NUMARICAL SUMMARY

2013

POWER UNIT

ELECTRICAL ENGINEERING

E_T = true value − approximation.

true error

absolute error

True fractional relative error : $\dfrac{\text{true value} - \text{approximation}}{\text{true value.}}$

$$\varepsilon_t = \frac{\text{true value} - \text{approximation}}{\text{true value}} \times 100\%$$

$$\varepsilon_a = \frac{\text{present approximation} - \text{previous approx}}{\text{present approximation}} \times 100\%$$

$\varepsilon_s = (.5 \times 10^{2-n})\%$ n = significat number

$|\varepsilon_a| < \varepsilon_s$.

round off error → computer

truncation error → calculation.

Taylor series expansion:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \cdots + \frac{f^{(n)}(x_i)h^n}{n!} + R_n$$

$R_n = \dfrac{f^{(n+1)}(\xi)\,h^{n+1}}{(n+1)!}$ → value of x that lies between x_i and x_{i+1}

$R_n = O(h^{n+1})$ → truncation error of the order of h^{n+1}

* The remainde for the taylor series

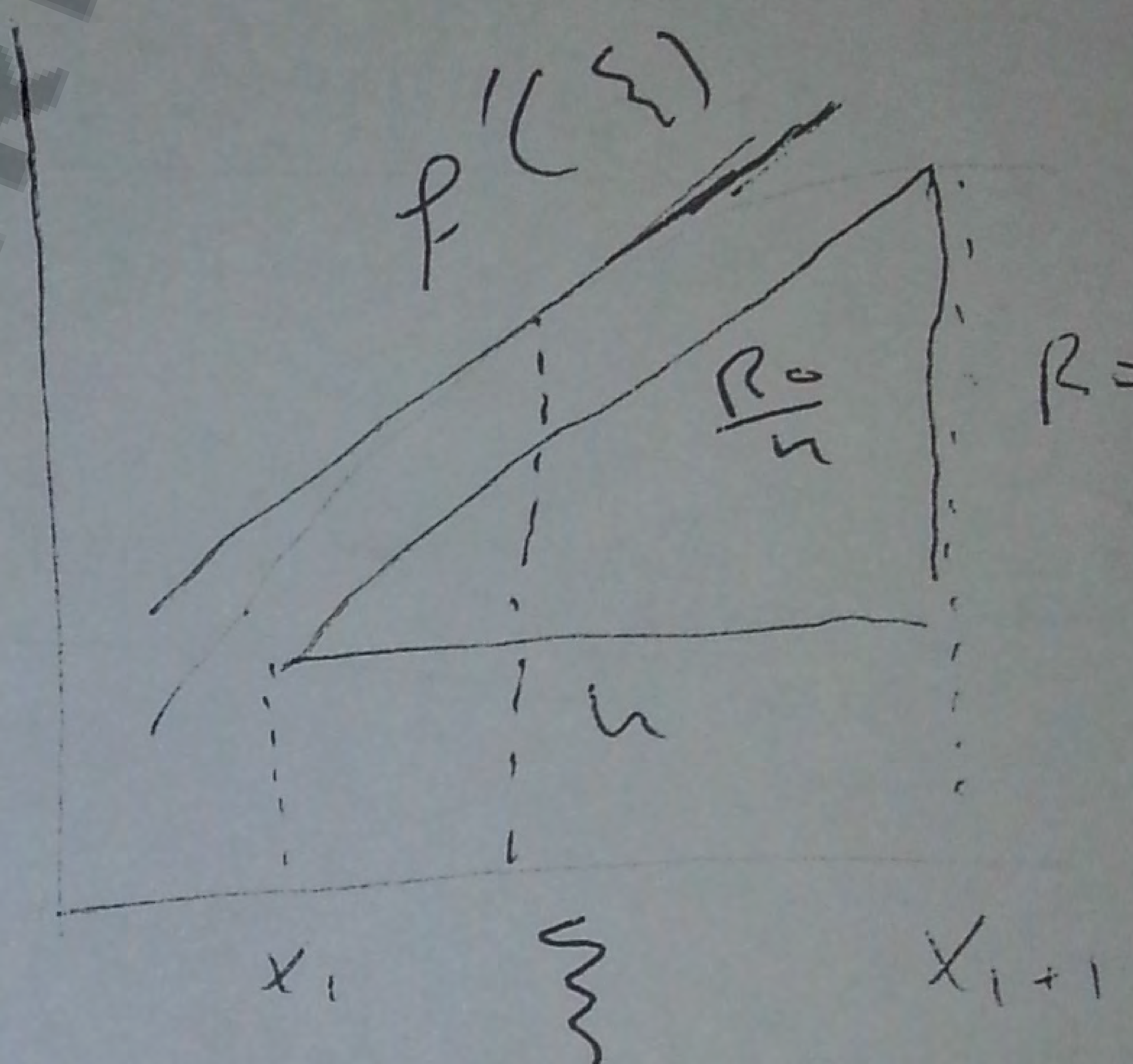
- suppose that we truncated the taylor

$$f(x_{i+1}) \cong f(x_i)$$

$$R_0 = f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \frac{f^{(3)}(x_i)}{3!}h^3 - - - -$$

$$R_0 \cong f'(x_i)$$

$$f'(\xi) = \frac{R_0}{h} \Rightarrow R_0 = f'(\xi)h$$



* using the taylor series to estimate truncation Errors: truncate the series after first derivative:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + R_1$$

$$v'(t_i) = \frac{v(t_{i+1}) - v(t_i)}{(t_{i+1} - t_i)} - \frac{R_1}{(t_{i+1} - t_i)}$$

$$\frac{R_1}{t_{i+1} - t_i} = \frac{v''(\xi)}{2!}(t_{i+1} - t_i) \quad \text{oR} \quad \frac{R_1}{t_{i+1} - t_i} = O(t_{i+1} - t_i)$$

Numerical Differentiation:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h) \quad \text{forward}$$

Backward difference :

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$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)h^2}{2!} \cdots$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + R_1$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + \underbrace{\frac{R_1}{h}}_{O(h)}$$

Centered Difference

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} \cdots$$

$$-\quad f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 \cdots$$

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)h + \frac{2f^{(3)}h^3}{3!} \cdots$$

$$\Rightarrow \quad f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1}) - \frac{2f^{(3)}h^3}{3!}}{2h}$$

$$f'(x_i) = \frac{f_{x_{i+1}} - f_{x_{i-1}}}{2h} - \frac{f^{(3)}}{3!}h^2$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - O(h^2)$$

Finite - Difference Approximation of Higher Derivatives.

$$+\quad f(x_{i+2}) = f(x_i) + f'(x_i)(2h) + \frac{f''(x_i)(2h)^2}{2!} + \cdots$$

$$-\quad 2\times\left(f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \cdots\right)$$

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)h^2 \cdots$$

which can be solved for

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

Similar manipulation can be employed to derive a backward version

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i+2})}{h^2} + O(h)$$

Centered:

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \cdots$$

$$+ \quad f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \cdots$$

$$f(x_{i-1}) + f(x_{i+1}) = 2f(x_i) + f''(x_i)h^2 + \cdots$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2)$$

Centered can be alternatively expressed as:

$$f''(x_i) \cong \frac{\dfrac{f(x_{i+1}) - f(x_i)}{h} - \dfrac{f(x_i) - f(x_{i-1})}{h}}{h}$$

ial umber error : 0

total error = truncation error + roundoff errors:

* the strategy for decreasing one component of the total error leads to an increase of other component:

Error Analysis of Numerical Differentiation:

$$\underbrace{f'(x_i)}_{\text{True value}} = \underbrace{\frac{f(x_{i+1}) - f(x_{i-1})}{2h}}_{\text{finite approximation}} - \underbrace{\frac{f^{(3)}(\xi) h^2}{6}}_{\text{truncation error}}$$

~~However~~ However:

$$f(x_{i-1}) = \tilde{f}(x_{i-1}) + e_{i-1}$$

$$f(x_{i+1}) = \tilde{f}(x_{i+1}) + e_{i+1}$$

$$\underbrace{f'(x_i)}_{\text{true}} = \underbrace{\frac{\tilde{f}(x_{i+1}) - \tilde{f}(x_{i-1})}{2h}}_{\text{approx.}} + \underbrace{\frac{e_{i+1} - e_{i-1}}{2h}}_{\text{Roundoff error}} - \underbrace{\frac{f^{(3)}(\xi) h^2}{6}}_{\text{truncation error}}$$

ε is the upper bound of round off error

so $e_{(i+1)} - e_{(i-1)} = 2\varepsilon$

M is max absolute value of the third derivative

total error $\leq \frac{\varepsilon}{h} + \frac{h^2 M}{6}$ (*)

diff. (*) and set the result to zero and solving for $h_{opt} = \sqrt[3]{\frac{3\varepsilon}{M}}$

Chapter 5: bracketing method:

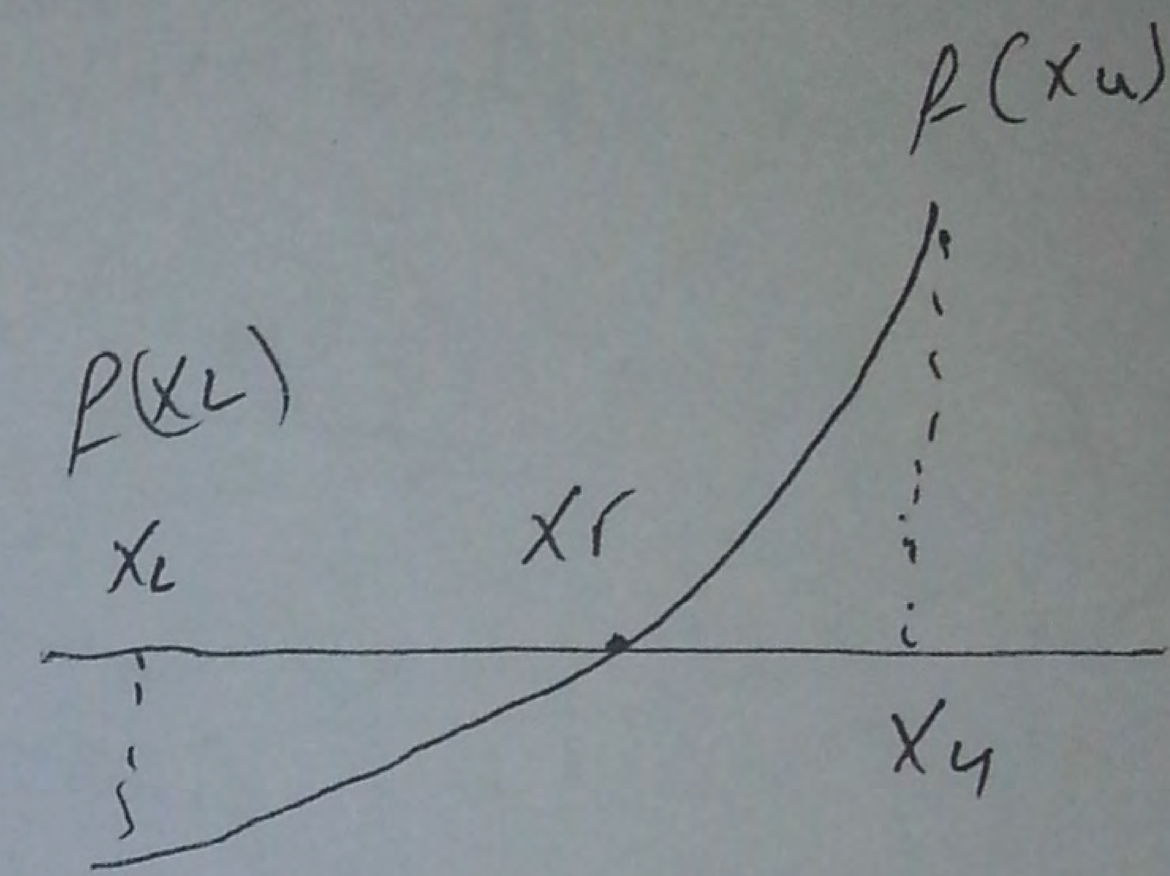
1- Graphical method → approximation

2- Bisection method:

3- False position: linear interpolation method:

$$x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}$$

$$\frac{f(x_L) - f(x_u)}{x_L - x_u} = \frac{f(x_u)}{x_u - x_r}$$



$$x_u - x_r = \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}$$

$$\boxed{x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}}$$

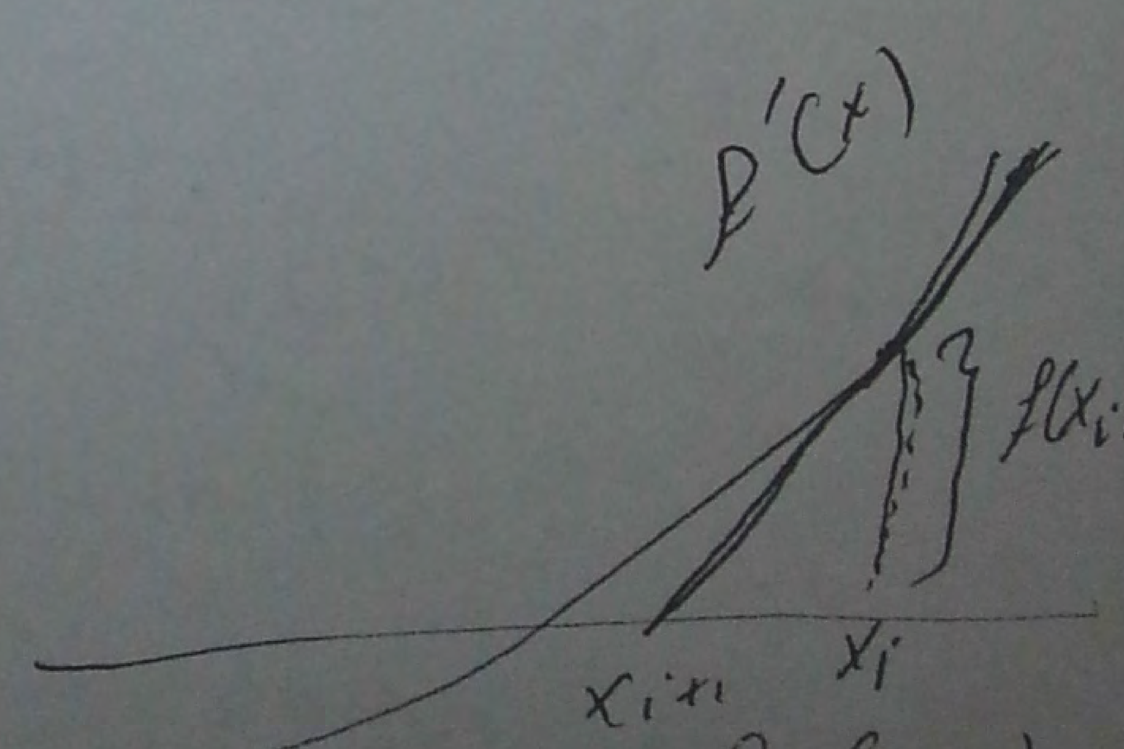
Chapter 6: Open methods

1- simple fixed-point

$$f(x) = 0$$

$$x = g(x)$$

$$\boxed{x_{i+1} = g(x_i)}$$



2- Newton-Raphson

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}} \Longrightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Secant method:

$$f'(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

modified

$$f'(x_i) = \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

$$x_{i+1} = x_i - \frac{\delta(x_i)\, f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

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chapter 8:

symmetric matrix: $a_{ij} = a_{ji}$

bounded matrix: $\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$

$$[A][B] \neq [B][A]$$

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

Chapter 9:

Chapter 10:

$$[A]\{x\} - \{b\} = 0 \sim (1)$$

suppose (1) can be expressed as an upper triangle

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \sim (2)$$

equation (2) can be expressed $[u][x] - \{d\} = 0 \sim (3)$

Now assume that there's

$$\{L\} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$ that has the property

that equation (3) is premultiplied by it - Eq. (1) is the result

$$[L]\{\{U\}\{x\} - \{d\}\} = [A]\{x\} - \{b\}$$

so $\{L\}[U] = \{A\}$

$\{L\}\{d\} = \{b\}$

$$l_{21} = \frac{a_{21}}{a_{11}} \qquad l_{32} = \frac{a_{32}'}{a_{22}'}$$

$$l_{31} = \frac{a_{31}}{a_{11}}$$

to solve the proplem

1) $\{L\}\{d\} = \{b\}$

$\{U\}\{x\} = \{d\}$

Cholesky factorization:

$$u_{11} = \sqrt{a_{11}} \qquad u_{22} = \sqrt{a_{22} - u_{12}^2}$$

$$u_{12} = \frac{a_{12}}{u_{11}} \qquad u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}}$$

$$u_{13} = \frac{a_{13}}{u_{11}}$$

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2}$$

$\{U\}^T [d] = \{b\}$

$\{U\} \{x\} = \{d\}$

The multiequation form is derived in an identical fashion

$$f_{1,i+1} = f_{1,i} + (x_{1,i+1} - x_{1,i})\frac{\partial f_{1,i}}{\partial x_1} + (x_{2,i+1} - x_{2,i})\frac{\partial f_{1,i}}{\partial x_2}$$

$$f_{2,i+1} = f_{2,i} + (x_{1,i+1} - x_{1,i})\frac{\partial f_{2,i}}{\partial x_1} + (x_{2,i+1} - x_{2,i})\frac{\partial f_{2,i}}{\partial x_2}$$

Just as for single - equation, the root estimate corresponds to the values of x_1 and x_2 where $f_{1,i+1}$ and $f_{2,i+1}$ equal zero.

For this situation (*) can be rearranged to give

$$\frac{\partial f_{1,i}}{\partial x_1} x_{1,i+1} + \frac{\partial f_{1,i}}{\partial x_2} x_{2,i+1} = -f_{1,i} + x_{1,i}\frac{\partial f_{1,i}}{\partial x_1} + x_{2,i}\frac{\partial f_{1,i}}{\partial x_2}$$

$$\frac{\partial f_{2,i}}{\partial x_1} x_{1,i+1} + \frac{\partial f_{2,i}}{\partial x_2} x_{2,i+1} = -f_{2,i} + x_{1,i}\frac{\partial f_{2,i}}{\partial x_1} + x_{2,i}\frac{\partial f_{2,i}}{\partial x_2}$$

$$x_{1,i+1} = x_{1,i} - \frac{f_{1,i}\frac{\partial f_{2,i}}{\partial x_2} - f_{2,i}\frac{\partial f_{1,i}}{\partial x_2}}{\frac{\partial f_{1,i}}{\partial x_1}\frac{\partial f_{2,i}}{\partial x_2} - \frac{\partial f_{1,i}}{\partial x_2}\frac{\partial f_{2,i}}{\partial x_1}}$$

$$x_{2,i+1} = x_{2,i} - \frac{f_{2,i}\frac{\partial f_{1,i}}{\partial x_1} - f_{1,i}\frac{\partial f_{2,i}}{\partial x_1}}{\frac{\partial f_{1,i}}{\partial x_1}\frac{\partial f_{2,i}}{\partial x_2} - \frac{\partial f_{1,i}}{\partial x_2}\frac{\partial f_{2,i}}{\partial x_1}}$$

2

* arithmetic mean $\bar{y} = \frac{\sum y_i}{n}$

<u>the median</u>: is the midpoint of a group of data

It's calculated by first putting the data in ascending

If the number of measurements is odd, the median is the middle value. If the number is even, it's the arithmetic mean of the two middle values.

<u>the mode</u>: is the value that occurs most frequently.

* Measures of Spread:
  - the range: the difference between the largest and the smallest value.
  - standard deviation (s_y) about the mean:

$$s_y = \sqrt{\frac{S_t}{n-1}} \quad \text{where } S_t = \sum (y_i - \bar{y})^2$$

$$\underbrace{s_y^2}_{\text{variance}} = \frac{S_t}{n-1} \qquad \text{another form} \quad s_y^2 = \frac{\sum y_i^2 - \frac{\sum y_i^2}{n}}{n-1}$$

the coefficient of variation (c.v.)

$$c.v. = \frac{s_y}{\bar{y}} \times 100\%$$

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2 \quad (*)$$ least squares

to determine values for a_0 and a_1 Eq $(*)$ is differentiated with respect to each unknown coefficien

$$\frac{\partial S_r}{\partial a_0} = -2 \sum (y_i - a_0 - a_1 x_i)$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum [(y_i - a_0 - a_1 x_i) x_i]$$

setting these to zero

$$0 = \sum y_i - \sum a_0 - \sum a_1 x_i$$

$$0 = \sum x_i y_i - \sum a_0 x_i - \sum a_1 x_i^2$$

* realizing that $\sum a_0 = n a_0$, we can express the equations as a set of two simultaneous linear equations with two unknowns (a_0 and a_1):

$$n a_0 + \left(\sum x_i\right) a_1 = \sum y_i$$

$$\left(\sum x_i\right) a_0 + \left(\sum x_i^2\right) a_1 = \sum x_i y_i$$

normal equations

they can be solved simultaneously for

$$a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a_0 = \frac{\sum y_i \sum x_i^2 - \sum x_i y_i \sum x_i}{n \sum x_i^2 - \left(\sum x_i\right)^2} = \bar{y} - a_1 \bar{x}$$

3 (4) Quantification of error of linea Reg

the sum of the squares is defined

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

Notic the similarity between this equation and

$$S_t = \sum (y_i - \bar{y})^2$$

$$\underbrace{S_{y/x}}_{\text{stantard error of the estimate}} = \sqrt{\frac{S_r}{n-2}}$$

standard error of the estimate } the subscript notation "y/x" designates that the error is for a predicted value of y corresponding to a particular value of x.

$r^2 = \frac{S_t - S_r}{S_t}$ → quantifies the improvement or error reduction due to describing the data in terms of a straight line. rather than an average value.

→ coefficient of determination

r: the correlation coefficient

$\boxed{\text{for a perfect fit } S_r = 0 \text{ , } r^2 = 1}$

$S_r = S_t$ $r^2 = 0$ ⇒ the fit represents No improvement.

An alternative formula for r

$$r = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{\sqrt{n \sum x_i^2 - (\sum x_i)^2} \sqrt{n \sum y_i^2 - (\sum y_i)^2}}$$

- Gauss-Seidal method
  initial guess = 0
- The equation must first be rearranged so that they ~~are~~ are diagonally dominant.

Eigenvalues:

$[A]\{x\} = 0$ only trivial case to x's $= 0$

$\{[A] - \lambda[I]\}\{x\} = 0$

power method initial guess $= 1$

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linearization of nonlinear Relations.

- exponential model.

$y = \alpha_1 e^{\beta_1 x}$

$\ln y = \ln \alpha_1 + \beta_1 x$

- Simple power equation:

$y = \alpha_2 x^{\beta_2}$

$\log y = \log \alpha + \beta_2 \log x$

- saturation-growth-rate equation,

$y = \alpha_3 \frac{x}{\beta_3 + x}$

$\frac{1}{y} = \frac{1}{\alpha} + \frac{\beta_3}{\alpha_3} \frac{1}{x}$

15) polynomial Regression

$$y = a_0 + a_1 x + a_2 x^2$$

$$n a_0 + \sum (x_i) a_1 + \sum (x_i)^2 a_2 = \sum y_i$$

$$\sum x_i a_0 + \sum x_i^2 a_1 + \sum x_i^3 a_2 = \sum y_i x_i$$

$$\sum x_i^2 a_0 \quad \sum x_i^3 a_1 + \sum x_i^4 a_2 = \sum y_i x_i^2$$

$$S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$$

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multiple linear regression:

$$y = a_0 + a_1 x_1 + a_2 x_2 + e$$

$$\begin{bmatrix} n & \sum x_{1,i} & \sum x_{2,i} \\ \sum x_{1,i} & \sum x_{1,i}^2 & \sum x_{1,i} x_{2,i} \\ \sum x_{2,i} & \sum x_{1,i} x_{2,i} & \sum x_{2,i}^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{matrix} \sum y_i \\ \sum x_{1,i} y_i \\ \sum x_{2,i} y_i \end{matrix}$$

ch. 17

the general formula for an $(n-1)$ th - order polynomial can be written as:

$$f(x) = a_1 + a_2 x + a_3 x^2 + \cdots a_n x^{n-1}$$

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{Bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{Bmatrix}$$

Newton interpolating polynomal.

17.2.1 linear interpolation.

$$\frac{f_1(x) - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

can be rearranged to yield

$$f_1(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

17.2.2 Quadratic interpolation.

$$f_2(x) = b_1 + b_2(x - x_1) + b_3(x - x_1)(x - x_2)$$

17.3 Lagrange interpolating polynomial.

$$f_1(x) = L_1 f(x_1) + L_2 f(x_2)$$

$$L_1 = \frac{x - x_2}{x_1 - x_2} , \quad L_2 = \frac{x - x_1}{x_2 - x_1}$$

$$f_2(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_1)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} f(x_3)$$

Chapter 19:

$$\text{Mean} = \frac{\int_a^b f(x)\,dx}{b-a}$$

- closed forms: where the data points at the beginning and end of the limits of integration are known
- open forms: have integration limits that extend beyond the range of data

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The trapezoidal rule:

first-order integration:

$$I = \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right] dx = \boxed{(b-a)\frac{f(a)+f(b)}{2}}$$

$$E_t = \frac{-1}{12} \underbrace{f''(\xi)}_{\text{the average value of second derivative}} (b-a)^3$$

* the composite trapezoidal Rule:

- $n+1$ point n segments of equal width

$$h = \frac{b-a}{n}$$

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx \cdots\cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$I = h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + h\frac{f(x_{n-1})+f(x_n)}{2}$$

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right] \Rightarrow I = (b-a)\frac{f(x_0) + 2\sum_i^{n-1} f(x_i) + f(x_n)}{2n}$$

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

$$\bar{f}'' \cong \frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \Rightarrow \sum f''(\xi_i) \cong n\bar{f}''$$

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

Simpson's Rules :

1- 1/3 Rule (three $f(x_i)$) , polynomial second

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

$$I = \frac{h}{3}\left[f(x_0) + 4 f(x_1) + f(x_2)\right] , \text{ where } h = \frac{b-a}{2}$$

$$\boxed{I = (b-a)\frac{f(x_0) + 4 f(x_1) + f(x_2)}{6}}$$

$$Et = \frac{-1}{90} h^5 f^{(4)}(\xi) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

2- the composite Simpson's 1/3 Rule: (even number of segment)

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$I = 2h \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + 2h \frac{f(x_2) + 4 f(x_3) + f(x_4)}{6} + \cdots + 2h \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}$$

$$I = (b-a) \frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

$$Ea = \frac{-(b-a)^5}{180 n^4} \bar{f}^{(4)}$$

Simpson's 3/8 Rule: requires (4 points), third order polynomial.
odd number of $\underline{n}$

write a third order lagrange polynomial can be fit to four points and integrated to yield:

$$I = \frac{3}{8} h \left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

where $h = \frac{3}{8}$

$$I = (b-a) \frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}$$

$$E_t = \frac{-3}{80} h^5 f^{(4)}(\xi) = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$$

Note that it's possible to use simpson's $\frac{1}{3}$ rule with conjuction with $\frac{3}{8}$ rule to estimate an integration for odd number of segments:

Higher order Newton-cotes formulas: (Boole's rule

5-points: $(b-a) \dfrac{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)}{90}$

$$(b-a) \frac{19f(x_0) + 75f(x_1) + 50f(x_2) + 50f(x_3) + 75f(x_4) + 19f(x_5)}{288}$$

Open methods:

segments	points	
2	1	$(b-a)\, f(x_1)$
3	2	$(b-a) \dfrac{f(x_1) + f(x_2)}{2}$
4	3	$(b-a) \dfrac{2f(x_1) - f(x_2) + 2f(x_3)}{3}$
5	4	$(b-a) \dfrac{11f(x_1) + f(x_2) + f(x_3) + 11f(x_4)}{24}$

chapter 20:

Romberg integration:

1- Richardson Extrapolation:

$$\underbrace{I}_{\text{exact}} = \underbrace{I(h)}_{\text{approximation}} + \underbrace{E(h)}_{\text{truncation error}}$$

for two estimates $I(h_1) + E(h_1) = I(h_2) + E(h_2)$ ---(1

recall that the error of the composite trapezoida[l] rule is $E = -\frac{(b-a)^3}{12n^2}\bar{f}''$

with $n = \frac{(b-a)}{h}$, E can expressed as

$$E \cong -\frac{(b-a)}{12} h^2 \bar{f}'' \quad \text{---- (20.2)}$$

* if it is assumed that $\bar{f}''$ is constant regardless of step size Eq. (20.2) can be used to determine that the ration of two errors

$$\frac{E(h_1)}{E(h_2)} \cong \frac{h_1^2}{h_2^2} \text{ --- (3)} \Longrightarrow E(h_1) = E(h_2)\left(\frac{h_1}{h_2}\right)^2$$

subtitute in (1)

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right)^2 = I(h_2) + E(h_2)$$

solve for

$$E(h_2) = \frac{I(h_1) - I(h_2)}{1 - \left(\frac{h_1}{h_2}\right)^2}$$

subtitute in $I = I(h_2) + E(h_2)$ to yield

$$I = I(h_2) + \frac{1}{\left(\frac{h_1}{h_2}\right)^2 - 1}[I(h_2) - I(h_1)]$$

$$I = \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1) \Longrightarrow \frac{16}{15} I_m - \frac{1}{15} I_l \Rightarrow \frac{64}{63} I_m - \frac{1}{63} I_l$$

$O(h^4)$ $O(h^6)$ $O(h^8)$

Chapter 21:

T.S $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 \ldots\ldots$

which can be solved for:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2!}h + O(h^2) \sim (*)$$

recall that: $f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$ — ①

substitute ① in *

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \left[\frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{2h^2}\right]h + O(h^2)$$

by collecting terms:

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)$$

Richardson extrapolation:

$$D = \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1)$$

Derivative of unequally spaced data:

* we can make a lagrange interpolating polynomial and then making analytical differentiation:

for three adjacent points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

the differentiation of largrange polynomial

$$f'(x) = f(x_0)\frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} + f(x_1)\frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} + f(x_2)\frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)}$$

partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x, y) - f(x-\Delta x, y)}{2\Delta x}$$

$$\frac{\partial f}{\partial y} = \frac{f(x, y+\Delta y) - f(x, y-\Delta y)}{2\Delta y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\frac{\partial f}{\partial y}(x+\Delta x, y) - \frac{\partial f}{\partial y}(x-\Delta x, y)}{2\Delta x}$$

$$= \frac{\frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y)}{2\Delta y} - \frac{f(x-\Delta x, y+\Delta y) - f(x-\Delta x, y-\Delta y)}{2\Delta y}}{2\Delta x}$$

$$= \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y) - f(x-\Delta x, y+\Delta y) + f(x-\Delta x, y-\Delta y)}{4\Delta x \Delta y}$$

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