Partial

Fall017

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* Vector Fields:

(ex) let $F: R^2 \to R^3$, be S.T:

* this is vector Field.

$(\hat{i}, \hat{j}, \hat{k}) \leftarrow$ يوجد

$$\underline{F}_{(x,y)} = 2x\hat{i} + xy\hat{j} + 2\hat{k}$$

$$\to \underline{F}_{(1,1)} = 2\hat{i} + \hat{j} + 2\hat{k} = \langle 2, 1, 2 \rangle$$

(ex) Consider $F: R^2 \to R$

$$\underline{F}_{(x,y)} = x^2 + y^2 + 5$$

* this is scaler Field

$$\underline{F}_{(2,1)} = 4 + 1 + 5 = 10$$

(ex) Consider $\phi_{(x,y,z)} = x^2 + y^2 + yz$, Then:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \to \text{(Vector)}$$

∇ → Del operator

$$\nabla\phi = 2x\hat{i} + (2y + z)\hat{j} + y\hat{k}$$

* Consider:

$$\underline{F}_{(x,y,z)} = f_{(x,y,z)}\hat{i} + g_{(x,y,z)}\hat{j} + h_{(x,y,z)}\hat{k}$$

Then the divergence of $\underline{F}$ is given by:

$$\operatorname{div} \underline{F} = \frac{dF}{dx} + \frac{dF}{dy} + \frac{dF}{dz} \to \text{(scaler)}$$

ex) let $\underline{F}_{(x,y,z)} = x^2 y\,\hat{i} + yz\,\hat{j} + x\,\hat{k}$, Then:

$\text{div}\,\underline{F} = 2xy + z + \text{zero} = 2xy + z.$ → (scaler)

→ Remark:

$$\underline{F}_{(x,y,z)} = f_{(x,y,z)}\,\hat{i} + g_{(x,y,z)}\,\hat{j} + h(x,y,z)\,\hat{k}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\rightarrow \quad \underline{\nabla}\cdot\underline{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \text{div}\,\underline{F}$$

* $\underline{\nabla}\cdot\underline{F} = \text{div}\,\underline{F}$

let $\underline{F}$, $\underline{G}$ be a vector field , ϕ be a scaler field.

Then:

1. $\text{div}\,(k\underline{F}) = k\,\text{div}\,\underline{F}$

2. $\text{div}\,(\underline{F} + \underline{G}) = \text{div}\,\underline{F} + \text{div}\,\underline{G}$

3. $\text{div}(\phi\underline{F}) = \phi\,\text{div}\underline{F} + \nabla\phi\cdot\underline{F}$

ex) Derive Rule number 3 → in the next page:

Sol:

let $\underline{F} = f\hat{i} + g\hat{j} + h\hat{k}$, Then:

$$\phi \underline{F} = \phi f\hat{i} + \phi g\hat{j} + \phi h\hat{k}$$

$$\rightarrow \operatorname{div}(\phi \underline{F}) = \frac{\partial}{\partial x}(\phi f) + \frac{\partial}{\partial y}(\phi g) + \frac{\partial}{\partial z}(\phi h)$$

$$= \phi f_x + \phi_x f + \phi g_y + \phi_y g + \phi h_z + \phi_z h$$

$$= \phi(f_x + g_y + h_z) + \nabla\phi \cdot \underline{F}$$

$$= \phi \operatorname{div} \underline{F} + \nabla\phi \cdot \underline{F} \quad \#$$

* let $\underline{F}_{(x,y,z)} = f_{(x,y,z)}\hat{i} + g_{(x,y,z)}\hat{j} + h_{(x,y,z)}\hat{k}$, Then:

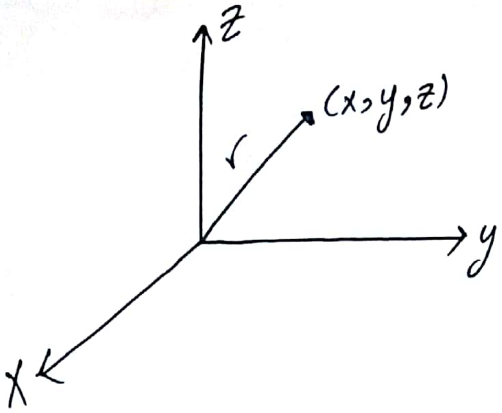
$\operatorname{div} \underline{F} = f_x + g_y + h_z$.

& let $\phi_{(x,y,z)}$ be ascaler vector , Then:

$\nabla\phi = \phi_x \hat{i} + \phi_y \hat{j} + \phi_z \hat{k}$.

* div of avector give a scaler Field.

* ∇ of ascaler give a vector Field.



$$\left.\begin{aligned} \underline{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \underline{r} &= \langle x, y, z \rangle \\ \underline{r} &= (x, y, z) \end{aligned}\right] \rightarrow \text{position vector.}$$

$\rightarrow \|\underline{r}\| = \sqrt{x^2+y^2+z^2}$, (norm) or (magnitude) $\longrightarrow$ scaler

$\rightarrow$ if $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Then:

$r = \sqrt{x^2+y^2+z^2} \longrightarrow$ same $\|\underline{r}\|$

$$\rightarrow \frac{dr}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\rightarrow \frac{dr}{dy} = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$

$$\rightarrow \frac{dr}{dz} = \frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

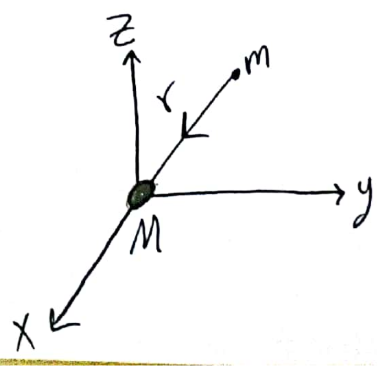
$$* \; f_{(x,y,z)} = \left(\sqrt{x^2+y^2+z^2}\right)^3 \longrightarrow f_{(r)} = r^3$$

Now: $\dfrac{d}{dx}(r^3) = 3r^2 \cdot \dfrac{x}{r} = 3xr$

* Gravitational Field:

$$\underline{F} = -\frac{GMm}{r^2} \cdot \frac{\underline{r}}{r}$$

$$\underline{F} = \frac{C}{r^3} \cdot \underline{r}$$



ex) General case: Consider $\underline{F} = f(r)\,\bar{r}$,

show that: $\operatorname{div} \underline{F} = 3f(r) + r f'(r)$:

<u>sol:</u>

$$\underline{F} = f(r) \cdot x\hat{i} + y\hat{j} + z\hat{k} = x f(r)\,\hat{i} + y f(r)\,\hat{j} + z f(r)\,\hat{k}$$

$$\rightarrow \operatorname{div} \underline{F} = \frac{\partial}{\partial x}(x f(r)) + \frac{\partial}{\partial y}(y f(r)) + \frac{\partial}{\partial z}(z f(r)).$$

$$* \ \frac{\partial}{\partial x}(x f(r)) = x \cdot f'(r) \cdot \frac{x}{r} + f(r) = \frac{x^2 f'(r)}{r} + f(r)$$

$$* \ \frac{\partial}{\partial y}(y f(r)) = y \cdot f'(r) \cdot \frac{y}{r} + f(r) = \frac{y^2 f'(r)}{r} + f(r)$$

$$* \ \frac{\partial}{\partial z}(z f(r)) = z \cdot f'(r) \cdot \frac{z}{r} + f(r) = \frac{z^2 f'(r)}{r} + f(r)$$

$$\text{Now} \rightarrow \operatorname{div} \underline{F} = \frac{\partial}{\partial x}(x f(r)) + \frac{\partial}{\partial y}(y f(r)) + \frac{\partial}{\partial z}(z f(r))$$

r^2

$$\operatorname{div} \underline{F} = 3f(r) + \frac{x^2 f'(r) + y^2 f'(r) + z^2 f'(r)}{r}$$

$$\operatorname{div} \underline{F} = 3f(r) + \frac{r^2 f'(r)}{r} = 3f(r) + r f'(r) \quad \#$$

ex Consider $\underline{F} = \frac{c}{r^3}\underline{r}$, Find $\operatorname{div} \underline{F}$?

sol:

$$\underline{F} = c\,\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} = \frac{cx}{r^3}\hat{i} + \frac{cy}{r^3}\hat{j} + \frac{cz}{r^3}\hat{k}$$

$$\operatorname{div} \underline{F} = \frac{\partial}{\partial x}\left(\frac{xc}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{cy}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{cz}{r^3}\right)$$

$$* \ \frac{\partial}{\partial x}\left(\frac{cx}{r^3}\right) = \frac{\partial}{\partial x}\left(cxr^{-3}\right) = cx \cdot -3r^{-4} \cdot \frac{x}{r} + cr^{-3}$$

$$= -3cr^{-5}x^2 + cr^{-3}$$

$$* \ \frac{\partial}{\partial y} = -3cr^{-5}y^2 + cr^{-3}$$

$$* \ \frac{\partial}{\partial z} = -3cr^{-5}z^2 + cr^{-3}$$

$$\text{Now} \rightarrow \operatorname{div} \underline{F} = -3cr^{-5}\left(\overbrace{x^2 + y^2 + z^2}^{r^2}\right) + 3cr^{-3}$$

$$\operatorname{div} \underline{F} = -3cr^{-3} + 3cr^{-3} = \text{Zero}.$$

$$* \ \frac{\partial}{\partial x}\left(f(r)\right) = f'(r) \cdot \frac{x}{r}$$

$$* \ \frac{\partial}{\partial x}\left(f(r^2)\right) = f'(r^2) \cdot 2r \cdot \frac{x}{r} = f'(r) \cdot 2x$$

$$* \ \|\underline{r}\| = r \longrightarrow \text{norm}$$

* Curl:

Consider : $\underline{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$

$$\text{div } \underline{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \longrightarrow \text{scaler}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \longrightarrow \text{vector}$$

$$\underline{F} = f\hat{i} + g\hat{j} + h\hat{k}$$

$$\rightarrow \underline{\nabla} \cdot \underline{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \text{div } \underline{F}$$

Now → let $\underline{F} = f\hat{i} + g\hat{j} + h\hat{k}$, Then:

$$\text{Curl } \underline{F} = \underline{\nabla} \times \underline{F} \longrightarrow (\text{vector Field})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

(ex) let $\underline{F} = (x^2 y)\hat{i} + (y^2)\hat{j} + (xz)\hat{k}$, Find Curl $\underline{F}$? (8)

<u>sol:</u>

$$\text{Curl } \underline{F} = \underline{\nabla} \times \underline{F}$$

$$= (0-0)\hat{i} - (z-0)\hat{j} + (0-x^2)\hat{k}$$

$$= -z\hat{j} - x^2\hat{k}$$

(ex) let $\underline{F}$, $\underline{G}$ be two vector fields, ϕ be a scaler field.

Then:

1. $\text{Curl } (k\underline{F}) = k \text{ Curl } \underline{F}$
2. $\text{Curl } (\underline{F} + \underline{G}) = \text{Curl } \underline{F} + \text{Curl } \underline{G}$
3. $\text{Curl } (\phi\underline{F}) = \phi \text{ Curl } \underline{F} + \underline{\nabla}\phi \times \underline{F}$

   (مشتقة الاول × الثاني + الاول × مشتقة الثاني) ← الترتيب مهم

4. $\text{div } (\text{Curl } \underline{F}) = \text{Zero}$
5. $\text{Curl } (\nabla\phi) = \text{Zero}$

ex) Consider the vector Field:

$\underline{F} = x\hat{i} + y\hat{j} + z\hat{k}$, Does there exist a vector Field $\underline{G}$ such that Curl $\underline{G} = \underline{F}$?

sol: Suppose there exist vector Field $\underline{G}$ such that

$$\text{Curl } \underline{G} = \underline{F}$$

Then take div to both sides:

$$\text{div}(\text{curl } \underline{G}) \overset{?}{=} \text{div} \underline{F}$$

$$\text{zero} \overset{?}{=} 3$$ $\longrightarrow$ No, Then there is No vector Field $\underline{G}$ such that Curl $\underline{G} = \underline{F}$ #

* We know from calculas II:

$$ds = \sqrt{1 + f'(x)^2}\, dx \quad , L = \int_a^b ds = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$ arc length

ex) Consider $y^2 = x^3$, Find the arc length from $(1,1)$ to $(4,8)$
$x_1\ y_1$; $x_2\ y_2$

sol: $y = x^{\frac{3}{2}} \longrightarrow \frac{dy}{dx} = 3/2\, x^{\frac{1}{2}}$

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx = \int_1^4 \left(1 + \frac{9}{4}x\right)^{\frac{1}{2}} dx = \left. \frac{\left(1 + \frac{9}{4}x\right)^{3/2}}{\frac{3}{2} \cdot \frac{9}{4}} \right]_1^4$$

* $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

→ if we have a circle equation $x^2 + y^2 = 4$, then the paramatric equation:

$$\left.\begin{aligned} x(t) &= 2\cos t \\ y(t) &= 2\sin t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

Now → $ds = \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\, dx$

$$ds = \sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\, dx = \frac{\sqrt{x'(t)^2 + y'(t)^2}}{\frac{\cancel{dx}}{dt}}\, \cancel{dx}$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

and arc length: $L = \int_{t_1}^{t_2} ds = \int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}\, dt$

ex Consider $x(t) = 2\cos t$, $y(t) = 2\sin t$, $0 \le t \le 2\pi$,

Find the arc length?

sol: $L = \int_{t_1}^{t_2} ds = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt$

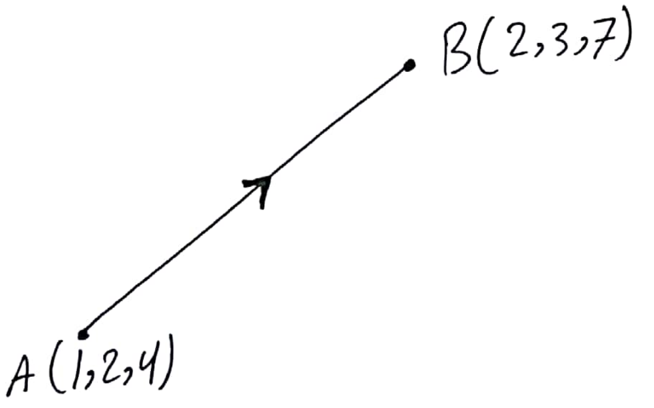
$$= \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t} = \int_0^{2\pi} \sqrt{4} = 2(2\pi) = 12.56$$

$\rightarrow x^2 + y^2 = 4$

$\rightarrow \frac{x^2}{4} + \frac{y^2}{4} = 1 \longrightarrow$ circle (special case from ellipse)

$$\left.\begin{aligned} x(t) &= 2\cos t \\ y(t) &= 2\sin t \end{aligned}\right\} \; 0 \le t \le 2\pi$$

* $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$
* $dx = x'(t)\, dt$
* $dy = y'(t)\, dt$
* $dz = z'(t)\, dt$



* parameterization for this line $= \boxed{A + (B-A)t}$

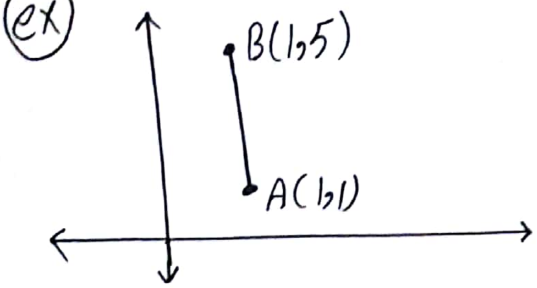
$= \langle 1,2,4 \rangle + \langle 1,1,3 \rangle t$

$= \langle 1+t, 2+t, 4+3t \rangle$

$$\left.\begin{aligned} x(t) &= 1+t \\ y(t) &= 2+t \\ z(t) &= 4+3t \end{aligned}\right\} \; 0 \le t \le 1$$

*أي parameterization بهذه
الطريقة تكون $(0 \le t \le 1)$

ex



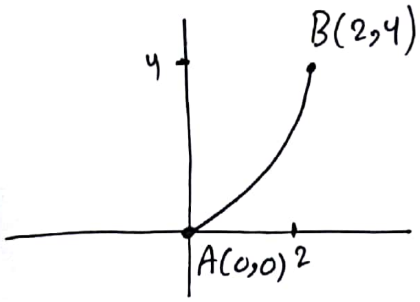
① $\langle 1,1\rangle + \langle 0,4\rangle t$

$= \langle 1, 1+4t\rangle$

$\left.\begin{array}{l} x(t) = 1 \\ y(t) = 1+4t \end{array}\right\} \quad 0 \le t \le 1$

② $\left.\begin{array}{l} x(t) = 1 \\ y(t) = t \end{array}\right\} \quad 1 \le t \le 5 \rightarrow$ (another parameterization)

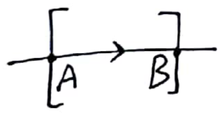
ex $y = x^2$



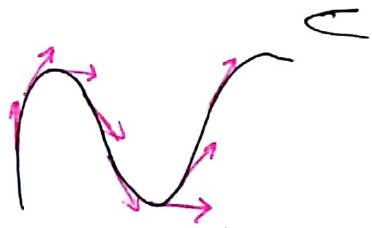
$\left.\begin{array}{l} x(t) = t \\ y(t) = t^2 \end{array}\right\} \quad 0 \le t \le 2$

** line integral:

$\int_a^b f(x)\,dx \longrightarrow$ one direction (x-axis)



$\int_C f(x,y,z)\,ds \longrightarrow$ integral on curve



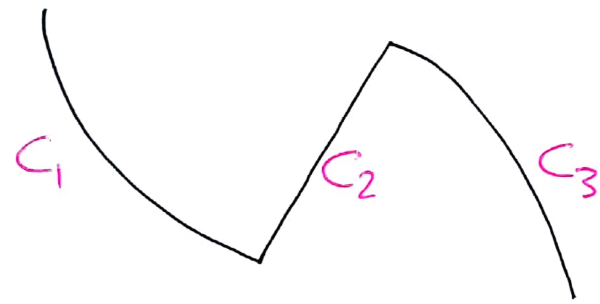
(on space)

$$\int_C f_{(x,y,z)}\, ds = \int_C \underbrace{x(t)\, y(t)\, z(t)}_{\text{come from parameterization}}\, dx\, dy\, dz$$

$\underline{\underline{C}}$: Smoth Curve or ($\underline{\text{piece wise smoth}}$)

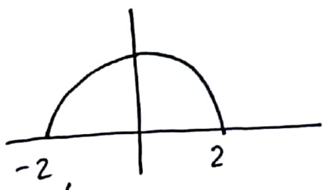
↳ (Continous & differential)   ↳ finite Union of smoth curves

C :



$$C = C_1 + C_2 + C_3 = \int_{C_1} ds + \int_{C_2} ds + \int_{C_3} ds$$

(ex) evaluate $\int_C [2xy + 4]\, ds$ where C : is the upper half of the circle : $x^2 + y^2 = 4$.



$\rightarrow$ $\begin{cases} x(t) = 2\cos t \\ y(t) = 2\sin t \end{cases}$ $\boxed{0 \le t \le \pi}$ (upper half circle)

$$= \int_{t_1}^{t_2} 2(2\cos t)(2\sin t) + 4 * \underbrace{\sqrt{x'(t)^2 + y'(t)^2}}\, dt$$

↳ $\sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$

→ follow

follow:

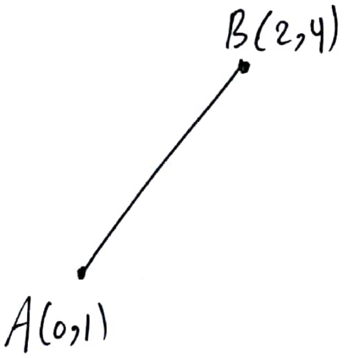
$$= \int_0^{\pi} \left[2\cdot(2\cos t)(2\sin t) + 4 \right] \cdot 2\, dt$$

$$= \int_0^{\pi} 16 \cos t \sin t + 8\, dt = \int_0^{\pi} 16 \cos t \sin t\, dt + \int_0^{\pi} 8\, dt$$

$$= \cancel{\frac{16 \sin^2 t}{2} \Big|_0^{\pi}} + 8\pi = \text{Zero} + 8\pi = 8\pi$$

Zero

(ex) evaluate: $\int_C 2xy\,dy + y\,dx$ where C: is the line segment from $(0,1)$ to $(2,4)$



$$\Rightarrow (\langle 0,1 \rangle + \langle 2,3 \rangle t)$$

$$= \langle 2t,\ 1+3t \rangle$$

$x(t) = 2t$
$y(t) = 1+3t$ $\Big\}$ $0 \le t \le 1$

Now $\rightarrow \int_{t_1}^{t_2} \left[2\, x(t)\, y(t)\, y'(t) + y(t)\, x'(t) \right] dt$

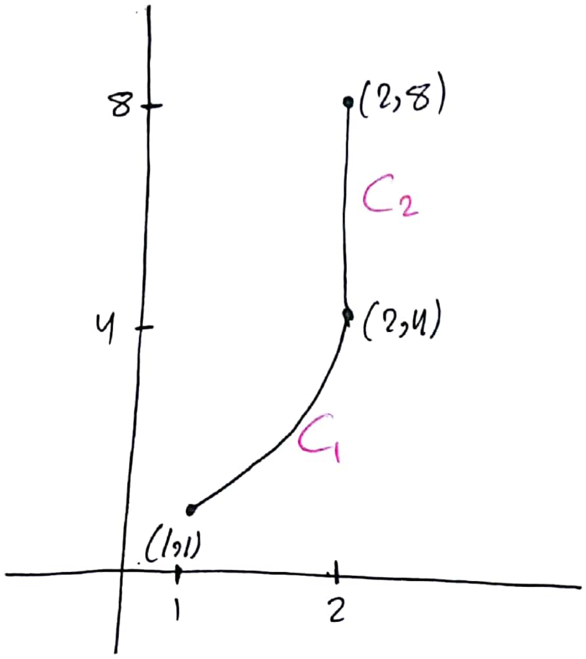
$* \, dx = x'(t)\, dt$
$* \, dy = y'(t)\, dy$

$$= \int_0^1 2(2t)(1+3t)\cdot 3 + (1+3t)(2)\, dt$$

$$= \int_0^1 12t + 36t^2 + 6t + 2\, dt = \frac{18t^2}{2} + \frac{36t^3}{3} + 2t \Big]_0^1$$

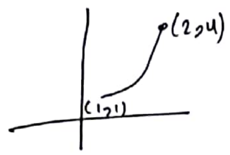
$$= (9+12+2) - (0+0+0) = 23$$

ex evaluate $\int_C 2xy\,dy$ where C is the part of the parabola $y=x^2$ from $(1,1)$ to $(2,4)$ followed by the segment from $(2,4)$ to $(2,8)$?



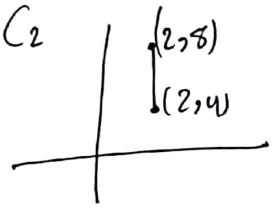
$$\int_C = \int_{C_1} ds + \int_{C_2} ds$$

for C_1:



$x(t) = t$
$y(t) = t^2$ $\Big\}$ $1 \le t \le 2$

for C_2



$= \langle 2,4\rangle + \langle 0,4\rangle t = \langle 2, 4+4t\rangle$

$x(t) = 2$
$y(t) = 4+4t$ $\Big\}$ $0 \le t \le 1$

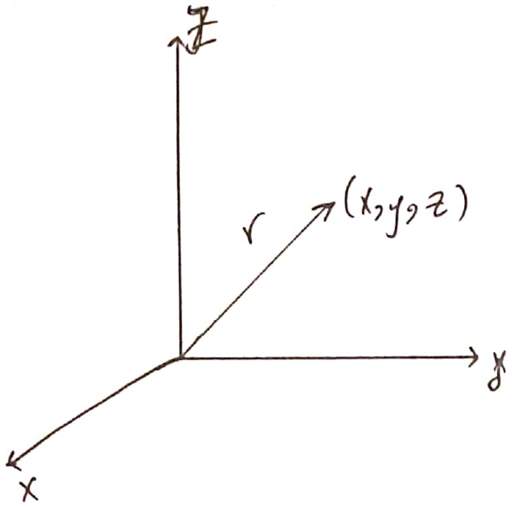
Now $\rightarrow \int_C = \int_{C_1} 2x(t)\,y(t)\,y'(t)\,dt + \int_{C_2} 2x(t)\,y(t)\,y'(t)\,dt$

$$= \int_1^2 2(t)(t^2)(2t)\,dt + \int_0^1 2(2)(4+4t)(4)\,dt$$

$$= \int_1^2 4t^4\,dt + \int_0^1 64+64t\,dt$$

$$\frac{4t^5}{5}\Big|_1^2 + \left(64t + \frac{64t^2}{2}\Big|_0^1\right) = \left(\frac{128}{5} - \frac{4}{5}\right) + (64+32)$$

* line integral:



$\underline{r} = (x, y, z)$

$\underline{r} = x\underline{\hat{i}} + y\underline{\hat{j}} + z\underline{\hat{k}}$

$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$

$\rightarrow \underline{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$

$* \underline{F}(x,y,z) = \underline{F}(r(t)) = \underline{F}(x(t), y(t), z(t))$

$\Rightarrow \int_c \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(r(t)) \cdot r'(t)\, dt$

ex) evaluate $\int_c \underline{F} \cdot d\underline{r}$, where: $\underline{F} = y\hat{i} + xz\hat{j} + y\hat{k}$,

$C: r(t) = \underbrace{2t}_{x(t)}\hat{i} + \underbrace{t}_{y(t)}\hat{j} + \underbrace{3}_{z(t)}\hat{k}$, $0 \le t \le 2$

sol:

$\underline{F}(r(t)) = [y(t), x(t)z(t), y(t)] = [t, 6t, t]$

$r'(t) = [2, 1, 0]$

follow

$\rightarrow F(r(t)) \cdot r'(t) = 2t + 6t + 0 = 8t$

$$\int_0^2 8t\,dt = 4t^2\Big|_0^2 = 16 - 0 = 16$$

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ex) Evaluate : $\int \underline{F} \cdot d\underline{r}$, where:

$\underline{F} = [y, z, x]$

$C: r(t) = [\underbrace{\cos(t)}_{x(t)}, \underbrace{\sin(t)}_{y(t)}, \underbrace{2t}_{z(t)}]$

from $(1, 0, 0)$ to $(-1, 0, 2\pi)$

sol:

$\left.\begin{array}{l}\cos(t) = 1 \\ \sin(t) = 0 \\ 2t = 0\end{array}\right\} \rightarrow \boxed{t = 0}$, $\left.\begin{array}{l}\cos(t) = -1 \\ \sin(t) = 0 \\ 2t = 2\pi\end{array}\right\} \rightarrow \boxed{t = \pi}$

$\rightarrow \underline{F}(r(t)) = [y(t), z(t), x(t)] = [\sin(t), 2t, \cos(t)]$

$\rightarrow r'(t) = [-\sin(t), \cos(t), 2]$ #

$* \ \underline{F}(r(t)) \cdot r'(t) = -\sin^2(t) + 2t\cos(t) + 2\cos(t)$

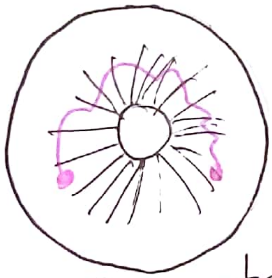
$$\rightarrow \int_C \underline{F} \cdot d\underline{r} = \int_0^{\pi} \left[-\sin^2(t) + 2t\cos(t) + 2\cos(t)\right] dt$$

Conservation Fields:

→ Independent of path → (Exact equation)

Def: $\underline{F}$ is Conservative: if there exist a scaler field ϕ • $\underline{F} = \nabla \phi$

simply connected (without holes)

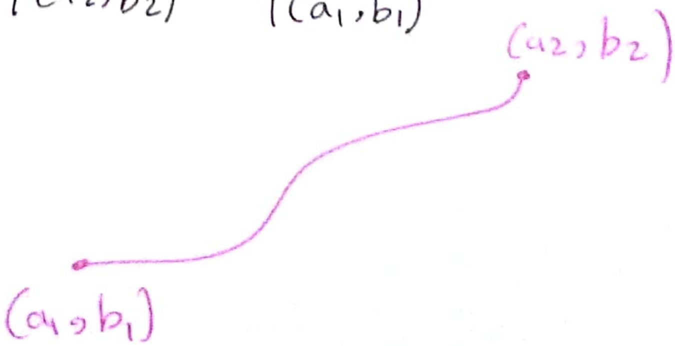


connected (with holes)

→ Theorem: let $\underline{F}(x,y) = f(x,y)\,\underline{i} + g(x,y)\,\underline{j}$ [The region D is simply connected]

If $\dfrac{df}{dy} = \dfrac{dg}{dx}$ → Then $\underline{F}$ is Conservative

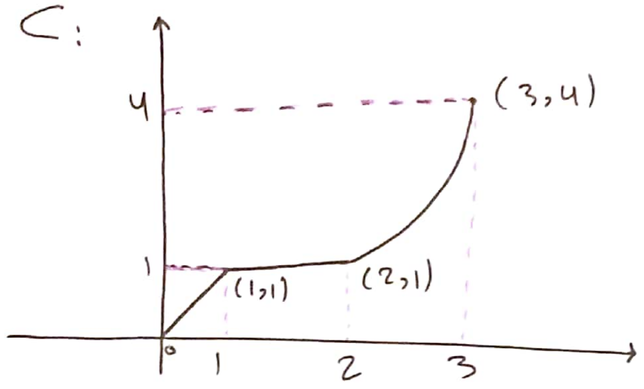
and $\int_c \underline{F}.d\underline{r} = \int_c \nabla\phi . dr = \phi_{(a_2,b_2)} - \phi_{(a_1,b_1)}$



ex) Evaluate $\int_C \underline{F} \cdot dr$

$$\underline{F} = [4xy^2 + 2]\,\underline{i} + [4x^2 y]\,\underline{j}$$

C:



sol:

$$\frac{df}{dy} = 8xy$$

$$\frac{dg}{dx} = 8xy$$

$\underline{F}$ is Conservative.

$$\rightarrow \underline{F} = \nabla\phi \rightarrow [4xy^2+2]\underline{i} + [4x^2y]\underline{j} = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j}$$

$$* \ \frac{\partial\phi}{\partial x} = 4xy^2+2 \quad , \quad * \ \frac{\partial\phi}{\partial y} = 4x^2y$$

$$\phi(x,y) = \int \frac{\partial\phi}{\partial x}\,dx = \int 4xy^2 + 2 \; dx$$

$$\phi(x,y) = 2x^2y^2 + 2x + k(y)$$

→ 20

#follow

$$\boxed{\text{Diff w.r.to } y}$$

$$\frac{\partial \phi}{\partial y} = 4x^2 y + k'(y)$$

$$\rightarrow 4x^2 y = 4x^2 y + k'(y)$$

so: $k'(y) = \text{zero}. \longrightarrow \int k'(y) = \int \text{zero} = C = k(y)$

$$\boxed{\phi(x,y) = 2x^2y^2 + 2x + C}$$

$$\int_c F \cdot dr = \int_c \nabla\phi \cdot dr$$

$$= \phi(3,4) - \phi(0,0) = (2)(9)(16) + (6) - \cancel{(C)} + \cancel{(C)}$$

$$= 288 + 6 = 294$$

OR $F = [4xy^2 + 2]\,i + [4x^2y]\,j$

$\int \quad dx \qquad\qquad \int dy$

Then: $\phi(x,y) = \int \; dx + \int \; dy \longrightarrow$ (من غير تكرار).

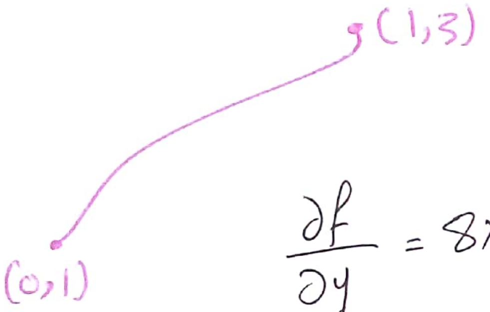
$$\underline{F}(x,y) = f(x,y)\,\underline{\hat{i}} + g(x,y)\,\underline{\hat{j}} \qquad , \underline{r} = x\underline{\hat{i}} + y\underline{\hat{j}}$$

$$d\underline{r} = dx\,\underline{\hat{i}} + dy\,\underline{\hat{j}}$$

$$\underline{F} \cdot d\underline{r} = f(x,y)\,dx + g(x,y)\,dy$$

$$\int_c \underline{F} \cdot d\underline{r} = \int f(x,y)\,dx + g(x,y)\,dy$$

ex Evaluate: $\displaystyle\int_{(0,1)}^{(1,3)} (2xy^4)\,dx + (4x^2y^3 - 4)\,dy$



* اذا ما حدد المسار بتأكد انو conservative.
* اذا طلع conservative ما بعتمد على شكل المسار.

$$\left.\begin{aligned} \frac{\partial f}{\partial y} &= 8xy^3 \\ \frac{\partial g}{\partial x} &= 8xy^3 \end{aligned}\right\} \quad \begin{aligned} &\underline{F} \text{ is conservative.} \\ &\underline{F} = \nabla \phi \end{aligned}$$

$$\int_{(0,1)}^{(1,3)} \underline{F} \cdot d\underline{r} = \int_{(0,1)}^{(1,3)} \nabla \phi \cdot d\underline{r} = \phi(1,3) - \phi(0,1)$$

$$\to \int 2xy^4\,dx = x^2y^4$$

$$\int 4x^2y^3 - 4\,dy = x^2y^4 - 4y$$

$$\phi(x,y) = x^2y^4 - 4y$$

* let $\underline{F}(x,y,z) = f_{(x,y,z)}\underline{i} + g_{(x,y,z)}\underline{j} + h_{(x,y,z)}\underline{k}$

<u>IF</u> $\boxed{\text{Curl } \underline{F} = \text{zero}}$, then $\underline{F}$ is Conservative.

* $\text{Curl } \underline{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$

* $\underline{F}$ Conservative, then:

① $\int_c \underline{F} \cdot d\underline{r} = \int_c \nabla\phi \cdot d\underline{r} = \phi_B - \phi_A$

② $\oint_c \underline{F} \cdot d\underline{r} = \text{zero}$ (Closed Curve)

ex Evaluate: $\int_c 3x^2\,dx + 2yz\,dy + y^2\,dz$

from A: (0,1,2) to B: (1,-1,7)

sol:

$\text{Curl } \underline{F} = (2y - 2y)\underline{i} - (0-0)\underline{j} + (0-0)\underline{k} = \text{Zero}$

$\rightarrow$ Curl $\underline{F}$ = zero $\rightarrow$ So $\underline{F}$ is Conservative.

$\int_A^B \underline{F} \cdot d\underline{r} = \phi_{(1,-1,7)} - \phi_{(0,1,2)}$

$x^3 \rightarrow \int dx$

$y^2 z \rightarrow \int dy$

$y^2 z \rightarrow \int dz$

$\phi_{(x,y,z)} = x^3 + y^2 z$

ex) Evaluate:

$$\int \underbrace{\frac{-y}{x^2+y^2}}_{f(x,y)}\,dx + \underbrace{\frac{x}{x^2+y^2}}_{g(x,y)}\,dy$$

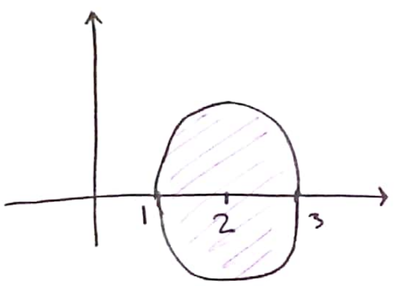
Where:

① C: $\boxed{(x-2)^2+y^2=1}$

② C: $\boxed{x^2+y^2=1}$

sol:

①



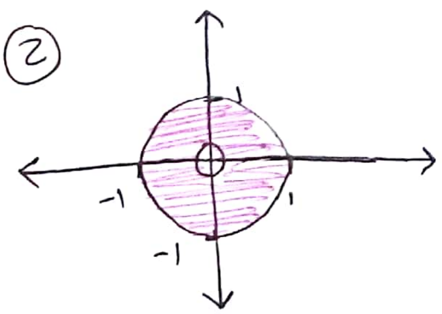
$$* \ \frac{df}{dy} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$* \ \frac{dg}{dx} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\oint \underline{F}\cdot d\underline{r} = 0 \implies \frac{df}{dy}=\frac{dg}{dx}$ / holes ما في $\implies$ Conservative

---

②



$x^2+y^2=1 \longrightarrow x(t)=\sin(t),\ y(t)=\cos(t)$

$0 \le t \le 2\pi$

$$\int \left[\frac{-y(t)}{x^2(t)+y^2(t)}\,x'(t) + \frac{x(t)}{x^2(t)+y^2(t)}\,y'(t)\right] dt$$

$$= \int_0^{2\pi} \left[\frac{-\sin(t)}{1}(-\sin(t)) + \frac{\cos(t)}{1}\cos(t)\right] dt = \int_0^{2\pi} 1\,dt = 2\pi$$

* هاد ما بحله مثل الطريقة ① لأنه في holes #

Green's theorem:

- +ve oriented (C.C.W)

(لما نمشي بكون ال Region على الشمال)

- f, g, and their first partial derivatives are continuous. Then:

$$\oint_C f\,dx + g\,dy = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA \quad dA \to dx\,dy,\ dy\,dx$$

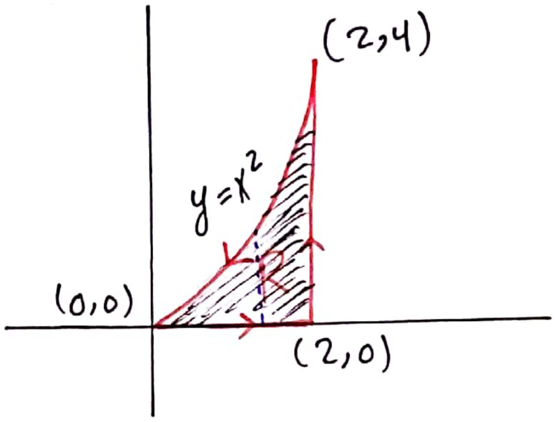
---

ex1 evaluate $\oint_C \underbrace{[x^2+y^3]}_{f}\,dx + \underbrace{[3xy^2]}_{g}\,dy$,

where C: consists of the portion of $y=x^2$ from $(2,4)$ to $(0,0)$ followed by the segments from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(2,4)$.

→ sol in page 25

sol:



اذا نلاقي ال curve مغلق بدون holes ← نطبق (green's theorem)

$$\oint \underline{F} \cdot d\underline{r} = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

$$= \iint_R [3y^2 - 3y^2]\, dA = \text{Zero} \therefore (\text{conservative})$$

لو ما كان (zero):

$$\int_0^2 \int_0^{x^2} [\ .....\ ]\, dy\, dx$$ (نفضل ان يكون x ثابت و y متغير)

---

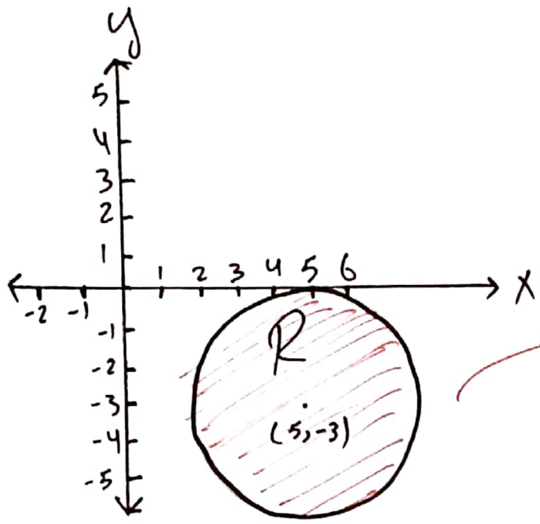
ex2 evaluate $\oint_c [\underbrace{7y - e^{\sin x}}_{f}]\, dx + [\underbrace{15x - \sin(y^3 + 8y)}_{g}]\, dy$

where C: $(x-5)^2 + (y+3)^2 = 9$

↳ circle with raduis = 3 & center = (5,-3)

→ Sol in page (26)

Sol:



- closed & no holes:

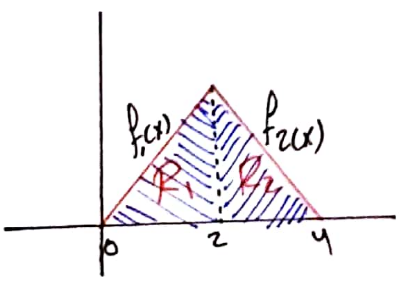
→ Apply green's theorem:

$$\oint_C \underline{F}\cdot d\underline{r} = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

$$= \iint_R [15 - 7]\, dA = 8 \iint_R dA$$

area of the region = area of circle with raduis = 3 $= 9\pi$

→ $= 8(9\pi) = 72\pi$



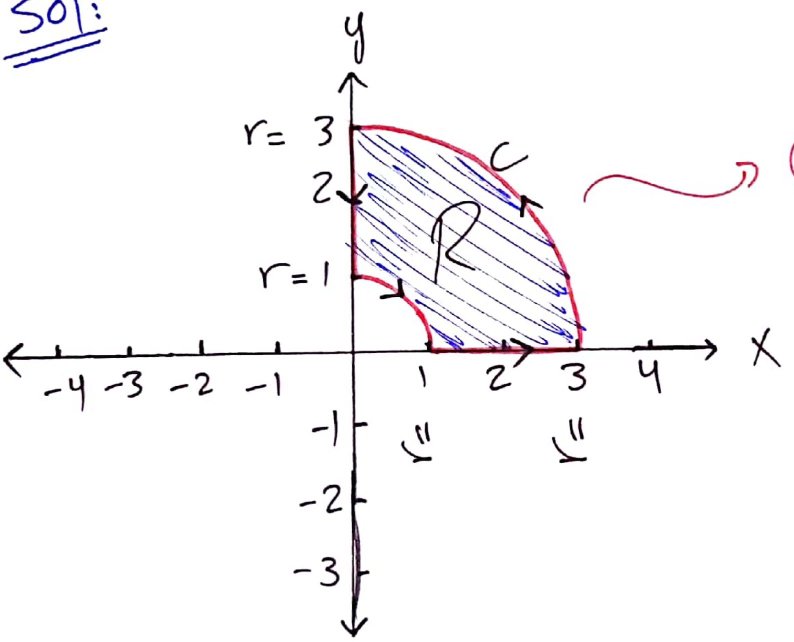
- في مثل هذه الحالة يكون الحل كالتالي :

$$\int_0^2\int_0^{f_1(x)} [\dots]\, dA + \int_2^4\int_0^{f_2(x)} [\dots]\, dA$$

(ex3) Evaluate $\oint \underbrace{[e^x + 6yx]}_{f} dx + \underbrace{[8x^2 + \sin y^2]}_{g} dy$, (27)

where C: is the boundary of the region enclosed by the Circles of raduises (1, 3) centered at the orgion and lying in the first quadrant.

Sol:



Closed → (green's theorem)

$$\oint_C F \cdot dr = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

$$= \iint_R [16x - 6x]\, dA = \iint_R 10x\, dA$$

Use poler coordenate : $x = r\cos\theta$, $y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$= \int_0^{\frac{\pi}{2}}\int_1^3 10\, r\cos\theta \cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} 10\cos\theta \boxed{\int_1^3 r^2\,dr}\, d\theta$$

$$\boxed{\int_1^3 r^2\,dr} = \frac{26}{3}$$

$$= \frac{260}{3}\underbrace{\int_0^{\frac{\pi}{2}} \cos\theta}_{=1} = \frac{260}{3}$$

note:

[1] $\int_{-C} f(x,y,z)\,ds = \int_C f(x,y,z)\,ds$

[2] $\int_{-C} f(x,y,z)\,dx = -\int_C f(x,y,z)\,dx$

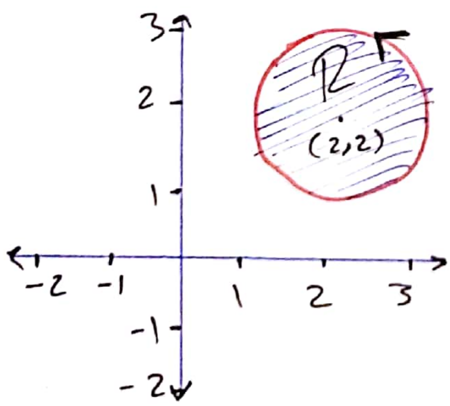
[3] $\int_{-C} \underline{F}\cdot d\underline{r} = -\int_C \underline{F}\cdot d\underline{r}$

ex evaluate:

$$\int_C \underbrace{\frac{-y}{x^2+y^2}}_{f}\,dx + \underbrace{\frac{x}{x^2+y^2}}_{g}\,dy \quad , \text{ where:}$$

(1) $C: (x-2)^2 + (y-2)^2 = 1$

sol:



$$\int_C f\,dx + g\,dy$$

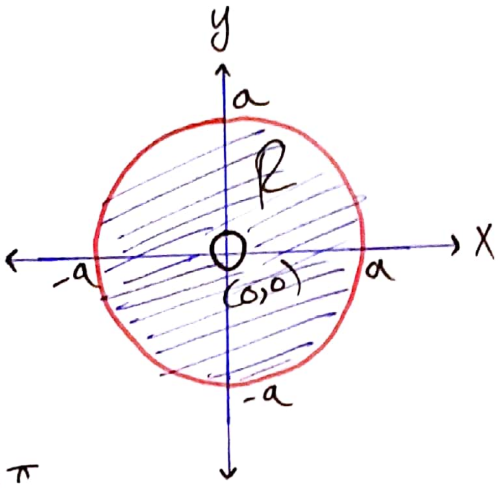
$$= \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA$$

$= Zero$

→ follow

② $C: y^2 + x^2 = a^2$

sol:



$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

ما بطبق green's theorem لأنو في holes

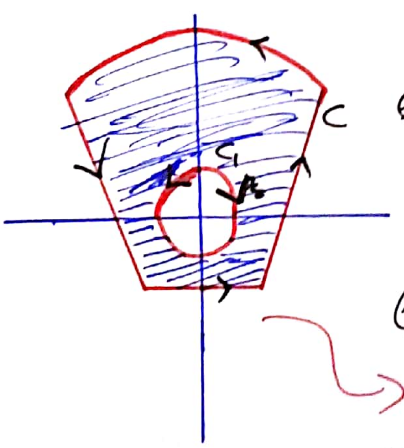
$$= \int_0^{2\pi} \left[\frac{-y(t)}{x^2(t)+y^2(t)} \cdot x'(t) + \frac{x(t)}{x^2(t)+y^2(t)} \cdot y'(t) \right] dt$$

$$= \int_0^{2\pi} \left[\frac{-a\sin t \cdot a\sin t}{a^2} + \frac{a\cos t \cdot a\cos t}{a^2} \right] dt$$

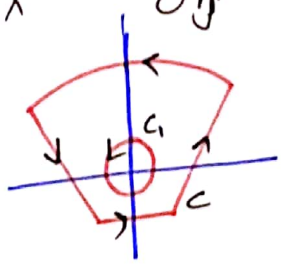
$= 2\pi$ → for any circle with any raduis.

③ C:

① $\int_C - \int_{C_1} = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA = 0$



② $\int_C = \int_{C_1} = 2\pi$

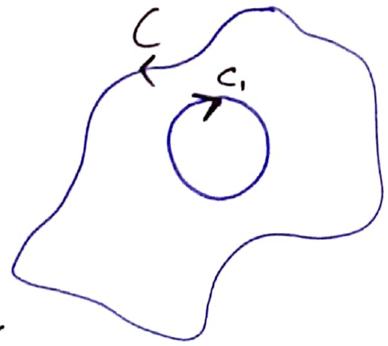


التكامل على اي دائرة مركزها $(0,0) = 2\pi$

③ $\int_C + \int_{C_1} = \text{zero} \longrightarrow \int_C = -\int_{C_1}$

- Remark:

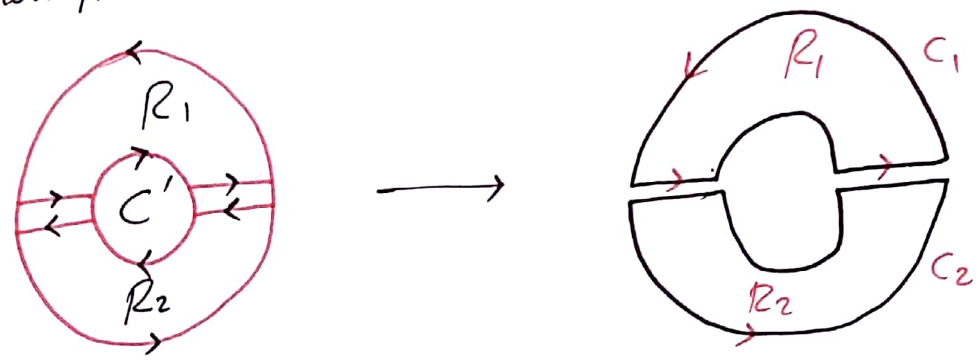
$$\int_C = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \rightarrow \text{for simply connected.}$$



$\rightarrow$ not simply connected

$$\int_C + \int_{C_1} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \rightarrow \text{for not simply connected.}$$

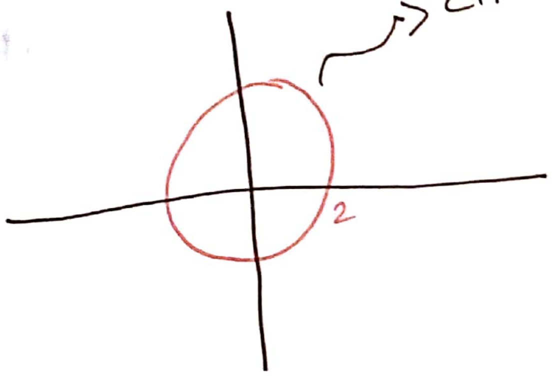
- Remark



$$C_1 = \iint_{R_1} \qquad C_2 = \iint_{R_2}$$

$$\int_{C_1} + \int_{C_2} = \iint_R \rightarrow \int_C + \int_{C'} = \iint_R$$

(31)

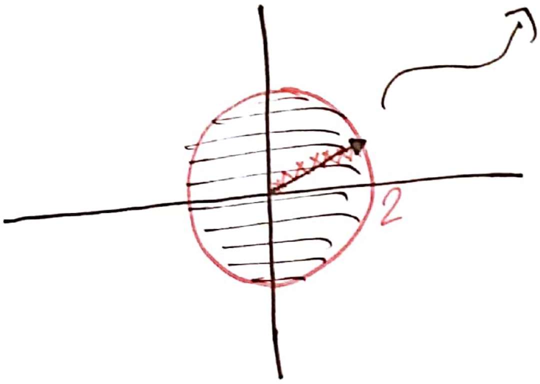


$x^2+y^2=4$

$x(t)=2\cos t$

$y(t)=2\sin t$

$\boxed{0\le t\le 2\pi}$



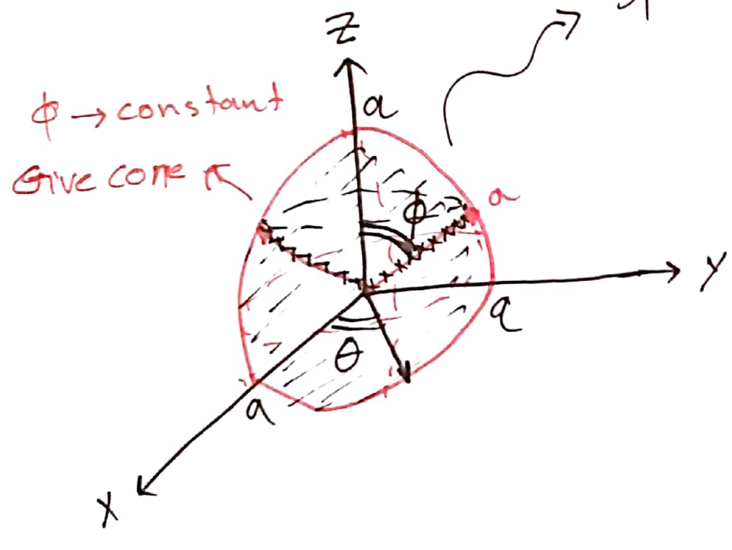
$x^2+y^2=4$

$x=r\cos\theta$

$y=r\sin\theta$

$0\le\theta\le 2\pi$

$0\le r\le 2$

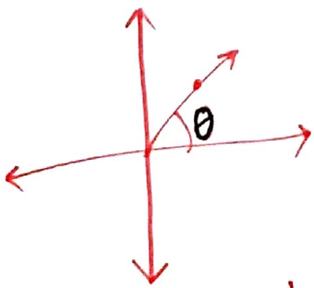


$x^2+y^2+z^2=a^2$

$x=\rho\cos\theta\sin\phi$

$y=\rho\sin\theta\sin\phi$

$z=\rho\cos\phi$



$0\le\rho\le a$

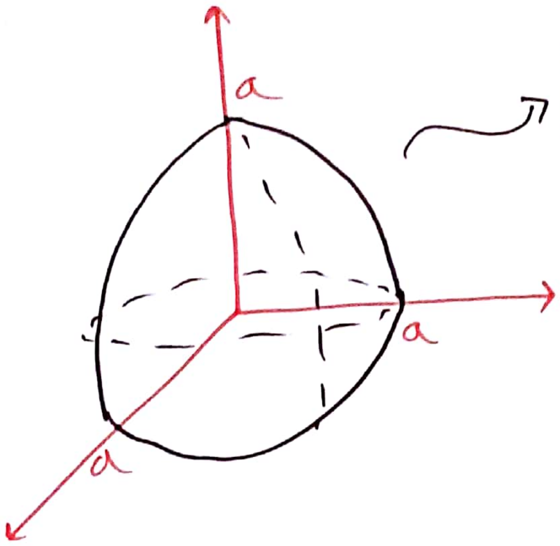
$0\le\theta\le 2\pi$

$0\le\phi\le\pi$

لما نسقط نقطة على

plane xy ← θ = الزاوية

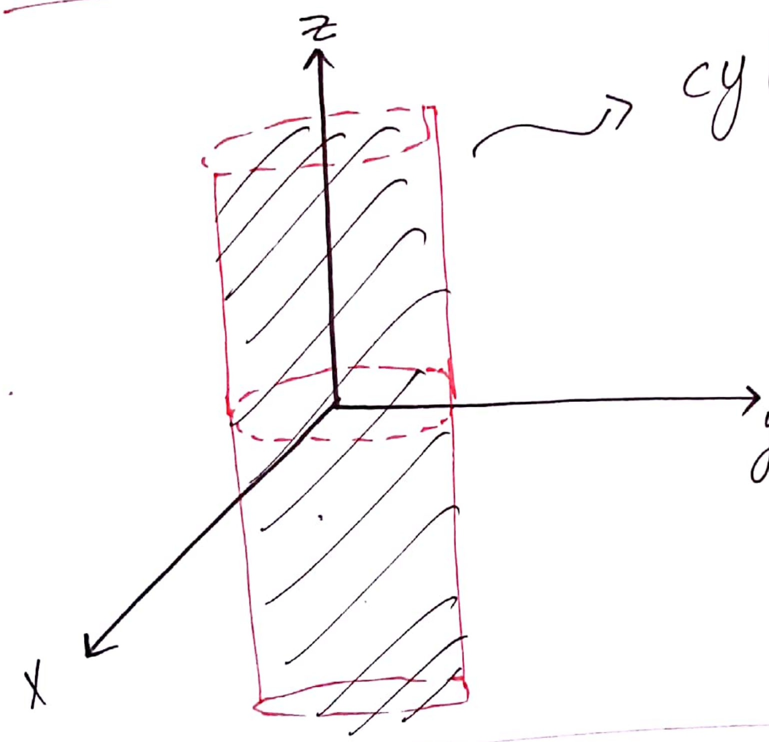
بين النقطة و X



(surface / hollow sphere)

$$x^2 + y^2 + z^2 = a^2$$

$$\boxed{\begin{aligned} x &= a\cos\theta\sin\phi \\ y &= a\sin\theta\sin\phi \\ z &= a\cos\phi \end{aligned}}$$



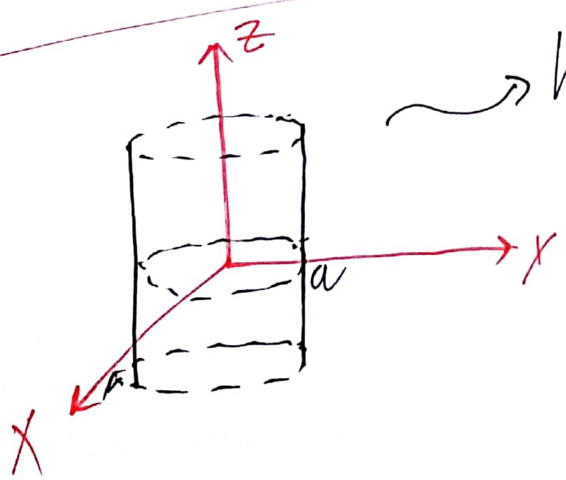
cylinder (solid)

$$x^2 + y^2 = a^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$



hollow cylinder (surface)

$$x^2 + y^2 = a^2$$

$$x = a\cos\theta$$

$$y = a\sin\theta$$

$$z = z$$

For a hollow sphere (surface)

$x = a\cos\theta\sin\phi$

$y = a\sin\theta\sin\phi$

$z = a\cos\phi$

we know:

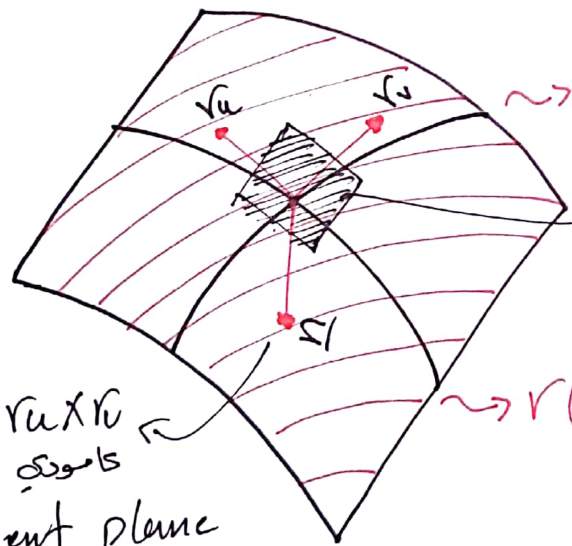
$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$\underline{r} = x(\theta,\phi)\underline{i} + y(\theta,\phi)\underline{j} + z(\theta,\phi)\underline{k}$

→ in general:

we can parameterize a surface as:

$\underline{r}(u,v) = x(u,v)\underline{i} + y(u,v)\underline{j} + z(u,v)\underline{k}$



the tangent plane at (x_0,y_0,z_0), with $\underline{n}(a,b,c) =$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Remark → $\underline{n} = r_u \times r_v$

Ex: Find the tangent plane of:

$$\boxed{r(u,v) = u^2\,\underline{i} + v^2\,\underline{j} + (u+2v)\,\underline{k}}$$ at $(1,1,3)$

sol:

$$a(x-1) + b(y-1) + c(z-3) = 0$$

$\rightarrow$ we know $n = r_u \times r_v \rightarrow$ Find $(r_u), (r_v)$

$$r_u = 2u\,\hat{i} + 0\,\hat{j} + \underline{k} = [2u, 0, 1]$$

$$r_v = 0\,\hat{i} + 2v\,\hat{j} + 2\,\underline{k} = [0, 2v, 2]$$

$$n = r_u \times r_v = \begin{vmatrix} \hat{i} & -\hat{j} & \underline{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = (-2v)\,\underline{i} - (4u)\,\hat{j} + (4uv)\,k$$

$$= \underbrace{[-2v, -4u, 4uv]}$$

نعوض النقطة الي معطينا ياها بالسؤال

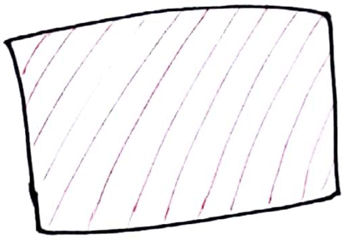
$r(u,v) = u^2\,\hat{i} + v^2\,\hat{j} + (u+2v)\,k$ $\Longleftarrow$

at $(1,1,3)$

$$\left.\begin{aligned} u^2 &= 1 \\ v^2 &= 1 \\ u+2v &= 3 \end{aligned}\right\} \rightarrow \begin{aligned} u &= 1 \\ v &= 1 \end{aligned}$$

Now $\rightarrow N = [-2, -4, 4]$

tangent plane $= -2(x-1) - 4(y-1) + 4(z-3) = 0$



S , σ $\longrightarrow$ (surface المقصود فيه)

$$\underline{r}(u,v) = x(u,v)\,\underline{i} + y(u,v)\,\underline{j} + z(u,v)\,\hat{\underline{k}}$$

$\rightarrow$ r_u , r_v : normal vector

---

sphere : $x^2 + y^2 + z^2 = a^2$

* $x = a\cos\theta\sin\phi$
* $y = a\sin\theta\sin\phi$
* $z = a\cos\phi$

$$\underline{r}(\phi,\theta) = x(\phi,\theta)\,\underline{\hat{i}} + y(\phi,\theta)\,\underline{\hat{j}} + z(\phi,\theta)\,\underline{\hat{k}}$$

$$= (a\cos\theta\sin\phi)\,\underline{\hat{i}} + (a\sin\theta\sin\phi)\,\underline{\hat{j}} + (a\cos\phi)\,\underline{\hat{k}}$$

$$= \langle a\cos\theta\sin\phi ,\ a\sin\theta\sin\phi ,\ a\cos\phi \rangle$$

$\rightarrow r_\phi = \langle a\cos\theta\cos\phi ,\ a\sin\theta\cos\phi ,\ -a\sin\phi \rangle$

$\rightarrow r_\theta = \langle -a\sin\theta\sin\phi ,\ a\cos\theta\sin\phi ,\ 0 \rangle$

$\underline{n} = r_\phi \times r_\theta =$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \end{vmatrix} \Longrightarrow$$

follow:

$$\underline{n} = (0 + a^2 \cos\theta \sin^2\phi)\underline{i} + (0 + a^2 \sin\theta \sin^2\phi)\underline{j} +$$

$$(\underbrace{a^2 \cos^2\theta \sin\phi\cos\phi + a^2 \sin^2\theta \sin\phi\cos\phi}) \underline{k}$$

$$= a^2 \sin\phi\cos\phi \, (\underbrace{\sin^2\theta + \cos^2\theta}_{1})$$

$$\rightarrow$$

$$\underline{n} = (a^2 \cos\theta \sin^2\phi)\underline{i} + (a^2 \sin\theta \sin^2\phi)\underline{j} + (a^2 \sin\phi\cos\phi)\underline{k}$$

$$\underline{n} = r_\phi \times r_\theta = \langle a^2 \cos\theta \sin^2\phi ,\ a^2 \sin\theta \sin^2\phi ,\ a^2 \sin\phi\cos\phi \rangle$$

هذه الصيغة ليست حفظ

في حالة sphere

طول الـ (vector) العامودي

$$\rightarrow \| r_\phi \times r_\theta \| = \sqrt{a^4 \cos^2\theta \sin^4\phi + a^4 \sin^2\theta \sin^4\phi + a^4 \sin^2\phi \cos^2\phi}$$

$$= a^2 \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$$

$$= a^2 \sqrt{\sin^2\phi (\underbrace{\sin^2\phi + \cos^2\phi}_{1})}$$

$$= a^2 \sqrt{\sin^2\phi}$$

$$= a^2 \sin\phi$$

حفظ

- $S: \underline{r}(u,v) = x(u,v)\underline{i} + y(u,v)\underline{j} + z(u,v)\underline{k}$

- $\underline{dS} = \|r_u \times r_v\|\, du\, dv \longrightarrow$ هذا القانون لأي plane وليس لل x-y plane فقط .

وحدة المساحة

- $S = z = f(x,y) \longrightarrow z - f(x,y) = 0 = G$

- $dS = \|\nabla G\|\, dx\, dy$

$\rightarrow \nabla G = \langle -f_x, -f_y, 1 \rangle \Rightarrow \|\nabla G\| = \sqrt{f_x^2 + f_y^2 + 1}$

$\rightarrow dS = \sqrt{f_x^2 + f_y^2 + 1}\; dx\,dy$ (حفظ)

$$\boxed{* \; S = \text{surface} \;,\; Area(S) = \iint_S dS}$$

$S: y = f(x,z)$

$\rightarrow dS = \sqrt{f_x^2 + f_z^2 + 1}\; dx\,dz$

ex: Find the surface area of:

$S = x^2 + y^2 + z^2 = a^2$ (sphere).

sol:

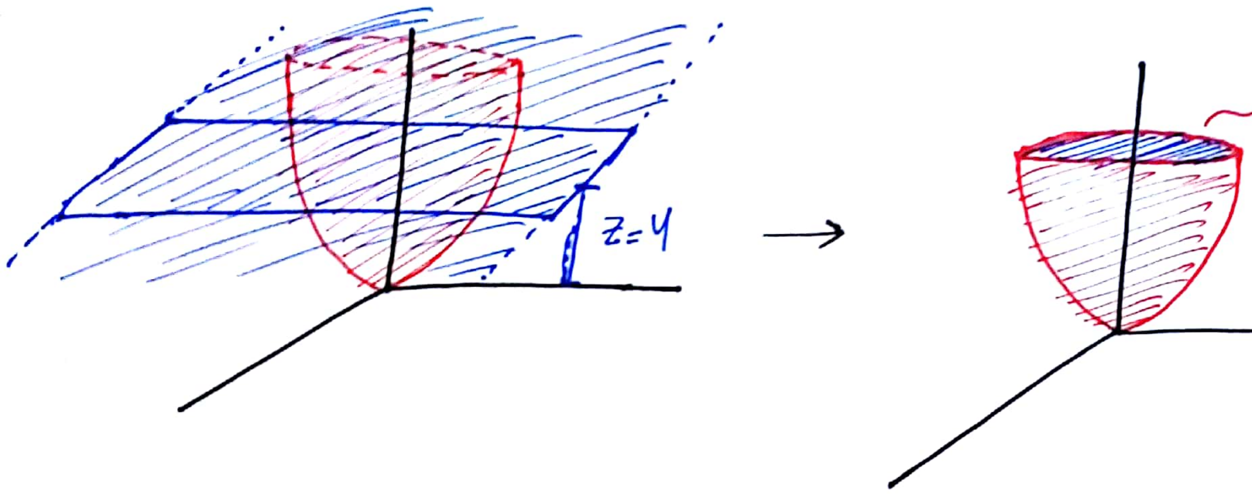
$$Area = \iint_S dS = \iint \|r_\phi \times r_\theta\|\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \underbrace{a^2 \sin\phi}_{\|r_\phi \times r_\theta\|}\, d\phi\, d\theta = a^2 \int_0^{2\pi} \cos\phi \Big|_{\pi}^{0}\, d\theta = \int_0^{2\pi} 2a^2\, d\theta$$

$$\boxed{= 4\pi a^2}$$

Ex: evaluate the surface area of the path of

$z = x^2 + y^2$ that lies under $z = 4$. (paraboloid)



curve

تقاطع الـ plane مع الشكل.

Sol:

$x^2 + y^2 = 4 \longrightarrow$ (التقاطع)

$z = x^2 + y^2 \longrightarrow f(x,y)$

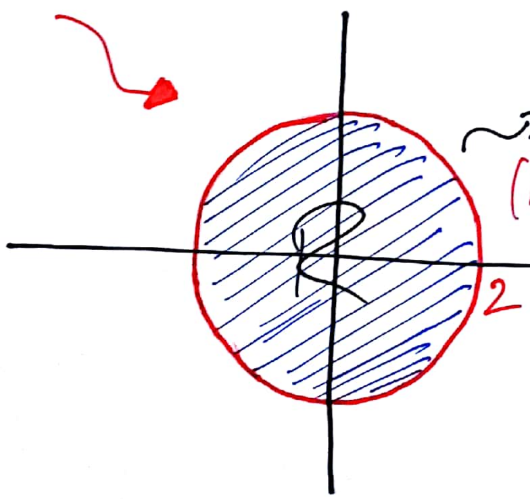
$$dS = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy = \sqrt{4x^2 + 4y^2 + 1}\,dx\,dy$$

$$= \sqrt{4(x^2+y^2)+1}\,dx\,dy$$

$$\rightarrow \text{Area}_{(S)} = \iint_S dS = \iint_R \sqrt{4(x^2+y^2)+1}\,dx\,dy$$

$R \rightarrow$ (plane x-y على الـ projection)

$$= \int_0^{2\pi}\int_0^{2} \sqrt{4r^2+1}\; r\,dr\,d\theta$$



Disk (not curve)

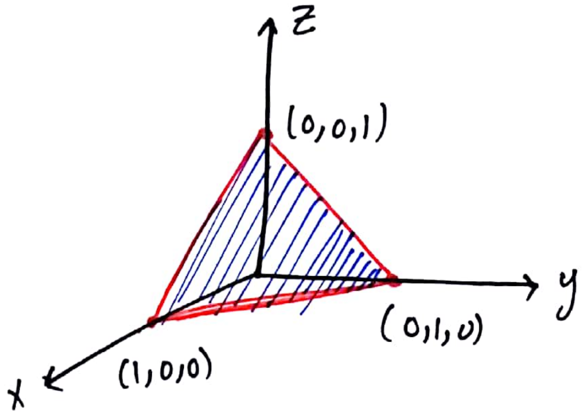
Ex: evaluate $\iint_S xz \, dS$, where

$xz \rightarrow$ function

(الثمن الأول)

S : the part of $x + y + z = 1$, in the first octant.

sol



$z = 1 - x - y \quad (f(x,y))$

$dS = \sqrt{f_x^2 + f_y^2 + 1} \; dx\,dy$

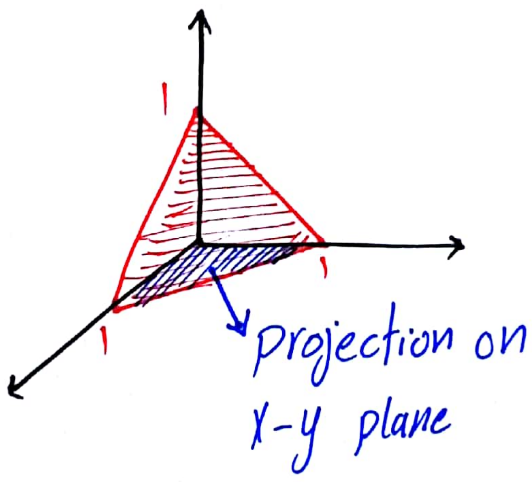
$dS = \sqrt{1 + 1 + 1} \; dx\,dy$

$dS = \sqrt{3} \; dx\,dy$

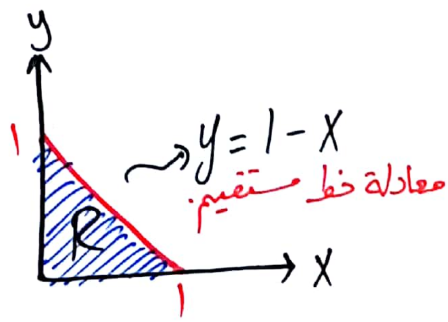
$$\rightarrow \iint_S xz \, dS = \iint x(1 - x - y) \, dS$$

$$= \iint_R x(1 - x - y) \underbrace{\sqrt{3} \, dx\,dy}_{dS}$$

$R \rightarrow$ projection on x-y plane



$\rightarrow$

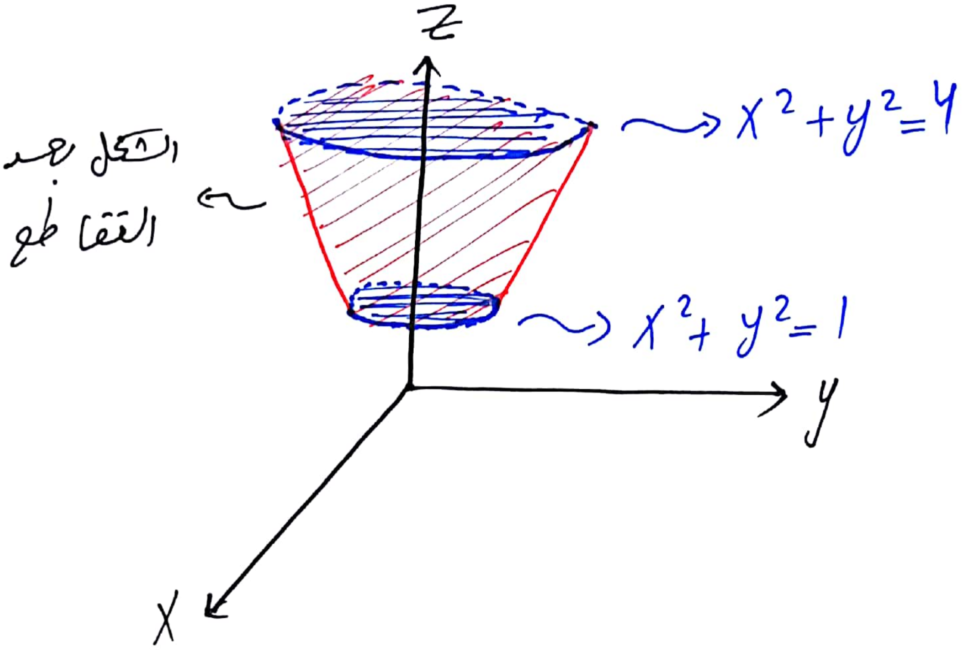


$$= \sqrt{3} \int_0^1 \int_0^{1-x} x(1 - x - y) \, dy \, dx$$

Ex: evaluate $\iint_S y^2 z^2 \, dS$, where:

S: the plane of $z = \sqrt{x^2+y^2}$, that lies between $z=1$, $z=2$.

$\hookrightarrow$ Cone (مخروط)



$z = \sqrt{x^2+y^2}$, $(f(x,y))$

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \longrightarrow f_x^2 = \frac{x^2}{x^2+y^2}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}} \longrightarrow f_y^2 = \frac{y^2}{x^2+y^2}$$

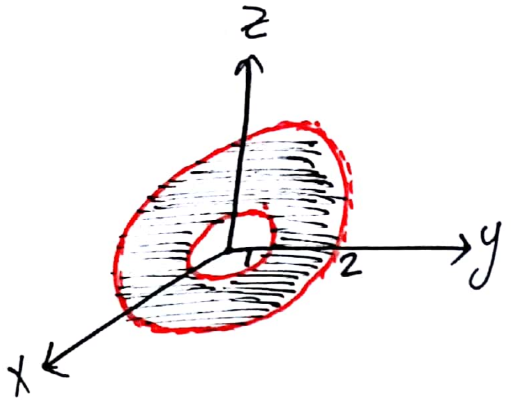
$$f_x^2 + f_y^2 = 1$$

$$dS = \sqrt{\underbrace{f_x^2 + f_y^2}_{=(1)} + 1} \, dx\,dy = \sqrt{2} \, dx\,dy$$

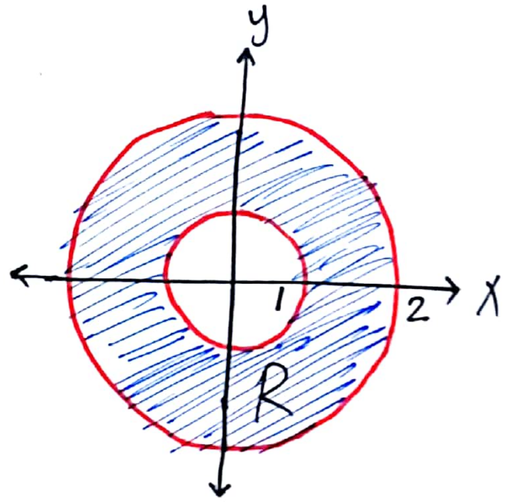
$$\iint_S y^2 z^2 \, ds \Longrightarrow \iint_S y^2 (x^2+y^2) \, dS$$

$$= \iint_R y^2 (x^2+y^2)\sqrt{2}\, dx\, dy$$

$R \longrightarrow$ projection on xy plane



$\longrightarrow$



$$= \sqrt{2} \int_0^{2\pi}\int_1^{2} r^2 \sin^2\theta \cdot r^2 \cdot r\, dr\, d\theta$$

Ex: evaluate $\iint_S x\, ds$, where:

$S : x^2 + y^2 + z^2 = 4$

* $S : \underline{r}(u,v) = \text{-----}$
  $ds = \| r_u \times r_v \|\, du\, dv$

* $S : z = f_{(x,y)}$
  $ds = \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy$

$\longrightarrow ds = \| r_\phi \times r_\theta \| d\phi\, d\theta$

$= a^2 \sin\phi\, d\phi\, d\theta = 4 \sin\phi\, d\phi\, d\theta$ (بحالة ال sphere)

* parametrization *

$x = 2\cos\theta \sin\phi$

$y = 2\sin\theta \sin\phi$

$z = 2\cos\phi$

$$= \iint 2\cos\theta \sin\phi * 4\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 8\cos\theta \sin^2\phi\, d\phi\, d\theta$$

• Flux:

→ Let $\underline{F}$ be a vector field, Then the Flux of $\underline{F}$ across S is given by:

$$\Phi = \iint_S \underbrace{\underline{F} \cdot \underline{n}}_{\text{Dot product.}} \, dS$$

$\underline{n}$ → unit normal vector. $= \dfrac{n}{\|n\|}$

Ex: find the Flux of $\underline{F} = z\hat{\underline{k}}$ across S, where:

$S: x^2 + y^2 + z^2 = a^2$

Sol:

- $r_\phi \times r_\theta = \langle a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\sin\phi\cos\theta \rangle$

- $\underline{n} = \dfrac{r_\phi \times r_\theta}{\|r_\phi \times r_\theta\|} = \dfrac{\langle a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\sin\phi\cos\phi \rangle}{a^2\sin\phi}$

always the same.

→ $dS = \|r_\phi \times r_\theta\| \, d\phi \, d\theta = a^2\sin\phi \, d\phi \, d\theta$

نا تج الضرب = صفر

dS

$\underline{F} \cdot \underline{n} \, dS = \langle 0, 0, \underset{a\cos\phi}{z} \rangle \cdot \dfrac{\langle \text{.....}, \text{.....}, a^2\sin\phi\cos\phi \rangle}{\cancel{a^2\sin\phi}} \cdot \cancel{a^2\sin\phi} \, d\phi \, d\theta$

$= a^3 \sin\phi \cos\phi^2 \, d\phi \, d\theta$

$\Rightarrow \Phi = \displaystyle\int_0^{2\pi}\int_0^{\pi} a^3 \sin\phi \cos^2\phi \, d\phi \, d\theta.$

Ex: find the flux of $\underline{F} = x\underline{i} + y\underline{j} + z\underline{k}$, across the surface S where S: the portion of $z = 1 - x^2 - y^2$ which lies above $z = 0$ (x-y plane).

Sol:

$$\Phi = \iint_S \underline{F} \cdot \underline{n} \, dS$$

- $\underbrace{z + x^2 + y^2 = 1}_{G} \longrightarrow \nabla G = \langle 2x, 2y, 1 \rangle$

- $\underline{n} = \dfrac{\nabla G}{\|\nabla G\|} = \dfrac{\langle 2x, 2y, 1 \rangle}{\sqrt{4x^2 + 4y^2 + 1}}$.

same

- $dS = \sqrt{f_x^2 + f_y^2 + 1} \, dx\,dy = \sqrt{4x^2 + 4y^2 + 1} \, dx\,dy$

$\to \underline{F} \cdot \underline{n} \, dS = \langle x, y, z \rangle \cdot \dfrac{\langle 2x, 2y, 1 \rangle}{\cancel{\sqrt{4x^2 + 4y^2 + 1}}} \cdot \cancel{\sqrt{4x^2 + 4y^2 + 1}} \, dx\,dy$

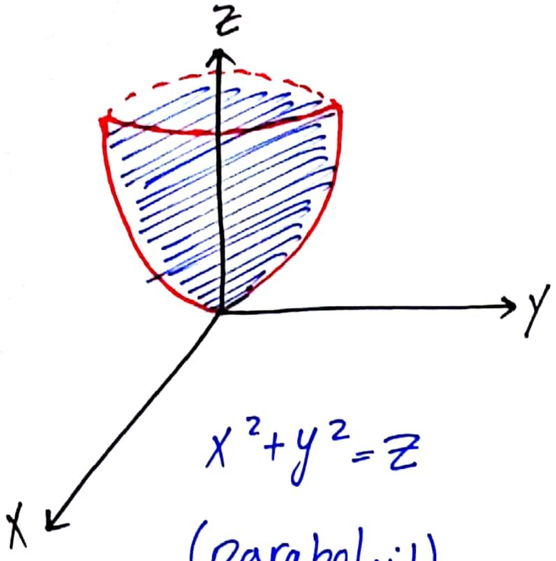
$= 2x^2 + 2y^2 + z \longrightarrow$ z نعوض قيمة

$= 2x^2 + 2y^2 + (1 - x^2 - y^2)$

$= x^2 + y^2 + 1$

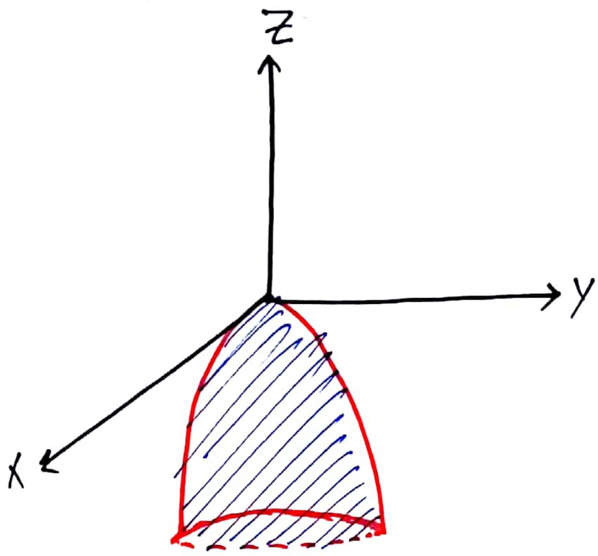
$$\to \Phi = \iint_R (x^2 + y^2 + 1) \, dx\,dy$$

now: for the previous example:

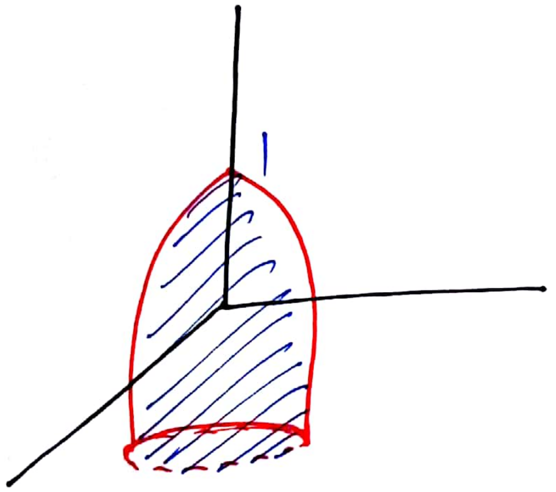


$x^2+y^2=z$

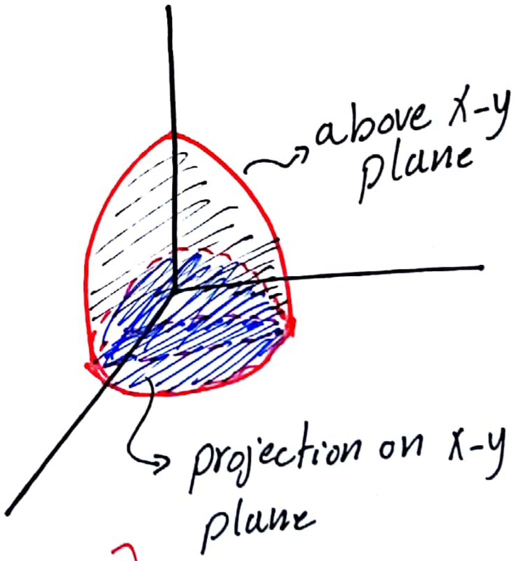
(paraboloid)



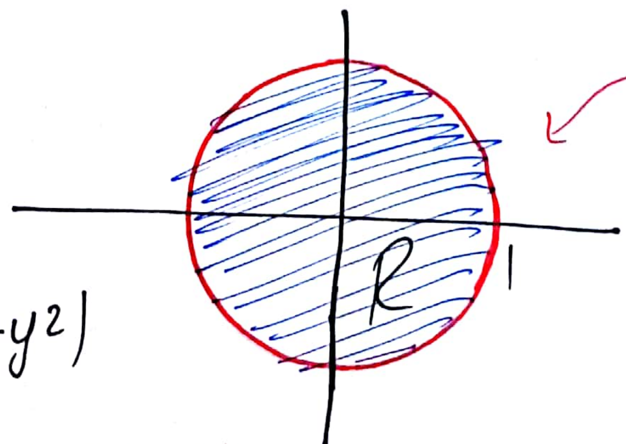
$-(x^2+y^2)=z$



$z=1-(x^2+y^2)$



الشكل المطلوب للسؤال السابق



$z=1-(x^2+y^2)$

$z=0$ $\quad x^2+y^2=1$ (disk)

Follow :

- $\Phi = \iint_R (\underbrace{x^2+y^2}_{r}+1)\,dx\,dy$

polar coordinate

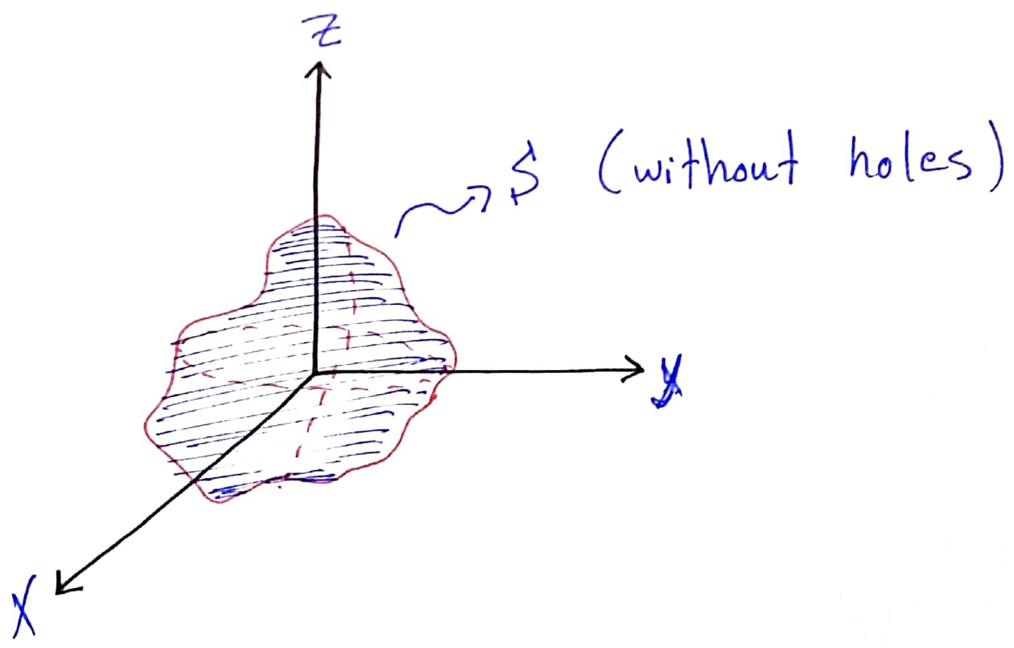
- $\Phi = \int_0^{2\pi}\int_0^1 (r^2+1)\, r\, dr\, d\theta$

Divergence theorem :

$$\iint_S \underline{F}\cdot\underline{n}\,ds = \iiint_G \operatorname{div} \underline{F}\,dv$$

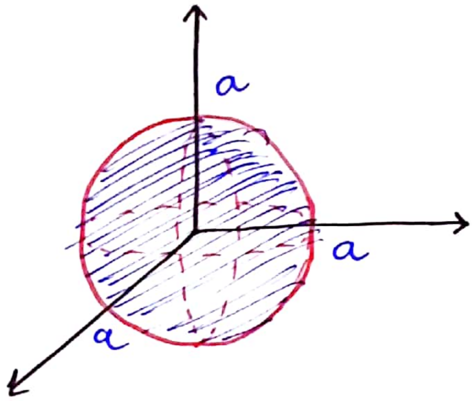
* where S: piece wise smoth Closed surface.

* $\underline{F} = f\underline{i} + g\underline{j} + h\underline{k}$, (f, g and h) have first partial derivatives (without holes)



Example 1: Find the outward flux of $\underline{F} = z\,\underline{k}$ across $x^2+y^2+z^2=a^2$.

Sol:



$$\iint_{S \text{ closed}} \underline{F}\cdot n \, ds = \iiint_G \operatorname{div} \underline{F} \, dv$$

→ volume of the solid sphere with radius (a).

$$= \iiint_G (1)\, dv = \frac{4\pi}{3} a^2$$

→ outward flux (التدفق الخارج من الكرة)

* inward flux (التدفق الداخل الى الكرة) $= \boxed{-\frac{4}{3}\pi a^2}$

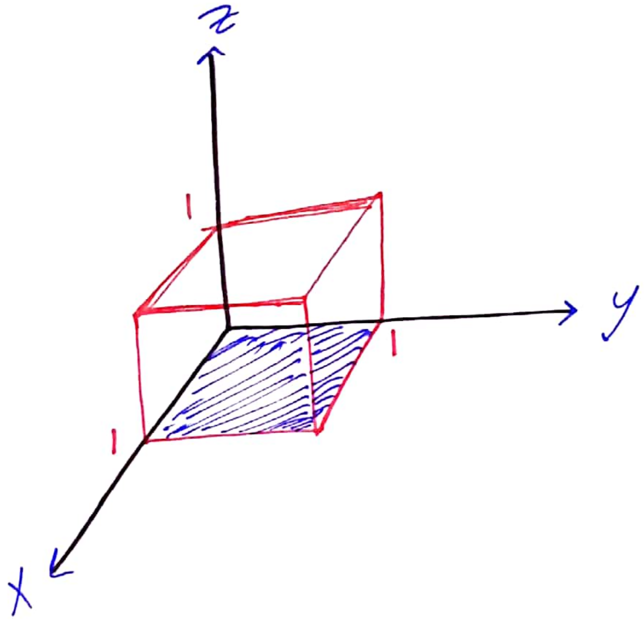
● $\boxed{\text{div} = \text{Zero}}$: means that the $\boxed{\text{outward flux}}$ from the region is equal to $\boxed{\text{Zero}}$

Scanned by CamScanner

Example 2: Find the outward flux of:

$\underline{F} = 2x\,\underline{i} + 3y\,\underline{j} + z^2\,\underline{k}$, across the unit cube in the first octant.

Sol:



$$\underset{S}{\iint} \underline{F}\cdot n\, ds = \underset{G}{\iiint} \operatorname{div} \underline{F}\, dv$$

S closed

G ⇝ solid cube

$$= \int_0^1\int_0^1\int_0^1 (2+3+2z)\, dx\,dy\,dz \rightsquigarrow dV$$

$$= \int_0^1 dx \int_0^1 dy \int_0^1 5+2z\, dz = \boxed{6}$$

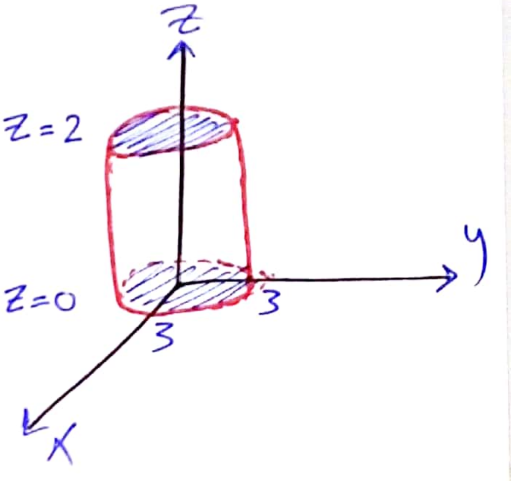
Example 3: Find the outward flux of:

$\underline{F} = x^3 \underline{i} + y^3 \underline{j} + z^3 \underline{k}$, a cross the surface of the region enclosed by: $x^2 + y^2 = 9$ and $z = 0$, $z = 2$

sol

$$\iint_S \underline{F} \cdot \underline{n} \, ds = \iiint_G \operatorname{div} \underline{F} \, dv$$

$$= \iiint_G (3x^2 + 3y^2 + 3z^2) \, dv$$

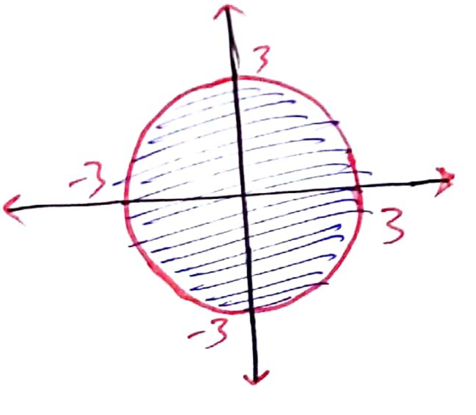


$$= 3 \iiint (x^2 + y^2 + z^2) \, dv$$

$$= 3 \int_0^{2\pi} \int_0^3 \int_0^2 (r^2 + z^2) \, r \, dz \, dr \, d\theta$$

where:

$$\left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array} \right\}$$



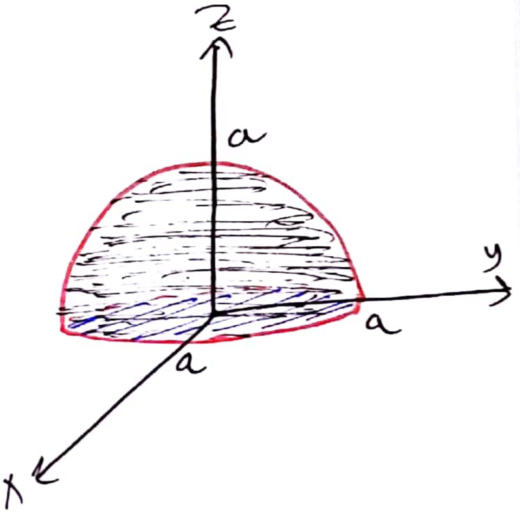
Example 4: Find the outward flux of:

$\underline{F} = \langle x^3, y^3, z^3 \rangle$, across the surface of the region enclosed By $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$

sol:

$$z = \sqrt{a^2 - x^2 - y^2}$$
$$z^2 = a^2 - x^2 - y^2$$
$$x^2 + y^2 + z^2 = a^2$$

the upper simi sphere



$$\Phi = \iint_S \underline{F} \cdot \underline{n} \, dS = \iiint_G \operatorname{div} \underline{F} \, dv$$

$$= \iiint_G 3(x^2 + y^2 + z^2) \, dv$$

$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$
$$dv = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a 3\rho^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\phi \, d\phi \int_0^a \rho^4 \, d\rho$$

$$= (3)(2\pi)(2)\left(\frac{a^5}{5}\right) = \frac{12}{5} \pi a^5$$

Example 5: Find the outward Flux of:

$\underline{F} = \dfrac{c\,\underline{r}}{r^3}$, across: $S = x^2 + y^2 + z^2 = a^2$

sol:

$$\underline{F} = c \cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$\underline{F} = c\left[\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\,\underline{\hat{i}} + \frac{y}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\,\underline{\hat{j}} + \frac{z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\,\underline{k}\right]$$

$\rightarrow$ $\text{flux} = \iint_S \underline{F} \cdot \underline{n}\, ds$

* ما بنحل على (divergence theorem)
بسبب وجود (hole) ← بنحل (parametrization)

$$\left\{ x = a\cos\theta\sin\phi \;,\quad y = a\sin\theta\sin\phi \;,\quad z = a\cos\phi \right\}$$

$$\rightarrow \underline{F} = c\left\langle \frac{a\cos\theta\sin\phi}{a^3}, \frac{a\sin\theta\sin\phi}{a^3}, \frac{a\cos\phi}{a^3} \right\rangle$$

$\rightsquigarrow \left(\sqrt{x^2+y^2+z^2}\right)^3 = \underline{r}^3$

$$\underline{F} = \frac{c}{a^2}\left\langle \cos\theta\sin\phi, \sin\phi\sin\theta, \cos\phi \right\rangle$$

unit normal vector

$$\rightarrow n = \frac{r_\theta \times r_\phi}{\|r_\theta \times r_\phi\|} = \frac{1}{a^2\sin\phi}\left\langle a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin\phi^2, a^2\sin\phi\cos\phi \right\rangle$$

$$\underline{F}\cdot\underline{n}\, ds = \underbrace{\frac{c}{a^2}\left\langle \cos\theta\sin\phi, \sin\phi\sin\theta, \cos\phi \right\rangle}_{F} \cdot \underbrace{\frac{a^2}{a^2\sin\phi}\left\langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi \right\rangle}_{n} \cdot \underbrace{a^2\sin\phi\, d\theta\, d\phi}_{ds}$$

$$= c\left[\cos^2\theta\sin^3\phi + \sin^2\theta\sin^3\phi + \sin\phi\cos^2\phi\right]$$

$\rightarrow$ follow

*Follow: example 5:

$$\underline{F}\cdot\underline{n}\, ds = C\left[\cos^2\theta\sin^3\phi + \sin^2\theta\sin^3\phi + \sin\phi\cos^2\phi\right]$$

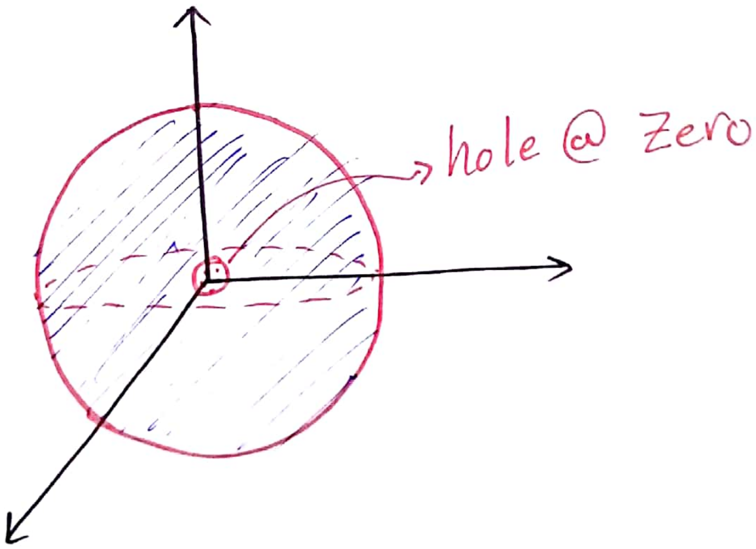
$$= C\left[\sin^3\phi\underbrace{(\cos^2\theta+\sin^2\theta)}_{1} + \sin\phi\cos^2\phi\right]$$

$$= C\left[\sin\phi\left[\underbrace{\sin^2\phi+\cos^2\phi}_{1}\right]\right] = C\sin\phi\, d\phi\, d\theta$$

$$\rightarrow \iint_S \underline{F}\cdot\underline{n}\, ds = C\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta$$

$$= C\cos\phi\Big]_0^{\pi}(2\pi) = \boxed{4\pi C}$$

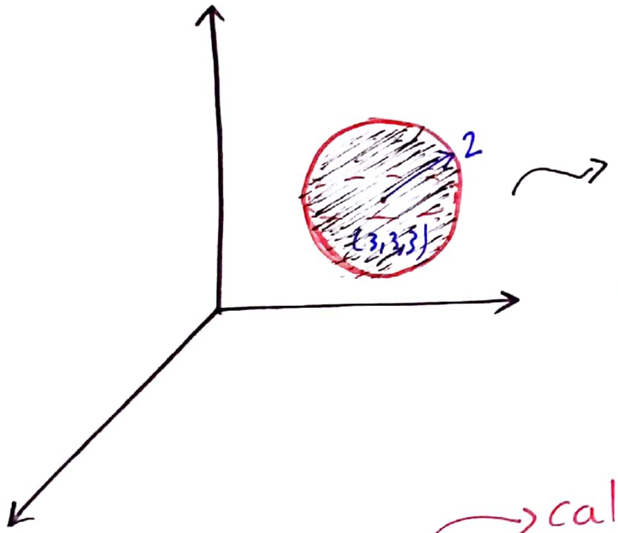
الرسمة للمثال السابق.



Example 6: Find the flux of: $\underline{F} = \frac{c\underline{r}}{r^3}$, across:

$S: (x-3)^2 + (y-3)^2 + (z-3)^2 = 4.$

sol:



لا يوجد hole لانه
ليست ضمن نقطة (0,0,0).

* نطبق (Divergence theorem).

calculated at page $\boxed{6}$

$$\iint_S \underline{F}\cdot\underline{n}\, ds = \iiint_G \underbrace{\operatorname{div} \underline{F}}_{\text{Zero}}\, dv = \text{Zero}$$

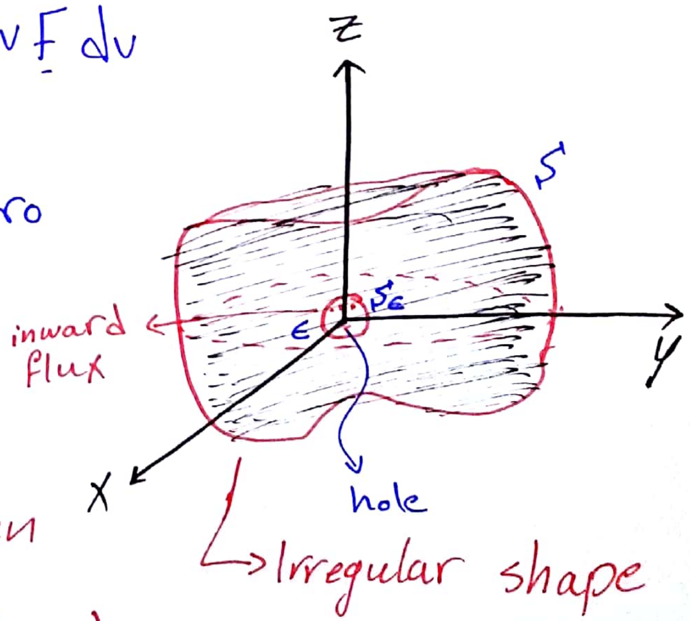
Example 7: Find the flux of: $F = \frac{c\underline{r}}{r^3}$ across any closed surface containing $(0,0,0) \rightarrow$ (hole)

sol:

$$\iint_S F\cdot n\, ds + \iint_{S_\epsilon} \underline{F}\cdot n\, ds = \iiint \operatorname{div} \underline{F}\, dv$$

$$\iint_S \underline{F}\cdot\underline{n}\, ds + (-4\pi c) = \text{Zero}$$

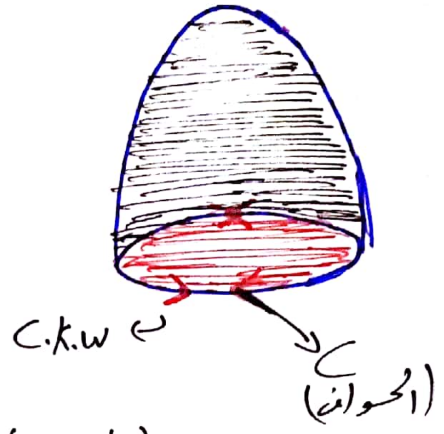
$$\therefore \iint_S \underline{F}\cdot\underline{n}\, ds = 4\pi c$$



* if there is a hole in the sphere then the outward flux is equal to: $(4\pi c)$

Stoke's theorem:

$$\int_C \underline{F}\cdot d\underline{r} = \iint_S \text{curl } \underline{F}\cdot\underline{n}\, ds ,$$



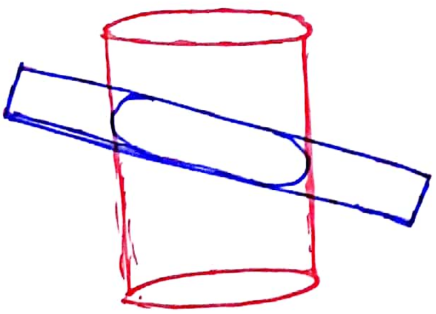
\# $\underline{F} = f\underline{i} + g\underline{j} + h\underline{k}$

\# S : pws surface (not necessary closed)

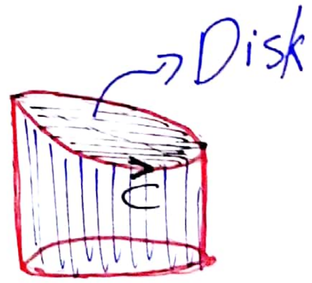
\# C: the boundary of S (simple and closed)

\# (f, g, h) : have continuous first partial derivatives

Example1: use stoke's theorem to evaluate : $\int_C \underline{F}\cdot d\underline{r}$, where $\underline{F} = -y^2\underline{i} + x\underline{j} + z^2\underline{k}$, C: the curve of intersection of $\underbrace{y+z=2}_{\text{plane}}$ & $\underbrace{x^2+y^2=1}_{\text{cylinder}}$.



$\rightarrow$



$S : y + z = 2$

$$\iint_S \text{curl } \underline{F}\cdot\underline{n}\, dS$$

Follow $\longrightarrow$

→ follow

$* \ \text{Curl } \underline{F} = (0-0)\underline{i} - (0-0)\underline{j} + (1+2y)\underline{k}$

$\text{curl } \underline{F} = \langle 0, 0, 1+2y \rangle$

$\rightarrow n = \dfrac{\nabla G}{\|\nabla G\|} = \dfrac{\langle 0,1,1 \rangle}{\sqrt{2}}$

$\rightarrow dS = \sqrt{f_x^2 + f_y^2 + 1}\ dx\,dy \rightarrow y + z = 2 \rightarrow z = 2 - y$

$dS = \sqrt{2}\ dx\,dy$

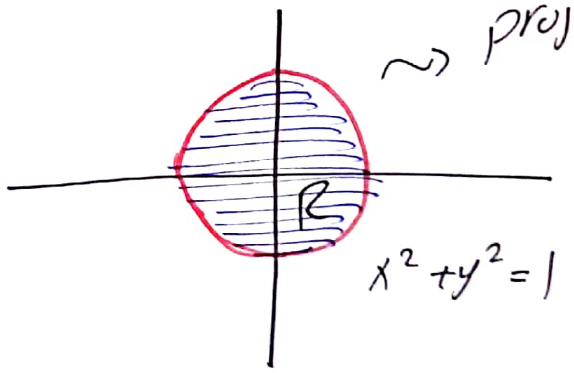
$\rightarrow \text{Curl } \underline{F}.\underline{n}\, dS = \underbrace{\langle 0, 0, 1+2y \rangle}_{\underline{F}} \cdot \underbrace{\langle 0,1,1 \rangle}_{\underline{n}}\ dx\,dy$

$= (1+2y)\, dx\,dy.$

$* \iint \text{curl } \underline{F}.\underline{n}\, ds =$

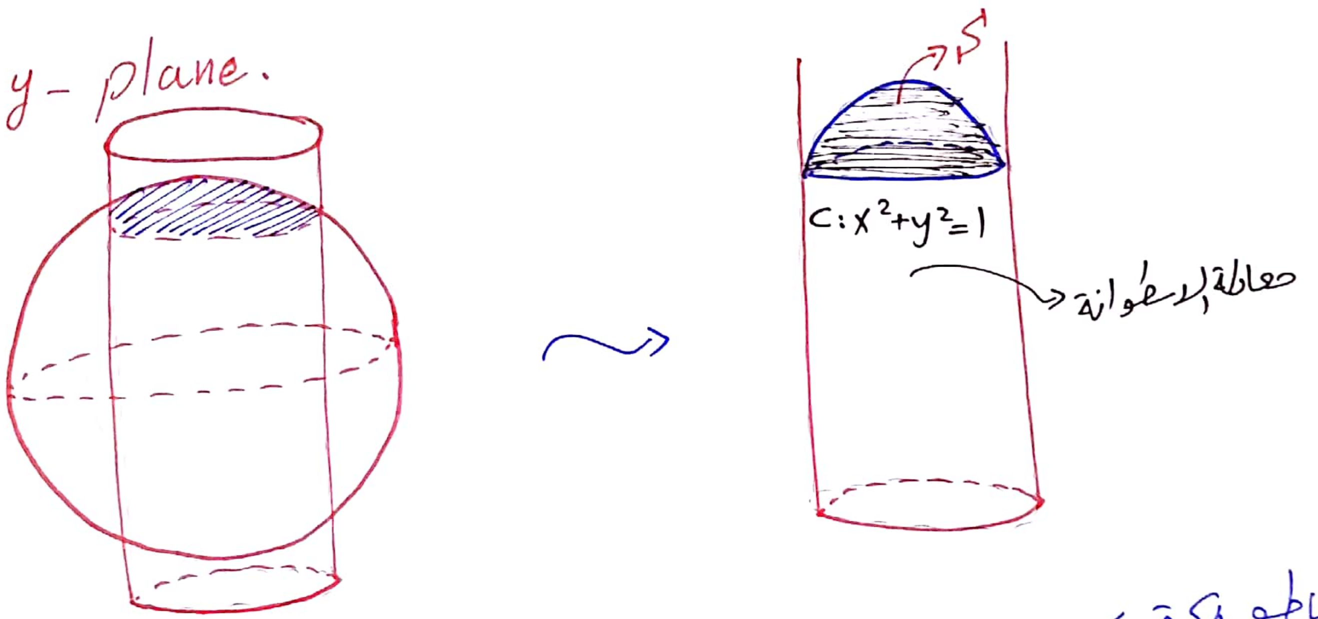
$\iint (1+2y) \underbrace{dx\,dy}_{\text{projection}}$

$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\,dr\,d\theta.$



Example 2: use stoke's theorem to evaluate :

$\iint \text{curl } \underline{F} \cdot \underline{n} \, ds$, where $\underline{F} = yz\,\underline{i} + xz\,\underline{j} + x\,\underline{k}$, S: the part of $x^2+y^2+z^2=4$ that lies inside $x^2+y^2=1$ and above xy-plane.



$1 + z^2 = 4 \rightarrow z^2 = 3 \rightarrow z = \sqrt{3} \leftarrow$ * تقاطع الكرة مع الاسطوانة

$\int_C \underline{F} \cdot d\underline{r}$, where $\underline{F} = \langle yz, xz, x \rangle$

$C: x^2+y^2=1$, $z=\sqrt{3}$

$\left\{ \begin{array}{l} x(t) = \cos t \\ y(t) = \sin t \\ z = \sqrt{3} \text{ (constant)} \end{array} \right\}$ $\{0 \le t \le 2\pi\}$

* $r(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$

$\int_C F \cdot dr = \int_a^b F_{(r(t))} \cdot r'(t)\, dt$

$\rightarrow$ follow

Follow

$\rightarrow \underline{F}(r(t)) = \langle \sqrt{3}\sin t, \sqrt{3}\cos t, \cos t \rangle$

$r'(t) = \langle -\sin t, \cos t, 0 \rangle$

$F(r(t)) \cdot r'(t) = -\sqrt{3}\sin^2 t + \sqrt{3}\cos^2 t + \text{Zero} = \underline{\sqrt{3}\cos 2t}$

$(\cos 2\alpha = \cos^2\alpha - \sin^2\alpha)$

$\int_c \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot r'(t)\, dt$

$= \sqrt{3}\int_0^{2\pi} \cos 2t\, dt = \underline{\text{Zero}}$

{* the integral for any sinusoidal on one or (n) period is equal to zero}

Example: suppose all conditions of divergence theorem are satisfied. show that:

$$\iint_S f\nabla g \cdot n\, ds = \iiint_G [f\nabla^2 g + \nabla f \cdot \nabla g]\, dv$$

sol:

$$\iint_S \underbrace{f\nabla g}_{F} \cdot n\, ds = \iiint \underline{\operatorname{div}(f\nabla g)}\, dv \quad \{\text{divergence theorem}\}$$

{ $* \nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\underline{k}$

$* \Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$\operatorname{div}(f\nabla g) = f\,\overbrace{\operatorname{div}(\nabla g)}^{= \nabla \cdot \nabla g = \nabla^2 g} + \nabla f \cdot \nabla g$

$\operatorname{div}(f\nabla g) = \underline{f\nabla^2 g + \nabla f \cdot \nabla g}$

So: $\iint f\nabla g \cdot n\, ds = \iiint [f\nabla^2 g + \nabla f \cdot \nabla g]\, dv$ #

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Example: suppose all the conditions of stokes theorem are satisfied, show that: $\boxed{57}$

$$\oint_c \underbrace{f\nabla g}_{F} \cdot dr = \iint_{S} \nabla f \times \nabla g \cdot n \, ds$$

sol

$$\oint_c f\nabla g \cdot dr = \iint_S curl(f\nabla g) \cdot n \, ds$$

$$Curl(f\nabla g) = \underbrace{f \, curl(\nabla g)} + \nabla f \times \nabla g$$

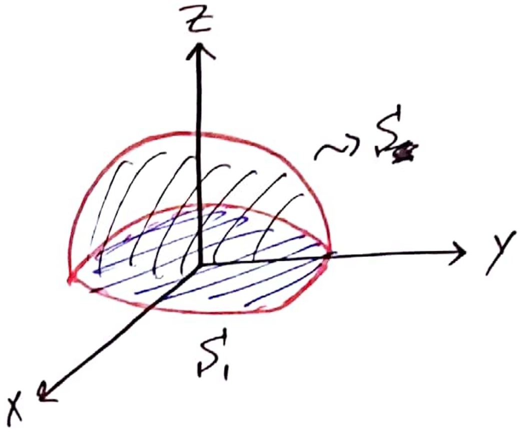
↳ will give (zero) → Curl of divergence is zero (see page 8)

$$\rightarrow curl(f\nabla g) = \nabla f \times \nabla g$$

So: $\oint f\nabla g \cdot dr = \iint_{S} \nabla f \times \nabla g \cdot n \, ds$ #

Example: evaluate: $\iint_S \underline{F}\cdot\underline{n}\, ds$, where:

$\underline{F} = (xz^2)\underline{i} + (\frac{1}{3}y^3 + \tan z)\underline{j} + (x^2z + y^2)\underline{k}$, and S: is the top half of: $x^2+y^2+z^2=1$



$$\iint_S \underline{F}\cdot\underline{n}\, ds + \iint_{S_1} \underline{F}\, n\, ds' - \iint_{S_1} F\cdot n\, ds$$

$$= \iint_{S\cup S_1} \underline{F}\cdot\underline{n}\, ds = \iiint_G \operatorname{div}\underline{F}\, dv$$

$$= \iiint_G [z^2+x^2+y^2]\, dv = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^2\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\underline{F} = \langle 0, \tfrac{1}{3}y^3, y^2 \rangle$$

$$\underline{n} = \frac{\nabla G}{\|\nabla G\|} = \frac{\langle 0,0,1\rangle}{\sqrt{1}} = \underline{k}, \quad ds = 1$$

$$\underline{F}\cdot\underline{n}\, ds = \langle 0, \tfrac{y^3}{3}, y^2\rangle \cdot \langle 0,0,1\rangle = \boxed{y^2}$$

$$\iint_{S_1} \underline{F}\cdot n\, ds = \iint_R y^2\, dA$$

$$= \int_0^{2\pi}\int_0^1 r^2 \sin\theta\, r\, dr\, d\theta$$