POWER UNIT

Control Notebook

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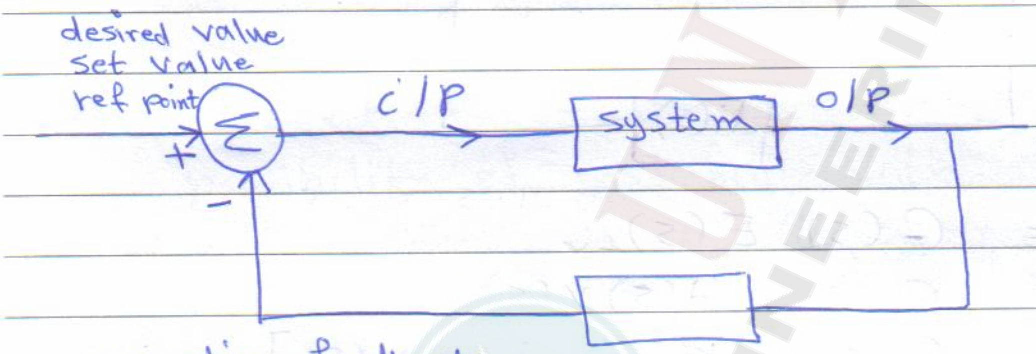
18/2

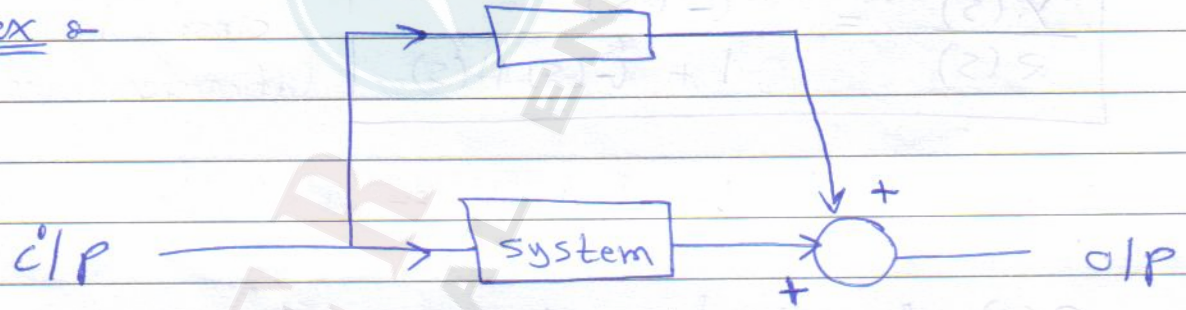
㊟ Feedback systems :-

The output or part of it is brought back to become the input or part of the input.



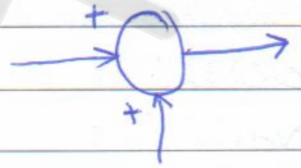
negative feedback

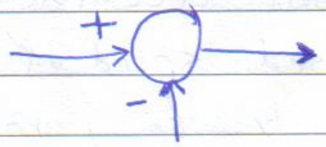
* ex :-



⇒ Not a feedback system, it is a feedforward system.

㊟ - with +ve feedback, the output is added to the set value.
- with -ve feedback, the output is subtracted from the set value.





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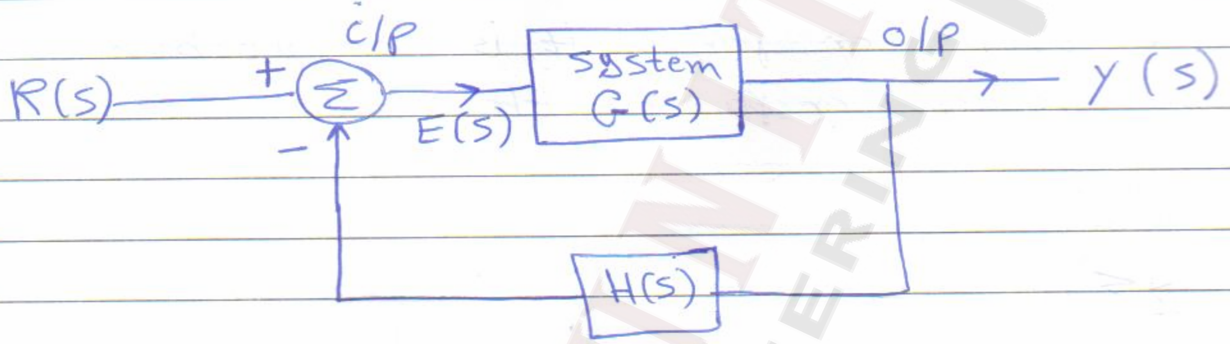
⊛ Basic feedback System



$$y(s) = G(s)\,E(s)$$
$$E(s) = R(s) - H(s)\,y(s)$$
$$y(s) = G(s)\left[R(s) - H(s)\,y(s)\right]$$

$$\boxed{\frac{y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}}$$

closed loop transfer function

⊛ A matter of terminology :-

- $G(s)$ is the forward T.F
- $H(s)$ is the feedback T.F
- $E(s)$ is the error T.F
- $G(s)\,R(s)$ is the open loop T.F
- $\dfrac{G(s)}{1 + G(s)H(s)}$ is the closed loop T.F.

⊛ Advantages of feedback :-

1. Increased accuracy.
2. Increased speed of response.
3. Insensitivity to parameter variations.
4. Insensitivity to external disturbanced.

5. Increased bandwidth.
6. Improved immunity to noise.
7. Improved linearity.

⊛ Disadvantages of feedback :

1. Reduced gain.
2. possibility of instability with careless design.
3. added complexity to the system which entails:-
   - extra cost.
   - extra weight.
   - extra size
   - truble shooting.

* ex :- Insensitivity to parameter variations
Suppose (G(s)) (system) experiences parameter variations.
i.e G(s) changes

⇒ If the system is open loop, then
$Y(s) = G(s)\,R(s)$
→ Y(s) changes, the same percentage as G(s)

⇒ If the system is closed loop, then

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

→ If $G(s)H(s) \gg 1$

$$\frac{Y(s)}{R(s)} = \frac{\cancel{G(s)}}{\cancel{G(s)}H(s)} = \frac{1}{H(s)}$$

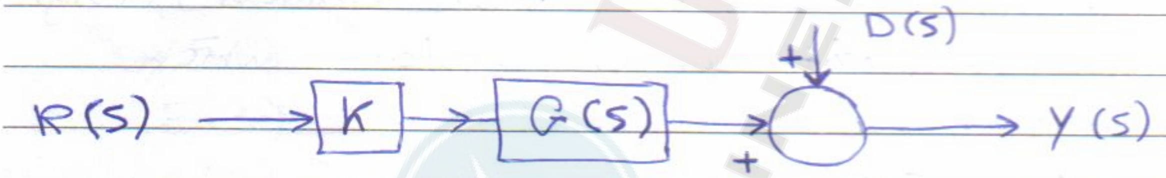
u

i.e the output dosen't change as $G(s)$ changes it depends on $H(s)$ can be arbitrarily closen as invariant.

i.e the system is insensitive to parameter variations.

* **ex** :- Insensitivity to external disturbances

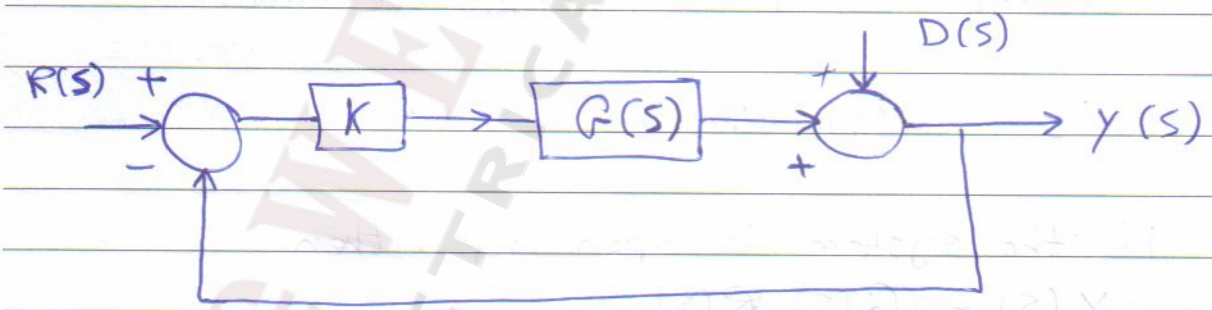
Consider the system shown.



$$Y(s) = K\,G(s)\,R(s) + \underbrace{D(s)}$$

All of $D(s)$ is affecting the output

Suppose now that unity feedback is introduced



$$Y(s) = \frac{K\,G(s)}{1+K\,G(s)}\,R(s) + \frac{1}{1+K\,G(s)}\,D(s)$$

If $K\,G(s) \gg 1$

$$Y(s) = R(s) + \frac{1}{1+K\,G(s)}\,D(s).$$

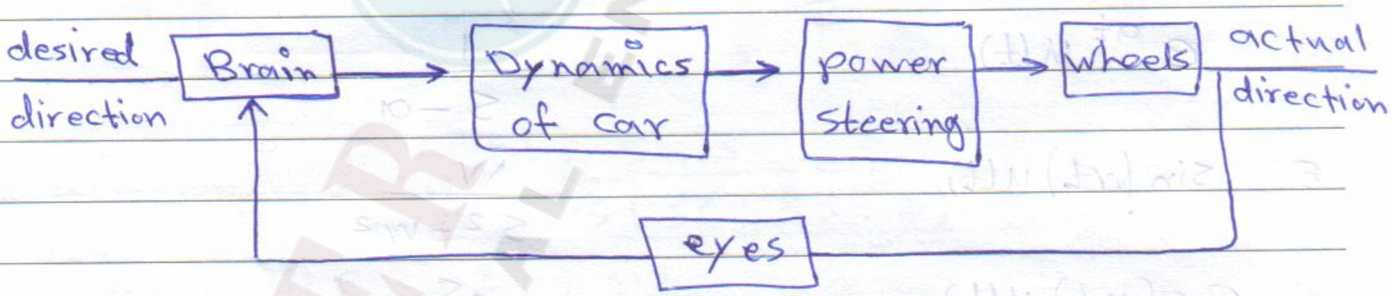
i.e a fraction of $D(s)$ is not affecting the output.

* **ex** :- Given the following $G(s) = \dfrac{10^3}{1 + \frac{s}{w_0}}$

① Determine the DC gain and the bandwidth.
② Introduce negative feedback and calculate the value of H such that the new feedback is new $10\,w_0$. What the new dc feedback gain.
③ Verify that gain × bandwidth is constant.

⊛ Schematic diagrams

⇒ Consider driving a car.



⊛ Modeling of systems.

Systems are usually modeled through differential equations.
To ease the solution, we convert differential equations to algebraic equations by using the Laplace transform.

⊛ Review of laplace transform.

Def :- $\mathcal{L}\, f(t) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

in linear operator.

$$\mathcal{L}\left(a f_1(t) + b f_2(t)\right) = a F_1(s) + b F_2(s)$$

④ The laplace transform (L.T) of certain functions :-

1	$u(t)$	$\frac{1}{s}$
2	$t\,u(t)$	$\frac{1}{s^2}$
3	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
4	$e^{at} u(t)$	$\frac{1}{s-a}$
5	$\sin(wt)\,u(t)$	$\frac{w}{s^2+w^2}$
6	$\cos(wt)\,u(t)$	$\frac{s}{s^2+w^2}$
7	$e^{at} f(t)$	$F(s-a)$
8	$\frac{d\,f(t)}{dt}$	$sF(s) - f(t)\Big\vert_{t=0}$
9	$\frac{d^2 f(t)}{dt^2}$	$s^2F(s) - s f(0) - f'(0)$
10	$\frac{d^3 f(t)}{dt^3}$	$s^3F(s) - s^2 f(0) - s f'(0) - f''(0)$

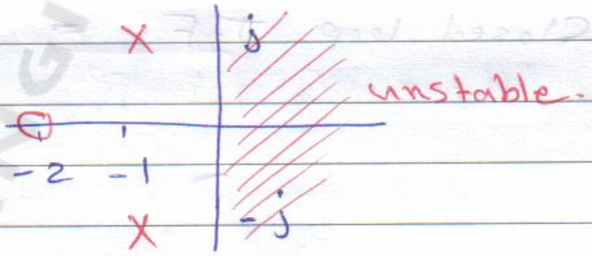
⊛ The initial value theorem

$$f(0) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$$

The final value theorem

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

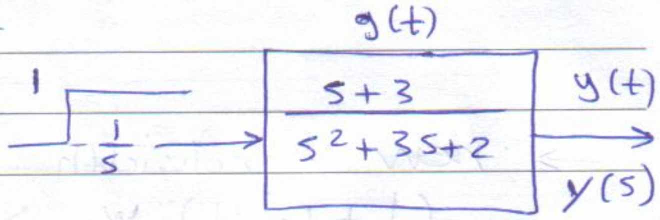
Valid provided sF(s) dosen't poles with +ve real parts



*ex: find $\mathcal{L}^{-1}\, \frac{s+3}{s(s^2+3s+2)}$

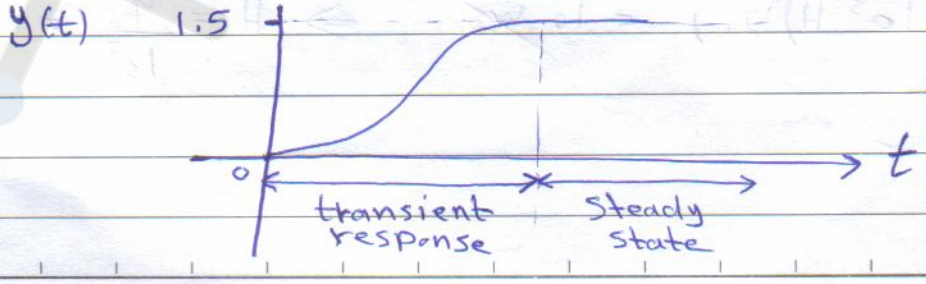
$$\Rightarrow\ y(s) = \frac{1.5}{s} + \frac{-2}{s+1} + \frac{0.5}{s+2}$$

$$y(t) = (1.5 - 2e^{-t} + 0.5e^{-2t})\, u(t)$$

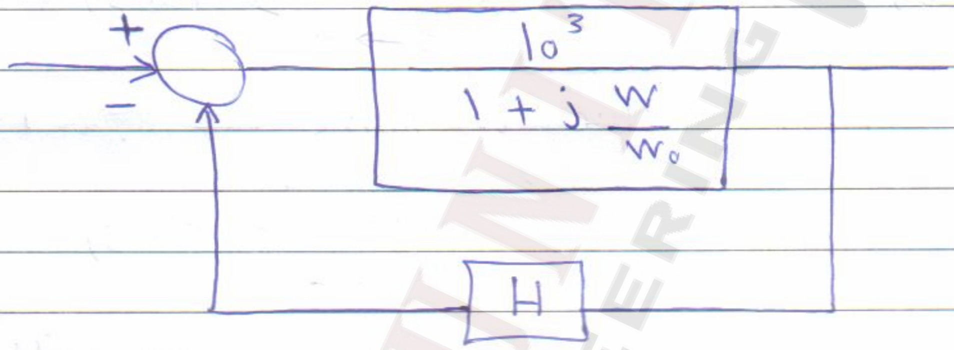


$$y_{ss} = y(t=\infty) = \lim_{t\to\infty} y(t) = 1.5$$

or $$y_{ss} = \lim_{s\to 0} \cancel{s}\, \frac{s+3}{\cancel{s}(s^2+3s+2)} = \frac{3}{2} = 1.5$$



⇒ Solve previous ex at page (5)



BW = w_o

Closed loop T.F = $$\frac{\frac{10^3}{1+j\frac{w}{w_o}}}{1+\frac{10^3}{1+j\frac{w}{w_o}}H}$$

$$= \frac{10^3}{1+j\frac{w}{w_o}+10^3 H}$$

$$= \frac{10^3}{(1+10^3H)\left(1+j\frac{w}{(1+10^3H)w_o}\right)}$$

→ new bandwidth

$(1+10^3H)\,w_o >> w_o$

→ new dc

$\frac{10^3}{1+10^3 H}$ → new gain = 100

$((1+10^3 H)\cancel{w_o}) = 10\cancel{w_o}$ ⇒ $H = \frac{9}{10^3} = 0.009$

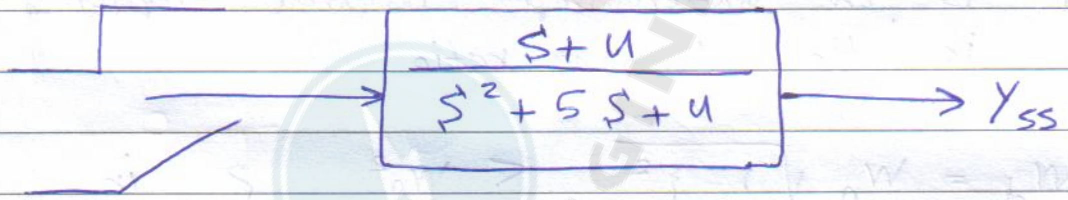
* ex :- A system is given by T.F

$$G(S) = \frac{S+4}{S^2+5S+4}$$

Calculate the output (t) due to

① a unit step and determine y_{ss}

② a unit ramp and determine y_{ss}.



(*) Second order systems :-

A particular second order system is represented by

$$G(S) = \frac{W_n^2}{S^2 + 2 \xi W_n S + W_n^2}$$

Where ξ & W_n result in complex poles given by

$$-\xi W_n \pm j W_n \sqrt{1-\xi^2}$$

Due to a unit step

$$Y(S) = \frac{W_n^2}{S(S^2 + 2 \xi W_n S + W_n^2)}$$

$$W_n^2 - \xi^2 W_n^2 = W_d^2$$

$$W_d^2 = W_n^2 (1 - \xi^2)$$

$$W_d = W_n \sqrt{1 - \xi^2}$$

$$Y(S) = \frac{W_n^2}{S \left[(S + \xi W_n)^2 + W_n^2 - \xi^2 W_n^2 \right]}$$

- W_n is the undamped natural frequency.
- W_d is the underdamped natural frequency.
- ξ is the damping ratio

$$\left.\begin{array}{l} W_d = W_n \sqrt{1 - \xi^2} < W_n^2 \\ 0 \leq \xi < 1 \end{array}\right\} \begin{array}{l} \text{in} \\ \text{underdamped} \end{array}$$

$$Y(S) = \frac{W_n^2}{S \left[(S + \xi W_n)^2 + W_d^2 \right]}$$

$$Y(S) = \frac{A}{S} + \frac{BS + C}{(S + \xi W_n)^2 + W_d^2}$$

$$= \frac{AS^2 + 2\xi W_n S A + W_n^2 A + BS^2 + CS}{S \left(\qquad\qquad \right)}$$

$$= \frac{W_n^2}{S \left(\qquad \right)}$$

$$A W_n^2 = W_n^2 \Rightarrow A = 1$$

$$(A + B) = 0 \Rightarrow B = -A = -1$$

$$(2 \xi W_n A + C) = 0 \Rightarrow C = -2 \xi W_n$$

$$Y(s) = \frac{1}{s} + \frac{-s - 2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} + \frac{-(s+\xi\omega_n) - \xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2} \cdot \frac{\xi\omega_n}{\omega_d}$$

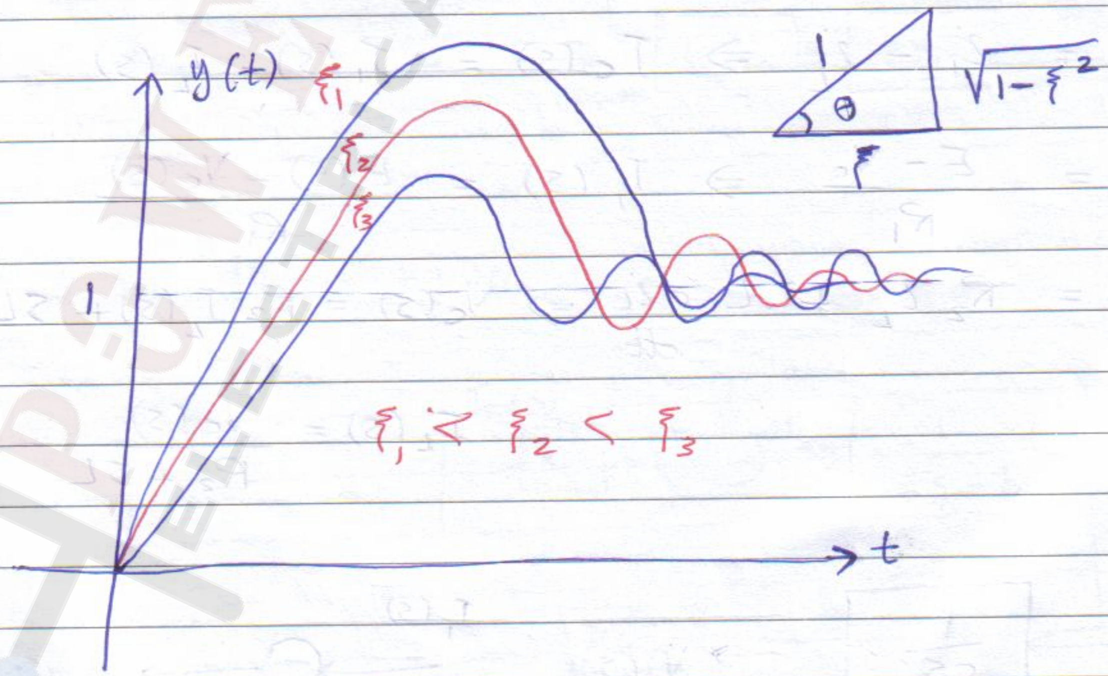
$$\frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi}{\sqrt{1-\xi^2}} \cdot \frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

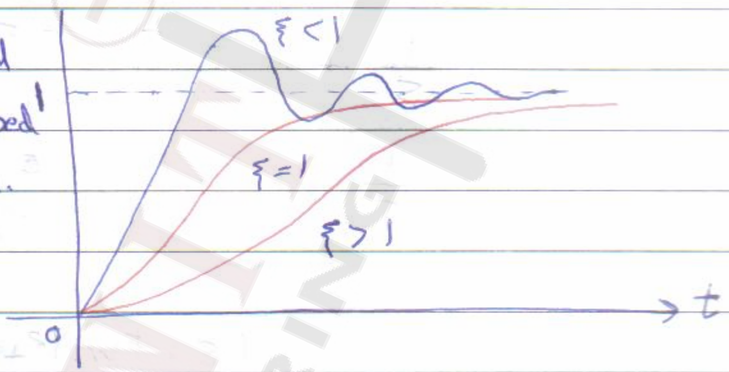
$$y(t) = \left[1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\omega_d t\right] u(t)$$

or Compactly

memo
$$y(t) = \left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_d t + \cos^{-1}\xi\right)\right] u(t)$$

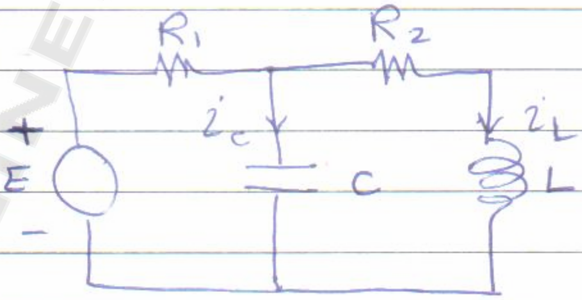


IF $\xi < 1$ underdamped
$\xi = 1$ Critically damped
$\xi > 1$ overdamped.



⊛ Modeling of systems and block diagram representation

⇒ Consider the following circuit
obtain a block diagram representation with $E(s)$ as the set value and $V_c(s)$ as the output.



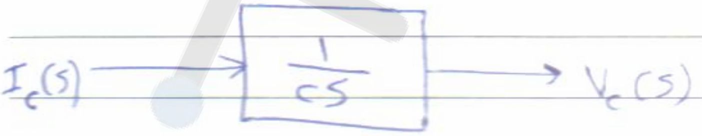
$$i_c = C\frac{dv_c}{dt} \Rightarrow I_c(s) = CS\,V_c(s)$$

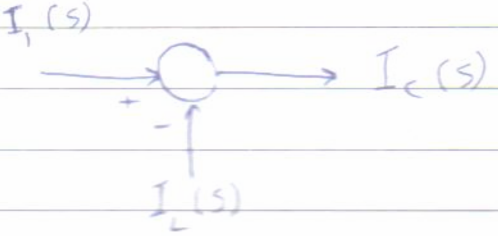
$$i_c = i_1 - i_L \Rightarrow I_c(s) = I_1(s) - I_L(s)$$

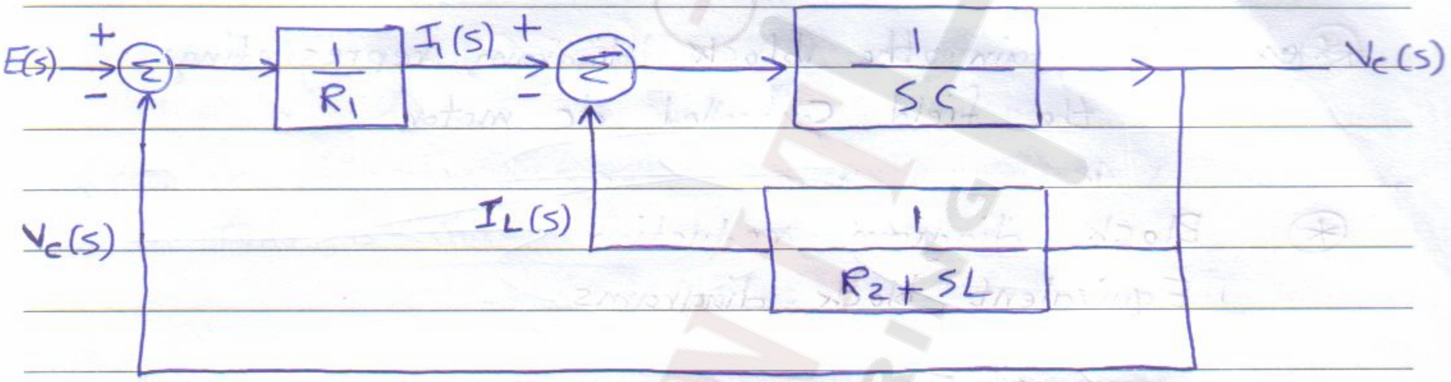
$$i_1 = \frac{E - V_c}{R_1} \Rightarrow I_1(s) = \frac{E(s) - V_c(s)}{R_1}$$

$$V_c = R_2 i_L + L\frac{di_L}{dt} \Rightarrow V_c(s) = R_2 I_L(s) + SL\,I_L(s)$$

$$I_L(s) = \frac{V_c(s)}{R_2 + SL}$$

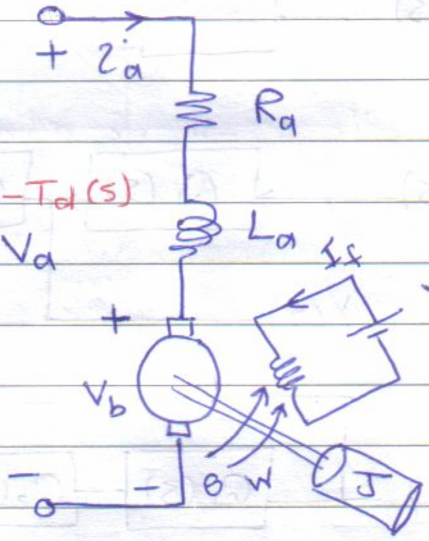






* ex : Armature - controlled dc motor.

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + V_b$$



$$T_m = K_m I_a$$

$$\underline{T_m} = \underline{T_L} + \underline{T_d} \Rightarrow T_L(s) = T_m(s) - T_d(s)$$

developed load disturbances

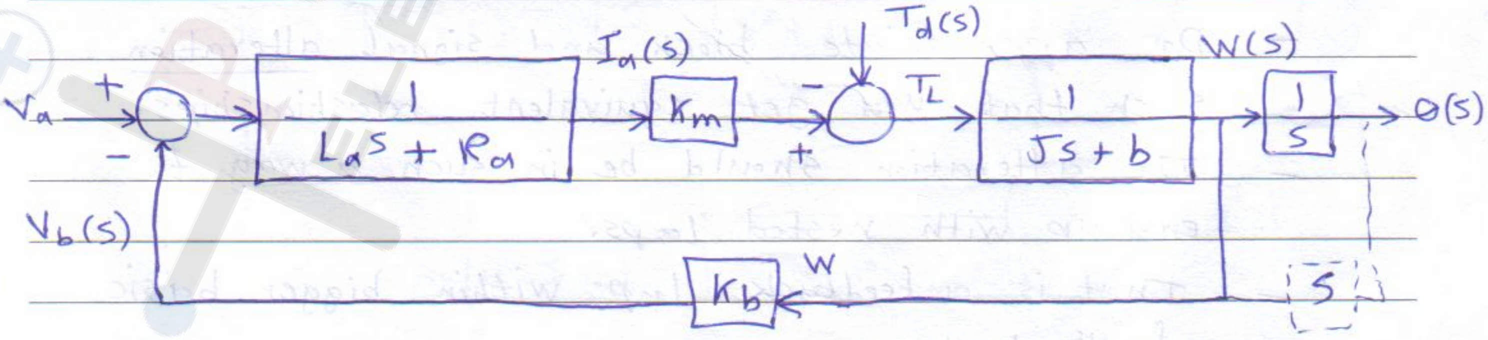
$$T_L = J \frac{d\theta^2}{dt^2} + b \frac{d\theta}{dt}$$

moment of inertia friction

$$= J \frac{dw}{dt} + bw$$

$$\Rightarrow W(s) = \frac{T_L}{JS + b}$$

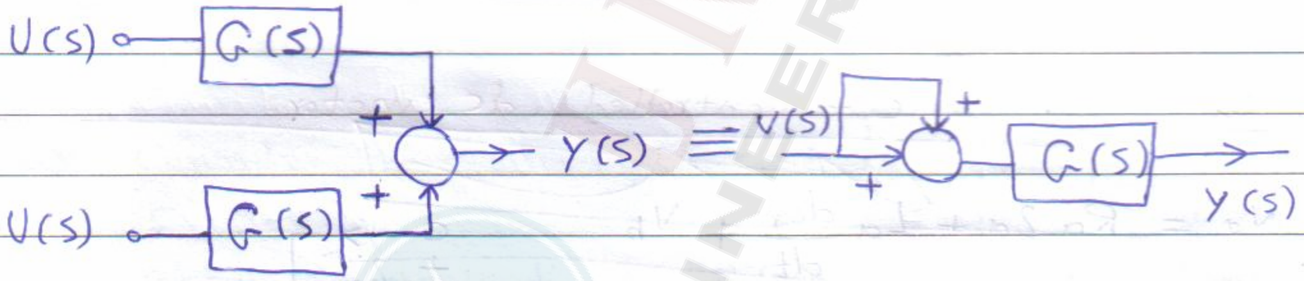
$$V_b = k_b \frac{d\theta}{dt} = K_b W \Rightarrow V_b(s) = K_b S \, \theta(s)$$



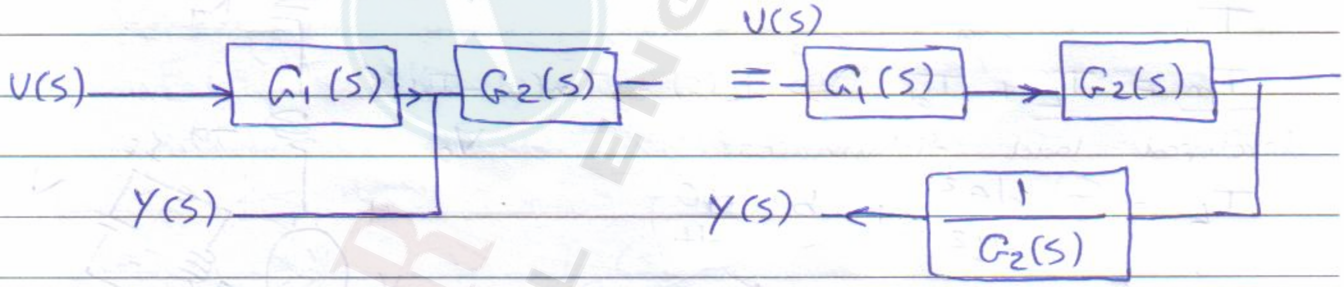
* ex :- obtain the block diagram representing the field Controlled dc motor.

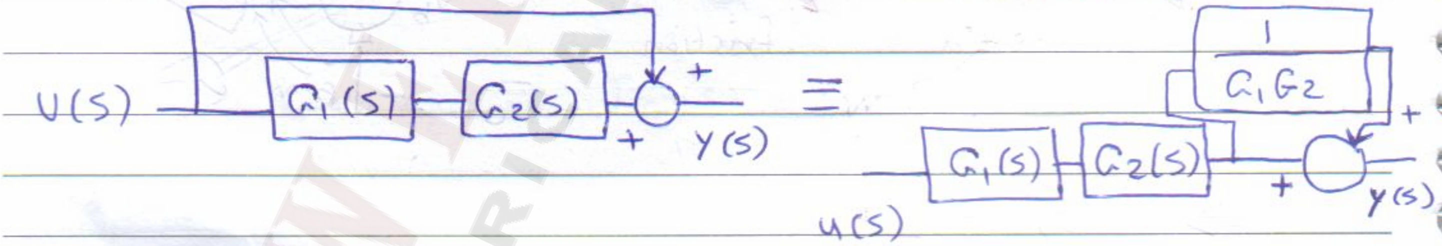
⊛ Block diagram reduction
Equivalent block diagrams

①



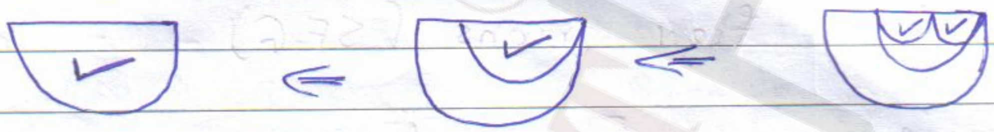
②





⇒ To reduce a block diagram :-

- Do appropriate block and signal alteration (تغير) Such that you get equivalent relashionships.
- The alteration should be in such a way to end up with nested loops.
- That is a feedback loop within bigger basic feedback loops.



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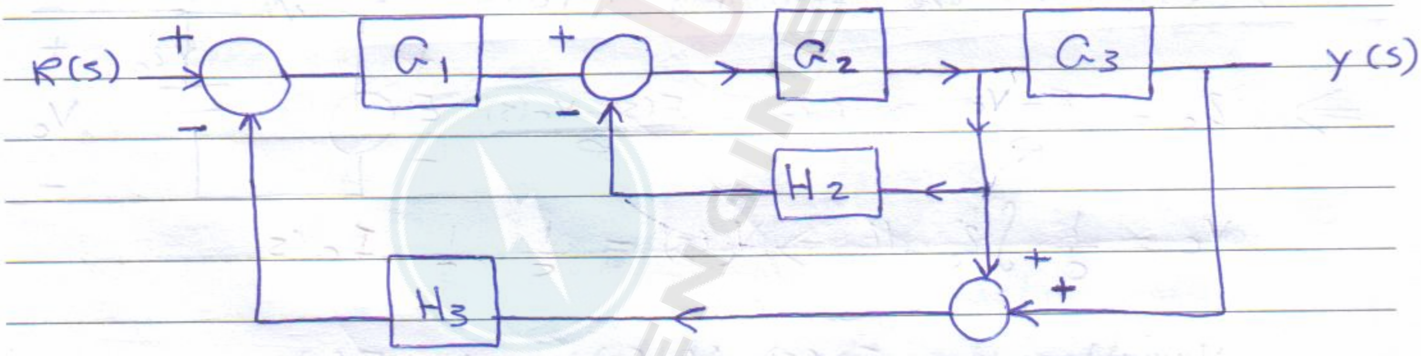
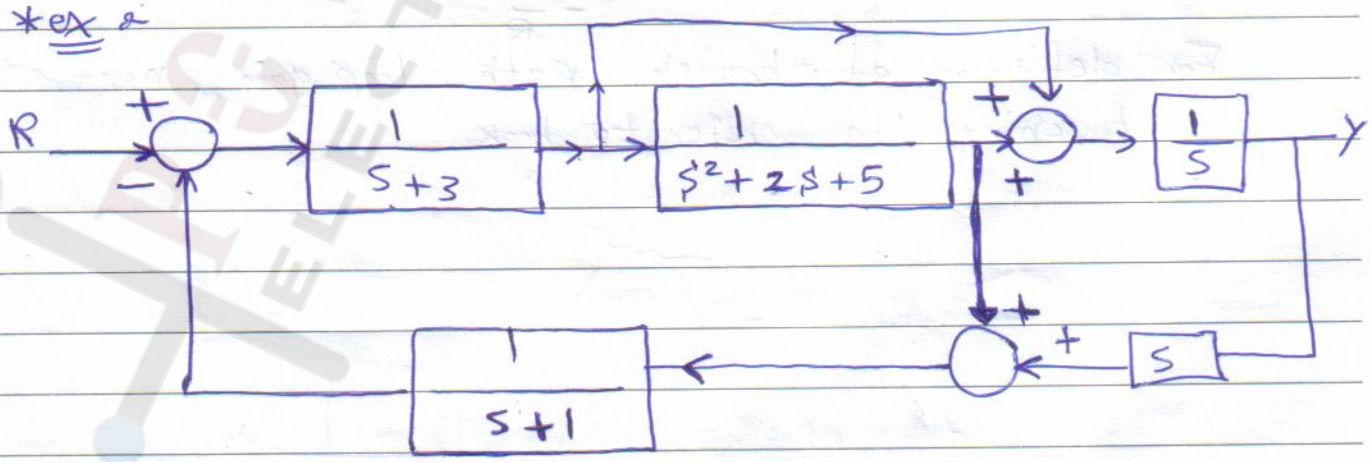
⇒ This is illustrated by an example

*ex: Consider the following block diagram



$$\frac{Y(s)}{R(s)} = \frac{\dfrac{G_1 G_2 G_3}{1+G_2 H_2}}{1 + \dfrac{G_1 G_2 G_3}{1+G_2 H_2}\left(H_3\left(1+\dfrac{1}{G_3}\right)\right)}$$

*ex:

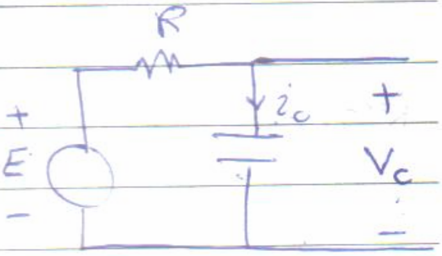


⊛ Signal flow graphs (SFG)

⇒ Another method for representing relashionship between variables.

<u>motivation example</u>

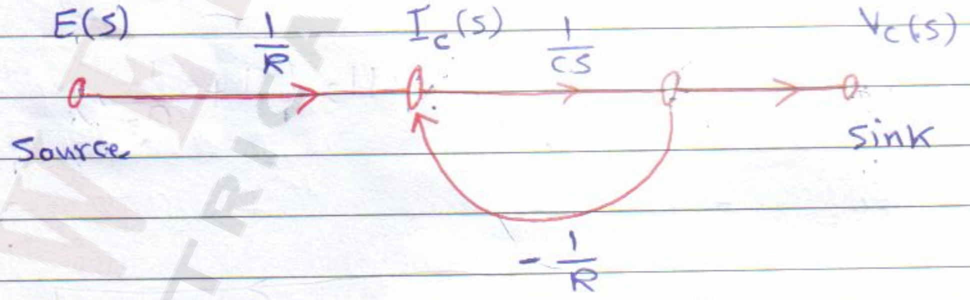
Consider the following circuit



$$\Rightarrow i_c = \frac{E - V_c}{R} \Rightarrow I_c(s) = \frac{E(s) - V_c(s)}{R}$$

$$V_c = \frac{1}{c}\int i_c \cdot dt \Rightarrow V_c(s) = \frac{1}{c} \cdot \frac{1}{s} I_c(s)$$

Variables are $I_c(S)$, $V_c(S)$ and $E(S)$ are assigned to node (0).



For definition of branch, path, loop gain, non-touching loop, refer to book.

⊛ Mason's gain rule

$$\frac{T}{R} = \frac{\sum P_i \, \Delta_i}{\Delta}$$

⇝ For previous example.

$$P_1 = \frac{1}{R} \cdot \frac{1}{CS} \,, \quad \Delta_1 = 1$$

$$\Delta = 1 - \left(- \frac{1}{RCS} \right)$$

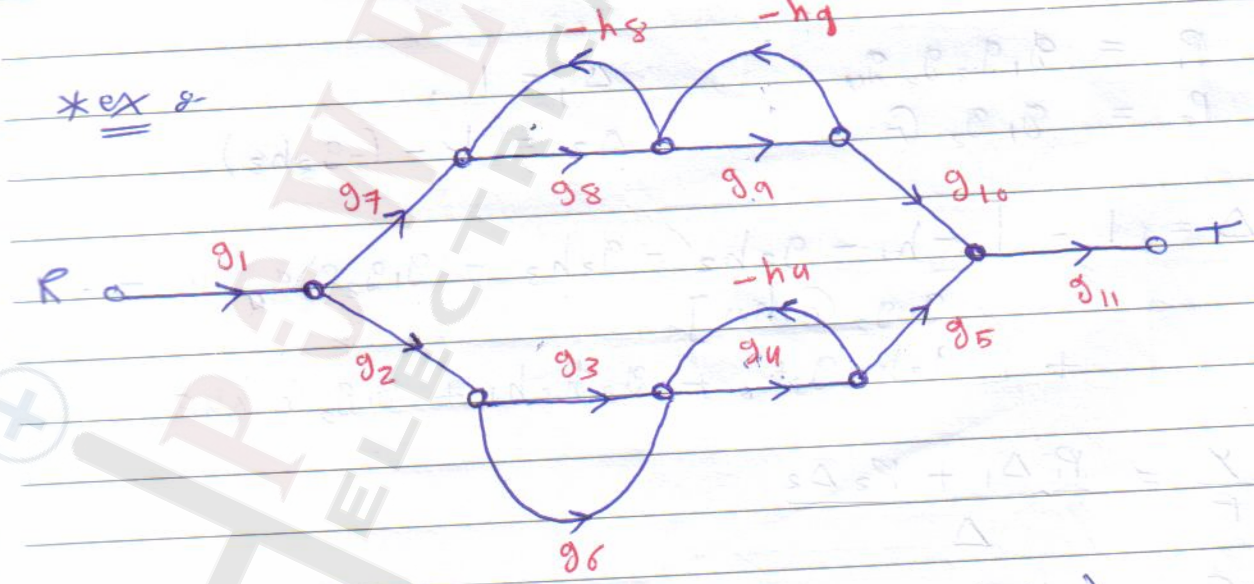
$$\frac{V_c(s)}{E(s)} = \frac{1/RCS}{1 + (1/RCS)} = \frac{1}{1 + RCS}$$

Where :-

Δ = 1 − Σ loop gains one at time
+ Σ product of non-touching loop gains taken two at a time.
− Σ product of non-touching loop gains taken three at a time.

⇒ N.B : Non-touching loops do not have nodes in common.

*ex :



$$\Rightarrow \Delta = 1 - (-g_4 h_4 - g_8 h_8 - g_9 h_9)$$
$$+ (-g_4 h_4 \cdot -g_8 h_8 + -g_4 h_4 \cdot -g_9 h_9)$$

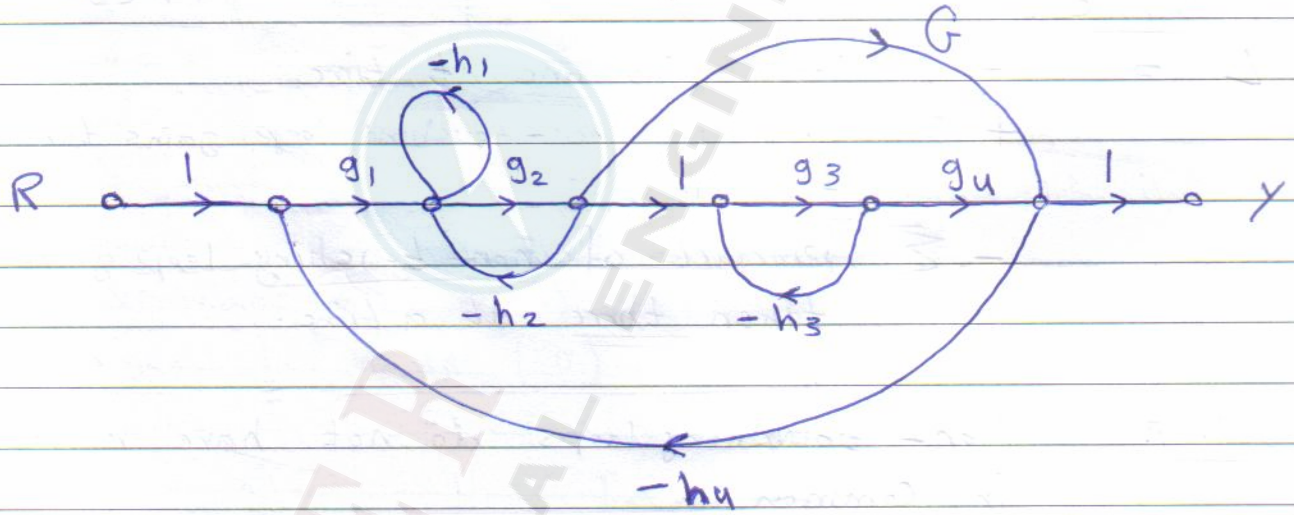
$P_1 = g_1 g_7 g_8 g_9 g_{10} g_{11}$, $\Delta_1 = 1 - (-g_4 h_4)$

$P_2 = g_1 g_2 g_3 g_4 g_5 g_{11}$, $\Delta_2 = 1 - (-g_8 h_8 - g_9 h_9)$

$P_3 = g_1 g_2 g_6 g_4 g_5 g_{11}$, $\Delta_3 = 1 - (-g_8 h_8 - g_9 h_9)$

$$\frac{T}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3}{\Delta}$$

* <u>ex</u> :- Consider the following SFG



$\Rightarrow$ $P_1 = g_1 g_2 g_3 g_4$, $\Delta_1 = 1$

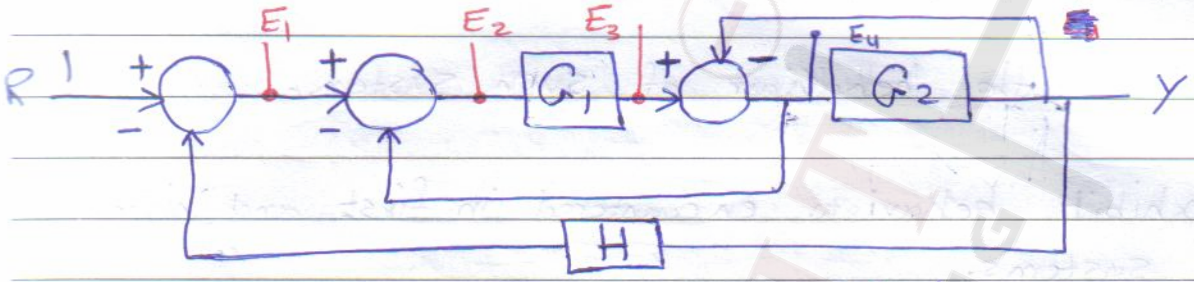
$P_2 = g_1 g_2 G$, $\Delta_2 = 1 - (-g_3 h_3)$

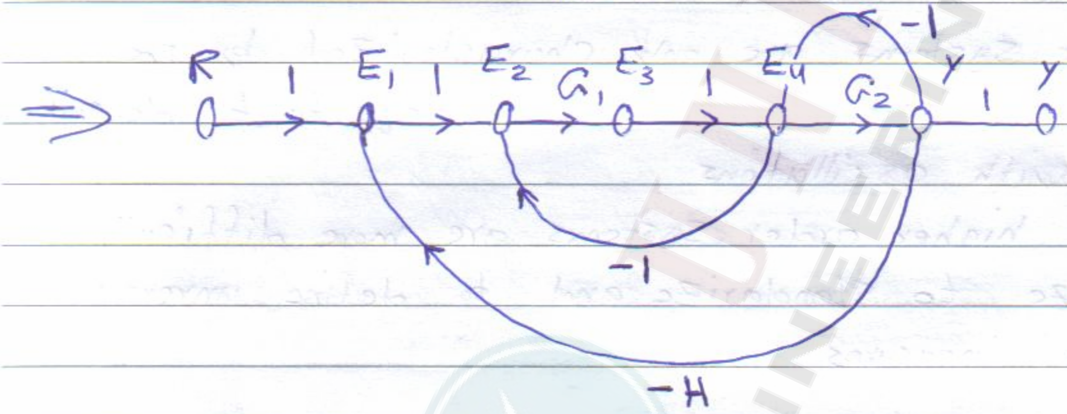
$$\Delta = 1 - [-h_1 - g_2 h_2 - g_3 h_3 - g_1 g_2 g_3 g_4 h_4 - g_1 g_2 G h_4] + [g_2 h_2 g_3 h_3 + g_3 h_3 h_1 + g_1 g_2 G h_4 g_3 h_3]$$

$$\frac{Y}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

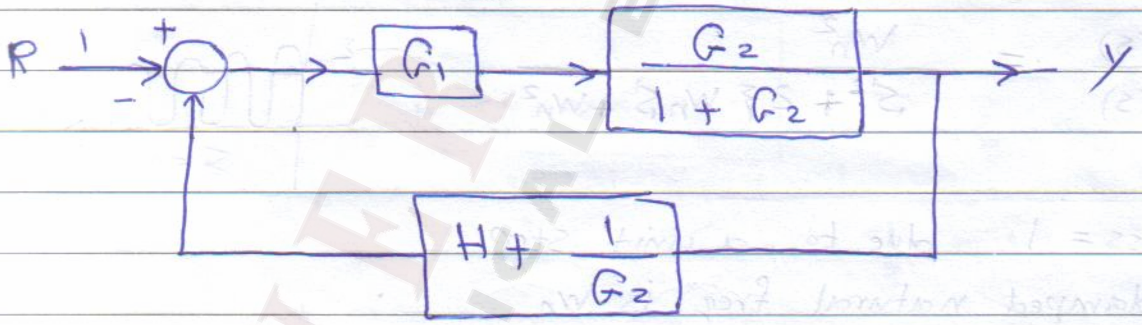
⊛ Converting a block diagram to a SFG

Best illustrated by an example.





$$\frac{Y}{R} = \frac{G_1 G_2 \times 1}{1 + G_1 + G_2 + G_1 G_2 H}$$



$$\frac{Y}{R} = \frac{G_1 \cdot \frac{G_2}{1+G_2}}{1 + \frac{G_1 G_2}{1+G_2} \times \left(H + \frac{1}{G_2}\right)} = \frac{G_1 G_2}{1 + G_2 + G_1 G_2 H + G_1}$$

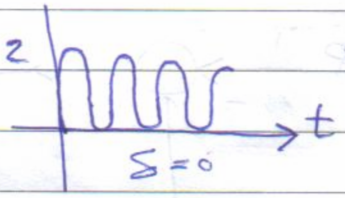
⊛ Time domain classification of a second order system.

Since Second order systems are characterized by the three parameters, gain, damping ratio and natural freq, it proves easily to define certain measures describing the behaviour of such system.

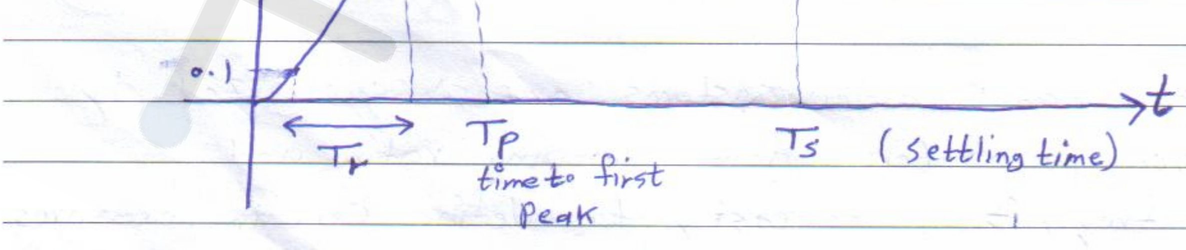
The exhibit behaviour encountered in first and higher order systems.
First order systems are only characterized by the gain and time constant . here, inadequate to describe systems with oscillations
Third or higher order systems are more difficult to analize , to standarize and to define unversually acseptable measures .

⟹ Assuming second order systems is described by the following transfer function

$$\frac{Y(s)}{R(s)} = \frac{W_n^2}{S^2 + 2\xi W_n S + W_n^2}$$



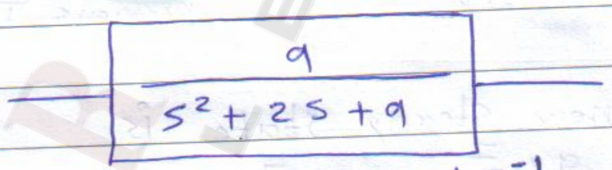
$y_{ss} = 1$ due to a unit step
undamped natural freq is W_n
damping ratio is ξ
specifications are for the case where $\xi < 1$



⇒ Definitions of M_p, T_r, T_p, T_s refer to book.

- percentage maximum overshoot $= \frac{M_p - f_V}{f_V} \times 100 \%$
- $M_p = 100\, e^{-\frac{\xi \pi}{\sqrt{1-\xi^2}}}$
- $T_p = \frac{\pi}{W_d} = \frac{\pi}{W_n\sqrt{1-\xi^2}}$
- $T_s \simeq \frac{4}{\xi W_n}$ for 2% criterion
- $T_r \simeq \frac{2.16\xi + 0.6}{W_n}$ $\quad 0.3 \leq \xi \leq 0.8$

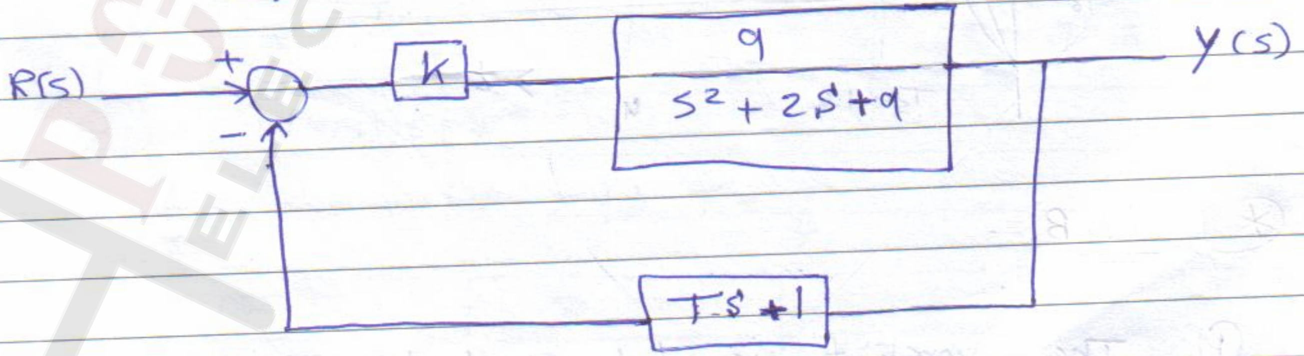
*__ex__: Consider the following system



$W_n = 3$ rad s^{-1}

$\xi = \frac{\cancel{2}}{\cancel{2}\times 3} = \frac{1}{3} < 1$ underdamped

Design a controller such that $\xi = 0.7$ & settling time is $\frac{1}{0.7}$



$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{9k}{s^2+2s+9+(9kTs+9k)}$$

$$= \frac{9k}{s^2+(2+9kT)s+9+9k}$$

$$T_s = \frac{1}{0.7} = \frac{4}{0.7\,w_n} \Rightarrow w_{nc} = 4 \text{ rad/s}$$

$$9+9k = 4^2 = 16 \Rightarrow k = \frac{7}{9}$$

$$\text{also, } \frac{2+9kT}{2\times \underset{4}{\cancel{w_n}}} = 0.7$$

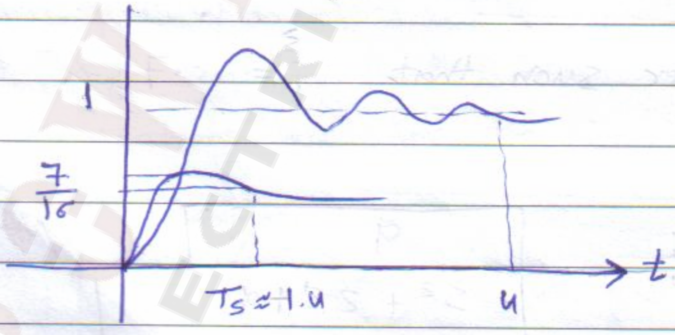
$$9kT = 5.6 \Rightarrow T = \frac{5.6}{9\times\frac{7}{9}} = 0.8$$

new $\xi = 0.7$ previous $\xi = \frac{1}{3}$

new $T_s = \frac{1}{0.7} \simeq 1.4$ previous $T_s = \frac{4}{\frac{1}{3}\times 3} = 4$ sec

however the new steady state is

$$y_{ss} = \frac{9k}{9+9k} = \frac{9\times\frac{7}{9}}{16} = \frac{7}{16}$$

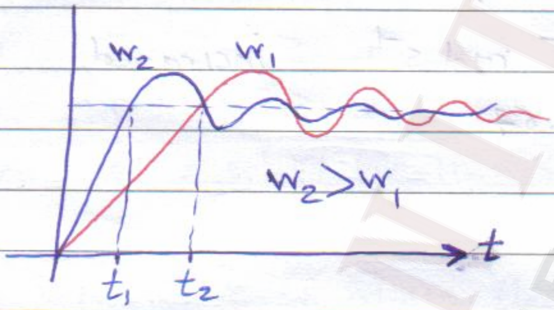


⊛ N.B :-

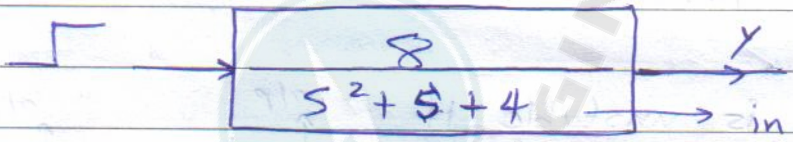
① The overshoot is determined by ξ only

$$M_p = 1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

② Natural freq Wn determine the speed of response



③ To increase damping, introduce a term involving ξ in the transfer function



→ in second order

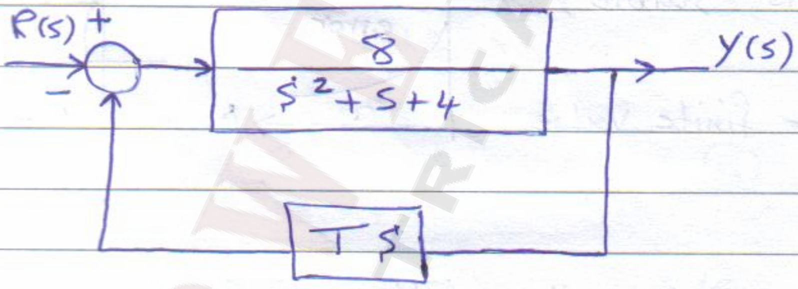
same sign → stable

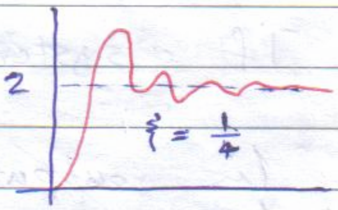
$y_{ss} = \frac{8}{4} = 2$

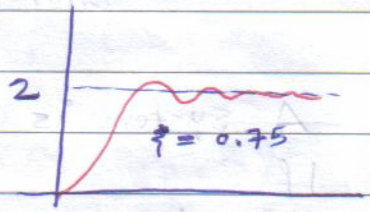
provided system is stable

$W_n = \sqrt{4} = 2 \text{ rad } s^{-1}$

$\xi = \frac{1}{2\times 2} = \frac{1}{4}$



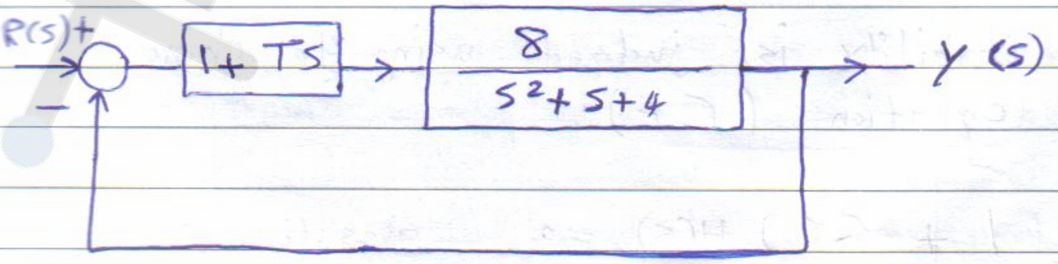




$$\frac{Y(s)}{R(s)} = \frac{8}{s^2+(8T+1)s+4}$$

$\xi = \frac{8T+1}{2\times 2} = 0.75$, If $T = \frac{1}{4}$

OR



$$\frac{Y(s)}{R(s)} = \frac{8(1+Ts)}{s^2+(1+8T)s+12}$$

$W_n = \sqrt{12} = 2\sqrt{3}$ rad s^{-1} increased

$$y_{ss} = \frac{8}{12} = 0.67$$

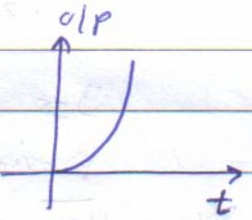
$$\xi = \frac{1+8T}{2 \times 2\sqrt{3}}$$

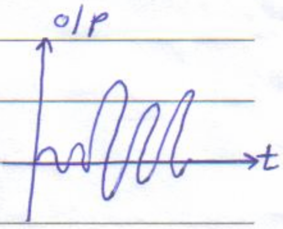
(*) Routh's Stability Criterian :-

Instability Definition :-

- A system is unstable If

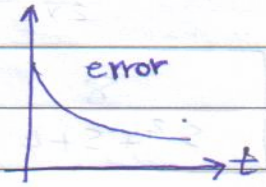
$\lim_{t \to \infty}$ ~~output~~ error $= \infty$

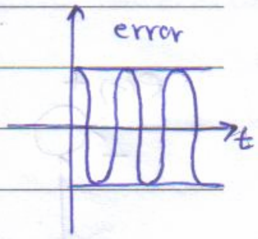




- If a system is stable,

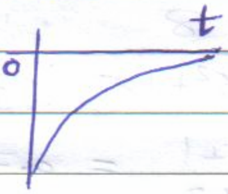
$\lim_{t \to \infty}$ ~~output~~ error = finite value

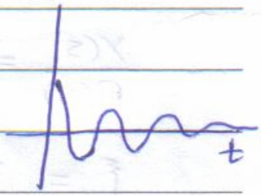




- A system is asymptotically stable If

$\lim_{t \to \infty}$ error $= 0$





(*) Stability is judged using the characteristic equation (CE)

$1 + G(s)\,H(s) = 0$ gives the poles

generally stability of a system is based on the poles (Zeros or roots of the denominator)

① - If all poles have negative real parts then the system is asymptotically stable.

② - * If the CE has a single Zero pole then the system is stable (where the remaining ones have all negative parts).

③ - If the CE has a pair of pure imaginary poles then the system is stable.

e.g $G(s) = \frac{1}{(s^2+4)(s+5)}$, $\frac{1}{(s^2+4)^2(s+5)}$ unstable

* $G(s) = \frac{1}{s^2(s+4)}$ unstable

*ex :- Determine stability, asymptotic stability or instability given the following transfer function.

① $\frac{s+3}{s^2+3s+2}$ ④ $\frac{s^2+3s+2}{s^2(s+8)^2}$

② $\frac{s+4}{(s+2)^3(s+5)}$?? ⑤ $\frac{1}{(s^2+2s+2)^2}$

③ $\frac{s}{s^3+6s^2+11s+80}$ ⑥ $\frac{s^2+9s+20}{s^3-6s^2+11s+16}$

Better than finding poles, we use rough's stability criterian which is applied to the CE (the denominator of the T.F)

The method is best illustrated by a numerical example.

* ex : Given $1 + G(s)\,H(s) = 0 + s^3 + 6s^2 + 11s + 6$

s^3	1	11	0
s^2	6	6	0
s^1	$\frac{66-6}{6}$	0	
s^0	$\frac{60-0}{10} = 6$		

$+\ s^3$	1	11	0
$+\ s^2$	6	6	0
$+\ s^1$	10	0	
$+\ s^0$	6		

Since there is no change in sgn, the system is stable.

لو نفس الإشارة : stable

$s^3 + 6s^2 + 11s + 6 = 0$

$(s+1)(s+2)(s+3) = 0$

poles $-1, -2, -3$

$\Rightarrow$ stable

* ex : $1 + G(s)\,H(s) = 0 = s^3 + 8s^2 + 11s - 20$

s^3	1	11
s^2	8	-20
s^1	13.5	0
s^0	-20	

Since there is a change is sign, the system is unstable $\Rightarrow$ a single polar with +ve real part.

$\Rightarrow$ Changing in sign $\longrightarrow$ stable

$(s+4)(s+5)(s-1) = 0$

polar $\underset{✓}{-4}, \underset{✓}{-5}, \underset{\times}{+1}$ unstable

* ex : $-s^3 - 8s^2 - 11s - 20 = 0$

$-s^3$	-1	-11
$-s^2$	-8	-20
$-s^1$	-68/8	0
$-s^0$	-20	

since the is no change in sign, the system is stable.

imag part, not include as in stable ←

x $\quad$ $\textcircled{x_1}$

x / 0

$\textcircled{x_1}$ same

<u>poles</u> $\quad -6.8$

$-0.59 \pm 1.6j$

* <u>ex</u> :- $1+G(s)H(s) = s^5 + 8s^4 + 2s^3 + 8s^2 + s + 10 = 0$

	$+\ s^5$	1	2	1
	$+\ s^4$	8	8	10
	$+\ s^3$	1	-0.25	0
①	$+\ s^2$	10	10	
	$-\ s^1$	-1.25	0	
②	$+\ s^0$	10		

two changes in sign
$\Rightarrow$ unstable
with two poles of +ve real parts

N.B : If the CE has a change in sign then the system is unstable.
If the CE has a missing power of s then the system is unstable.

$\Rightarrow$ The case where the potential element is zero.

In this case either
1) replace the zero (0) element by a +ve ϵ.
2) replace the (s) by $\left(\frac{1}{s}\right)$
3) multiply the CE by $(s+1)$

* <u>ex</u> : $CE = 2s^4 + 10s^3 + s^2 + 5s + 8 = 0$

+	s^4	2	1	8
+	s^3	10	5	⓪
+	s^2	$\cancel{0}\ \epsilon$	8	
−	s^1	$\boxed{\frac{5\epsilon-80}{\epsilon}}$	⓪	
+	s^0	8		

two poles +ve real parts ⇒ unstable

$$\lim_{\epsilon\to0} \frac{5\epsilon-80}{\epsilon}$$

$$\lim_{\epsilon\to0} 5-\frac{80}{\epsilon} = -\infty$$

⇒ Exercise :- Apply the other two procedure to conform instability

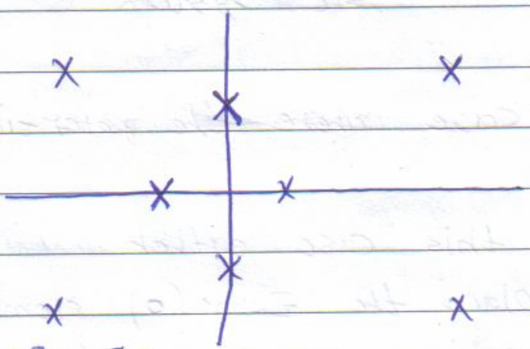
$$\frac{2}{s^4}+\frac{10}{s^3}+\frac{1}{s^2}+\frac{5}{s}+8=0$$

$$8s^4+5s^3+s^2+10s+2=0$$

⇒ The case where the all element in a row are Zero

This case arises when some poles have symmetry about the origin.

This method is best illustrated by an example



$$CE = 5s^5+4s^4+10s^3+8s^2+5s+4=0$$

s^5	5	10	5
s^4	4	8	4
s^3	$\cancel{0}\ 16$	$\cancel{0}\ 16$	0
s^2	4	4	
s^1	$\cancel{0}\ 8$	0	
s^0	4		

no change in sign hence stable

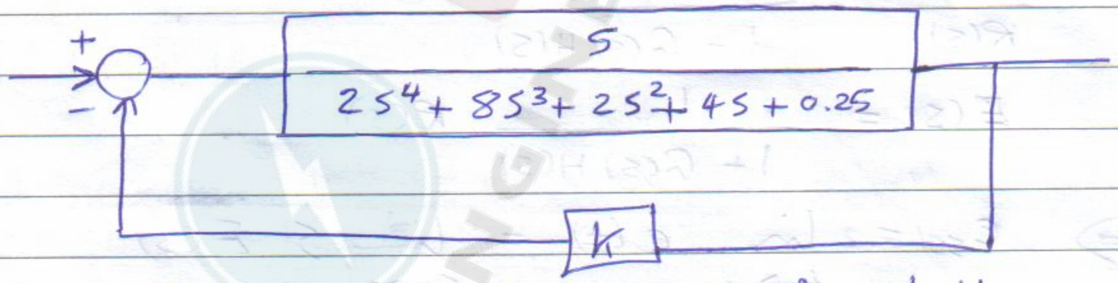
auxiliary equation ⇒ زوجي

$A(s) = 4s^{④} + 8s^{②} + 4s^{⓪}$ $\qquad$ $A(s) = 4s^2 + 4$

$\dfrac{dA(s)}{ds} = 16s^3 + 16s$ $\qquad$ $\dfrac{dA(s)}{ds} = 8s$

* using Rough's to determine k for stable

Best illustrated by an example.



⇒ prove that the open loop system is stable

$$\Rightarrow 1 + H(s)G(s) = 0 = 1 + \frac{5k}{2s^4 + 8s^3 + 2s^2 + 4s + 0.25} = 0$$

$$2s^4 + 8s^3 + 2s^2 + 4s + 5k + 0.25 = 0$$

+	2	2	5k+ 0.25
+	8	4	0
+	1	5k+ 0.25	
→	$\boxed{4 - 40k - 2} \rightarrow 2 - 40k$		
	5 k + 0.25		

We require

$2 - 40k > 0 \Rightarrow k < 0.05$

$5k + 0.25 > 0$

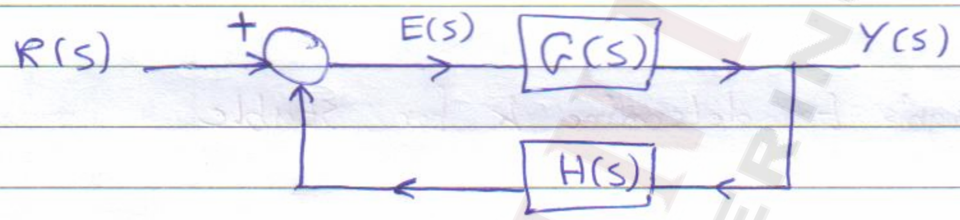
$k > -0.05$

⇒ For stability

$-0.05 < k < 0.05$

⊛ Static error Coefficient

Consider the following closed loop system



$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$E(s) = \frac{1}{1+G(s)H(s)} R(s)$$

$$\Rightarrow \quad e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s)$$

valid provided the closed loop system is stable

$$\Rightarrow \quad \text{Suppose} \quad G(s) = \frac{K \prod_{i=1}^{m} (s+z_i)}{s^n \prod_{j=1}^{n} (s+p_j)}$$

N is known as type of the system

⊛ Position error Coefficient

associated with the unit - step input

i.e $R(s) = \frac{1}{s}$

$$e_{ss} = \lim_{s\to 0} \cancel{s} \frac{1}{1+G(s)H(s)} \frac{1}{\cancel{s}}$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\lim_{s\to 0} G(s)H(s)}$$

$$= \frac{1}{1+\boxed{K_p}}$$ Position error constant

⟹ If $N=0$, no integrator

$$K_p = \frac{K Z_1 Z_2 --- Z_m}{P_1 P_2 --- P_m} = K' = \text{finite value}$$

$$e_{ss} = \frac{1}{1+K'} = \text{finite error.}$$

⟹ If $N \geq 1$, one integrator

$$K_p = \infty$$

$$e_{ss} = \frac{1}{1+\infty} = 0 \Rightarrow \underline{\underline{i.e}}$$ Zero steady state error.

⊛ Error Coeffients due to a unit ramp.

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+G(s)H(s)} \cdot \frac{1}{s^{\cancel{2}}}$$

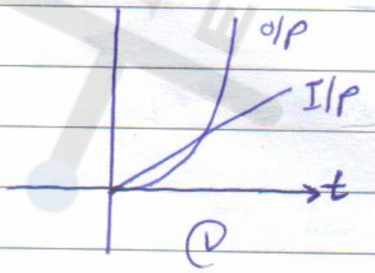
$$e_{ss} = \lim_{s\to 0} \frac{1}{s\,G(s)H(s)} = \frac{1}{\boxed{K_v}} \rightarrow$$, velocity error constant

where $K_v = \lim_{s\to 0} s\,G(s)\,H(s)$

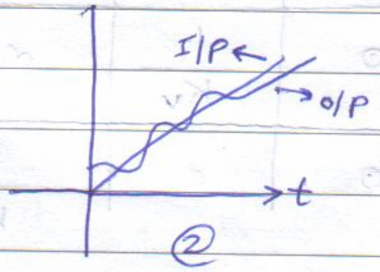
If $N=0 \Rightarrow K_v = 0 \Rightarrow e_{ss} = \frac{1}{0} = \infty$ --- ①

$N=1 \Rightarrow K_v = \text{finite} \Rightarrow \frac{1}{K_v} = \text{finite}$ --- ②

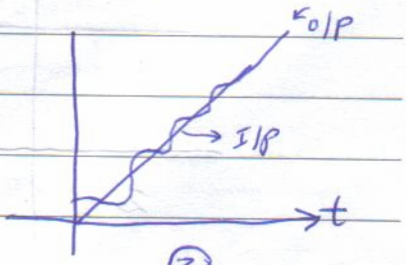
$N \geq 2 \Rightarrow K_v = \infty \Rightarrow \frac{1}{\infty} = 0$, --- ③



①



②



③

⊛ Error Coefficients due to a unit acceleration

$$e_{ss} = \lim_{s \to 0} \cancel{s} \frac{1}{1+G(s)H(s)} \frac{1}{s^{\cancel{3}2}}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{s^2 G(s)H(s)} = \frac{1}{\lim_{s \to 0} s^2 G(s)H(s)}$$

$$e_{ss} = \frac{1}{K_a} \quad \text{where}$$

$$K_a = \lim_{s \to 0} s^2 G(s)H(s)$$

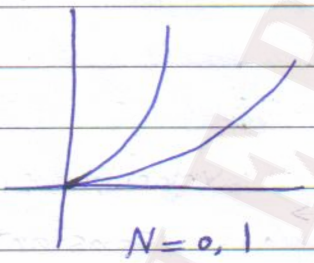
If

$N=0 \Rightarrow K_a = 0 \Rightarrow e_{ss} = \frac{1}{0} = \infty$

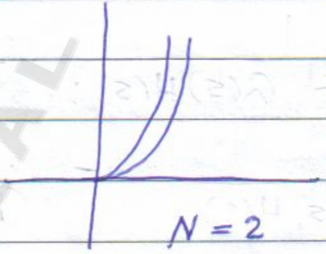
$N=1 \Rightarrow K_a = 0 \Rightarrow e_{ss} = \frac{1}{0} = \infty$

$N=2 \Rightarrow K_a = \text{finite} \Rightarrow e_{ss} = \frac{1}{\text{finite}} = \text{finite}$

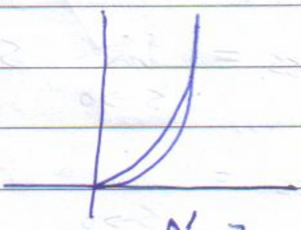
$N \geq 3 \Rightarrow K_a = \infty \Rightarrow e_{ss} = \frac{1}{\infty} = 0$



$N=0,1$



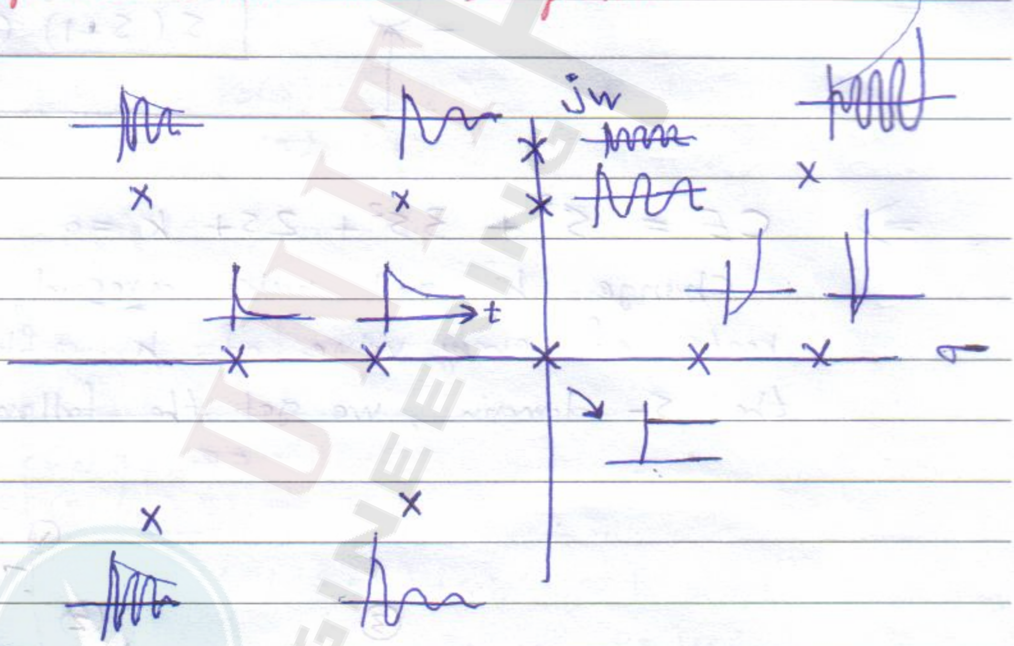
$N=2$



$N=3$

N	error due to step	ramp	acceleration
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$

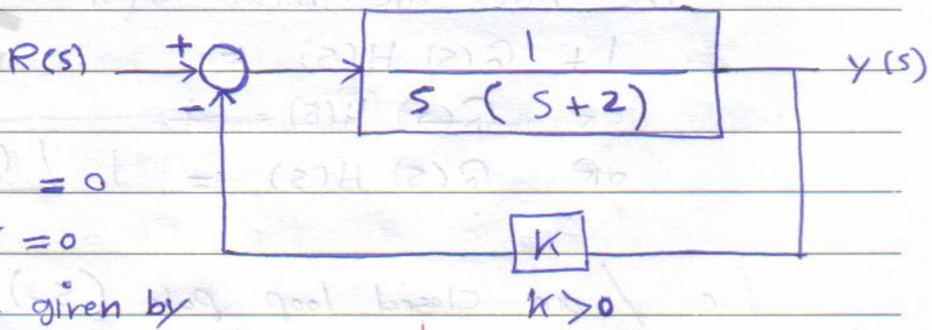
⊛ Time response over the s-plane.



⊛ The root locus (RL) المحل الهندسي

Is the locus of all root of the CE as a function of a certain parameter.

motivational example : Consider the following system.

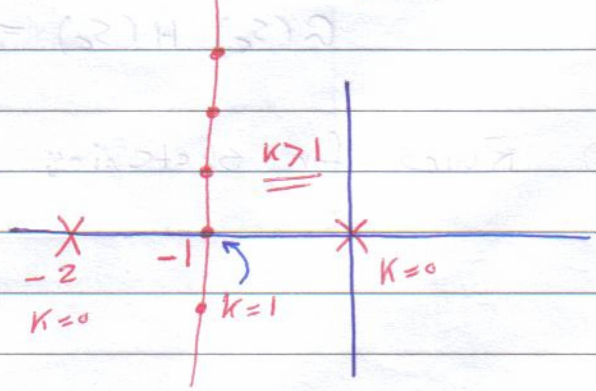


CE = $1 + G(S)H(S) = 0$

$S^2 + 2S + K = 0$

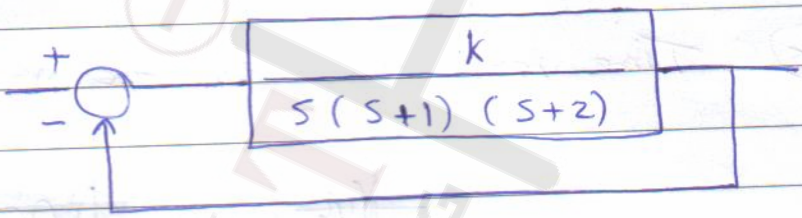
Closed loop poles are given by

$S = -1 \pm \sqrt{1-K}$



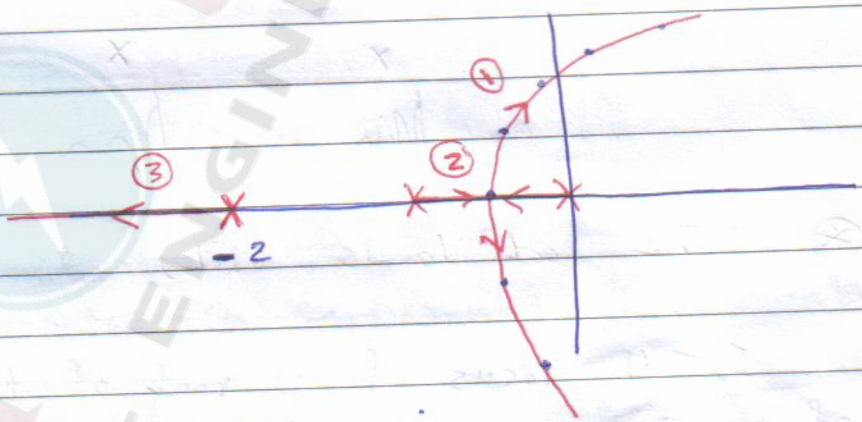
The system is stable for all K>0

since the RL lies in the left side of the S-plane *ex :



⇒ $CE = S^3 + 3S^2 + 2S + k = 0$

Change k and make a record of the three roots of every value of k after plotting on the S- domain, we get the following.



⊛ The root locus is plotted using certain rules without resolting to root finding techniques

The rules are based upon

$1 + G(S)\,H(S) = 0$

oR $G(S)\,H(S) = -1$

oR $G(S)\,H(S) = 1 \;\angle\underline{(1 \pm 2h)180^{\circ}}$

i.e / an closed loop pole (S_c) make

$G(S_c)\,H(S_c) = -1$

⊛ Rules for sketching the root locus (RL)

Starting with the CE of the system represent it as

$$1 + k\frac{Z(s)}{p(s)} = 0$$

where k is the variable parameter.

rearranging

$$K\frac{Z(s)}{p(s)} = -1$$

hence $K\left|\frac{Z(s)}{p(s)}\right| = 1$

$$\angle K\frac{Z(s)}{P(s)} = (1 \pm 2h)\,180^\circ \;;\; h = 0,1,2,\ldots$$ angle Condition

odd multiples of 180°

*ex: $CE = \underline{s^3 + 6s^2 + 11s + 6} + k = 0$

$$\Rightarrow \quad 1 + \frac{K}{(s+1)(s+2)(s+3)} = 0$$

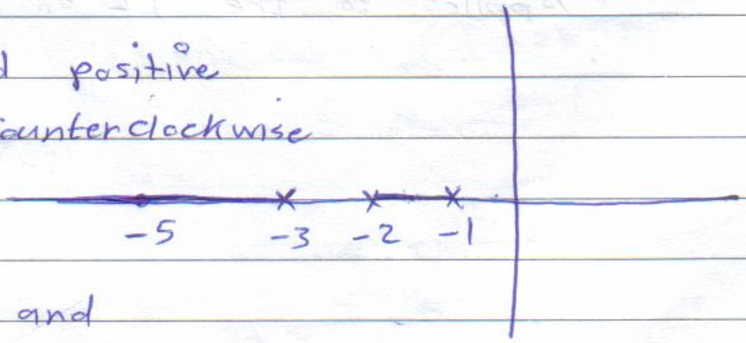
$$\frac{K}{(s+1)(s+2)(s+3)} = -1$$

−5 is a closed loop pole since

$$\frac{|K|}{|-5+1||-5+2||-5+3|} = 1 \quad \text{when } k = 24$$

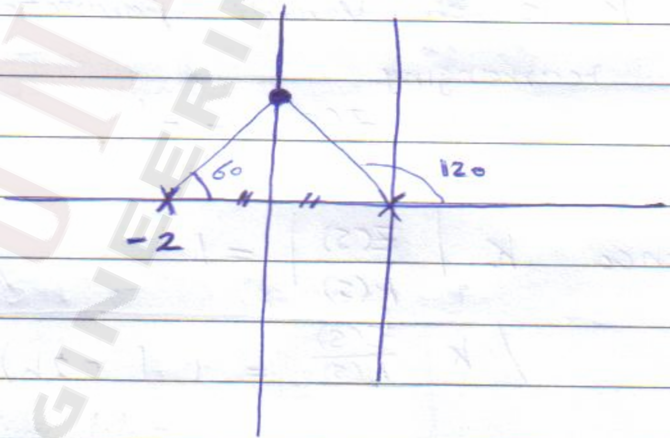
angles are considered positive measured in the Counterclockwise direction

They are measured from the other poles and Zero to the point under consideration.



* **ex:** $CE = s^2 + 2s + k = 0$

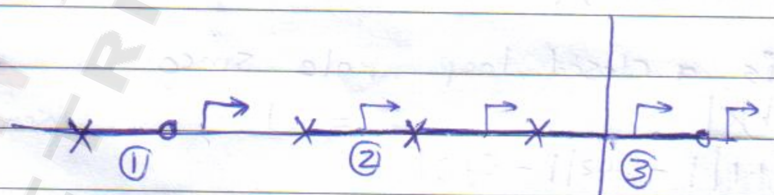
$$\Rightarrow \quad 1 + \frac{K}{s(s+2)} = 0$$



Rule: A point lies on the RL if the sum of angles due to the Zeros and poles is odd multiple of 180°

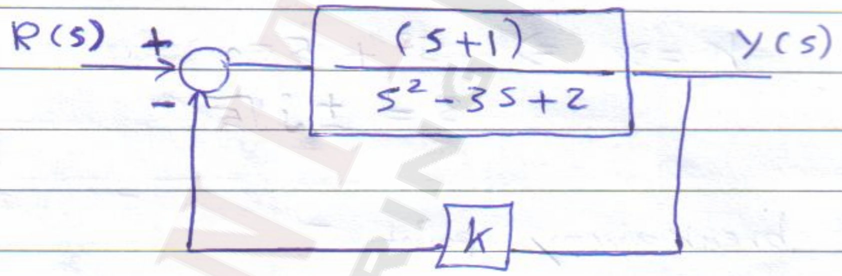
Based on this If the number of poles and Zero to the right of the point is odd then the point lies on the root locus.

⇒



(Applies to the RL on the real axis)

*ex : Consider the following system



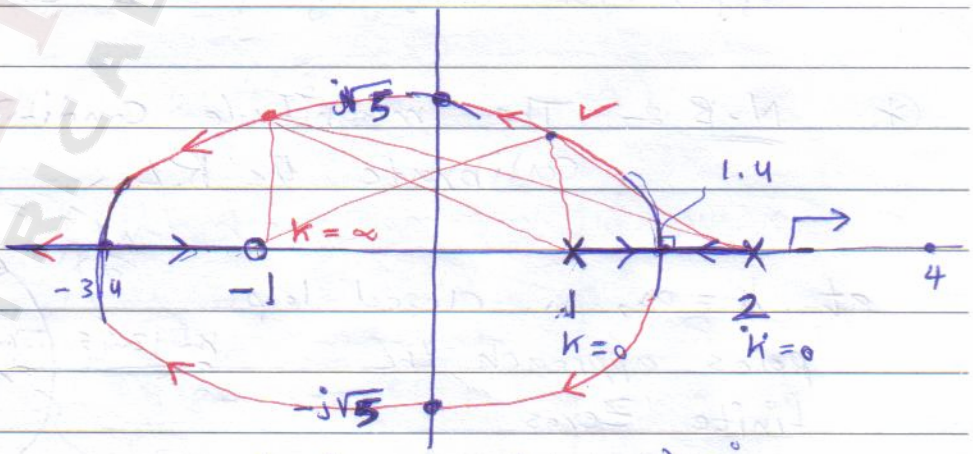
$$\Rightarrow \quad CE = 1 + \frac{k(s+1)}{s^2 - 3s + 2} = 0$$

$$\underline{s^2 - 3s + 2} + k(s+1) = 0$$

$$1 + k\,\frac{s+1}{(s-1)(s-2)}$$

Zeros @ -1

Poles @ 1 , 2



- assymptotes makes angle 180° $\frac{(1 \pm 2h)180^\circ}{2-1}$
  (on real axis)
- $\sigma = \frac{(1+2) - (-1)}{2-1} = 4$
- intersection of the RL with the jw-axis
  (use Rouths)

$s^2 + (k-3)s + k+2 = 0$

$k = 3 \Rightarrow s^2 + 5 = 0$

$s = \pm j\sqrt{5}$

s^2	1	$k+2$
s^1	$k-3$	0
s^0	$k+2$	

- breakaway point

$$k = \frac{-s^2 - 3s + 2}{s+1}$$

$$\frac{dk}{ds} = \frac{(2s-3)(s+1) - 1 \times (s^2 - 3s + 2)}{(s+1)^2} = 0$$

$s^2 + 2s - 5 = 0$

$s = 1.4, -3.4$

- departure angle dosen't apply (because it is apply for imaginary poles & Zeros).

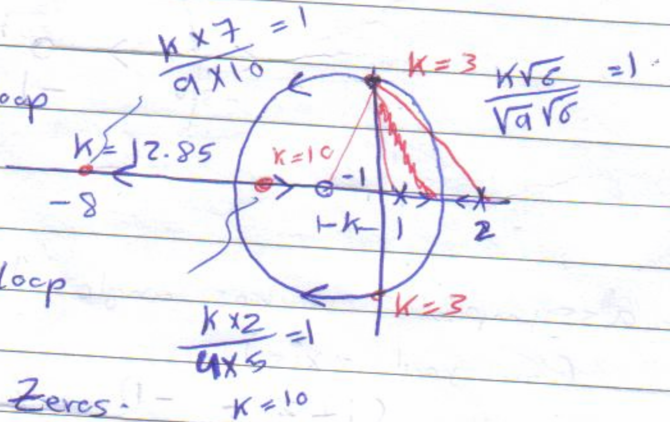
(*) **N.B**:- The magnitude Condition is needed to calibrate the R.L.

at $k = \infty$, m closed loop poles approach the finite Zeros

n-m closed loop poles approach ± ∞

where m is no of Zeros.

n is no of poles.



The other closed loop associated with k = 12.85 is

$\frac{12.85 \times (z-1)}{(z+1)(z+2)} = 1$ Solve for z to get the second closed loop pole.

⊛ Time response graphically

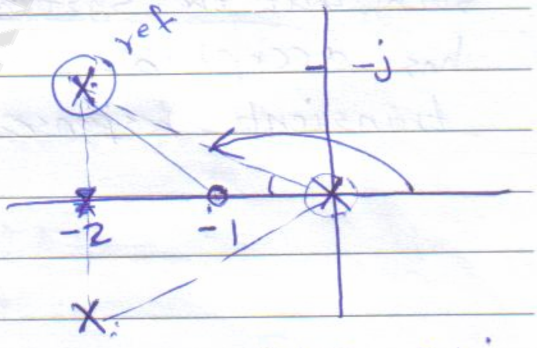
The case of complex poles

$$\mathcal{L}^{-1}\frac{s+1}{s(s^2+4s+5)} = \mathcal{L}^{-1}\frac{|A|}{s} + 2|B|\,e^{-2t}\cos(\overset{1}{\cancel{\ }}t+\theta)$$

$$|B| = \frac{\sqrt{2}}{\sqrt{5}\times 2} = \frac{1}{\sqrt{10}}$$

$$\theta = (180^\circ - 45^\circ) - (180^\circ - \tan^{-1}\frac{1}{2} + 90^\circ)$$

$$= -135^\circ + 26^\circ = -109^\circ$$



Poles = −2 ± j

$$A = \frac{1}{\sqrt{5}\sqrt{5}} = \frac{1}{5}$$

***ex** :- find $\mathcal{L}^{-1}\dfrac{s}{(s+1)(s^2+4s+5)}$ graphically.

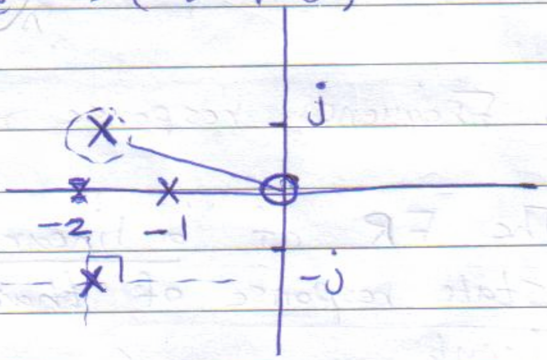
$$\Downarrow$$
$$-2\pm j$$

$$\mathcal{L}^{-1}(-) = \mathcal{L}^{-1}\frac{A}{s+1} + 2|B|\,e^{-2t}\cos(t+\theta)$$

$$A = \frac{1\angle 180^\circ}{\sqrt{2}\angle\psi\ \sqrt{2}\angle-\psi} = \frac{-1}{2}$$

$$B = \frac{\sqrt{5}}{\sqrt{2}\times 2} = \frac{\sqrt{5}}{\sqrt{8}}$$

$$= \sqrt{\frac{5}{8}}$$



$$\theta = (180^\circ - \tan^{-1}\frac{1}{2}) - (180^\circ - \tan^{-1}\frac{1}{1} + 90^\circ)$$

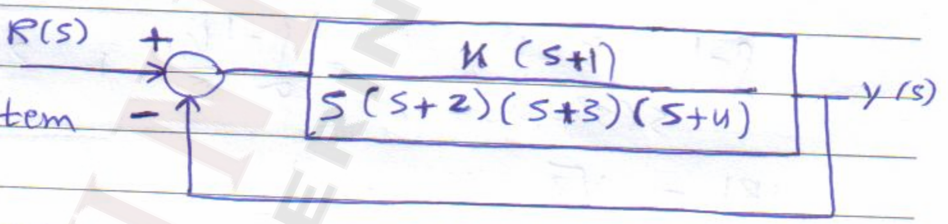
$$= 180^\circ - 26^\circ - 18^\circ + 45^\circ - 90^\circ = -71^\circ$$

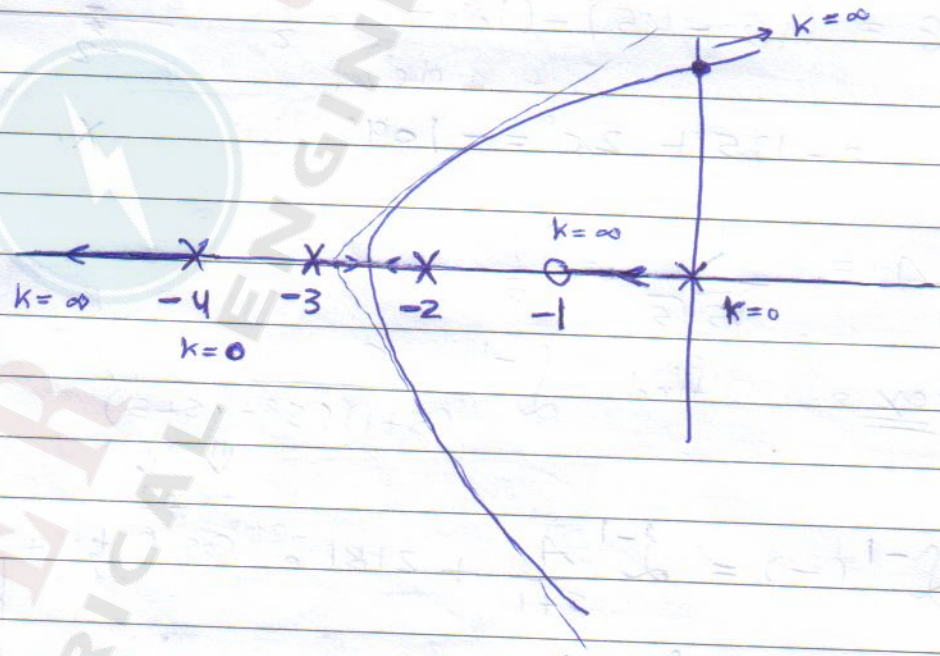
$$\therefore \mathcal{L}^{-1} - = \frac{1}{2}e^{-t} + 2\sqrt{\frac{5}{8}}\cos(1t - 71^\circ)$$

* ex 8 Design using the Root locus.

Consider the following system

Choose a k such that the system has acceptable transient response.



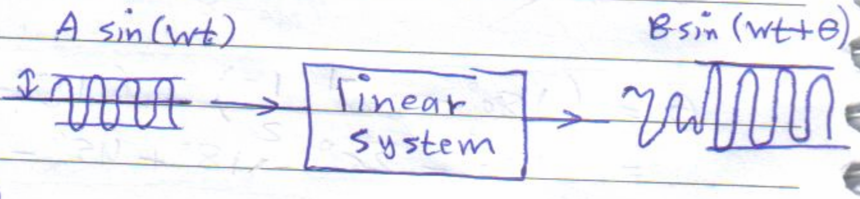


Frequency response methods (FR).

The FR of a linear system is the steady state response of that system due to a sinusoidal input

i/p = A sin (wt)

steady state o/p = B sin (wt + θ)



⇒ Freq response uses

gain = $\frac{B}{A}$, phase shift = θ

A table is generated

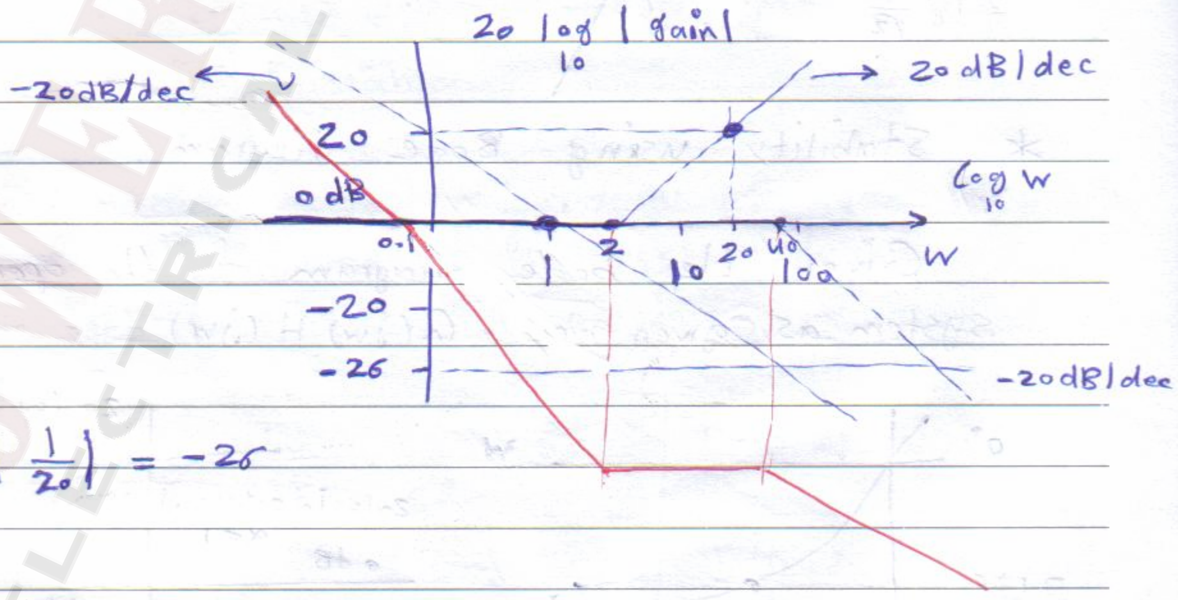
W	gain	phase shift
W_1	G_1	θ_1
W_2	G_2	θ_2
|	|	|

⟹ A suitable graphing method is sought to represent the system information as given by the date points.
The bode diagram is a Convenient one.

*ex: Let $G(s) = \dfrac{s+2}{s(s+40)}$

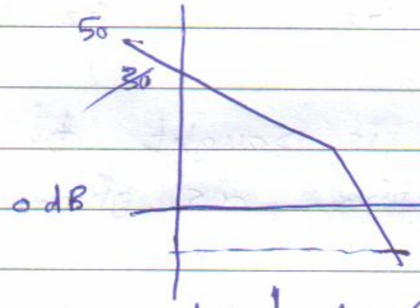
$$\Rightarrow G(s) = \frac{2(1+\frac{s}{2})}{40(1+\frac{s}{40})s} = \frac{1}{20} \cdot \frac{(1+\frac{s}{2})}{(1+\frac{s}{40})s}$$

$$G(jw) = \frac{1}{20} \cdot \frac{1+j\frac{w}{2}}{j\frac{w}{1}(1+\frac{jw}{40})}$$



$$20\log_{10}\left|\frac{1}{20}\right| = -26$$

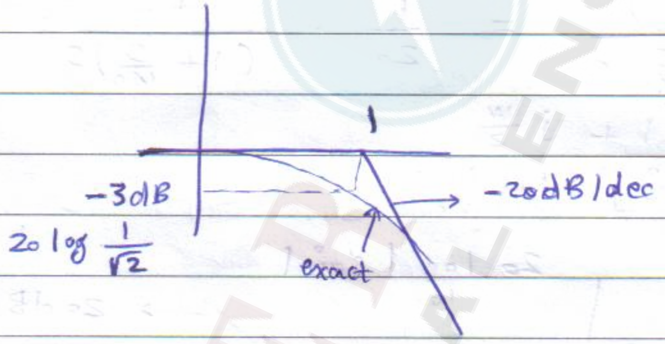
***ex :-** given $20 \log_{10} |G(jw)|$ as shown



$20 \log |e^{jwT} G(jw)|$ plot is the same as the $20 \log_{10} |G(w)|$ plot

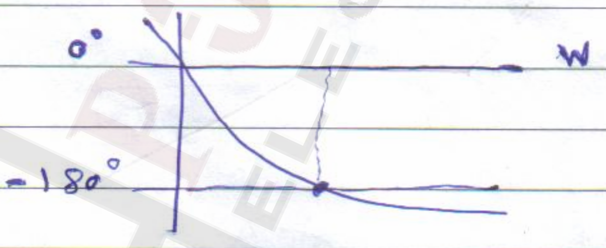
$20 \log_{10} |-10 G(jw)|$

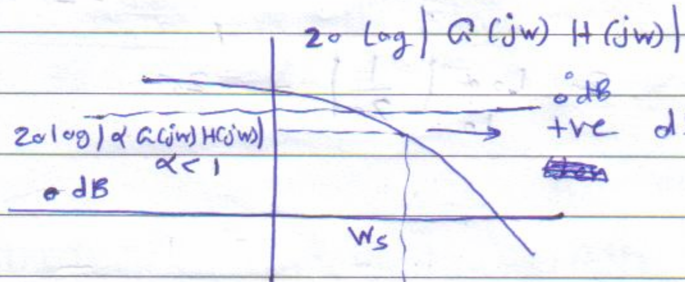
***ex :-** given $\frac{1}{1+jw}$



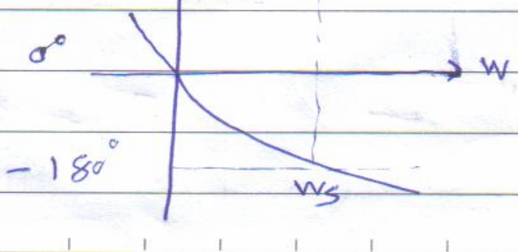
* Stability using Bode diagram

Given the bode diagram of the open loop system as given by $G(jw)\, H(jw)$





at $w = w_s$ where $\angle G(jw)\, H(jw) = -180°$



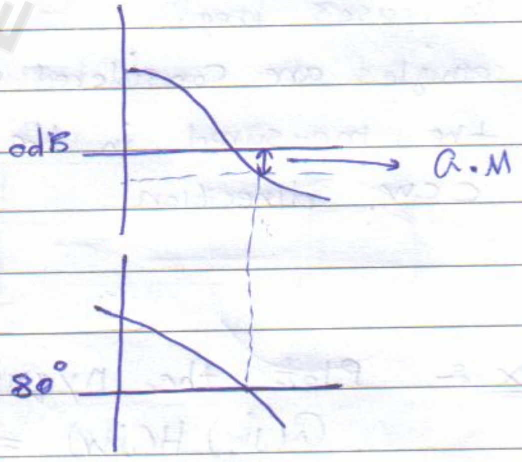
U3

If $20 \log_{10} |G(jw)\, H(jw)| \geq 0$ dB then the closed loop system is unstable.

⊛ Gain and phase margins :-

⟹ Gain margin is the amount in dB by which the system gain can be increased before the onset of instability.

عتبة



Suppose $GM = 6\,dB$

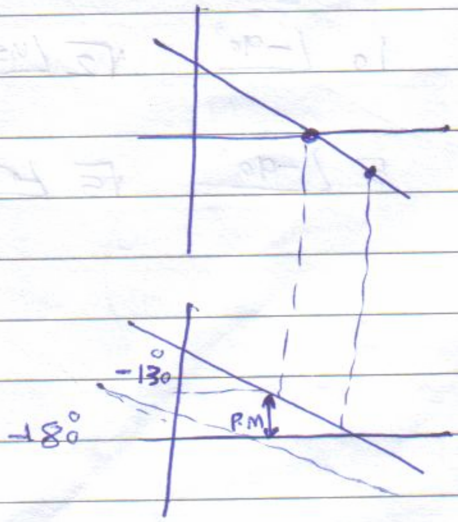
⟹ $6 = 20 \log k$

$k = 10^{\frac{6}{20}} \cong 2$

So, If the system gain multiplied by 2, the closed loop system will be unstable.

⟹ Phase margin :- is the amount of negative phase that can be involed to the system to make the closed loop system unstable.

$P.M = -130 - (-180°) = 50°$

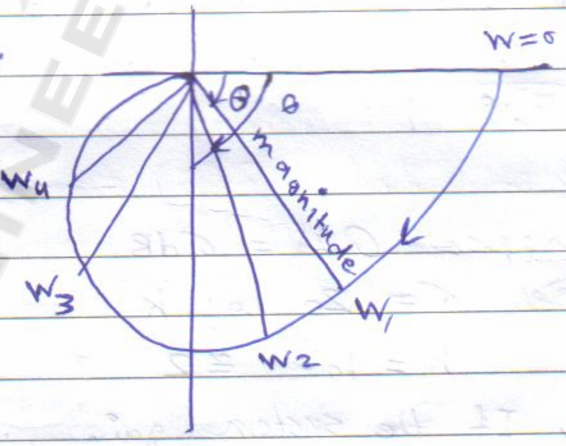


(*) Another method for representing frequency response data is the nyquist diagram.

Nyquist diagram

⇒ It's a polar plot with w as a parameter

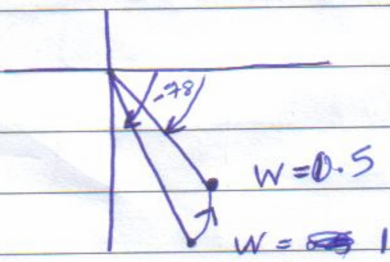
- direction of arrow indicates increases freq.
- angles are considered the measured in the ccw direction.



*ex: Plot the nyquist Diagram (ND) given

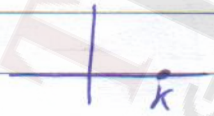
$$G(jw)\,H(jw) = \frac{10\,(1+jw)}{jw\,(1+j\frac{w}{2})}$$

W	$\frac{10}{jw}$	$1+j\frac{w}{1}$	$\frac{1}{1+j\frac{w}{2}}$	$G(jw)\,H(jw)$
0.5	$20\angle -90^\circ$	$\sqrt{1.25}\angle 26^\circ$	$\frac{1}{\sqrt{\frac{17}{16}}}\angle -14^\circ$	$20\sqrt{1.25}\,\frac{4}{\sqrt{17}}\angle -90+26-14^\circ$
1	$10\angle -90^\circ$	$\sqrt{2}\angle 45^\circ$	$\frac{2}{\sqrt{5}}\angle -26^\circ$	$\frac{20\sqrt{2}}{\sqrt{5}}\angle -90^\circ+45^\circ-26^\circ$
2	$5\angle -90^\circ$	$\sqrt{5}\angle 63^\circ$	$\frac{1}{\sqrt{2}}\angle -45^\circ$	—



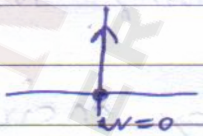
⊛ ND of certain transfer function.

$G(j\omega) = K > 0$

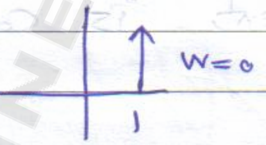


$G(j\omega) = K < 0$

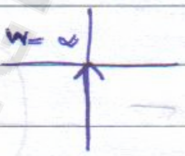
$G(j\omega) = j\omega$



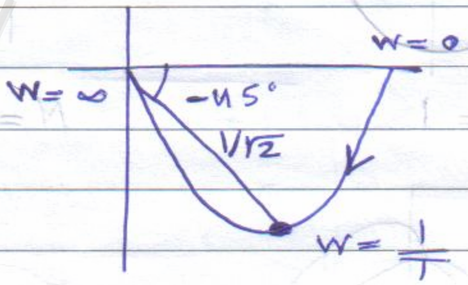
$G(j\omega) = 1 + j\omega T$



$G(j\omega) = \frac{1}{j\omega}$

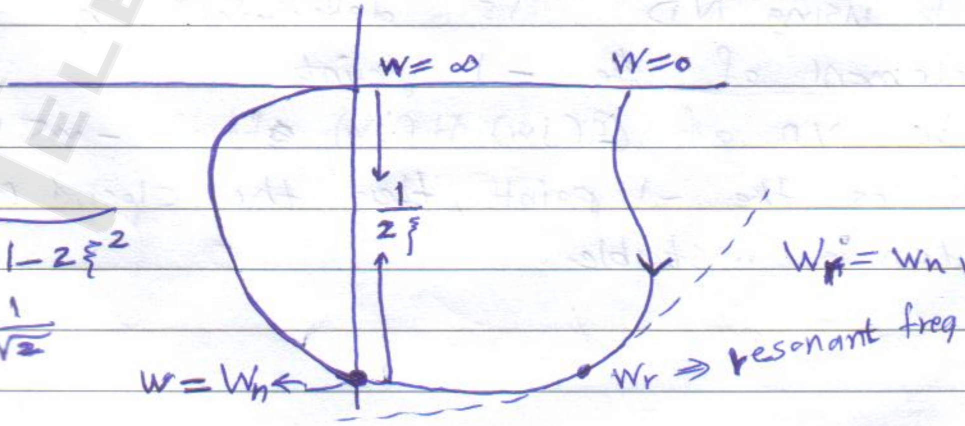


$G(j\omega) = \frac{1}{1 + j\omega T}$



$\Rightarrow$ $G(s) = \frac{W_n^2}{s^2 + 2\xi W_n s + W_n^2}$

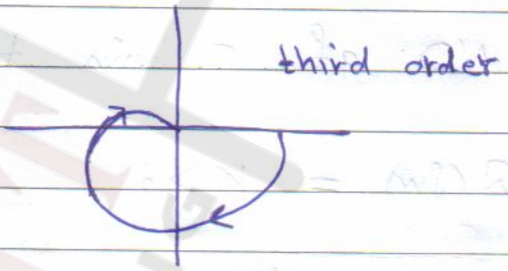
$G(j\omega) = \frac{W_n^2}{W_n^2 - W^2 + j2\xi W_n W}$, $\xi < 1$



$W_r = W_n\sqrt{1-2\xi^2}$

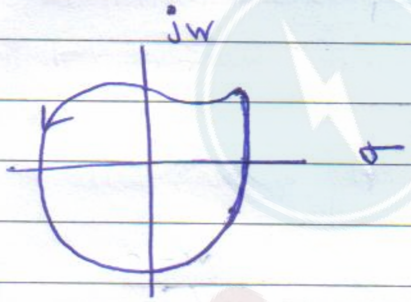
$\xi < \frac{1}{\sqrt{2}}$

$W_r = W_n\sqrt{1-2\xi^2}$

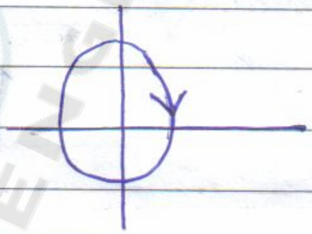


⊛ Stability using ND &

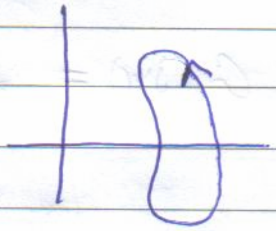
Determined by encirclement (N) of the 1- point encirclement. is considered positive in CCW direction.



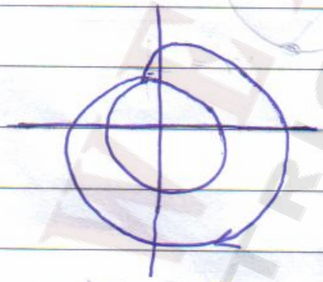
N= 1



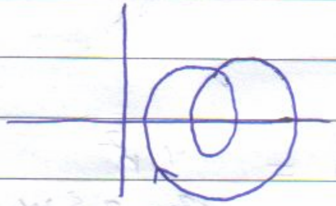
N= -1



N=0



N= -2

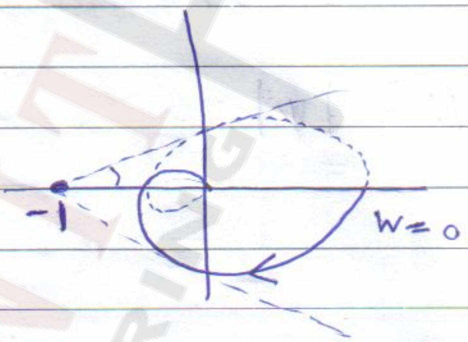


N = 0

Stability using ND is determined by the encirclement of the -1 point.
If the ND of G(jw) H(jw) for $-\infty < w < \infty$ in circles the -1 point, then the closed loop system is unstable.

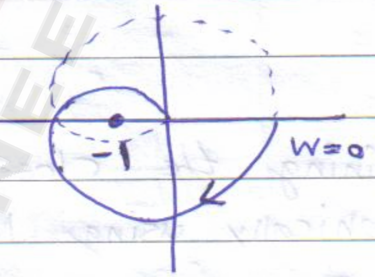
***ex 8-**

N=0, closed loop system is Stable.



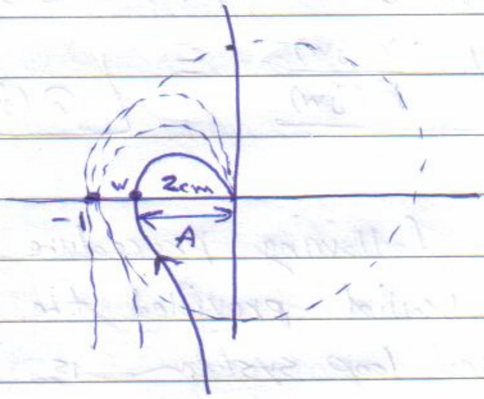
N=-2, 2 poles with +ve real parts

⇒ unstable



⊛ Gain & phase margins.

⇒ No encirclement of the -1 point hence CL system is stable.



$G.M = \frac{1}{A}$

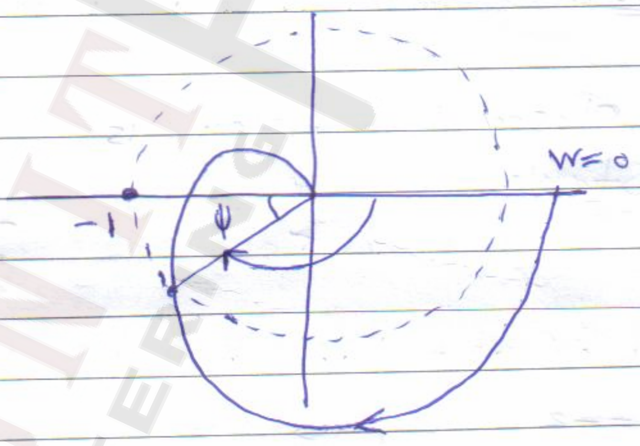
So, if 1 unit = 5cm

$A = \frac{2cm}{5cm} = 0.4$ units

$G.M = \frac{1}{0.4} = 2.5$

*ex :

$P.M = |\psi|$



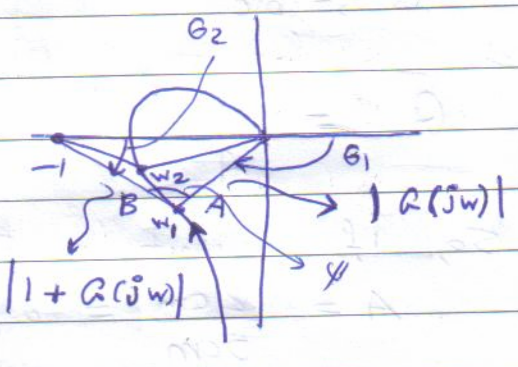
⊛ Obtaining the closed loop freq Response graphically using ND.

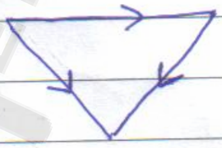
⇒ Valid when $H(jw) = 1$. , unity feedback.

$$\left|\frac{Y(jw)}{R(jw)}\right| = \frac{|G(jw)|}{|1+G(jw)|}$$

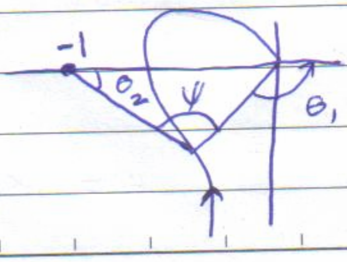
$$\angle\frac{Y(jw)}{R(jw)} = \angle G(jw) - \angle 1+G(jw)$$

The following procedure is valid provided the closed loop system is stable.



① 

$$\frac{|Y(jw)|}{|R(jw)|} = \frac{A}{B}$$



② $\theta_1 - \theta_2 = \psi$ ✓

⊛ State space presentation of linear systems. (SSR)

⟹ SSR is needed to :

① Deal with non linear systems.

② Deal with systems with multi inputs & multi outputs.

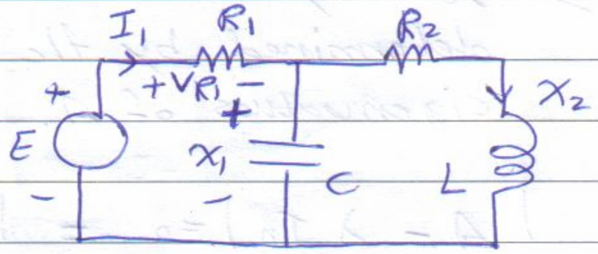
③ Deal with systems with no zero initial conditions

④ Deal with systems with time delay.

⑤ Deal with optimal control system.

⟹ SSR is a time domain approach to systems. State variable are asigned & first order Differential equations are written to describe system behaviour, for linear and non linear

The SSR approach is best illustrated by an example.



Choose $x_1 = V_C$ and $x_2 = i_L$ as states.

Choose $y = V_{R_1}$ as output

$$I_C = I_1 - i_L$$

$$C \frac{dx_1}{dt} = \frac{E - x_1}{R_1} - x_2$$

$$x_1 = R_2 x_2 + L \frac{dx_2}{dt}$$

$$y = E - x_1$$

$$\begin{bmatrix} x_1^{\circ} \\ x_2^{\circ} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C} \\ 0 \end{bmatrix} E$$

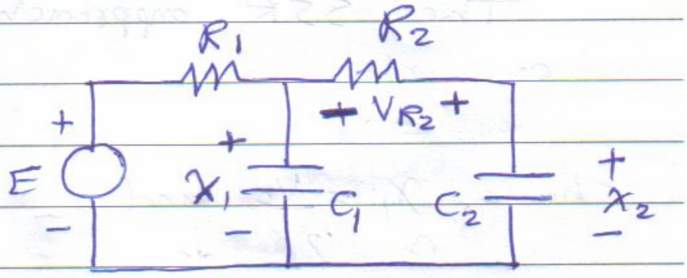
$$\boxed{u = E} \qquad \overset{\circ}{x} = \begin{bmatrix} -\frac{1}{R_1C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C} \\ 0 \end{bmatrix} E$$

$$\overset{\circ}{x} = Ax + Bu \qquad x \in R^n,\ u \in R^m$$

$$y = [-1 \ \ 0]\, x + 1\, E$$

$$y = Cx + Du \qquad y \in R^{l}$$

*ex :- obtain the A, B, C and D matricies for the following circuit.



⇒ Stability determined by the eigenvalues of A.

$$|A - \lambda I_n| = 0 = |\lambda I_n - A|$$

λ_i should all have negative real parts for a system to be assymptotically Stable.

Time Response

It can be shown than

$$x(t) = e^{At} x(0) + \int_{t_0}^{t} e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

e^{At} is known as the exponential matrix or fundamental matrix.

⇒ e^{At} is calculated as :

① $$e^{At} = I_n + At + \frac{(At)^2}{2!} --- + \frac{(At)^n}{n!} + --$$

Suitable for computer determine.

OR

② $$e^{At} = \mathcal{L}^{-1}\left[s I_n - A \right]^{-1}$$

Suitable for closed form determination.

*example : See notes.

*ex : determine e^{At} when

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \; ; \; A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \; ; \; A = \begin{bmatrix} 0 & -8 \\ 1 & -4 \end{bmatrix}$$

⊛ Properties of e^{At} :-

① $e^{At}\Big|_{t=0} = I_n$

② $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$

③ $[e^{At}]^{-1} = e^{A(-t)}$

④ $e^{(A+B)t} \neq e^{At} e^{Bt}$ unless $AB = BA$

⊛ It can be shown If $u(t)$ is a unit step then & $|A| \neq 0$

$$x(t) = e^{At} x(0) + A^{-1}[e^{At} - I_n]B$$

If the system is asymptotically stable, then the steady state value of x due to a unit step is

$$x_{ss} = -A^{-1}B$$

*ex :- $\dot{x} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 40 \end{bmatrix} u$

$\Rightarrow$ $|A - \lambda I_n| = \begin{vmatrix} -\lambda & 1 \\ -20 & -\lambda-9 \end{vmatrix} = 0$

$\lambda^2 + 9\lambda + 20 = 0$

$(\lambda + 4)(\lambda + 5) = 0$

$\lambda_1 = -4$, $\lambda_2 = -5$ $\Rightarrow$ asymptotically stable

$$x_{ss} = -\begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 40 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

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