INSTRUCTOR'S SOLUTIONS MANUAL FOR

ELEMENTS OF ELECTROMAGNETICS

INTERNATIONAL FOURTH EDITION

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New York - Oxford
Oxford University Press
2007

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CHAPTER 1

Prob. 1.1

Let $D = \alpha A + \beta B + C$

$$= (5\alpha - \beta + 8)a_x + (3\alpha + 4\beta + 2)a_y + (2\alpha + 6\beta)a_z$$

$$D_x = 0 \rightarrow 5\alpha - \beta + 8 = 0 \qquad (1)$$

$$D_z = 0 \rightarrow 2\alpha + 6\beta = 0 \rightarrow \alpha = -3\beta \qquad (2)$$

Substituting (2) into (1),

$$-15\beta - \beta + 8 = 0 \rightarrow \beta = \frac{1}{2}$$

Thus

$$\alpha = -\frac{3}{2}, \beta = \frac{1}{2}$$

Prob. 1.2

(a) $A \cdot B = \text{AB}\cos\theta_{AB}$

$A \times B = \text{AB}\sin\theta_{AB} a_n$

$$(A \cdot B)^2 + |A \times B|^2 = (AB)^2(\cos^2\theta_{AB} + \sin^2\theta_{AB}) = (\text{AB})^2$$

(b) $\boldsymbol{a}_x \cdot (\boldsymbol{a}_y \times \boldsymbol{a}_z) = \boldsymbol{a}_x \cdot \boldsymbol{a}_x = 1$. Hence,

$$\frac{\boldsymbol{a}_y \times \boldsymbol{a}_z}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_x}{1} = \boldsymbol{a}_x$$

$$\frac{\boldsymbol{a}_z \times \boldsymbol{a}_x}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_y}{1} = \boldsymbol{a}_y$$

$$\frac{\boldsymbol{a}_x \times \boldsymbol{a}_y}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_z}{1} = \boldsymbol{a}_z$$

Prob. 1.3

(a) $A_B = \boldsymbol{A} \cdot \boldsymbol{a}_B = \dfrac{\boldsymbol{AB}}{|\boldsymbol{B}|} = \dfrac{-1+12+15}{\sqrt{1+4+9}} = \dfrac{26}{\sqrt{14}} = \underline{\underline{6.95}}$

(b) $\boldsymbol{B}_A = (\boldsymbol{B} \cdot \boldsymbol{a}_A)\boldsymbol{a}_A = \dfrac{(\boldsymbol{B} \cdot \boldsymbol{A})\boldsymbol{A}}{|\boldsymbol{A}|^2} = \dfrac{26(-1,6,5)}{(1+36+25)}$

$$= -0.4193\boldsymbol{a}_x + 2.516\boldsymbol{a}_y + 2.097\boldsymbol{a}_z$$

(c) $\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} -1 & 6 & 5 \\ 1 & 2 & 3 \end{vmatrix} = 8\boldsymbol{a}_x + 8\boldsymbol{a}_y - 8\boldsymbol{a}_z$

A unit vector perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{a}_{A\times B} = \frac{8\boldsymbol{a}_x + 8\boldsymbol{a}_y - 8\boldsymbol{a}_z}{8\sqrt{1+1+1}} = \frac{\boldsymbol{a}_x + \boldsymbol{a}_y - \boldsymbol{a}_z}{\sqrt{3}} = \underline{\underline{0.577\boldsymbol{a}_x + 0.577\boldsymbol{a}_y - 0.577\boldsymbol{a}_z}}$$

Prob. 1.4

(a) Using the fact that

$$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A},$$

we get

$$\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{B}) = -(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{A} = \underline{\underline{(\boldsymbol{B} \cdot \boldsymbol{A})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{A})\boldsymbol{B}}}$$

(b)
$$\begin{aligned} A \times (A \times (A \times B)) &= A \times [(A \times B)A - (A \times A)B] \\ &= (A \cdot B) \quad (A \times A) - (A \cdot A) \quad (A \times B) \\ &= \underline{\underline{-A^2 (A \times B)}} \end{aligned}$$

since AxA = 0

Prob. 1.5

(a) $\boldsymbol{r}_{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_P = (2,-1,3) - (-1,4,8) = (3,-5,-5)$

$r_{PQ} = |\boldsymbol{r}_{PQ}| = \sqrt{9+25+25} = \underline{\underline{7.681}}$

(b) $\boldsymbol{r}_{PR} = \boldsymbol{r}_R - \boldsymbol{r}_P = (-1,2,3) - (-1,4,8) = (0,-2,-5) = \underline{\underline{-2\boldsymbol{a}_y - 5\boldsymbol{a}_z}}$

(c) $r_{QP} = -r_{PQ} = -3a_x + 5a_y + 5a_z$

$r_{QR} = r_R - r_Q = (-1,2,3) - (2,-1,3) = -3a_x + 3a_y$

$$\cos\theta = \frac{r_{QP} \cdot r_{QR}}{|r_{QP}||r_{QR}|} = \frac{9+15}{\sqrt{9+25+25}\sqrt{9+9}} = \frac{24}{\sqrt{18}\sqrt{59}}$$

$$\underline{\underline{\theta = 42.57^\circ}}$$

(d) Area $= \frac{1}{2}|r_{QP} \times r_{QR}|$

$$r_{QP} \times r_{QR} = \begin{vmatrix} -3 & 5 & 5 \\ -3 & 3 & 0 \end{vmatrix} = -15a_x - 15a_y + 6a_z$$

$$\text{Area} = \frac{1}{2}\sqrt{15^2 + 15^2 + 6^2} = \underline{\underline{11.02}}$$

(e) Perimeter $= QP + PR + RQ = |r_{QP}| + |r_{PR}| + |r_{QR}|$

$$= \sqrt{59} + \sqrt{4+25} + \sqrt{18}$$
$$= 7.681 + 5.385 + 4.243$$
$$= \underline{\underline{17.31}}$$

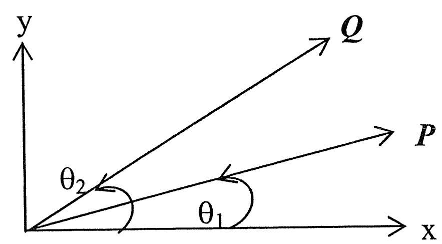
Prob.1.6

Let R be the midpoint of PQ.

$$r_R = \frac{1}{2}\{(2,4,-1) + (12,16,9)\} = (7,10,4)$$
$$OR = \sqrt{49+100+16} = \sqrt{165} = 12.845$$
$$t = \frac{OR}{v} = \frac{12.845}{300} = \underline{\underline{42.82 \text{ ms}}}$$

Prob. 1.7

(a) Let P and Q be as shown below:



$|\boldsymbol{P}| = \cos^2\theta_1 + \sin^2\theta_1 = 1, |\boldsymbol{Q}| = \cos^2\theta_2 + \sin^2\theta_2 = 1,$

Hence **P** and **Q** are unit vectors.

(b) $P \cdot Q = (1)(1)\cos(\theta_2 - \theta_1)$

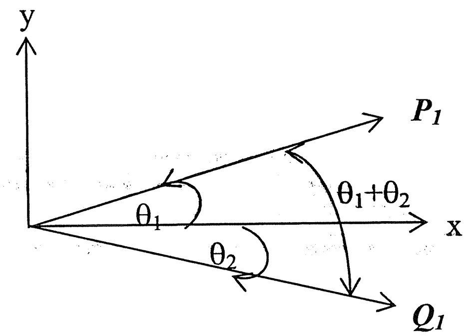
But $P \cdot Q = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$. Thus,

$$\underline{\underline{\cos(\theta_2 - \theta_1) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}}$$

Let $\boldsymbol{P}_1 = \boldsymbol{P} = \cos\theta_1\boldsymbol{a}_x + \sin\theta_1\boldsymbol{a}_y$ and

$\boldsymbol{Q}_1 = \cos\theta_2\boldsymbol{a}_x - \sin\theta_2\boldsymbol{a}_y$.

$\boldsymbol{P_1}$ and $\boldsymbol{Q_1}$ are unit vectors as shown below:



$P_1 \cdot Q_1 = (1)(1)\cos(\theta_1 + \theta_2)$

But $P_1 \cdot Q_1 = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$,

$\cos(\theta_2 + \theta_1) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

Alternatively, we can obtain this formula from the previous one by replacing θ_2 by $-\theta_2$ in $\boldsymbol{Q}$.

(c)

$$\frac{1}{2}|P - Q| = \frac{1}{2}|(\cos\theta_1 - \cos\theta_2)a_x + (\sin\theta_1 - \sin\theta_2)a_y$$

$$= \frac{1}{2}\sqrt{\cos^2\theta_1 + \sin^2\theta_1 + \cos^2\theta_2 + \sin^2\theta_2 - 2\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2}$$

$$= \frac{1}{2}\sqrt{2 - 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)} = \frac{1}{2}\sqrt{2 - 2\cos(\theta_2 - \theta_1)}$$

Let $\theta_2 - \theta_1 = \theta$, the angle between **P** and **Q.**

$$\frac{1}{2}|P - Q| = \frac{1}{2}\sqrt{2 - 2\cos\theta}$$

But $\cos 2A = 1 - 2\sin^2 A$.

$$\frac{1}{2}|P - Q| = \frac{1}{2}\sqrt{2 - 2 + 4\sin^2\theta/2} = \sin\theta/2$$

Thus,

$$\frac{1}{2}|P - Q| = |\sin\frac{\theta_2 - \theta_1}{2}|$$

Prob. 1.8

$$w = \frac{w(1,-2,2)}{3} = (1,-2,2), \quad r = r_p - r_o = (1,3,4) - (2,-3,1) = (-1,6,3)$$

$$u = w \times r = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 6 & 3 \end{vmatrix} = (-18,-5,4)$$

$$\underline{\underline{u = -18a_x - 5a_y + 4a_z}}$$

Prob.1.9

(a) $H(1,3,-2) = 6a_x + a_y + 4a_z$

$$a_H = \frac{(6,1,4)}{\sqrt{36+1+16}} = \underline{\underline{0.8242a_x + 0.1374a_y + 0.5494a_z}}$$

(b) $|H| = 10 = \sqrt{4x^2y^2 + (x+z)^2 + z^4}$

or

$$\underline{\underline{100 = 4x^2y^2 + x^2 + 2xz + z^2 + z^4}}$$

Prob. 1.10

(a) At (1,2,3), $E = (2,1,6)$

$$|E| = \sqrt{4+1+36} = \sqrt{41} = \underline{\underline{6.403}}$$

(b) At (1,2,3), $F = (2,-4,6)$

$$E_F = (E \cdot a_F)a_F = \frac{(E \cdot F)F}{|F|^2} = \frac{36}{56}(2,-4,6)$$

$$= \underline{\underline{1.286a_x - 2.571a_y + 3.857a_z}}$$

(c) At (0,1,-3), $E = (0,1,-3)$, $F = (0,-1,0)$

$$E \times F = \begin{vmatrix} 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = (-3,0,0)$$

$$a_{E\times F} = \pm\frac{E \times F}{|E \times F|} = \underline{\underline{\pm a_x}}$$

CHAPTER 2

Prob. 2.1
(a)

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$

$$V = \underline{\underline{\rho z\cos\phi - \rho^2\sin\phi\cos\phi + \rho z\sin\phi}}$$

(b)

$$U = x^2 + y^2 + z^2 + y^2 + 2z^2$$
$$= r^2 + r^2\sin^2\theta\sin^2\phi + 2r^2\cos^2\theta$$
$$= \underline{\underline{r^2[1 + \sin^2\theta\sin^2\phi + 2\cos^2\theta]}}$$

Prob. 2.2 (a)

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x}{\sqrt{\rho^2+z^2}} \\ \dfrac{y}{\sqrt{\rho^2+z^2}} \\ \dfrac{4}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$F_\rho = \frac{1}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho}{\sqrt{\rho^2+z^2}};$$

$$F_\phi = \frac{1}{\sqrt{\rho^2+z^2}}[-\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi] = 0;$$

$$F_z = \frac{4}{\sqrt{\rho^2+z^2}};$$

$$\underline{\underline{\bar{F} = \frac{1}{\sqrt{\rho^2+z^2}}(\rho\bar{a}_\rho + 4\bar{a}_z).}}$$

In Spherical:

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{4}{r} \end{bmatrix}$$

$$F_r = \frac{r}{r}\sin^2\theta\cos^2\phi + \frac{r}{r}\sin^2\theta\sin^2\phi + \frac{4}{r}\cos\theta \quad = \sin^2\theta + \frac{4}{r}\cos\theta;$$

$$F_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \frac{4}{r}\sin\theta = \sin\theta\cos\theta - \frac{4}{r}\sin\theta;$$

$$F_\phi = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0;$$

$$\therefore \bar{F} = \underline{\underline{(\sin^2\theta + \frac{4}{r}\cos\theta)\bar{a}_r + \sin\theta(\cos\theta - \frac{4}{r})\bar{a}_\theta}}.$$

(b)

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x\rho^2}{\sqrt{\rho^2+z^2}} \\ \frac{y\rho^2}{\sqrt{\rho^2+z^2}} \\ \frac{z\rho^2}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$G_\rho = \frac{\rho^2}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho^3}{\sqrt{\rho^2+z^2}};$$

$$G_\phi = 0;$$

$$G_z = \frac{z\rho^2}{\sqrt{\rho^2+z^2}};$$

$$\bar{G} = \underline{\underline{\frac{\rho^2}{\sqrt{\rho^2+z^2}}(\rho\bar{a}_\rho + z\bar{a}_z)}}.$$

Spherical :

$$\boldsymbol{G} = \frac{\rho^2}{r}(x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z) = \frac{r^2\sin^2\theta}{r}r\boldsymbol{a}_r = \underline{\underline{r^2\sin^2\theta\boldsymbol{a}_r}}$$

Prob. 2.3 (a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho(z^2+1) \\ -\rho z\cos\phi \\ 0 \end{bmatrix}$$

$$A_x = \rho(z^2+1)\cos\phi + \rho z\sin\phi\cos\phi$$

$$= \sqrt{x^2+y^2}\,(z^2+1)\frac{x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\,(\frac{zxy}{x^2+y^2})$$

$$= x(z^2+1) + \frac{xyz}{\sqrt{x^2+y^2}}.$$

$$A_y = \rho(z^2+1)\sin\phi - \rho z\cos^2\phi$$

$$= \sqrt{x^2+y^2}\,(z^2+1)\frac{y}{\sqrt{x^2+y^2}} - \frac{x^2 z}{\sqrt{x^2+y^2}}$$

$$= y(z^2+1) - \frac{x^2 z}{\sqrt{x^2+y^2}};$$

$$A_z = 0;$$

$$\therefore \bar{A} = [x(z^2+1) + \frac{xyz}{\sqrt{x^2+y^2}}]\bar{a}_x + [y(z^2+1) - \frac{x^2 z}{\sqrt{x^2+y^2}}]\bar{a}_y.$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 2x \\ r\cos\theta\cos\phi \\ -r\sin\phi \end{bmatrix}$$

$$B_x = 2x\sin\theta\cos\phi + r\cos^2\theta\cos^2\phi + r\sin^2\phi$$

$$= \frac{2x^2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} + \frac{\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}\left(\frac{x^2z^2}{x^2+y^2}\right) + \sqrt{x^2+y^2+z^2}\left(\frac{y^2}{x^2+y^2}\right)$$

$$= \frac{2x^2}{\sqrt{x^2+y^2+z^2}} + \frac{x^2z^2}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} + \frac{y^2\sqrt{x^2+y^2+z^2}}{x^2+y^2};$$

$$B_y = 2x\sin\theta\sin\phi + r\cos^2\theta\sin\phi\cos\phi - r\sin\phi\cos\phi$$

$$= \frac{2xy\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} + \frac{\sqrt{x^2+y^2+z^2}\,(xyz^2)}{(x^2+y^2+z^2)(x^2+y^2)} - \sqrt{x^2+y^2+z^2}\left(\frac{xy}{x^2+y^2}\right)$$

$$= \frac{2xy}{\sqrt{x^2+y^2+z^2}} + \frac{xyz^2}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} - \frac{xy\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}};$$

$$B_z = 2x\cos\theta - r\sin\theta\cos\theta\cos\phi$$

$$= \frac{2xz}{\sqrt{x^2+y^2+z^2}} - \frac{\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)}\frac{(xz)\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \frac{2xz}{\sqrt{x^2+y^2+z^2}} - \frac{xz}{\sqrt{x^2+y^2+z^2}} = \frac{xz}{\sqrt{x^2+y^2+z^2}};$$

$$\therefore \bar{B} = \left[\frac{2x^2}{\sqrt{x^2+y^2+z^2}} + \frac{x^2z^2}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} + \frac{y^2\sqrt{x^2+y^2+z^2}}{x^2+y^2}\right]\bar{a}_x +$$

$$\left[\frac{2xy}{\sqrt{x^2+y^2+z^2}} + \frac{xyz^2}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} - \frac{xy\sqrt{x^2+y^2+z^2}}{x^2+y^2}\right]\bar{a}_y +$$

$$\left[\frac{xz}{\sqrt{x^2+y^2+z^2}}\right]\bar{a}_z.$$

Prob. 2.4 (a)

$$\bar{a}_x \bullet \bar{a}_\rho = (\cos\phi \bar{a}_\rho - \sin\phi \bar{a}_\phi) \bullet \bar{a}_\rho = \cos\phi$$

$$\bar{a}_x \bullet \bar{a}_\phi = (\cos\phi \bar{a}_\rho - \sin\phi \bar{a}_\phi) \bullet \bar{a}_\phi = -\sin\phi$$

$$\bar{a}_y \bullet \bar{a}_\rho = (\sin\phi \bar{a}_\rho + \cos\phi \bar{a}_\phi) \bullet \bar{a}_\rho = \sin\phi$$

$$\bar{a}_y \bullet \bar{a}_\phi = (\sin\phi \bar{a}_\rho + \sin\phi \bar{a}_\phi) \bullet \bar{a}_\phi = \cos\phi$$

(b)

In spherical system:

$$\bar{a}_x = \sin\theta\cos\phi\, \bar{a}_r + \cos\theta\cos\phi\, \bar{a}_\theta - \sin\phi\, \bar{a}_\phi .$$

$$\bar{a}_y = \sin\theta\sin\phi\, \bar{a}_r + \cos\theta\sin\phi\, \bar{a}_\theta - \cos\phi\, \bar{a}_\phi .$$

$$\bar{a}_z = \cos\theta\, \bar{a}_r - \sin\theta\, \bar{a}_\theta .$$

Hence,

$$\bar{a}_x \bullet \bar{a}_r = \sin\theta\cos\phi;$$

$$\bar{a}_x \bullet \bar{a}_\theta = \cos\theta\cos\phi;$$

$$\bar{a}_y \bullet \bar{a}_r = \sin\theta\sin\phi;$$

$$\bar{a}_y \bullet \bar{a}_\theta = \cos\theta\sin\phi;$$

$$\bar{a}_z \bullet \bar{a}_r = \cos\theta;$$

$$\bar{a}_z \bullet \bar{a}_\theta = -\sin\theta;$$

Prob 2.5 (a)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

$$\theta = \tan^{-1}\frac{\rho}{z}; \qquad \phi = \phi.$$

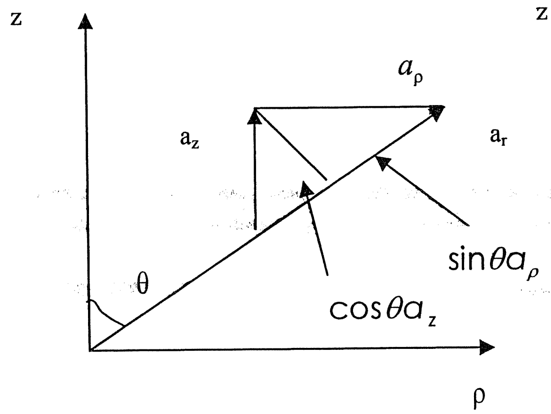
or

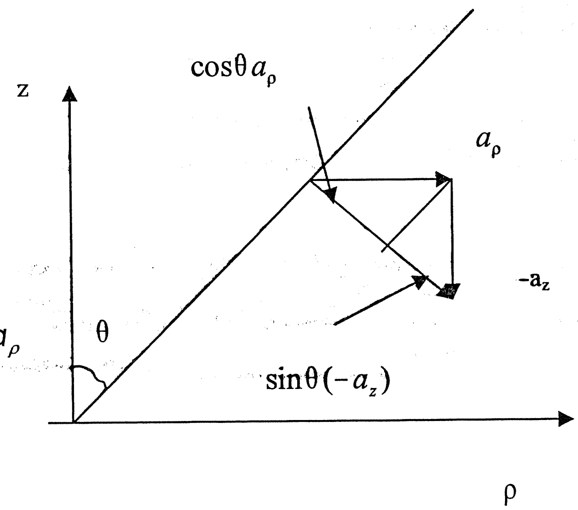
$$\rho = \sqrt{x^2 + y^2} = \sqrt{r^2 \sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}.$$

$$= r\sin\theta;$$

$$z = r\cos\theta; \qquad \phi = \phi.$$

(b) From the figures below,





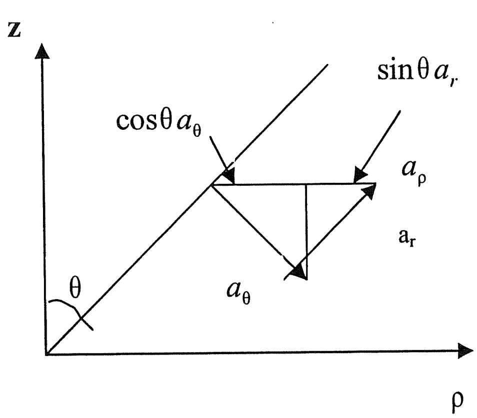
$\bar{a}_r = \sin\theta \bar{a}_\rho + \cos\theta \bar{a}_z$; $\bar{a}_\theta = \cos\theta \bar{a}_\rho - \sin\theta \bar{a}_z$; $\bar{a}_\phi = \bar{a}_\phi$.

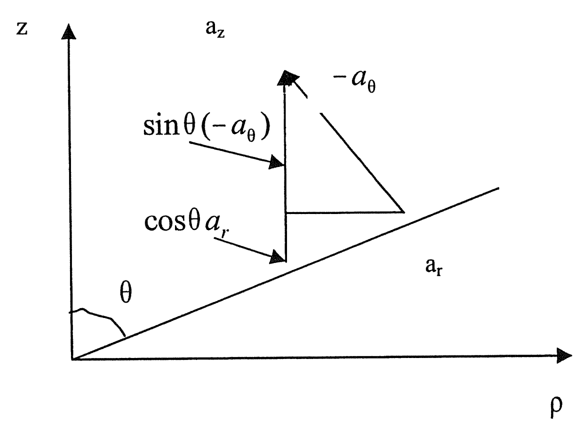
Hence,

$$\begin{bmatrix} \bar{a}_r \\ \bar{a}_\theta \\ \bar{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_\rho \\ \bar{a}_\phi \\ \bar{a}_z \end{bmatrix}$$

From the figures below,

$\bar{a}_\rho = \cos\theta \bar{a}_\theta + \sin\theta \bar{a}_r$; $\bar{a}_z = \cos\theta \bar{a}_r - \sin\theta \bar{a}_\theta$; $\bar{a}_\phi = \bar{a}_\phi$.





$$\begin{bmatrix} \bar{a}_\rho \\ \bar{a}_\phi \\ \bar{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_r \\ \bar{a}_\theta \\ \bar{a}_z \end{bmatrix}$$

Prob 2.6

(a) $d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$

(b)
$$d^2 = 3^2 + 5^2 - 2(3)(5)\cos\pi + (-1-5)^2 = 100$$
$$d = \sqrt{100} = \underline{\underline{10}}$$

(c)
$$d^2 = 10^2 + 5^2 - 2(10)(5)\cos\frac{\pi}{4}\cos\frac{\pi}{6} - 2(10)(5)\sin\frac{\pi}{4}\sin\frac{\pi}{6}\cos(7\frac{\pi}{4} - \frac{3\pi}{4})$$
$$= 125 - 100(\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}) = 125 - 100\cos 75^\circ = 99.12$$
$$d = \sqrt{99.12} = \underline{\underline{9.956.}}$$

Prob. 2.7

(a) $\mathbf{A} \bullet \mathbf{B} = (5,2,-1) \bullet (1,-3,4) = \underline{\underline{-5}}$

(b) $\mathbf{A}\text{x}\mathbf{B} = \begin{vmatrix} 5 & 2 & -1 \\ 1 & -3 & 4 \end{vmatrix} = \underline{\underline{5\mathbf{a}_\rho - 21\mathbf{a}_\phi - 17\mathbf{a}_z}}$

(c) $\cos\theta_{AB} = \frac{\mathbf{A} \bullet \mathbf{B}}{AB} = \frac{-5}{\sqrt{25+4+1}\sqrt{1+9+16}} = -0.179 \longrightarrow \theta_{AB} = \underline{\underline{100.31^\circ}}$

(d)
$$\mathbf{a}_n = \frac{\mathbf{A}\text{x}\mathbf{B}}{|\mathbf{A}\text{x}\mathbf{B}|} = \frac{(5,-21,-17)}{\sqrt{5^2 + 21^2 + 17^2}} = \frac{(5,-21,-17)}{\sqrt{755}}$$
$$= \underline{\underline{0.182\mathbf{a}_\rho - 0.7643\mathbf{a}_\phi - 0.6187\mathbf{a}_z}}$$

(e) $\mathbf{A}_B = (\mathbf{A} \bullet \mathbf{a}_B)\mathbf{a}_B = \frac{(\mathbf{A} \bullet \mathbf{B})\mathbf{B}}{B^2} = \frac{-5\mathbf{B}}{26} = \underline{\underline{-0.1923\mathbf{a}_\rho + 0.5769\mathbf{a}_\phi - 0.7692\mathbf{a}_z}}$

Prob.2.8

At $P(0,2,-5)$, $\phi = 90°$;

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

$$\bar{B} = -\bar{a}_x - 5\bar{a}_y - 3\bar{a}_z$$

(a) $\bar{A} + \bar{B} = (2,4,10) + (-1,-5,-3)$

$$= \underline{\underline{\bar{a}_x - \bar{a}_y + 7\bar{a}_z.}}$$

(b) $\cos\theta_{AB} = \dfrac{\bar{A} \bullet \bar{B}}{AB} = \dfrac{-52}{\sqrt{4200}}$

$$\theta_{AB} = \cos^{-1}\left(\frac{-52}{\sqrt{4200}}\right) = \underline{\underline{143.36°.}}$$

(c) $A_B = \bar{A} \bullet \bar{a}_B = \dfrac{\bar{A} \bullet \bar{B}}{B} = -\dfrac{52}{\sqrt{35}} = \underline{\underline{-8.789.}}$

Prob. 2.9

(a) At T, $x = -3, y = 4, z = 1, \rho = 5, \cos\phi = -\dfrac{3}{5}$

$$\bar{A} = 0\bar{a}_\rho - 5(1)\left(-\frac{3}{5}\right)\bar{a}_\phi + 25(1)\bar{a}_z$$

$$= \underline{\underline{3\bar{a}_\phi + 25\bar{a}_z}}$$

$r = \sqrt{26}$, $\quad \sin\theta = \dfrac{5}{\sqrt{26}}$, $\quad \cos\theta = \dfrac{1}{\sqrt{26}}$

$$\bar{B} = 26\left(\frac{-3}{5}\right)\bar{a}_r + 2(\sqrt{26})\frac{5}{\sqrt{26}}\bar{a}_\phi$$

$$= \underline{\underline{-15.6\bar{a}_r + 10\bar{a}_\phi}}$$

(b) In cylindrical coordinates,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} -15.6 \\ 0 \\ 10 \end{bmatrix}$$

$$B_\rho = -15.6\sin\theta = -15.6\left(\frac{5}{\sqrt{26}}\right) = -15.3$$

$$B_\phi = 10, \quad B_z = -15.6\cos\theta = -3.059$$

$$\bar{B}(\rho,\phi,z) = (-15.3, 10, -3.059)$$

$$\bar{A}_B = (\bar{A}\bullet\bar{a}_B)\bar{a}_B = (\bar{A}\bullet\bar{B})\bar{B}\frac{1}{\left|\bar{B}\right|^2} = \frac{(30-76.485)(-15.3,10,-3.059)}{343.45}$$

$$= \underline{\underline{2.071\bar{a}_\rho - 1.354\bar{a}_\phi + 0.4141\bar{a}_z}}.$$

(c) In spherical coordinates,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 25 \end{bmatrix}$$

$$A_r = 25\cos\theta = \frac{25}{\sqrt{26}} = 4.903$$

$$A_\theta = -25\sin\theta = -25\left(\frac{5}{\sqrt{26}}\right) = -24.51$$

$$A_\phi = 3$$

$$\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\phi \\ 4.903 & -24.51 & 3 \\ -15.6 & 0 & 10 \end{vmatrix} = -245.1\mathbf{a}_r - 95.83\mathbf{a}_\theta - 382.43\mathbf{a}_\phi$$

$$\bar{a}_{A\times B} = \frac{\pm\bar{A}\times\bar{B}}{464.23} = \underline{\underline{\pm(0.528\bar{a}_r - 0.2064\bar{a}_\theta + 0.8238\bar{a}_\phi}}.$$

Prob. 2.10

(*a*) $\bar{J}_z = (\bar{J} \bullet \bar{a}_z)\bar{a}_z$.

$At(2,\ \pi/2,\ 3\pi/2),\ \bar{a}_z = \cos\theta\,\bar{a}_r - \sin\theta\,\bar{a}_\theta = -\bar{a}_\theta$.

$\bar{J}_z = -\cos 2\theta \sin\phi\,\bar{a}_\theta = -\cos\pi \sin(3\pi/2)\,\bar{a}_\theta = -\bar{a}_\theta$.

(*b*) $\bar{J}_\phi = \tan\frac{\theta}{2}\ln r\,\bar{a}_\phi = \tan\frac{\pi}{4}\ln 2\,\bar{a}_\phi = \ln 2\,\bar{a}_\phi = 0.6931\,\bar{a}_\phi$.

(*c*) $\bar{J}_t = \bar{J} - \bar{J}_n = \bar{J} - \bar{J}_r = -\bar{a}_\theta + \ln 2\,\bar{a}_\phi = \underline{\underline{-\bar{a}_\theta + 0.6931\,\bar{a}_\phi}}$.

(*d*) $\bar{J}_P = (\bar{J} \bullet \bar{a}_x)\bar{a}_x$

$\bar{a}_x = \sin\theta\cos\phi\,\bar{a}_r + \cos\theta\cos\phi\,\bar{a}_\theta - \sin\phi\,\bar{a}_\phi = \bar{a}_\phi$.

At $(2,\ \pi/2,\ 3\pi/2)$,

$\bar{J}_P = \underline{\underline{\ln 2\,\bar{a}_\phi}}$.

CHAPTER 3

Prob. 3.1

(a)

$$dl = \rho d\phi; \qquad \rho = 3$$

$$L = \int dl = 3\int_{\pi/4}^{\pi/2} d\phi = 3(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{3\pi}{4} = \underline{\underline{2.356}}$$

(b)

$$dl = r\sin\theta d\phi; \qquad r = 1, \quad \theta = 30^\circ;$$

$$L = \int dl = r\sin\theta \int_0^{\pi/3} d\phi = (1)\sin 30^\circ[(\frac{\pi}{3}) - 0] = \underline{\underline{0.5236.}}$$

(c)

$$dl = rd\theta$$

$$L = \int dl = r\int_{\pi/6}^{\pi/2} d\theta = 4(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{4\pi}{3} = \underline{\underline{4.189}}$$

Prob.3.2

$$\int \bar{H} \bullet d\bar{l} = \int_{x=1}^{0} (x-y)dx \Big|_{y=0,\, z=0} + \int_{z=0}^{1} 5yzdz \Big|_{x=0,\, y=0}$$

$$+ \int (x^2 + zy)dy + 5yzdz \Big|_{x=0,\, z=1-y/2}$$

$$= \int_1^0 xdx + \int_0^2 (y - \frac{y^2}{2})dy + \int_1^0 (10z - 10z^2)dz$$

$$= \underline{\underline{-1.5}}$$

Prob. 3.3

(a) $\nabla \bullet A = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(-y) = 2y$

$$\int \nabla \bullet Adv = \iiint 2ydxdydz = 2\int_0^2 dx \int_0^2 ydy \int_0^2 dz$$

$$= 2(2)\frac{y^2}{2}\Big|_0^2 (2) = 4(4-0) = \underline{\underline{16}}$$

(b) $\nabla \bullet A = 2y = 2\rho \sin\phi$

$$\int \nabla \bullet A dv = \iiint 2\rho \sin\phi \rho d\rho d\phi dz = 2\int_0^3 \rho^2 d\rho \int_0^{2\pi} \sin\phi d\phi \int_0^5 dz = \underline{\underline{0}}$$

since integrating sin ϕ over $0 < \phi < 2\pi$ is 0.

(c) $\nabla \bullet A = 2y = 2r\sin\theta\sin\phi$

$$\int \nabla \bullet A dv = \iiint 2r\sin\theta\sin\phi r^2 \sin\theta d\theta d\phi dr = 2\int_0^4 r^3 dr \int_0^{2\pi} \sin\phi d\phi \int_0^{\pi} \sin^2 d\theta = \underline{\underline{0}}$$

Prob. 3.4

(a)

$$\nabla U = \frac{\partial U}{\partial x}\bar{a}_x + \frac{\partial U}{\partial y}\bar{a}_y + \frac{\partial U}{\partial z}\bar{a}_z$$

$$= \underline{\underline{4z^2 \bar{a}_x + 3z\bar{a}_y + (8xz + 3y)\bar{a}_z}}$$

(b)

$$\nabla W = \frac{\partial W}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial W}{\partial \phi}\bar{a}_\phi + \frac{\partial W}{\partial z}\bar{a}_z$$

$$= \underline{\underline{2(z^2+1)\cos\phi\,\bar{a}_\rho - 2(z^2+1)\sin\phi\,\bar{a}_\phi + 4\rho z\cos\phi\,\bar{a}_z}}$$

(c)

$$\nabla H = \frac{\partial H}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial H}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial H}{\partial \phi}\bar{a}_\phi$$

$$= \underline{\underline{2r\cos\theta\cos\phi\,\bar{a}_r - r\sin\theta\cos\phi\,\bar{a}_\theta - r\cot\theta\sin\phi\,\bar{a}_\phi}}$$

Prob. 3.5

We convert A to cylindrical coordinates; only the ρ-component is needed.

$$A_\rho = A_x\cos\phi + A_y\sin\phi = 2x\cos\phi - z^2\sin\phi$$

But $x = \rho\cos\phi$,

$$A_\rho = 2\rho\cos^2\phi - z^2\sin\phi$$

$$\Psi = \int_S A \cdot dS = \iint A_\rho \rho d\phi dz = \iint \left[2\rho^2\cos^2\phi - \rho z^2\sin\phi\right] d\phi dz$$

$$= 2(2)^2 \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\phi)d\phi \int_0^1 dz - 2\int_0^1 z^2 dz \int_0^{\pi/2} \sin\phi d\phi$$

$$= 4(\phi + \frac{1}{2}\sin 2\phi)\Big|_0^{\pi/2} - 2\frac{z^3}{3}\Big|_0^1 (-\cos\phi)\Big|_0^{\pi/2} = 2\pi - 2/3 = \underline{\underline{5.6165}}$$

Prob 3.6

(a)

$$\oint \bar{A} \bullet d\bar{S} = \int_V \nabla \bullet \bar{A} dv, \quad \nabla \bullet \bar{A} = y + z + x$$

$$\oint \bar{A} \bullet d\bar{S} = \int_0^1\int_0^1\int_0^1 (x+y+z)dxdydz$$

$$= 3\int_0^1 xdy \int_0^1 dy \int_0^1 dz = 3(\frac{x^2}{2}\Big|_0^1)(1)(1)$$

$$= \underline{\underline{1.5}}$$

(b)

$\nabla \bullet \bar{A} = 0.$ Hence, $\oint \bar{A} \bullet d\bar{S} = \underline{\underline{0}}$

Prob. 3.7

(a) $$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & -xz \end{vmatrix} = z\mathbf{a}_y - x\mathbf{a}_z$$

$$\nabla \times \mathbf{B} = \left(\frac{1}{\rho}2\rho z 2\sin\phi\cos\phi - 0\right)\mathbf{a}_\rho + (2\rho z - 2z\sin^2\phi)\mathbf{a}_\phi + \frac{1}{\rho}\left(2\rho\sin^2\phi - 0\right)\mathbf{a}$$

(b) $$= 4z\sin\phi\cos\phi\mathbf{a}_\rho + 2(\rho z - z\sin^2\phi)\mathbf{a}_\phi + 2\sin^2\phi\mathbf{a}_z$$

$$= \underline{\underline{2z\sin 2\phi\cos\phi\mathbf{a}_\rho + 2z(\rho - \sin^2\phi)\mathbf{a}_\phi + 2\sin^2\phi\mathbf{a}_z}}$$

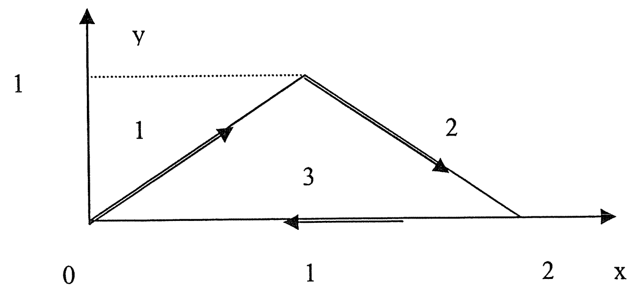
$$\nabla x \mathbf{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(r\cos^2\theta\sin\theta\right]\mathbf{a}_r - \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos^2\theta\right]\mathbf{a}_\theta$$

(c)

$$= \frac{r}{r\sin\theta}\left[(2\cos\theta)(-\sin\theta)\sin\theta + \cos\theta(\cos^2\theta)\right]\mathbf{a}_r - \frac{\cos^2\theta}{r}(2r)\mathbf{a}_\theta$$

$$= \frac{(\cos^3\theta - 2\sin^2\theta\cos\theta)}{\sin\theta}\mathbf{a}_r - 2\cos^2\theta\mathbf{a}_\theta$$

Prob. 3.8



(a)

$$\oint_L \bar{F}\bullet d\bar{l} = (\int_1 + \int_2 + \int_3)\bar{F}\bullet d\bar{l}$$

For 1, $y = x \quad dy = dx, d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y,$

$$\int_1 \bar{F}\bullet d\bar{l} = \int_0^1 x^3 dx - x dx = -\frac{1}{4}$$

For 2, $y = -x + 2, dy = -dx, d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y,$

$$\int_2 \bar{F}\bullet d\bar{l} = \int_1^2 (-x^3 + 2x^2 - x + 2)dx = \frac{17}{12}$$

For 3,

$$\int_3 \bar{F}\bullet d\bar{l} = \int_2^0 x^2 y dx\Big|_{y=0} = 0$$

$$\oint_L \bar{F}\bullet d\bar{l} = -\frac{1}{4} + \frac{17}{12} + 0 = \frac{7}{6}$$

(b)

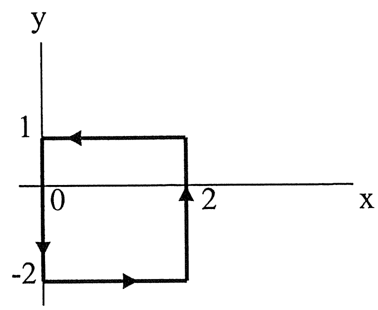
$$\nabla \times \bar{F} = -x^2 \bar{a}_z \ ; \qquad d\bar{S} = dxdy(-\bar{a}_z)$$

$$\int (\nabla \times \bar{F}) \bullet d\bar{S} = -\iint (-x^2) dxdy = \int_0^1 \int_0^x x^2 dydx + \int_1^2 \int_{y=0}^{-x+2} x^2 dydx$$

$$= \int_0^1 x^2 y \Big|_0^x dx + \int_1^2 x^2 y \Big|_0^{-x+2} dx = \frac{x}{4}\Big|_0^1 + \int_1^2 x^2(-x+2)dx = \underline{\underline{\frac{7}{6}}}$$

(c) Yes

Prob. 3.9



$$\oint_L \mathbf{F} \cdot dl = \int_0^2 3y^2 z dx \Big|_{y=-2, z=1} + \int_{-2}^1 6x^2 y dy \Big|_{x=2, z=1}$$

$$+ \int_2^0 3y^2 z dx \Big|_{y=1, z=1} + \int_1^{-2} 6x^2 y dy \Big|_{x=0, z=1}$$

$$= 3(4)(1)(2) + 6(4)\frac{y^2}{2}\Big|_{-2}^{1} + 3(1)(1)(-2) + 0 = \underline{\underline{-18}}$$

$$\nabla x\mathbf{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^2 z & 6x^2 y & 9xz^2 \end{vmatrix} = (12xy - 6yz)\boldsymbol{a}_z + \ldots$$

$$\int_S (\nabla x F) \cdot dS = \iint (12xy - 6yz) dxdy \Big|_{z=1} = 12\int_0^2 xdx \int_{-2}^1 ydy - 6\int_{-2}^1 ydy \int_0^2 dx$$

$$= 3x^2 \Big|_0^2 y^2 \Big|_{-2}^1 - 3y^2 \Big|_{-2}^1 (2) = 3(4)((1-4) - 6(1-4) = \underline{\underline{-18}}$$

Prob.3.10

$$\bar{Q} = \frac{r}{r\sin\theta} r\sin\theta[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y]$$

$$= r(\cos\phi - \sin\phi)\bar{a}_x + r(\cos\phi + \sin\phi)\bar{a}_y$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

$$\bar{Q} = r\sin\theta\bar{a}_r + r\cos\theta\bar{a}_\theta + r\bar{a}_\phi$$

(a)

$$d\bar{l} = \rho d\phi\bar{a}_\phi, \quad \rho = r\sin 30° = 2(\frac{1}{2}) = 1$$

$$z = r\cos 30° = \sqrt{3}$$

$$Q_\phi = r = \sqrt{\rho^2 + z^2}$$

$$\oint \bar{Q} \bullet d\bar{l} = \int_0^{2\pi} \sqrt{\rho^2 + z^2}\,\rho d\phi = 2(1)(2\pi) = \underline{\underline{4\pi}}$$

(b)

$$\nabla \times \bar{Q} = \cot\theta\,\bar{a}_r - 2\bar{a}_\theta + \cos\theta\,\bar{a}_\phi$$

For S_1, $d\bar{S} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$

$$\int_{S_1} (\nabla \times \bar{Q}) \bullet d\bar{S} = \int r^2 \sin\theta \cot\theta\, d\theta\, d\phi \Big|_{r=2}$$

$$= 4\int_0^{2\pi} d\phi \int_0^{30°} \cos\theta\, d\theta = \underline{\underline{4\pi}}$$

(c)

For S_2, $d\bar{S} = r\sin\theta\, d\theta\, dr\, \bar{a}_\theta$

$$\int_{S_2} (\nabla \times \bar{Q}) \bullet d\bar{S} = -2\int r\sin\theta\, d\phi\, dr \Big|_{\theta=30°}$$

$$= -2\sin 30 \int_0^2 r dr \int_0^{2\pi} d\phi$$

$$= \underline{\underline{-4\pi}}$$

(d)

For S_1, $d\bar{S} = r^2 \sin\theta \, d\phi \, d\theta \, \bar{a}_r$

$$\int_{S_1} \bar{Q} \bullet d\bar{S} = r^3 \int \sin^2\theta \, d\theta \, d\phi \Big|_{r=2}$$

$$= 8 \int_0^{2\pi} d\phi \int_0^{30^\circ} \sin^2\theta \, d\theta$$

$$= 4\pi\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right] = \underline{\underline{2.2767}}$$

(e)

For S_2, $d\bar{S} = r \sin\theta \, d\phi \, dr \, \bar{a}_\theta$

$$\int_{S_2} \bar{Q} \bullet d\bar{S} = \int r^2 \sin\theta \cos\theta \, d\phi \, dr \Big|_{\theta=30^\circ}$$

$$= \frac{4\pi\sqrt{3}}{3} = \underline{\underline{7.2552}}$$

(f)

$$\nabla \bullet \bar{Q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 \sin\theta) + \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + 0$$

$$= 2\sin\theta + \cos\theta\cot\theta$$

$$\int \nabla \bullet \bar{Q} \, dV = \int (2\sin\theta + \cos\theta\cot\theta) r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{r^3}{3}\Big|_0^2 (2\pi) \int_0^{30} (1 + \sin^2\theta) d\theta$$

$$= \frac{4\pi}{3}\left(\pi - \frac{\sqrt{3}}{2}\right) = \underline{\underline{9.532}}$$

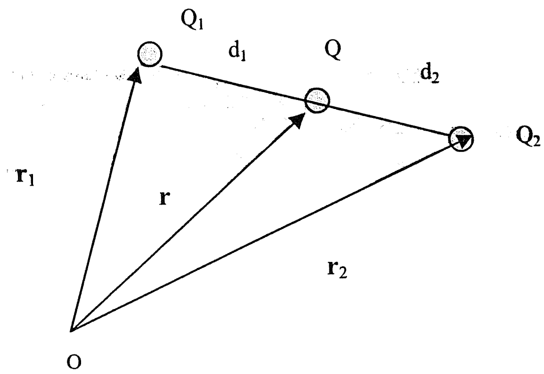
$$Check: \int \nabla \bullet \bar{Q} \, dV = \left(\int_{S_1} + \int_{S_2}\right) \bar{Q} \bullet d\bar{S}$$

$$= 4\pi\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}\right]$$

$$= \frac{4\pi}{3}\left[\pi - \frac{\sqrt{3}}{2}\right] \quad \text{(It checks!)}$$

CHAPTER 4

Prob 4.1



Let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}$ be the position vectors of Q_1, Q_2 and Q_3 respectively.
Given $Q_1 = Q$, $Q_2 = 3Q$, $d_1+d_2 = 2$.
Any positive charge would not keep equilibrium. There must be brought a negative charge Q3 and should be placed collinear with Q_1 and Q_2. For Q_3 to be in equilibrium,

$$\frac{Q_1 Q_3}{4\pi\varepsilon_o d_1^2} = \frac{Q_2 Q_3}{4\pi\varepsilon_o d_2^2}$$

$$\frac{d_1}{d_2} = \sqrt{\frac{Q_1}{Q_2}} = \frac{1}{\sqrt{3}}$$

Therefore, $\mathbf{r} = \dfrac{d_1\mathbf{r}_2 + d_2\mathbf{r}_1}{d_1 + d_2} = \dfrac{\sqrt{Q_1}\ \mathbf{r}_2 + \sqrt{Q_2}\ \mathbf{r}_1}{\sqrt{Q_1} + \sqrt{Q_2}} = \dfrac{\mathbf{r}_2 + \sqrt{3}\ \mathbf{r}_1}{1+\sqrt{3}}$

If Q2 is at equilibrium,

$$\frac{Q_1 Q_2}{4\pi\varepsilon_o (d_1 + d_2)^2} = \frac{Q_2 Q_3}{4\pi\varepsilon_o d_2^2}$$

This gives,

$$Q_3 = \frac{Q_1 d_2^2}{(d_1 + d_2)^2} = \frac{Q_1}{(\frac{d_1}{d_2} + 1)^2} = \frac{Q}{(\frac{1}{\sqrt{3}} + 1)^2} = \frac{3Q}{(1+\sqrt{3})^2}$$

The third *negative* charge must be $\dfrac{-3Q}{(1+\sqrt{3})^2}$ and must be located at $\dfrac{\mathbf{r}_2 + \sqrt{3}\ \mathbf{r}_1}{1+\sqrt{3}}$

Prob. 4.2

$$\boldsymbol{E}_R = \int \frac{\rho_L d\boldsymbol{l}\boldsymbol{R}}{4\pi\varepsilon_o R^3}, \qquad \boldsymbol{R} = -2\boldsymbol{a}_\rho, \qquad \rho_L = \frac{Q_L}{2\pi} = \frac{10}{2\pi} \text{ nC/m}$$

But $\boldsymbol{a}_\rho = \cos\phi \boldsymbol{a}_x + \sin\phi \boldsymbol{a}_y$

Due to symmetry, contributions along y add up, while contributions along x cancel out.

$$E_R = \frac{\rho_L}{4\pi\varepsilon_o R^3}\int_0^\pi (-2\sin\phi)\boldsymbol{a}_y 2d\phi = \frac{\rho_L(2)}{4\pi\varepsilon_o 2^3}(-1-1)2\boldsymbol{a}_y = -\frac{\rho_L}{4\pi\varepsilon_o}\boldsymbol{a}_y = -\frac{Q_L}{8\pi^2\varepsilon_o}\boldsymbol{a}_y, \qquad \rho_L = \frac{Q_L}{2\pi}$$

$$\boldsymbol{E}_Q = \frac{Q\boldsymbol{R}}{4\pi\varepsilon_o R^3} = \frac{Q[(0,0,0)-(0,-4,0)]}{4\pi\varepsilon_o |(0,0,0)-(0,-4,0)|^3} = \frac{Q\boldsymbol{a}_y}{64\pi\varepsilon_o}$$

$$E = E_R + E_Q = -\frac{Q_L}{8\pi^2\varepsilon_o} + \frac{Q}{64\pi\varepsilon_o} = 0 \longrightarrow Q = 8\frac{Q_L}{\pi}$$

$$Q = \frac{8x10nC}{\pi} = \underline{\underline{25.47 \text{ nC}}}$$

Prob. 4.3

(a)

Due to symmetry, $\bar{E}$ has only z – component given by

$$dE_z = dE\cos\alpha$$

$$= \frac{\rho_S dx\, dy}{4\pi\varepsilon_0(x^2+y^2+h^2)}\frac{h}{(x^2+y^2+h^2)^{1/2}}$$

$$E_z = \frac{\rho_S h}{4\pi\varepsilon_0}\int_{-a}^{a}\int_{-b}^{b}\frac{dxdy}{(x^2+y^2+h^2)^{3/2}}$$

$$= \frac{\rho_S h}{\pi\varepsilon_0}\int_0^a\int_0^b\frac{dxdy}{(x^2+y^2+h^2)^{3/2}}$$

$$= \frac{\rho_S h}{\pi\varepsilon_0}\int_0^a\frac{y\,dx}{(x^2+h^2)(x^2+y^2+h^2)^{1/2}}\Bigg|_0^b$$

$$= \frac{\rho_S h}{\pi\varepsilon_0}\int_0^a\frac{b\,dx}{(x^2+h^2)(x^2+b^2+h^2)^{1/2}}$$

By changing variables, we finally obtain

$$E_z = \frac{\rho_s}{\pi\,\varepsilon_0}\tan^{-1}\left\{\frac{ab}{h(a^2+b^2+h^2)^{1/2}}\right\}\bar{a}_z$$

(b) $Q = \int \rho_s dS = \rho_s (2a)(2b) = 10^{-5}(4)(10) = \underline{\underline{0.4 \text{ mC}}}$

$$\boldsymbol{E} = \frac{10^{-5}}{\pi \times \frac{10^{-9}}{36\pi}} \tan^{-1}\left[\frac{10}{10(4+25+100)^{1/2}}\right]\boldsymbol{a}_z = 36\times10^4(0.0878 \text{ radians})\boldsymbol{a}_z$$

$$= \underline{\underline{31.61\boldsymbol{a}_z \text{ kV/m}}}$$

Prob 4.4

Let Q_1 be located at the origin. At the spherical surface of radius r,

$$Q_1 = \oint \boldsymbol{D} \cdot d\boldsymbol{S} = \varepsilon E_r (4\pi r^2)$$

Or

$$\boldsymbol{E} = \frac{Q_1}{4\pi\varepsilon r^2}\boldsymbol{a}_r \quad \text{by Gauss's law}$$

If a second charge Q_2 is placed on the spherical surface, Q_2 experiences a force

$$\boldsymbol{F} = Q_2\boldsymbol{E} = \frac{Q_1 Q_2}{4\pi\varepsilon r^2}\boldsymbol{a}_r$$

which is Coulomb's law.

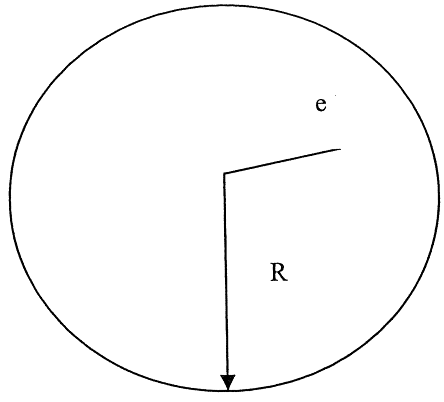
Prob. 4.5

(a) $\rho_v = \nabla \bullet \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \underline{\underline{2y \text{ C/m}^3}}$

(b) $\Psi = \int \boldsymbol{D} \cdot d\boldsymbol{S} = \iint x^2 dxdz \Big|_{y=1} = \int_0^1 x^2\, dx \int_0^1 dz = \underline{\underline{\frac{1}{3} \text{ C}}}$

(c) $Q = \int_v \rho_v dv = \iiint 2ydxdydz = 2\int_0^1 dx \int_0^1 ydy \int_0^1 dz = \underline{\underline{1 \text{ C}}}$

Prob. 4.6



$$F = eE$$

$$\rho_0 = \frac{e}{4\pi \frac{R^3}{3}} = \frac{3e}{4\pi R^3}$$

$$\rho_V = \begin{cases} \rho_0, & 0 < r < R \\ 0, & \text{elsewhere} \end{cases}$$

$$\oint \bar{D} \bullet d\bar{S} = Q_{enc} = \int \rho_V dV = \frac{3e}{4\pi R^3}\frac{4\pi r^3}{3} = D_r(4\pi r^2)$$

$$E_r = \frac{3e\,r}{12\pi\varepsilon_0 R^3}$$

$$F = eE = \underline{\underline{\frac{e^2\,r}{4\pi\,\varepsilon_0\,R^3}}}$$

Prob 4.7

(a)

$$\psi = Q_{enc} \quad \text{at } r = 2$$

$$Q_{enc} = \int \rho_V\,dV = \iiint \frac{10}{r^2} r^2 \sin\theta\,d\theta\,dr\,d\phi$$

$$= 10 \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta\,d\theta dr d\phi$$

$$= 10(2)\,(2\pi)\,(2) = (80\,\pi)\ \text{mC}$$

Thus, $\psi = \underline{\underline{251.3\ \text{mC}}}$

At $r = 6$;

$$Q_{enc.} = 10 \int_{r=0}^{4} dr \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta\,d\theta$$

$$= 10\,(4)(2\pi)\,(2) = 160\pi\ \text{mC}$$

$\psi = \underline{\underline{502.6\ \text{mC}}}$

(b)

$$\psi = Q_{enc}$$

But $\psi = \oint \bar{D} \bullet d\bar{S} = D_r \oint dS = D_r (4\pi r^2)$

At $r = 1$,

$$Q_{enc} = 10 \int_{r=0}^{1} dr \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$Q_{enc} = 10\,(1)(2\pi)\,(2) = 40\pi \text{ mC}$$

Thus,

$$D_r = \frac{Q_{enc}}{4\pi r^2} = \frac{40\pi}{4\pi(1)} = 10$$

$$\bar{D} = \underline{\underline{10\,\bar{a}_r \quad \text{mC/m}^2}}$$

At $r = 5$, $Q_{enc} = 160\,\pi$

$$D_r = \frac{Q_{enc}}{4\pi r^2} = \frac{160\pi}{4\pi(5)^2} = 1.6$$

$$= \underline{\underline{1.6\,\bar{a}_r \quad m\text{C/m}^2}}$$

Prob. 4.8

(a)

$$W_{AB} = q \int \bar{E} \bullet d\bar{l}, \qquad d\bar{l} = d\rho\, \bar{a}_\rho$$

$$\frac{-W_{AB}}{q} = \int (z+1)\sin\phi \, d\rho \Big|_{\phi=0, z=0} = 0$$

$$W_{AB} = 0$$

(b)

$$\frac{-W_{BC}}{q} = \int_{\phi=0}^{30} (z+1)\cos\phi\, \rho\, d\phi \Bigg|_{\rho=4, z=0} = 4\sin\phi \Big|_{0}^{30^\circ} = 2$$

$$W_{BC} = -2q = \underline{\underline{-8 \text{ nJ}}}$$

(c)

$$\frac{-W_{CD}}{q} = \int_{z=0}^{-2} \rho \sin\phi \, dz \Big|_{\substack{\phi=30^\circ \\ \rho=4}} = 4\sin 30^\circ \left(z \Big|_0^{-2}\right) = -4$$

$$W_{CD} = 4q = \underline{\underline{16 \text{ nJ}}}$$

(d)

$$W_{AD} = W_{AB} + W_{BC} + W_{CD} = 0 - 8 + 16 = \underline{\underline{8 \text{ nJ}}}$$

Prob. 4.9

(a)

$m\dfrac{d^2y}{dt^2} = eE$; divide by m, and integrate once, one obtains:

$$u = \frac{dy}{dt} = \frac{eEt}{m} + c_0$$

$$y = \frac{eEt^2}{2m} + c_0 t + c_1 \qquad (1)$$

"From rest" implies $c_1 = 0 = c_0$

At $t = t_0$, $y = d$, $E = \dfrac{V}{d}$ or $V = Ed$.

Substituting this in (1) yields:

$$t^2 = \frac{2md}{eE}$$

Hence:

$$u = \frac{eE}{m}\sqrt{\frac{2md}{eE}} = \sqrt{\frac{2eEd}{m}} = \sqrt{\frac{2eV}{m}}$$

that is, $u \ \alpha \ \sqrt{V}$

or $u = k\sqrt{V}$

(b)

$$k = \sqrt{\frac{2e}{m}} = \sqrt{\frac{2\,(1.603)\,10^{-19}}{9.1066\ (10^{-31})}}$$

$$= \underline{\underline{5.933 \times 10^5}}$$

(c)

$$V = \frac{u^2 m}{2e} = \frac{9(10^{16})\,\frac{1}{100}}{2(1.76)\,(10^{11})} = \underline{\underline{2.557 \ k\,\text{V}}}$$

Prob. 4.10

(a)

This is similar to Example 4.3.

$$u_y = \frac{eEt}{m}, \quad u_x = u_0$$

$$y = \frac{eEt^2}{2m}, \quad x = u_0 t$$

$$t = \frac{x}{u_0} = \frac{10(10^{-2})}{10^7} = 10 \text{ ns}$$

Since $x = 10$ cm when $y = 1$cm,

$$E = \frac{2my}{et^2} = \frac{2(10^{-2})}{1.76(10^{11})(10^{-16})} = 1.136 \text{ kV/m}$$

$$\underline{\underline{E = -\ 1.136\ \bar{a}_y \text{ kV/m}}}$$

(b)

$$u_x = u_0 = 10^7,$$

$$u_y = \frac{2000}{1.76}(1.76)10^{11}(10^{-8}) = 2(10^6)$$

$$\bar{u} = \underline{\underline{(\bar{a}_x + 0.2\bar{a}_y)(10^7) \text{ m/s}}}$$

CHAPTER 5

Prob. 5.1

$I = \int J \bullet dS, \quad dS = r\sin\theta d\phi dr a_\theta$

$$I = -\int_{r=0}^{2}\int_{\phi=0}^{2\pi} r^3 \sin^2\theta d\phi dr \,\Big|_{\theta=30^\circ} = -(\sin 30^\circ)^2 \left.\frac{r^4}{4}\right|_0^2 (2\pi) = -2\pi = \underline{\underline{-6.283}} \text{ A}$$

Prob. 5.2 (a) $R = \frac{l}{\sigma S} = \frac{8x10^{-2}}{3x10^4 x\pi x 25x10^{-6}} = \frac{8}{75\pi} = \underline{\underline{33.95 \text{m}\Omega}}$

(b) $I = V / R = 9x\frac{75\pi}{8} = \underline{\underline{265.1}}$ A

(c) P = IV = $\underline{\underline{2.386 \text{ kW}}}$

Prob. 5.3 (a) $S_i = \pi r_i^2 = \pi(1.5)^2 x10^{-4} = 7.068x10^{-4}$

$S_o = \pi(r_o^2 - r_i^2) = \pi(4 - 2.25)x10^{-4} = 5.498x10^{-4}$

$$R_I = \frac{\rho_I l}{S_I} = \frac{11.8\times10^{-8}\times10}{7.068X10^{-4}} = 16.69\times10^{-4}$$

$$R_o = \frac{\rho_o l}{S_o} = \frac{1.77\times10^{-8}\times10}{5.498X10^{-4}} = 3.219\times10^{-4}$$

$$R = R_i // R_o = \frac{R_i R_o}{R_i + R_o} = \frac{16.69\times3.219\times10^{-4}}{16.69+3.219} = \underline{\underline{0.27\text{m}\Omega}}$$

(b) $V = I_i R_i = I_o R_o \longrightarrow \frac{I_i}{I_o} = \frac{R_o}{R_i} = \frac{0.3219}{1.669} = 0.1929$

$I_i + I_o = 1.1929 I_o = 60$ A

$I_o = \underline{\underline{50.3 \text{ A}}}$ (copper), $I_i = \underline{\underline{9.7 \text{ A}}}$ (steel)

(c) $R = \frac{10x1.77x10^{-8}}{1.75\pi x10^{-4}} = \underline{\underline{0.322\text{m}\Omega}}$

Prob. 5.4

(a) Applying Coulomb's law,

$$E_r = \begin{cases} \dfrac{D_r}{\varepsilon_o} = \dfrac{Q}{4\pi\varepsilon_o r^2}, & r > b \\ \dfrac{D_r}{\varepsilon} = \dfrac{Q}{4\pi\varepsilon r^2}, & a < r < b \end{cases}$$

$$P = \frac{\varepsilon_r - 1}{\varepsilon_r} D \qquad (= D - \varepsilon_o E)$$

Hence

$$P_r = \frac{\varepsilon_r - 1}{\varepsilon_r} \cdot \frac{Q}{4\pi r^2}, \qquad a < r < b$$

(b) $$\rho_{pv} = -\nabla \bullet \mathbf{P} = -\frac{1}{r^2}\frac{d}{dr}(r^2 P_r) = 0$$

(c)

$$\rho_{ps} = \mathbf{P} \bullet (-\mathbf{a}_r) = -\frac{Q}{4\pi a^2}\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right), \qquad r = a$$

$$\rho_{ps} = \mathbf{P} \bullet (\mathbf{a}_r) = -\frac{Q}{4\pi b^2}\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right), \qquad r = b$$

Prob. 5.5

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_o E_o \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\boldsymbol{D}_x = \varepsilon_o E_o (4\boldsymbol{a}_x + \boldsymbol{a}_y - \boldsymbol{a}_z)$$
$$\boldsymbol{D}_y = \varepsilon_o E_o (\boldsymbol{a}_x + 3\boldsymbol{a}_y - \boldsymbol{a}_z)$$
$$\boldsymbol{D}_z = \varepsilon_o E_o (\boldsymbol{a}_x + \boldsymbol{a}_y - 2\boldsymbol{a}_z)$$
$$\boldsymbol{D} = \varepsilon_o E_o (6\boldsymbol{a}_x + 5\boldsymbol{a}_y - 4\boldsymbol{a}_z) \text{ C/m}^2$$

Prob. 5.6

$$Q = Q_o e^{-t/T_r} \longrightarrow \frac{1}{3}Q_o = Q_o e^{-t_1/T_r} \longrightarrow e^{t_1/T_r} = 3$$

(a) But $T_r = \frac{\varepsilon_r \varepsilon_o}{\sigma}$, $\varepsilon_r = \frac{\sigma T_r}{\varepsilon_o} = \frac{10^{-4} \text{x} 18.2 \text{x} 10^{-6}}{\frac{10^{-9}}{36\pi}} = \underline{\underline{205.8}}$

(b) $T_r = \frac{t_1}{\ln 3} = \frac{20\mu s}{\ln 3} = \underline{\underline{18.2\mu s}}$

(c) $\frac{Q}{Q_o} = e^{-t_o/T_r} = e^{-30/18.3} = 0.1923$ i.e. $\underline{\underline{19.23\%}}$

Prob. 5.7

f(x,y)= 4x +3y –10=0

$$\nabla f = 4\mathbf{a}_x + 3\mathbf{a}_y \longrightarrow \mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{4\mathbf{a}_x + 3\mathbf{a}_y}{5} = 0.8\mathbf{a}_x + 0.6\mathbf{a}_y$$

$$\mathbf{D}_{1n} = (\mathbf{D}_1 \cdot \mathbf{a}_n)\mathbf{a}_n = (1.6 - 2.4)\mathbf{a}_n = -0.64\mathbf{a}_x - 0.48\mathbf{a}_y$$

$$\mathbf{D}_{1t} = \mathbf{D}_1 - \mathbf{D}_{1n} = 2.64\mathbf{a}_x - 3.52\mathbf{a}_y + 6.5\mathbf{a}_z$$

$$\mathbf{D}_{2n} = \mathbf{D}_{1n} = -0.64\mathbf{a}_x - 0.48\mathbf{a}_y$$

$$E_{2t} = E_{1t} \longrightarrow \frac{D_{2t}}{\varepsilon_1} = \frac{D_{1t}}{\varepsilon_2}$$

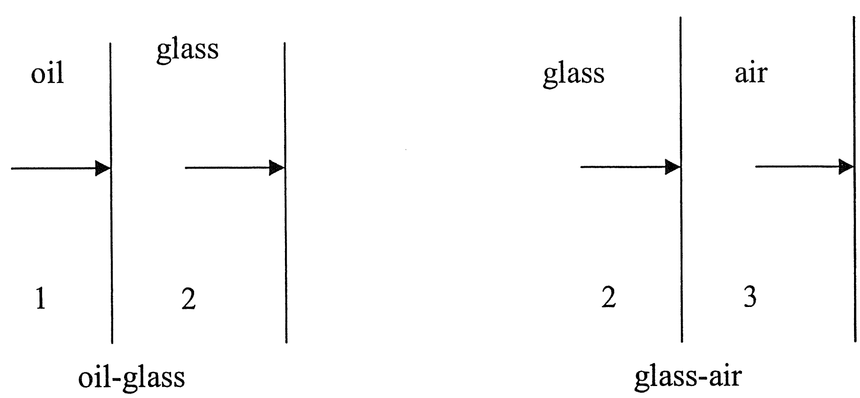
$$\mathbf{D}_{2t} = \frac{\varepsilon_1}{\varepsilon_2}\mathbf{D}_{1t} = \frac{1}{2.5}(2.64, -3.52, 6.5) = (1.056, -1.408, 2.6)$$

$$\mathbf{D}_2 = \mathbf{D}_{2n} + \mathbf{D}_{2t} = \underline{\underline{0.416\mathbf{a}_x - 1.888\mathbf{a}_y + 2.6\mathbf{a}_z \text{ nC/m}^2}}$$

$$\mathbf{D}_2 \bullet a_n = |\mathbf{D}_2| \cos\theta_2 \longrightarrow \cos\theta_2 = \frac{D_{2n}}{D} = \frac{0.8}{3.24} = 0.2469$$

$$\theta_2 = \underline{\underline{75.71^o}}$$

Prob. 5.8 (a) The two interfaces are shown below



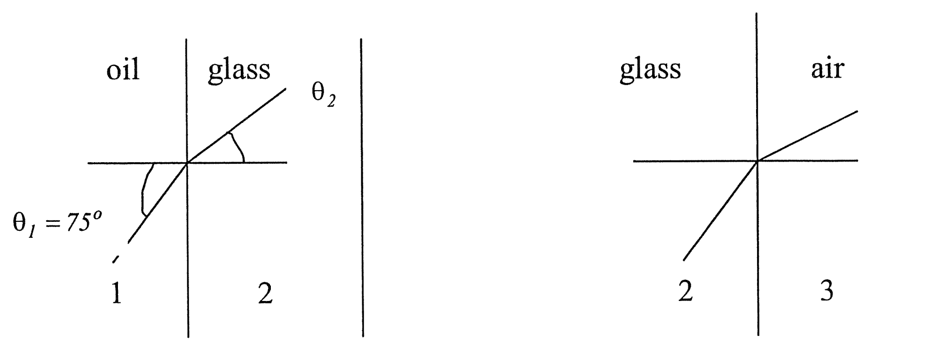
$$E_{1n} = 2000, \quad E_{1t} = 0 = E_{2t} = E_{3t}$$

$$D_{1n} = D_{2n} = D_{3n} \longrightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} = \varepsilon_3 E_{3n}$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{3.0}{8.5}(2000) = \underline{\underline{705.9 \text{ V/m}, \ \theta_2 = 0^o}}$$

$$E_{3n} = \frac{\varepsilon_1}{\varepsilon_3} E_{1n} = \frac{3.0}{1.0}(2000) = \underline{\underline{6000 \text{ V/m}, \ \theta_3 = 0^o}}$$

(b)



$$E_{1n} = 2000\cos 75^o = 517.63, \quad E_{1t} = 2000\sin 75^o = E_{2t} = E_{3t} = 1931.85$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{3}{8.5}(517.63) = 182.7, \quad E_{3n} = \frac{\varepsilon_1}{\varepsilon_3} E_{1n} = \frac{3}{1}(517.63) = 1552.89$$

$$E_2 = \sqrt{E_{2n}^2 + E_{2t}^2} = 1940.5, \quad \theta_2 = \tan^{-1}\frac{E_{2t}}{E_{2n}} = \underline{\underline{84.6^o}},$$

$$E_3 = \sqrt{E_{3n}^2 + E_{3t}^2} = 2478.6, \quad \theta_3 = \tan^{-1}\frac{E_{3t}}{E_{3n}} = \underline{\underline{51.2^o}}$$

Prob.5.9

$$\rho_s = D_n = \varepsilon_o E_n$$

$$Q = \rho_s 4\pi r^2 = \varepsilon_o E_n 4\pi r^2 = \frac{10^{-9}}{36\pi} x4x10^6 x4\pi x0.1^2 = \underline{\underline{4.444 \ \mu\text{C}}}$$

CHAPTER 6

Prob. 6.1

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon_o} = -\frac{10}{\rho}$$

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{dV}{d\rho}) = -\frac{10}{\rho}$$

$$\rho\frac{dV}{d\rho} = -10\rho + A \quad \longrightarrow \quad \frac{dV}{d\rho} = -10 + \frac{A}{\rho}$$

$$V = -10\rho + A\ln\rho + B$$

$$V(\rho = a) = V_o = -10a + A\ln a + B \qquad (1)$$

$$V(\rho = b) = 0 = -10b + A\ln b + B \qquad (2)$$

From (1) and (2),

$$A = \frac{10(b-a) - V_o}{\ln b/a}, \qquad B = 10b - A\ln b$$

a=4.5mm, b=2mm, V_o =40. Substituting these gives
A=49.3569, B=306.754

$$V = \underline{\underline{-10\rho + 49.36\ln\rho + 306.754 \text{ V}}}$$

Prob. 6.2 (a)

$$\nabla^2 V_1 = \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = -e^{-y}\sin x + e^{-y}\sin x = 0$$

(b) $$\nabla^2 V_2 = \frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} = -\left(\frac{\pi}{a}\right)^2 V_2 + \left(\frac{\pi}{a}\right)^2 V_2 = 0$$

(c) $$\nabla^2 V_3 = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot\frac{-\cos\phi}{\rho^2}) - \frac{1}{\rho^2}\frac{\cos\phi}{\rho} = \frac{\cos\phi}{\rho^3} - \frac{\cos\phi}{\rho^3} = 0$$

(d)

$$\nabla^2 V_4 = \frac{1}{r^2}\frac{\partial}{\partial r}(-20r^{-1}\cos\theta) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(-\sin\theta\cdot 10r^{-2}\sin\theta)$$

$$= \frac{20\cos\theta}{r^4} + \frac{10}{r^4\sin\theta}(-2\sin\theta\cos\theta) = 0$$

Prob. 6.3 From Example 6.8, solving $\nabla^2 V = 0$ when $V = V(\rho)$ leads to

$$V = \frac{V_o \ln \rho / a}{\ln b / a} = V_o \frac{\ln(a/\rho)}{\ln(a/b)}$$

$$\mathbf{E} = -\nabla V = -\frac{V_o}{\rho \ln b/a}\mathbf{a}_\rho = \frac{V_o}{\rho \ln a/b}\mathbf{a}_\rho, \qquad \mathbf{D} = \varepsilon \mathbf{E} = -\frac{\varepsilon_o \varepsilon_r V_o}{\rho \ln b/a}\mathbf{a}_\rho$$

$$\rho_s = D_n = \pm \frac{\varepsilon_o \varepsilon_r V_o}{\rho \ln b/a}\bigg|_{\rho=a,b}$$

In this case, V_o=100 V, b=5mm, a=15mm, $\varepsilon_r = 2$. Hence at $\rho = 10$mm,

$$V = \frac{100 \ln(10/15)}{\ln(5/15)} = \underline{\underline{36.91\ \text{V}}}$$

$$\mathbf{E} = \frac{100}{10 \text{x} 10^{-3} \ln 3}\mathbf{a}_\rho = \underline{\underline{9.102\mathbf{a}_\rho\ \text{kV/m}}}$$

$$\mathbf{D} = 9.102 \text{x} 10^3 \text{x} \frac{10^{-9}}{36\pi} 2\mathbf{a}_\rho = \underline{\underline{161\mathbf{a}_\rho\ \ \text{nC/m}^2}}$$

$$\rho_s(\rho = 5\text{mm}) = \frac{10^{-9}}{36\pi}(2)\frac{10^5}{5 \ln 3} = \underline{\underline{322\ \text{nC/m}^2}}$$

$$\rho_s(\rho = 15\text{mm}) = -\frac{10^{-9}}{36\pi}(2)\frac{10^5}{15 \ln 3} = \underline{\underline{-107.3\ \text{nC/m}^2}}$$

Prob. 6.4 (a)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0 \quad \longrightarrow \quad V = A \ln \rho + B$$

$$V(\rho = b) = 0 \quad \longrightarrow \quad 0 = A \ln b + B \quad \longrightarrow \quad B = -A \ln b$$

$$V(\rho = a) = V_o \quad \longrightarrow \quad V_o = A \ln a/b \quad \longrightarrow \quad A = -\frac{V_o}{\ln b/a}$$

$$V = -\frac{V_o}{\ln b/a} \ln \rho / b = \frac{V_o \ln b/\rho}{\ln b/a}$$

$$V(\rho = 15\text{mm}) = 70 \frac{\ln 2}{\ln 50} = \underline{\underline{12.4\ \text{V}}}$$

(b) As the electron decelerates, potential energy gained = K.E. loss

$$e[70-12.4]=\frac{1}{2}m[(10^7)^2-u^2] \longrightarrow 10^{14}-u^2=\frac{2e}{m}\times 57.6$$

$$u^2=10^{14}-\frac{2\times 1.6\times 10^{-19}}{9.1\times 10^{-31}}\times 57.6=10^{12}(100-20.25)$$

$$\underline{\underline{u=8.93\times 10^6 \text{ m/s}}}$$

Prob. 6.5 (a) As in Example 6.5, $X(x)=A\sin(n\pi x/b)$

For Y,

$$Y(y)=c_1\cosh(n\pi y/b)+c_2\sinh(n\pi y/b)$$

$$Y(a)=0 \longrightarrow 0=c_1\cosh(n\pi a/b)+c_2\sinh(n\pi a/b) \longrightarrow c_1=-c_2\tanh(n\pi a/b)$$

$$V=\sum_{n=1}^{\infty} a_n\sin(n\pi x/b)\left[\sinh(n\pi y/b)-\tanh(n\pi a/b)\cosh(n\pi y/b)\right]$$

$$V(x,y=0)=V_o=-\sum_{n=1}^{\infty} a_n\tanh(n\pi a/b)\sin(n\pi x/b)$$

$$-a_n\tanh(n\pi a/b)=\frac{2}{b}\int_0^b V_o\sin(n\pi x/b)dx=\begin{cases}\dfrac{4V_o}{n\pi}, & n=\text{odd}\\ 0, & n=\text{even}\end{cases}$$

Hence,

$$V=-\frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\sin(n\pi x/b)\left[\frac{\sinh(n\pi y/b)}{n\tanh(n\pi a/b)}-\frac{\cosh(n\pi y/b)}{n}\right]$$

$$=-\frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\frac{\sin(n\pi x/b)}{n\sinh(n\pi a/b)}\left[\sinh(n\pi y/b)\cosh(n\pi a/b)-\cosh(n\pi y/b)\sinh(n\pi a/b)\right]$$

$$\underline{\underline{=\frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\frac{\sin(n\pi x/b)\sinh[n\pi(a-y)/b]}{n\sinh(n\pi a/b)}}}$$

Alternatively, for Y

$$Y(y) = c_1 \sinh n\pi(y - c_2)/b$$

$$Y(a) = 0 \longrightarrow 0 = c_1 \sinh[n\pi(a - c_2)/b] \longrightarrow c_2 = a$$

$$V = \sum_{n=1}^{\infty} b_n \sin(n\pi x/b) \sinh[n\pi(y-a)/b]$$

where

$$b_n = \begin{cases} -\dfrac{4V_o}{n\pi \sinh(n\pi a/b)}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

(b) This is the same as Example 6.5 except that we exchange y and x. Hence

$$V(x,y) = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{\sin(n\pi y/a)\sinh(n\pi x/a)]}{n\sinh(n\pi b/a)}$$

(c) This is the same as part (a) except that we must exchange x and y. Hence

$$V(x,y) = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{\sin(n\pi y/a)\sinh[n\pi(b-x)/a]}{n\sinh(n\pi b/a)}$$

Prob. 6.6 (a) X(x) is the same as in Example 6.5. Hence

$$V(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x/b)\left[a_n \sinh(n\pi y/b) + b_n \cosh(n\pi y/b)\right]$$

At y=0, $V = V_1$

$$V_1 = \sum_{n=1}^{\infty} b_n \sin(n\pi x/b) \longrightarrow b_n = \begin{cases} \dfrac{4V_1}{n\pi}, & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

At y=a, $V = V_2$

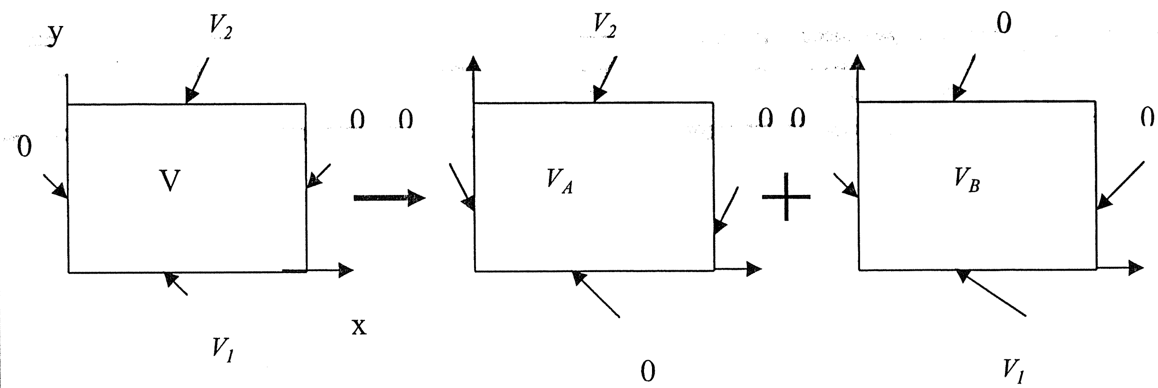
$$V_2 = \sum_{n=1}^{\infty} \sin(n\pi x/b)\left[a_n \sinh(n\pi a/b) + b_n \cosh(n\pi a/b)\right]$$

$$a_n \sinh(n\pi a/b) + b_n \cosh(n\pi a/b) = \begin{cases} \dfrac{4V_2}{n\pi}, & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

or

$$a_n = \begin{cases} \dfrac{4V_2}{n\pi \sinh(n\pi a/b)}\left(V_2 - V_1 \cosh(n\pi a/b)\right), & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

Alternatively, we may apply superposition principle.



i.e. $V = V_A + V_B$

V_A is exactly the same as Example 6.5 with $V_o = V_2$, while V_B is exactly the same as Prob. 6.19(a). Hence

$$V = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x/b)}{n\sinh(n\pi a/b)}\left[V_1 \sinh[n\pi(a-y)/b] + V_2 \sinh(n\pi y/b)\right]$$

(b)

$$V(x,y) = (a_1 e^{-\alpha x} + a_2 e^{+\alpha x})(a_3 \sin\alpha y + a_4 \cos\alpha y)$$

$$\lim_{x \longrightarrow \infty} V(x,y) = 0 \quad \longrightarrow \quad a_2 = 0$$

$$V(x, y=0) = 0 \quad \longrightarrow \quad a_4 = 0$$

$$V(x, y=a) = 0 \quad \longrightarrow \quad \alpha = n\pi/a, \quad n = 1,2,3,\ldots$$

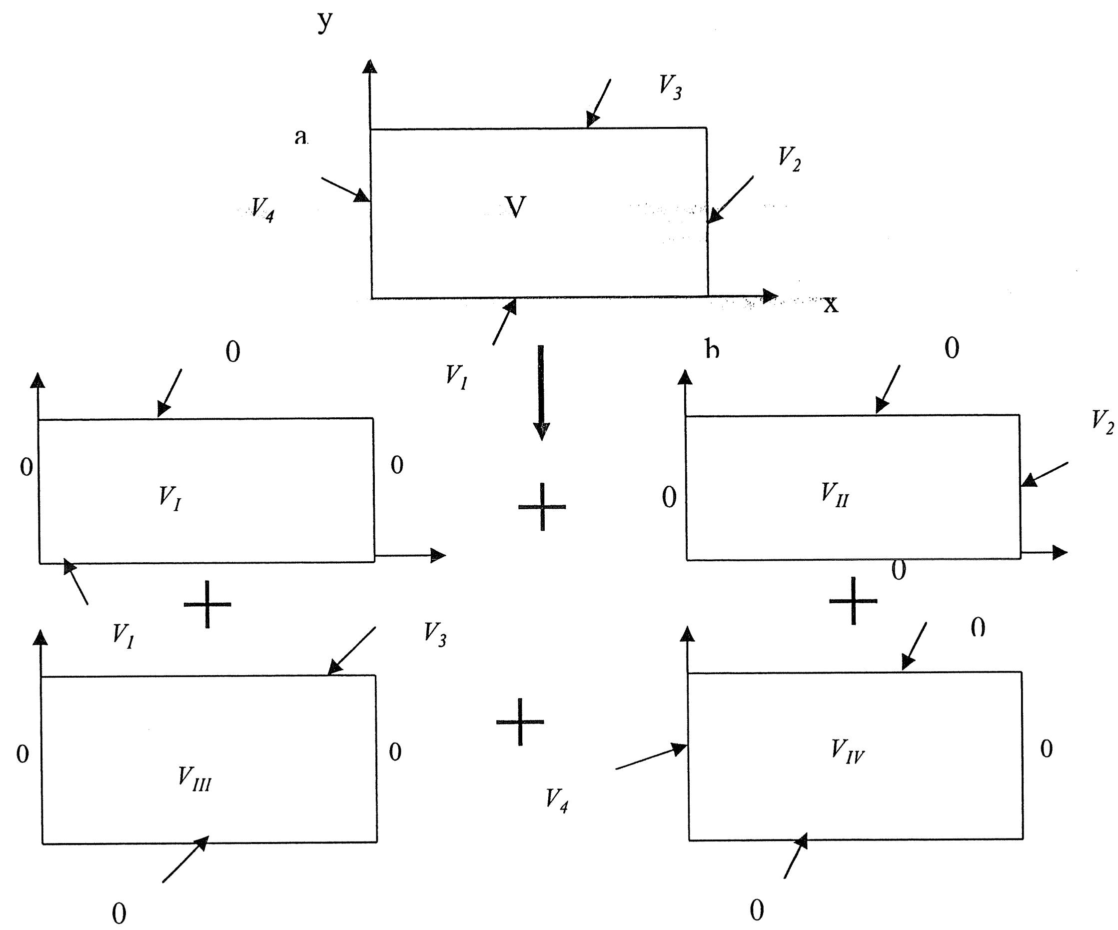
Hence,

$$V(x,y) = \sum_{n=1}^{\infty} a_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$V(x=0,y) = V_o = \sum_{n=1}^{\infty} a_n \sin(n\pi y/a) \quad \longrightarrow \quad a_n = \begin{cases} \dfrac{4V_o}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$V(x,y) = \frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi y/a)}{n}\exp(-n\pi x/a)$$

(c) The problem is easily solved using superposition theorem, as illustrated below.



Therefore,

$$V = V_I + V_{II} + V_{III} + V_{IV}$$

$$= \frac{4}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n} \left\{ \begin{array}{l} \dfrac{\sin(n\pi x/b)}{\sinh(n\pi a/b)} \left[V_1 \sinh(n\pi(a-y)/b) + V_3 \sinh(n\pi y/b) \right] \\ + \dfrac{\sin(n\pi x/a)}{\sinh(n\pi b/a)} \left[V_2 \sinh(n\pi y/a) + V_4 \sinh(n\pi(b-x)/a) \right] \end{array} \right\}$$

where

$$V_I = \frac{4V_I}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x/b)\sinh[n\pi(a-y)/b]}{n\sinh(n\pi a/b)}$$

$$V_{II} = \frac{4V_2}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x / a)\sinh(n\pi y / a)}{n\sinh(n\pi b / a)}$$

$$V_{III} = \frac{4V_3}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x / b)\sinh(n\pi y / b)}{n\sinh(n\pi a / b)}$$

$$V_{IV} = \frac{4V_4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi y / a)\sinh[n\pi(b-x) / a]}{n\sinh(n\pi b / a)}$$

Prob. 6.7 If $V(r=a)=0$, $V(r=b)=V_o$, from Example 6.9,

$$E = \frac{V_o}{r^2(1/a - 1/b)}, \qquad J = \sigma E$$

Hence,

$$I = \int J \bullet dS = \frac{V_o\sigma}{1/a - 1/b} \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} \frac{1}{r^2} r^2 \sin\theta d\theta d\phi = \frac{2\pi V_o \sigma}{1/a - 1/b}(-\cos\theta)\Big|_0^{\alpha}$$

$$R = \frac{V_o}{I} = \frac{\frac{1}{a} - \frac{1}{b}}{2\pi\sigma(1-\cos\alpha)}$$

Prob. 6.8 For a spherical capacitor, from Eq. (6.38),

$$R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\sigma}$$

For the hemisphere, $R' = 2R$ since the sphere consists of two hemispheres in parallel. As $b \longrightarrow \infty$,

$$R' = \lim_{b \longrightarrow \infty} \frac{2\left[\frac{1}{a} - \frac{1}{b}\right]}{4\pi\sigma} = \frac{1}{2\pi a\sigma}$$

$$G = 1/R' = 2\pi a\sigma$$

Alternatively, for an isolated sphere, $C = 4\pi\varepsilon a.$ But

$$RC = \frac{\varepsilon}{\sigma} \longrightarrow R = \frac{1}{4\pi a\sigma}$$

$$R' = 2R = \frac{1}{2\pi a\sigma} \quad \text{or} \quad G = 2\pi a\sigma$$

Prob. 6.9

$$F dx = dW_E \longrightarrow F = \frac{dW_E}{dx}$$

$$W_E = \int \frac{1}{2}\varepsilon |E|^2\, dv = \frac{1}{2}\varepsilon_o \varepsilon_r E^2 xad + \frac{1}{2}\varepsilon_o E^2 da(1-x)$$

where $E = V_o / d$.

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^2}{d^2}(\varepsilon_r - 1)da \longrightarrow F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^2 a}{2d}$$

Alternatively, $W_E = \frac{1}{2}CV_o^2$, where

$$C = C_1 + C_2 = \frac{\varepsilon_o \varepsilon_r ax}{d} + \frac{\varepsilon_o \varepsilon_r (1-x)}{d}$$

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^2 a}{d}(\varepsilon_r - 1)$$

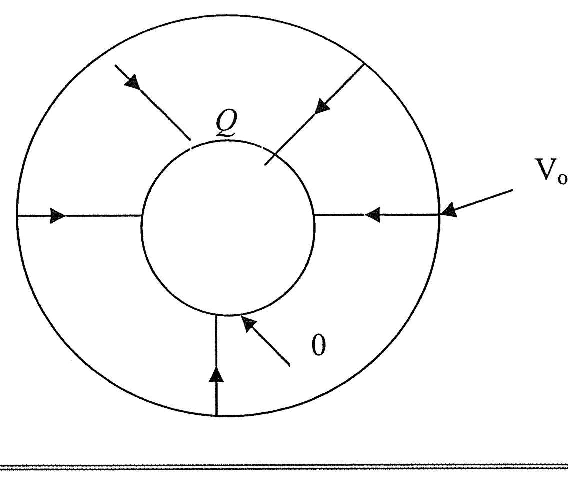
$$\underline{\underline{F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^2 a}{2d}}}$$

Prob. 6.10

$$C = \frac{2\pi\varepsilon_o L}{\ln(b/a)} = \frac{2\pi \times \frac{10^{-9}}{36\pi} \times 100 \times 10^{-6}}{\ln(600/20)} = 1.633 \times 10^{-15} \text{ F}$$

$$V = Q/C = \frac{50 \times 10^{-15}}{1.633 \times 10^{-15}} = \underline{\underline{30.62 \text{ V}}}$$

Prob. 6.11 $\quad E = \dfrac{Q}{4\pi\varepsilon r^2}\mathbf{a}_r$



$$W = \frac{1}{2}\int \varepsilon |E|^2 dv = \iiint \frac{Q^2}{32\pi^2\varepsilon^2 r^4}\varepsilon r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{Q^2}{32\pi^2\varepsilon}(2\pi)(2)\int_c^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$W = \frac{Q^2(b-c)}{8\pi\varepsilon bc}$$

Prob. 6.12 (a) Method 1: $\boldsymbol{E} = \frac{\rho_s}{\varepsilon}(-\boldsymbol{a}_x)$, where ρ_s is to be determined.

$$V_o = -\int \mathbf{E} \bullet dl = -\int \frac{-\rho_s}{\varepsilon}dx = \rho_s \int_0^d \frac{1}{\varepsilon_o}\frac{d}{d+x}dx = \frac{\rho_s}{\varepsilon}d\ln(x+d)\Big|_0^d$$

$$V_o = \rho_s d \ln\frac{2d}{d} \longrightarrow \rho_s = \frac{V_o\varepsilon_o}{d\ln 2}$$

$$\mathbf{E} = -\frac{\rho_s}{\varepsilon}\mathbf{a}_x = -\underline{\underline{\frac{V_o}{(x+d)\ln 2}}}\mathbf{a}_x$$

Method 2: We solve Laplace's equation

$$\nabla \bullet (\varepsilon \nabla V) = \frac{d}{dx}\left(\varepsilon\frac{dV}{dx}\right) = 0 \longrightarrow \varepsilon\frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\varepsilon} = \frac{Ad}{\varepsilon_o(x+d)} = \frac{c_1}{x+d}$$

$$V = c_1\ln(x+d) + c_2$$

$$V(x=0) = 0 \longrightarrow 0 = c_1\ln d + c_2 \longrightarrow c_2 = -c_1\ln d$$

$$V(x=d) = V_o \longrightarrow V_o = c_1\ln 2d - c_1\ln d = c_1\ln 2$$

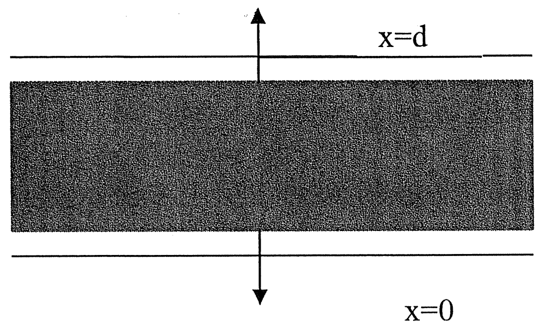
$$c_1 = \frac{V_o}{\ln 2}$$

$$V = c_1\ln\frac{x+d}{d} = \underline{\underline{\frac{V_o}{\ln 2}\ln\frac{x+d}{d}}}$$

$$\mathbf{E} = -\frac{dV}{dx}\mathbf{a}_x = -\frac{V_o}{(x+d)\ln 2}\mathbf{a}_x$$

(b) $$\mathbf{P} = (\varepsilon_r - 1)\varepsilon_o \mathbf{E} = -\left(\frac{x+d}{d} - 1\right)\frac{\varepsilon_o V_o}{(x+d)\ln 2}\mathbf{a}_x = -\frac{\varepsilon_o x V_o}{d(x+d)\ln 2}\mathbf{a}_x$$

(c)



$$\rho_{ps}\,|_{x=0} = \mathbf{P} \bullet (-\mathbf{a}_x)\,|_{x=0} = 0$$

$$\rho_{ps}\,|_{x=d} = \mathbf{P} \bullet \mathbf{a}_x\,|_{x=d} = -\frac{\varepsilon_o V_o}{2d\ln 2}$$

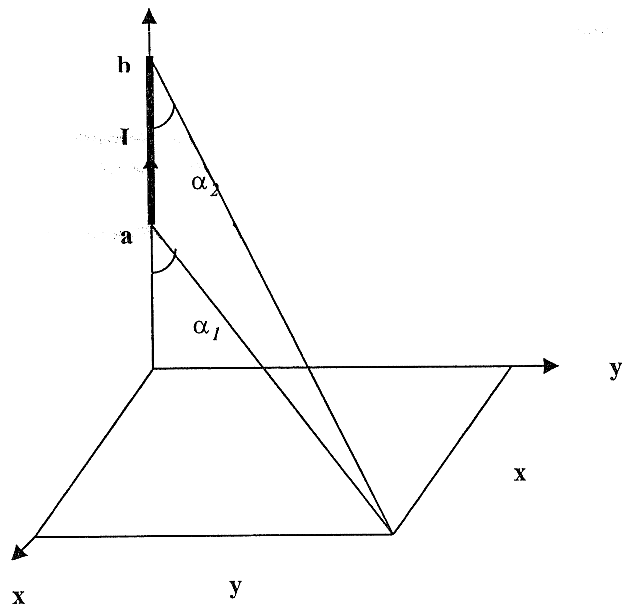
Prob. 6.13 We have 7 images as follows: -Q at (-1,1,1), -Q at (1,-1,1), -Q at (1,1,-1), -Q at (-1,-1,-1), Q at (1,-1,-1), Q at (-1,-1,1), and Q at (-1,1,-1). Hence,

$$\mathbf{F} = \frac{Q^2}{4\pi\varepsilon_o}\left[\begin{array}{l} -\frac{2}{2^3}\mathbf{a}_x - \frac{2}{2^3}\mathbf{a}_y - \frac{2}{2^3}\mathbf{a}_z - \frac{(2\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)}{12^{3/2}} + \frac{(2\mathbf{a}_y + 2\mathbf{a}_z)}{8^{3/2}} \\ + \frac{(2\mathbf{a}_x + 2\mathbf{a}_y)}{8^{3/2}} + \frac{(2\mathbf{a}_x + 2\mathbf{a}_z)}{8^{3/2}} \end{array}\right]$$

$$= 0.9(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\left(-\frac{1}{4} - \frac{1}{12\sqrt{3}} + \frac{1}{4\sqrt{2}}\right) = -0.1092(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\text{ N}$$

CHAPTER 7

Prob. 7.1



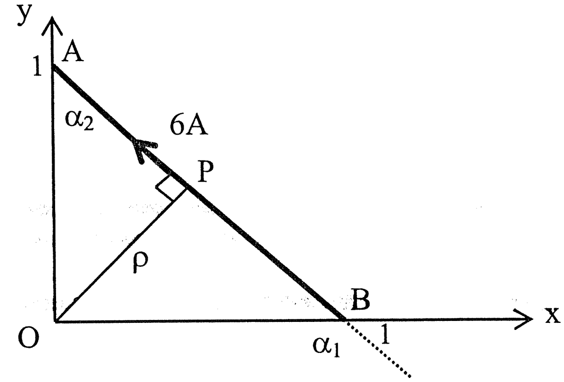
$$H = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)a_\phi$$

$$\rho = \sqrt{x^2 + y^2}, \cos\alpha_1 = \frac{a}{\sqrt{a^2 + \rho^2}}, \cos\alpha_2 = \frac{b}{\sqrt{b^2 + \rho^2}}$$

$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$. Hence,

$$H = \frac{I}{4\pi\sqrt{x^2 + y^2}}\left[\frac{b}{\sqrt{x^2 + y^2 + b^2}} - \frac{a}{\sqrt{x^2 + y^2 + a^2}}\right]\mathbf{a}_\phi$$

Prob. 7.2



$$\bar{H} = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)\bar{a}_\phi$$

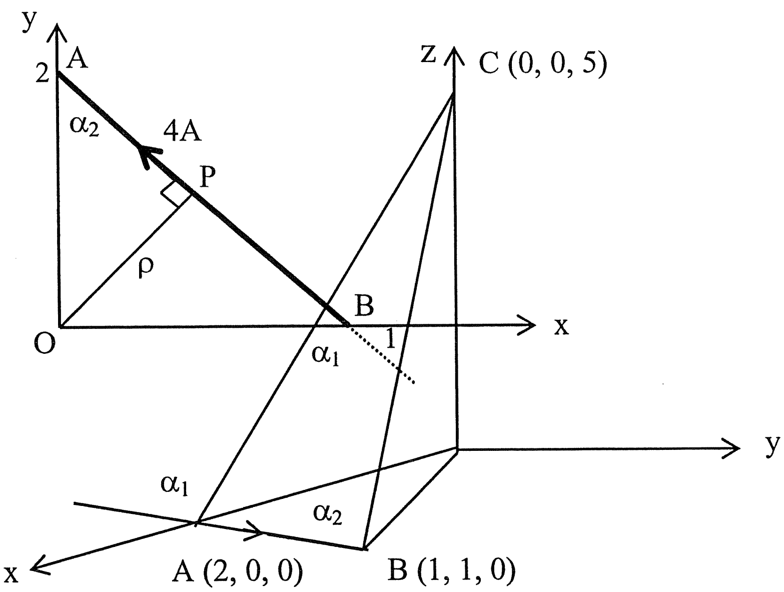
$$\alpha_1 = 135^\circ,\ \ \alpha_2 = 45^\circ,\ \ \rho = \frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$$

$$\bar{a}_\phi = \bar{a}_l \times \bar{a}_\rho = \left(\frac{-\bar{a}_x + \bar{a}_y}{\sqrt{2}}\right) \times \left(\frac{-\bar{a}_x - \bar{a}_y}{\sqrt{2}}\right) = \frac{1}{2}\begin{vmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = \bar{a}_z$$

$$\bar{H} = \frac{6}{4\pi\frac{\sqrt{2}}{2}}\left(\cos 45^\circ - \cos 135^\circ\right)\bar{a}_z = \frac{3}{\pi}\bar{a}_z$$

$$\bar{H}\,(0,0,0) = \underline{\underline{0.954\bar{a}_z \text{ A/m}}}$$

Prob. 7.3



(*a*) Consider the figure above.

$$AB = (1, 1, 0) - (2, 0, 0) = (-1, 1, 0)$$

$$AC = (0, 0, 5) - (2, 0, 0) = (-2, 0, 5)$$

$AB \cdot AC = 2$, i.e AB and AC are not perpendicular.

$$\cos(180^\circ - \alpha_1) = \frac{AB \cdot AC}{|AB||AC|} = \frac{2}{\sqrt{2}\sqrt{29}} \rightarrow \cos\alpha_1 = -\sqrt{\frac{2}{29}}$$

$$BC = (0, 0, 5) - (1, 1, 0) = (-1, -1, 5)$$

$$BA = (1, -1, 0)$$

$$\cos\alpha_2 = \frac{\bar{BC} \cdot \bar{BA}}{|BC||BA|} = \frac{-1+1}{|BC||BA|} = 0$$

i.e. $BC = \bar{\rho} = (-1, -1, 5), \quad \rho = \sqrt{27}$

$$\bar{a}_\phi = \bar{a}_l \times \bar{a}_\rho = \frac{(-1, 1, 0)}{\sqrt{2}} \times \frac{(-1, -1, 5)}{\sqrt{27}} = \frac{(5, 5, 2)}{\sqrt{54}}$$

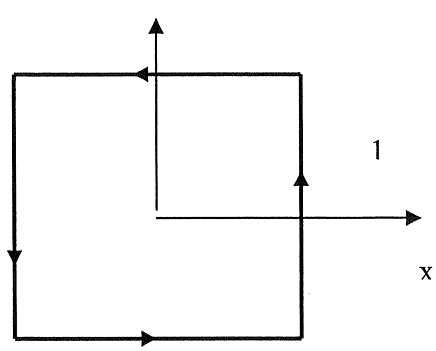
$$\bar{H}_2 = \frac{10}{4\pi\sqrt{27}}\left(0 + \sqrt{\frac{2}{29}}\right)\frac{(5, 5, 2)}{\sqrt{2}\sqrt{27}} = \frac{5}{2\pi\sqrt{29}} \cdot \frac{(5, 5, 2)}{27} \text{ A/m}$$

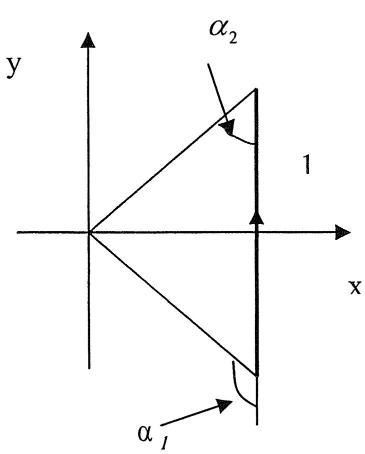
$$= \underline{\underline{27.37\,\bar{a}_x + 27.37\,\bar{a}_y + 10.95\,\bar{a}_z \text{ mA/m}}}$$

(b) $\bar{H} = \bar{H}_1 + \bar{H}_2 + \bar{H}_3 = (0, -59.1, 0) + (27.37, 27.37, 10.95) + (-30.63, 30.63, 0)$

$$= \underline{\underline{-3.26\,\bar{a}_x - 1.1\,\bar{a}_y + 10.95\,\bar{a}_z \text{ mA/m}}}$$

Prob. 7.4





H = 4**H**$_1$, where **H**$_1$ is due to side 1.

$$\boldsymbol{H}_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\boldsymbol{a}_\phi$$

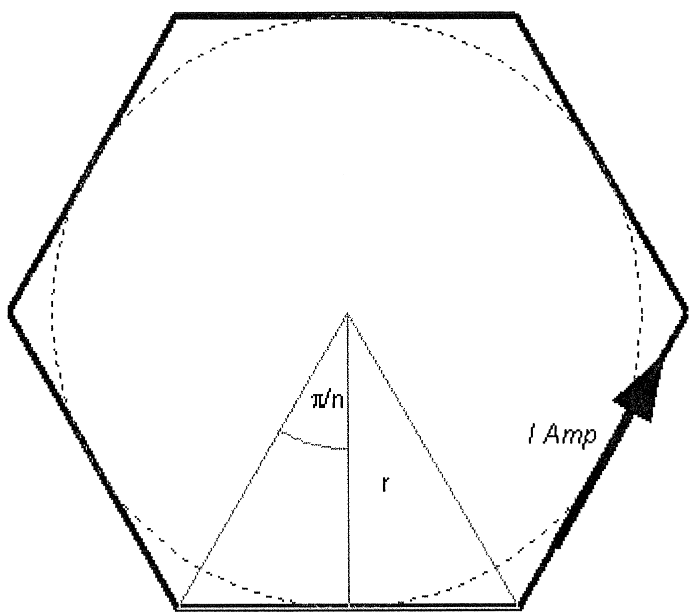
$\rho = a, \quad \alpha_2 = 45^o, \quad \alpha_1 = 135^o, \quad a_\phi = a_y x - a_x = a_z$

$$\boldsymbol{H}_1 = \frac{I}{4\pi\rho}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\boldsymbol{a}_z = \frac{2I}{4\pi a\sqrt{2}}\boldsymbol{a}_z$$

Therefore,

$$\boldsymbol{H} = 4\boldsymbol{H}_1 = \frac{I}{\pi a}\sqrt{2}\boldsymbol{a}_z \text{ A/m}$$

Prob. 7.5



(a) Consider one side of the polygon as shown. The angle subtended by the Side At the center of the circle

$$\frac{360^\circ}{n} = \frac{2\pi}{n}$$

The field due to this side is

$$H_1 = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)$$

where $\rho = r, \quad \cos\alpha_2 = \cos(90 - \frac{\pi}{n}) = \sin\frac{\pi}{n}$

$$\cos\alpha_1 = -\sin\frac{\pi}{n}$$

$$H_1 = \frac{I}{4\pi r} 2\sin\frac{\pi}{n}$$

$$\bar{H} = n\bar{H}_1 = \frac{nI}{2\pi r}\sin\frac{\pi}{n}$$

(*b*) For $n = 3, \quad H = \frac{3I}{2\pi r}\sin\frac{\pi}{3}$

$$r\cot 30^\circ = 2 \rightarrow r = \frac{2}{\sqrt{3}}$$

$$H = \frac{3\times 5}{2\pi\, {}^{2}\!/_{\sqrt{3}}} \cdot \frac{\sqrt{3}}{2} = \frac{45}{8\pi} = 1.79 \text{ A/m.}$$

For n = 4, $H = \frac{4I}{2\pi r}\sin\frac{\pi}{4} = \frac{4\times 5}{2\pi(2)}\cdot\frac{1}{\sqrt{2}}$

$= 1.128$ A/m.

(c) As $n \to \infty$,

$$H = \lim_{n\to\infty}\frac{nI}{2\pi r}\sin\frac{\pi}{n} = \frac{nI}{2\pi r}\cdot\frac{\pi}{n} = \frac{I}{2r}$$

From Example 7.3, when h = 0,

$$H = \frac{I}{2r}$$

which agrees.

Prob. 7.6

From Example 7.3, $\bar{H}$ due to circular loop is

$$\bar{H}_1 = \frac{I\rho^2}{2\left(\rho^2+z^2\right)}\bar{a}_z$$

(a) $$\bar{H}(0,0,0) = \frac{5\times 2^2}{2\left(2^2+0^2\right)^{3/2}}\bar{a}_z + \frac{5\times 2^2}{2\left(2^2+4^2\right)^{3/2}}\bar{a}_z$$

$$= \underline{\underline{1.36\,\bar{a}_z \text{ A/m}}}$$

(b) $$\bar{H}(0,0,2) = 2\frac{5\times 2^2}{2\left(2^2+2^2\right)^{3/2}}\bar{a}_z$$

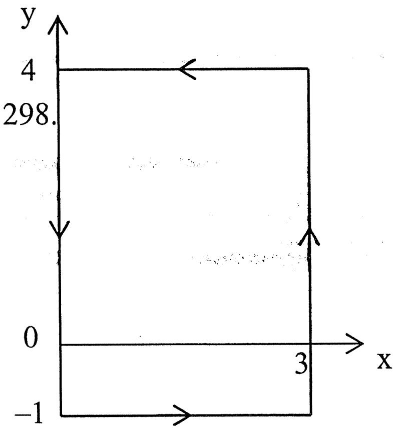
$$= \underline{\underline{0.884\,\bar{a}_z \text{ A/m}}}$$

Prob. 7.7

$$\bar{B} = \mu_o \bar{H} = \frac{\mu_o NI}{L}$$

$$N = \frac{Bl}{\mu \cdot I} = \frac{5\times10^{-3}\times3\times10^{-2}}{4\pi\times10^{-7}\times400\times10^{-3}} = 298.$$

$$N \approx 300 \text{ turns.}$$



(*a*) $$\bar{J} = \nabla\cdot\bar{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$$

$$\bar{J} = \underline{\underline{-2\,\bar{a}_z \text{ A/m}^2}}$$

(*b*) $$\oint \bar{H}.d\bar{l} = I_{enc}$$

$$I_{enc} = \int \bar{J}.d\bar{s} = \int_{x=0}^{3}\int_{y=-1}^{4}(-2)\,dx\,dy = (-2)(3)(5) = -30\text{ A}$$

$$\oint \bar{H}.d\bar{l} = \int_0^3 y\,dy\Big|_{y=-1} + \int_{y=-1}^{4}(-x)\,dy\Big|_{x=3} + \int_3^0 y\,dx\Big|_{y=4}$$

$$+ \int_4^{-1}(-x)\,dy\Big|_{x=0} = (-1)(3) + (-3)(5) + (4)(-3)$$

$$= -30\text{ A}$$

Thus, $$\oint \bar{H}.d\bar{l} = I_{enc} = \underline{\underline{-30\text{ A}}}$$

Prob. 7.8

$$\psi = \int \bar{B}.d\bar{s} = \mu_o \int_{z=0}^{0.2}\int_{\phi=0}^{50^o}\frac{10^6}{\rho}\sin 2\phi\;\rho\,d\phi\,dz$$

$$\psi = 4\pi\times10^{-7}\times10^6(0.2)\left(-\frac{\cos 2\phi}{2}\right)\Bigg|_0^{50^o}$$

$$= 0.04\pi\left(1-\cos 100^o\right)$$

$$= \underline{\underline{0.1475\text{ Wb}}}$$

Prob.7.9

On the slant side of the ring, $z = \frac{h}{6}(\rho - a)$

where $\bar{H}_1$ and $\bar{H}_2$ are due to the wires centered at $x = 0$ and $x = 10\text{cm}$ respectively

$$\psi = \int \bar{B}.d\bar{s} = \int \frac{\mu_o I}{2\pi\rho}\, d\rho\, dz$$

$$= \frac{\mu_o I}{2\pi} \int_{\rho=a}^{a+b} \int_{z=0}^{\frac{h}{b}(\rho-a)} \frac{dz\, d\rho}{\rho} = \frac{\mu_o I h}{2\pi b} \int_{\rho=a}^{a+b} \left(1 - \frac{a}{\rho}\right) d\rho$$

$$= \underline{\underline{\frac{\mu_o I h}{2\pi b}\left(b - a \ln \frac{a+b}{a}\right)}} \quad \text{as required.}$$

If $a = 30$ cm, $b = 10$ cm, $h = 5$ cm, $I = 10$ A,

$$\psi = \frac{2\pi \times 10^{-7} \times 10 \times 0.05}{2\pi\left(5 \times 10^{-2}\right)} \left(0.1 - 0.3 \text{ In } \frac{4}{3}\right)$$

$$= \underline{\underline{1.37 \times 10^{-8} \text{ Wb}}}$$

Prob. 7.10

For the outer conductor,

$$J_z = -\frac{I}{\pi(c^2 - b^2)} = -\frac{I}{\pi(16-9)a^2} = -\frac{I}{7\pi a^2}$$

Let $\bar{A} = A_z \, \bar{a}_z$. Using Poisson's equation,

$$\nabla^2 A_z = -\mu_o J_z$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) = \frac{\mu_o I}{7a^2\pi}$$

or $$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) = \frac{\mu_o I\rho}{7\pi a^2}$$

Integrating once,

$$\rho\frac{\partial A_z}{\partial \rho} = \frac{\mu_o I\rho^2}{14\pi a^2} + c_1$$

or $$\frac{\partial A_z}{\partial \rho} = \frac{\mu_o I\rho}{14\pi a^2} + \frac{c_1}{\rho}$$

Integrating again,

$$A_z = \frac{\mu_o I\rho^2}{28\pi a^2} + c_1 \, In\, \rho + c_2$$

But $A_z = 0$ when $\rho = 3a$.

$$0 = \frac{9}{28\pi}\mu_o I + c_1 \ln 3a + c_2$$

$$c_2 = -c_1 \, \text{In } 3a - \frac{9}{28\pi}\mu_o I$$

i.e. $$A_z = \frac{\mu_o I}{28\pi}\left(\frac{p^2}{a^2} - a\right) + c_1 \ln\frac{\rho}{3a}$$

But $\nabla \times \bar{A} = \bar{B} = \mu_o \bar{H}$

$$\nabla \times \bar{A} = -\frac{\partial A_z}{\partial \rho}\bar{a}_\phi = -\left(\frac{\mu_o I\rho}{14\pi a^2} + \frac{c_1}{\rho}\right)\bar{a}_\phi$$

At $\rho = 3a$, $\int \bar{H}.d\bar{l} = I \rightarrow 2\pi(3a)\, H_\phi = I$

or $$H_\phi = \frac{I}{6\pi a}$$

Thus $\nabla \times \bar{A}\Big|_{\rho=3a} = \mu_o \bar{H}\ (\rho = 3a)$ implies that

$$-\left(\frac{3\mu_o I}{14\pi a} + \frac{c_1}{3a}\right) = \frac{\mu_o I}{6\pi a}$$

$$\text{or } c_1 = -\frac{I\mu_o}{2\pi} - \frac{9\mu_o I}{14\pi} = -\frac{16\mu_o I}{14\pi}$$

Thus,

$$A_z = \frac{\mu_o I}{28\pi}\left(\frac{\rho^2}{a^2} - 9\right) - \frac{8\mu_o I}{7\pi} \ln \frac{\rho}{3a}$$

Prob. 7.11

(a)
$$\nabla \times \nabla V = \nabla \times \left(\frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z\right)$$

$$= \left(\frac{1}{\rho}\frac{\partial^2 V}{\partial\phi\partial z} - \frac{1}{\rho}\frac{\partial^2 V}{dz\partial\phi}\right)\bar{a}_\rho + \left(\frac{\partial^2 V}{\partial z\partial\rho} - \frac{\partial^2 V}{\partial\rho\partial z}\right)\bar{a}_\phi$$

$$+ \frac{1}{\rho}\left(\frac{\partial^2 V}{\partial\rho\partial\phi} - \frac{\partial^2 V}{\partial\phi d\rho}\right)\bar{a}_z = 0$$

(b)
$$\nabla\cdot\left(\nabla\times\bar{A}\right) = \nabla\cdot\left[\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\rho}{\partial z}\right)\bar{a}_\rho\right.$$

$$\left. + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\bar{a}_\phi + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)\bar{a}_z\right]$$

$$= \frac{1}{\rho}\frac{\partial^2 A_z}{\partial\rho\partial\phi} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial z}\right) + \frac{1}{\rho}\frac{\partial^2 A_\rho}{\partial\phi\partial z} - \frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial\rho}$$

$$+ \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)\right) - \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)$$

$$= -\frac{\partial^2 A_\phi}{\partial\rho\partial z} - \frac{1}{\rho}\frac{\partial A_\phi}{\partial z} + \frac{\partial^2 A_\phi}{\partial z\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial z} = 0$$

CHAPTER 8

Prob. 8.1

(a) $F = m\vec{a} = Q(\vec{E} + \vec{u} \times \vec{B})$

$$\frac{d}{dt}(u_x, u_y, u_z) = 2\left[-4\vec{a}_y + \begin{vmatrix} u_x & u_y & u_z \\ 5 & 0 & 0 \end{vmatrix}\right] = -8\vec{a}_y + 10u_z\vec{a}_y - 10u_y\vec{a}_z$$

i.e. $\frac{du_x}{dt} = 0 \rightarrow u_x = A_1$ (1)

$$\frac{du_y}{dt} = -8 + 10u_z \qquad (2)$$

$$\frac{du_z}{dt} = -10u_y \qquad (3)$$

$$\frac{d^2u_y}{dt^2} = 0 + 10\frac{du_z}{dt} = -100u_y$$

$$\ddot{u}_y + 100u_y = 0 \rightarrow u_y = B_1\cos 10t + B_2\sin 10t$$

From (2),

$$10u_z = 8 + \dot{u}_y = 8 - 10B_1\sin 10t + 10B_2\cos 10t$$

$u_z = 0.8 - B_1\sin 10t + B_2\cos 10t$

At t=0, $\vec{u} = 0 \rightarrow A_1 = 0, B_1 = 0, B_2 = -0.8$

Hence,

$$\vec{u} = (0,\ -0.8\sin 10t,\ 0.8 - 0.8\cos 10t) \qquad (4)$$

$$u_x = \frac{dx}{dt} = 0 \rightarrow x = c_1$$

$$u_y = \frac{dy}{dt} = -0.8\sin 10t \rightarrow y = 0.08\cos 10t + c_2$$

$$u_z = \frac{dz}{dt} = 0.8 - 0.8\cos 10t \rightarrow z = 0.8t + c_3 - 0.08\sin 10t$$

At t=0, (x, y, z) = (2, 3, -4) $\Rightarrow$ $c_1=2$, $c_2=2.92$, $c_3=-4$

Hence (x, y, z) = (2, 2.92 + 0.08cos10t, 0.8t – 0.08sin10t – 4)

At t=1,

(x, y, z) = <u>(2, 2.853, -3.156)</u>

(a) From (4), at t=1, $\vec{u} = \underline{\underline{(0, 0.435, 1.471)}}$ m/s

$$\text{K.E.} = \frac{1}{2}m|\vec{u}|^2 = \frac{1}{2}(1)(0.435^2 + 1.471^2) = \underline{\underline{1.177\text{J}}}$$

Prob. 8.2

(a) $m\vec{a} = -e(\vec{u} \times \vec{B})$

$$-\frac{m}{e}\frac{d}{dt}(u_x, u_y, u_z) = \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & B_o \end{vmatrix} = u_y B_o \vec{a}_x - B_o u_x \vec{a}_y$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = c = 0$$

$$\frac{du_x}{dt} = -u_y \frac{B_o e}{m} = -u_y w, \text{ where } w = \frac{B_o e}{m}$$

$$\frac{du_y}{dt} = u_x w$$

Hence,

$$\ddot{u}_x = -w\dot{u}_y = -w^2 u_x$$

$$\text{or } \ddot{u}_x + w^2 u_x = 0 \rightarrow u_x = A\cos wt + B\sin wt$$

$$u_y = -\frac{\dot{u}_x}{w} = A\sin wt - B\cos wt$$

$$u_y = \frac{\dot{u}_x}{w} = -A\sin wt + B\cos wt$$

At t=0, $u_x = u_o$, $u_y = 0 \rightarrow A = u_o$, B=0

Hence,

$$u_x = u_o \cos wt = \frac{dx}{dt} \rightarrow x = \frac{u_o}{w}\sin wt + c_1$$

$$u_y = u_o \sin wt = \frac{dy}{dt} \rightarrow y = -\frac{u_o}{w}\cos wt + c_2$$

At t=0, x = 0 = y → c_1=0, $c_2 = \frac{u_o}{w}$. Hence,

$$x = \frac{u_o}{w}\sin wt, y = \frac{u_o}{w}(1 - \cos wt)$$

$$\frac{u^2{}_o}{w^2}(\cos^2 wt + \sin^2 wt) = \left(\frac{u_o}{w}\right)^2 = x^2 + (y - \frac{u_o}{w})^2$$

showing that the electron would move in a circle centered at (0, $\frac{u_o}{w}$). But since the field does not exist throughout the circular region, the electron passes through a semi-circle and leaves the field horizontally.

(b) d = twice the radius of the semi-circle

$$= \frac{2u_o}{w} = \frac{2u_o m}{B_o e}$$

Prob. 8.3

$$\vec{\Im} = I\vec{L} \times \vec{B} \rightarrow \vec{\Im} = \frac{\vec{F}}{L} = I_1 \vec{a}_l \times \vec{B}_2 = \frac{\mu_o I_1 I_2 a_l \times \vec{a}_\phi}{2\pi\rho}$$

(a) $$\boldsymbol{F}_{21} = \frac{a_z \times (-a_y) 4\pi \times 10^{-7} (-100)(200)}{2\pi} = \underline{\underline{a_x \text{ mN/m}}} \quad \text{(repulsive)}$$

(b) $$F_{12} = -F_{21} = -a_x \text{ mN/m} \quad \text{(repulsive)}$$

(c) $$\vec{a}_l \times \vec{a}_\phi = \vec{a}_z \times (-\frac{4}{5}\vec{a}_x + \frac{3}{5}\vec{a}_y) = -\frac{3}{5}\vec{a}_x - \frac{4}{5}\vec{a}_y, \rho = 5$$

$$\vec{F}_{31} = \frac{4\pi \times 10^{-7}(-3 \times 10^4)}{2\pi(5)}\left(-\frac{3}{5}\vec{a}_x - \frac{4}{5}\vec{a}_y\right)$$

$$= \underline{\underline{0.72\vec{a}_x + 0.96\vec{a}_y}} \text{ mN/m (attractive)}$$

(d) $$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32}$$

$$\vec{F}_{32} = \frac{4\pi \times 10^{-7} \times 6 \times 10^4)}{2\pi(3)}(\vec{a}_z \times \vec{a}_y) = -4\vec{a}_x \text{ mN/m(attractive)}$$

$$\vec{F}_3 = \underline{\underline{-3.28\vec{a}_x + 0.96\vec{a}_y}} \text{ mN/m}$$

(attractive due to L_2 and repulsive due to L_1)

Prob. 8.4

(a) $$\vec{F}_1 = \int_{\rho=2}^{6} \frac{\mu_o I_1 I_2}{2\pi\rho} d\rho \vec{a}_\rho \times \vec{a}_\phi = \frac{4\pi \times 10^{-7}}{2\pi}(2)(5)\ln \tfrac{6}{2}\vec{a}_z$$

$$= 2 \ln 3\, \vec{a}_z\ \mu\text{N} = \underline{\underline{2.197\, \vec{a}_z\ \mu\text{N}}}$$

(b) $$\vec{F}_2 = \int I_2 d\vec{l}_2 \times \vec{B}_1$$

$$= \frac{\mu_o I_1 I_2}{2\pi} \int \frac{1}{\rho}\left[d\rho \vec{a}_\rho + dz \vec{a}_z\right] \times \vec{a}_\phi$$

$$= \frac{\mu_o I_1 I_2}{2\pi} \int \frac{1}{\rho}\left[d\rho \vec{a}_z - dz\vec{a}_\rho\right]$$

But ρ = z+2, dz=dρ

$$\vec{F}_2 = \frac{4\pi \times 10^{-7}}{2\pi}(5)(2) \int_{\rho=4}^{2} \frac{1}{\rho}\left[d\rho \vec{a}_z - dz\vec{a}_\rho\right]$$

$$2\ln \tfrac{2}{4}(\vec{a}_z - \vec{a}_\rho)\mu N = 1.386\vec{a}_\rho - 1.386\vec{a}_z \mu N$$

$$\vec{F}_3 = \frac{\mu_o I_1 I_2}{2\pi} \int \frac{1}{\rho}\left[d\rho \vec{a}_z - dz\vec{a}_\rho\right]$$

But z = -ρ + 6, dz = -dρ

$$\vec{F}_3 = \frac{4\pi \times 10^{-7}}{2\pi}(5)(2) \int_{\rho=6}^{4} \frac{1}{\rho}\left[d\rho \vec{a}_z - dz\vec{a}_\rho\right]$$

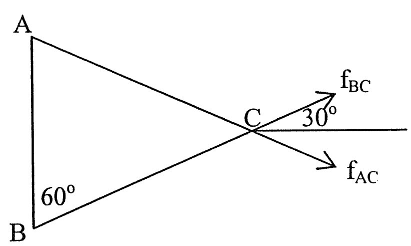
$$2\ln \tfrac{4}{6}(\vec{a}_z + \vec{a}_\rho)\mu N = -0.8109\vec{a}_\rho - 0.8109\vec{a}_z \mu N$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$= \boldsymbol{a}_\rho(\ln 4 + \ln 4 - \ln 9) + \boldsymbol{a}_z(\ln 9 - \ln 4 + \ln 4 - \ln 9)$$

$$= \underline{\underline{0.575\boldsymbol{a}_\rho \ \mu\text{N}}}$$

Prob. 8.5



From Prob. 8.7,

$$f = \frac{\mu_o I_1 I_2}{2\pi\rho}\vec{a}_\rho$$

$$\vec{f} = \vec{f}_{AC} + \vec{f}_{BC}$$

$$|\vec{f}_{AC}| = |\vec{f}_{BC}| = \frac{4\pi \times 10^{-7} \times 75 \times 150}{2\pi \times 2} = 1.125 \times 10^{-3}$$

$$\vec{f} = 2 \times 1.125 \cos 30^o \vec{a}_x \text{ mN/m}$$

$$= \underline{\underline{1.949\vec{a}_x}} \text{ mN/m}$$

Prob. 8.6

$$\vec{F} = \int L d\vec{l} \times \vec{B} = \int \vec{J} dv \times \vec{B}$$

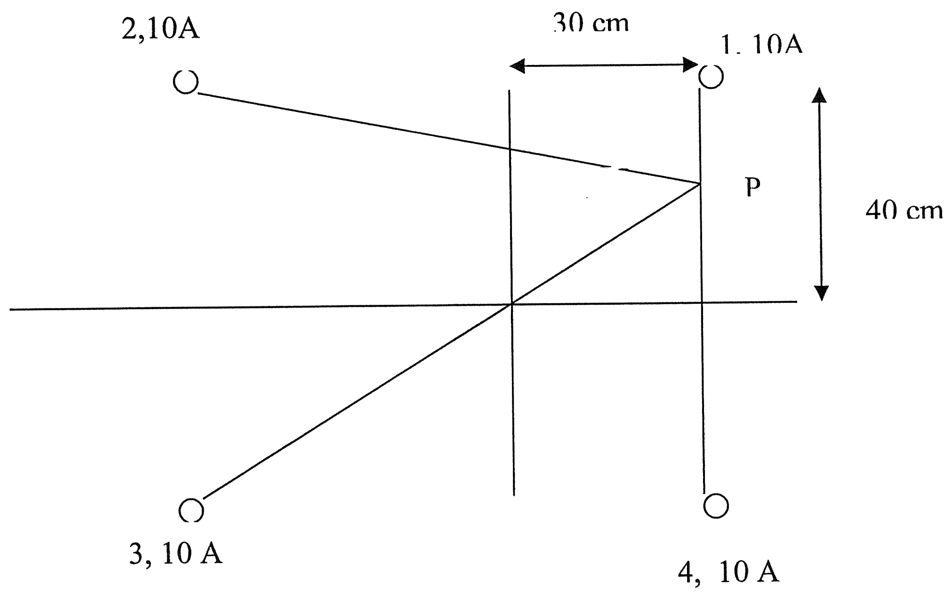
$$\vec{J} = \frac{I}{\pi(b^2 - a^2)}\vec{a}_z, \ \vec{B} = B_o\vec{a}_\rho$$

$$F = \frac{I}{\pi(b^2 - a^2)} \int a_z dv x B_o a_\rho = \frac{IBa_\phi}{\pi(b^2 - a^2)} \int dv$$

$$= \frac{-IB_o}{\pi(b^2 - a^2)} \pi(a^2 - b^2) l$$

$$\vec{f} = \frac{\vec{F}}{l} = \underline{\underline{IB_o\vec{a}_\phi}}$$

Prob. 8.7



Let $\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4$

where $\vec{B}_n = \dfrac{\mu_o \mu_r I}{2\pi\rho}\vec{a}_\phi$

For (1), $a_\phi = a_\ell x a_\rho = a_z x(-a_y) = a_x$

$$\vec{B}_1 = \frac{4\pi \times 10^{-7} \times 2000 \times 10}{2\pi \times 20 \times 10^{-3}}\vec{a}_x = 0.2\vec{a}_x$$

For (2), $\vec{\rho} = 6\vec{a}_x - 2\vec{a}_y$,

$$\vec{a}_\phi = -\vec{a}_z \times \frac{(6\vec{a}_x - 2\vec{a}_y)}{\sqrt{40}} = \frac{(-2\vec{a}_x - 6\vec{a}_y)}{\sqrt{40}}$$

$$\vec{B}_2 = \frac{4\pi \times 10^{-7} \times 2000 \times 10}{2\pi \times 400 \times 10^{-3}}(-2\vec{a}_x - 6\vec{a}_y)$$

$$= -0.02\vec{a}_x - 0.06\vec{a}_y$$

For (3), $\vec{\rho} = 6\vec{a}_x + 6\vec{a}_y$,

$$\vec{a}_\phi = \vec{a}_z \times \frac{(6\vec{a}_x + 6\vec{a}_y)}{\sqrt{72}} = \frac{(-6\vec{a}_x + 6\vec{a}_y)}{\sqrt{72}}$$

$$\vec{B}_3 = \frac{4\pi \times 10^{-7} \times 2000 \times 10}{2\pi \times 720 \times 10^{-3}}(-6\vec{a}_x + 6\vec{a}_y)$$

$$= -0.03333\vec{a}_x + 0.03333\vec{a}_y$$

For (4), $\vec{a}_\phi = -\vec{a}_z \times \vec{a}_y = \vec{a}_x$,

$$\vec{B}_4 = \frac{4\pi \times 10^{-7} \times 2000 \times 10}{2\pi \times 60 \times 10^{-3}}\vec{a}_x = 0.06667\vec{a}_x$$

$$\vec{B} = (2 + 2/3 - 1/5 - 1/3)x10^{-1}\vec{a}_x + (-3/5 + 1/3)x10^{-1}\vec{a}_y$$

$$= 0.21333\vec{a}_x - 0.02667\vec{a}_y \text{ Wb/m}^2$$

Prob. 8.8

(a) From $H_{1t} - H_{2t} = K$ and $M = \chi_m H$, we obtain:

$$\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = K$$

Also from $B_{1n} - B_{2n} = 0$ and $B = \mu H = (\mu/\chi_m)M$, we get:

$$\frac{\mu_1 M_{1n}}{\chi_{m1}} = \frac{\mu_2 M_{2n}}{\chi_{m2}}$$

(b) From $B_1\cos\theta_1 = B_{1n} = B_{2n} = B_2\cos\theta_2$ (1)

and $\frac{B_1 \sin\theta_1}{\mu_1} = H_{1t} = K + H_{2t} = K + \frac{B_2 \sin\theta_2}{\mu_2}$ (2)

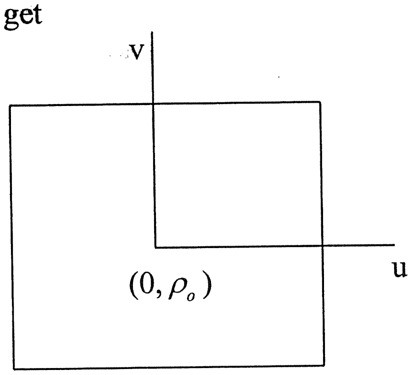
Dividing (2) by (1) gives

$$\frac{\tan\theta_1}{\mu_1} = \frac{k}{B_2\cos\theta_2} + \frac{\tan\theta_2}{\mu_2} = \frac{\tan\theta_2}{\mu_2}\left(1 + \frac{k\mu_2}{B_2\sin\theta_2}\right)$$

i.e. $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}\left(1 + \frac{k\mu_2}{B_2 \sin\theta_2}\right)$$

Prob. 8.9

(a) The square cross-section of the toroid is shown below. Let (u,v) be the local coordinates and ρ_o=mean radius. Using Ampere's law around a circle passing through P, we get



$$H(2\pi)(\rho_o + v) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + v)}$$

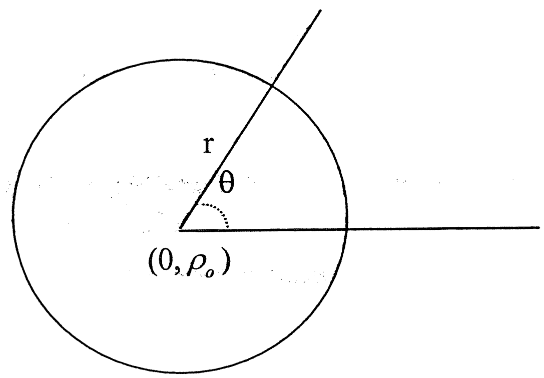
The flux per turn is

$$\Psi = \int_{u=-a/2}^{a/2} \int_{v=-a/2}^{a/2} B du dv = \frac{\mu_o NIa}{2\pi} \ln\left(\frac{\rho_o + a/2}{\rho_o - a/2}\right)$$

$$L = \frac{N\Psi}{I} = \frac{\mu_o N^2 Ia}{2\pi} \ln\left(\frac{2\rho_o + a}{2\rho_o - a}\right)$$

(b) The circular cross-section of the toroid is shown below. Let (r,θ) be the local coordinates. Consider a point P($r\cos\theta, \rho_o + r\sin\theta$) and apply Ampere's law around a circle that passes through P.

$$H(2\pi)(\rho_o + r\sin\theta) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + r\sin\theta)} \approx \frac{NI}{2\pi\rho_o}\left(1 - \frac{r\sin\theta}{\rho_o}\right)$$



Flux per turn $\Psi = \int_{r=0}^{a}\int_{\theta}^{2\pi} \frac{\mu NI}{2\pi\rho_o}\left(1 - \frac{r\sin\theta}{\rho_o}\right) r\,dr\,d\theta = \frac{\mu NI}{2\pi\rho_o}\frac{a^2}{2}(2\pi)$

$$L = \frac{N\Psi}{I} = \frac{\mu N^2 a^2}{2\rho_o}$$

Or from Example 8.10,

$$L = L'l = \frac{\mu_o N^2 l S}{l^2} = \frac{\mu_o N^2 \pi a^2}{2\pi\rho_o} = \frac{\mu_o N^2 a^2}{2\rho_o}$$

Prob. 8.10

We may approximate the longer solenoid as infinite so that $B_1 = \frac{\mu_o N_1 I_1}{l_1}$. The flux linking the second solenoid is:

$$\psi_2 = N_2 B_1 S_1 = \frac{\mu_o N_1 I_1}{l_1} \bullet \pi r_1^{\,2} \Box N_2$$

$$M = \frac{\psi_2}{I_1} = \frac{\mu_o N_1 N_2}{l_1} \bullet \pi r_1^{\,2}$$

Prob. 8.11

$$F = \frac{B^2 S}{2\mu_o} = \frac{\psi^2}{2\mu_o S} = \frac{4\times10^{-6}}{2\times4\pi\times10^{-7}\times0.3\times10^{-4}} = \underline{\underline{53.05}}\text{ kN}$$

CHAPTER 9

Prob. 9.1

(a) $v = \int(\bar{u}\times\bar{B})\cdot d\bar{l}, \quad d\bar{l} = dy\bar{a}_y$

$\bar{u}\times\bar{B} = 2\bar{a}_x \times 0.1\bar{a}_z = -0.2\bar{a}_y$

y = x since the angle of the v-shaped conductor is 45°. Hence

y = x = ut. At t = 0, x = 0 = y

$v = -\int 0.2\, dy = -0.2y, \quad y = ut = 2t$

$\underline{\underline{v = -0.4t \text{ V}}}$

(b) $v = \int(\bar{u}\times\bar{B})\cdot d\bar{l}, \quad d\bar{l} = dy\bar{a}_y$

$\bar{u}\times\bar{B} = 2\bar{a}_x \times 0.5x\bar{a}_z = -x\bar{a}_y$

But y = x and x = ut. When t = 0, x = 0 = y

$v = -\int x\, dy = -\int y\, dy = -y^2/2$

But x = y = ut = 2t

$\underline{\underline{v = -2t^2 \text{ V}}}$

Prob. 9.2

$$\bar{B} = \frac{\mu_o I}{2\pi y}(-\bar{a}_x)$$

$$\psi = \int \vec{B}\bullet d\vec{S} = \frac{\mu_o I}{2\pi}\int_{z=0}^{a}\int_{y=\rho}^{\rho+a}\frac{dzdy}{y} = \frac{\mu_o Ia}{2\pi}\ln\frac{\rho+a}{\rho}$$

$$V_{emf} = -\frac{\partial\psi}{\partial t} = -\frac{\partial\psi}{\partial\rho}\bullet\frac{\partial\rho}{\partial t} = -\frac{\mu_o Ia}{2\pi}u_o\frac{d}{d\rho}[\ln(\rho+a)-\ln\rho]$$

$$= -\frac{\mu_o Ia}{2\pi}u_o\left[\frac{1}{\rho+a}-\frac{1}{\rho}\right] = \underline{\underline{\frac{\mu_o a^2 Iu_o}{2\pi\rho(\rho+a)}}}$$

where $\rho = \rho_o + u_o t$

Prob. 9.3

$$V_{emf} = \int_{\rho}^{\rho+a} 3\boldsymbol{a}_z\times\frac{\mu_o I}{2\pi\rho}\boldsymbol{a}_\phi\bullet d\rho\boldsymbol{a}_\rho = -\frac{3\mu_o I}{2\pi}\ln\frac{\rho+a}{\rho}$$

$$= -\frac{4\pi\times10^{-7}}{2\pi}\times15\times3\ln\frac{60}{20} = -9.888\mu V$$

Thus the induced emf = <u>9.888μV, point A at higher potential.</u>

Prob. 9.4

$$V_{emf} = -\int \frac{\partial \vec{B}}{\partial t} \bullet dS + \int (\vec{u} \times \vec{B}) \bullet d\vec{l}$$

where $\vec{B} = B_o \cos \omega t \vec{a}_x, \vec{u} = u_o \cos \omega t \vec{a}_y, d\vec{l} = dz\vec{a}_z$

$$V_{emf} = \int_{z=0}^{l} \int_{y=-a}^{y} B_o \omega \sin \omega t dy dz - \int_{0}^{l} B_o u_o \cos^2 \omega t dz$$

$= B_o \omega l(y+a)\sin\omega t - B_o u_o l\cos^2\omega t$

Alternatively,

$$\psi = \int \vec{B} \bullet d\vec{s} = \int_{z=0}^{l} \int_{y=-a}^{y} B_o \cos \omega t \vec{a}_x \bullet dy dz \vec{a}_x = B_o(y+a)l\cos wt$$

$$V_{emf} = -\frac{\partial \psi}{\partial t} = B_o(y+a)l\omega \sin \omega t - B_o \frac{dy}{dt} l \cos \omega t$$

But $\frac{dy}{dt} = u = u_o \cos \omega t \rightarrow y = \frac{u_o}{\omega} \sin \omega t$

$V_{emf} = B_o \omega l(y+a)\sin\omega t - B_o u_o l\cos^2\omega t$

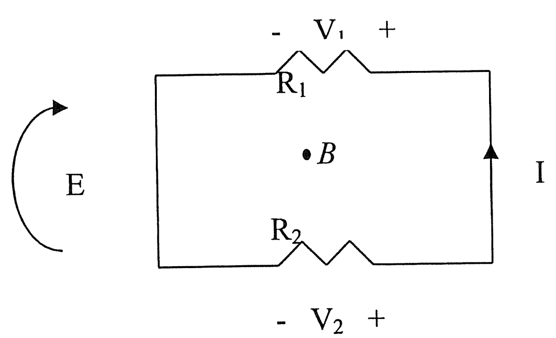
$= B_o u_o l\sin^2\omega t + B_o \omega a l\sin\omega t - B_o u_o l\cos^2\omega t$

$= -B_o u_o l\cos 2\omega t + B_o \omega a l\sin\omega t$

$= 6 \times 10^{-3} \times 5[10 \times 10\sin 10t - 2\cos 20t]$

$V_{emf} = \underline{\underline{3\sin 10t - 0.06\cos 20t \text{ V}}}$

Prob. 9.5



$$\oint \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \int \vec{B} \bullet dS$$

$= I(R_1 + R_2)$

$$\frac{dB}{dt} \bullet S = I(R_1 + R_2) \qquad (1)$$

Also, $\oint \vec{E} \bullet d\vec{l} = V_1 - V_2 = -\frac{dB}{dt} \bullet S$ (2)

Hence, $V_1 = IR_1 = -\frac{SR_1}{R_1 + R_2}\frac{dB}{dt}$

$V_2 = -IR_2 = \frac{SR_2}{R_1 + R_2}\frac{dB}{dt}$

$$V_1 = \frac{10 \times 10^{-4} \times 10}{15} \times 0.2 \times 150\pi \sin 150\pi t = \underline{\underline{0.0628 \sin 150\pi t}} \text{ V}$$

$$V_2 = \frac{-10 \times 10^{-4} \times 5}{15} \times 0.2 \times 150\pi \sin 150\pi t = \underline{\underline{-0.0314 \sin 150\pi t}} \text{ V}$$

Prob. 9.6

$V = \int (\vec{u} \times \vec{B}) \bullet d\vec{l}$, where $\vec{u} = \rho\omega\vec{a}_\phi$, $\vec{B} = B_o\vec{a}_z$

$$V = \int_{\rho_1}^{\rho_2} \rho\omega B_o d\rho = \frac{\omega B_o}{2}({\rho^2}_2 - {\rho^2}_1)$$

$$V = \frac{60 \times 15}{2} \bullet 10^{-3}(100 - 4) \bullet 10^{-4} = \underline{\underline{4.32 \text{ mV}}}$$

Prob. 9.7

If $\vec{J} = 0 = \rho_v$, then

$$\nabla \bullet \vec{B} = 0 \quad (1)$$

$$\nabla \bullet \vec{D} = \rho_v \quad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (3)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (4)$$

Since $\nabla \bullet \nabla \times \vec{A} = 0$ for any vector field $\vec{A}$,

$$\nabla \bullet \nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \bullet \vec{B} = 0$$

$$\nabla \bullet \nabla \times \vec{H} = -\frac{\partial}{\partial t}\nabla \bullet \vec{D} = 0$$

showing that (1) and (2) are incorporated in (3) and (4). Thus Maxwell's equations can be reduced to (3) and (4), i.e.

$$\underline{\underline{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{H} = -\frac{\partial \vec{D}}{\partial t}}}$$

Prob. 9.8

$$\nabla \bullet J = (0+0+3z^2)\sin 10^4 t = -\frac{\partial \rho_v}{\partial t}$$

$$\rho_v = -\int \nabla \bullet J dt = -\int 3z^2 \sin 10^4 t dt = \frac{3z^2}{10^4}\cos 10^4 t + C_o$$

If $\rho_v |_{z=0} = 0$, then $C_o = 0$ and

$$\rho_v = 0.3z^2 \cos 10^4 t \text{ mC/m}^3$$

Prob. 9.9

$$\nabla \bullet \boldsymbol{E} = 0$$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(y,z,t) & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z}\boldsymbol{a}_y - \frac{\partial E_x}{\partial y}\boldsymbol{a}_z$$

$$= \frac{\beta\omega\mu b}{\pi} H_o \sin(\pi y / b)\cos(\omega t - \beta z)\boldsymbol{a}_y + \omega\mu H_o \cos(\pi y / b)\sin(\omega t - \beta z)\boldsymbol{a}_z$$

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \longrightarrow \boldsymbol{H} = -\frac{1}{\mu}\int \nabla \times \boldsymbol{E} dt$$

$$\boldsymbol{H} = -\frac{\beta b}{\pi} H_o \sin(\pi y / b)\sin(\omega t - \beta z)\boldsymbol{a}_y + H_o \cos(\pi y / b)\cos(\omega t - \beta z)\boldsymbol{a}_z$$

which is the given H field.

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \partial x & \partial y & \partial z \\ 0 & H_y & H_z \end{vmatrix} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right)\boldsymbol{a}_x$$

$$= [-\frac{\pi}{b} H_o \sin(\pi y / b)\cos(\omega t - \beta z) - \frac{\beta^2 b}{\pi} H_o \sin(\pi y / b)\cos(\omega t - \beta z)]\boldsymbol{a}_x$$

$$\nabla \times \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \longrightarrow \boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt$$

$$\boldsymbol{E} = [-\frac{\pi}{\omega b \varepsilon} H_o \sin(\pi y / b)\sin(\omega t - \beta z) - \frac{\beta^2 b}{\pi\omega\varepsilon} H_o \sin(\pi y / b)\sin(\omega t - \beta z)]\boldsymbol{a}_x$$

Setting this equal to the given E,

$$\frac{\omega\mu b}{\pi} H_o = \frac{\pi}{\omega b \varepsilon} H_o + \frac{\beta^2 b}{\pi\omega\varepsilon} H_o \longrightarrow \beta^2 = -\frac{\pi^2}{b^2} + \omega^2 \mu\varepsilon$$

$$\beta = \sqrt{\omega^2 \mu\varepsilon - \frac{\pi^2}{b^2}}$$

Prob. 9.10 From Maxwell's equations,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (1)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (2)$$

Dotting both sides of (2) with $\vec{E}$ gives:

$$\vec{E} \bullet (\nabla \times \vec{H}) = \vec{E} \bullet \vec{J} + \vec{E} \bullet \frac{\partial \vec{D}}{\partial t} \quad (3)$$

But for any arbitrary vectors $\vec{A}$ and $\vec{B}$,

$$\nabla \bullet (\vec{A} \times \vec{B}) = \vec{B} \bullet (\nabla \times \vec{A}) - \vec{A} \bullet (\nabla \times \vec{B})$$

Applying this on the left-hand side of (3) by letting $\vec{A} \equiv \overline{H}$ and $\vec{B} \equiv \vec{E}$, we get

$$\vec{H} \bullet (\nabla \times \vec{E}) + \nabla \bullet (\vec{H} \times \vec{E}) = \vec{E} \bullet \vec{J} + \tfrac{1}{2}\frac{\partial}{\partial t}(\vec{D} \bullet \vec{E}) \quad (4)$$

From (1),

$$\vec{H} \bullet (\nabla \times \vec{E}) = \vec{H} \bullet \left(-\frac{\partial \vec{B}}{\partial t} \right) = \tfrac{1}{2}\frac{\partial}{\partial t}(\vec{B} \bullet \vec{H})$$

Substituting this in (4) gives:

$$-\tfrac{1}{2}\frac{\partial}{\partial t}(\vec{B} \bullet \vec{H}) - \nabla \bullet (\vec{E} \times \vec{H}) = \vec{J} \bullet \vec{E} + \tfrac{1}{2}\frac{\partial}{\partial t}(\vec{D} \bullet \vec{E})$$

Rearranging terms and then taking the volume integral of both sides:

$$\int_v \nabla \bullet (\vec{E} \times \vec{H}) dv = -\frac{\partial}{\partial t} \tfrac{1}{2} \int_v (\vec{E} \bullet \vec{D} + \vec{H} \bullet \vec{B}) dv - \int_v \vec{J} \bullet \vec{E} dv$$

Using the divergence theorem, we get

$$\oint_s (\vec{E} \times \vec{H}) \bullet dS = -\frac{\partial W}{\partial t} - \int_v \vec{J} \bullet \vec{E} dv$$

or $\dfrac{\partial W}{\partial t} = -\oint_s (\vec{E} \times \vec{H}) \bullet dS - \int_v \vec{E} \bullet \vec{J} dv$ as required.

Prob. 9.11

$$\nabla \times \vec{E} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi)\vec{a}_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 t e^{-\rho-t})\vec{a}_z$$

$$= (2-\rho) t e^{-\rho-t}\vec{a}_z$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} \rightarrow \vec{B} = -\int \nabla \times \vec{E} dt = \int \frac{(\rho-2)t}{V}\frac{e^{-\rho-t}dt}{du}\vec{a}_z$$

Integrating by parts yields

$$\vec{B} = [-(\rho-2)te^{-\rho-t} + \int(\rho-2)e^{-\rho-t}dt]\vec{a}_z$$

$$= \underline{\underline{(2-\rho)(1+t)e^{-\rho-t}\vec{a}_z}} \ \text{Wb/m}^2$$

$$\vec{J} = \nabla\times\vec{H} = \nabla\times\frac{\vec{B}}{\mu_o} = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial\rho}\vec{a}_\phi$$

$$= -\frac{1}{\mu_o}(1+t)(-1-2+\rho)e^{-\rho-t}\vec{a}_\phi$$

$$\vec{J} = \underline{\underline{\frac{(1+t)(3-\rho)e^{-\rho-t}}{4\pi x10^{-7}}\vec{a}_\phi}} \ \text{A/m}^2$$

Prob. 9.12

With the given **A**, we need to prove that

$$\nabla^2 A = \mu\varepsilon\frac{\partial^2 A}{\partial t^2}$$

$\nabla^2 A = \mu\varepsilon(j\omega)(j\omega)A = -\omega^2\mu\varepsilon A$

Let $\beta^2 = \omega^2\mu\varepsilon$, then $\nabla^2 A = -\beta^2 A$ is to be proved. We recognize that

$$A = \frac{\mu_o}{4\pi r}e^{j\omega t}e^{-j\beta r}\mathbf{a}_z$$

Assume $\varphi = \dfrac{e^{-j\beta r}}{r}$, $\quad A = \dfrac{\mu_o}{4\pi}e^{j\omega t}\varphi\mathbf{a}_z$

$$\nabla^2\varphi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\frac{\partial\varphi}{dr})\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^2)\left(\frac{-j\beta}{r}-\frac{1}{r^2}\right)e^{-j\beta r}\right]$$

$$= \frac{1}{r^2}\left(-\beta^2 r + j\beta - j\beta\right)e^{-j\beta r} = -\beta^2\frac{e^{-j\beta r}}{r} = -\beta^2\varphi$$

Therefore, $\quad \nabla^2 A = -\beta^2 A$

We can find V using Lorentz gauge.

$$V = \frac{-1}{\mu_o\varepsilon_o}\int\nabla\bullet A dt = \frac{-1}{j\omega\mu_o\varepsilon_o}\nabla\bullet A$$

$$= \frac{-1}{j\omega\mu_o\varepsilon_o}\frac{\partial}{\partial r}\left(\frac{\mu_o}{4\pi r}e^{-j\beta r}e^{j\omega t}\right) = \frac{-1}{j\omega\varepsilon_o(4\pi)}\left(\frac{-j\beta}{r}-\frac{1}{r^2}\right)e^{-j\beta r}e^{j\omega t}\cos\theta$$

$$V = \underline{\underline{\frac{\cos\theta}{j4\pi\omega\varepsilon_o r}\left(j\beta+\frac{1}{r}\right)e^{j(\omega t-\beta r)}}}$$

CHAPTER 10

Prob. 10.1 If

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = -\omega^2\mu\varepsilon + j\omega\mu\sigma \quad \text{and} \ \gamma = \alpha + j\beta, \ \text{then}$$

$$|\gamma^2| = \sqrt{(\alpha^2 - \beta^2) + 4\alpha^2\beta^2} = \sqrt{(\alpha^2 + \beta^2)^2} = \alpha^2 + \beta^2$$

i.e.

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{(\sigma^2 + \omega^2\varepsilon^2)} \qquad (1)$$

$$\text{Re}(\gamma^2) = \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

$$\beta^2 - \alpha^2 = \omega^2\mu\varepsilon \qquad (2)$$

Subtracting and adding (1) and (2) lead respectively to

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

(b) From eq. (10.25), $\boldsymbol{E}_s(z) = E_o e^{-\gamma z}\boldsymbol{a}_x$.

$$\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H}_s \longrightarrow \boldsymbol{H}_s = \frac{j}{\omega\mu}\nabla \times \boldsymbol{E}_s = \frac{j}{\omega\mu}(-\gamma E_o e^{-\gamma z}\boldsymbol{a}_y)$$

But $\boldsymbol{H}_s(z) = H_o e^{-\gamma z}\boldsymbol{a}_y$, hence $H_o = \dfrac{E_o}{\eta} = -\dfrac{j\gamma}{\omega\mu}E_o$

$$\eta = \frac{j\omega\mu}{\gamma}$$

(c) From (b),

$$\eta = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - j\dfrac{\sigma}{\omega\varepsilon}}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}, \tan 2\theta_\eta = \left(\frac{\omega\varepsilon}{\sigma}\right)^{-1} = \frac{\sigma}{\omega\varepsilon}$$

Prob. 10.2 (a) $\frac{\sigma}{\omega\varepsilon} = \tan 2\theta_\eta = \tan 60^o = \underline{\underline{1.732}}$

(b) $|\eta| = 240 = \frac{\frac{120\pi}{\sqrt{\varepsilon_r}}}{\sqrt[4]{1+3}} = \frac{120\pi}{\sqrt{2\varepsilon_r}} \longrightarrow \varepsilon_r = \frac{\pi^2}{8} = \underline{\underline{1.234}}$

(c) $\varepsilon_c = \varepsilon(1 - j\frac{\sigma}{\omega\varepsilon}) = 1.234 \times \frac{10^{-9}}{36\pi}(1 - j1.732) = \underline{\underline{(1.091 - j1.89) \times 10^{-11}}}$ F/m

(d)

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} = \frac{2\pi \times 10^6}{3\times 10^8}\sqrt{\frac{1}{2}\frac{\pi^2}{8}\left[\sqrt{1+3} - 1\right]} = \underline{\underline{0.0164}} \text{ Np/m}$$

Prob. 10.3 (a)

$$T = 1/f = 2\pi/\omega = \frac{2\pi}{\pi x 10^8} = \underline{\underline{20 \text{ ns}}}$$

(b) Let $x = \sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1}\right)^{1/2}$$

But $\alpha = \frac{\omega}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}}\sqrt{x-1}$

$$\sqrt{x-1} = \frac{\alpha c}{\omega\sqrt{\frac{\mu_r\varepsilon_r}{2}}} = \frac{0.1\times 3\times 10^8}{\pi\times 10^8\sqrt{2}} = 0.06752 \longrightarrow x = 1.0046$$

$$\beta = \left(\frac{x+1}{x-1}\right)^{1/2}\alpha = \left(\frac{2.0046}{0.0046}\right)^{1/2} 0.1 = 2.088$$

$$\lambda = 2\pi / \beta = \frac{2\pi}{2.088} = \underline{\underline{3}}\,\text{m}$$

(c) $$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{x}} = \frac{377}{2\sqrt{1.0046}} = 188.1$$

$$x = \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} = 1.0046$$

$$\frac{\sigma}{\omega\varepsilon} = 0.096 = \tan 2\theta_\eta \longrightarrow \theta_\eta = 2.74^o$$

$$\eta = 188.1\angle 2.74^o \quad \Omega$$

$$E_o = \eta H_o = 12x188.1 = 2257.2$$

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \longrightarrow \mathbf{a}_E \times \mathbf{a}_x = \mathbf{a}_y \longrightarrow \mathbf{a}_E = \mathbf{a}_z$$

$$\mathbf{E} = \underline{\underline{2.257e^{-0.1y}\sin(\pi \times 10^8 t - 2.088y + 2.74^o)\mathbf{a}_z}} \ \text{kV/m}$$

(d) The phase difference is $\underline{\underline{2.74^o}}$.

Prob. 10.4 (a) $$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-6}}{2\pi \times 10^7 \times 5 \times \frac{10^{-9}}{36\pi}} = 3.6 \times 10^{-4} << 1$$

Thus, the material is <u>lossless</u> at this frequency.

(b) $$\beta = \omega\sqrt{\mu\varepsilon} = \frac{2\pi \times 10^7}{3 \times 10^8}\sqrt{5 \times 750} = \underline{\underline{12.83}} \text{ rad/m}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{12.83} = \underline{\underline{0.49}} \text{ m}$$

(c) Phase difference = βl = $\underline{25.66 \text{ rad}}$

(d) $$\eta = \sqrt{\mu/\varepsilon} = 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{750}{5}} = \underline{\underline{4.62\ \text{k}\Omega}}$$

Prob. 10.5

Let $\boldsymbol{E}_s = \boldsymbol{E}_r + j\boldsymbol{E}_i$ and $\boldsymbol{H}_s = \boldsymbol{H}_r + j\boldsymbol{H}_i$

$$\boldsymbol{E} = \text{Re}(\boldsymbol{E}_s e^{j\omega t}) = \boldsymbol{E}_r \cos \omega t - \boldsymbol{E}_i \sin \omega t$$

Similarly,

$$\boldsymbol{H} = \boldsymbol{H}_r \cos \omega t - \boldsymbol{H}_i \sin \omega t$$

$$\mathcal{P} = \boldsymbol{E} \times \boldsymbol{H} = \boldsymbol{E}_r \times \boldsymbol{H}_r \cos^2 \omega t + \boldsymbol{E}_i \times \boldsymbol{H}_i \sin^2 \omega t - \frac{1}{2}(\boldsymbol{E}_r \times \boldsymbol{H}_i + \boldsymbol{E}_i \times \boldsymbol{H}_r) \sin\ 2\omega t$$

$$\mathcal{P}_{\text{ave}} = \frac{1}{T}\int_0^T \mathcal{P} dt = \frac{1}{T}\int_0^T \cos^2 \omega dt (\boldsymbol{E}_r \times \boldsymbol{H}_r) + \frac{1}{T}\int_0^T \sin^2 \omega dt (\boldsymbol{E}_i \times \boldsymbol{H}_i) - \frac{1}{2T}\int_0^T \sin 2\omega dt (\boldsymbol{E}_i \times \boldsymbol{H}_i + \boldsymbol{E}_i \times \boldsymbol{H}_r)$$

$$= \frac{1}{2}(\boldsymbol{E}_r \times \boldsymbol{H}_r + \boldsymbol{E}_i \times \boldsymbol{H}_i) = \frac{1}{2}\text{Re}[(\boldsymbol{E}_r + j\boldsymbol{E}_i) \times (\boldsymbol{H}_r - j\boldsymbol{H}_i)]$$

$$\mathcal{P}_{\text{ave}} = \frac{1}{2}\ \text{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s^*)$$

as required.

Prob. 10.6

(a) $$\boldsymbol{H}_s = \frac{j30\beta I_o dl}{120\pi r} \sin \theta e^{-j\beta r} \boldsymbol{a}_H$$

where $\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_\theta \times \boldsymbol{a}_H = \boldsymbol{a}_r \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_\phi$

$$\boldsymbol{H}_s = \frac{j\beta I_o dl}{4\pi r} \sin \theta e^{-j\beta r} \boldsymbol{a}_\phi$$

(b) $$\boldsymbol{P}_{ave} = \frac{1}{2}\text{Re}[\boldsymbol{E}_s \times \boldsymbol{H}_s^*] = \frac{1}{2}\text{Re}[\frac{30\beta^2 I_o^2 dl^2 \sin^2\theta}{4\pi r^2}\boldsymbol{a}_r] = \frac{15\beta^2 I_o^2 dl^2 \sin^2\theta}{4\pi r^2}\boldsymbol{a}_r$$

Prob. 10.7 (a) $P_{i,ave} = \dfrac{E_{io}^2}{2\eta_1}$, $\quad P_{r,ave} = \dfrac{E_{ro}^2}{2\eta_1}$, $\quad P_{t,ave} = \dfrac{E_{to}^2}{2\eta_2}$

$$R = \frac{P_{r,ave}}{P_{i,ave}} = \frac{E_{ro}^2}{E_{io}^2} = \Gamma^2 = \underline{\underline{\left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)^2}}$$

$$R = \left(\frac{\sqrt{\frac{\mu_o}{\varepsilon_2}} - \sqrt{\frac{\mu_o}{\varepsilon_1}}}{\sqrt{\frac{\mu_o}{\varepsilon_2}} + \sqrt{\frac{\mu_o}{\varepsilon_1}}}\right)^2 = \left(\frac{\sqrt{\mu_o\varepsilon_1} - \sqrt{\mu_o\varepsilon_2}}{\sqrt{\mu_o\varepsilon_1} + \sqrt{\mu_o\varepsilon_2}}\right)^2$$

Since $n_1 = c\sqrt{\mu_1\varepsilon_1} = c\sqrt{\mu_o\varepsilon_1}$, $\quad n_2 = c\sqrt{\mu_o\varepsilon_2}$,

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$T = \frac{P_{t,ave}}{P_{i,ave}} = \frac{\eta_1}{\eta_2}\frac{E_{to}^2}{E_{io}^2} = \frac{\eta_1}{\eta_2}\tau^2 = \frac{\eta_1}{\eta_2}(1+\Gamma)^2 = \frac{4n_1n_2}{(n_1+n_2)^2}$$

(b) If $P_{r,ave} = P_{t,ave} \longrightarrow RP_{i,ave} = TP_{i.ave} \longrightarrow R = T$

i.e. $(n_1 - n_2)^2 = 4n_1n_2 \quad \longrightarrow \quad n_1^2 - 6n_1n_2 + n_2^2 = 0$

or $\left(\dfrac{n_1}{n_2}\right)^2 - 6\left(\dfrac{n_1}{n_2}\right) + 1 = 0$, so

$$\frac{n_1}{n_2} = 3 \pm \sqrt{8} = \underline{\underline{5.828 \quad \text{or} \quad 0.1716}}$$

(Note that these values are mutual reciprocals, reflecting the inherent symmetry of the problem.)

Prob. 10.8 (a) $\omega = \beta c = 3 \times 3 \times 10^8 = \underline{\underline{9 \times 10^8 \text{ rad/s}}}$

(b) $\lambda = 2\pi / \beta = 2\pi / 3 = \underline{\underline{2.094 \text{ m}}}$

(c) $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{4}{9 \times 10^8 \times 80 \times 10^{-9} / 36\pi} = 2\pi = \underline{\underline{6.288}}$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 6.288 \quad \longrightarrow \quad \theta_\eta = 40.47^o$$

$$|\eta_2| = \frac{\sqrt{\mu_2/\varepsilon_2}}{\sqrt[4]{1+\left(\dfrac{\sigma_2}{\omega\varepsilon_2}\right)^2}} = \frac{377/\sqrt{80}}{\sqrt[4]{1+4\pi^2}} = 16.71$$

$$\eta_2 = \underline{\underline{16.71\angle 40.47^o \ \Omega}}$$

(d) $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \dfrac{16.71\angle 40.47^o - 377}{16.71\angle 40.47^o + 377} = 0.935\angle 179.7^o$

$$E_{or} = \Gamma E_{oi} = 9.35\angle 179.7^o$$

$$\underline{\underline{\mathbf{E}_r = 9.35\sin(\omega t - 3z + 179.7)\mathbf{a}_x \text{ V/m}}}$$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{\mu_{r2}\varepsilon_{r2}}{2}\left[\sqrt{1+\left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2} - 1\right]} = \frac{9\times 10^8}{3\times 10^8}\sqrt{\frac{80}{2}\left[\sqrt{1+4\pi^2} - 1\right]} = 43.94 \text{ Np/m}$$

$$\beta_2 = \frac{9\times 10^8}{3\times 10^8}\sqrt{\frac{80}{2}\left[\sqrt{1+4\pi^2} + 1\right]} = 51.48 \text{ rad/m}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\times 16.71\angle 40.47^o}{16.71\angle 40.47^o + 377} = 0.0857\angle 38.89^o$$

$$E_{ot} = \tau E_o = 0.857\angle 38.89^o$$

$$\underline{\underline{E_t = 0.857 e^{43.94z}\sin(9\times 10^8 t + 51.48z + 38.89^o) \text{ V/m}}}$$

Prob. 10.9
If **A** is a uniform vector and $\Phi(r)$ is a scalar,

$$\nabla \times (\Phi A) = \nabla\Phi \times A + \Phi(\nabla \times A) = \nabla\Phi \times A$$

since $\nabla \times A = 0$.

$$\nabla \times E = (\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z) \times E_o e^{j(k_x x + k_y y + k_z z - \omega t)} = j(k_x a_x + k_y a_y + k_z a_z)e^{j(k \bullet r - \omega t)} \times E_o$$

$$= jk \times E_o e^{j(k \bullet r - \omega t)} = jk \times E$$

Also, $-\frac{\partial B}{\partial t} = j\omega\mu H$. Hence $\nabla \times E = -\frac{\partial B}{\partial t}$ becomes $k \times E = \omega\mu H$

From this, $\underline{\underline{a_k \times a_E = a_H}}$

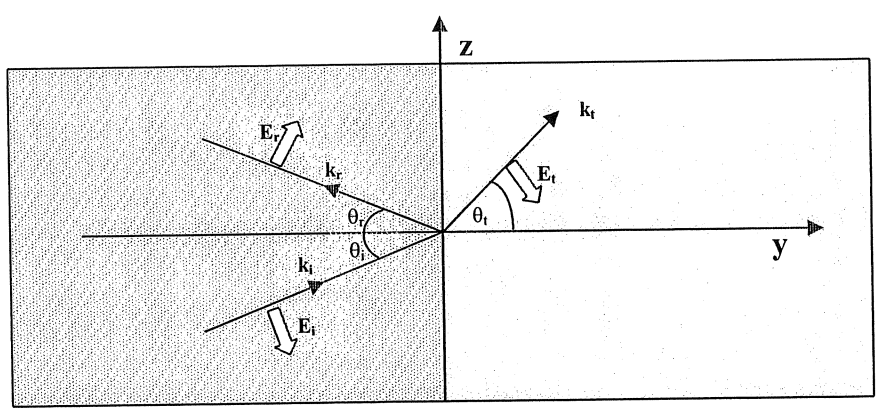
Prob. 10.10 (a) $k_i = 4a_y + 3a_z$

$k_i \bullet a_n = k_i \cos\theta_i \longrightarrow \cos\theta_i = 4/5 \longrightarrow \underline{\underline{\theta_i = 36.87^o}}$

(b)

$$P_{ave} = \frac{1}{2}\text{Re}(E_s \times H_s^*) = \frac{E_o^2}{2\eta}a_k = \frac{(\sqrt{8^2+6^2})^2}{2 \times 120\pi}\frac{(4a_y + 3a_z)}{5} = \underline{\underline{106.1a_y + 79.58a_z \text{ mW/m}^2}}$$

(c) $\theta_r = \theta_i = 36.87^o$. Let



From the figure, $\mathbf{k}_r = k_{rz}\mathbf{a}_z - k_{ry}\mathbf{a}_y$. But $k_r = k_i = 5$

$$k_{rz} = k_r \sin\theta_r = 5(3/5) = 3, \qquad k_{ry} = k_r \cos\theta_r = 5(4/5) = 4,$$

Hence, $\mathbf{k}_r = -4\mathbf{a}_y + 3\mathbf{a}_z$

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{c\sqrt{\mu_1\varepsilon_1}}{c\sqrt{\mu_2\varepsilon_2}}\sin\theta_i = \frac{3/5}{\sqrt{4}} = 0.3$$

$$\theta_t = 17.46, \quad \cos\theta_t = 0.9539, \qquad \eta_1 = \eta_o = 120\pi, \eta_2 = \eta_o/2 = 60\pi$$

$$\Gamma_{//} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\frac{\eta_o}{2}(0.9539) - \eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = -0.253$$

$$E_{ro} = \Gamma_{//}E_{io} = -0.253(10) = -2.53$$

But $(E_{ry}\mathbf{a}_y + E_{rz}\mathbf{a}_z) = E_{ro}(\sin\theta_r\mathbf{a}_y + \cos\theta_r\mathbf{a}_z) = -2.53(\frac{3}{5}\mathbf{a}_y + \frac{4}{5}\mathbf{a}_z)$

$$\mathbf{E}_r = -(1.518\mathbf{a}_y + 2.024\mathbf{a}_z)\sin(\omega t + 4y - 3z) \text{ V/m}$$

Similarly, let

$$\mathbf{E}_t = (E_{ty}\mathbf{a}_y + E_{tz}\mathbf{a}_z)\sin(\omega t - \mathbf{k}_t \bullet \mathbf{r})$$

$$k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \omega\sqrt{4\mu_o\varepsilon_o}$$

But $k_i = \beta_1 = \omega\sqrt{\mu_o\varepsilon_o}$

$$\frac{k_t}{k_i} = 2 \quad \longrightarrow \quad k_t = 2k_i = 10$$

$$k_{ty} = k_t\cos\theta_t = 9.539, \qquad k_{tz} = k_t\sin\theta_t = 3,$$

$$k_t = 9.539a_y + 3a_z$$

Note that $k_{iz} = k_{rz} = k_{tz} = 3$

$$\tau_{\backslash\backslash} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = \frac{\eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = 0.6265$$

$$E_{to} = \tau_{\backslash\backslash} E_{io} = 6.265$$

But

$$(E_{ty}\boldsymbol{a}_y + E_{tz}\boldsymbol{a}_z) = E_{to}(\sin\theta_t \boldsymbol{a}_y - \cos\theta_t \boldsymbol{a}_z) = 6.256(0.3\boldsymbol{a}_y - 0.9539\boldsymbol{a}_z)$$

Hence,

$$\boldsymbol{E}_t = (1.879\boldsymbol{a}_y - 5.968\boldsymbol{a}_z)\sin(\omega t - 9.539y - 3z) \text{ V/m}$$

Prob. 10.11

$$\beta_1 = \sqrt{3^2 + 4^2} = 5 = \omega / c \longrightarrow \omega = \beta_1 c = 15\times 10^8 \text{ rad/s}$$

Let $\boldsymbol{E}_r = (E_{ox}, E_{oy}, E_{oz})\sin(\omega t + 3x + 4y)$. In order for

$$\nabla \bullet \boldsymbol{E}_r = 0, \qquad 3E_{ox} + 4E_{oy} = 0 \qquad (1)$$

Also, at y=0, $E_{1\tan} = E_{2\tan} = 0$

$$E_{1\tan} = 0, \qquad 8a_x + 5a_z + E_{ox}a_x + E_{oz}a_z = 0$$

Equating components, $E_{ox} = -8$, $E_{oz} = -5$

From (1), $4E_{oy} = -3E_{ox} = 24$ $\quad E_{oy} = 6$

Hence,

$$\boldsymbol{E}_r = (-8\boldsymbol{a}_x + 6\boldsymbol{a}_y - 5\boldsymbol{a}_z)\sin(15\times 10^8 t + 3x + 4y) \text{ V/m}$$

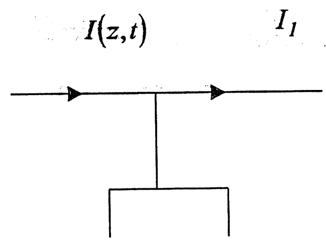
CHAPTER 11

Prob. 11.1

(a) Applying Kirchhoff's voltage law to the loop yields

$$V(z+\Delta z,t)=V(z,t)-R\Delta z I_1 - L\Delta z\frac{\partial I_1}{\partial t}$$

But $I_1 = I(z,t)-\frac{C}{2}\Delta z\frac{\partial V(z,t)}{\partial t}-\frac{G}{2}\Delta z V(z,t)$



Hence,

$$V(z+\Delta z,t)= V(z,t)- R\Delta z\left[I(z,t)-\frac{C}{2}\Delta z\frac{\partial V}{\partial t}-\frac{G}{2}\Delta z V\right]- L\Delta z\left[\frac{\partial I}{\partial t}-\frac{C}{2}\Delta z\frac{\partial^2 V}{\partial t^2}-\frac{G}{2}\Delta z\frac{\partial V}{\partial t}\right]$$

Dividing by Δz and taking limits as $\Delta t\to 0$ give

$$\lim_{\Delta z\to 0}\frac{V(z+\Delta z,t)-V(z,t)}{\Delta z}=\lim_{\Delta z\to 0}\left[-RI-L\frac{\partial I}{\partial t}+\frac{RC}{2}\Delta z\frac{\partial V}{\partial t}+\frac{RG}{2}\Delta z V+\frac{LC}{2}\Delta z\frac{\partial^2 V}{\partial t^2}+\frac{LG}{2}\Delta z\frac{\partial V}{\partial t}\right]$$

or $-\frac{\partial V}{\partial z}=RI+L\frac{\partial I}{\partial t}$

Similarly, applying Kirchhoff's law to the node leads to

$$I(z+\Delta z,t)-I(z,t)=-G\Delta z\left(\frac{V(z,t)+V(z+\Delta z)}{2}\right)-C\Delta z\frac{\partial}{\partial t}\left(\frac{V(z,t)+V(z+\Delta z,t)}{2}\right)$$

Let $\Delta z\longrightarrow 0$, we get

$$-\frac{\partial I}{\partial z}=GV+C\frac{\partial V}{\partial t}$$

(b) Applying Kirchhoff's voltage law,

$$V(z,t)= R\frac{\Delta l}{2}I(z,t)+ L\frac{\Delta l}{2}\frac{\partial I}{\partial t}(z,t)+ V(z+\Delta l/2,t)$$

or

$$-\frac{V(z+\Delta l/2,t)-V(z,t)}{\Delta l/2}= RI+ L\frac{\partial I}{\partial t}$$

As $\Delta l \to 0, \quad -\dfrac{\partial V}{\partial z} = RI + L\dfrac{\partial I}{\partial t}$

Here, we take $\Delta l = \Delta z$. Applying Kirchhoff's current law,

$$I(z,t) = I(z+\Delta l,t) + G\Delta l V(z+\Delta l/2,t) + C\Delta l \frac{\partial V\left(z+\Delta l/2,t\right)}{\partial t}$$

or

$$-\frac{I(z+\Delta l,t) - I(z,t)}{\Delta l} = GV(z+\Delta l/2,t) + C\frac{\partial V(z+\Delta l/2,t)}{\partial t}$$

As $\Delta l \to 0, \quad -\dfrac{\partial I(z,t)}{\partial z} = GV(z,t) + C\dfrac{\partial V(z,t)}{\partial t}$

Prob.11.2

(a)

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = j\omega\sqrt{LC}\sqrt{(1+\frac{R}{j\omega L})(1+\frac{G}{j\omega C})}$$

$$= j\omega\sqrt{LC}\sqrt{1-\frac{RG}{\omega^2 LC}+\frac{R}{j\omega L}+\frac{G}{j\omega C}}$$

As R<< ωL and G<<ωC, dropping the ω^2 term gives

$$\gamma \cong j\omega\sqrt{LC}\sqrt{1+\frac{R}{j\omega L}+\frac{G}{j\omega C}} \cong j\omega\sqrt{LC}\left[1+\frac{R}{2j\omega L}+\frac{G}{2j\omega C}\right]$$

$$= \underline{\underline{\left(\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}\right)+j\omega\sqrt{LC}}}$$

(b)

$$Z_o = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{L}{C}}\sqrt{\frac{1+\frac{R}{j\omega L}}{1+\frac{G}{j\omega C}}} = \sqrt{\frac{L}{C}}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{-1/2}$$

$$\cong \sqrt{\frac{L}{C}}\left(1+\frac{R}{2j\omega L}+\ldots\right)\left(1-\frac{G}{j2\omega C}+\ldots\right) = \sqrt{\frac{L}{C}}\left(1-j\frac{R}{2\omega L}+j\frac{G}{2\omega C}+\ldots\right)$$

$$\cong \underline{\underline{\sqrt{\frac{L}{C}}\left[1+j\left(\frac{G}{2\omega C}-\frac{R}{2\omega L}\right)\right]}}$$

Prob. 11.3

(a) $\frac{R}{L} = \frac{G}{C} \rightarrow G = \frac{R}{L}C = \frac{20 \times 63 \times 10^{-12}}{0.3 \times 10^{-6}}$

$G = 4.2 \times 10^{-3}$ S/m

$\alpha = \sqrt{RG} = \sqrt{20 \times 4.2 \times 10^{-3}} = 0.2898$

$\beta = \omega\sqrt{LC} = 2\pi \times 120 \times 10^{6}\sqrt{0.3 \times 10^{-6} \times 63 \times 10^{-12}} = 3.278$

$\underline{\underline{\gamma = 0.2898 + j3.278 \ \ /\text{m}}}$

$$u = \frac{\omega}{\beta} = \frac{2\pi x120x10^{6}}{3.278} = \underline{\underline{2.3x10^{8}\ \text{m/s}}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.3x10^{-6}}{63x10^{-12}}} = \underline{\underline{69\ \Omega}}$$

(b) Let V_o be its original magnitude

$V_o e^{-\alpha z} = 0.2V_o \rightarrow e^{\alpha z} = 5$

$z = \frac{1}{\alpha}\ln 5 = \underline{\underline{5.554\ \text{m}}}$

(c) $\beta l = 45^{o} = \pi/4 \rightarrow l = \frac{\pi}{4\beta} = \frac{4}{4 \times 3.278}$

$$\underline{\underline{l = 0.2396\ \text{m}}}$$

Prob. 11.4

(a) $$T_L = \frac{V_L}{V_o^+} = \frac{Z_L I_L}{\frac{1}{2}(V_L + Z_o I_L)} = \frac{2Z_L I_L}{Z_L I_L + Z_o I_L}$$

$$= \frac{2Z_L}{Z_L + Z_o}$$

$$1 + \Gamma_L = 1 + \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{2Z_L}{Z_L + Z_o}$$

(b) (i) $$\tau_L = \frac{2nZ_o}{nZ_o + Z_o} = \frac{2n}{n+1}$$

(ii) $$\tau_L = _{Y_L} \xrightarrow{\lim} 0 = \frac{2}{1 + Z_o/Z_L} = 2$$

(iii) $$\tau_L = _{Z_L} \xrightarrow{\lim} 0 = \frac{2Z_L}{Z_L + Z_o} = 0$$

(iv) $$\tau_L = \frac{2Z_o}{2Z_o} = 1$$

Prob. 11.5

From eq. (11.33)

$$Z_{sc} = Z_{in}\Big|_{Z_L=0} = Z_o \tanh \gamma l$$

$$Z_{oc} = Z_{in}\Big|_{Z_L=\infty} = \frac{Z_o}{\tanh \gamma l} = Z_o \coth(\gamma l)$$

For lossless line, $\gamma = j\beta, \tan(\gamma l) = \tanh(j\beta l) = j\tan(\beta l)$

$$Z_{sc} = jZ_o \tan(\beta l), Z_{oc} = -jZ_o \cot(\beta l)$$

Prob. 11.6

$$V_1 = V_s(z=0) = V_o^+ + V_o^- \qquad (1)$$

$$V_2 = V_s(z=l) = V_o^+ e^{-\gamma l} + V_o^- e^{\gamma l} \qquad (2)$$

$$I_1 = I_s(z=0) = \frac{V_o^+}{Z_o} - \frac{V_o^-}{Z_o} \qquad (3)$$

$$I_2 = -I_s(z=l) = -\frac{V_o^+}{Z_o} e^{-\gamma l} + \frac{V_o^-}{Z_o} e^{\gamma l} \qquad (4)$$

$$(1) + (3) \rightarrow V_o^+ = \frac{1}{2}(V_1 + Z_o I_1)$$

$$(1) - (3) \rightarrow V_o^- = \frac{1}{2}(V_1 - Z_o I_1)$$

Substituting V_o^+ and V_o^- in (2) gives

$$V_2 = \frac{1}{2}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2}(V_1 - Z_o I_1)e^{\gamma l}$$

$$= \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})V_1 + \frac{1}{2}Z_o(e^{-\gamma l} - e^{\gamma l})I_1$$

$$V_2 = \cosh \gamma l V_1 - Z_o \sinh \gamma l I_1 \qquad (5)$$

Substituting V_o^+ and V_o^- in (4),

$$I_2 = -\frac{1}{2Z_o}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2Z_o}(V_1 - Z_o I_1)e^{\gamma l}$$

$$= \frac{1}{2Z_o}(e^{\gamma l} - e^{-\gamma l})V_1 + \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})I_1$$

$$I_2 = -\frac{1}{Z_o}\sinh \gamma l V_1 - \cosh \gamma l I_1 \qquad (6)$$

From (5) and (6)

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} \cosh \gamma l & -Z_o \sinh \gamma l \\ -\frac{1}{Z_o}\sinh \gamma l & -\cosh \gamma l \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

But

$$\begin{bmatrix} \cosh \gamma l & -Z_o \sinh \gamma l \\ -\frac{1}{Z_o}\sinh \gamma l & -\cosh \gamma l \end{bmatrix}^{-1} = \begin{bmatrix} \cosh \gamma l & Z_o \sinh \gamma l \\ \frac{1}{Z_o}\sinh \gamma l & \cosh \gamma l \end{bmatrix}$$

Thus

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh \gamma l & Z_o \sinh \gamma l \\ \frac{1}{Z_o}\sinh \gamma l & \cosh \gamma l \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Prob. 11.7

Using the Smith chart, $z_L = \dfrac{60-j40}{75} = 0.8 - j0.533$

$$l = \frac{3}{4}\lambda \longrightarrow \frac{3}{4}x720^o = 540^o$$

At C, $Z_{in} = 75(0.8654 + j0.5769) = 65 + j43\ \Omega$

$$z_{in} = \frac{65+j43}{100} = 0.65 + j0.43$$

$$\frac{\lambda}{2} \longrightarrow \frac{720^o}{2} = 360^o$$

At B, $Z_{in} = 65 + j43$

$$z_{in} = \frac{65+j43}{50} = 1.2981 + 0.8654$$

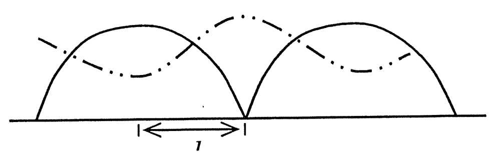
$$\frac{\lambda}{4} \longrightarrow \frac{720^o}{4} = 180^o$$

At A,

$$Z_{in} = 50(0.53 - j0.35) = \underline{\underline{26.7 - j17.8\ \Omega}}$$

Prob. 11.8

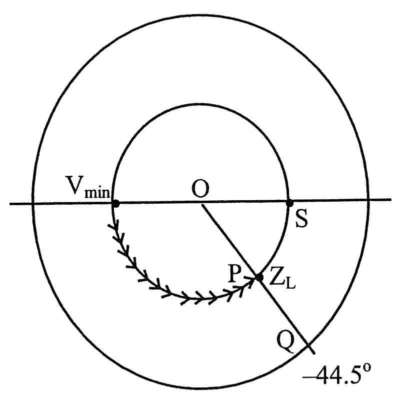
$$s = \frac{V_{max}}{V_{min}} = \frac{0.95}{0.45} = \underline{\underline{2.11}}$$



$$\frac{\lambda}{2} = 22.5 - 14 = 8.5 \quad \rightarrow \quad \lambda = 17 \text{ cm}$$

$$f = \frac{c}{\lambda} = \frac{3\times10^8}{0.17} = \underline{\underline{1.764 \text{ GHz}}}$$

$$l = 3.2 \text{ cm} = \frac{3.2}{17}\lambda \rightarrow 135.5^o$$



At P, $z_L = 1.4 - j0.8$

$$Z_L = 50(1.4 - j0.8) = \underline{\underline{70 - j40\Omega}}$$

(Exact value = 70.606-j40.496 Ω)

$$|\Gamma| = \frac{s-1}{s+1} = \frac{1.11}{3.11} = 0.357, \quad \theta_\Gamma = -44.5^o$$

$$\Gamma = \underline{\underline{0.357\angle -44.5^o}}$$

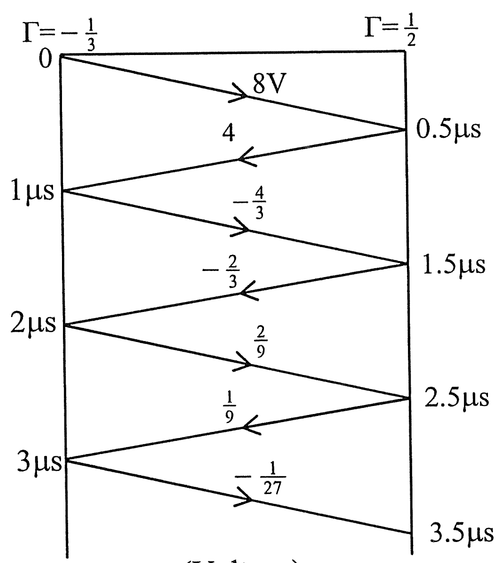
(Exact value = $0.3571\angle\text{-}44.471^o$)

Prob. 11.9

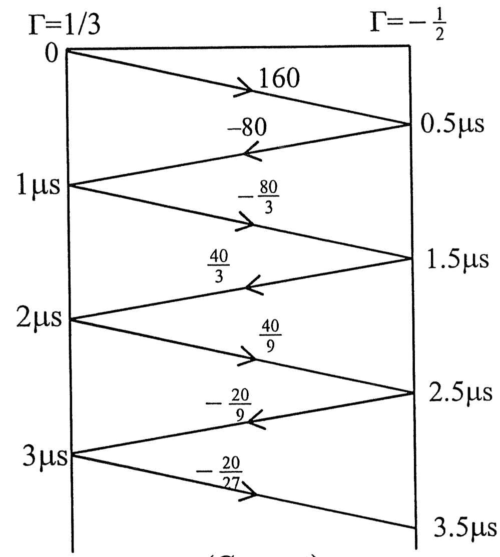
(*a*) $t_1 = \frac{l}{u} = \frac{150}{3\times10^8} = 0.5\mu s,$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{150-50}{150+150} = \frac{1}{2}, \quad \Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{25-50}{75} = -\frac{1}{3},$$

$$V_o = \frac{Z_o V_g}{Z_o + Z_g} = \frac{50(12)}{75} = 8V, \quad I_o = \frac{V_g}{Z_g + Z_o} = \frac{12}{75} = 160\,\text{mA}$$



(Voltage)



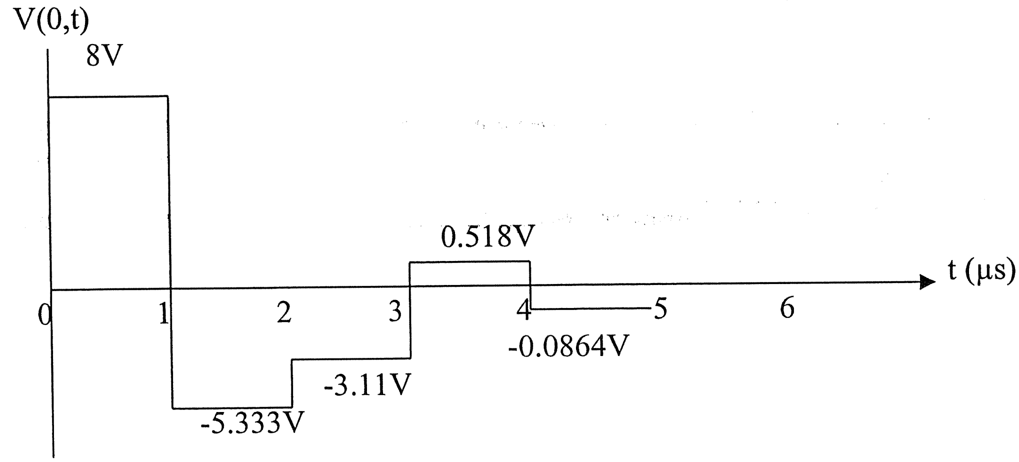
(Current)

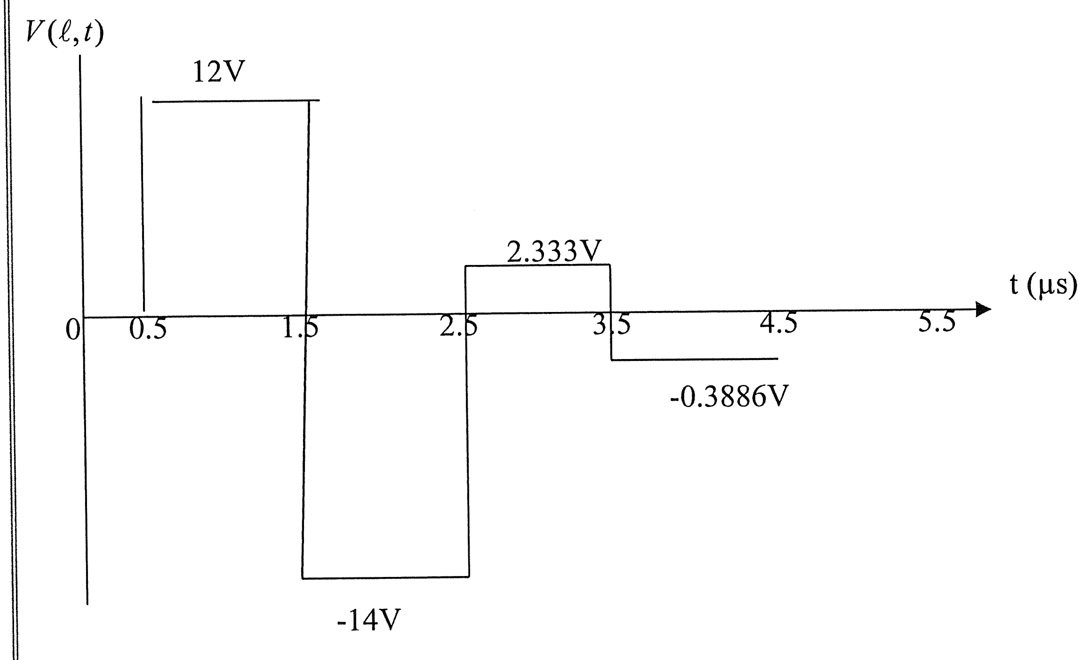
The bounce diagrams are for the leading pulse. The bounce diagrams for the second pulse is delayed by 1μs and negated because of -12V.

(b) For each time interval, we add the contributions of the two pulses together.
For $0 < t < 1\mu s$, V(0,t) = 8V
For $1 < t < 2\mu s$, V(0,t) = -8 + 4 – 4/3 = -5.331V
For $2 < t < 3\mu s$, V(0,t) = -(4-4/3)-2/3 + 2/9 = -2.667 -0.444 = -3.11V
For $3 < t < 4\mu s$, V(0,t) = 0.444 + 1/9 -1/27 = 0.444+0.0741 = 0.518V
For $4 < t < 5\mu s$, V(0,t)= -0.0741 – 0.0124 = -0.0864V

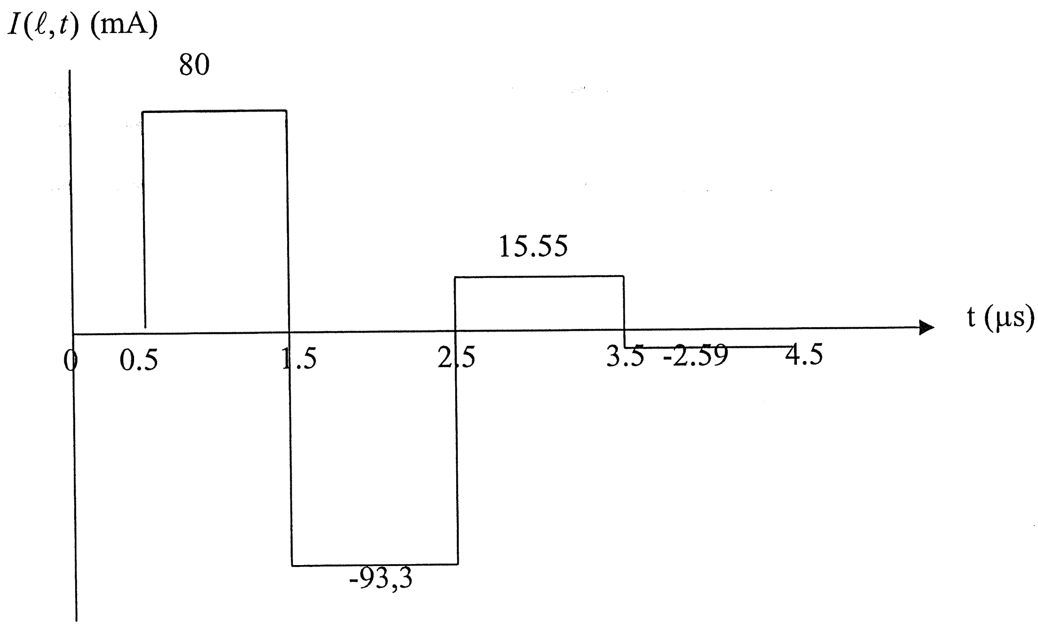
We do the same thing at the load end.
For $0 < t < 0.5\,\mu s$, $\quad V(\ell,t) = 0$
For $0.5 < t < 1.5\mu s$, $V(\ell,t)$=8+4 = 12
For $1.5 < t < 2.5\mu s$, $V(\ell,t)$=-12 + (-4/3 -2/3)=-12-2=-14
For $2.5 < t < 3.5\mu s$, $V(\ell,t)$=2+2/9 +1/9=2.333
For $3.5 < t < 4.5\mu s$, $V(\ell,t)$=-0.333-1/27-1/54=-0.3886V

The results are shown below.





Since $I(\ell,t) = \dfrac{V(\ell,t)}{Z_L} = \dfrac{V(\ell,t)}{150}$, we scale $V(\ell,t)$ by a factor of 1/150 as shown below.



Prob. 11.10

For w = 0.4 mm, $\frac{w}{h} = \frac{0.4\,\text{mm}}{2\,\text{m}} = 0.2 \rightarrow$ narrow strip

For $\frac{w}{h} = 0.2, \quad \varepsilon_{\text{eff}} = 5.851, \quad Z_o = 91.53\Omega$

For $\frac{w}{h} = 0.4, \quad \varepsilon_{\text{eff}} = 6.072, \quad Z_o = 73.24\Omega$

Hence,

$$\underline{\underline{73.24\Omega < Z_o < 91.53\Omega}}$$

CHAPTER 12

Prob. 12.1 a/b = 3 $\longrightarrow$ a = 3b

$$f_{c10} = \frac{u'}{2a} \longrightarrow a = \frac{u'}{2f_{c10}} = \frac{3x10^8}{2x18x10^9} \text{ m} = 0.833\text{cm}$$

A design could be <u>a = 9mm, b = 3mm.</u>

Prob. 12.2

$$u = \frac{\omega}{\beta} = \frac{u'}{\sqrt{1-(f_c/f)^2}} = \frac{3\times10^8}{\sqrt{1-(6.5/7.2)^2}} = 6.975\times10^8 \text{ m/s}$$

$$u_g = \frac{9\times10^{16}}{u} = \underline{\underline{1.2903\times10^8 \text{ m/s}}}$$

$$t = \frac{2l}{u_g} = \frac{300}{1.2903\times10^8} = \underline{\underline{2.325}}\ \mu\text{s}$$

Prob.12.3

In evanescent mode,

$$k^2 = \omega^2\mu\varepsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\beta = 0, \quad \gamma = \alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} = \sqrt{4\pi^2\mu\varepsilon f_c^2 - \omega^2\mu\varepsilon}$$

$$\alpha = \sqrt{\mu\varepsilon}\sqrt{4\pi^2 f_c^2 - 4\pi^2 f^2} = \underline{\underline{2\pi\sqrt{\mu\varepsilon} f_c\sqrt{1-\left(\frac{f}{f_c}\right)^2}}}$$

Prob. 12.4 (a)

$$f_c = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2}, \; \beta = \beta'\sqrt{1-(f_c/f)^2}$$

$$u = \omega/\beta = \frac{u'}{\sqrt{1-(f_c/f)^2}}, \quad \lambda = 2\pi/\beta = \frac{\lambda'}{\sqrt{1-(f_c/f)^2}}$$

(b) If a = 2b = 2.5cm, $f_c = \frac{u'}{2a}\sqrt{m^2+4n^2}$. For TE_{11},

$$f_c = \frac{3\times10^8}{2\times2.5\times10^{-2}}\sqrt{1+4} = 13.42 \text{ GHz}, \quad u = \frac{3\times10^8}{\sqrt{1-(13.42/20)^2}} = \underline{\underline{4.06\times10^8}} \text{ m/s}$$

$$\lambda = u/f = \frac{4.046\times10^8}{200\times10^8} = \underline{\underline{2.023}} \text{ cm}$$

For TE_{21},

$$f_c = \frac{3\times10^8}{2\times2.5\times10^{-2}}\sqrt{4+4} = 16.97 \text{ GHz}, \quad u = \frac{3\times10^8}{\sqrt{1-(16.97/20)^2}} = \underline{\underline{5.669\times10^8}} \text{ m/s}$$

$$\lambda = u/f = \frac{5.669\times10^8}{200\times10^8} = \underline{\underline{2.834}} \text{ cm}$$

Prob. 12.5 Substituting $E_z = R\Phi Z$ into the wave equation,

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}(\rho R') + \frac{RZ}{\rho^2}\Phi'' + R\Phi Z'' + k^2 R\Phi Z = 0$$

Dividing by $R\Phi Z$,

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho R') + \frac{\Phi''}{\Phi\rho^2} + k^2 = -\frac{Z''}{Z} = -k_z^2$$

i.e. $\underline{\underline{Z'' - k_z^2 Z = 0}}$

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho R') + \frac{\Phi''}{\Phi\rho^2} + (k^2 + k_z^2) = 0$$

$$\frac{\rho}{R}\frac{d}{d\rho}(\rho R') + (k^2 + k_z^2)\rho^2 = -\frac{\Phi''}{\Phi} = k_\phi^2$$

or

$$\Phi'' + k_\phi^2 \Phi = 0$$

$\rho \frac{d}{d\rho}(\rho R') + (k_\rho^2 \rho^2 - k_\phi^2)R = 0$, where $k_\rho^2 = k^2 + k_z^2$. Hence

$$\rho^2 R'' + \rho R' + (k_\rho^2 \rho^2 - k_\phi^2)R = 0$$

Prob. 12.6

$$f_c = \frac{3x10^8}{2}\sqrt{(m/0.025)^2 + (n/0.01)^2} = 15\sqrt{n^2 + (m/2.5)^2} \quad \text{GHz}$$

$f_{c10} = 6$ GHz, $f_{c20} = 12$ GHz, $f_{c01} = 15$ GHz.
Since f_{c20}, $f_{c10} > 11$ GHz, only the dominant TE_{10} mode is propagated.

(a) $\frac{u_p}{u} = \frac{1}{\sqrt{1-(f_c/f)^2}} = \frac{1}{\sqrt{1-(6/11)^2}} = \underline{\underline{1.193}}$

(b) $\frac{u_g}{u} = \sqrt{1-(6/11)^2} = \underline{\underline{0.8381}}$

Prob. 12.7

$$P_{ave} = \frac{1}{2\eta}\int_{y=0}^{b}\int_{0}^{a}(|E_{xs}|^2 + |E_{ys}|^2)dxdy$$

But

$$E_{xs} = \frac{-j\beta}{h^2}(\pi/a)E_o\cos(\pi x/a)\sin(\pi y/b)e^{-j\beta z}$$

$$E_{ys} = \frac{-j\beta}{h^2}(\pi/b)E_o\sin(\pi x/a)\cos(\pi y/b)e^{-j\beta z}$$

$$P_{ave} = \frac{1}{2\eta_{TM11}}\frac{\beta^2\pi^2}{h^4}E_o^2[\frac{1}{a^2}\int_0^a\cos^2(\pi x/a)dx\int_0^b\sin^2(\pi x/b)dy + \frac{1}{b^2}\int_0^a\sin^2(\pi x/a)dx\int_0^b\cos^2(\pi x/b)dy]$$

$$= \frac{1}{2\eta_{TM11}}\frac{\beta^2\pi^2}{h^4}E_o^2[\frac{1}{a^2} + \frac{1}{b^2}](a/2)(b/2)$$

Note that $h^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} = \frac{a^2 + b^2}{a^2 b^2}\pi^2$

$$P_{ave} = \frac{\beta^2 E_o^2}{8\pi^2 \eta_{TM11}} \frac{a^3 b^3}{a^2 + b^2}$$

Prob. 12.8 (a) For TE_{10} mode,

$$f_c = \frac{u'}{2a}, \qquad u' = \frac{c}{\sqrt{2.11}}$$

$$f_c = \frac{3\times10^8}{\sqrt{2.11}(2\times 2.25\times10^{-2})} = \underline{\underline{4.589 \text{ GHz}}}$$

(b) $$\alpha_{cTE10} = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2} + \frac{b}{a}(f_c/f)^2\right]$$

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi\times5\times10^9\times4\pi\times10^{-7}}{1.37\times10^7}} = 3.796\times10^{-2}$$

$$\eta' = \frac{377}{\sqrt{2.11}} = 259.54$$

$$\alpha_c = \frac{2\times3.796\times10^{-2}[0.5 + \frac{1.5}{2.25}(4.589/5)^2]}{1.5\times10^{-4}(259.54)\sqrt{1-(4.589/5)^2}} = \underline{\underline{0.05217}} \text{ Np/m}$$

Prob. 12.9 For TE_{10} mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

But a = b, $R_s = \frac{1}{\sigma_c\delta} = \sqrt{\frac{\pi f\mu}{\sigma_c}}$

$$\alpha_c = \frac{2\sqrt{\frac{\pi f\mu}{\sigma_c}}}{a\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\left(\frac{f_c}{f}\right)^2\right] = \frac{k\sqrt{f}\left[\frac{1}{2}+\left(\frac{f_c}{f}\right)^2\right]}{\sqrt{1-(f_c/f)^2}}$$

where k is a constant.

$$\frac{d\alpha_c}{df} = \frac{k[1-(\frac{f_c}{f})^2]^{1/2}[\frac{1}{4}f^{-1/2}-\frac{3}{2}f_c^2 f^{-5/2}] - \frac{k}{2}[\frac{1}{2}f^{1/2}+f_c^2 f^{-3/2}](2f_c^2 f^{-3})[1-(\frac{f_c}{f})^2]^{-1/2}}{1-(f_c/f)^2}$$

For minimum value, $\frac{d\alpha_c}{df} = 0$. This leads to $\underline{f = 2.962\ f_c}$.

Prob. 12.10 For the TE mode to z,

$$E_{zs} = 0, H_{zs} = H_o\cos(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/c)$$

$$E_{ys} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} = \underline{\underline{-\frac{j\omega\mu}{h^2}(m\pi/a)H_o\sin(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/c)}}$$

as required.

$$E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} = \frac{j\omega\mu}{h^2}(n\pi/b)H_o\cos(m\pi x/a)\sin(n\pi y/b)\sin(p\pi z/c)$$

From Maxwell's equation,

$$-j\omega\mu\mathbf{H}_s = \nabla\times\mathbf{E}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs} & E_{ys} & 0 \end{vmatrix}$$

$$H_{xs} = \frac{1}{j\omega\mu}\frac{\partial E_{ys}}{\partial z} = \underline{\underline{-\frac{1}{h^2}(m\pi/a)(p\pi/c)H_o\sin(m\pi x/a)\cos(n\pi y/b)\cos(p\pi z/c)}}$$

Prob. 12.11 Maxwell's equation can be written as

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x}$$

For a rectangular cavity,

$$h^2 = k_x^{\ 2} + k_y^{\ 2} = (m\pi / a)^2 + (n\pi / b)^2$$

For TM mode, $H_{zs} = 0$ and

$$E_{zs} = E_o \sin(m\pi x / a)\sin(n\pi y / b)\cos(p\pi z / c)$$

Thus

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} = \underline{\underline{\frac{j\omega\varepsilon}{h^2}(n\pi / b)E_o \sin(m\pi x / a)\cos(n\pi y / b)\cos(p\pi z / c)}}$$

as required.

$$H_{xs} = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y}$$

$$= -\frac{j\omega\varepsilon}{h^2}(m\pi / a)E_o \cos(m\pi x / a)\sin(n\pi y / b)\cos(p\pi z / c)$$

From Maxwell's equation,

$$j\omega\varepsilon \mathbf{E}_s = \nabla \times \mathbf{H}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_{xs} & H_{ys} & 0 \end{vmatrix}$$

$$E_{ys} = \frac{1}{j\omega\varepsilon}\frac{\partial H_{xs}}{\partial z} = \underline{\underline{\frac{1}{h^2}(n\pi / b)(p\pi / c)E_o \sin(m\pi x / a)\cos(n\pi y / b)\cos(p\pi z / c)}}$$

CHAPTER 13

Prob. 13.1

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \longrightarrow j\omega\varepsilon \boldsymbol{E}_s = \nabla \times \boldsymbol{H}_s$$

$$\boldsymbol{E}_s = \frac{1}{j\omega\varepsilon} \nabla \times \boldsymbol{H}_s$$

But $\boldsymbol{B}_s = \mu \boldsymbol{H}_s = \nabla \times \boldsymbol{A}_s \longrightarrow \boldsymbol{H}_s = \frac{1}{\mu} \nabla \times \boldsymbol{A}_s$

$$\boldsymbol{E}_s = \frac{1}{j\omega\mu\varepsilon} \nabla \times \nabla \times \boldsymbol{A}_s = \frac{1}{j\omega\mu\varepsilon} \left[\nabla(\nabla \cdot \boldsymbol{A}_s) - \nabla^2 \boldsymbol{A}_s \right]$$

But $\nabla^2 \boldsymbol{A}_s + \omega^2 \mu\varepsilon \boldsymbol{A}_s = -\mu \boldsymbol{J}_s = 0 \longrightarrow \nabla^2 \boldsymbol{A}_s = -\omega^2 \mu\varepsilon \boldsymbol{A}_s$

$$\boldsymbol{E}_s = \frac{1}{j\omega\mu\varepsilon} \omega^2 \mu\varepsilon \boldsymbol{A}_s + \frac{\nabla(\nabla \cdot \boldsymbol{A}_s)}{j\omega\mu\varepsilon} = -j\omega \boldsymbol{A}_s + \frac{\nabla(\nabla \cdot \boldsymbol{A}_s)}{j\omega\mu\varepsilon}$$

as required.

Prob. 13.2

(a) $$A_{zs} = \frac{e^{-j\beta r}}{4\pi r} \int_{-l/2}^{l/2} I_o (1 - \frac{2|z|}{l}) e^{j\beta z \cos\theta} \partial z$$

$$= \frac{e^{-j\beta r}}{4\pi r} I_o \left[\int_{-l/2}^{l/2} (1 - \frac{2|z|}{l}) \cos(\beta z \cos\theta) dz + j \int_{-l/2}^{l/2} (1 - \frac{2|z|}{l}) \sin(\beta z \cos\theta) dz \right]$$

$$= \frac{e^{-j\beta r}}{4\pi r} 2 I_o \int_0^{l/2} (1 - \frac{2z}{l}) \cos(\beta z \cos\theta) dz$$

$$= \frac{I_o e^{-j\beta r}}{2\pi r \beta^2 \cos^2\theta} \cdot \frac{2}{l} \left[1 - \cos(\beta l/2 \cos\theta) \right]$$

$$\boldsymbol{E}_s = -j\omega\mu \boldsymbol{A}_s \longrightarrow E_{\theta s} = j\omega\mu \sin\theta A_{zs} = j\beta\eta \sin\theta A_{zs}$$

$$E_{\theta s} = \frac{j\eta I_o e^{-j\beta r}}{\pi r l} \frac{\sin\theta \left[1 - \cos(\beta l/2 \cos\theta) \right]}{\beta \cos^2\theta}$$

If $\beta l/2 \ll 1$, $\cos(\frac{\beta l}{2}\cos\theta) = 1 - \frac{(\frac{\beta l}{2}\cos\theta)^2}{2!}$. Hence

$$E_{\theta s} = \frac{j\eta I_o}{8\pi r}\beta l e^{-j\beta r}\sin\theta \ , \quad H_{\phi s} = E_{\theta s}/\eta$$

$$P_{ave} = \frac{|E_{\theta s}|^2}{2\eta}, \quad P_{rad} = \int P_{ave} dS$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} \frac{n}{2}\left(\frac{I_o\beta l}{8\pi}\right)^2 \frac{1}{r^2}\sin^2\theta r^2 \sin\theta d\theta d\phi$$

$$= 10\pi^2 I_o^2\left(\frac{l}{\lambda}\right)^2 = \frac{1}{2} I_o^2 R_{rad}$$

$$\text{or} \quad R_{rad} = 20\pi^2\left(\frac{l}{\lambda}\right)^2$$

(b) $0.5 = 20\pi^2\left(\frac{l}{\lambda}\right)^2 \longrightarrow l = 0.05\lambda$

Prob. 13.3

Change the limits in Eq. (13.16) to $\pm l/2$ i.e.

$$A_s = \frac{\mu I_o e^{j\beta z Cos\theta}}{4\pi r} \frac{(j\beta\cos\theta\cos\beta t + \beta\sin\beta t)}{-\beta^2\cos^2\theta + \beta^2}\Bigg|_{-l/2}^{l/2}$$

$$= \frac{\mu I_o e^{j\beta r}}{2\pi r}\frac{1}{\beta\sin^2\theta}\left[\sin\frac{\beta l}{2}\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta\cos\frac{\beta l}{2}\sin\left(\frac{\beta l}{2}\cos\theta\right)\right]$$

But $\boldsymbol{B} = \mu\boldsymbol{H} = \nabla \times \boldsymbol{A}$

$$H_{\phi s} = \frac{1}{\mu r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right],$$

where $A_o = -A_z\sin\theta, A_r = A_z\cos\theta$

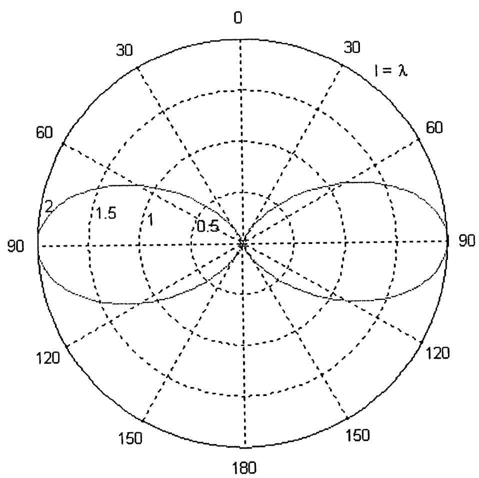
$$H_{\phi s} = \frac{I_o}{2\pi r}\frac{e^{-j\beta r}}{\beta}\left(\frac{j\beta}{\sin\theta}\right)\left[\sin\frac{\beta l}{2}\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta\cos\frac{\beta l}{2}\sin\left(\frac{\beta l}{2}\cos\theta\right)\right] + \frac{I_o}{2\pi r^2}e^{-j\beta r}(......)$$

For far field, only the $\frac{1}{r}$-term remains. Hence

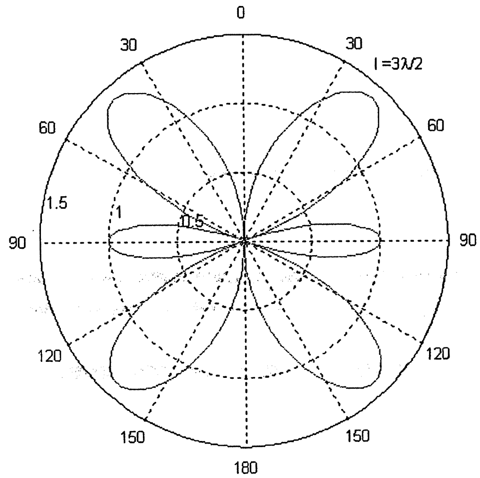
$$H_{\phi s} = \frac{jI_o}{2\pi r} e^{-j\beta r} \frac{\left[\sin\frac{\beta l}{2} \cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta \cos\frac{\beta l}{2} \sin\left(\frac{\beta l}{2}\cos\theta\right) \right]}{\sin\theta}$$

(b) $$f(\theta) = \frac{\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\frac{\beta l}{2}}{\sin\theta}$$

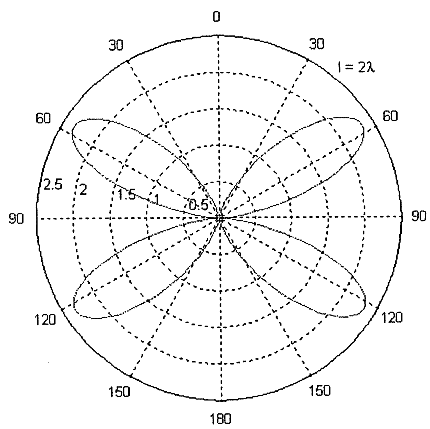
For $l = \lambda, f(\theta) = \dfrac{\cos(\pi\cos\theta) + 1}{\sin\theta}$



For $l = \dfrac{3\lambda}{2}, f(\theta) = \dfrac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}$



For $l = 2\lambda,\ f(\theta) = \dfrac{\cos\theta \sin(2\pi\cos\theta)}{\sin\theta}$



Prob. 13.4

(a) $P_{rad} = \int P_{rad} . \partial s = P_{ave} . 2\pi r^2$ (hemisphere)

$$P_{ave} = \frac{P_{rad}}{2\pi r^2} = \frac{200\times10^3}{2\pi(2500\times10^6)} = 12.73\mu W / m^2$$

$$\underline{\underline{\boldsymbol{P}_{ave} = 12.73\boldsymbol{a}_r\ \mu\text{W/m}^2}}\ .$$

(b) $P_{ave} = \dfrac{(E_{\max})^2}{2\eta}$

$$E_{\max} = \sqrt{2\eta P_{ave}} = \sqrt{240\pi \times 12.73 \times 10^{-6}}$$

$$= \underline{\underline{0.098\ \text{V/m}}}$$

Prob. 13.5

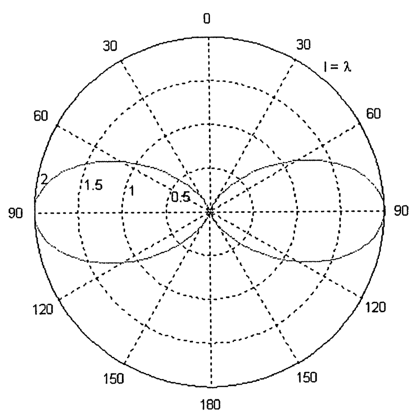
From Prob. 13.10, $E_{\theta s} = \dfrac{j\eta I_o e^{-j\beta r}\left[\cos\left(\dfrac{\beta l}{2}\cos\theta\right) - \cos\dfrac{\beta l}{2}\right]}{2\pi r\sin\theta}$

For $l = \lambda, \dfrac{\beta l}{2} = \dfrac{2\pi}{\lambda}.\dfrac{\lambda}{2} = \pi$

$$|E_{\theta s}| = \frac{\eta I_o\left[\cos(\pi\cos\theta)+1\right]}{2\pi r\sin\theta}$$

$$f(\theta) = \frac{|E_{\theta s}|}{|E_{\theta s}|_{max}} = \frac{\cos(\pi\cos\theta)+1}{\sin\theta}$$

It is sketched below.



Prob. 13.6

$$\mathbf{P}_{ave} = \frac{|E_r|^2}{2\eta}\mathbf{a}_r = \frac{I_o^2 \sin^2\theta}{2\eta r^2}\mathbf{a}_r$$

$$P_{rad} = \frac{I_o^2}{2\eta}\iint \frac{\sin^2\theta}{r^2} r^2 \sin\theta d\theta d\phi = \frac{I_o^2}{240\pi}(2\pi)\int_0^{2\pi}(1-\cos^2\theta)d(-\cos\theta)$$

$$= \frac{I_o^2}{120}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Bigg|_0^{\pi} = \frac{I_o^2}{120}(-1/3+1-1/3+1) = \frac{I_o^2}{90}$$

$$I_o^2 = 90P_{ave} = 90\times 50\times 10^{-3} \longrightarrow I_o = \underline{\underline{2.121 \text{ A}}}$$

Prob. 13.7

This is similar to Fig. 13.10 except that the elements are z-directed.

$$\mathbf{E}_s = \mathbf{E}_{s1} + \mathbf{E}_{s2} = \frac{j\eta\beta I_o dl}{4\pi}\left[\sin\theta_1 \frac{e^{-j\beta r_1}}{r_1}\mathbf{a}_{\theta 1} + \sin\theta_2 \frac{e^{-j\beta r_2}}{r_2}\mathbf{a}_{\theta 2}\right]$$

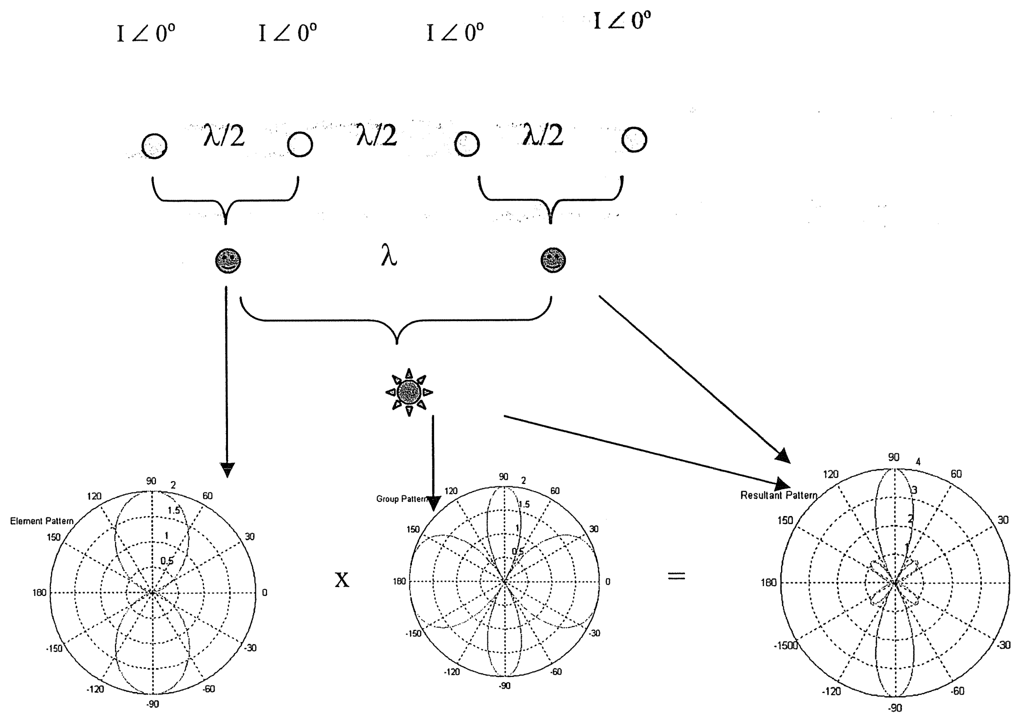
where $r_1 \cong r - \frac{d}{2}\cos\theta,\quad r_2 \cong r + \frac{d}{2}\cos\theta,\quad \theta_1 \cong \theta_2 \cong \theta,\quad \mathbf{a}_{\theta 1} \cong \mathbf{a}_{\theta 2} = \mathbf{a}_\theta$

$$\mathbf{E}_s = \frac{j\eta\beta I_o dl}{4\pi}\sin\theta\, \mathbf{a}_\theta\left[e^{j\beta d\cos\theta/2} + e^{-j\beta d\cos\theta/2}\right]$$

$$\underline{\underline{\mathbf{E}_s = \frac{j\eta\beta I_o dl}{2\pi}\sin\theta\, \cos(\frac{1}{2}\beta d\cos\theta)\mathbf{a}_\theta}}$$

Prob. 13.8

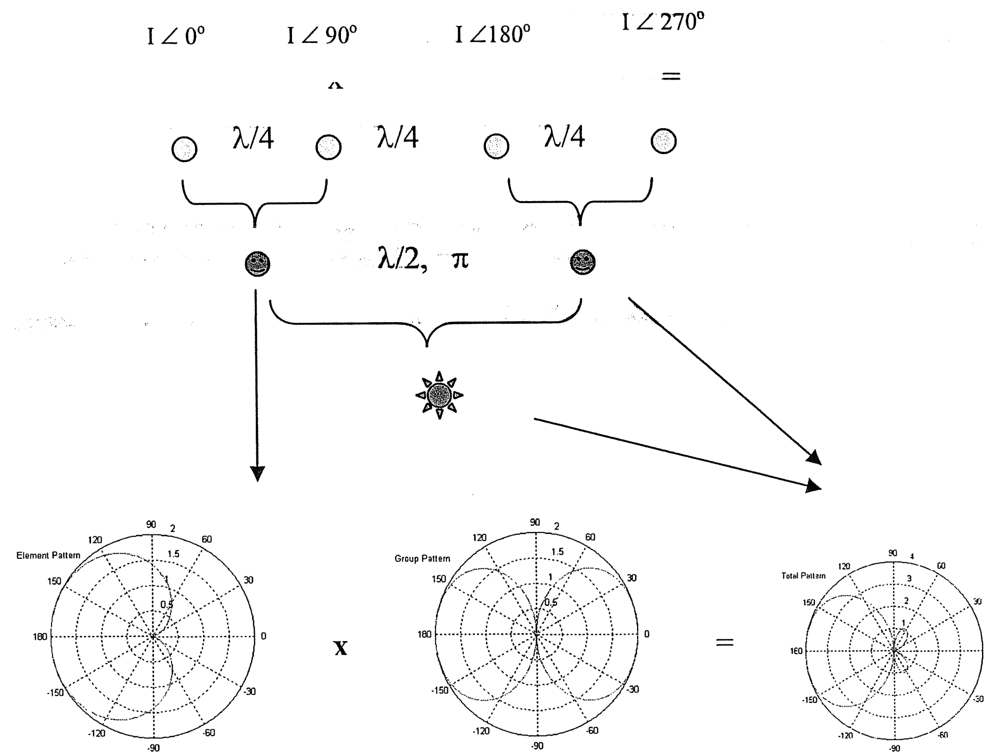
(a) The resultant pattern is obtained as follows.



(b) The array is replaced by by + $\frac{\lambda}{4}$ +, with phases $\angle 0^o$ and $\angle \pi/2$

where + stands for . ⟷ . with phases $\angle 0^o$ and $\angle \pi$

Thus the resultant pattern is obtained as shown.



Prob. 13.9

(a) $A_{er} = \frac{\lambda^2}{4\pi} G_{dr}, \qquad A_{et} = \frac{\lambda^2}{4\pi} G_{dt}$

$$P_r = G_{dr} G_{dt} \left(\frac{\lambda}{4\pi r} \right)^2 P_t = \left(\frac{4\pi}{\lambda^2} A_{er} \right) \left(\frac{4\pi}{\lambda^2} A_{et} \right) \left(\frac{\lambda}{4\pi r} \right)^2 P_t$$

or $\frac{P_r}{P_t} = \frac{A_{er} A_{et}}{\lambda^2 r^2}$

(b) $P_{r,\max} = \frac{A_{cr} A_{ct}}{\lambda^2 r^2} P_t \quad , \quad A_{cr} = A_{ct} = \frac{\lambda^2}{4\pi}(1.68)$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{100 \times 10^6} = 3m,$$

$$P_{r,\max} = \frac{(0.13\lambda^2)^2 (80)}{\lambda^2 (10^3)^2} = \underline{\underline{12.8\mu W}}$$

Prob. 13.10

$$P_{rad} = \frac{4\pi}{G_{dt}G_{dr}}\left(\frac{4\pi r_1 r_2}{\lambda}\right)^2 \frac{P_r}{\sigma}$$

But $G_{dt} = 36dB = 10^{3.6} = 3981.1$

$$G_{dr} = 20dB = 10^2 = 100$$

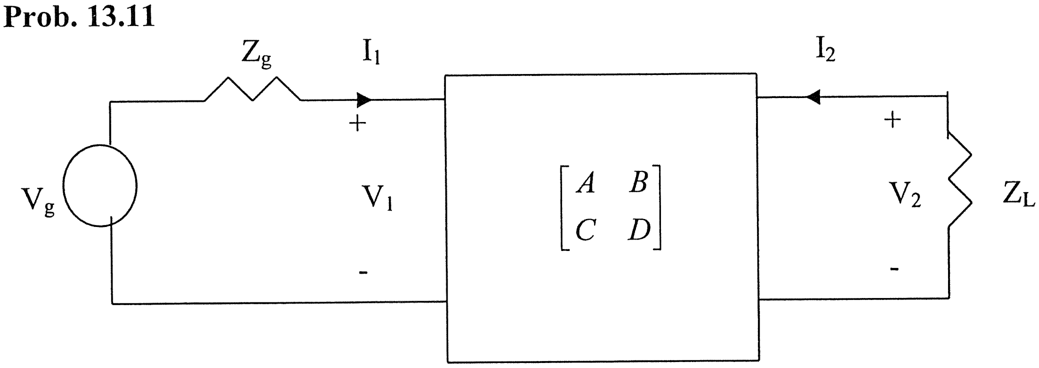
$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^9} = 0.06$$

$r_1 = 3km$, $r_2 = 5km$

$$P_{rad} = \frac{4\pi}{3981.1\times 100}\left(\frac{4\pi \times 15\times 10^6}{6\times 10^{-2}}\right)^2 \frac{8\times 10^{-12}}{2.4}$$

$= \underline{\underline{1.038}}$ kW

Prob. 13.11



By definition,

$$V_1 = AV_2 - BI_2 \quad (1)$$
$$I_1 = CV_2 - DI_2 \quad (2)$$

Let V_2 and $\overline{V}_2$ be respectively the load voltages when the filter circuit is present and when it is absent.

$$V_2 = -I_2 Z_L = \frac{I_1 Z_L}{CZ_L + D}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{V_1}{I_1}\right)(CZ_L + D)} = \frac{V_g Z_L}{\left(Z_g + \frac{AV_2 - BI_2}{CV_2 - DI_2}\right)(CZ_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{AZ_L + B}{CZ_L + D}\right)(CZ_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g(CZ_L + D) + AZ_L + B\right)}$$

$$\overline{V}_2 = \frac{V_g Z_L}{\left(Z_g + Z_L\right)}$$

Ratio and modulus give

$$\left|\frac{\overline{V}_2}{V_2}\right| = \frac{\left(Z_g(CZ_L + D) + AZ_L + B\right)}{Z_g + Z_L}$$

Insertion loss =

$$\text{IL} = 20\log_{10}\left|\frac{\overline{V}_2}{V_2}\right| = 20\log_{10}\left|\frac{\left(Z_g(CZ_L + D) + AZ_L + B\right)}{Z_g + Z_L}\right|$$

which is the required result

CHAPTER 14

Prob. 14.1

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0$$

The equivalent finite difference expression is

$$\frac{V(\rho_o+\Delta\rho, z_o) - 2V(\rho_o, z_o) + V(\rho_o - \Delta\rho, z_o)}{(\Delta\rho)^2} + \frac{1}{\rho_o}\frac{V(\rho_o+\Delta\rho, z_o) - V(\rho_o-\Delta\rho, z_o)}{2\Delta\rho}$$

$$+ \frac{V(\rho_o, z_o + \Delta z) - 2V(\rho_o, z_o) + V(\rho_o, z_o - \Delta z)}{(\Delta z)^2} = 0$$

If $\Delta z = \Delta\rho = h$, rearranging terms gives

$$V(\rho_o, z_o) = \frac{1}{4}V(\rho_o, z_o + h) + \frac{1}{4}V(\rho_o, z_o - h) + (1 + \frac{h}{2\rho_o})V(\rho + h, z_o)$$

$$+ (1 - \frac{h}{2\rho_o})V(\rho - h, z_o)$$

as expected.

Prob. 14.2

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0, \qquad (1)$$

$$\frac{\partial^2 V}{\partial \rho^2} = \frac{V_{m+1}{}^n - 2V_m{}^n + V_{m+1}{}^n}{(\Delta\rho)^2}, \qquad (2)$$

$$\frac{\partial^2 V}{\partial \phi^2} = \frac{V_m{}^{n+1} - 2V_m{}^n + V_m{}^{n-1}}{(\Delta\phi)^2}, \qquad (3)$$

$$\left.\frac{\partial V}{\partial \rho}\right|_{m,n} = \frac{V^n{}_{m+1} - V^n{}_{m-1}}{2\Delta\rho}. \qquad (4)$$

Substituting (2) to (4) into (1) gives

$$\nabla^2 V = \frac{V^n{}_{m+1} - V^n{}_{m-1}}{m\Delta\rho(2\Delta\rho)} + \frac{V_{m+1}{}^n - 2V_m{}^n + V_{m+1}{}^n}{(\Delta\rho)^2} + \frac{V_m{}^{n+1} - 2V_m{}^n + V_m{}^{n-1}}{(m\Delta\rho\Delta\phi)^2}$$

$$= \frac{1}{(\Delta\rho)^2}\left[(1 - \frac{1}{2m})V_{m-1}{}^n - 2V_m{}^n + (1 + \frac{1}{2m})V_{m-1}{}^n + \frac{1}{(m\Delta\phi)^2}(V_m{}^{n+1} - 2V_m{}^n + V_m{}^{n-1})\right]$$

as required.

Prob. 14.3 (a) Matrix [A] remains the same. To each term of matrix [B], we add

$-h^2\rho_s/\varepsilon$.

(b) Let $\Delta x = \Delta y = h = 0.25$ so that NX = 5= NY.

$$\frac{\rho_v}{\varepsilon} = \frac{x(y-1)10^{-9}}{10^{-9}/36\pi} = 36\pi x(y-1)$$

Modify the program in Fig. 14.16 as follows.

```
  H=0.1;
for  I=1:nx -1
  for  J=1: ny-1
  X = H*I;
  Y=H*J;
  RO = 36.0*pi*X*(Y-1);
  V(I,J) = 0.25*( V(I+1,J) + V(I-1,J) + V(I,J+1) + V(I,J-1) + H*H*RO );
   end
 end
```

This is the major change. However, in addition to this, we must set

```
v1 = 0.0;
v2 = 10.0;
v3 = 20.0;
v4 = -10.0;
nx = 5;
ny = 5;
```

The results are:

V_a = 4.6095 V_b= 9.9440 V_c= 11.6577
V_d = -1.5061 V_e =3.5090 V_f= 6.6867
V_g= -3.2592 V_h = 0.2366 V_i = 3.3472

Prob. 14.4

$$\frac{1}{c^2}\frac{\Phi^{j+1}_{m,n} + \Phi^{j-1}_{m,n} - 2\Phi^{j}_{m,n}}{(\Delta t)^2} = \frac{\Phi^{j}_{m+1,n} + \Phi^{j}_{m-1,n} - 2\Phi^{j}_{m,n}}{(\Delta x)^2}$$
$$+ \frac{\Phi^{j}_{m,n+1} + \Phi^{j}_{m,n-1} - 2\Phi^{j}_{m,n}}{(\Delta z)^2}$$

If $h = \Delta x = \Delta z$, then after rearranging we obtain

$$\Phi^{j+1}_{m,n} = 2\Phi^{j}_{m,n} - \Phi^{j-1}_{m,n} + \alpha(\Phi^{j}_{m+1,n} + \Phi^{j}_{m-1,n} - 2\Phi^{j}_{m,n})$$
$$+\alpha(\Phi^{j}_{m,n+1} + \Phi^{j}_{m,n-1} - 2\Phi^{j}_{m,n})$$

where $\alpha = (c\Delta t / h)^2$.

Prob. 14.5

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2} \qquad \longrightarrow \qquad \frac{V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t)}{(\Delta x)^2} =$$
$$\frac{V(x,t+\Delta t) - 2V(x,t) + V(x,t-\Delta t)}{(\Delta t)^2}$$

$$V(x,t+\Delta t) = \left(\frac{\Delta t}{\Delta x}\right)^2 \left[V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t\right] + 2V(x,t) - V(x,t-\Delta t)$$

or

$$V(i,j+1) = \alpha\left[V(i+1,j) + v(i-1,j)\right] + 2(1-\alpha)V(i,j) - V(i,j-1)$$

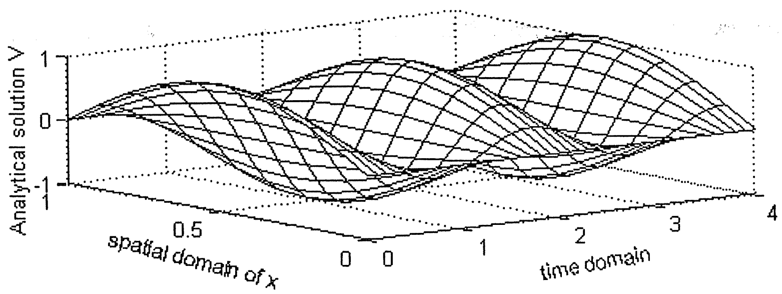
where $\alpha = \left(\frac{\Delta t}{\Delta x}\right)^2$. Applying the finite difference formula derived above, the following

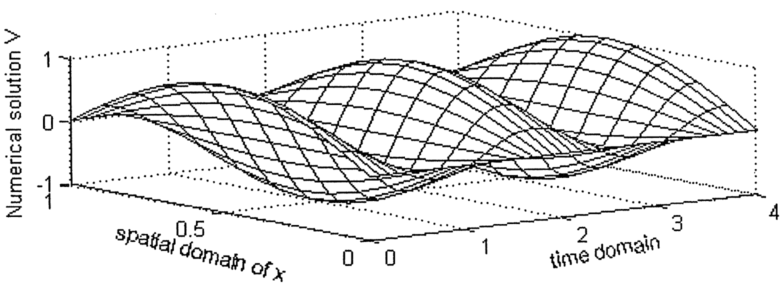
programs was developed.

```
xd=0:.1:1;td=0:.1:4;
[t,x]=meshgrid(td,xd);
Va=sin(pi*x).*cos(pi*t);%Analytical result
subplot(211) ;mesh(td,xd,Va);colormap([0 0 0])
%%%% Numerical result
N=length(xd);M=length(td);
v(:,1)=sin(pi*xd');
v(2:N-1,2)=(v(1:N-2,1)+v(3:N,1))/2;
for k=2:M-1
```

```
   v(2:N-1,k+1)=-v(2:N-1,k-1)+v(1:N-2,k)+v(3:N,k);
end
subplot(212);mesh(td,xd,v);colormap([0 0 0])
```

The results of the finite difference algorithm agree perfectly with the exact solution as shown below.





Prob. 14.6

(a)Points 1, 3, 5, and 7 are equidistant from O. Hence

$$V_o = \frac{1}{4}(V_1 + V_3 + V_5 + V_7) \qquad (1)$$

Also points 2, 4, 6, and 8 are equidistant from O so that

$$V_o = \frac{1}{4}(V_2 + V_4 + V_6 + V_8) \qquad (2)$$

Adding (1) and (2) gives

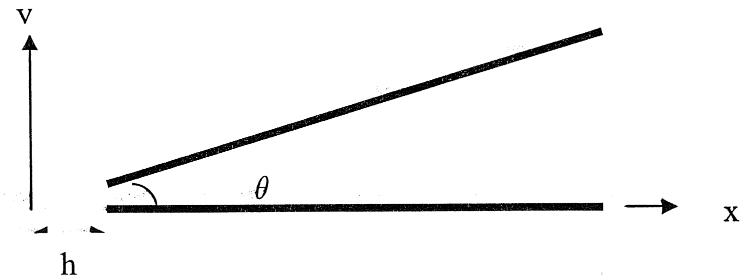
$$2V_o = \frac{1}{4}(V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8)$$

or

$$V_o = \frac{1}{8}\sum_{i=1}^{8} V_i$$

as required.

Prob. 14.7



To find C, take the following steps:

(1)Divide each line into N equal segments. Number the segments in the lower conductor as 1, 2, …, N and segments in the upper conductor as N+1, N+2, …, 2N,

(2) Determine the coordinate (x_k, y_k) for the center of each segment.

For the lower conductor, $y_k = 0$, k=1, …, N, $x_k = h + \Delta\,(k-1/2)$, k = 1,2,… N

For the upper conductor, $x_k = [h + \Delta\,(k-1/2)]\sin\theta$, k=N+1, N+2, …,2 N,

$x_k = [h + \Delta\,(k-1/2)]\cos\theta$, k = N+1,N+2,… 2N

where h is determined from the gap g as

$$h = \frac{g}{2\sin\theta/2}$$

(3)Calculate the matrices [V] and [A] with the following elements

$$V_k = \begin{cases} V_o, k = 1,\ldots,N \\ -V_o, k = N+1,\ldots 2N \end{cases}$$

$$A_{ij} = \begin{cases} \dfrac{\Delta}{4\pi\varepsilon R_{ij}}, i \neq j \\ 2\ln\Delta / a, i = j \end{cases}$$

where $R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

(4) Invert matrix [A] and find [ρ] = $[A]^{-1}$ [V].

(5) Find the charge Q on one conductor

$$Q = \sum \rho_k \Delta = \Delta \sum_{k=1}^{N} \rho_k$$

(6) Find C = |Q|/2V$_o$

Taking N= 10, V$_o$ = 1.0, a program was developed to obtain the following result.

θ	C (in pF)
10	8.5483
20	9.0677
30	8.893
40	8.606
50	13.004
60	8.5505
70	9.3711
80	8.7762
90	8.665
100	8.665
110	10.179
120	8.544
130	9.892
140	8.7449
150	9.5106
160	8.5488
170	11.32
180	8.6278

Prob. 14.8

We make use of the formulas in Problem 14.21.

$$V_i = \sum_{j=1}^{2N} A_{ij} \rho_i$$

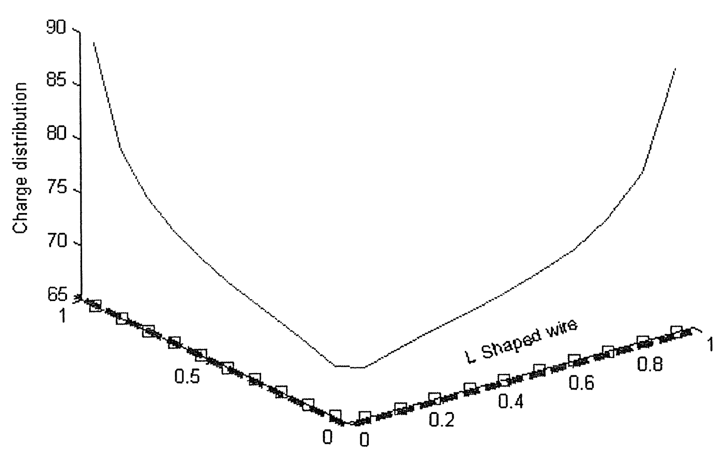
where N is the number of divisions on each arm of the conductor.

The MATLAB codes is as follows:

```
aa=0.001;
L=2.0;
N=10; %no.of divisions on each arm
NT=N*2;
delta=L/(NT);
```

```
x=zeros(NT,1);
y=zeros(NT,1);
%Second calculate the elements of the coefficient matrix
for i=1:N-1
   y(i)=0;
   x(i)=delta*(i-0.5)
end
for i=N+1:NT
   x(i)=0;
   y(i)=delta*(i-N-0.5);
end
for  i=1:NT
   for j=1:NT
      if (i ~=j)
         R=sqrt( (x(i)-x(j))^2 + (y(i)-y(j))^2 )
      A(i,j)=-delta*R;
     else
      A(i,j)=-delta*(log(delta)-1.5);
     end
   end
end
%Determine the matrix of constant vector B and find rho
B=2*pi*eo*vo*ones(NT,1);
rho=inv(A)*B;
```

The result is presented below.



Segment	X	y	ρ in pC/m
1	0.9500		89.6711
2	0.8500	0	80.7171
3	0.7500	0	77.3794
4	0.6500	0	75.4209
5	0.5500	0	74.0605
6	0.4500	0	73.0192
7	0.3500	0	72.1641
8	0.2500	0	71.4150
9	0.1500	0	70.6816
10	0.0500	0	69.6949
11	0	0	69.6949
12	0	0.0500	70.6816
13	0	0.1500	71.4150
14	0	0.2500	72.1641
15	0	0.3500	73.0192
16	0	0.4500	74.0605
17	0	0.5500	75.4209
18	0	0.6500	77.3794
19	0	0.7500	80.7171
20	0	0.8500	89.6711

Prob. 14.9 (a) Exact solution yields

$C = 2\pi\varepsilon / In(\Delta / a) = 8.02607x10^{-11}$ F/m and $\underline{\underline{Z_o = 41.559\Omega}}$

where a = 1cm and Δ = 2cm. The numerical solution is shown below.

N	C (pF/m)	$Z_o(\Omega)$
10	82.386	40.486
20	80.966	41.197
40	80.438	41.467
100	80.025	41.562

(b)For this case, the numerical solution is shown below.

N	C (pF/m)	$Z_o(\Omega)$
10	109.51	30.458
20	108.71	30.681
40	108.27	30.807
100	107.93	30.905

Prob. 14.10 We modify the MATLAB code in Fig. 14.26 (for Example 14.5) by changing the input data and matrices [A] and [B]. We let

$x_i = h + \Delta\,(i\text{-}1/2),\ \ i = 1,2,\ldots N, \qquad \Delta = L/N$

$y_i = h/2,\ \ j = 1,2,\ldots N, \quad z_k = t/2,\ \ k = 1,2,\ldots N$

and calculate

$$R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

We obtain matrices [A] and [B]. Inverting [A] gives

$$[q] = [A]^{-1}\,[B], \quad [\rho_v] = [q]/(ht\Delta), \quad C = \frac{\sum_{i=1}^{N} q_i}{10}$$

The computed values of $[\rho_v]$ and C are shown below.

i	$\rho_{vi}(\times 10^{-6})C/m^3$
1, 20	0.5104
2, 19	0.4524
3, 18	0.4324
4, 17	0.4215
5, 16	0.4144
6, 15	0.4096
7, 14	0.4063
8, 13	0.4041
9, 12	0.4027
10,11	0.4020

C = 17.02 pF

Prob. 14.11 From the given figure, we obtain

$$\alpha_1 = \frac{A_1}{A} = \frac{1}{2A}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

as expected. The same applies for α_2 and α_3.