}_{\text{zer}} + 4r^4\cos\theta + 4r^4 \frac{\cos^3\theta}{3} \Big]_0^{\pi} dr$$

$$\int -4r^4(-1-1) + \frac{4r^4}{3}(-1+1)\, dr$$

$$\int_0^2 8r^4 - \frac{8}{3}r^4 \, dr$$

$$\frac{8r^5}{5} - \frac{8}{15}r^5 \Big]_0^2$$

$$8\frac{2^5}{5} - \frac{8}{15}2^5$$

$$2^5\left(\frac{8}{5} - \frac{8}{15}\right)$$

$$\frac{16}{15}\cdot 2^5$$

$$\boxed{\frac{512}{15}}$$

2
4
8

8 − 3×8
(1−3)
8+2

2
4
8
16
32

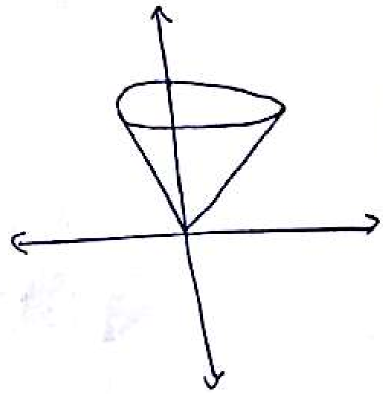
32
16 ×
192
320
512

Q4. (4 point
bou

Q5. (6 points) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dA$, where $\vec{F}(x,y,z) = x\hat{i} + \hat{y}j + z^4\hat{k}$,

S is the part of the cone $z=\sqrt{x^2+y^2}$, $0\le z\le 1$
with downward orientation.

$$r(u,v) = [u\cos v,\ u\sin v,\ u]$$



$$\hat{N} = \vec{r}_v \times \vec{r}_u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -u\sin v & u\cos v & 0 \\ \cos v & \sin v & 1 \end{vmatrix}$$

$$= u\cos v\,\hat{i} + u\sin v\,\hat{j} - u\,\hat{k}$$

$$\iint \vec{F}\cdot\hat{n}\,dA = \iint \vec{F}\cdot\hat{N}\,du\,dv$$

$$= \iint u^2\cos^2 v + u^2\sin^2 v + u^4\cdot(-u)\,du\,dv$$

$$\int_0^{2\pi}\int_0^1 u^2 - u^5\,du\,dv$$

$$\int_0^{2\pi} \frac{u^3}{3} - \frac{u^6}{6}\Big]_0^1\,du\,dv$$

$$\int_0^{2\pi} \frac{1}{3} - \frac{1}{6}\,dv$$

$$2\pi\left(\frac{1}{3}-\frac{1}{6}\right)$$

$$\frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$$

6

$u^2\cos^2 v + u^2\sin^2 v$

$\sqrt{u^2} = u$

$\sqrt{x^2+y^2}$

$\cos u,\ \sin v$