Numerical

NoteBook

POWER UNIT
ELECTRICAL ENGINEERING

No. Numerical

Lecture "1" 10/2/2014

Introduction :-

Dependent variable = f (Independent variable + parameters + forcing fn.)

a mathmatical model: Is defined as a formulation or equation that express the Essential features of a physical system or process In mathmatical ~~form~~ (curve / formulation) terms.

* The Solution of an Engineering problems can be obtained using Analytical or numerical methods.
* The complexity of many Practical Eng. problems makes it nesecery to use numerical methods.

* In general, the analytical or ~~the~~ closed form ~~~~ Solution gives the exact solutions of the Original mathmatical model, So if it's not possible to use the Analytical method then we use the numerical method which Approximates the exact solution.

* Numerical methods are ~~teqniques~~ technique by which mathmatical problems are formulated so they can be solved with Arithmatic & Logical operation So it reffers as computer mathmatics.

* Ex
Find the minimum of the fn.:
$y = x^2 - 3x + 2$ using Analytical & numerical methods.

FIVE APPLE

Sol

[1] Using Analytical method

$y' = 2x - 3$

$y' = 0 = 2x - 3 \longrightarrow x = 3/2 = 1.5$ (exact value)

[2] Using Numerical method

A. Let use the Interval from (1-2) with a constant Increment $\Delta x = 0.2$

x	1	1.2	1.4	1.6	1.8	2
y	0	-0.16	-0.24	-0.24	-0.16	0

* التعويض في المعادلة الأصلية *

B. To Improve the accuracy of the Sol. the search Interval of (x) can be stablished as $(1.4 < x < 1.6)$ and $\Delta x = 0.04$

X	1.4	1.44	1.48	1.52	1.56	1.6
Y	-0.24	-0.2464	-0.2496	-0.2496	-2464	-0.24

* لاحظ إذا أخذ Interval أقرب من (1.48 - 1.52) هنقترب من exact value *

* Characterestics of Numerical method *

-1- The solution procedure is Iterative where the accuracy of the Estimated sol. Improving with each Itteration.

-2- The solution procedure provides only an Amproximation to the true value.

-3- An Intial Istimate of the solution may be required

-4- The solution procedure is simple with Algorithmis representing the procedure that can be easily programmed on a digital computers

-5- The solution procedure may occasionly diverge from - rather than converge - to the true solution.

* "end of CH. 1" *

* CH. 2 * Errors *

→ Numerical errors arised from the use of approximation to represent exact mathmatical operations and quantities.

→ The relation ship between the exact or true value & the Approximation is :-

True value = Approximation + error ——— **

Then The error is :

Error (E_t) = True value - Approximation value —— **,

E_t is called the : true error / Absolute error.

→ True fraction relative error = $\dfrac{\text{True value} - \text{App. value}}{\text{True value}}$.

→ True percent relative error (ε_T) is :

$$\varepsilon_T = \frac{\text{True value} - \text{App. value}}{\text{True value}} \times 100\%$$ —— **

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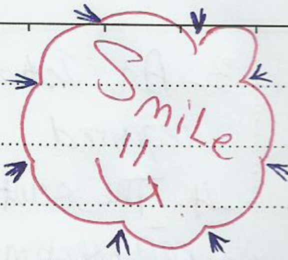
→ Percent relative error is :

$$\varepsilon_a = \frac{\text{App. error}}{\text{Approximation}} * 100\%$$



$$= \frac{\text{Present Approximation} - \text{Previous Approximation}}{\text{Present approximation}} * 100\%$$

* الخطأ ممكن أن يكون (+ve) أو (-ve) ولذلك نأخذ القيمة المطلقة للنسبة
إذاً *.

→ We may not concern ~~[crossed out]~~ with sign of the relative error but the absolute value of the error, which should be less than the prespecified Tolerance "ε_s" where :-

$\varepsilon_s = (0.5 * 10^{2-n})\ \%$; where:
(n = is the number of significant figures)

Lecture "2" 2/24/14

ex

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

add terms until the abs value of the App. error estimate (ε_a) falls below a prespecified error criterion (ε_s) using 3 significant figures: ~~for~~ at $x = 0.5$

Sol

$\varepsilon_s = (0.5 \times 10^{2-3})\% = 0.05\%$

the Computed value is $e^{0.5} = 1.648721$

$e^{0.5} = 1$ (using 1 term)

$$\varepsilon_t = \frac{1.648721 - 1}{1.648721} \times 100\% = 39.3\%$$

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2 terms :-

$$e^{0.5} = 1 + 0.5 = 1.5$$

$$\epsilon_t = \frac{1.648721 - 1.5}{1.648721} \times 100\% = 9.02\%$$

$$\epsilon_a = \frac{1.5 - 1}{1.5} \times 100\% = 33.3\%$$

3 terms :-

$$e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{2} = 1.625$$

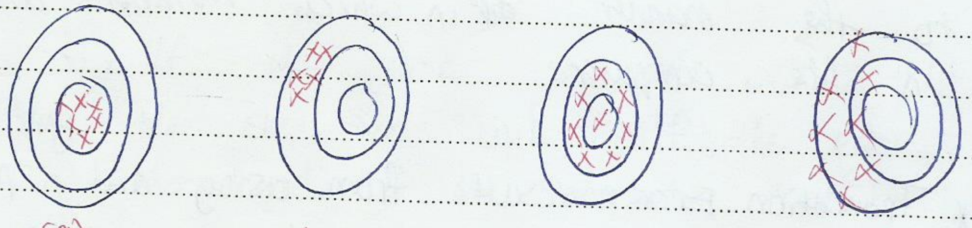
$\epsilon_t = 1.44\%$ $\quad\quad$ $\epsilon_a = 7.69\%$

~~[illegible]~~

"n (number of terms)"	"result"	"ϵ_t%"	"ϵ_a%"
1	1	39.3	—
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.645833	0.175	1.27
5	1.6484375	0.0172	0.15
6	1.6486979167	0.00142	0.0158

note that : 0.0158 < 0.05 , then after uses 6 terms we will get the result

* Accuracy and precision :-



(a) $\quad$ b $\quad$ c $\quad$ d

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Target	Bias	Accurate	precise
a	non	high	high.
b	high	non	high
c	non.	low	non.
d	high	non.	non.

* Accuracy : how closely a measured value with a true value.
* precision : refers to how closely individual meas. values agree with each other.
* Bias : systematic ~~[illegible]~~ deviation from the true value

* Errors types *

There are 2 types of errors.

-1- round_off error : due to computer Approximation
-2 Trancation error : due to mathmatics App.

notes :- * Round_off error ~~[illegible]~~ arrises because digital computer can't represent some quantities exactly which can lead to huge error.

* Round _off errors are directly related ~~to the~~ to the manner ~~[illegible]~~ in which numbers are stored in the computer.

* Trancation Errors : results from using and App. in place an ~~[illegible]~~ exact mathmatical procedure.

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Smile :)

* Taylor's series :-

→ States that any smooth fn can be Approximated as a polynomial :
The Taylor's series expansion is given by :

→ ** $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \cdots + \frac{f^{(n)}(x_i)h^n}{n!} + R_n$

→ where h is the distance between (x_i) and (x_{i+1}), $f', f'', f^{(n)}$ are the derivatives of $f(x)$ and R_n is the Remainder which includes all terms from $(n+1)$ to (∞), So :

→ ** $R_n = \frac{f^{n+1}(\xi)\, h^{n+1}}{(n+1)!}$; where :

→ (ξ) is a value of (x) that lies somewhere between (x_i) & (x_{i+1}) also
$R_n = O(h^{n+1})$

→ $O(h^{n+1})$: means that the trancation error is of the order of $(h^{(n+1)})$.

* So, if the error is ~~of~~ $O(h)$ then halving the step size will half the error.
In other hand if the error is $O(h^2)$ halving the step size will quarter the error.

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ex

Use Taylor's series expansion with $n=0$ to 6 to Approximate $f(x) = \cos(x)$ at $x_{i+1} = \pi/3$ on the basis of the value of $f(x)$ and it's deviation at $x_i = \pi/4$.

Solution :

$x_i = \pi/4$, $x_{i+1} = \pi/3$, $h = \pi/3 - \pi/4 = \pi/12$

-1- $f(x_{i+1}) = f(x_i)$ for 1st terms :

$\cos(\pi/3) = \cos(\pi/4) = 0.707106781$

$$\epsilon_t = \left| \frac{0.5 - 0.707106781}{0.5} \right| \times 100\% = 41\%$$

-2- Using two terms : for 1st order App.

$$f(x_{i+1}) = f(x_i) + f'(x_i)h$$

$\cos(\pi/3) = \cos(\pi/4) - \sin(\pi/4)\,\pi/12 = 0.52198665$

$|\epsilon_t| = 4.4\%$

"order (n)"	"$f^{(n)}(x)$"	"$f(\pi/3)$"	"$\epsilon_t\%$"
0	$\cos x$	0.707106781	41%
1	$-\sin x$	0.52198665	4.4%
2	$-\cos x$	0.497754491	0.449%
3	$\sin x$	0.49986914	2.62×10^{-2}
4	$\cos x$	0.500001551	1.51×10^{-3}
5	$-\sin x$	0.500000304	6.08×10^{-5}
6	$-\cos x$	0.499999988	2×10^{-6}

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Lecture "3" :-

2 / 26 / 14

The Remainder for the Taylor series expansion:-

suppose that we trancate the Taylor series expansion after the zero order term:-

$f(x_{i+1}) \cong f(x_i)$ then,

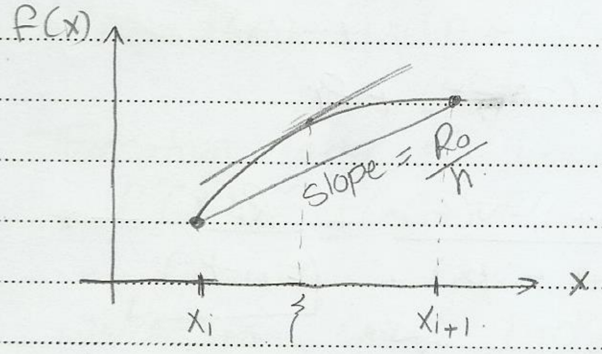
The Remainder or error consists of the infinite series of terms that's been trancated

So :

$$R_0 = f'(x_i)\, h + \frac{f''(x_i)\, h^2}{2!} + \cdots + \frac{f^{(n)}(x_i)\, h^n}{n!}$$

* It is inconvenient to deal with the Remainder in this infinite series format so we can trancate it as:

$R_0 \cong f'(x_i)h$ — * * *



* Using the derivative mean value Thm which states that if a fn f(x) & it's derivatives are cont over an interval from (x_i to x_{i+1}) then there exist at least one pnt on the fn that has a slope given by $f'(\xi)$ that is parallel to the line joining $f(x_i)$ & $f(x_{i+1})$, & the parameter (ξ) mark x value where slope is occured.

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thus :

$$f'(\xi) = \frac{R_0}{h} \longrightarrow R_0 = f'(\xi)h$$

→ Using The Same procedures So:

$$R_1 = f''(\xi)\frac{h^2}{2!} \quad —— ***$$

(*) Using The Taylor's series to estimate Trancation errors:-

The Taylor's series expansion for $v(t)$ is:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + \frac{v''(t_i)}{2!}(t_{i+1} - t_i)^2 + \dots + R_n$$

→ Assume that the Trancation happens after the 1st derivatives, then

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + R_1$$

$$\text{So: } v'(t_i) = \underbrace{\frac{v(t_{i+1}) - v(t_i)}{(t_{i+1} - t_i)}}_{\text{1st order Approximation}} - \underbrace{\frac{R_1}{(t_{i+1} - t_i)}}_{\text{Trancation error}} \quad —— *$$

$$\text{So: } \frac{R_1}{(t_{i+1} - t_i)} = \frac{v''(\xi)(t_{i+1} - t_i)}{2!}$$

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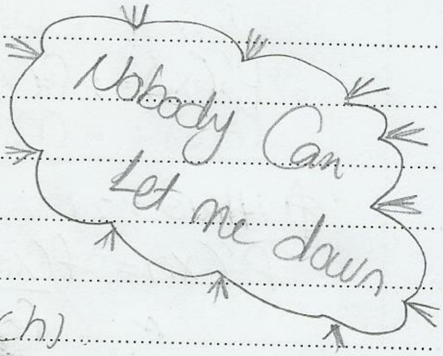
* Numerical diffrentiation :-



① it can be represented as

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(x_{i+1} - x_i)$$

OR

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

→ This is called Forward difference because it utilizes data at i & $i+1$ to estimate the derivative.

② Taylor's series can be expanded backward to calculate a previous value on the basis of a present value as follows:

$$f(x_{i-1}) = f(x_i) - f'(x_i)\,h + \frac{f''(x_i)h^2}{2!} - \cdots + \cdots - \cdots$$

Trancation after 1^{st} derivative, then:

$$f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{h}$$

which is called backward difference Approximation.

③ A Third way to Approximate the 1^{st} derivative by subtraction backword taylor series from the forward taylor's series, thus:

$$f(x_{i+1}) = f(x_{i-1}) + 2f'(x_i)h + 2\frac{f''(x_i)h^3}{3!}$$, then:-

"Check from the book"

* Note * $h = \frac{x_{i+1} - x_{i-1}}{2} = x_{i+1} - x_i = x_i - x_{i-1}$

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$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - 6\frac{f'''(x_i)h^2}{6} + \dots$$

& thus:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - O(h^2)$$

→ This is called The Centered finite difference Approximation of 1st order derivative.

* ex

Use forward & backward difference Approximation of $O(h)$ & a central difference Approximation of $O(h^2)$ to estimate the 1st derivative of
$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$
at $x = 0.5$ using $h = 0.5$, Repeat using $h = 0.25$.

sol

$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$

$f'(0.5) = -0.9125$ "True value"

* For $h = 0.5$

$x_{i-1} = 0$, $x_i = 0.5$, $x_{i+1} = 1$

$f(x_{i-1}) = 1.2$, $f(x_i) = 0.925$, $f(x_{i+1}) = 0.2$

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-1- For /using Forward difference :

$$f'(0.5) = \frac{f(x_{i+1}) - f(x_i)}{h} = \frac{f(1) - f(0.5)}{0.5}$$

$$= \frac{0.2 - 0.925}{0.5} = -1.45$$

$|\epsilon_t| = 58.9\ \%$

-2- Using Backward difference

$$f'(0.5) = \frac{f(0.5) - f(0)}{0.5} = \frac{0.925 - 1.2}{0.5}$$

$$= -0.55$$

$|\epsilon_t| = 39.7\ \%$

-3- Using Centered :

$$f'(0.5) = \frac{f(1) - f(0)}{2(0.5)} = \frac{0.2 - 1.2}{1} = -1$$

$|\epsilon_t| = 9.8\ \%$

* For $h = 0.25$

$x_{i-1} = 0.25$ $\quad f(x_{i-1}) = 1.10351363$

$x_i = 0.5$ $\quad f(x_i) = 0.925$

$x_{i+1} = 0.75$ $\quad f(x_{i+1}) = 0.63632813$

-1- Using Forward difference :

$$f'(0.5) = \frac{0.63632813 - 0.925}{0.25} = -1.155$$

$|\epsilon_t| = 26.5\%$ (half the error)

No.

2- Using Backward:

$$f'(0.5) = \frac{0.925 - 1.10351563}{0.25} = -0.714$$

$|\epsilon_t| = 21.7\ \%$ (half the error)

3- Using Centered:

$$f'(0.5) = \frac{0.63632813 - 1.10351363}{2(0.25)} = -0.934$$

$|\epsilon_t| = 2.4\ \%$ (quarter the error)

***

* The Second Forward finite difference is:

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

* The backward finite difference is: "Second"

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h)$$

* The second centered finite difference:

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h)$$

OR

$$f''(x_i) = \frac{\dfrac{f(x_{i+1}) - f(x_i)}{h} - \dfrac{f(x_i) - f(x_{i-1})}{h}}{h}$$

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*Lecture "4" Total numerical Error :- Mar/03/14

is the summation of the trancation & round of error where the round of error can be minimized by increase the number of significant figures of the computer & while the trancation error can be minimized by decreasing the step size

* The 1st Order Centered difference approximation is -

$$\underbrace{f'(x_i)}_{\text{true value}} = \underbrace{\frac{f(x_{i+1}) - f(x_{i-1})}{2h}}_{\text{Approximation value}} - \underbrace{\frac{f^{(3)}(\xi)\, h^2}{6}}_{\text{Trancation error}}$$

The fn $f(x_{i+1})$ & $f(x_{i-1})$ ~~make~~ may have round of error so :-

$$f(x_{i-1}) = \bar{f}(x_{i-1}) + e_{i-1}$$

$$f(x_{i+1}) = \bar{f}(x_{i+1}) + e_{i+1}$$ where

(e_{i-1}, e_{i+1}) are the associated rand off error. thus :-

$$\underbrace{\bar{f}(x_i)}_{\text{True value}} = \underbrace{\frac{\bar{f}(x_{i+1}) - \bar{f}(x_{i-1})}{2h}}_{\text{finite difference Approximation}} + \underbrace{\frac{e_{i+1} - e_{i-1}}{2h}}_{\text{round off erros}} - \underbrace{\frac{f^{(3)}(\xi)\, h^2}{6}}_{\text{Trancation error}}$$

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* اشتق بالنسبة لـ H لإيجاد القيمة الأفضل لل "step size".

Total error is :

$$\left| f'(x_i) - \frac{\bar{f}(x_{i+1}) - \bar{f}(x_{i-1})}{2h} \right| \leq \frac{\epsilon}{h} + \frac{h^2 M}{6}$$ where ,

ϵ : is the upper bound of the absolute value of each component of the round-off error so the maximum value of the difference is :
$e_{i+1} - e_{i-1} = 2\epsilon$.
M : is the maximum absolute value of the third derivative.

*** An Optimal step size is :

$$h_{opti} = \sqrt[3]{\frac{3\epsilon}{M}}$$

* ROOTS *

the roots are the zero's of equation which are the Input values that make the output equal zero & they can be found using :

-1- Backeting method : which envolves two intial guesses that bracket the root & it always converges but slowly.

-2- Open Method : which envolves one or more intial guesses but there is no need to bracket the root , this methods doesn't always work & may diverge but it converges quicker.

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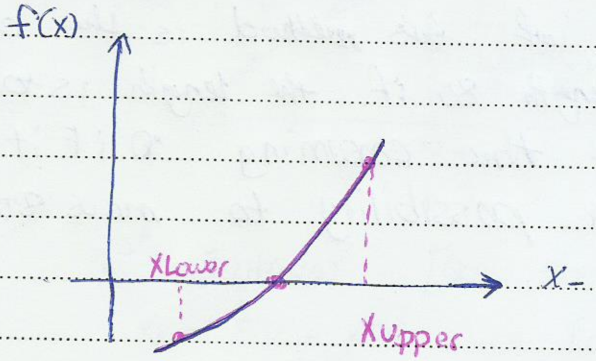
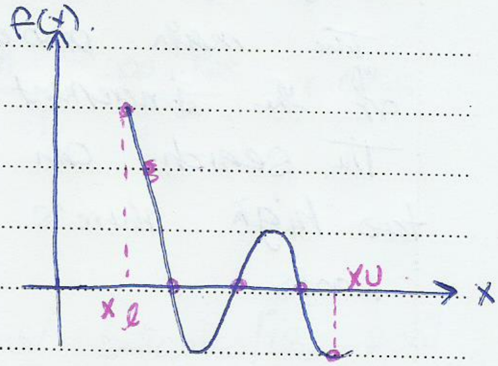
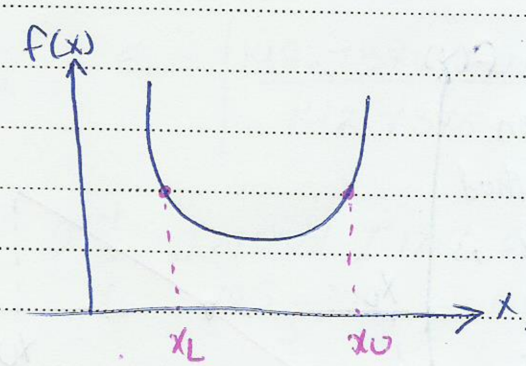
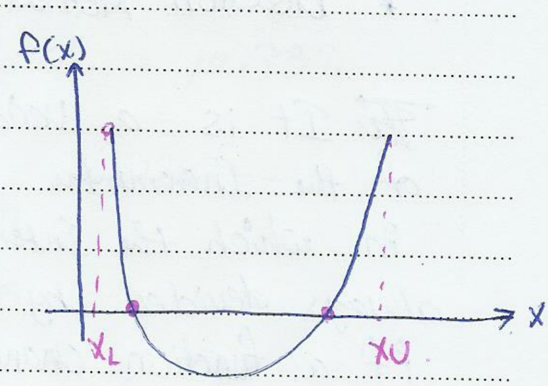
* **Graphical method :-**

It is a simple method to obtain a estimate of the root of the fn. by plotting the fn. & observe where it crosses x-axis. which provides a rough a proximation of ~~the~~ the root which can be employed as starting guess for numerical method.







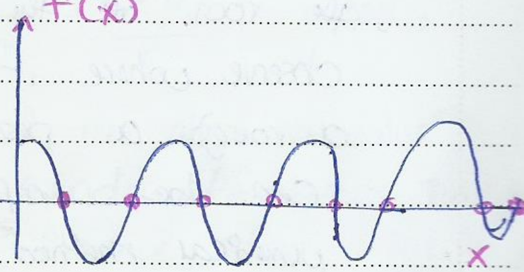


* If $f(X_L)$ & $f(X_U)$ have oppisite signs, there are odd number of roots In the Interval. Other wise, If $f(X_L)$ & $f(X_U)$ have the Same signs then there are either ~~no~~ no roots or an even num. of roots in the Interval. This conditions don't hold if the fn. ~~is~~ are tangintial to x-axis. & for discontinous fns.

* Bracketing method & Initial guesses :-

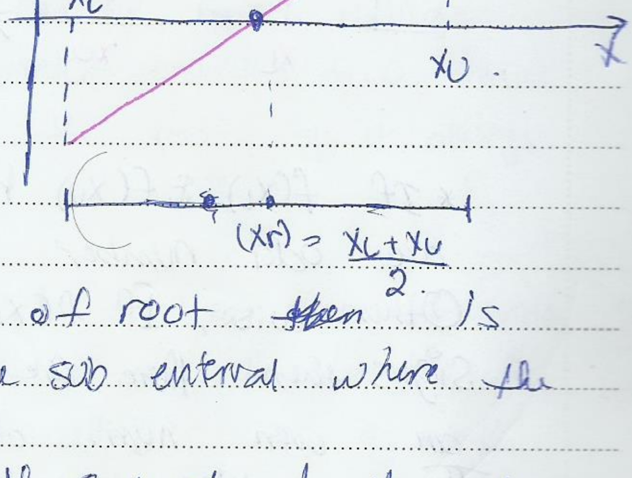
* The Incremental Search :-

In general ~~the~~ Incremental search method depends on observation by taking an Interval where the function change sign occure



The main problem of this method is the choise of the Increment length so if the length is too small The search can be time consuming & if it is too high there's a possibility to miss some Roots.

* Bisection Method.



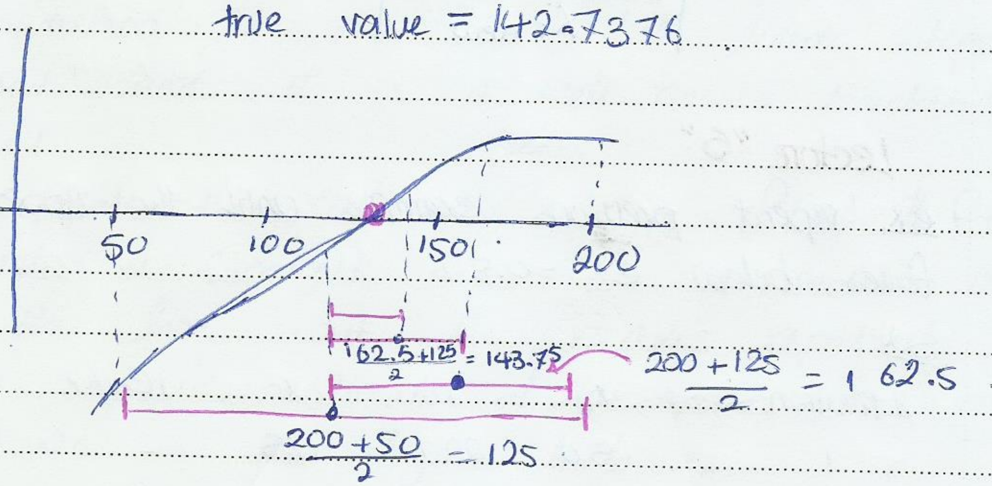
$(X_r) = \frac{X_L + X_U}{2}$

~~The~~ It is a Variation of the Incremental method in which the Interval is always devided by 2 if a function changes sign over an Interval the fn value at the mid point is evaluated and ~~to~~ the location of root ~~then~~ is determined within the sub enterval where the sign change occur.

The sub interval becomes the Interval of the next itteration & it reapeted until we get the root.

ex. Use Bisection method to find the root of figure below. "check the book fig (5.5)".

true value = 142.7376



sol/

1st Interval [50 - 200]

$x_r = \frac{50+200}{125}$ → Interval becomes [125 - 200].

$\epsilon_t = \left| \frac{142.7376 - 125}{142.7376} \right| \times 100\% = 12.43\%$

$f(125) = -ve$.

2nd Interval [125 - 200].

$x_r = \frac{125+200}{2} = 162.5$.

$f(162.5) = +ve$.

$\epsilon_t = 13.85\%$.

becomes → [125 - 162.5].

3rd Interval [125 - 162.5)

$x_r = \frac{162.5+125}{2} = 143.75$

$\epsilon(t) = 0.709\%$.

$f(143.75) = -ve$.

and so on.

No.

* Approximate relative error (ϵ_a) =

$$\epsilon_a = \left| \frac{X_{r\,new} - X_{r\,old}}{X_{r\,new}} \right| \times 100\%$$

Lecture "5"

<u>ex</u> repeat previous example until the Approximate error falles below $\epsilon_s = 0.5\%$

<u>sol</u>

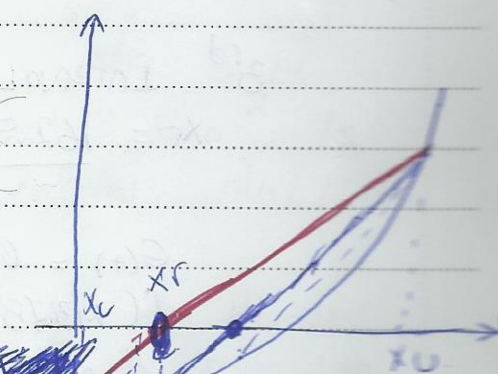
Iteration	X_L	X_U	X_r	$\lvert \epsilon_a \rvert \%$	$\lvert \epsilon_t \rvert \%$
1	50	200	125	_	12.43
2	125	200	162.5	23.08	13.85
3	125	162.5	143.75	13.04	0.71
4	125	143.75	134.375	6.98	5.86
5	134.375	143.75	139.0625	3.37	2.58
6	139.0625	143.75	141.4063	1.66	0.95
7	141.4063	143.75	142.5781	0.82	0.11
8	142.5781	143.75	143.1641	$\boxed{0.41}$ less than 0.5%	0.3

to remember :-

* $\epsilon_a\% = \frac{\text{new value} - \text{old value}}{\text{new value}} \times 100\%$

* False position :-

~~$X_r = X_u - f(X_u)(X_l - X_u)$~~



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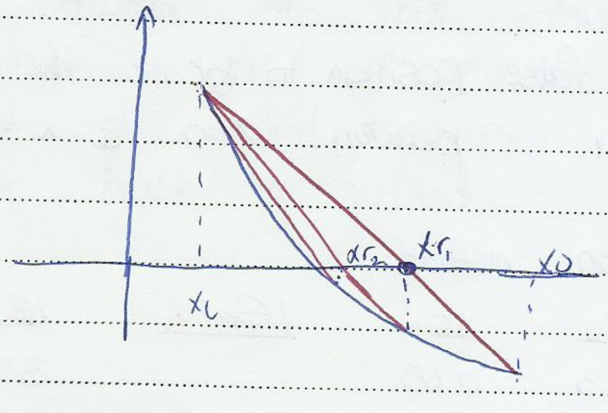
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \quad — \; (*)$$

* This method is also called "the linear interpolation method" and it is a well known bracketing method
* It locates the root by joining $f(x_l)$ & $f(x_u)$ with a straight line the intersection of this line with the x-axis represents an improve estimate of the root (x_r) as shown in the figure before.
* then replace which-ever of the two intial guesses x_l OR x_u yields a function value with the same sign as $f(x_r)$ & the process is repeated until the root is estimated probably.
*



ex repeat the previous example using False position method.

sol $x_l = 50 \qquad x_u = 200$

$f(x_l) = -4.579387$

$f(x_u) = 0.860291$

(1) $$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \Rightarrow$$

$$x_r = 200 - \frac{(0.860291)(50-200)}{-4.579387 - 0.860291}$$

$x_r = 176.22773$ $\varepsilon_t = 23.5$

(2) $f(x_r) = f(176.22773) = 0.566174$ then:

$x_r = x_u$ thus.

$x_u = 176.22773$ $f(x_u) = 0.566174$ "معطى من كتاب"

$x_l = 50$ $f(x_l) = -4.579887$

(3) ~~xr = 176.22773~~ as before we find x_r :-

$x_r = 162.3828$

$\varepsilon_t = 13.76\%$

$\varepsilon_a = 8.56\%$

and so on....

$\boxed{*}$ note (*) في الأسئلة بعطي fn. الـ $f(x_u)$, $f(x_l)$, $f(x_r)$ (*)

$\boxed{ex}$ Use bisection & false position to locate the root of $f(x) = x^{10} - 1$ betwen $x=0$ & $x=1.3$

true value $= 1$

sol(1) Using Bisection method.

Iteration	x_l	x_u	x_r	$\lvert\varepsilon_a\rvert\%$	$\lvert\varepsilon_t\rvert\%$
1	0	1.3	0.65	—	35%
2	0.65	1.3	0.975	33.3%	2.5%
3	0.975	1.3	1.1375	14.3%	13.8%
4	0.975	1.1375	1.05625	7.7%	5.6%
5	0.975	1.05625	1.015625	4%	1.6%

No.

Sol (2) Using False position:- ✓ using the formula

Iteration	x_L	x_u	x_r	$\epsilon_a\%$	$\epsilon_t\%$
1	0	1.3	0.09430	–	90.6%
2	0.09430	1.3	0.18176	48.1	81.8
3	0.18176	1.3	0.26287	30.9	73.7
4	0.26287	1.3	0.33811	22.3	66.2
5	0.33811	1.3	0.40788	17.1	59.2

→ and so on until we get $\epsilon_a < 5\%$.

So in this fn. Bisection theorem converges quicker than False position ...

Lecture "6"

CH. 4 Open method.

Be always number one

It requires a single starting value OR more, but donot necessarly to bracket the root, this method sometimes divarges " or move away from the true root ". When this method converges usually quicker than the bracketing method.

1- Simple fixed Point Iteration:-

Open methods employ the formula to predict the root, In Simple fixed point Iteration (one pnt Iteration or successive substitution " we arrange the fn. $f(x)$ So that x is in the left hand side of the equation.

$$x = g(x)$$

This can be done using algebric mainpulation or by adding (x) to both sides of the original equation.

No.

So:

x_{i+1} can be calculated from x_i,

So:

$x_{i+1} = g(x_i)$. thus the approximation error will be:

$$\rightarrow \varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%$$

ex Use Simple point Iteration to Locate the root of $f(x) = e^{-x} - x$ & the true value is 0.56714329 with intial guess $x_0 = 0$.

Sol

$$f(x) = e^{-x} - x = 0$$

$$x = e^{-x}$$

$$x_{i+1} = e^{-x_i}$$

Iteration	x_i	x_{i+1}	$\lvert\varepsilon_t\rvert\%$	$\lvert\varepsilon_a\rvert\%$
0	0	1	100%	–
1	1	0.3679	76.322	100
2	0.3679	0.6922	35.135	171.828
3	0.6922	0.5005	22.05	46.854
4	0.5005	0.6062	11.755	38.309
5	0.6062	0.5454	6.894	17.447
6	0.5454	0.5796	3.835	11.157
7	0.5796	0.5601	2.199	5.903
8	0.5601	0.5711	1.239	3.481

check

and so on "check the numbers from the book"

FIVE APPLE

10:12

No. ....

* remember *

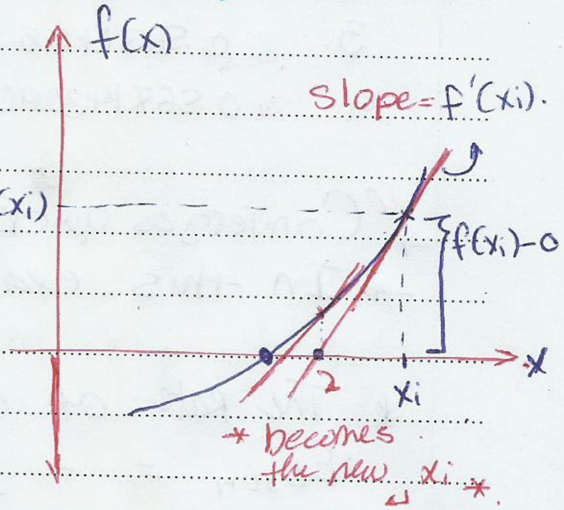
" always put the fonction as $x = g(x)$ no mater what the fn. is"

(*) Newton Raphson's method :- "NRM"

→ General formula :-

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$



* In "N.R.M" the Intial guess at (x_i) a tangent can be extended from the pnt x_i & $f(x_i)$, then the pnt where this tangent crosses the x-axis; usually represent an Improve Estimate of the Root.

ex 1

Use NRM to Solve the previous example "to estimate the Root of $f(x) = e^{-x} - x$ with an intial guess $x_0 = 0$.

Sol

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$f(x_i) = e^{-x_i} - x_i$$

$$f'(x_i) \quad -e^{-x_i} - 1$$

$$\longrightarrow \quad x_{i+1} = x_i - \left(\frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}\right).$$

No.

Iteration	x_i	x_{i+1}	ϵ_t	~~ϵ_a~~
0	0	0.5	100%	
1	0.5	0.5663101	11.8%	
2	0.5663101	0.567144165	0.147%	
3	0.567144165	0.567143290	0.000022%	
4	0.567143290	$\boxed{}$	$< 10^{-8}$	

* "Converges quicker than Simple point Iteration" #
→ In this example ←

* The Rate of convergence for this method is

$$E_{t,i+1} = -\frac{f''(x_r)}{2f'(x_r)} E_{t,i}^2$$

ex2

Use NRM to ~~locate~~ estimate the positive root of $f(x) = x^{10} - 1$ with an intial guess $x_0 = 0.5$.

Sol

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$f'(x_i) = 10x^9$$

$$x_{i+1} = x_i - \left(\frac{x_i^{10} - 1}{10x_i^9}\right)$$

FIVE APPLE

No.

i	x_i	$\lvert \epsilon_a \rvert \%$
0	0.5	—
1	51.65	99.032
2	46.485	11.111
3	41.8365	11.111
4	37.65285	11.111
⋮	⋮	⋮
40	1.002316	2.13
41	1.000024	0.229
42	1	0.002

"In this example it's converges slowly" So it depends on the intial guess and on the function it self.

Lecture "7" Secant method

12/3/2014

The problem In "NRM" is the evaluation of the derivative, Since there are certain functions whose derivatives may be very difficult to evalute, So the derivative Can be approximated by back word finite difference:

$$f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

FIVE APPLE

No.

* From "NRM":

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

* By substitution:

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

This is the Secant Formula.

* Secant method requires two intial estimates/(guesses) but there's no need to bracket the root.

* Instead of using two intial estimates we can involves a fractional Perterbation (δ) of the independent variables to estimate $f'(x)$

$$f'(x_i) \approx \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

where: δ is a small perturbration fraction

So:

$$x_{i+1} = x_i - \frac{\delta x_i f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

which is the modified secant method.

No.

ex

Use the Modified Secant Method to determine a mass of a Jumper with drag coefficient $C_d = 0.25$ kg/m to a volicity of $v(t) = 36$ m/s after 4 seconds of free fall,

→ note that the accelaration of gravity is $g = 9.81$ m/s^2, use intial guess of $m = 50$ kg & $\delta = 10^{-6}$.

$$f(m) = \sqrt{\frac{gm}{C_d}} \tanh\left(\sqrt{\frac{gC_d}{m}}\, t\right) - v(t)$$

Sol

~~$x_0 = 50$~~

$$x_{i+1} = x_i - \frac{\delta x_i \, f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

$x_i = 50$

$x_i + \delta x_i = 50 + 10^{-6} \times 50 = 50.00005$

$f(x_i) = -4.57938708$

$f(x_i + \delta x_i) = -4.579381118$

So:

$$x_{i+1} = 50 - \frac{(10^{-6})(50)(-4.57938708)}{-4.57938118 - ^{-}4.57938708}$$

$= 88.39931$

$|\epsilon_t| = 38.1\ \%$

$|\epsilon_a| = 43.4\ \%$

No.

i	x_i	$\lvert\epsilon_t\rvert$ %	$\lvert\epsilon_a\rvert$ %
0	50	64.971	—
1	88.3993	38.069	43.438
2	124.0897	13.064	28.762
3	140.5417	~~0.021~~ 1.538	11.706
4	142.7072	0.021	1.517
5	142.736	4.1×10^{-6}	0.021

" check the number from the book"

& " we do the same procedore for each Iteration".

For 2^{nd} Iteration:

$x_1 = 88.3993$ $\quad$ $f(x_1) = -1.69220771$

$x_1 + \delta x_i = 88.39940$ $\quad$ $f(x_1 + \delta x_1) = -1.69220516$

thus:

$$x_2 = 88.3993 - \frac{(10^{-6})(88.39931)(-1.69220771)}{-1.692203516 - -1.69220771}$$

$$= 124.08970$$

$\lvert\epsilon_t\rvert = 13.064\%$

$\lvert\epsilon_a\rvert = 28.76\%$

* Linear Systems *

* CH.5 * Linear Algabric equations & Matrices :-

→ Matrix consist of a rectangular Array of elements:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \leftarrow \text{row}$$

↑ coloum

& dimension of A is $(m \times n)$.

→ if $m = n$ then it is called a square matrix

eg if $m = n = 3$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{bmatrix}$$

the main diagonal of matrix.

* types of Matrix:

-1- Symmetric matrix → $a_{ij} = a_{ji}$

eg ⇒

$$A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$$

No.

-2- Diagonal Matrix:

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

-3- Identity Matrix (I)

$$(I) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[A][I] = [I][A] = [A]$ — *

-4- Upper- ~~Diagonal~~ triangular Matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

-5- Lower- Triangular Matrix

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

FIVE APPLE

No.

-6- Tridiagonal Matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}.$$

* Lecture "8" *

* Matrix's Operations :-

→ matrix operating Rules :-

Always happy

$[A] = [B]$; if and only if the dimension of ~~all matrix~~ both matrices are equal & each element in $[A]$ equal to its Similar element in $[B]$, thus:
corresponding
($A_{ij} = B_{ij}$).

-1- Addition and Subtraction :-

(Addition / subtraction) of all matrices $[A]$ & $[B]$ is done by adding / subtracting Corresponding term in each matrix So,

$c_{ij} = a_{ij} + b_{ij}$, and the Subtraction is:

$d_{ij} = e_{ij} - f_{ij}$

No.

* The addition is commutative :

$[A]+[B]=[B]+[A]$.

* The addition is associative :

$([A]+[B])+[C]=[A]+([B]+[C])$.

-2- Multiplication :-

* multiplication of matrix [A] by a scalar (g) is done by

$$D = g[A] = \begin{bmatrix} ga_{11} & ga_{12} \\ ga_{21} & ga_{22} \end{bmatrix}.$$

* The product of two matrices is

$[C]=[A]\cdot[B] \Rightarrow$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$; where :

$n \equiv$ is the coloum dimension of [A]. and the row dimension of [B].

* Matrix multiplication is associative :

$([A][B])[C] = [A]([B][C])$, and distributive :

$[A]([B]+[C]) = [A][B]+[A][C]$

No.

& generally not commutative.

$$[A][B] \neq [B][A].$$

3- The inverse of the Matrix :-

for (2x2) matrix is :

$$[A]^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

4- The transpose of the matrix :

it is involving its rows into coloums and it's ~~coloums~~ coloums into rows.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

note → (it doesn't depends on the dimension of the matrix)

$$\rightarrow [A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

* Representing Linear Algabric equ. in matrix Form :-

$$a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 = b_1$$
$$a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 = b_2$$
$$a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 = b_3$$

* It can be represented as

$$[A]\,[x] = [b]$$ where :

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \& \quad [b] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

* thus the system can be written as :

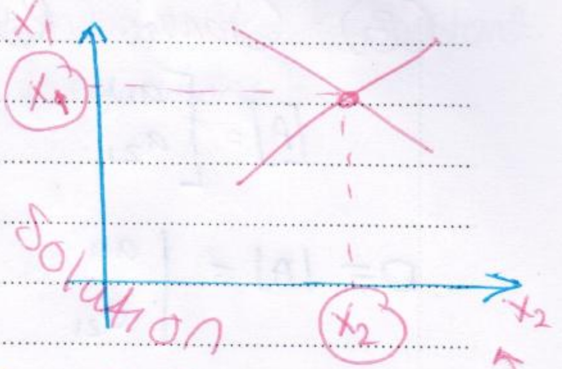
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

* CH.6 : Gauss Elimination :-

* Solving Small numbers of equations :-

1. Graphical methord.
2. Cramer's rule.
3. Elimination of unknown.

(A) Graphical method:



It's used to solve two linear equ.; by plotting them on Cartisian coordinates; with one axis corresponding to (x_1) & the other to (x_2), & the intersection pnt is the solution. 2-D system:

Also for 3-D system.

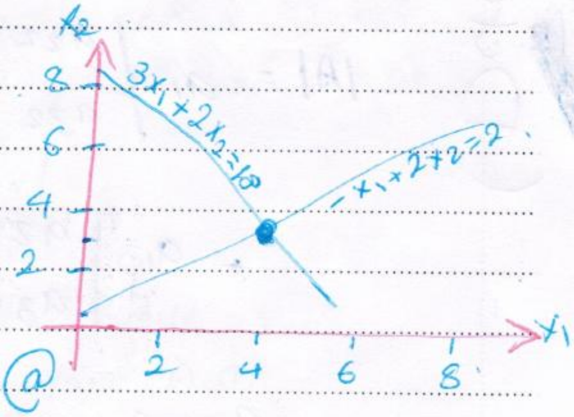
ex Use Graphical method to solve:

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

sol

$$x_2 = -\frac{3}{2}x_1 + 9$$

$$x_2 = \frac{1}{2}x_1 + 1$$



note that: the intersection pnt @ $x_1 = 4$ & $x_2 = 3$.

* See figures from the book * Cases where i can't use graphical method.

(B) Cramer's Rule & Determinant :-

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ (2x2) matrix

$$D = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{21} a_{12}.$$

For (3x3) matrix :

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(Determinant)

$$\Rightarrow |A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

* Cramer's rule states that each unknown in a system of linear Algabric eq. may be expressed as a fraction of two determinants with the denomenator (D) & the ~~B~~nominator obtained from (D) by replacing the Colum of Coefficient

No.

of the unknown (in question) by the constant (b_1, ... to ..., b_n), So for 3 equations :-

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

ex Use Cramer's Rule to Solve :-

$$0.3\,x_1 + 0.52\,x_2 + x_3 = -0.01$$
$$0.5\,x_1 + x_2 + 1.9\,x_3 = 0.67$$
$$0.1\,x_1 + 0.3\,x_2 + 0.5\,x_3 = -0.44$$

No.

Sol

$$\begin{bmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.01 \\ 0.67 \\ -0.44 \end{bmatrix}$$

$\overline{D} = -0.0022$.

$x_1 \Rightarrow$ replace Colum ① with b_1 & b_2 & b_3 to get :-

$x_1 = -14.9$. & the Same to Find x_2 & x_3 :-

colum ② $\leftarrow x_2 = -29.5$.

colum ③ $\leftarrow x_3 = 19.8$.

No.

Lecture "9" :-

Naive Gauss Elemination :-

it based on forward elemination & back substitution.

assume a set of equations:

$$a_{11} x_1 + a_{12} x_2 + \text{------} + a_{1n} x_n = b_1 \quad (1.a)$$

$$a_{21} x_1 + a_{22} x_2 + \text{------} + a_{2n} x_n = b_2 \quad (1.b)$$

$$\vdots$$

$$a_{n1} x_1 + a_{n2} x_2 + \text{------} + a_{nn} x_n = b_n \quad (1.c)$$

The 1^{st} step to eleminate the 1^{st} unknown (x_1) so multiply eq. (1.a) by $\left(\frac{a_{21}}{a_{11}}\right)$ to give:-

$$a_{21} x_1 + \frac{a_{21}}{a_{11}} a_{12} x_2 + \text{----} + \frac{a_{21}}{a_{11}} a_{1n} x_n = b \frac{a_{21}}{a_{11}}$$

Then Subtracte it from equ. (1.b) to give:-

$$\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) x_2 + \text{----} + \left(a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}\right) x_n$$

$$= b_2 - \frac{a_{21}}{a_{11}} b_1$$

OR $\quad a_{22}' x_2 + \dots + a_{2n}' x_n = b_2'$

where the prime (') indicates the values after change.

Then the procedures are repeated for the remaining equations, so equ. (1.a) is called "The pivot equation" & a_{11} is called "The pivot element".

No. ....

The next step is to eleminate (x_2) & so on. After that the sys. will have been transform to an upper triangular sys.

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$$
$$a_{22}' x_2 + \dots + a_{2n}' x_n = b_2'$$
$$a_{33}'' x_3 + \dots + a_{3n}'' x_n = b_3''$$

& so on.

It will goes until we get:

$$a_{nn}^{(n-1)} x_n = b_n^{(n-1)}$$

where $(n-1)$ is the # of primes & then we make back substitution.

So; $x_n = \dfrac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$; In general:-

$$x_i = \frac{b^{(i-1)} - \sum_{i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ij}^{(i-1)}}$$

for $i = (n-1, n-2, \dots, 1)$.

ex

Use Gauss elemination to Solve:

$$3x_1 - 0.1 x_2 - 0.2 x_3 = 7.85 \quad (1)$$
$$0.1 x_1 + 7 x_2 - 0.3 x_3 = -19.3 \quad (2)$$
$$0.3 x_1 - 0.2 x_2 + 10 x_3 = 71.4 \quad (3)$$

Sol

A. Multiply (1) by $(\frac{0.1}{3})$ & subtract it from (2) this gives:-

$$7.0033 x_2 - 0.29333 x_3 = -19.5617 \quad (4)$$

B. Multiply (1) by $(\frac{0.3}{3})$ & then subtract it from (3) that gives.

No.

$-0.19\,x_2 + 10.02\,x_3 = 70.615$.

The sys. is :-

$3x_1 - 0.1\,x_2 - 0.2\,x_3 = 7.85$ — ④

$7.0033\,x_2 + 0.293333\,x_3 = -19.5617$ — ⑤

$-0.19\,x_2 + 10.02\,x_3 = 70.615$ — ⑥

multiply ⑤ by $\left(\frac{-0.9}{7.0033}\right)$ & then subtract it from ⑥ to give :-

$10.012\,x_3 = 70.0843$ so, the new sys is.

$3x_1 - 0.1\,x_2 - 0.2\,x_3 = 7.85$ — ①

$7.0033\,x_2 - 0.293333\,x_3 = -19.5617$ — ②

$10.012\,x_3 = 70.0843$ — ③

now we do back substitution :-

$$x_3 = \frac{70.0843}{10.012} = 7.00003$$

$$x_2 = \frac{-19.5617 + (0.29333 * x_3)}{7.00333}$$

$x_2 = -2.5$ &

$x_1 = 3$.

(*) The reason for calling the pre method Naive is the possibility of division by zero if $a_{11} = 0$ OR if the pivot element is v. close to zero.

then Round of error can be produced, So; before division it's good to determine the coefficient with Largest abs value in the coloum below the pivot element

No.

then, the row can be switched so that the Largest element is the pivot element, this is called "Partial pivoting".

If coloums as well as rows are searched for the Largest element & then switched the ~~[illegible]~~ procedure is called "Complete pivoting.

ex Use Gauss elemination to Solve:

$$0.0003 x_1 + 3 x_2 = 2.0001 \quad \text{— (1)}$$

$$x_1 + x_2 = 1 \quad \text{— (2)}$$

repeat Using Partial pivoting ~~[illegible]~~ where the exact values are:-

$$x_1 = 1/3 \quad \& \quad x_2 = 2/3$$

Sol Multiply (1) by $\frac{1}{0.0003}$ & sub it from (2) to give:

$$-9999 x_2 = -6666$$

$$\longrightarrow x_2 = 2/3$$

The result is very sensitive to the number of significant figures:-

Sig. fig.	x_2	x_1	error
3	0.667	-3	1099
4	0.6667	0	100
5	0.66667	0.3	10
6	0.666667	0.33	1
7	0.6666667	0.333	0.1

No.

Using Pivoting

$x_1 + x_2 = 1$

$0.0003 x_1 + 3x_2 = 2.0001$

→ Do the same procedures so :-

$x_2 = 2/3$

$x_1 = \frac{1 - 2/3}{1} =$

Sig. fig	x_2	x_1	error
3	0.667	0.333	0.1
4	0.6667	0.3333	0.01
5	0.66667	0.33333	0.001
6	0.666667	0.333333	0.0001
7	0.6666667	0.3333333	0.00001

Lecture "10": LU Factorization:

LU Factorization requires pivoting to avoid devision by zero, So the system of (3x3) can be arraged as $[a][X] - [b] = 0$ ... eq(1)

Suppose that eq(1) can be represented as an upper triangular system:-

$$\begin{bmatrix} u_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

which can be done as the 1st Step of gauss elimination:

$[U][x] - [d] = 0$

* So there is a lower diagonal matrix which is:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

$L\{[U][x] - [d]\} = [A][x] - [b]$

So:

$[L][U] = [A]$ —— *

$[L][d] = [b]$

* So to obtain the solution there are two steps:

1. LU Factorization step
2. Substitution step to generate [d] by forward substitution then find [x] by back substitution.

* Gauss elemination as LU Factorization:
gauss elemination can be used to decompose matrix A into matrix U & L where matrix U can be found by forward elemination

No.

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \quad \&$$

From the forward elemination :-

$$f_{21} = \frac{a_{21}}{a_{11}} \qquad f_{31} = \frac{a_{31}}{a_{11}}$$

$$f_{32} = \frac{a_{32}'}{a_{22}'} \qquad \text{so :}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

After that to find the solution :

-1- a forward substitution to find (d) which is:

$$d_i = b_i - \sum_{j=1}^{i-1} f_{ij}\, d_j \quad ; \text{for } i = 1, 2, \ldots, n$$

-2- Back substitution to find (x) where :

$$x_n = \frac{d_n}{U_{nn}}$$

$$x_i = \frac{d_i - \sum_{j=i+1}^{n} U_{ij}\, x_j}{U_{ii}} \quad ; \text{for } i = n-1, n-2, \ldots, 1$$

ex Solve the following system using LU-Factorization :

$$3x_1 - 0.1\,x_2 - 0.2\,x_3 = 7.85$$

$$0.1\,x_1 + 7\,x_2 - 0.3\,x_3 = -19.3$$

$$0.3\,x_1 - 0.2\,x_2 + 10\,x_3 = 71.4$$

No.

Sol

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{bmatrix}$$

Using Gauss Elimination:

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.012 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7.85 \\ -19.5617 \\ 70.088 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.012 \end{bmatrix}$$

$$f_{21} = \frac{0.1}{3} = 0.0333333$$

$$f_{31} = \frac{0.3}{3} = 0.1$$

$$f_{32} = -\frac{0.19}{7.00333} = -0.02713$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.03333333 & 1 & 0 \\ 0.1 & -0.02713 & 1 \end{bmatrix}$$

No.

$[L][d] = [b]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.1 & -0.02713 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{bmatrix}$$

$d_1 = 7.85$

$d_2 = -19.5617$

$d_3 = 70.0843$

$[U][x] = [d]$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.012 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{bmatrix}$$

$x_1 = 3$

$x_2 = -2.5$

$x_3 = 7.00003$

No. #

Lecture "11"

LU Factorization with pivoting :-

~~Solution with~~ partial pivoting is nesecceary to obtain reliable solution with LU factorization.

& it can be done using :-

1- Elimination, with pivoting of matrix [A], So that

$$[P][A] = [L][U] \quad \text{where :}$$

[P] is the permutation matrix.

2- Forward Substitution

$$[L][d] = [P][b]$$

3- Back Substitution

$$[U][X] = [d]$$

ex Compute LU Factorization and Find the Solution for the system:-

$$\begin{bmatrix} 0.0003 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2.0001 \\ 1 \end{bmatrix}$$

Sol

1-

$$\begin{bmatrix} 1 & 1 \\ 0.0003 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.0001 \end{bmatrix}$$

No.

→ After Elimination:-

2-

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 2.9997 \end{bmatrix} \quad \&$$

$$L = \begin{bmatrix} 1 & 0 \\ 0.0003 & 1 \end{bmatrix}$$

3 → Doing Forward substitution to find d :-

$$\begin{bmatrix} 1 & 0 \\ 0.0003 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.0001 \end{bmatrix}$$

$d_1 = 1$ $\quad$ $d_2 = 1.9998$

4 & Now back substitution:-

$$\begin{bmatrix} 1 & 1 \\ 0 & 2.9997 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.9998 \end{bmatrix}$$ thus

$x_2 = 0.66667$

$x_1 = 0.33333$

No.

* Cholesky Factorization:-

For a system that can be represented as a symmetric matrix & $[a_{ij} = a_{ji}]$ for all (i) & (j) thus: $[A] = [A]^T$; So a special solution ~~tech~~ techniques are avaible for such sys so the symmetric matrix can be decomposed as :-

$$[A] = [U]^T [U]$$

The factorization can be generated use:

$$U_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} U_{ki}^2}$$

$$U_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} U_{ki} U_{kj}}{U_{ii}}$$

for $i = i+1, \ldots, n$.

then we use:-

$[U]^T [d] = [b]$ and

$[U][X] = [d]$ to find x

ex

Compute the cholesky factorization for the symmetric matrix:-

$$[A] = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}$$

* we want U to be $\Rightarrow [U] = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{21} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$.

No.

Sol

For the 1st row of U :-

$U_{11} = \sqrt{a_{11}} = \sqrt{6} = 2.44949$

$U_{12} = \dfrac{a_{12}}{U_{11}} = \dfrac{15}{\sqrt{6}} = \dfrac{15}{2.44949} = 6.123724$

$U_{13} = \dfrac{a_{13}}{U_{11}} = \dfrac{55}{2.44949} = 22.45366$

For the 2nd row :- $i = 2$

$U_{22} = \sqrt{a_{22} - \sum_{k=1}^{1} U_{k1}} = \sqrt{a_{22} - (U_{12})^2}$

$= \sqrt{55 - (6.123724)^2} = \cancel{\sqrt{49} = \sqrt{7}}$.

$\cancel{\text{or } (2.44949)^2}$

$U_{22} = 4.1833$

$U_{23} = \dfrac{a_{23} - \sum_{k=1}^{1} U_{k2} U_{13}}{U_{22}}$

$= \dfrac{225 - (6.12372 * 22.45366)}{4.1833} = 20.9165$

for $\boxed{i = 3}$ 3rd row :-

$U_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} U_{k3}^2}$

$U_{33} = \sqrt{a_{33} - [(U_{13})^2 + (U_{23})^2]} =$

$= \sqrt{979 - [(22.45366)^2 + (20.9165)^2]}$

$= 6.110101$

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No.

$$[U] = \begin{bmatrix} 2.44949 & 6.123724 & 22.45366 \\ 0 & 4.1833 & 20.9165 \\ 0 & 0 & 6.110101 \end{bmatrix}$$

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← من نجدسارة الونرست

Subject: Chapter # 8 2014 / 4 / 2

* Matrix Inverse :-

For a square matrix A there is another matrix A^{-1} called the inverse matrix of matrix A, For

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

→ The inverse can be found in a coloumn by coloume method by generating solutions with unit vectors as the Right and Side constant.

So, the best way to find in inverse is LU Factoraization

*example :- employ LU Factorization to determine the matrix inverse for :-

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0 & 0 & 10.012 \end{bmatrix}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1 & -0.02713 & 1 \end{bmatrix}$$

→ The first colum in the matrix inverse can be found using forward subistitution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.1 & -0.02713 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So, using Forward Subist. :-

$$[d]^T = [1 \quad -0.0333 \quad -0.1009]$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.03333 \\ -0.1009 \end{bmatrix}$$

Using backword subistitution.

$$[X]^T = [0.33249 \quad -0.00518 \quad -0.01008]$$

→ First الـ inverse هيك طلعنا الـ coloum

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بدل $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ بحط $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

To determin the second coloum

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ -0.1 & -0.02713 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

→ using Forword Subistitution.

$$[d]^T = [0 \quad 1 \quad 0.02713]$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0.02713 \end{bmatrix}$$

→ using Backword Subistitution

$$[X]^T = [0.004944 \quad 0.142903 \quad 0.002710]$$

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To determin the 3rd coloum

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.03333333 & 1 & 0 \\ 0.1 & -0.02713 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

→ Using forword Subistitution

$$[d]^T = [0 \ 0 \ 1]$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0333 & -0.293333 \\ 0 & 0 & 10.012 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

→ Using Backword Subistitution

$$[X]^T = [0.00679 \ \ 0.004182 \ \ 0.09986]$$

So, The inverse Matrix is :

$$[A]^{-1} = \begin{bmatrix} 0.33449 & 0.00494 & 0.006798 \\ -0.00519 & 0.142903 & 0.004183 \\ -0.01008 & 0.002710 & 0.099880 \end{bmatrix}$$

* To check the result we have to multiply [A] [A]$^{-1}$ and it should give the identity matrix [I].

* Vector and matrix Norm :

A Norm → is a real valued function that provide a measure or length or size of multicomponent of mathmatical Identeties, such that vectors and Matrices.

For a vector in that dimenional Euclidean space, $\vec{F} = [a, b, c]$
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→ where a, b, & c are distances along x, y & z axis.

The length of this vector is the distance from the point (0, 0, 0) to (a, b, c)
So,

$$\|F\|_e = \sqrt{a^2 + b^2 + c^2}$$

When $\|F\|_e$ indicates that the length is referred to the Euclidean norm of $\vec{F}$.

Subject: ______ / /

→ For n-dimentional vector

$$X = [x_1, x_2, \ldots, x_n]$$

the Euclideam norm is

$$\|X\|_e = \sqrt{\sum_{i=1}^{n} x_i^2}$$

this concept can be extended to matrix [A], then

$$\|A\|_f = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2}$$

→ (معتبرين انه ال A is square matrix عشان هيك اخذت i & j من 1 → n)

Which is called the Forbenious norm.

ZOOM Notebook

Wed 9/4. numerical

* Iterative method :-

Linear systems : Gauss Seidel :-

It's the most commonly used iterative method to solve linear system consist of n-equations.

For 3x3 ~~systems~~ set of equations with diagonal elements all non zero, then.

$$x_1^j = \frac{b_1 - a_{12} x_2^{j-1} - a_{13} x_3^{j-1}}{a_{11}}$$

$$x_2^j = \frac{b_2 - a_{21} x_1^j - a_{23} x_3^{j-1}}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31} x_1^j - a_{32} x_2^j}{a_{33}}$$

where j & $j-1$ are the present & previous iterations.

to start the solution process intial guess must be made for the values of (x) A simple approach to assume that they are all zero's

So since $x_2 = x_3 = 0$ then

$$x_1 = \frac{b_1}{a_{11}}$$ then the process

is repeated. to find x_2 & x_3.

$$|\varepsilon_{a,i}| = \left| \frac{x_i^j - x_i^{j-1}}{x_i^j} \right| \times 100\%.$$

<u>example</u> :- Solve the system follows using gauss seidel method:-

$$3x_1 - 0.1\,x_2 - 0.2\,x_3 = 7.85.$$

$$0.1\,x_1 + 7x_2 - 0.3\,x_3 = -19.3.$$

$$0.3\,x_1 - 0.2\,x_2 + 10\,x_3 = 71.4.$$

where the true solutions are:.

$$x_1 = 3, \quad x_2 = -2.5, \quad x_3 = 7.$$

<u>Sol</u> 1st Iteration:-

$$x_1 = \frac{7.85 + 0.1\,x_2 + 0.2\,x_3}{3}.$$

$$x_2 = \frac{-19.3 - 0.1\,x_1 + 0.3\,x_3}{7}.$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

Using initial guesses :

Let $x_2 = x_3 = 0$.

thus $x_1 = \frac{07.85}{3} = 2.616667$.

$$x_2 = \frac{-19.3 - 0.1 * (2.616667)}{7}.$$

$\therefore x_2 = -2.794524$.

$$x_3 = \frac{71.4 - 0.3(2.616667) + 0.2(-2.794524)}{10}$$

$= 7.005610$.

2nd Iteration :-

$$x_1 = \frac{7.85 + 0.1(-2.794324) + 0.2(7.005610)}{3}.$$

$= 2.990557$.

x_2 using $x_1 = 2.990557$. &

$x_3 = 7.00561$.

$$x_2 = \frac{-19.3 - 0.1(2.990557) + 0.3(7.005610)}{7}.$$

$\therefore x_2 = -2.149925$

$$x_3 = \frac{71.4 - 0.3(2.990557) + 0.2(-2.149925)}{10}.$$

$= 7.000291$.

$$\varepsilon_{a,1} = \left| \frac{2.990557 - 2.61667}{2.990557} \right| \times 100\% = 12.5\%$$

$\varepsilon_{a,2} = 11.8\%$ (for x_2)

$\varepsilon_{a,3} = 9.076\%$ (for x_3).

* Review page 287 fig. (12.1)

→ Note that each new (x) value for the gauss seidel method, it is immediatly used in the next equ. to determine another (x) value. So if the solution is converging the best avaible estimate will be employed.

An Alternative approach called "(Jacobi) Iteration" in this method rather than Using the latest avaible x values in the Same iteration. So as the new values are generated & they are not immediatly used but ~~they~~ returned to the next iteration.

* Non - Linear Systems :-

- 1- Graphical method :-

<u>example</u> : Solve the following non Linear system.

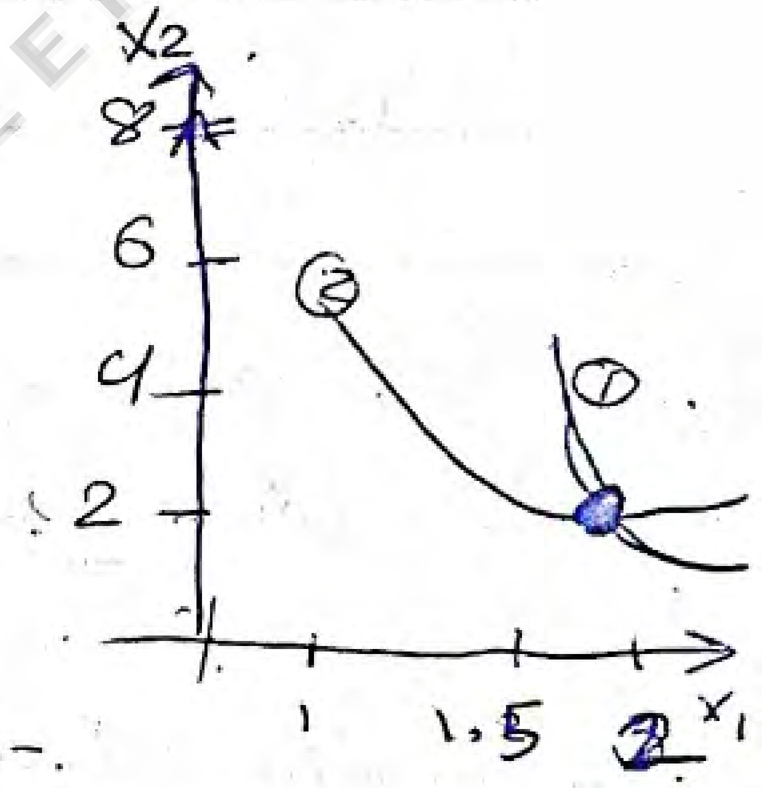
$$x_1^2 + x_1 x_2 = 10$$

$$x_2 + 3x_1 x_2^2 = 57$$

<u>Solution</u> :

by plotting these fn.

$$\begin{bmatrix} x_1 = 2 \\ x_2 = 3 \end{bmatrix}$$



- 2- Successive Substitution :-

In this method each one of the nonlinear equ. can be solved for one of the unknowns.

These equ. can be implemented itteretenvly to compute the ~~the~~ new values which will converges to the true value.

This method requires intial guess.

example :-

Use successive Substitution to Solve:

$$x_1^2 + x_1 x_2 = 10$$

$$x_2 + 3x_1 x_2^2 = 57$$

where the true values are

$x_1 = 2$ & $x_2 = 3$.

Use the initial guess $x_1 = 1.5$, $x_2 = 3.5$

Solution :-

$$x_1 = \frac{10 - x_1^2}{x_2}$$ are the $(j-1)$ values

$$x_2 = 57 - 3x_1 . x_2^2$$

$$\rightarrow x_1 = \frac{10 - (1.5)^2}{3.5} = 2.21429$$

$$x_2 = 57 - 3(2.21429)(3.5)^2$$
$$= -24.37516$$

2nd Iteration:-

$$x_1 = \frac{10 - (2.21429)^2}{-24.37516} = -0.20910$$

$$x_2 = 57 - 3(-0.20910)(-24.37516)^2$$
$$= 429.704$$

Note that the solution are diverging.

So; Let $x_1 = \sqrt{10 - x_1 x_2}$ &

$$x_2 = \sqrt{\frac{57 - x_2}{3x_1}}$$

$$x_1 = \sqrt{10 - (1.5)(3.5)} = 2.17945.$$

$$x_2 = \sqrt{\frac{57 - 3.5}{3(2.17945)}} = 2.86051.$$

2^{nd} Iteration :

$$x_1 = \sqrt{10 - (2.17945)(2.86051)} = 1.94053.$$

$$x_2 = \sqrt{\frac{57 - 2.86051}{3(1.94053)}}$$

$$x_2 = 3.04955.$$

→ Note that : the Solutions are converging to the true values..

" depends on the intial guess & the Formula of the unknown".

→ Note that : for Successive Substitution converging depends on the manner in which equation are formulated also on the intial guesses.

Lecture" " Mon 14/4/2014

(*) Newton Raphson :-

$$X_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$ "For single ~~sigle~~ eq."

It can be generalized for multi-equations. , So : for two variables function:

$$x_{1,i+1} = x_{1,i} - \frac{\left(f_{1,i}\frac{\partial f_{2,i}}{\partial x_2} - f_{2,i}\frac{\partial f_{1,i}}{\partial x_2}\right)}{\left(\frac{\partial f_{1,i}}{\partial x_1}\frac{\partial f_{2,i}}{\partial x_2} - \frac{\partial f_{1,i}}{\partial x_2}\frac{\partial f_{2,i}}{\partial x_1}\right)}$$

and:

$$x_{2,i+1} = x_{2,i} - \frac{\left(f_{2,i}\frac{\partial f_{1,i}}{\partial x_1} - f_{1,i}\frac{\partial f_{2,i}}{\partial x_1}\right)}{\left(\frac{\partial f_{1,i}}{\partial x_1}\frac{\partial f_{2,i}}{\partial x_2} - \frac{\partial f_{1,i}}{\partial x_2}\frac{\partial f_{2,i}}{\partial x_1}\right)}$$

example Use (N.R) method to Solve :

$$x_1^2 + x_1 x_2 = 10 \quad \Rightarrow f_1(x)$$

$$x_2 + 3 x_1 x_2^2 = 57 \quad \Rightarrow f_2(x)$$

Use intial guesses of : $x_1 = 1.5$ & $x_2 = 3.5$.

Sol :

$$\frac{\partial f_{1,0}}{\partial x_1} = 2x_1 + x_2 = 2(1.5) + 3.5 = 6.5 .$$

$$\frac{\partial f_{1,0}}{\partial x_2} = x_1 = 1.5 .$$

$$\frac{\partial f_{2,0}}{\partial x_1} = 3x_2^2 = 3(3.5)^2 = 36.75 .$$

$$\frac{\partial f_{2,0}}{\partial x_2} = 1 + 6 x_1 x_2 = 1 + 6(1.5)(3.5) = 32.5$$

So the denominator is :

$$(6.5)(32.5) - (1.5)(36.75) = 156.125 .$$

$$f_{1,0} = (1.5)^2 + (1.5)(3.5) - 10 = -2.5 .$$

$$f_{2,0} = 3.5 + 3(1.5)(3.5)^2 - 57 = 1.625 .$$

So :

$$x_1 = 1.5 - \frac{(-2.5)(32.5) - (1.625)(1.5)}{156.125} .$$

$$x_1 = 2.03603 .$$

& $$x_2 = 3.5 - \frac{(1.65)(6.5) - (-2.5)(36.75)}{156.125} = 2.84388$$

Then the process can be repeated.

* Eigen Values :-
  → $[A][x] = [b]$;
* This system is nonhomogeneous since the right hand side isn't equal to zero.
* For $[A][x] = 0$, it's a homogeneous system. where the trivial solutions are ~~where~~ all x's = 0 .

Eigen values are typically of the form:
$$[\,[A] - [\lambda][I]\,]\,[x] = 0 .$$

where λ is the eign values:-
So ; rather than sitting the values of x to zero we can determine the value of λ that drives the L.H side to zero by making the determinante of the matrix equal to zero :-
$$|\,[A] - \lambda[I]\,| = 0 .$$
By expanding the determinante to give a polynomial in (λ) which is called the characteristic polynomial , then we find the roots of the characteristic polynomial which are called "The Eigen Values"

So; For two equations system;

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

thus the system becomes:-

$$(a_{11}-\lambda)\,x_1 + a_{12}\,x_2 = 0$$

$$a_{21}\,x_1 + (a_{22}-\lambda)\,x_2 = 0$$

So the determinante is:-

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - (a_{12} * a_{21})$$

$$D = \lambda^2 - (a_{11}+a_{22})\lambda - a_{12}a_{21}.$$

This equation is called the characteristic, which can solved using:-

$$\lambda_1, \lambda_2 = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2 - 4a_{12}a_{21}}}{2}$$

So this method is called the polynomial method.

<u>example:</u> Using Polynomial method solve:

$$(10-\lambda)x_1 - 5x_2 = 0$$

$$-5x_1 + (10-\lambda)x_2 = 0$$

<u>sol</u>

$$\begin{vmatrix} 10-\lambda & -5 \\ -5 & (10-\lambda) \end{vmatrix} = \lambda^2 - 20\lambda + 75$$

$$\cancel{\lambda_1, \lambda_2 =}\ \lambda^2 - 20\lambda + 75 = 0$$

$$(\lambda_2 - 5)(\lambda_2 - 15) = 0$$

$$\lambda_2 = 5 \quad \& \quad \lambda_1 = 15$$

For $\lambda_1 = 15$:-

$$-5x_1 - 5x_2 = 0 \rightarrow x_1 = -x_2$$

So ; $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which is called the Eigen vector for the Eign value ($\lambda = 15$).

For $\lambda_2 = 5$:-

$$-5x_1 + 5x_2 = 0 \longrightarrow x_1 = x_2$$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & it's the Eigen vector for the eigen value ($\lambda = 5$).

Lecture " " Wed 16/4/2014

* The power method :- it is an Iterative approach that can be used to determine the largest or dominant Eigen value.
Also it can be used to determine the smallest eigen value with slight modification, & this method based on:

$$[A].[x] = \lambda [x]$$

<u>ex</u>

Employ the power Method to determine the highest eigen value & it's associated eigen vector For :

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} - \lambda [I] = 0$$

<u>sol</u>: ① we should write as $[A][x] = \lambda [x]$:

$$40x_1 - 20x_2 = \lambda x_1$$

$$-20x_1 + 40x_2 - 20x_3 = \lambda x_2$$

$$-20x_2 + 40x_3 = \lambda x_3$$

② Assume the intial guesses are:

$x_1 = x_2 = x_3 = 1$ "if we don't have intial guess".

$$40(1) - 20(1) = \lambda(1) \quad \text{—— ①}$$

$$-20(1) + 40(1) - 20(1) = \lambda(1) \quad \text{—— ②}$$

$$-20(1) + 40(1) = \lambda(1) \quad \text{—— ③}$$

$\lambda = 20 \rightarrow$ From ①

$\lambda = 0 \rightarrow$ From ②

$\lambda = 20 \rightarrow$ from ③.

$$\begin{bmatrix} 20 \\ 0 \\ 20 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$ thus:

$\lambda = 20$ & the eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

2^{nd} Iteration:- it consist of multiplying the matrix by the eigen vector. $[1 \quad 0 \quad 1]^T \rightarrow$

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 40 \\ -40 \\ 40 \end{bmatrix}$$

$$\begin{bmatrix} 40 \\ -40 \\ 40 \end{bmatrix} = 40 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \leftarrow \text{new eigen vector.}$$

new eigen value

* Calculate the approximate error :-

$$|\varepsilon_a| = \left| \frac{40 - 20}{40} \right| * 100\% = 50\%.$$

3rd Iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 60 \\ -80 \\ 60 \end{bmatrix}$$

$$\begin{bmatrix} 60 \\ -80 \\ 60 \end{bmatrix} = -80 \begin{bmatrix} -0.75 \\ 1 \\ -0.75 \end{bmatrix}$$

"we take the largest number".

So the new eigen value is -80 & the eigen vector is $\begin{bmatrix} -0.75 & 1 & -0.75 \end{bmatrix}^T$

$$|\varepsilon_a| = \left| \frac{-80 - 40}{-80} \right| \times 100\% = 150\%$$

change in sign.

→ Fourth Iteration :-

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{bmatrix} -0.75 \\ 1 \\ -0.75 \end{bmatrix} = \begin{bmatrix} -50 \\ 70 \\ -50 \end{bmatrix}$$

$$\begin{bmatrix} -50 \\ 70 \\ -50 \end{bmatrix} = 70 \begin{bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{bmatrix}$$

$|\epsilon_a| = 2.14\%$

→ Fifth Iteration :-

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{bmatrix} = \begin{bmatrix} -48.51714 \\ 68.51714 \\ -48.51714 \end{bmatrix}$$

$$\begin{bmatrix} -48.51714 \\ 68.51714 \\ -48.51714 \end{bmatrix} = 68.51714 \begin{bmatrix} -0.70833 \\ 1 \\ -0.70833 \end{bmatrix}$$

$|\epsilon_a| = 2.08\%$

So the eigen value is converging.

* Note * To find the smallest eigen value the power ~~m~~ method is Applied to the matrix Inverse.

⊛ Curve Fitting :-

Linear Regression :-

Data are often given for discreate values, & we ~~make~~ may require to estimate @ points between the discreate values; So there are ~~tecni~~ techniques to Fit curves to such data to obtain intermediate estimates.

So we need to compute values of the function @ number of discreate values along the range of interest then a simple function may be derived to fit these values.

This is called "Curve fitting".

* There are two Approaches for Curve Fitting :

-1- Least square regression (as shown in a)

-2- Inter polation (as shown in b & c)

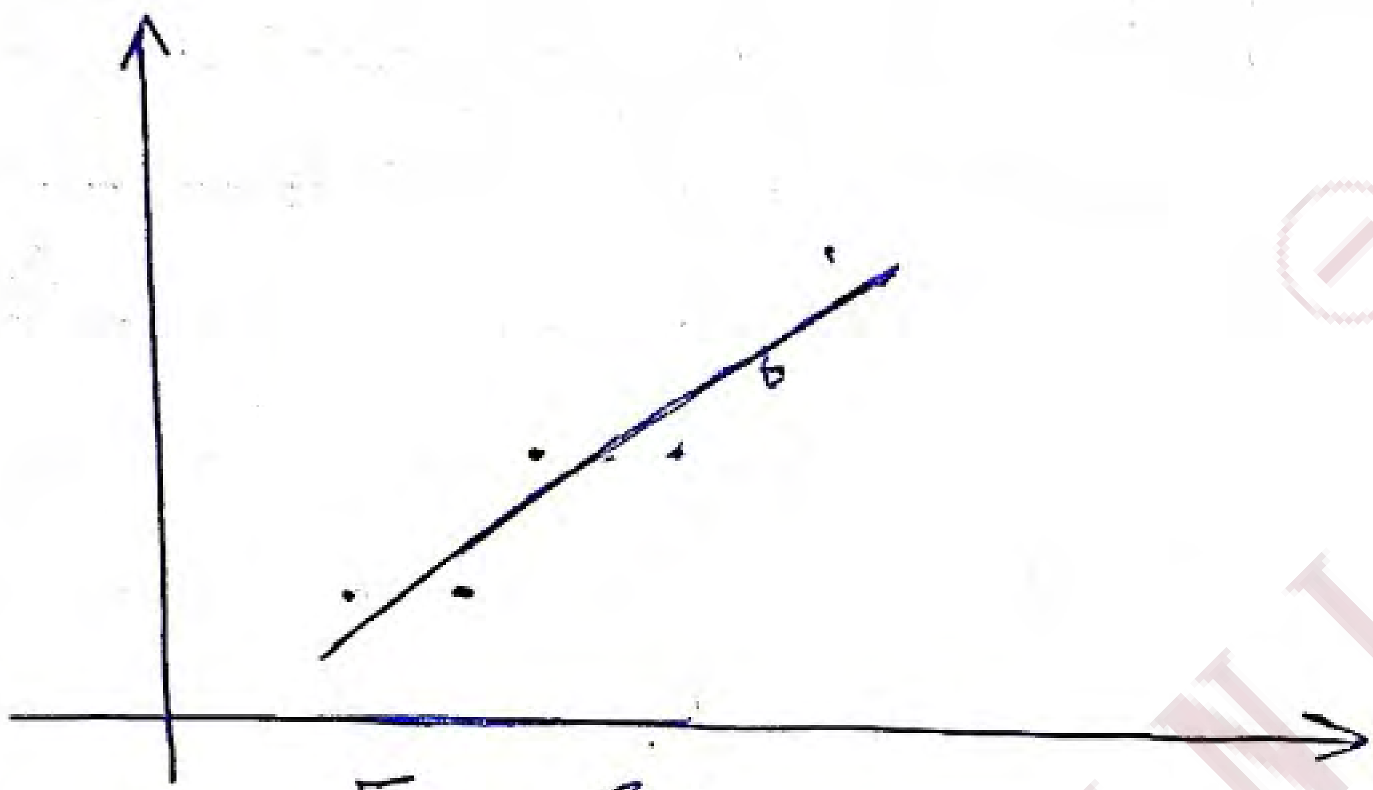


Fig (a).

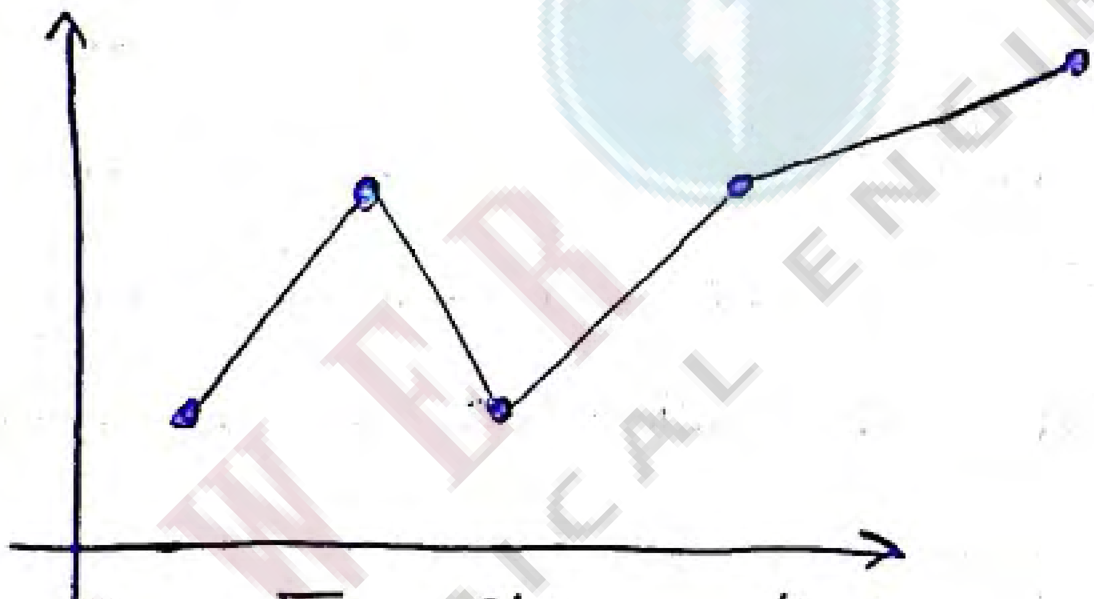


Fig (b). " Linear Interpolation "

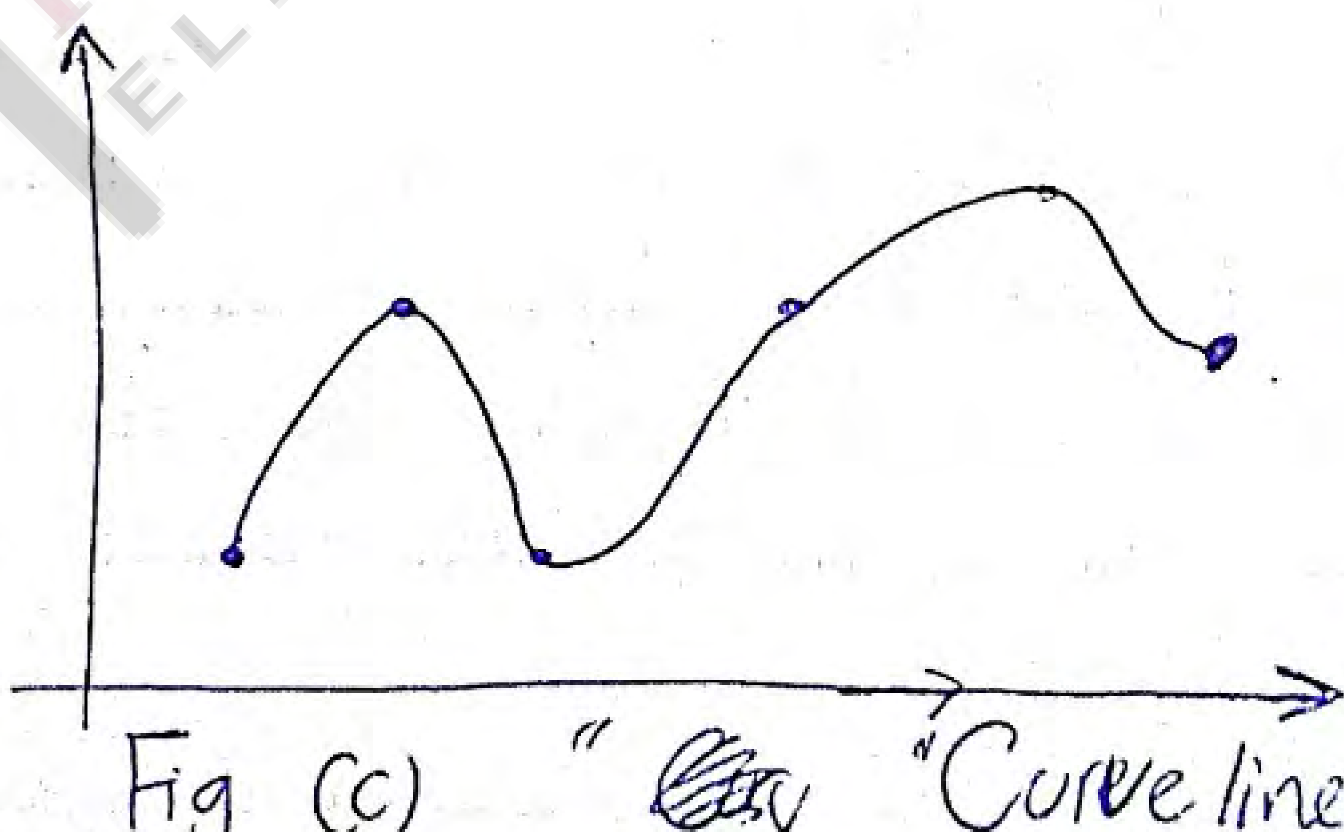


Fig (c) " "Curve linear Interpolation "

* Statistic Review:-

-1- The arithmatic mean value:

mean $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$.

-2- median : is the mid point of the group of data & it's calculated by :

Putting the data in order if the number of the data is odd, then it's the middle value. & if it's even its the Arithmatic mean of the two middle values.

-3- mode: the value that aqqures most frequently .

-4- Range : the difference between the largest & smallest value .

-5- Standard deviation:- "S_y" is .

$$S_y = \sqrt{\frac{S_t}{n-1}}$$ where :

S_r is the total sum of the square of the residual between the data pnt

& the mean :-

$$S_t = \sum (y_i - \bar{y})^2$$

-6- the variance : is the square of the Standard deviation :-

$$S_y^2 = \frac{\sum (y_i - \bar{y})^2}{n-1},$$

& the coefficient of variation (C.V) is :-

$$C.V = \frac{S_y}{\bar{y}} \times 100\%$$

See the example in page 328.

الاثنين

Subject:

2014 / 4 / 21

* Linear Leas-sequare Requession

Note that → ~~when~~ one approch for fit the shape of the data is to visiually expect the plot of data and then sketch the best line through, this approch is valid for some cases but not for all. So, we need to establishe a bases for the fit.

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ then, the mathmatical expresion for the straight line is

$$y = a_0 + a_1 x + e$$

→ where a_0 and a_1 are coefficiants represent the intercept point and the slope, ~~resp~~ and e is the error or residual between the model and the observation, so

$$e = y - a_0 - a_1 x$$

Subject: ______ / /

Least square Fit of a straight line:-

to minimize the sum of ~~t~~ squares of the residuals :-

$$S_r = \sum_{i=1}^{n} e_i^2$$

$$= \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

this criteria is called the least square, So, we need to find a_0 & a_1 that minimize the error.

So, by partial observation with respect to a_0 and a_1, then

$$\frac{\partial S_r}{\partial a_0} = -2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum [(y_i - a_0 - a_1 x_i) x_i]$$

which can be simplified, and put them equal to zero to find the minimum of S_r, ~~then~~ so

$$0 = \sum y_i - \sum a_0 - \sum a_1 x_i$$

$$0 = \sum x_i y_i - \sum x_i a_0 - \sum x_i a_1$$

Subject: ______ / /

note that $\sum a_0 = n a_0$

then

$$n a_0 + (\sum x_i) a_1 = \sum y_i$$

$$(\sum x_i) a_0 + (\sum x_i^2) a_1 = \sum x_i y_i$$

$$a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

and

$$a_0 = \bar{y} - a_1 \bar{x}$$

when $\bar{y}$ & $\bar{x}$ are the mean values of y and x.

* example: fit the straigh line to values in the table:

i	x_i	y_i	x_i^2	$x_i y_i$
1	10	25	100	250
2	20	70	400	1400
3	30	380	900	11900
4	40	550	1600	22000
5	50	610	2500	30500
6	60	1220	3600	73200
7	70	830	4900	58100
8	80	1450	6400	116000
$\sum =$	360	5135	20400	312850

Subject: / /

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$$= \frac{8(312850) - (360)(5135)}{8(20400) - (360)^2}$$

$$= 19.47024 \quad \text{(slope)}$$

$$a_0 = \bar{y} - a_1\bar{x}$$

$$\bar{y} = \frac{5135}{8} = 641.875$$

$$\bar{x} = \frac{360}{8} = 45$$

$$a_0 = 641.875 - (19.47024 \times 45)$$

$$= -234.2857$$

$$y = -234.2857 + \cancel{69}\ 19.47024x$$

from the

Subject: ____ / /

* Qualification of error of linear requassion :

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x)^2$$

$$S_t = \sum (y_i - \bar{y})^2$$

where, S_t is the square of the residual represented the square of the difference between the data and the mean.

* the standare error of the estimate ($s_{y,x}$) is

$$S_{y,x} = \sqrt{\frac{S_r}{n-2}}$$

شرح

y, x means that the error is for apredicted value of y corrosponding for aparticular value of x.

coefficient ~~for~~ of determination (r^2) is

$$r^2 = \frac{S_t - S_r}{S_t}$$

$$r = \sqrt{r^2}$$

when r is the corrolation coefficent

$S_r = 0$ → highly corrolation

لما تربيع ال r ال one

كان اكثر corrolation

if $r = 1$ $(S_r = 0)$ then the values are corrolated and if $r = 0$ $(S_r = S_t)$ then there is <u>no</u> improvement

*examples: compute the total standard deviation, the standard error of estimate & the corrolation coeff. for the fit of the previous example.

i	x_i	y_i	$a_0 + a_1 x$	$(y_i - \bar{y})^2$	$(y_i - a_0 - a_1 x_i)^2$
1	10	25	-39.58	380.575	4171
2	20	70	155.12	327.041	7249
3	30	380	349.82	68.579	911
4	40	550	544.52	8441	30
5	50	610	739.23	1016	16.699
6	60	1220	933.93	3341224	81837
7	70	830	1128.63	635391	89180
8	80	1450	1323.8	687066	16044
$\Sigma =$	360	5135		1808293	26118

Subject: / /

The standard diviation $= \sqrt{\frac{1808297}{8-1}}$

$= 508.26$

$S_{y,x} = \sqrt{\frac{216,118}{8-2}} = 189.79$

$r^2 = \frac{1,808,297 - 216,118}{1808297} = 0.8805$

$r = \sqrt{r^2} = 0.9383$

Wed 23/4.

Linearization of non-linear relationship:-

→ Linear regression can be used to express data of non Linear relationship in the form that is compatable with Linear regression.

$$y = \alpha_1 e^{\beta_1 x} \text{ — * ; where :}$$

α_1 & β_1 are constants & $\beta_1 \neq 0$.

→ it can be Linearized by taking (Ln)

$$\ln(y) = \ln|\alpha| + \beta_1 x \text{ — *}$$

$$y = \alpha_2 x^{\beta_2} \text{ — * ; where :}$$

α_2 & β_2 are constants and $\beta_2 \neq 0$.

→ & it can be Linearized by taking ($\log_{10}$) →

$$\log(y) = \log(\alpha_2) + \beta_2 \log(x) \text{ — *}$$

For $y = \alpha_3 \dfrac{x}{\beta_3 + x}$ — *

it can be linearized by inverting:

$$\frac{1}{y} = \frac{1}{\alpha_3} + \frac{\beta_3}{\alpha_3} \frac{1}{x} \text{ — *}$$

example :- Fit the following equations to the data that given in the below table using Algorithim Transfer.

i	x_i	y_i	$\log x_i$	$\log y_i$	$(\log(x_i))^2$
1	10	25	1	1.398	1
2	20	70	1.301	1.845	1.693
3	30	380	1.477	2.580	2.182
4	40	550	1.602	2.740	2.567
5	50	610	1.699	2.785	2.886
6	60	1220	1.778	3.086	3.162
7	70	830	1.845	2.919	3.404
8	80	1450	1.903	3.161	3.622
			12.606	20.515	20.516

$\log y_i \log x_i$
1.396
2.401
3.811
4.390
4.732
5.488
5.386
6.016
33.622

$$\log y = \log \alpha_2 + \beta_2 \log x \ ; \ y = a_0 + a_1 x.$$

$$\overline{\log x} = \frac{12.606}{8} = 1.5757$$

$$\overline{\log y} = \frac{20.515}{8} = 2.5644$$

$$a_1 = \frac{8(33.622) - (12.606)(20.515)}{8(20.516) - (12.606)^2}$$

$$= 1.9842.$$

$$a_0 = 2.5644 - (1.9842)(1.5757)$$

$$= -0.5620$$

$$\log y = -0.5620 + 1.9842 \log x.$$

then $\alpha_2 = 10^{-0.5620} = 0.2741.$

$\beta_2 = 1.9842.$

$$y = 0.2741 \, x^{1.9842}.$$

General Linear Least-square & non Linear regression:

→ Polynomial Regression:

In some cases a curve would be better to fit the data; so, a polynomial can be used to Fit data using polynomial regression based on Least square procedures that can be extended to fit higher order polynomial.

Suppose that we fit a second order Polynomial:

$$y = a_0 + a_1 x + a_2 x^2 + e$$

So, the sum of the square of the residual is:

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$$

→ to generate the Least square Fit, we take the derivatives with respect to unknown OR unknowns.

$$\frac{d S_r}{d a_0} = -2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

$$\frac{d S_r}{d a_1} = -2 \sum (y_i - a_0 - a_1 x_i - a_2 x_i^2)(x_i)$$

$$\frac{d S_r}{d a_2} = -2 \sum (y_i - a_0 - a_1 x_i - a_2 x_i^2)(x_i^2)$$

* These equations can be set equals to zero & rearranged :

$$n(a_0) + (\sum x_i) a_1 + (\sum x_i^2) a_2 = \sum y_i$$

$$(\sum x_i) a_0 + (\sum x_i^2) a_1 + (\sum x_i^3) a_2 = \sum x_i y_i$$

$$(\sum x_i^2) a_0 + (\sum x_i^3) a_1 + (\sum x_i^4) a_2 = \sum x_i^2 y_i$$

* n could be any number from one to ∞

* the polynomial regrission can be extended to a (n^{th}) order.

$$y = x_0 + a_1 x + a_2 x^2 + e$$ *

$$S_{y,x} = \sqrt{\frac{S_r}{n-(m+1)}}$$

$$r^2 = \frac{S_t - S_r}{S_t}$$ } qualification the error.

"See the example from the book".

Mon 28/4 :-

example : Fit a second order polynomial to the data in the first two coloum.

x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
0	2.1					
1	7.7					
2	13.6					
3	27.6					
4	40.9					
5	61.1					
15	152.6					

Calculate them & the (Σ).

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 .$$

$$\begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix}$$

$n = 6$, $\Sigma x_i = 15$, $\Sigma x_i^2 = 55$

$\Sigma x_i^3 = 225$, $\Sigma x_i^4 = 979$

$\Sigma y_i = 152.16$, $\Sigma x_i y_i = 585.6$

$\Sigma x_i^2 y_i = 2488.8$

$\bar{y} = 25.431$

$\bar{x} = 2.5$.

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{bmatrix}$$

$a_0 = 2.4786$

$a_1 = 2.3593$

$a_2 = 1.8607$.

x_i	y_i	$(y_i-\bar{y})^2$	$(y_i-a_0-a_1x_i-a_2x_i^2)^2$
0	2.1	544.44	0.14332
1	7.7	314.47	1.00286
2	13.6	140.03	1.08160
3	27.6	3.12	0.80487
4	40.9	239.22	0.61959
5	61.1	1272.11	0.09434
15	152.6	2513.30	3.74657

$$y = 2.4786 + 2.3593x + 1.8607x^2$$

$$S_{y,x} = \sqrt{\frac{3.74657}{6-(2+1)^2}} = 1.1175.$$

$$r^2 = \frac{2513.3 - 3.74657}{2513.3} = 0.9985$$

⊛ Multiple Linear regression:

The linear regression can be extended for two or more indep. variables such as:

$$y = a_0 + a_1x_1 + a_2x_2 + e$$

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1x_{1,i} - a_2x_{2,i})^2$$

then differentiating with respect to each of the unknown, then:

$$\rightarrow \frac{dS_r}{da_0} = -2 \sum (y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i})$$

$$\rightarrow \frac{dS_r}{da_1} = -2 \sum (y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i})\, x_{1,i}$$

$$\rightarrow \frac{dS_r}{da_2} = -2 \sum (y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i})\, x_{2,i}$$

then, the minimum sum of the squares of the residual is obtained by setting the partial derivative equal zero.

$$\begin{bmatrix} n & \sum x_{1,i} & \sum x_{2,i} \\ \sum x_{1,i} & \sum x_{1,i}^2 & \sum x_{1,i} \sum x_{2,i} \\ \sum x_{2,i} & \sum x_{1,i} \sum x_{2,i} & \sum x_{2,i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sum y_i \\ \sum x_{1,i}\, y_i \\ \sum x_{2,i}\, y_i \end{bmatrix}$$

This can be extended for (n) independent variables as:

$$y = a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + e$$

$$S_{y,x} = \sqrt{\frac{S_r}{n-(m+1)}}$$

* Multiple Linear regression can be used in the derivative of Power equation such that :-

$$y = a_0 \, x_1^{a_1} \, x_2^{a_2} \cdots x_m^{a_m}.$$

So; to use multiple regression, the equation transformed ~~by~~ taking it's Logarithim, then:

$$\log(y) = \log(a_0) + a_1 \log(x_1) + \cdots + a_m \log(x_m)$$

*

Example: The following data were created from the equation:

$$y = 5 + 4x_1 - 3x_2.$$

use multiple Linear regression to Fit this data.

$x_{1,i}$	$x_{2,i}$	y_i	$x_{1,i}^2$	$x_{2,i}^2$	$x_{1,i}x_{2,i}$
0	0	5	0	0	0
2	1	10	4	1	2
2.5	2	9	6.25	4	5
1	3	0	1	9	3
4	6	3	16	36	24
7	2	27	49	4	14
16.5	14	54	7625	54	48

$x_{1,i}\, y_i$	$x_{2,i}\, y_i$
0	0
20	10
22.5	18
0	0
12	18
189	54
243.5	100

Cont.

$$\begin{bmatrix} 6 & 16.5 & 14 \\ 16.5 & 76.25 & 48 \\ 14 & 48 & 54 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 54 \\ 243.5 \\ 100 \end{bmatrix}$$

$a_0 = 5$, $a_1 = 4$, $a_2 = -3$.

$y = 5 + 4x_1 + -3x_2$.

Wed (30/4)

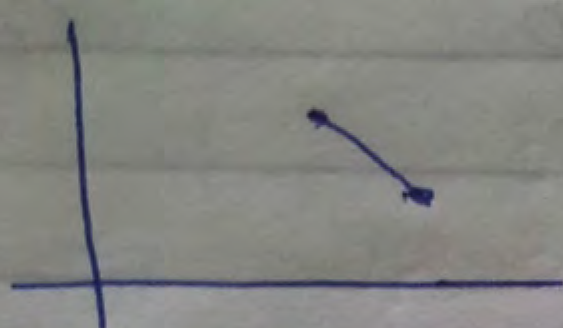
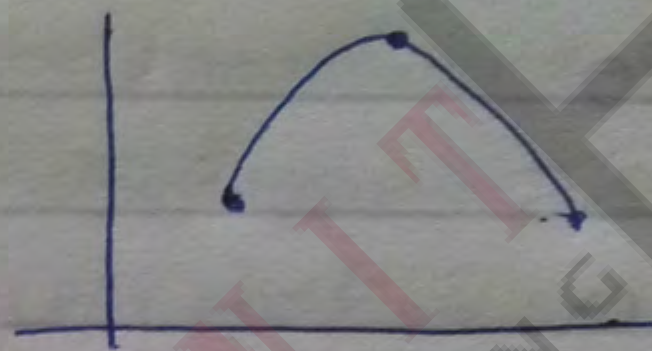
"Polynomial Interpolation"

Introduction for interpolation:

The general formula for $(n-1)$ order polynomial is

$$f(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}$$

For (n) data pnt there is only one polynomial

ex : determine the difference of the parabola

$$F(x) = P_3 + P_2(x) + P_1 x^2$$

that pass the three pnts :

$f(300) = 0.616$ $\quad$ $f(400) = 0.525$

$f(500) = 0.457$.

Sol

$0.616 = P_3 + 300 P_2 + 90\,000 P_1$.

$0.525 = P_3 + 400 P_2 + 160000 P_1$.

$0.457 = P_3 + 500 P_2 + 250000 P_1$

$P_1 = 0.0000015$

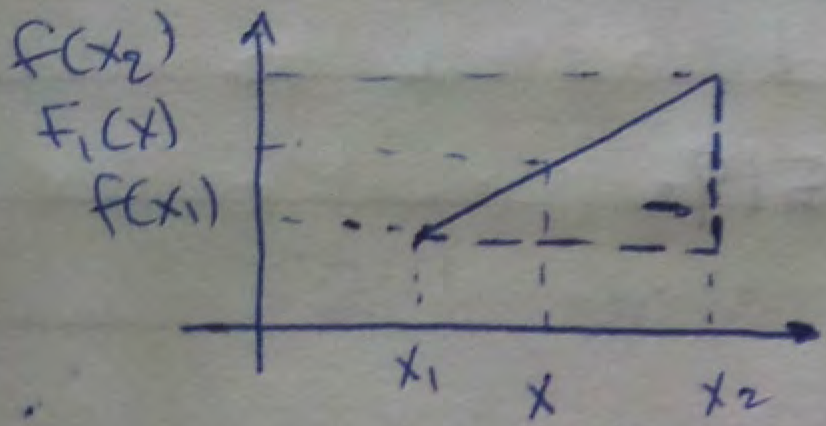
$P_2 = 0.001715$

$P_3 = 1.027$.

$$F(x) = 1.027 + 0.001715 x + 0.0000015 x^2 .$$

* Newton Interpolation polynomial :-

→ Linear interpolation



Using Similar triangle:

$$\frac{F_1(x) - F(x_1)}{x - x_1} = \frac{F(x_2) - F(x_1)}{x_2 - x_1}$$

$$F_1(x) = F(x_1) + \frac{F(x_2) - F(x_1)}{x_2 - x_1}(x - x_1).$$

which is the Newton ~~Raphson~~ Interpolation formula: when $F_1(x)$ is a 1st order interpolation polynomial.

ex Estimate $\ln(2)$ using Linear Interpolation using the pnts.

1- $\ln|1| = 0$, $\ln|6| = 1.791759$

2- $\ln|1| = 0$ $\ln|4| = 1.386294$.

true Value is 0.6931472.

Sol.

1. $F_1(2) = 0 + \frac{1.791759 - 0}{6-1}(2-1)$

$= 0.3583519$

$\epsilon_t = 48.3\%$.

2. $F_2(2) = 0 + \frac{1.386294 - 0}{4-1}(2-1) =$

0.4620981

$\epsilon_t = 33.3\%$.

⊛ Quadratic Interpolation:

to reduce the error for approximating a curve we use a 2^{nd} order polynomial (quadratic polynomial / parabola) which need 3 pnts.

$$f_2(x) = b_1 + b_2(x - x_1) + b_3(x - x_1)(x - x_2)$$

to determine the coefficients For b_1, Let:

$x = x_1$, then $b = f(x_1)$

For $b_2 \rightarrow$

$x = x_2$ then $b_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

~~b_3~~ = For $b_3 \rightarrow$ let: $x = x_3$.

$$b_3 = \frac{\left[\dfrac{F(x_3) - F(x_2)}{x_3 - x_2}\right] - \left[\dfrac{F(x_2) - f(x_1)}{x_2 - x_1}\right]}{x_3 - x_1}$$

"2nd order Finite difference approximation".

<u>ex.</u> empoly a second order N.I.P. method to estimate Ln|2| with data points:

$\ln|1| = 0$ (x_1) $\quad \ln|4| = 1.386294$ (x_2)

$\ln|6| = 1.791759$ (x_3)

<u>sol</u>

$b_1 = F(x_1) = \ln|1| = 0$.

$$b_2 = \frac{\ln|4| - \ln|1|}{4-1} = 0.4620981$$

$$b_3 = \frac{\dfrac{\ln|6| - \ln|4|}{6-4} - \dfrac{\ln|4| - \ln|1|}{4-1}}{6-1}$$

$b_3 = -0.0518731$

$F_2(x) = 0.4620981(x-1) - 0.0518731(x-1)(x-4)$

substitue $\underset{=}{2}$ in $f_2(x) \rightarrow x = 2$.

$F_2(2) = 0.5658444 \qquad \epsilon_t = 18.4\%$.

General form of N.I.P :

this method can be generalized to fit (n-1) order polynomial to n data pnts, so to (n-1) order polynomial:

$$f_{n-1}(x) = b_1 + b_2(x-x_1) + \cdots + b_n(x-x_1)\cdots(x-x_n)$$

for n pnts.

$b_1 = f(x_1)$

$b_2 = f(x_2, x_1)$

$b_3 = f(x_3, x_2, x_1)$

$\vdots$

$b_n = f(x_n, x_{n-1}, \ldots, x_1)$. where:

$$F[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j} \quad \&$$

$$F[x_i, x_j, x_k] = \frac{F[x_i, x_j] - F[x_j, x_k]}{x_i - x_k}$$

$$F[x_n, \ldots, x_1] = \frac{F[x_n, \ldots, x_2] - F[x_{n-1}, \ldots, x_1]}{x_n - x_1}$$

then $F_{(n-1)}(x) = F(x_1) + F[x_2, x_1](x - x_1)$
$+ F[x_3, x_2, x_1](x - x_1)(x - x_2)$ ~~[illegible]~~ $+$
$\cdots + F[x_n, x_{n-1}, \cdots, x_2, x_1](x - x_1)\cdots$
$(x - x_2)\cdots\cdots$ ~~[illegible]~~ $\cdot (x - x_{n-1})$.

ex employ the 3rd N.I.P method to estimate $\ln|2|$; using the points:
$x_1 = 1$, $x_2 = 4$, $x_3 = 6$, $x_4 = 5$,
$\ln|5| = 1.6$ → check the sol from the book.

(Mon 5/5)

Lagrage Interpolating polynomial:

→ for Linear Interpolating polynomial the straight line is given by:

$f(x) = L_1 f(x_1) + L_2 f(x_2)$ where:

the $L(s)$ are the weighting fn.

$L_1 = \dfrac{x - x_2}{x_1 - x_2}$ ~~[illegible]~~ & $L_2 = \dfrac{x - x_1}{x_2 - x_1}$.

$$f(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2).$$

This is called Linear Lagrange Interpolating polynomial

~~This~~

The same strategy can be used to Fit a parabola through 3-pnts. So the 2nd order Lagrage Interpolating polynomial can be written as

$$f_2(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} f(x_3)$$

→ The General Lagrange Interpolating polynomial Can be written as:

$$f_{n-1}(x) = \sum_{i=1}^{n} L_i(x)\, f(x_i) \quad ; \text{where:}$$

$$L_i(x) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \quad \text{where}$$

n is: the number of data pnts.

__example__ Use (L.I.P) of the 1^{st} & 2^{nd} order to evaluate $f(x)$ @ $x=15$ based on the following data.

$x_1 = 0 \qquad f(x_1) = 3.85$

$x_2 = 20 \qquad f(x_2) = 0.8$

$x_3 = 40 \qquad f(x_3) = 0.212$

__sol__

for the 1^{st} order :-

$$f_1(15) = \frac{(15-20)}{0-20}(3.85) + \frac{(15-0)}{20}(0.8)$$

$$= 1.5626$$

$$f_2(15) = \frac{(15-20)(15-40)}{(0-20)(0-40)}(3.85)$$

$$+ \frac{(15-0)(15-40)}{(20-0)(20-40)}(0.8)$$

$$+ \frac{(15-0)(15-20)}{(40-0)(40-20)}(0.212)$$

$$f_2(15) = 1.3316875$$

→ we can do 1^{st} order between x_1 & x_3 because 15 is in the Interval but not between x_2 & x_3.

Integration & Differentiation :-

Numerical Integration Formula :

$$I = \int_a^b f(x)\, dx \quad , \text{~~the mean va~~}$$

& the mean value is:

$$= \frac{\int_a^b f(x)\, dx}{b-a}.$$

→ 1 Newton _ Cotes Formulas :-

They are the most common numerical Integration theorems which are based on replacing a complicated function ~~with~~ a polynomial that is easy to Integrate

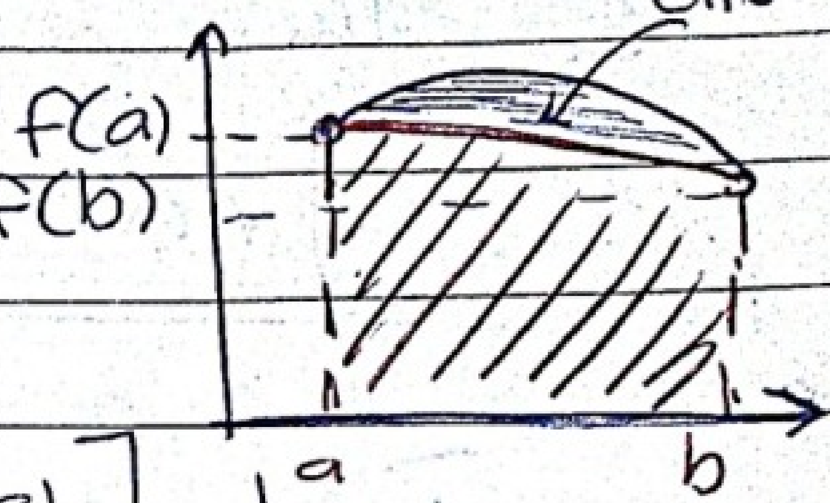
$$I = \int_a^b f(x)\, dx \simeq \int_a^b f_n(x)\, dx \quad \text{where:}$$

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

Note : it can be classified as closed OR open formulas ; the closed form. are those where the data points at the beginning & end of Integration are known while the open formulas have Integration Limits that extend beyond the range of data.

A. The Trapezoidal Rule :-

It is the 1^{st} of the newton-cotes Formulas " closed Formulas", & it's based on one polynomial of the 1^{st} order.



$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b-a} (x-a) \right] dx.$$

the result of the Integration is

$$I = (b-a) \frac{f(a) + f(b)}{2}$$ which is

called " The Trapezoidal Rule".

⇒ Note I = width * avg height ⊛

The error of Trapezoidal Rule is:

$$E_t = \frac{-1}{12} f''(\xi)(b-a)^3$$

where (ξ) lies somewhere between a & b.

→ If the function being integrated is Linear then the Trapezoidal Rule will be exact because the 2^{nd} derivative is zero.

Otherwise there will be some error.

<u>example</u>:- Use Trapezoidal Rule to Integrate:

$$f(x) = 0.2 + 0.25x - 200x^2 + 675x^3 - 900x^4 + 400x^5.$$

From $a = 0$ to $b = 0.8$ where the true value is 1.640533.

<u>Sol</u>

$$I = (b-a)\,\frac{f(a)+f(b)}{2}$$

$$= (0.8-0)\,\frac{(0.2)+(0.232)}{2}$$

$$= \frac{0.8 * 0.432}{2}$$

$$I = 0.1728.$$

$$\epsilon_t = \frac{1.640533 - 0.1728}{1.640533} * 100\% = 89.5\%$$

$$f''(x) = -400 + 4050x - 10800x^2 + 8000x^3$$

$$f''(\xi) = \frac{\int_0^{0.8} (-400 + 4050x - 10800x^2 + 8000x^3)}{0.8}$$

$$= -60$$

$$E_t = \frac{-1}{12} * -60\,(0.8)^3 = 2.56$$

The composite Trapezoidal Rule:-

To improve the accuracy of the Trapezoidal Rule the integration Interval from a to b can be devided into a number of segments & then apply to each segment & then the areas of Individual segments can be added This is called "The composite Trapezoidal Rule"

Note :- ~~There~~ There are $(n+1)$ equally spaced points with (n) segments now of equal width (h).

$$h = \frac{b-a}{n}$$

If $a = x_0$, $b = x_n$; then the total integral is

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx.$$

$$I = h\,\frac{f(x_0)+f(x_1)}{2} + h\,\frac{f(x_1)+f(x_2)}{2} + \cdots + h\,\frac{f(x_{n-1})+f(}{2}$$

$$= \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n} f(x_i) + f(x_n) \right].$$

" Check the example from the book "

$$I = \underbrace{(b-a)}_{\text{width}} \; \underbrace{\frac{f(x_0) + 2\sum_{i=1}^{n} f(x_i) + f(x_n)}{2n}}_{\text{Avg height}}$$

$$E_t = \frac{(b-a)^3}{12n^3} \sum_{i=1}^{n} f''(\xi_i),$$

(ξ_i) is located in segment ;

$$E_a = \frac{-(b-a)^3}{12n^3} \bar{f}''$$

* Simpson's Rule :-

To obtain more accurate estimation of an integral we can use higher order polynomial, So for 3-points with equally spaced they can be connected with parabola, & for (4) points can be connected with 3rd order.

The formulas that results from taking the (I) under these polynomials are called "Simpson's Rule".

→ Simpson's 1/3 Rule :-

It corresponds to the case when the 2nd order polynomial is used.

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

where x_0, x_1 & x_2 are the corresponding points where:

$x_0 = a$, $x_2 = b$ then

$$\therefore I = \frac{h}{3} \left[f(x_0) + 4 f(x_1) + f(x_2) \right],$$

where $h = \frac{b-a}{2}$.

$$I = \frac{b-a}{6} \left[f(x_0) + 4 f(x_1) + f(x_2) \right] \quad (*)$$

$$E_t = \frac{-1}{90} h^5 f^{(4)}(\xi)$$

Note:- Simpson's 1/3 Rule is more accurate than the Trapezoidal Rule since it's 3rd order accurate

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⊛ Composite Simpson's 1/3 Rule:

It can be improved by dividing the (I) interval into a number of segments of equally width.

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= 2h\,\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + 2h\,\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}$$

$$+ \dots + 2h\,\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}$$

$$I = \frac{b-a}{3n}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)\right]$$

$$E_a = -\frac{(b-a)^5}{180\,n^4}\,\bar{f}^{(4)}(\xi)$$

" Check the example from the book "

x

* Simpsons 3/8 Rule : It's used for (4) points for the estimation of Integration, using the same procedures for Simpson's $\frac{1}{3}$ Rule then :-

$$I = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

where $h = \frac{b-a}{3}$.

$$I = \frac{(b-a)}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$E_t = -\frac{3}{80} h^5 f^{(4)}(\xi)$$

$$= -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$$

note The dominonator is larger than the dominonator in Simpson's $\frac{1}{3}$ Rule then it's more accurate.

note It is possible to use simpson's $\frac{1}{3}$ Rule & Simpson's $\frac{3}{8}$ Rule to estimate for 5 segments (6 pnts) so:

Simpson's $\frac{1}{3}$ Rule can be applied for the 1st (2 segments) & Simpson's $\frac{3}{8}$ Rule can be applied for the last 3 segments.

"Check the example from the book".

* <u>Higher order Newton Cote's Formulas</u>:-

(Check page 481) → The table

For greater than (4) pnts we need to use "higher order newton Cote's Formulas" such as:

Boole's rule: for (5) points which is:

$$I = \frac{(b-a)}{90}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]$$

$$E_t = \frac{-8h^7}{945} f^{(6)}(\xi)$$

& For (6) points:-

$$I = \frac{b-a}{288}\left[19f(x_0) + 75f(x_1) + 50f(x_2) + 50f(x_3) + 75f(x_4) + 19f(x_5)\right]$$

$$E_t = \frac{-275 h^7}{12096} f^{(6)}(\xi)$$

* Integration with unequal segments :-

If the segments are not equally spaced then we Apply "The Trapezoidal Rule" to each segment & ~~sum~~ sum the results.

$$I = h_1 \frac{f(x_0)+f(x_1)}{2} + h_2 \frac{f(x_1)+f(x_2)}{2} + \dots + h_n \frac{f(x_{n-1})+f(x_n)}{2}$$

where (h_i) is the width of segment i

"check the example from the book"

→ hint for the example :-

$$I = (0.12)\frac{(0.2+1.309729)}{2} + 0.1\frac{(1.309729+1.30524}{2}$$

$$+ \dots + (0.11)\frac{(2.303+0.232)}{2}$$

$$I = 1.594801$$

* Open method *

The general Formula for the open method is : $I = (b-a) \times$ avg height.

"Check table 19.4 P 480"

* Numerical Integration function *

In previous chapter we noted that the fun. to be integrated numerically will be of two forms:
A Table of values or a function where the form of data has an important influence on the approach that can be used to evaluate the integral.

→ Roberg Integration:

A. Richardson extrapolation:

it is used to improve the results of numerical integration on the basis of the integral estimates themselves
It uses 2-estimates of an integral to compute a 3rd.
The estimates & the error associated with the composite Trapezoidal Rule is

$I = I(h) + E(h)$ where

I is the exact value of the integral & I(h) is the

Approximation from an n segments of Trapezoidal Rule with stepsize h.

$E(h) \rightarrow$ the error.

For two separate estimates with stepsize h_1 & h_2:

$$I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

$$E = \frac{b-a}{12} h^2 \bar{f}''$$

$$\frac{E(h_1)}{E(h_2)} = \left(\frac{h_1}{h_2}\right)^2 \Rightarrow E(h_1) = E(h_2)\left(\frac{h_1}{h_2}\right)^2$$

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right)^2 = I(h_2) + E(h_2)$$

$$E(h_2) = \frac{I(h_1) - I(h_2)}{1 - \left(\frac{h_1}{h_2}\right)^2}$$

then:-

$$I = I(h_2) + \frac{1}{\left(\frac{h_1}{h_2}\right)^2 - 1}\left(I(h_2) - I(h_1)\right)$$

then the error is $O(h^4)$

For a special case where $h_2 = \frac{h_1}{2}$

$$I = \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1)$$

example: Use Richardson Extrapolation to evaluate the integral of

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

From $a = 0$ to $b = 0.8$

Sol:

For $h_1 = 0.8$

$$I(h_1) = (b-a) \frac{f(a) + f(b)}{2}$$

$$= 0.8 \frac{(0.2) + (0.232)}{2}$$

$I(h_1) = 0.1728 \quad \epsilon_t = 89.5\%$

For $h_2 = 0.4$; $x_0 = 0$; $x_1 = 0.4$, $x_2 = 0.8$

Using composite Trapezoidal Rule:-

$f(x_0) = 0.2 \quad f(x_1) = 2.456 \quad f(x_2) = 0.232$

$$I_2 = (0.8) \frac{0.2 + 2(2.456) + 0.232}{4}$$

$I_2 = 1.0688 \quad \epsilon_t = 34.5\%$

For $h_3 = 0.2$

$x_0 = 0 \quad x_1 = 0.2 \quad x_2 = 0.4 \quad x_3 = 0.6 \quad x_4 = 0.8$

$$I_3 = 0.8\,\frac{(0.2 + 2(1.288) + 2.456 + 3.464 + 0.232)}{8}$$

$I_3 = 1.4848$

$\epsilon_t = 9.5\%$

For the 1st two estimates:-

$$I = \frac{4}{3}(1.0688) - \frac{1}{3}(0.1728) = 1.367467$$

$\epsilon_t = 16.6\%$

For the last 2 estimates:-

$$I = \frac{4}{3}(1.4848) - \frac{1}{3}(1.0688) = 1.623467$$

$\epsilon_t = 1\%$

- For special case where the original Trapezoidal estimates are based on successive halfing of the stepsize. the equation used is:

$$I = \frac{16}{15} I_m - \frac{1}{15} I_L \quad \text{where}$$

I_m & I_L are the more & less accurate estimates respectively $O(h^6)$

Similarly to $O(h^8)$

$$I = \frac{64}{63} I_m - \frac{1}{63} I_l$$

example: Combine two $O(h^4)$ estimates to estimate the pre. Integral with $O(h^6)$.

Sol

From the pre. example:

$I_m = 1.623467$

$I_l = 1.367467$

$$I = \frac{16}{15}(1.623467) + \frac{1}{15}(1.367467)$$

$I = 1.640533 \qquad \epsilon_t = 0\%$

* Numerical Differentiation:-

Taylor series expansion were employed to derived finite differentiation approx. of derivative.

There were forward, Backward & Difference Approximation of the 1^{st} & higher derivative.

the level of accuracy depends on the number of terms of the taylor series.

→ 1^{st} derivative :-

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2!} h + O(h^2)$$

OR

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

where $f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$

then :-

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{2h^2} + O(h^2).$$

then:-

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)$$

The same procedure can be used to derive f'', $f^{(3)}$ & so on.

" check figures (21-3, 21-4, 21-5) "

& solve the example from the book.

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→ Richardson extrapolation:-

To improve the derivative estimate we decrease the stepsize or use higher order formula.

It uses two derivative estimates to compute a 3rd more accurate Approximations.

$$D = \frac{4}{3} D(h_2) - \frac{1}{3} D(h_1) \quad ; \text{where}$$

$$h_2 = \frac{h_1}{2}$$

example: Estimate the derivative of $f(x)$ using Richardson extrapolation.

$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$

at $x = 0.5$; where $h_1 = 0.5$, $h_2 = 0.25$ where the true value is -0.9125.

Sol

$$D(0.5) = \frac{0.2 - 1.2}{1} = -1 \qquad \epsilon_t = 9.6\%$$

$$D(0.25) = \frac{0.6336281 - 1.103516}{0.5}$$

$$= -0.934375 \qquad \epsilon_t = 2.4\%$$

$$D = \frac{4}{3} D(h_2) - \frac{1}{3} D(h_1) =$$

$$= \frac{4}{3}(-0.934375) - \frac{1}{3}(-1)$$

$$= -0.9125$$

$\varepsilon_t = 0\%$.

* Derivative of unequally spaced data:-
In this case we can use a lagrange Interpolating polynomial to fit the data & then we can differentiate ~~analytically~~ ~~to yield a~~ analytically to yield a Formula that can be used to estimate the derivative.
For 3 adjacent point :
(x_0, y_0), (x_1, y_1) & (x_2, y_2) we can fit a 2nd order lagrange Polynomial & differentiate it so:

$$f'(x) = f(x_0)\frac{(2x - x_1 - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1)\frac{(2x - x_0 - x_2)}{(x_1 - x_2)(x_1 - x_0)} + f(x_2)\frac{(2x - x_1 - x_0)}{(x_2 - x_0)(x_2 - x_1)}.$$

Where x is the value at which we want to estimate the derivative.

example: For the following data find $f'(0)$:

x	0	0.0125	0.0375
$f(x)$	13.5	12	10

$$f'(0) = f(0)\frac{(2\times0 - 0.0125 - 0.0375)}{(0-0.0125)(0-0.0375)} + f(0.0125)\frac{(2\times0 - 0 - 0.0375)}{(0.0125-0)(0.0125-0.0375)} + f(0.0375)\frac{(2\times0 - 0 - 0.0125)}{(0.0375-0)(0.0375-0.0125)}$$

$$f'(0) = -133.333$$

the point should be in the interval or we can't use it.

→ Partial derivative:-

The partial derivative of two dimensionless function $f(x,y)$ of equally spaced data with centered difference is

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x, y) - f(x-\Delta x, y)}{2\Delta x}$$

$$\frac{\partial f}{\partial y} = \frac{f(x, y+\Delta y) - f(x, y-\Delta y)}{2\Delta y}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\frac{\partial f}{\partial y}(x+\Delta x, y) - \frac{\partial f}{\partial y}(x-\Delta x, y)}{2\Delta x}$$

$$= \frac{\frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y)}{2\Delta y} - \frac{f(x-\Delta x, y+\Delta y) - f(x-\Delta x, y-\Delta}{\Delta y}}{\Delta x}$$

$$= \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y) - f(x-\Delta x, y+\Delta y) - f(x-\Delta x, y+\Delta}{4\Delta x \Delta y}$$

Ordinary Differential Equation (ODE) :-

→ Intial Value problem:

The general form to Solve the ODE

$\frac{\partial y}{dt} = f(y, t)$ is

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$$y_{i+1} = y_i + \underset{\uparrow \text{slope}}{\phi}\, \overset{\text{stepsize}}{h} \quad \text{———} \quad (*)$$

Where the slope ϕ is called an increment fn. & h is the stepsize. So the slope estimate of ϕ is used to extrapolate from an old value (y_i) to a new value (y_{i+1}) over a stepsize (h) this formula is called "One Step Method" because the Increment function based on Information at a single point (1).

→ All one step method can be expressed in the General form of equation $(*)$.

with the only difference being the manner ~~of~~ in which the slope is estimated.

* Euler method.

The 1st derivative provide a direct estimate of the slope at t_i :-

$\phi = F(t_i, y_i)$ where $f(t_i, y_i)$ is the D.E evaluated at t_i, & y_i then :.

$$y_{i+1} = y_i + f(t_i, y_i)\, h \quad \text{—} \ (*)$$

this is the Euler formula or the Euler Cochy method.

example

Use Euler method to integrate $f(t,y) = y' = 4e^{0.8t} - 0.5y$, from $t=0$ to 4 with $h=1$ & $y(0)=2$ the true answer is :

$$y = \frac{4}{1.3}\left(e^{0.8t} - e^{-0.5t}\right) + 2e^{-0.5t}.$$

$$f(0,2) = 3 \rightarrow 4e^{0.8t} - 0.25(2).$$

Sol

$y(1) = y(0) + f(0,2) \quad = 5$

The true value @ $t=1$ is 6.19463 $\quad \epsilon_t = 19.28\%$

$y(2) = y(1) + f(1,5)(1) =$

$= 5 + 4e^{0.8(1)} - 0.5(5)(1).$

$= 11.40216.$

& The True value is @ $t=2$
is 14.84292.
$\epsilon_t = 23.19\%$.

t	y_{true}	y_{Euler}	$\epsilon_t\ \%$
0	2	2	0
1	6.14463	5	19.28%
2	14.84392	11.40216	23.19%
3	33.67717	25.51321	24.24%
4	57.33896	56.8931	24.54%

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* Error analysis for Eular Method:-

[1] Trancatio error

↳ due to nature of that Tech employed to approx. the value of y it composed of 2 parts

① local trancation error
that results from an application method

② progapated trancation error

↳ that result from the approx. reduce during the previous step

→ The sum of 2 error called global trancation error

[2] Round off error.

→ The approx. Local trancation error :

$$E_a = \frac{f'(t_i, y_i) h^2}{2!}, \quad E_a = O(h^2)$$

* Improvement of Eular method :

ح رح نستخدم طريقتين :

A fundemental source of method is that the derevative of interval to modification using two estimate one at the begining and one at the end.

→ Ulian's Method :-

to improve the estimate of the slope involve the determination of two derivative one at the beginning and one ~~and~~ at the end, then avarge the result, to obtain an improv estimate of the slop.

$$\bar{y}_i = f(t_i, y_i)$$

is used to extrapolate linearly to y_{i+1}

$$\bar{y}_{i+1} = y_i + f(t_i, y_i)\,h$$

This equation is called a ~~periodic~~ prediction equation. , it provids an estimate that allows the calculations of a slop at the end of the interval. So,

$$\bar{y}_{i+1} = f(t_{i+1}, y_{i+1})$$

So, the two slope are avareged as:-

$$\bar{y}' = \frac{f(t_i, y_i) + f(t_{i+1}, y_{i+1})}{2}$$

This avarage slope is then used to extrapolate linearly from $y_i \xrightarrow{to} y_{i+1}$ using Eular method

$$y_{i+1} = y_i + \frac{f(t_i, y_i) + f(t_{i+1}, y^0_{i+1})}{2} h$$

which is called the corrector equation

So, the Euler method is predictor corrector approch

Prediction

$$y^0_{i+1} = y^m_i + f(t_i, y_i) h$$

corrector

$$y^j_{i+1} = y^m_i + \frac{f(t_i, y^m_i) + f(t_{i+1}, y^{j-1}_{i+1})}{2} h$$

for $i = 1, 2, \ldots, m$

$$|\varepsilon_a| = \left| \frac{y^j_{i+1} - y^{j-1}_{i+1}}{y^j_{i+1}} \right| \times 100\%$$

if ODE is only a function of the indep. varabale, then the predictor step is not required and the corrector is applied onee for each itteration.

$$y_{i+1} = y_i + \frac{f(t_i) + f(t_{i+1})}{2} h$$

with Local tranction error.

$$E_t = -\frac{f''(\xi)}{12} h^3$$

(في Examples بالكتاب طارح عليهم الدكتور.)
حلوهم ::

The Midpoint Method

this method uses Eular method to predict a value of y at the mid point of the interval.

$$y_{i+\frac{1}{2}} = y_i + f(t_i, y_i)\frac{h}{2}$$

Then this predicted value is used to calculate a slope at the midpoint →

$$y'_{i+1/2} = f(t_{i+1/2}, y_{i+1/2})$$

Then the slope is used to extrapolate linearly from t_i to t_{i+1}

So,

$$y_{i+1} = y_i + f(t_{i+1/2}, y_{i+1/2})h$$

كما في example في الكتاب

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)},\quad x_{i+1}=x_i-\frac{f(x_i)(x_{i-1}-x_i)}{f(x_{i-1})-f(x_i)},\quad x_{i+1}=x_i-\frac{\delta x_i f(x_i)}{f(x_i+\delta x_i)-f(x_i)}$$

$$\begin{aligned}&na_0+(\sum x_i)a_1=\sum y_i\\&(\sum x_i)a_0+(\sum x_i^2)a_1=\sum x_i y_i\end{aligned},\qquad \begin{aligned}&na_0+(\sum x_i)a_1+(\sum x_i^2)a_2=\sum y_i\\&(\sum x_i)a_0+(\sum x_i^2)a_1+(\sum x_i^3)a_2=\sum x_i y_i\\&(\sum x_i^2)a_0+(\sum x_i^3)a_1+(\sum x_i^4)a_2=\sum x_i^2 y_i\end{aligned}$$

$$\begin{aligned}&na_0+(\sum x_{1,i})a_1+(\sum x_{2,i})a_2=\sum y_i\\&(\sum x_{1,i})a_0+(\sum x_{1,i}^2)a_1+(\sum x_{1,i}x_{2,i})a_2=\sum x_{1,i}y_i\\&(\sum x_{2,i})a_0+(\sum x_{1,i}x_{2,i})a_1+(\sum x_{2,i}^2)a_2=\sum x_{2,i}y_i\end{aligned}$$

$$f_1(x)=f(x_1)+\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1),$$

$$f_{n-1}(x)=b_1+b_2(x-x_1)+\dots\dots+b_n(x-x_1)(x-x_2)\dots\dots(x-x_{n-1})$$

$$b_1=f(x_1),b_2=f[x_2,x_1],b_3=f[x_3,x_2,x_1],f[x_n,x_{n-1},\dots\dots,x_3,x_2,x_1]'$$

$$f_{n-1}(x)=\sum_{i=1}^{n}L_i(x)f(x_i),L_i(x)=\prod_{\substack{j=1\\j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}$$

$$I=(b-a)\frac{f(a)+f(b)}{2},I=(b-a)\frac{f(x_0)+2\sum_{i=1}^{n-1}f(x_i)+f(x_n)}{2n}$$

$$I=(b-a)\frac{f(x_0)+4f(x_1)+f(x_2)}{6},I=(b-a)\frac{f(x_0)+4\sum_{i=1,3,5}^{n-1}f(x_i)+2\sum_{j=2,4,6}^{n-2}f(x_j)+f(x_n)}{3n}$$

$$I=(b-a)\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8},I=I(h_2)+\frac{1}{(h_1/h_2)^2-1}[I(h_2)-I(h_1)]$$

$$f'(x_i)=\frac{f(x_{i+1})-f(x_i)}{h},f'(x_i)=\frac{-f(x_{i+2})+4f(x_{i+1})-3f(x_i)}{2h}$$

$$f'(x_i)=\frac{f(x_i)-f(x_{i-1})}{h},f'(x_i)=\frac{3f(x_i)-4f(x_{i-1})+f(x_{i-2})}{2h}$$

$$f'(x_i)=\frac{f(x_{i+1})-f(x_{i-1})}{2h},f'(x_i)=\frac{-f(x_{i+2})+8f(x_{i+1})-8f(x_{i-1})+f(x_{i-2})}{12h}$$

$$\frac{\partial f}{\partial x}=\frac{f(x+\Delta x,y)-f(x-\Delta x,y)}{2\Delta x},\frac{\partial f}{\partial y}=\frac{f(x,y+\Delta y)-f(x,y-\Delta y)}{2\Delta y}$$

$$y_{i+1}=y_i+\phi h,y_{i+1}^0=y_i^m+f(t_i,y_i)h,y_{i+1}^j=y_i^m+\frac{f(t_i,y_i^m)+f(t_{i+1},y_{i+1}^{j-1})}{2}h,(j=1,2,\dots,m)$$

$$\phi=a_1k_1+a_2k_2+\dots..+a_nk_n,k_1=f(t_i,y_i),k_2=f(t_i+p_1h,y_i+q_{11}k_1h),k_3=f(t_i+p_2h,y_i+q_{21}k_1h+q_{22}k_2h)$$

$$\dots\dots\dots\dots,k_n=f(t_i+p_{n-1}h,y_i+q_{n-1,1}k_1h+q_{n-1,2}k_2h+\dots..+q_{n-1,n-1}k_{n-1}h)$$

$$a_1=1-a_2,p_1=q_{11}=\frac{1}{2a_2}$$

$$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h,y_{i+1}=y_i+\frac{1}{90}(7k_1+32k_3+12k_4+32k_5+7k_6)h,$$