CONTENTS

PREFACE

This manual has been prepared as an aid to instructors in courses using the book Probability, Random Variables, and Random Signal Principles, 4th edition. It contains solutions to all the problems in the book, and instructors are cautioned that the problem numbers are different than in all prior editions. The reason is that problems have been renumbered according to the book's section to which they apply.

The other structure of the manual is the same as with prior editions. The principal developments needed to understand the solutions of the problems are given. However, for reasonable length, many straightforward algebraic reductions are omitted, and well-known integrals and trigonometric identities are also used without comment. In some important steps in developments suitable references are given for clarity. These references are typically one of the following types:

1. Literature references; these are found either in the text or at the end of this manual.
2. Equation references; for example, (5.3-4) refers to equation (5.3-4) in Chapter 5, Section 5.3, of the text.
3. Pair references; for example, pair 15 is used to refer to pair 15 in the table of Fourier transforms in Appendix E of the text.
4. Other references; these are usually clear enough to be unambiguous. For example, references to other equations within the manual.

As usual, efforts have been made to minimize the number or errors that may exist in the manual on publication. Since some errors may remain, I invite anyone using the manual to advise me of any errors they may identify.

Finally, I am grateful to many students who have helped reduce the number of errors through their having worked many of the problems as course assignments over the last 20 years. Dr. T. V. Blalock independently worked a number of the original problems (first edition) and offered many improvements. Special thanks are extended to Mr. Kenneth Hild, a doctoral student, who worked and plotted the results for all the computer (MATLAB-based) problems (and exercises in the text).

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Gainesville, Florida
June, 2000

CHAPTER 1

1.1-1. $A = \{0 < \text{integers} < 4\}$, $B = \{6 < \text{even integers} < 16\}$, $C = \{0 < \text{odd integers}\}$. Other definitions also are possible.

1.1-2. Class $= \{A, B, C\}$ or class $= \{\{1,2,3\}, \{8,10,12,14\}, \{1,3,5,7,\cdots\}\}$.

1.1-3. A, B, D, E, and F are countable and finite. C is countable and infinite. G, H, and I are uncountable and infinite.

1.1-4. $A = B$, $A \subset C$, $A \subset G$, $A \subset I$.

$B = A$, $B \subset C$, $B \subset G$, $B \subset I$.

C is not equal to, or a subset of, any of the other sets. The same applies to D.

$E \subset D$.

$F \subset D$, $F \subset E$ (Note that F may be a null set.)

G is not equal to, or a subset of, any of the other sets.

$H \subset G$, if x is in meters and negative length is allowed.

I is not equal to, or a subset of, any of the other sets.

1.1-5. $\{\cdot\}$ (null set); $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$; $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$; $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$; $\{a, b, c, d\}$.

1.1-6. (a) $S = \{-40°F \le t \le 130°F\}$, t is temperature. (b) $\{t \le 32°F\}$. (c) $\{32°F < t \le 100°F\}$.

★ **1.1-7.** The null set is one subset. There are N subsets with one element each. Taking elements by pairs corresponds to $\binom{N}{2} = N!/2!(N-2)!$ subsets, which is the number of combinations possible of N things taken 2 at a time. For subsets with three elements the number is $\binom{N}{3} = N!/3!(N-3)!$. Continuing the logic gives

$$\text{Subsets} = \sum_{i=0}^{N} \binom{N}{i} = \sum_{i=0}^{N} \frac{N!}{i!(N-i)!} = 2^N$$

where a series has been used from Jolley (1961, p. 36).

1.1-8. (a) $S = \{-10 \le s \le 10\}$. (b) $V = \{0 \le s \le 10\}$. (c) $S = \{-13 \le s \le 7\}$, $V = \{0 \le s \le 7\}$.

1.1-9. $A = \{-7, -6, -5, \cdots, -1, 0, 1, \cdots, 567\}$, $B = \{-1, 1, 3, 5\}$, $A \subset B$ is not true, $B \subset A$ is true.

1.1-10. $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_3\}$, $A_4 = \{a_1, a_2\}$, $A_5 = \{a_1, a_3\}$, $A_6 = \{a_2, a_3\}$, $A_7 = \{a_1, a_2, a_3\} = A$, $A_8 = \phi$.

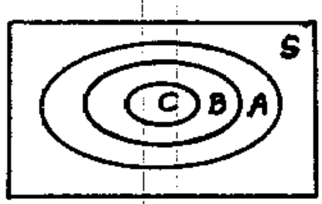
1.1-11. (a) $\{2, 3, 4, 5, 6, 7, 8\}$, $\{1 < I = \text{integers} < 9\}$.

(b) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $\{1 \le I = \text{integers} \le 9\}$.

(c) $R_{equivalent} = R/n$, $n = 1, 2, \cdots, 5$, where $R = 10\,\Omega$, so $\{10, 5, 10/3, 2.5, 2\}$, $\{R = 10/n,\ n = 1, 2, \cdots, 5\}$. (d) $R_{equivalent} = nR$, $n = 1, 2, \cdots, 6$, where $R = 2.2\,\Omega$, so $\{2.2, 4.4, 6.6, 8.8, 11.0, 13.2\}$, $\{R = 2.2\,n,\ n = 1, 2, \cdots, 6\}$.

1.1-12. (a) 60, (b) 100, (c) 35.

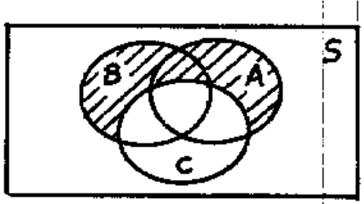
1.2-1. A Venn diagram proves $C \subset A$ if $C \subset B$ and $B \subset A$.



1.2-2. (a) $A - B = \{-6, -4, 1.6, 8\}$. (b) $B - A = \{1, 2, 4\}$.
(c) $A \cup B = \{-6, -4, -0.5, 0, 1, 1.6, 2, 4, 8\}$.
(d) $A \cap B = \{-0.5, 0\}$.

1.2-3. (a) $\bar{A} = S - A = \{6, 8, 12\}$. (b) $A - B = \{2\}$; $B - A = \{6, 8\}$. (c) $A \cup B = \{2, 4, 6, 8, 10\}$.
(d) $A \cap B = \{4, 10\}$. (e) $\bar{A} \cap B = \{6, 8\}$.

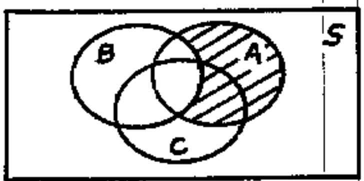
1.2-4. (a)



$(A \cup B) - C$ shaded

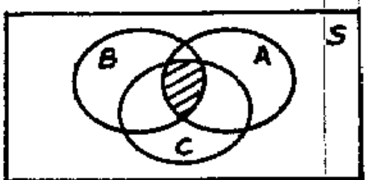
1.2-4. (Continued)

(b)



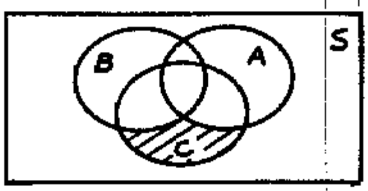
$\bar{B} \cap A$ shaded

(c)



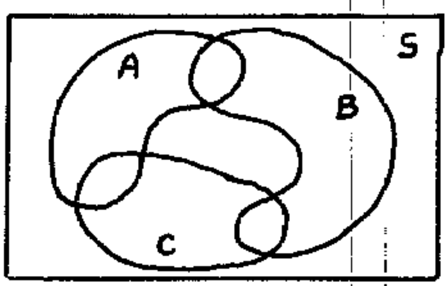
$A \cap B \cap C$ shaded

(d)



$\overline{(A \cup B)} \cap C$ shaded

1.2-5.



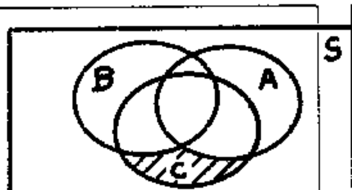
$A \cap B \neq \emptyset$

$B \cap C \neq \emptyset$

$A \cap C \neq \emptyset$

$A \cap B \cap C = \emptyset$.

1.2-6. (a)

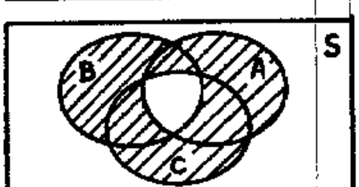


$\overline{(A \cup B)} \cap C$ or

$C - [(A \cap C) \cup (B \cap C)]$

shaded.

(b)

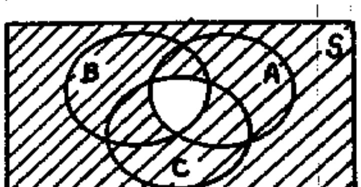


$(A \cup B \cup C) - (A \cap B \cap C)$ or

$(\bar{A} \cap B) \cup (\bar{B} \cap C) \cup (\bar{C} \cap A)$

Shaded.

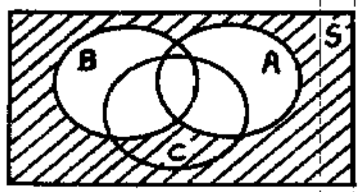
(c)



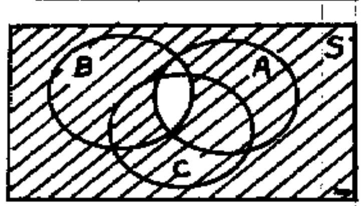
$\overline{A \cap B \cap C}$ or $\bar{A} \cup \bar{B} \cup \bar{C}$

shaded.

(1.2-7.)



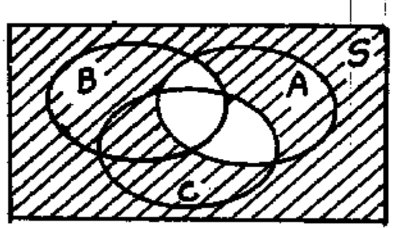
$\overline{A \cup B}$ or $\bar{A} \cap \bar{B}$ Shaded.



$\overline{A \cap B}$ or $\bar{A} \cup \bar{B}$ shaded.

(1.2-8.) (a) $A \cup B = \{-10 \le s < -4\}$. (b) $A \cap B = \{-7 < s \le -5\}$. (c) The set $C = \{-9 \le s \le N\}$ satisfies making $A \cap C$ as large as possible for any $-5 \le N \le -4$. The set $B \cap C$ is largest if $C = \{-9 \le s \le -4\}$. The set satisfying both requirements is therefore $C = \{-9 \le s \le -4\}$. The set $C = \{-9 \le s < -4\}$ is also a valid solution. (d) $A \cap B \cap C = \{-7 < s \le -5\}$.

(1.2-9.) (a) To prove $\overline{A \cap (B \cup C)} = (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \bar{C})$ work with the left side. Use (1.2-13) to get $\overline{A \cap (B \cup C)} = \bar{A} \cup \overline{(B \cup C)}$. Next, use (1.2-12) to get $\overline{A \cap (B \cup C)} = \bar{A} \cup (\bar{B} \cap \bar{C})$. Finally, (1.2-9) is used to obtain the desired equation.

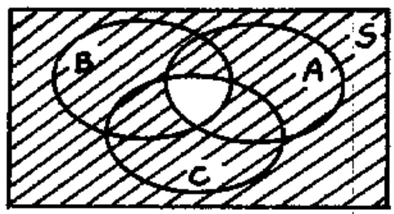


$\overline{A \cap (B \cup C)}$ and $(\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \bar{C})$ shaded.

(b) By using (1.2-13) on the left side of $\overline{A \cap B \cap C} =$

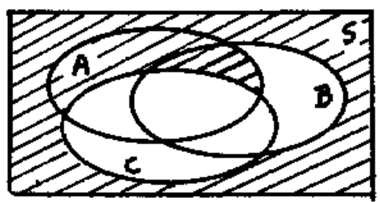
1.2-9. (Continued)

$\bar{A} \cup \bar{B} \cup \bar{C}$ we have $\overline{A \cap B \cap C} = \bar{A} \cup \overline{(B \cap C)} = \bar{A} \cup (\bar{B} \cup \bar{C}) = \bar{A} \cup \bar{B} \cup \bar{C}$.

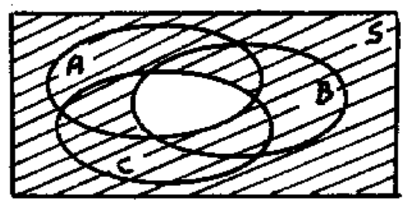


$\overline{A \cap B \cap C}$ and $\bar{A} \cup \bar{B} \cup \bar{C}$ shaded.

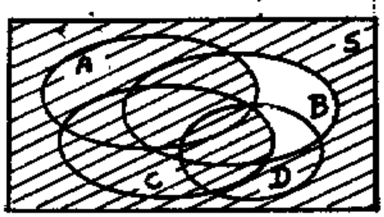
1.2-10.



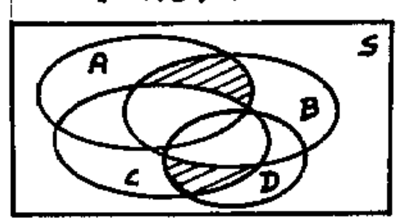
(a) $(A \cup \bar{B}) \cap \bar{C}$



(b) $\overline{(A \cap B)} \cup \bar{C}$

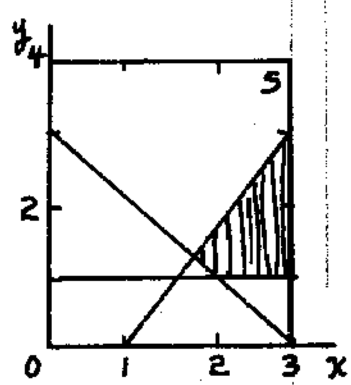


(c) $(A \cup \bar{B}) \cup (C \cap D)$

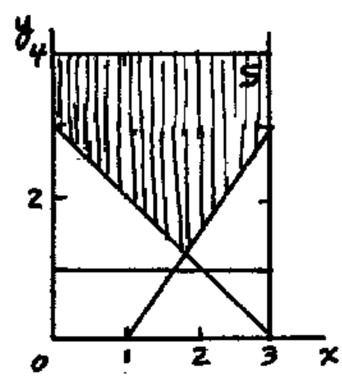


(d) $(A \cap B \cap \bar{C}) \cup (\bar{B} \cap C \cap D)$

1.2-11.



(a) $A \cap B \cap C$



(b) $C \cap B \cap \bar{A}$

(1.2-12.) (a) $A = \{80\,m \le d \le 1050\,m\}$, $B = \{950\,m \le d \le 1750\,m\}$, $C = \{1750\,m \le d \le 2000\,m\}$. (b) $A \cap B = \{950\,m \le d \le 1050\,m\}$ is the set of distances where both propeller and jet aircraft take off. (c) $\overline{A \cup B} = \{0 \le d \le 80\,m$ and $1750\,m < d \le 2000\,m\}$ is the portion of the runway used by both types of aircraft and the portion used by neither (safety margin). (d) $\overline{A \cup B \cup C} = \{0 \le d < 80\,m\}$ is portion of runway used by all aircraft in takeoff. $A \cup B = \{80\,m \le d \le 1750\,m\}$ is the runway distances used by all aircraft in taking off.

(1.2-13.) $\overline{A_1 \cap A_2} = \bar{A}_1 \cup \bar{A}_2 \triangleq C$ so $A_1 \cap A_2 = \bar{C}$

$\overline{A_1 \cap A_2 \cap A_3} = \overline{\bar{C} \cap A_3} = C \cup \bar{A}_3 = \bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3$

Continuation of the iteration proves the desired result.

(1.2-14.) Apply (1.2-12) and define C as

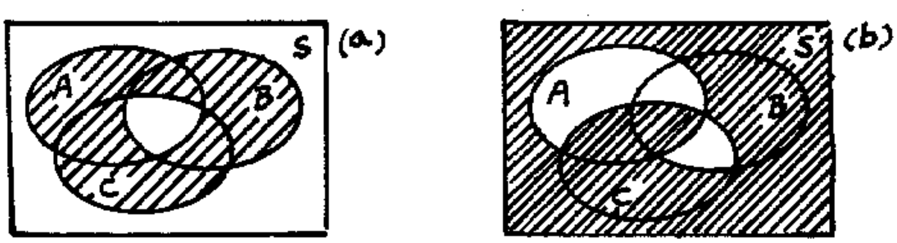
$\overline{A_1 \cup A_2} = \bar{A}_1 \cap \bar{A}_2 \triangleq C$ so $\bar{C} = A_1 \cup A_2$

$\overline{A_1 \cup A_2 \cup A_3} = \overline{\bar{C} \cup A_3} = C \cap \bar{A}_3 = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$

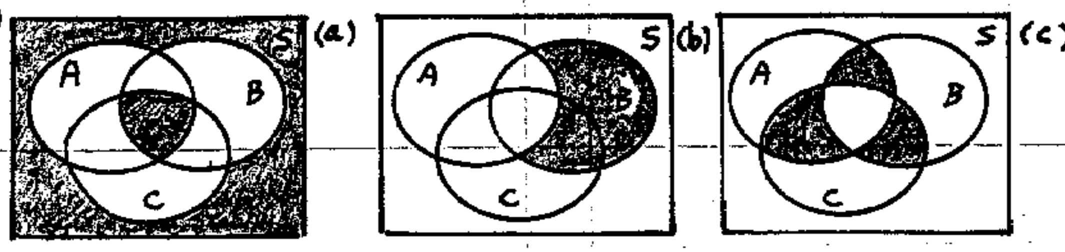
Continued iteration proves the desired result.

(1.2-15.) (a) False, (b) true, (c) false, (d) false, (e) true, (f) true, and (g) false.

(1.2-16.)



1.2-17.



1.3-1. The events are $A = \{1, 3, 5\}$, $B = \{4, 5, 6\}$, $A \cup B = \{1, 3, 4, 5, 6\}$, $A \cap B = \{5\}$. By assuming a fair die the probability of each of the six mutually exclusive outcomes is $1/6$. Thus, from axiom 3: $P(A) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1/2$, $P(B) = 1/2$, $P(A \cup B) = 5/6$ and $P(A \cap B) = 1/6$.

1.3-2. There are 36 possible mutually exclusive possible outcomes, each with probability $1/36$. Only six produce a seven: $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$ and $(6,1)$. Two produce an eleven: $(5,6)$ and $(6,5)$. Thus, eight outcomes satisfy the event "7 or 11." Therefore $P(\text{sum is 7 or 11}) = 8/36 = 2/9$.

1.3-3. (a) $S = \{0 < s \le 100\}$. (b) $P\{20 < s \le 35\} = (35-20)/100 = 15/100$. (c) $P\{s = 58\} = 0$ since the number 58 is only one of an infinite number of numbers in S.

1.3-4. (a) $P(A \cup C) = P\{a_1, a_5, a_6, a_9\} = 4/10$. (b) $P(B \cup \bar{C}) = P\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\} = P(S) = 1$. (c) $P[A \cap (B \cup C)] = P\{a_1, a_9\} = 2/10$.

(1.3-4.) (Continued)

(d) $P(\overline{A \cup B}) = P\{a_3, a_4, a_7, a_8, a_{10}\} = 5/10 = 1/2$.

(e) $P[A \cup (B \cap C)] = P\{a_6, a_9\} = 2/10 = 1/5$.

(1.3-5.) A and $\bar{A}$ are mutually exclusive and span the entire sample space. Therefore, from axiom 3: $P(A \cup \bar{A}) = P(S) = 1 = P(A) + P(\bar{A})$ or $P(\bar{A}) = 1 - P(A)$.

(1.3-6.) (a) $P(A) = P\{6\} = 1/6$; $P(B) = P\{2,5\} = 2/6 = 1/3$.
(b) $P(A) + P(B) + P(C) = 1$ is guaranteed if C is exclusive of A and B and comprises the balance of the sample space. Thus, $C = \overline{A \cup B} = \{1, 3, 4\}$.

(1.3-7.) $P(\text{black}) = 75/500$, $P(\text{green}) = 150/500$, $P(\text{red}) = 175/500$, $P(\text{white}) = 70/500$ and $P(\text{blue}) = 30/500$.

(1.3-8.) (a) $P(\text{jack}) = 4 \text{ jacks}/52 \text{ cards} = 4/52 = 1/13$.
(b) $P(5 \text{ or smaller}) = [4 \text{ fives} + 4 \text{ fours} + 4 \text{ threes} + 4 \text{ twos}]/52 \text{ cards} = 16/52 = 4/13$. (C) $P(\text{red ten}) = [1 \text{ ten of hearts} + 1 \text{ ten of diamonds}]/52 \text{ cards} = 2/52$.

(1.3-9.) (a) $P(A \text{ wins}) = P(2,4) + P(1,4) + P(4,1) + P(4,2) = 4/36$, (b) $P(B \text{ wins}) = P(4,1) + P(4,2) + P(4,3) + P(4,4) + P(4,5) + P(4,6) + P(1,4) + P(2,4) + P(3,4) + P(5,4) + P(6,4) = 11/36$. (c) $P(A \text{ and } B \text{ win}) = P(A \text{ wins}) = 4/36$ because $A \subset B$.

(1.3-10.) (a) P(stay in after one toss) $= P(H,H,T) + P(H,T,H) + P(T,T,H) + P(T,H,T) = 4/8$. (b) P(out after first toss) $= P(H,T,T) + P(T,H,H) = 2/8$. (c) P(no "odd man") $= P(H,H,H) + P(T,T,T) = 2/8$.

(1.3-11.)

Ω \ W	0.25	0.50	1.0
10	0.08	0.10	0.01
22	0.20	0.26	0.05
48	0.12	0.15	0.03
	0.40	0.51	0.09

$S = \{$ all resistors $\}$

All resistors span S.

From table: (a) $P(48\Omega$ and $0.25W) = 0.12$, (b) $P(48\Omega$ and $0.5W) = 0.15$, (c) $P(48\Omega$ and $1.0W) = 0.03$.

(1.3-12.)

(a) P(one is a two and other is 3 or more) $= P(2,3) + P(2,4) + P(2,5) + P(2,6) + P(3,2) + P(4,2) + P(5,2) + P(6,2) = 8(1/36) = 2/9$. (b) $P(10 \le \text{sum}$ and $\text{sum} \le 4) = P(4,6) + P(5,5) + P(5,6) + P(6,4) + P(6,5) + P(6,6) + P(1,1) + P(1,2) + P(1,3) + P(2,1) + P(2,2) + P(3,1) = 12/36 = 1/3$.

(1.3-13.)

Probabilities of sums showing up are:

	1	2	3	4	5	6
1			2/49			
2			2/49			
3			2/49			
4	2/49	2/49	4/49	2/49	2/49	2/49
5			2/49			
6			2/49			

all other probabilities not shown are 1/49

$P\{7\} = P(6,1) + P(5,2) + P(4,3) + P(3,4) + P(2,5) + P(1,6)$

$= 9/49 \approx 0.1837$

For fair dice each outcome has probability $= 1/36$ so $P\{7\} = 6/36 = 1/6 \approx 0.1667$. Improvement $= (9/49)/(6/36) \approx 1.102$, (10.2%).

(1.4-1.) (a) P(second is queen | first is queen) $= 3/51$.

(b) P(second is seven | first is queen) $= 4/51$.

(c) $P(\text{queen} \cap \text{queen}) = P(\text{queen}|\text{queen})\,P(\text{queen})$

$= \frac{3}{51} \cdot \frac{4}{52} = 1/221.$

(1.4-2.) P(four sevens) $= P$(fourth is seven | first three sevens) $\cdot P$(first three sevens) $= \frac{1}{49} P$(first three sevens) $=$ $\frac{1}{49} P$(third is seven | first two sevens) P(first two sevens) $= \frac{1}{49} \cdot \frac{2}{50} \cdot P$(first two sevens) $= \frac{1}{49} \cdot \frac{2}{50} P$(second is seven | first is seven) P(first is seven) $= \frac{1}{49} \cdot \frac{2}{50} \cdot \frac{3}{51} \cdot \frac{4}{52} = \frac{1}{270725}$

$= 3.694\,(10^{-6})$.

(1.4-3.) $P(D) = 24/100$, $P(E) = 38/100$,

$P(D \cap E) = 14/100$, $P(D|E) = 14/38$,

$P(E|D) = 14/24$.

(1.4-4.) Define B_1 = "draw a 5% resistor" and B_2 = "draw a 10% resistor." (a) From (1.4-10):

$$P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2)$$
$$= \frac{10}{62}\left(\frac{62}{100}\right) + \frac{14}{38}\left(\frac{38}{100}\right) = 24/100.$$

(b) $P(5\% \mid 22\,\Omega) = P(B_1|D) = P(D|B_1)P(B_1)/P(D)$

$$= \frac{10}{62}\left(\frac{62}{100}\right)\Big/\frac{24}{100} = 10/24.$$

(1.4-5.) (a) $P(0.01\mu F \mid \text{box } 2) = 95/210.$

(b) $P(0.01\mu F \mid \text{box } 3)\,P(\text{box } 3) = P(\text{box } 3 \mid 0.01\mu F)\,P(0.01\mu F)$

Thus,
$$P(\text{box } 3 \mid 0.01\mu F) = \frac{P(0.01\mu F \mid \text{box } 3)\,P(\text{box } 3)}{P(0.01\mu F)}$$

From the total probability theorem:

$$P(0.01\mu F) = P(0.01\mu F \mid \text{box } 1)\,P(\text{box } 1)$$
$$+ P(0.01\mu F \mid \text{box } 2)\,P(\text{box } 2) + P(0.01\mu F \mid \text{box } 3)\,P(\text{box } 3)$$
$$= \frac{20}{145}\left(\frac{1}{3}\right) + \frac{95}{210}\left(\frac{1}{3}\right) + \frac{25}{245}\left(\frac{1}{3}\right) = \frac{5903}{3(29)42(7)}.$$

Thus,
$$P(\text{box } 3 \mid 0.01\mu F) = \frac{\frac{25}{245}\left(\frac{1}{3}\right)}{\frac{5903}{3(29)42(7)}} = \frac{870}{5903} \approx 0.1474.$$

(1.4-6.) $P(0.01\mu F \mid \text{box } 1) = \frac{20}{145}$, $P(0.01\mu F \mid \text{box } 2) = \frac{95}{210}$,
$P(0.01\mu F \mid \text{box } 3) = \frac{25}{245}$, $P(0.1\mu F \mid \text{box } 1) = \frac{55}{145}$, $P(0.1\mu F \mid \text{box } 2) = \frac{35}{210}$,
$P(0.1\mu F \mid \text{box } 3) = \frac{75}{245}$, $P(1.0\mu F \mid \text{box } 1) = \frac{70}{145}$, $P(1.0\mu F \mid \text{box } 2) = \frac{80}{210}$,
$P(1.0\mu F \mid \text{box } 3) = \frac{145}{245}$.

(1.4-7.) From equations in Example 1.4-2: $P(A_1) = 0.95(0.6) + 0.05(0.4) = 0.59$, $P(A_2) = 0.05(0.6) + 0.95(0.4) = 0.41$, $P(B_1|A_1) = 0.95(0.6)/0.59 = 0.966$, $P(B_2|A_2) = 0.95(0.4)/0.41 = 0.927$, $P(B_1|A_2) = 0.05(0.6)/0.41 = 0.073$, $P(B_2|A_1) = 0.05(0.4)/0.59 = 0.034$.

(1.4-8.) From equations in Example 1.4-2: $P(A_1) = 1.0(0.7) + 0.0(0.3) = 0.7$, $P(A_2) = 0.0(0.7) + 1.0(0.3) = 0.3$, $P(B_1|A_1) = 1.0(0.7)/0.7 = 1.0$, $P(B_2|A_2) = 1.0(0.3)/0.3 = 1.0$, $P(B_1|A_2) = 0$, $P(B_2|A_1) = 0$. The system is ideal (or noise-free) since the probabilities of an error in symbol reception are zero.

(1.4-9.)

Event	10 W	25 W	50 W	Totals
D (Defective)	15	7	3	25
G (Good)	85	63	27	175
Totals	100	70	30	200

(a) $P(D|10W) = 15/100$. (b) $P(D \cap 50W) = P(D|50W) \cdot P(50W) = \frac{3}{30}\left(\frac{1}{3}\right) = 1/30$. (c) $P(D) =$

$$P(D|10W)P(10W) + P(D|25W)P(25W) + P(D|50W)P(50W)$$

$$= \frac{15}{100}\left(\frac{1}{3}\right) + \frac{7}{70}\left(\frac{1}{3}\right) + \frac{3}{30}\left(\frac{1}{3}\right) = 0.35/3 \approx 0.1167.$$

1.4-10. (a) $P(\text{launch}) = P(A \text{ fails} \cap B \text{ fails})$

$= P(B \text{ fails} \mid A \text{ fails}) P(A \text{ fails}) = 0.06(0.01)$

$= 0.0006.$

(b) $P(A \text{ fails} \mid B \text{ fails}) = P(A \text{ fails} \cap B \text{ fails}) / P(B \text{ fails}) =$

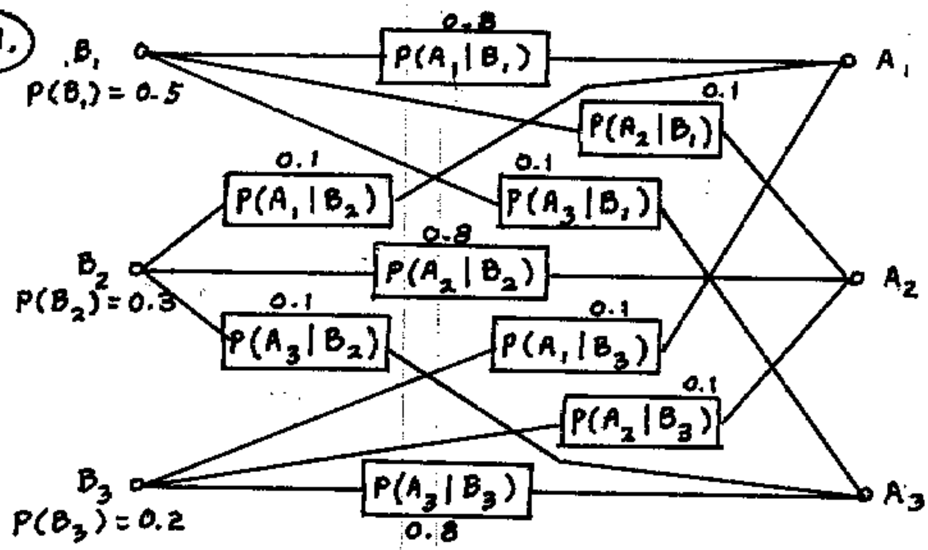
$6(10^{-4}) / 3(10^{-2}) = 0.02.$

(c) $P(A \text{ fails}) P(B \text{ fails}) = 0.01(0.03) = 3(10^{-4}) \neq 6(10^{-6}) =$

$P(A \text{ fails} \cap B \text{ fails})$ so events "A fails" and "B fails" are not independent.

★ 1.4-11.

(a)



(b) $P(A_1) = P(A_1|B_1)P(B_1) + P(A_1|B_2)P(B_2) + P(A_1|B_3)P(B_3)$

$= 0.8(0.5) + 0.1(0.3) + 0.1(0.2) = 0.45$, $P(A_2) =$

$0.1(0.5) + 0.8(0.3) + 0.1(0.2) = 0.31$, $P(A_3) = 0.1(0.5)$

$+ 0.1(0.3) + 0.8(0.2) = 0.24.$ (c)

(c) $P(B_1|A_1) = 0.8(0.5)/0.45 = 0.8889$, Bayes' rule.

$P(B_1|A_2) = 0.1(0.5)/0.31 = 0.1613$

$P(B_1|A_3) = 0.1(0.5)/0.24 = 0.2083$

★1.4-11. (Continued)

$P(B_2 | A_1) = 0.1(0.3)/0.45 = 0.0667$

$P(B_2 | A_2) = 0.8(0.3)/0.31 = 0.7742$

$P(B_2 | A_3) = 0.1(0.3)/0.24 = 0.1250$

$P(B_3 | A_1) = 0.1(0.2)/0.45 = 0.0444$

$P(B_2 | A_2) = 0.1(0.2)/0.31 = 0.0645$

$P(B_3 | A_3) = 0.8(0.2)/0.24 = 0.6667$

(d) When $P(B_i) = 1/3$, $i = 1, 2, 3$, then $P(A_1) = 1/3[P(A_1|B_1) + P(A_1|B_2) + P(A_1|B_3)] = \frac{1}{3}[0.8 + 0.1 + 0.1] = 1/3$. Similarly $P(A_2) = P(A_3) = P(A_1) = 1/3$ and also $P(B_i | A_k) = 0.1$, $k \neq i$, and $P(B_i | A_i) = 0.8$, $i = 1, 2, 3$.

1.4-12.

	No. Pills	Line L: L_1	Line L: L_2
A_1	102	0.02	0.03
A_2	101	0.06	0.08
A_3	100	0.88	0.83
A_4	99	0.03	0.04
A_5	98	0.01	0.02

$P(L_1) = 0.45$, $P(L_2) = 0.55$

$P(102|L_1) = 0.02$, $P(102|L_2) = 0.03$

$P(101|L_1) = 0.06$, $P(101|L_2) = 0.08$

$P(100|L_1) = 0.88$, $P(100|L_2) = 0.83$

$P(99|L_1) = 0.03$, $P(99|L_2) = 0.04$

$P(98|L_1) = 0.01$, $P(98|L_2) = 0.02$

(a) $P(102) = P(102|L_1)P(L_1) + P(102|L_2)P(L_2) = 0.02(0.45) + 0.03(0.55) = 0.0255$. Similarly, $P(101) = 0.0710$, $P(100) = 0.8525$, $P(99) = 0.0355$, and $P(98) = 0.0155$. (b) $P(L_1|100) = \dfrac{P(100|L_1)P(L_1)}{P(100)} =$

(1.4-12.) (Continued) $\frac{0.88(0.45)}{0.8525} \approx 0.4645$. (c) $P(\text{pills} < 100) =$
$P(98) + P(99) = 0.0355 + 0.0155 = 0.0510$.

(1.4-13.)

Define:	Then:	
$A = \{IC \text{ from source } A\}$	$P(D\|A) = 0.001$,	$P(A) = 1/3$
$B = \{IC \text{ from source } B\}$	$P(D\|B) = 0.003$,	$P(B) = 1/3$
$C = \{IC \text{ from source } C\}$	$P(D\|C) = 0.002$,	$P(C) = 1/3$
$D = \{\text{defective } IC\}$		

(a) $P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) = (0.001/3) + (0.003/3) + (0.002/3) = 0.006/3 = 0.002$. (b) $P(A|D) = P(D|A)P(A)/P(D) = 0.001(1/3)/0.002 = 1/6$. Similarly, $P(B|D) = 1/2$, $P(C|D) = 1/3$.

(1.4-14.)

(a) $P(\text{ace}) = P(A|D_1)P(D_1) + P(A|D_2)P(D_2) + P(A|D_3)P(D_3) = \frac{4}{52}\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) + 0\left(\frac{1}{6}\right) = \frac{19}{156} \approx 0.1218$. (b) $P(3) = P(3|D_1)P(D_1) + P(3|D_2)P(D_2) + P(3|D_3)P(D_3) = \frac{4}{52}\left(\frac{1}{2}\right) + 0\left(\frac{1}{3}\right) + \frac{4}{36}\left(\frac{1}{6}\right) = \frac{20}{351} \approx 0.0570$. (c) Similarly $P(\text{red card}) = \frac{26}{52}\left(\frac{1}{2}\right) + \frac{8}{16}\left(\frac{1}{3}\right) + \frac{18}{36}\left(\frac{1}{6}\right) = \frac{1}{2} = 0.50$.

(1.5-1.) $P(A \cap B) = 28/100 = 0.28 \neq P(A)P(B) = (44/100) \cdot (62/100) = 0.273$ so A and B are not independent. $P(A \cap C) = 0.0$ while $P(A)P(C) \neq 0$ so A and C are not independent. $P(B \cap C) = 0.24 \neq P(B)P(C) = (62/100)(32/100) = 0.198$ so B and C are not

(1.5-1.) (Continued)

independent. Finally, $P(A \cap B \cap C) = 0.0 \neq P(A) \cdot P(B) P(C)$ so A, B, and C are not statistically independent, even by pairs.

(1.5-2.) $P(A_1 \cap A_2) = P(A_1)P(A_2)$, $\quad P(A_1 \cap A_3) = P(A_1)P(A_3)$,

$P(A_1 \cap A_4) = P(A_1)P(A_4)$, $\quad P(A_2 \cap A_3) = P(A_2)P(A_3)$,

$P(A_2 \cap A_4) = P(A_2)P(A_4)$, $\quad P(A_3 \cap A_4) = P(A_3)P(A_4)$,

$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$, $P(A_1 \cap A_2 \cap A_4) = P(A_1)P(A_2)P(A_4)$,

$P(A_1 \cap A_3 \cap A_4) = P(A_1)P(A_3)P(A_4)$, $P(A_2 \cap A_3 \cap A_4) = P(A_2)P(A_3)P(A_4)$,

and $P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2)P(A_3)P(A_4)$.

★ (1.5-3.) (a) First form the product $P(A_1)P(\bar{A}_2) = P(A_1)[1-P(A_2)] = P(A_1) - P(A_1)P(A_2)$. Since A_1 and A_2 are independent this becomes

$$P(A_1)P(\bar{A}_2) = P(A_1) - P(A_1 \cap A_2). \qquad (1)$$

A Venn diagram shows that $A_1 = (A_1 \cap \bar{A}_2) \cup (A_1 \cap A_2)$ so $P(A_1) = P[(A_1 \cap \bar{A}_2) \cup (A_1 \cap A_2)]$. From (1.4-2): $P[(A_1 \cap \bar{A}_2) \cup (A_1 \cap A_2)] = P(A_1 \cap \bar{A}_2) + P(A_1 \cap A_2) - P[(A_1 \cap \bar{A}_2) \cap (A_1 \cap A_2)]$. However, a Venn diagram will show that $(A_1 \cap \bar{A}_2) \cap (A_1 \cap A_2) = \emptyset$ so these last two results give

$$P(A_1) = P(A_1 \cap \bar{A}_2) + P(A_1 \cap A_2). \qquad (2)$$

By combining (1) and (2) we have $P(A_1)P(\bar{A}_2) = P(A_1 \cap \bar{A}_2)$ so A_1 and $\bar{A}_2$ are statistically indep-

*(1.5-3.) (Continued)

endent from (1.5-3).

(b) The procedure is the same as in (a): $P(\bar{A}_1)P(A_2) = P(A_2)[1-P(A_1)] = P(A_2) - P(A_1 \cap A_2)$. But $A_2 = (A_2 \cap \bar{A}_1) \cup (A_2 \cap A_1)$ and $(A_2 \cap \bar{A}_1) \cap (A_2 \cap A_1) = \emptyset$ so $P(A_2) = P[(A_2 \cap \bar{A}_1) \cup (A_2 \cap A_1)] = P(A_2 \cap \bar{A}_1) + P(A_2 \cap A_1) - P[(A_2 \cap \bar{A}_1) \cap (A_2 \cap A_1)] = P(A_2 \cap \bar{A}_1) + P(A_2 \cap A_1)$. Thus, from the first and last equations:

$$P(\bar{A}_1)P(A_2) = P(\bar{A}_1 \cap A_2)$$

so $\bar{A}_1$ and A_2 are independent.

(c) Again repeat the procedures of (a) and (b):

$$P(\bar{A}_1)P(\bar{A}_2) = [1-P(A_1)][1-P(A_2)] = 1 - P(A_1) - P(A_2)$$

$$+ P(A_1)P(A_2) = 1 - P(A_1) - P(A_2) + P(A_1 \cap A_2)$$

$= 1 - [P(A_1) + P(A_2) - P(A_1 \cap A_2)] = 1 - P(A_1 \cup A_2)$ from (1.4-2). But since $1 - P(A_1 \cup A_2) = P(\overline{A_1 \cup A_2})$, then $P(\bar{A}_1)P(\bar{A}_2) = P(\overline{A_1 \cup A_2})$. However, a Venn diagram shows that $\overline{A_1 \cup A_2} = \bar{A}_1 \cap \bar{A}_2$ so

$$P(\bar{A}_1)P(\bar{A}_2) = P(\bar{A}_1 \cap \bar{A}_2)$$

which, from (1.5-3) shows $\bar{A}_1$ and $\bar{A}_2$ to be independent.

(1.5-4.) Comb. N things 2 at time $= \binom{N}{2}$ (pairs)

Comb. N things 3 at time $= \binom{N}{3}$ (triple)

$\vdots$ $\vdots$

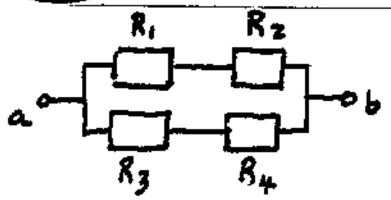
Comb. N things N at time $= \binom{N}{N}$ (N-tuple)

1.5-4. (Continued)

$$\text{Total} = \text{sum} = \sum_{i=2}^{N}\binom{N}{i} = -\binom{N}{0} - \binom{N}{1} + \sum_{i=0}^{N}\binom{N}{i}$$

Use (C-61). $\text{Total} = -1 - N + \sum_{i=0}^{N}\binom{N}{i} = 2^N - N - 1.$

1.5-5.



$R_i = \{\text{relay } R_i \text{ fails}, i = 1,2,3,4\} = \{R_i \text{ opens}\}$

$$P(R_i) = \begin{cases} p_1 = 0.005, & i = 1,2 \\ p_2 = 0.008, & i = 3,4 \end{cases}$$

$P(\text{signal does not arrive}) = P\{(R_1 \text{ or } R_2 \text{ opens}) \text{ and } (R_3 \text{ or } R_4 \text{ opens})\}$
$= P\{(R_1 \cup R_2) \cap (R_3 \cup R_4)\} = P(R_1 \cup R_2)P(R_3 \cup R_4)$ (independent failures) $= [P(R_1) + P(R_2) - P(R_1 \cap R_2)][P(R_3) + P(R_4) - P(R_3 \cap R_4)]$
$= (2p_1 - p_1^2)(2p_2 - p_2^2) = [0.01 - 25(10^{-6})][0.016 - 64(10^{-6})] \approx 0.00016.$

1.5-6.

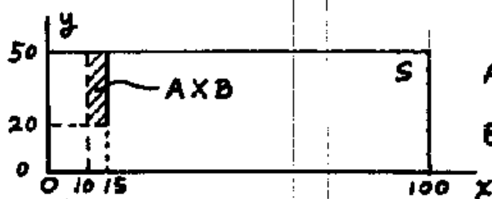
~~P{signal does not arrive} = P{(upper path fails) and (lower~~ path fails)} = P{(upper path fails) $\cap$ (lower path fails)}
= P(upper path fails) P(lower path fails). But P(upper path fails)
$= P(R_1) + P(R_2 \cap R_3) - P(R_1 \cap R_2 \cap R_3) = p_1 + p_2^2 - p_1 p_2^2 = 5.0995(10^{-3})$, and
P(lower path fails) $= p_2 + p_3^2 - p_2 p_3^2 = 12.475(10^{-3})$. Thus,
P{signal does not arrive} $= 5.0995(10^{-3})\,12.475(10^{-3}) = 63.6163(10^{-6})$.

1.6-1. Let c_1, c_2, c_3, c_4 and c_5 represent the five cities and let m_1, m_2, m_3 and m_4 represent the motels. Then $S_1 = \{c_1, c_2, c_3, c_4, c_5\}$ and $S_2 = \{m_1, m_2, m_3, m_4\}$. The combined sample space becomes

(1.6-1.) (Continued)

$$S = S_1 \times S_2 = \{(c_1,m_1),\ (c_1,m_2),\ (c_1,m_3),\ (c_1,m_4),$$
$$(c_2,m_1),\ (c_2,m_2),\ (c_2,m_3),\ (c_2,m_4),$$
$$(c_3,m_1),\ (c_3,m_2),\ (c_3,m_3),\ (c_3,m_4),$$
$$(c_4,m_1),\ (c_4,m_2),\ (c_4,m_3),\ (c_4,m_4),$$
$$(c_5,m_1),\ (c_5,m_2),\ (c_5,m_3),\ (c_5,m_4)\}.$$

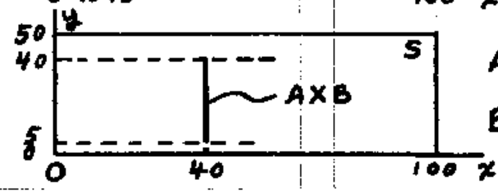
(1.6-2.) (a)



$A = \{10 < x \le 15\}$

$B = \{20 < y \le 50\}$

(b)



$A = \{x = 40\}$

$B = \{5 < y \le 40\}$

(1.6-3.) Sequences $= 6(6)6(6) = 6^4 = 1296.$

(1.6-4.) Number of poker hands $= \binom{52}{5} = \dfrac{52(51)50(49)48}{5(4)3(2)}$

$= 2{,}598{,}960.$

(1.7-1.) This is a Bernoulli trials experiment with $N=4$,

$p = P(\text{a can is out of tolerance}) = 0.03$.

(a) $P(4 \text{ out of tolerance}) = \binom{4}{4}(0.03)^4(1-0.03)^0 = 8.1(10^{-7}).$

(b) $P(2 \text{ out of tolerance}) = \binom{4}{2}(0.03)^2(1-0.03)^2$

$= \dfrac{4!}{2!2!}(9)10^{-4}(0.97)^2 \approx 5.081(10^{-3}).$

(c) $P(\text{all in tolerance}) = P(\text{none is out of tolerance})$

$= \binom{4}{0}(0.03)^0(1-0.03)^4 = (0.97)^4 \approx 0.8853.$

1.7-2. This is a Bernoulli trials experiment with $N = 6$, $p = P(\text{land in recovery zone}) = 0.8$.

(a) $P(\text{none in zone}) = \binom{6}{0}(0.8)^0(1-0.8)^6 = (0.2)^6 = 6.4(10^{-5})$.

(b) $P(\text{at least one in zone}) = 1 - P(\text{none in zone}) =$

$1 - 6.4(10^{-5}) = 0.999936$.

(c) $P(\text{success}) = P(3 \text{ in zone}) + P(4 \text{ in zone}) + P(5 \text{ in zone})$

$+ P(6 \text{ in zone}) = \binom{6}{3}(0.8)^3(0.2)^3 + \binom{6}{4}(0.8)^4(0.2)^2$

$+ \binom{6}{5}(0.8)^5(0.2)^1 + \binom{6}{6}(0.8)^6(0.2)^0 \approx 0.983$.

Yes, the program is successful.

1.7-3. For $N=2$: $P\{\text{carrier sunk}\} = P\{2 \text{ hits}\} = \binom{2}{2}(0.4)^2 \cdot (1-0.4)^0 = 0.16$. For $N=4$: $P\{\text{carrier sunk}\} = P\{2 \text{ hits}\} + P\{3 \text{ hits}\} + P\{4 \text{ hits}\} = \binom{4}{2}(0.4)^2(0.6)^2 + \binom{4}{3}(0.4)^3(0.6)^1 + \binom{4}{4}(0.4)^4(0.6)^0 = 0.5248$.

1.7-4. $P(\text{late } k \text{ of } N \text{ times}) = \binom{N}{k} 0.4^k (0.6)^{N-k}$, $N=5$.

(a) $P(\text{late 3 or more times in one week}) = \binom{5}{3} 0.4^3 (0.6)^2 + \binom{5}{4} 0.4^4 (0.6)^1 + \binom{5}{5} 0.4^5 = 0.2304 + 0.0768 + 0.01024 = 0.31744$. (b) $P(\text{not late at all}) = \binom{5}{0} 0.4^0 (0.6)^5 = 0.07776$.

1.7-5. (a) $P(0 \text{ late}) = \binom{5}{0}(0.3)^0(0.7)^5 = 0.16807$. (b) $P(\text{all late}) = \binom{5}{5} 0.3^5 (0.7)^0 = 0.00243$. (c) $P(3 \text{ or more on time}) = P(0 \text{ late}) + P(1 \text{ late}) + P(2 \text{ late}) = 0.16807 + \binom{5}{1} 0.3 (0.7)^4 + \binom{5}{2} 0.3^2 (0.7)^3 = 0.83692$.

(1.7-6.) $P(\text{all on time}) = P(0 \text{ late}) = \binom{5}{0} x^0 (1-x)^5 = (1-x)^5$
$= 0.9$. Thus, $x = P(\text{a flight is late}) = 1 - 0.9^{1/5} = 0.020852$.
Yes — to reduce 0.3 to 0.020852!

(1.7-7.) (a) $P(\text{win}) = P(2 \text{ heads}) = \binom{5}{2} 0.7^2 (0.3)^3 = 0.13230$. No!
(b) $P(\text{win}) = P(4 \text{ heads}) + P(5 \text{ heads}) = \binom{5}{4} 0.7^4 (0.3)^1$
$+ \binom{5}{5} 0.7^5 (0.3)^0 = 0.36015 + 0.16807 = 0.52822$. Yes!

★(1.7-8.) (a) $P(\text{win set}) = P(5 \text{ hits in } 6) + P(6 \text{ hits in } 6)$
$= \binom{6}{5}(0.8)^5(0.2)^1 + \binom{6}{6}(0.8)^6(0.2)^0 = 0.393216 +$
$0.262144 = 0.65536$. (b) Second Bernoulli trial problem. $P(3 \text{ sets of } 3) = \binom{3}{3}(0.65536)^3(1-0.65536)^0$
$= 0.28147$.

(1.7-9.)

$P(\text{successful arrival}) = P\{\text{engine survives and navigation system survives}\} = P\{\text{engine survives}\} P\{\text{navigation system survives}\}$
$= [1 - P(\text{engine fails})][1 - P(\text{navigation system fails})] = 0.95(0.999)$
$= 0.94905$.

(1.7-10.)

(a) $P(\text{all operate}) = P(0 \text{ fail}) = \dfrac{6!}{0!(6-0)!}(0.06)^0(0.94)^{6-0} = 0.6899$.
(b) $P(\text{all fail}) = P(6 \text{ fail}) = \dfrac{6!}{6!(6-6)!}(0.06)^6(0.94)^{6-6} = 4.666(10^{-8})$.
(c) $P(1 \text{ fails}) = \dfrac{6!}{1!(6-1)!}(0.06)^1(0.94)^{6-1} = 0.2642$.

1.7-11.

(a) P(2 or more not repaired) $= 1 - P$(0 not repaired) $- P$(1 not repaired) $= 1 - \dfrac{8!}{0!\,(8-0)!}(0.1)^0(0.9)^{8-0} - \dfrac{8!}{1!\,(8-1)!}(0.1)^1(0.9)^{8-1}$ $\approx 1 - 0.4305 - 0.3826 = 0.1869$. (b) P(8 properly repaired) $=$ P(0 not repaired) ≈ 0.4305.

1.7-12.

P(0 fail to return key) $= \dfrac{(Np)^{k=0}\, e^{-Np}}{(k=0)!} = \dfrac{2.5^0 e^{-2.5}}{0!} = 0.0821$.

P(1 fails to return key) $= \dfrac{2.5^1 e^{-2.5}}{1!} = 0.2052$.

P(2 fail to return keys) $= \dfrac{2.5^2 e^{-2.5}}{2!} = 0.2565$.

P(3 fail to return keys) $= \dfrac{2.5^3 e^{-2.5}}{3!} = 0.2138$.

P(no more than 3 fail to return keys) $= 0.0821 + 0.2052 + 0.2565$ $+ 0.2138 = 0.7576$.

CHAPTER
2

2.1-1. Let S_X denote the set of values that X can have. (a) For $X = 2s$: $S_X = \{0, 2, 5, 12\}$.

(b) For $X = 5s^2 - 1$: $S_X = \{-1, 4, 30.25, 179\}$.

(c) For $X = \cos(\pi s)$: $S_X = \{1, -1, 0\}$

(d) For $X = 1/(1-3s)$: $S_X = \{1, -0.5, -1/6.5, -1/17\}$.

2.1-2. Let S_X denote the set of values that X can have.

(a) For $X = 2s$: $S_X = \{-4 < x \le 10\}$.

(b) For $X = 5s^2 - 1$: $S_X = \{-1 \le x \le 124\}$.

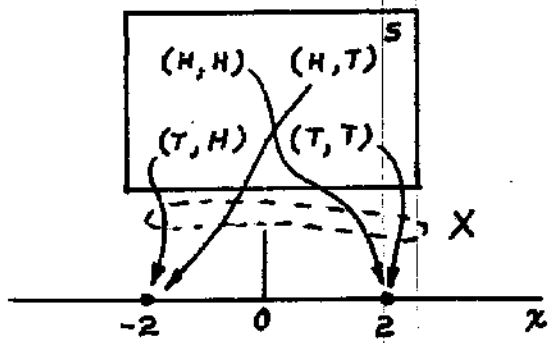
(c) For $X = \cos(\pi s)$: $S_X = \{-1 \le x \le 1\}$.

(d) For $X = 1/(1-3s)$: $S_X = \{-\infty < x \le -\frac{1}{14}$ and $1/7 < x < \infty\}$.

2.1-3. (a) Continuous. (b) Mixed. (c) Discrete. (d) Discrete.

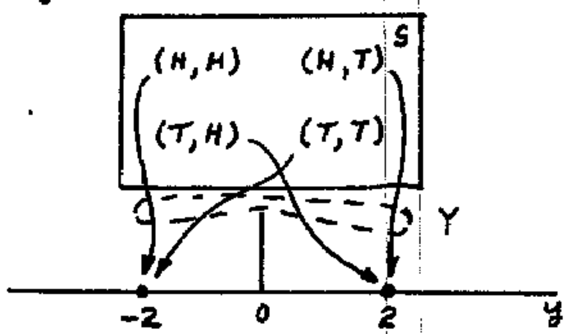
2.1-4. Domain of P is all subsets defined on S. The domain of X is S. The range of P is $0 \le P \le 1$. The range of X is infinite (X can have any value $-\infty < x < \infty$).

(2.1-5.) For the man:

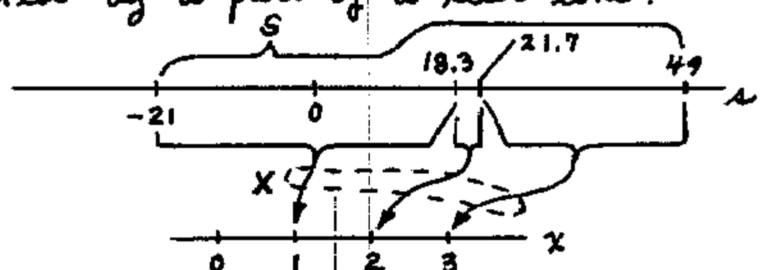


$x = 2$ is the "positive" winning value of X.
$x = -2$ is the "negative" winning value of X.

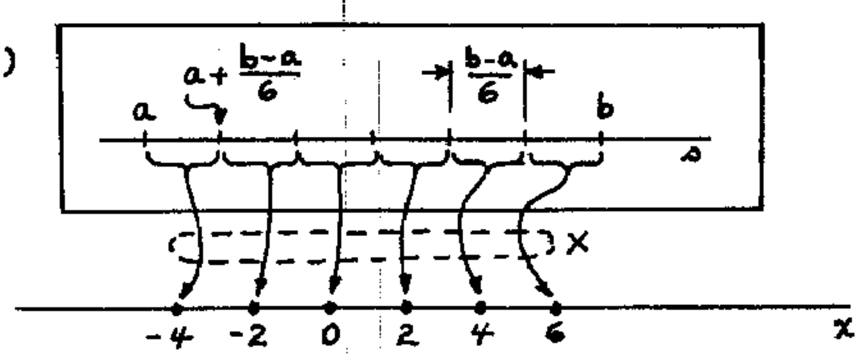
For the friend:



(2.1-6.) Here $S = \{-21°C \le s \le 49°C\}$ which can be represented by a part of a real line.



(2.1-7.) (a)



2.1-7. (Continued) (b) From (a) it is obvious that $(b-a)/6 = 2$ and $a + [(b-a)/12] = -4$. Solving for a and b gives $a = -5$ and $b = 7$.

★2.1-8. Here

$$a_1 - a_0 = \Delta$$
$$a_2 - a_1 = 2\Delta$$
$$\vdots$$
$$a_N - a_{N-1} = 2^{N-1}\Delta .$$

Thus,

$$a_N = a_0 + \Delta + 2\Delta + \cdots + 2^{N-1}\Delta = a_0 + \Delta \sum_{i=0}^{N-1} 2^i . \quad (1)$$

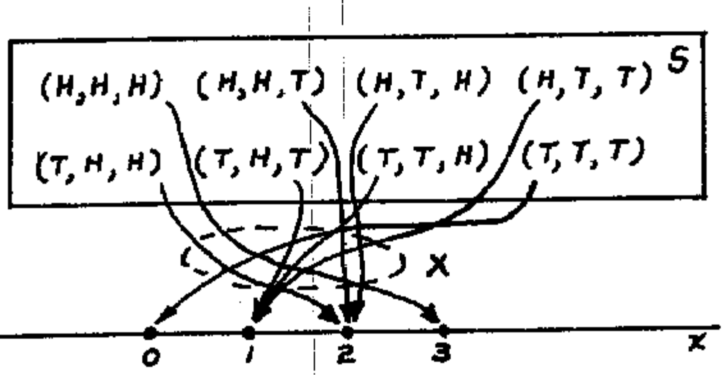
But from (C-58): $\sum_{i=0}^{N-1} 2^i = \frac{2^N - 1}{2-1} = 2^N - 1$ so (1) gives

$$\Delta = \frac{a_N - a_0}{2^N - 1} .$$

For the nth term above $a_n = a_0 + \Delta \sum_{i=0}^{n-1} 2^i$ so

$$a_n = a_0 + \left(\frac{2^n - 1}{2^N - 1}\right)(a_N - a_0) .$$

2.1-9. (a)



(b) $P\{X=0\} = 1/8$, $P\{X=1\} = 3/8$

$P\{X=2\} = 3/8$, $P\{X=3\} = 1/8$.

2.1-10. (a) Mapping is identical to that in (a) of Problem 2-9. (b) This is a Bernoulli trials problem.

$$P\{X=0\} = \binom{3}{0}(0.6)^0(0.4)^3 = (0.4)^3 = 0.064$$

$$P\{X=1\} = \binom{3}{1}(0.6)^1(0.4)^2 = 0.288$$

$$P\{X=2\} = \binom{3}{2}(0.6)^2(0.4)^1 = 0.432$$

$$P\{X=3\} = \binom{3}{3}(0.6)^3(0.4)^0 = 0.216.$$

2.1-11. For $180\,\Omega$: $E_2 = 12\left(\frac{180}{820+180}\right) = 2.16$ V.

For $470\,\Omega$: $E_2 = 12\left(\frac{470}{820+470}\right) \approx 4.372$V.

For $1000\,\Omega$: $E_2 = 12\left(\frac{1000}{820+1000}\right) = 6.593$ V.

For $2200\,\Omega$: $E_2 = 12\left(\frac{2200}{820+2200}\right) = 8.742$ V.

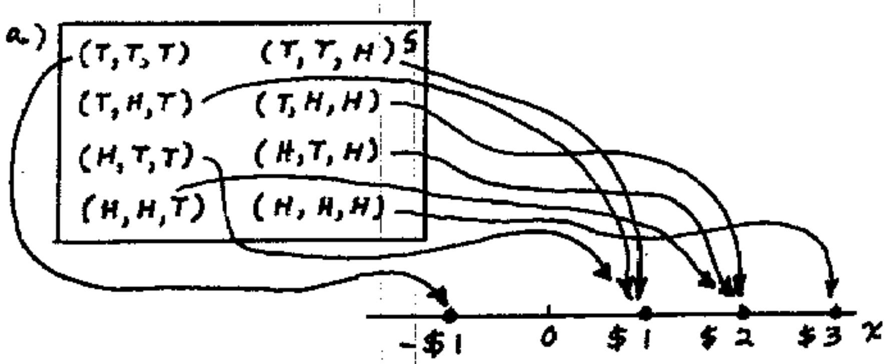
Thus,

$$E_2 = \{2.16, 4.372, 6.593, 8.742\}.$$

Since all resistor values are equally probable so are the voltage values. The set P of the probabilities is $P = \{1/4, 1/4, 1/4, 1/4\}$.

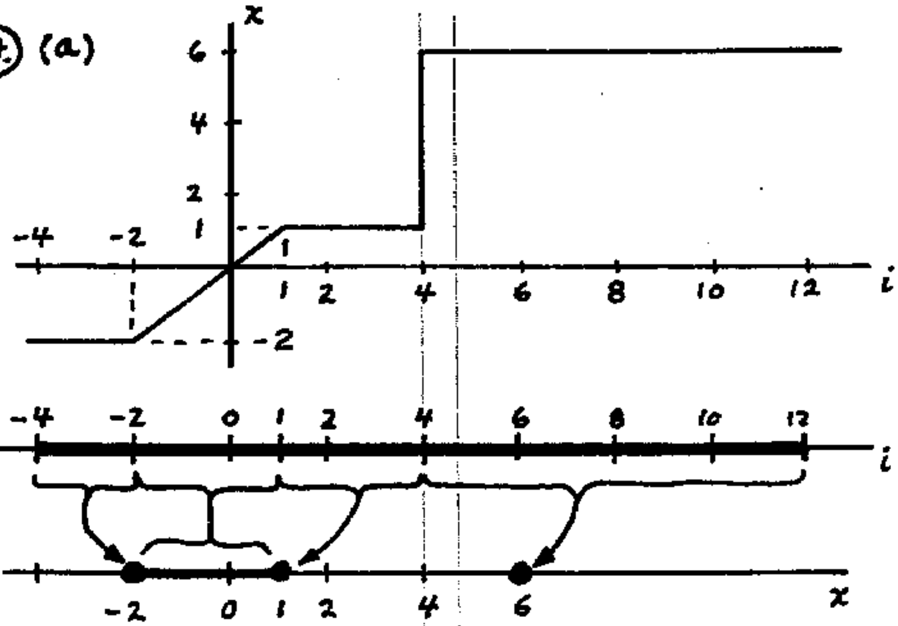
2.1-12. (a) X is discrete. (b) $X = \{2, 3, 5\}$.

2.1-13. (a)



(b) $P(-\$1) = 1/8$, $P(\$1) = P(\$2) = 3/8$, $P(\$3) = 1/8$.

2.1-14. (a)



(b) Mixed random variable.

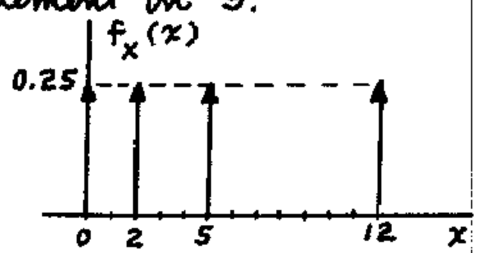
2.2-1. Fraction of the production run that is purchased is equal to the probability bolt length is above 0.95 (760) mm but not above 1.05 (760) mm. Thus, from (2.2-2e),

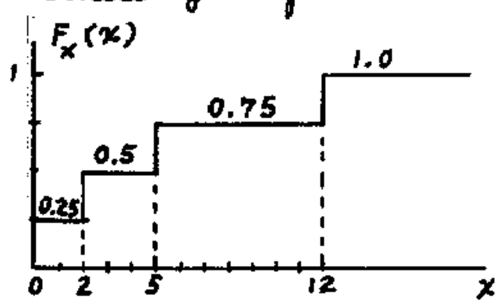
$$P(0.95(760) < X \le 1.05(760)) = \int_{0.95(760)}^{1.05(760)} \frac{dx}{920-650}$$

$$= \frac{798-722}{920-650} = \frac{76.0}{270} = 0.2815$$

or 28.15% of the production run.

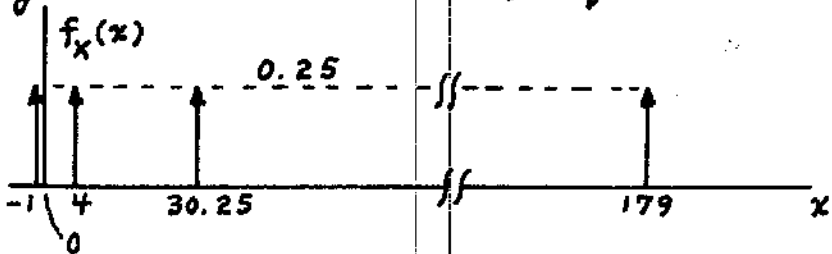
2.2-2. (a) Here there is one value of X for each element in S.

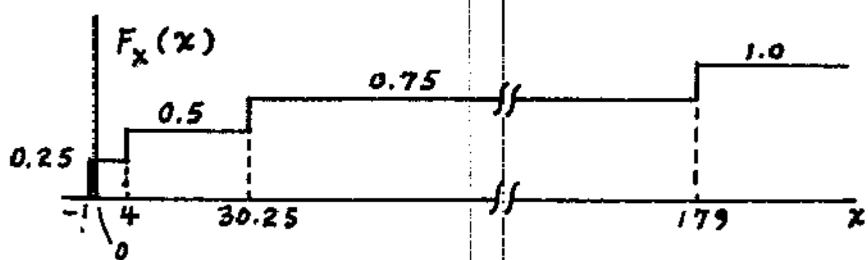




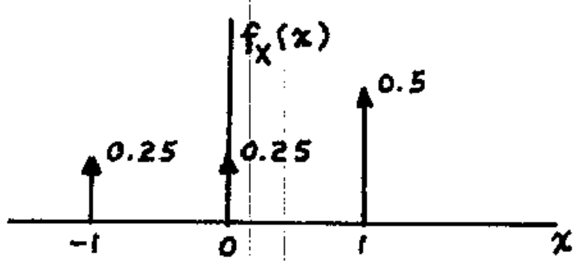
2.2-2 (Continued)

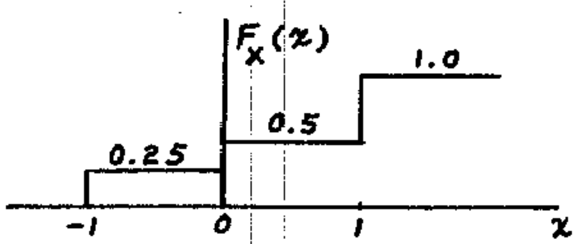
(b) Again there is one value of X for each value of S.





(c) Here the points 0, 1, 2.5, 6 map, respectively, to the points 1, -1, 0, 1 so $P(1) = 0.5$ and $P(-1) = P(0) = 0.25$.



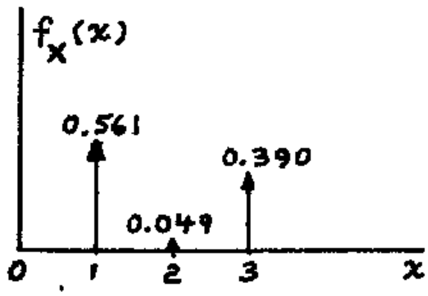


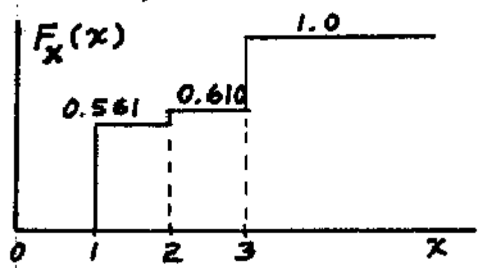
2.2-3. $P\{X=1\} = P\{-21 \le s \le 18.3\} = \dfrac{18.3-(-21)}{49-(-21)} = \dfrac{39.3}{70}$

$\approx 0.561.$

$P\{X=2\} = P\{18.3 < s \le 21.7\} = \dfrac{21.7-18.3}{49-(-21)} = \dfrac{3.4}{70} \approx 0.049.$

$P\{X=3\} = P\{21.7 < s \le 49\} = \dfrac{49-21.7}{49-(-21)} = \dfrac{27.3}{70} \approx 0.390.$





2.2-4. (a) Write $P\{0.9a + 0.1b < X \le 0.7a + 0.3b\}$ as $P\{x_1 < X \le x_2\}$. It is necessary to determine the possible values of x_1 and x_2. By direct substitution we find $a < x_1 < b$ is true so long as $a < b$, which is true. Similarly, $a < x_2 < b$ is true if $a < b$. Finally, $x_1 < x_2$ if $a < b$. Thus,

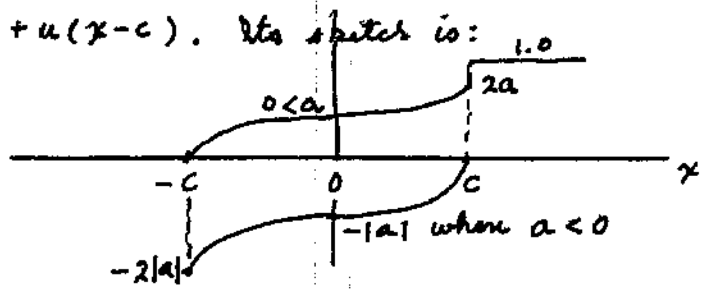
$$P\{0.9a + 0.1b < X \le 0.7a + 0.3b\} = \int_{0.9a+0.1b}^{0.7a+0.3b} \frac{dx}{(b-a)}$$

$$= \frac{0.2(b-a)}{b-a} = 0.2$$

(b) $P\{(a+b)/2 < X \le b\} = \displaystyle\int_{\frac{a+b}{2}}^{b} \frac{dx}{b-a} = 1/2.$

(2.2-5.) $G_X(x)$ must satisfy (2.2-2 a, b, d and f) to be valid. (a) $G_X(-\infty)=0$, $G_X(\infty)=1$, $G_X(x_2)>G_X(x_1)$ if $x_2>x_1$ as seen from a sketch which also shows $G_X(x^+)=G_X(x)$. Therefore $G_X(x)$ is a valid distribution. (b) From calculations and a sketch $G_X(x)$ is a valid distribution. (c) $G_X(\infty)\neq 1$ so it is not a valid distribution.

(2.2-6.) Test 1: $G_X(-\infty)=0$ is true. Test 2: $G_X(\infty)=1$ is true if $(a+b)=1$ or $b=1-a$. Thus $G_X(x)=a[1+\frac{2}{\pi}\sin^{-1}(\frac{x}{c})]$ $\text{rect}(\frac{x}{2c})+u(x-c)$. Its sketch is:



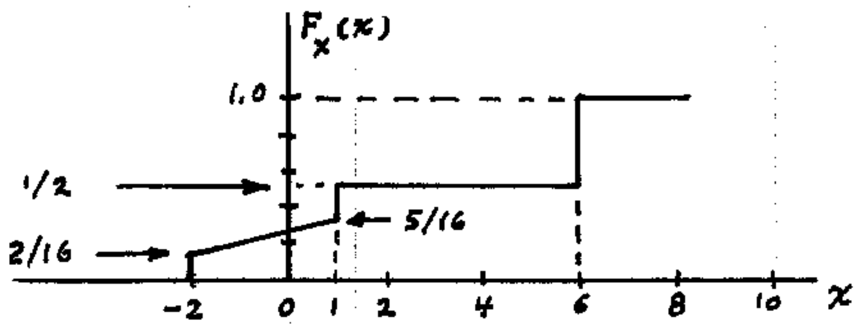
Test 3: $G_X(x_1)\leq G_X(x_2)$ for $x_1<x_2$ is true if $a\geq 0$ and $2a\leq 1$ so $0\leq a\leq 1/2$ and $b=1-a$ are required. Test 4: $G_X(x^+)=G_X(x)$ is true. Discussion — If: (a) $a=b=1/2$ X is continuous, (b) $a=0$, $b=1$, then X is discrete with one value $x=c$, (c) $0<a<1/2$, then X is mixed with one discrete value $x=c$.

(2.2-7.) (a) Tests: 1. $G_X(-\infty)=0$ is true due to $u(x)$. 2. $G_X(\infty)=1$ is true if $b>0$. 3. $G_X(x_2)\geq G_X(x_1)$ if $x_2>x_1$ is true if $0\leq a\leq 1$ from a sketch. 4. $G_X(x^+)$

2.2-7. (Continued)

$= G_X(x)$ is true. Thus, $G_X(x)$ is valid if $b>0$ and $0<a<1$. (b) If $0 \le a \le 1$ there is a step at $x=0$ so X is mixed. X is continuous only when $a=1$; it is discrete only when $a=0$.

2.2-8. (a) Assume all values of i are equally probable. $P(x_1=-2) = P\{-4 \le i \le -2\} = 2/16$, $P(x_2=1) = P\{1 < i \le 4\} = 3/16$, $P(x_3=6) = P\{4 < i \le 12\} = 8/16$. Continuous values of X are $\{-2 < x \le 1\}$ and have equal probabilities to the values of 5 defined by $\{-2 < i \le 1\}$



2.2-9. (a) $P\{-\infty < X \le 6.5\} = F_X(6.5) = \frac{1}{650} + \frac{2^2}{650} + \frac{3^2}{650} + \cdots + \frac{6^2}{650} = \frac{91}{650} = 0.140$. (b) $P\{X>4\} = 1 - P\{X \le 4\} = 1 - F_X(4) = 1 - \frac{1}{650} - \frac{2^2}{650} - \frac{3^2}{650} - \frac{4^2}{650} = 1 - \frac{30}{650} = \frac{620}{650} \approx 0.9538$. (c) $P\{6 < X \le 9\} = F_X(9) - F_X(6) = \sum_{n=1}^{9}(n^2/650) - \sum_{n=1}^{6}(n^2/650) = \frac{7^2}{650} + \frac{8^2}{650} + \frac{9^2}{650} = \frac{194}{650} \approx 0.2985$.

2.2-10. $F_X(+\infty) = 1$ for a valid distribution. $G_X(\infty) = K\sum_{n=1}^{N} n^3 = K\frac{N^2(N+1)^2}{4} \overset{\text{must}}{=} 1$ from (C-57). Thus, $K = \frac{4}{N^2(N+1)^2}$.

(2.3-1.) The required tests are (2.3-6 a and b). From a sketch, $f_X(x) \geq 0$ if $a > 0$, and $b > 0$ is necessary for noninfinite area under $f_X(x)$. For these assumed true:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 2\int_m^{\infty} a\, e^{-(x-m)/b}\,dx = 2ab\int_0^{\infty} e^{-\xi}\,d\xi$$

$$= 2ab\left[-e^{-\xi}\right]\Big|_0^{\infty} = 2ab \overset{\text{must}}{=\!=\!=} 1$$

Thus, we require $b > 0$, $a = 1/2b$.

(2.3-2.) This is a Bernoulli trials problem where $N=6$ and $p = 0.4$. Here

$$P(0W) = \binom{6}{0}(0.4)^0(0.6)^6 \approx 0.0467$$

$$P(0.5W) = \binom{6}{1}(0.4)^1(0.6)^5 \approx 0.1866$$

$$P(1.0W) = \binom{6}{2}(0.4)^2(0.6)^4 \approx 0.3110$$

$$P(1.5W) = \binom{6}{3}(0.4)^3(0.6)^3 \approx 0.2765$$

$$P(2.0W) = \binom{6}{4}(0.4)^4(0.6)^2 \approx 0.1382$$

$$P(2.5W) = \binom{6}{5}(0.4)^5(0.6)^1 \approx 0.0369$$

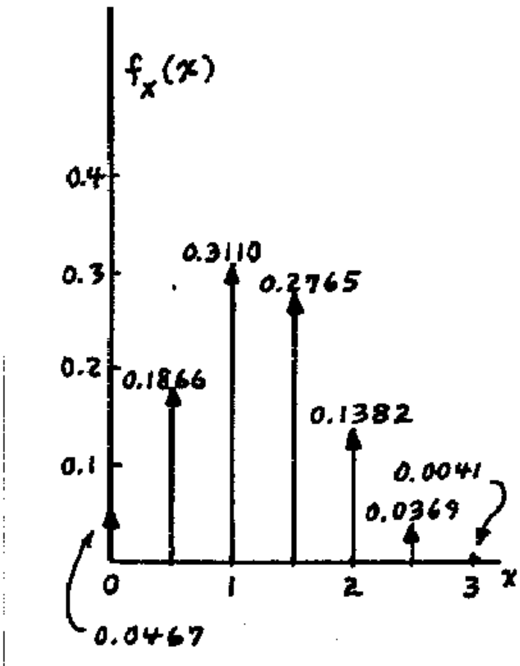
$$P(3.0W) = \binom{6}{6}(0.4)^6(0.6)^0 \approx 0.0041.$$

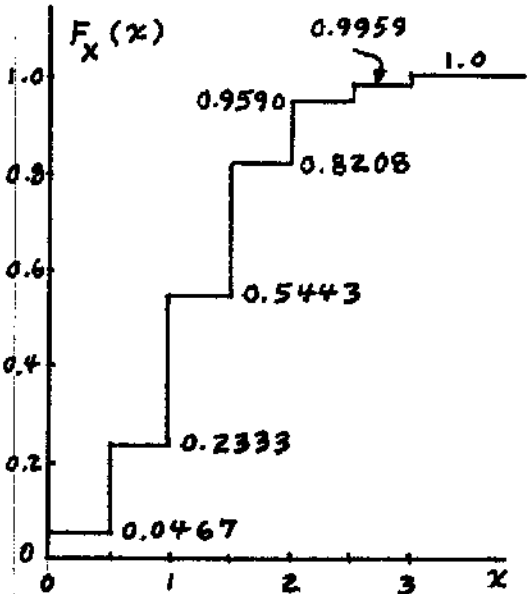
(a) Let X be a random variable representing power delivered. From (2.3-5) and (2.2-6):

$$f_X(x) = 0.0467\,\delta(x) + 0.1866\,\delta(x-0.5)$$
$$+ 0.3110\,\delta(x-1.0) + 0.2765\,\delta(x-1.5)$$
$$+ 0.1382\,\delta(x-2.0) + 0.0369\,\delta(x-2.5)$$
$$+ 0.0041\,\delta(x-3.0)$$

(2.3-2) (Continued)

$$F_X(x) = 0.0467\,u(x) + 0.1866\,u(x-0.5) + 0.3110\,u(x-1.0) + 0.2765\,u(x-1.5) + 0.1382\,u(x-2.0) + 0.0369\,u(x-2.5) + 0.0041\,u(x-3.0)$$





(b) $P(\text{Overload}) = P(2.5\,W) + P(3.0\,W) = 0.041$ or 4.1%.

★ (2.3-3.) As in Problem 2.3-2 this is a Bernoulli trials problem. For the 4 W unit: By use of (1.7-5)

$$P(0W) = \binom{12}{0}(0.4)^0(1-0.4)^{12} \approx 0.00218$$

$$P(0.5W) = \binom{12}{1}(0.4)^1(1-0.4)^{11} \approx 0.01741.$$

Similarly, other probabilities of drawing various power levels are $P(1.0W) \approx 0.06385$, $P(1.5W) \approx 0.14189$, $P(2.0W) \approx 0.21284$, $P(2.5W) \approx 0.22703$, $P(3.0W) \approx 0.17658$, $P(3.5W) \approx 0.10090$, $P(4.0W) \approx$

★2.3-3. (Continued)

0.04204, $P(4.5W) \approx 0.01246$, $P(5.0W) \approx 0.00249$, $P(5.5W) \approx 0.00030$, $P(6.0W) \approx 0.00002$. Thus,

$$P(\text{overload}) = P(4.5W) + P(5.0W) + P(5.5W) + P(6.0W) \approx 0.01527 \text{ or } 1.53\%.$$

For the two independent 2W units define events O_1 = "first 2W unit overloads" and O_2 = "second 2W unit overloads." Overload of the system occurs when the first, second, or both units overloads; that is,

$$P(\text{overload}) = P(O_1 \cup O_2) = P(O_1) + P(O_2) - P(O_1 \cap O_2)$$
$$= P(O_1) + P(O_2) - P(O_1)P(O_2).$$

However, $P(O_1) = 0.041$ from Problem 2-18, and since $P(O_1) = P(O_2)$ here because the two units are identical we get

$$P(\text{overload}) = 2(0.041) - (0.041)^2 = 0.0803 \text{ or } 8.03\%.$$

Thus, use of the 4W unit is better than two 2W units.

2.3-4. By definition $F_X(x) = P\{X \le x\}$. If $x = -\infty$ the event $\{X \le -\infty\}$ has no elements so $F_X(-\infty) = 0$. Similarly, $\{X \le \infty\} = S$ so its probability must be unity; that is, $F_X(\infty) = 1$. Finally, for any $-\infty < x < \infty$ the event $\{X \le x\} \subset S$ and its probability must lie between 0 and 1 and (2.2-2c) results.

(2.3-5.) We use (A-2) for (a), (b), (c) and (d). We use (A-1) for (e).

(a) $\int_3^4 (3x^2+2x-4)\delta(x-3.2)\,dx = 3(3.2)^2+2(3.2)-4$

$= 33.12$

(b) $\int_{-\infty}^{\infty} \cos(6\pi x)\,\delta(x-1)\,dx = \cos(6\pi) = 1.0$

(c) $\int_{-\infty}^{\infty} \frac{24\,\delta(x-2)\,dx}{x^4+3x^2+2} = \frac{24}{(2)^4+3(2)^2+2} = \frac{4}{5} = 0.8$

(d) $\int_{-\infty}^{\infty} \delta(x-x_0)\,e^{-j\omega x}\,dx = e^{-j\omega x_0}$

(e) $\int_{-3}^{3} u(x-2)\,\delta(x-3)\,dx = \frac{1}{2}u(3^+) = \frac{1}{2}$.

(2.3-6.) 1. Since $f_X(x) = dF_X(x)/dx$ is the slope of $F_X(x)$ we use the fact that $F_X(x)$ is a non-decreasing function of x, from (2.2-2d), to establish $f_X(x) \geq 0$.

2. We integrate $f_X(x) = dF_X(x)/dx$ from x_1 to x_2 to obtain

$$\int_{x_1}^{x_2} f_X(x)\,dx = F_X(x_2) - F_X(x_1). \quad (1)$$

Next, use (2.2-2a,b) to establish

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

by letting $x_1 = -\infty$ and $x_2 = \infty$.

3. Let $x_1 = -\infty$ and $x_2 = x$ in (1) above and use (2.2-2a) to get

2.3-6. (Continued)

$$\int_{-\infty}^{x} f_X(x)\,dx = F_X(x).$$

4\. Use (2.2-2e) with (1) above to prove

$$P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx.$$

2.3-7. (a) $P\{x_0 - 0.6\alpha < X \le x_0 + 0.3\alpha\} = G_X(x_0 + 0.3\alpha) - G_X(x_0 - 0.6\alpha) = \left[\frac{1}{2} + \frac{1}{\alpha}(x_0 + 0.3\alpha - x_0) - \frac{1}{2\alpha^2}(x_0 + 0.3\alpha - x_0)^2\right] - \left[\frac{1}{2\alpha^2}(x_0 - 0.6\alpha - x_0 + \alpha)^2\right] = 0.675.$

(b) $P\{X = x_0\} = 0$ because the probability of a continuous random variable having a discrete value is zero.

2.3-8. Test 1: $f_X(x) \ge 0$ is true. Test 2: Area $= \int_0^b \frac{e^{3x/4}}{4}\,dx = \frac{1}{4}\left(-\frac{1}{3} + \frac{e^{3b}}{3}\right) \overset{\text{must}}{=\!=} 1$. Thus $b = \frac{1}{3}\ln(13)$ is required.

2.3-9. Test 1: $g_X(x) \ge 0$ is true for any b. Test 2: Area $= \int_{-b}^{b} 4\cos\left(\frac{\pi x}{2b}\right)dx = \frac{16b}{\pi} \overset{\text{must}}{=\!=} 1$ so $b = \pi/16$.

2.3-10. $C = A \cap B = \{1 < X \le 2.5\}$. For $0 \le x$: $F_X(x) = \int_0^x \frac{1}{2} e^{-\alpha/2}\,d\alpha = [1 - e^{-x/2}]u(x)$. (a) $P(A) = F_X(3) - F_X(1) = e^{-1/2} - e^{-3/2} = 0.3834$. (b) $P(B) = F_X(2.5) = 1 - e^{-1.25} = 0.7135$. (c) $P(C) = F_X(2.5) - F_X(1) = e^{-1/2} - e^{-1.25} = 0.3200$.

★2.3-11. Two cases. Case 1: $a>0$. Let $\alpha = ax+b$, $d\alpha = a\,dx$. $G(a,b)=\int_{-\infty}^{\infty}\phi\left(\frac{\alpha-b}{a}\right)\delta(\alpha)\frac{d\alpha}{a} = \frac{1}{a}\phi\left(\frac{-b}{a}\right)$, $a>0$.
Case 2: $a<0$. Let $\alpha = -|a|x+b$, $d\alpha = -|a|dx$.
$G(a,b)=\int_{\infty}^{-\infty}\phi\left(\frac{\alpha-b}{-|a|}\right)\delta(\alpha)\frac{d\alpha}{-|a|} = \frac{1}{|a|}\phi\left(\frac{b}{|a|}\right)$, $a<0$.
Thus,

$$G(a,b)=\frac{1}{|a|}\phi\left(\frac{-b}{a}\right),\ -\infty<a<\infty,\ \text{any } b.$$

2.3-12. Test 1: $f_X(x) = a\left[1-\frac{x}{b}\right]\geq 0$ for $b>0$ and $x>0$ if $a>0$ and $0\leq x\leq c<b$ (or $c/b<1$).
Test 2: Area $=\int_0^c a\left[1-\frac{x}{b}\right]dx = a\left[c-\frac{c^2}{2b}\right]\overset{\text{must}}{=\!=}1$.
This occurs for $c = b\pm b\sqrt{1-(2/ab)}$. Only the root with negative corresponds to $c/b<1$ so final conditions are: $a>0$, $c<b$, and $c/b = 1-\sqrt{1-(2/ab)}$.

2.3-13. (a) $(-5)^2/[1+(-5)^2] = 25/26$. (b) $\cos(\pi 3/6)=0$. (c) $e^{-4(-1+1)} = e^0 = 1$.

2.3-14. (a) $1+0+1=2$. (b) 0 (impulse outside integral's limits.)
(c) $e^{-2(1^2)}/[1+1^2+4^2] - e^{-2(-2)^2}/[1+(-2)^2+(-2)^4] = \frac{e^{-2}}{3}-\frac{e^{-8}}{21}\approx 0.0451$.

2.3-15. From use of (C-39) and (C-41):
$\int_{-\infty}^{\infty}f_X(x)\,dx = K\int_{-1}^{1}(1-x^2)\cos(\pi x/2)\,dx = K\left[\frac{4}{\pi}-\frac{8}{\pi^3}\left(\frac{\pi^2}{2}-4\right)\right] = K\frac{32}{\pi^3}$
$\overset{\text{must}}{=\!=} 1$ so $K = \pi^3/32\approx 0.9689$.

2.4-1. (a) $P\{|X|>2\} = P\{2<X\} + P\{X<-2\} = 1 - P\{X \le 2\} + P\{X<-2\}$. From (2.4-3) this equals $P\{|X|>2\} = 1 - F(2) + F(-2) = 1 - F(2) + 1 - F(2) = 2 - 2F(2)$. From Appendix B: $P\{|X|>2\} = 2 - 2(0.9772) = 0.0456$.

(b) $P\{X>2\} = 1 - P\{X \le 2\} = 1 - F(2) = 1 - 0.9772 = 0.0228$.

2.4-2. (a) $P\{|X|>2\} = P\{X>2\} + P\{X<-2\} = 1 - P\{X \le 2\} + P\{X<-2\}$. Use (2.4-6), (2.4-7) and Appendix B to get $P\{|X|>2\} = 1 - F\left(\frac{2-a_X}{\sigma_X}\right) + F\left(\frac{-2-a_X}{\sigma_X}\right) = 1 - F\left(\frac{2-4}{2}\right) + F\left(\frac{-2-4}{2}\right) = 1 - F(-1) + F(-3) = 1 - [1-F(1)] + [1-F(3)] = 1 + F(1) - F(3) = 1 + 0.8413 - 0.9987 = 0.8426$.

(b) $P\{X>2\} = 1 - P\{X \le 2\} = 1 - F\left(\frac{2-4}{2}\right) = 1 - F(-1) = 1 - [1-F(1)] = F(1) = 0.8413$.

2.4-3. $\int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\,\sigma_X} e^{-(x-a_X)^2/2\sigma_X^2}\,dx$

$$= \int_{-\infty}^{\infty} \frac{(x-a_X)}{\sqrt{2\pi}\,\sigma_X} e^{-(x-a_X)^2/2\sigma_X^2}\,dx + a_X \int_{-\infty}^{\infty} \frac{e^{-(x-a_X)^2/2\sigma_X^2}}{\sqrt{2\pi}\,\sigma_X}\,dx.$$

On letting $\xi = (x-a_X)/\sqrt{2}\,\sigma_X$ in the first right-side integral we obtain an odd integrand so the integral is zero. The second integral is a_X times the area under the Gaussian density function which is unity. Thus,

$$\int_{-\infty}^{\infty} x f_X(x)\,dx = a_X$$

(2.4-3.) (Continued)

for a Gaussian random variable defined by (2.4-1).

(2.4-4.) $\int_{-\infty}^{\infty}(x-a_X)^2 f_X(x)\,dx = \int_{-\infty}^{\infty}\frac{(x-a_X)^2}{\sqrt{2\pi}\,\sigma_X}e^{-(x-a_X)^2/2\sigma_X^2}dx$

Let $\xi = (x-a_X)/\sqrt{2}\,\sigma_X$, $d\xi = dx/\sqrt{2}\,\sigma_X$ so

$$\int_{-\infty}^{\infty}(x-a_X)^2 f_X(x)\,dx = \int_{-\infty}^{\infty}\frac{2\sigma_X^2}{\sqrt{\pi}}\xi^2 e^{-\xi^2}d\xi = \frac{4\sigma_X^2}{\sqrt{\pi}}\int_0^{\infty}\xi^2 e^{-\xi^2}d\xi.$$

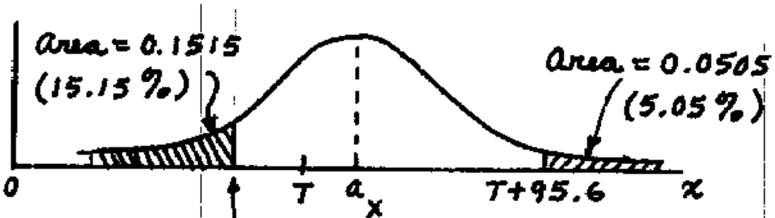
On using (C-52) this becomes

$$\int_{-\infty}^{\infty}(x-a_X)^2 f_X(x)\,dx = \frac{4\sigma_X^2}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{4} = \sigma_X^2.$$

(2.4-5.) (a) Rejected resistors correspond to $\{X<900\Omega\}$ and $\{X>1100\Omega\}$. Fraction rejected corresponds to probability of rejection $P\{\text{resistor rejected}\} = P\{X<900\}+P\{X>1100\} = F_X(900)+[1-F_X(1100)]$. From (2.4-7) this becomes $P\{\text{resistor rejected}\} = F\left(\frac{900-1000}{40}\right)+1-F\left(\frac{1100-1000}{40}\right) = 1+F(-2.5)-F(2.5)$ $= 1+[1-F(2.5)]-F(2.5) = 2-2F(2.5) = 2-2(0.9938)$ $= 0.0124$ or 1.24% are rejected.

(b) $P\{\text{resistor rejected}\} = P\{X<900\}+P\{X>1100\} = F_X(900)+1-F_X(1100) = F\left(\frac{900-1050}{40}\right)+1-F\left(\frac{1100-1050}{40}\right) = F\left(\frac{-150}{40}\right)+1-F\left(\frac{50}{40}\right) = 1-F\left(\frac{150}{40}\right)+1-F(1.25) = 2-F(3.75) - F(1.25) = 2-0.9999-0.8944 = 0.1057$ or 10.57% are rejected.

2.4-6. We use the sketch as a guide where T = target.



$$P\{X > T+95.6\} = 0.0505 = 1 - F\left(\frac{T+95.6-a_X}{\sigma_X}\right)$$

occurs when
$$\frac{T+95.6-a_X}{\sigma_X} = 1.64 \qquad (1)$$
from Appendix B.

$$P\{X \le T-11.2\} = 0.1515 = F\left(\frac{T-11.2-a_X}{\sigma_X}\right)$$

$= 1 - F\left(-\frac{T-11.2-a_X}{\sigma_X}\right)$ occurs when
$$-\frac{T-11.2-a_X}{\sigma_X} = 1.03. \qquad (2)$$

On solving (1) and (2) for a_X and σ_X we get $a_X = T + 30$ m and $\sigma_X = 40$ m.

2.4-7. (a) $P\{X > 1.0\} = 1 - P\{X \le 1.0\} = 1 - F_X(1.0) = 1 - F\left(\frac{1.0-2.0}{2.0}\right) = 1 - F(-1/2) = F(1/2) = 0.6915$. (b) $P\{X \le -1.0\} = F\left(\frac{-1.0-2.0}{2.0}\right) = F(-3/2) = 1 - F(3/2) = 1.0 - 0.9332 = 0.0668$.

2.4-8. (a) $P\{X \le 26\text{ m}\} = F_X(26) = F\left(\frac{26-30}{5}\right) = F(-0.8) = 1 - F(0.8) = 1 - 0.7881 = 0.2119$. (b) $P\{X > 42\} = 1 - F_X(42) = 1 - F\left(\frac{42-30}{5}\right) = 1 - F(2.4) = 1 - 0.9918 = 0.0082$ (0.82%).

2.4-9. (a) $P(A) = F_X(3300\text{ m}) - F_X(1000\text{ m}) = F\left(\frac{3300-4000}{1000}\right) - F\left(\frac{1000-4000}{1000}\right) = F(-0.7) - F(-3.0) = [1 - F(0.7)] - [1 - F(3.0)] = F(3) - F(0.7) = 0.9987 - 0.7580 = 0.2407$. (b) $P(B) =$

(2.4-9.) (Continued)

$F_X(4200) - F_X(2000) = F(0.2) - F(-2.0) = F(0.2) - 1 + F(2)$

$= 0.5793 - 1.0 + 0.9772 = 0.5565$. (c) P(both correct) $=$

$P(A \cap B) = P\{2000\text{ m} < X \le 3300\text{ m}\} = F_X(3300) - F_X(2000)$

$= F(-0.7) - F(-2.0) = [1 - 0.7580] - [1 - 0.9772] = 0.2192.$

(2.4-10.) (a) $P(\text{mistake}) = P\{X > 0.45\} = 1 - P\{X \le 0.45\}$

$= 1 - F_X(0.45) = 1 - F\left(\frac{0.45 - 0}{0.3}\right) = 1 - F(1.5) = 0.0668$

from Table B-1. (b) P(correct decision) $= P\{X \le 0.45\}$

$= F(1.5) = 0.9332$ (Table B-1).

(2.4-11.) 0.25 W exceeded when $|X|^2/100 > 0.25$ or $|X| > 5$ V. P(0.25 W exceeded) $= P\{|X| > 5\} = P\{X > 5\} + P\{X < -5\} = 1 - P\{X \le 5\} + P\{X < 5\}$

$= 1 - F\left(\frac{5-0}{4.2}\right) + F\left(\frac{-5-0}{4.2}\right) = 1 - F(5/4.2) + F(-5/4.2) = 2[1 - F(5/4.2)] = 2Q\left(\frac{5}{4.2}\right)$

$$\approx \frac{2}{\left[0.661\left(\frac{5}{4.2}\right) + 0.339\sqrt{(5/4.2)^2 + 5.51}\right]} \frac{1}{\sqrt{2\pi}} e^{-(5/4.2)^2/2} \approx 0.2339.$$

(2.4-12.) $|X|^2/100 > 0.50$ when $|X| > \sqrt{50}$. Similar to the solution ~~of Prob. 2-70: P(0.5 W exceeded) $= 2Q(\sqrt{50}/4.2)$~~

$$\approx \frac{2}{\left[0.661(\sqrt{50}/4.2) + 0.339\sqrt{(\sqrt{50}/4.2)^2 + 5.51}\right]} \cdot \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{50}/4.2)^2/2} \approx 0.09244.$$

(2.4-13.) $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-(x-a_X)^2/2\sigma_X^2}$

$$\frac{df_X(x)}{dx} = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-(x-a_X)^2/2\sigma_X^2}\left[\frac{-(x-a_X)}{\sigma_X^2}\right] = \frac{-(x-a_X)}{\sigma_X^2} f_X(x)$$

$$\frac{d^2 f_X(x)}{dx^2} = \frac{-(x-a_X)}{\sigma_X^2}\frac{df_X(x)}{dx} + f_X(x)\left(\frac{-1}{\sigma_X^2}\right) = \left[\frac{(x-a_X)^2}{\sigma_X^4} - \frac{1}{\sigma_X^2}\right] f_X(x) = 0 \text{ when}$$

2.4-13. (Continued) $(x-a_x)^2 - \sigma_x^2 = 0$ or $x = a_x \pm \sigma_x$.

2.4-14. (a) $P\{1.4 < X \le 2.0\} = F_X(2.0) - F_X(1.4) = F\left(\frac{2.0-1.6}{0.4}\right) - F\left(\frac{1.4-1.6}{0.4}\right)$

$= F(1.0) - F(-0.5) = F(1.0) - [1 - F(0.5)] = 0.8413 + 0.6915 - 1.0 = 0.5328.$

(b) $P\{-0.6 < (X-1.6) \le 0.6\} = P\{1.0 < X \le 2.2\} = F_X(2.2) - F_X(1.0)$

$= F\left(\frac{2.2-1.6}{0.4}\right) - F\left(\frac{1.0-1.6}{0.4}\right) = F(1.5) - F(-1.5) = 2F(1.5) - 1 = 2(0.9332)$

$-1.0 = 0.8664.$

2.4-15. (a) $P\{1980-68 < X \le 1980+68\} = F_X(2048) - F_X(1912) = F\left(\frac{2048-2000}{40}\right) -$

$F\left(\frac{1912-2000}{40}\right) = F(1.2) - F(-2.2) = 0.8849 - [1 - 0.9861] = 0.8710.$

(b) $P\{X \ge 2050\} = 1 - P\{X < 2050\} = 1 - F_X(2050) = 1 - F\left(\frac{2050-2000}{40}\right)$

$= 1 - F(1.25) = 0.1056.$

2.4-16. (a) $P\{160 < X \le 210\} = F_X(210) - F_X(160) = F\left(\frac{210-200}{20}\right) - F\left(\frac{160-200}{20}\right) =$

$F(0.5) - F(-2) = F(0.5) - [1 - F(2.0)] = 0.6915 + 0.9772 - 1 = 0.6687.$

(b) $P\{X > 231\} = 1 - P\{X \le 231\} = 1 - F_X(231) = 1 - F\left(\frac{231-200}{20}\right) = 1 - F(1.55)$

$= 1 - 0.9394 = 0.0606.$

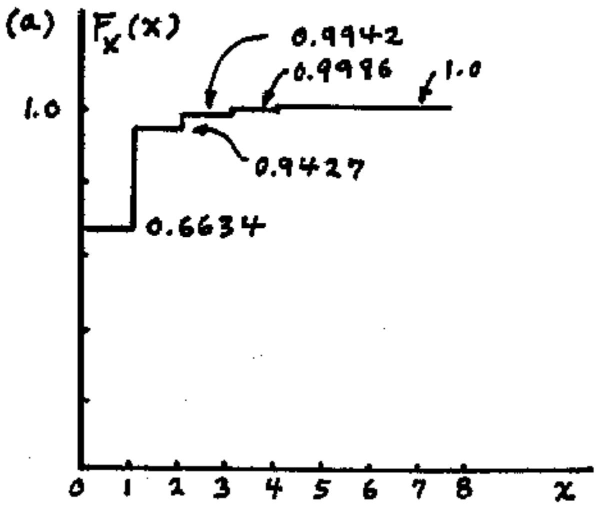
2.4-17. Here $P\{0 \text{ inoperable}\} = \frac{8!}{0!\,8!}(0.05)^0(0.95)^8 = 0.6634$

$P\{1 \text{ inoperable}\} = \frac{8!}{1!\,7!}(0.05)^1(0.95)^7 = 0.2793$

Similarly, $P\{2 \text{ inoperable}\} = 0.0515$, $P\{3 \text{ inoperable}\} = 0.0054$, $P\{4 \text{ inoperable}\} = 0.0004$, $P\{5 \text{ inoperable}\} = 0.00002$, $P\{6 \text{ inoperable}\} \approx 0$, $P\{7 \text{ inoperable}\} \approx 0$, and $P\{8 \text{ inoperable}\} \approx 0$.

(2.4-17.) (Continued)

(a)



(b) $P\{\text{exactly 1 lamp is inoperable}\} = 0.2793.$

(c) $P\{\text{8 are functional}\} = P\{\text{0 inoperable}\} = 0.6634.$

(d) $P\{\text{1 or more inoperable}\} = 1 - P\{\text{0 inoperable}\} = 1 - 0.6634 = 0.3366.$

(2.5-1.) (a) $I_2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \frac{1}{b}\int_a^{\infty} x^2 e^{-(x-a)/b}\,dx$.

$= \frac{e^{a/b}}{b}\int_a^{\infty} x^2 e^{-x/b}\,dx$. Use (C-47). $I_2 = \frac{e^{a/b}}{b}$

$\cdot\left\{ e^{-x/b}\left[\frac{x^2}{-1/b} - \frac{2x}{(-1/b)^2} + \frac{2}{(-1/b)^3}\right]\Big|_a^{\infty}\right\} = (a+b)^2 + b^2.$

(b) $I_1 = \int_{-\infty}^{\infty} x f_X(x)\,dx = \frac{e^{a/b}}{b}\int_a^{\infty} x e^{-x/b}\,dx$. Use (C-46). $I_1 = \frac{e^{a/b}}{b}\left\{ e^{-x/b}\left[\frac{x}{(-1/b)} - \frac{1}{(-1/b)^2}\right]\Big|_a^{\infty}\right\} = (a+b).$

(c) $I_2 - I_1^2 = (a+b)^2 + b^2 - (a+b)^2 = b^2.$

2.5-2. For $x \geq a$: $df_X(x)/dx = \frac{2(x-a)}{b} e^{-(x-a)^2/b}\left[\frac{-2(x-a)}{b}\right]$
$+ e^{-(x-a)^2/b}\left[\frac{2}{b}\right] = 0$ when $-\frac{2(x-a)^2}{b} + 1 = 0$ or
$x = a + \sqrt{b/2}$. By substitution: $f_X(a+\sqrt{\frac{b}{2}})$
$= \frac{2}{b}\sqrt{\frac{b}{2}}\, e^{-1/2} = \sqrt{2/b}\, e^{-1/2} \approx 0.607\sqrt{2/b}$.

Explanation: The mode is the point where $f_X(x)$ is maximum. There is no restriction that a function $f_X(x)$ cannot have more than one such maximum and therefore have more than one mode.

2.5-3. We use the Rayleigh distribution, given by (2.5-7) with $a=0$ and $b=400$, for probability.

(a) $P\{X \leq 1\} = F_X(1) = 1 - e^{-1/400} \approx 0.0025$ or 0.25%.

(b) $P\{X > 52\} = 1 - F_X(52) = 1 - [1 - e^{-(52)^2/400}]$
$= e^{-(52)^2/400} \approx 0.00116$ or 0.12%.

2.5-4. $$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{-\infty}^{x} \frac{(b/\pi)\,du}{b^2 + (u-a)^2}.$$

Let $\xi = u - a$, $d\xi = du$ to get
$$F_X(x) = \frac{b}{\pi}\int_{-\infty}^{x-a} \frac{d\xi}{b^2+\xi^2} = \frac{b}{\pi}\left[\frac{1}{b}\tan^{-1}(\xi/b)\Big|_{-\infty}^{x-a}\right]$$
$$= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-a}{b}\right),$$
where (C-25) has been used.

2.5-5. $$F_X(x) = \int_b^x \frac{e^{-[\ln(u-b) - a_X]^2/2\sigma_X^2}}{\sqrt{2\pi}\,\sigma_X(u-b)}\,du, \quad x \geq b$$
$$= 0, \quad x < b$$

2.5-5. (Continued) For $x \geq b$ let $(u-b) = e^{\xi}$ or $\xi = \ln(u-b)$, $du = e^{\xi} d\xi = (u-b)\,d\xi$.

$$F_X(x) = \int_{-\infty}^{\ln(x-b)} \frac{e^{-(\xi - a_X)^2/2\sigma_X^2}}{\sqrt{2\pi}\,\sigma_X}\,d\xi, \quad x \geq b.$$

Now let $\eta = (\xi - a_X)/\sigma_X$, $d\eta = d\xi/\sigma_X$, and use (2.4-3):

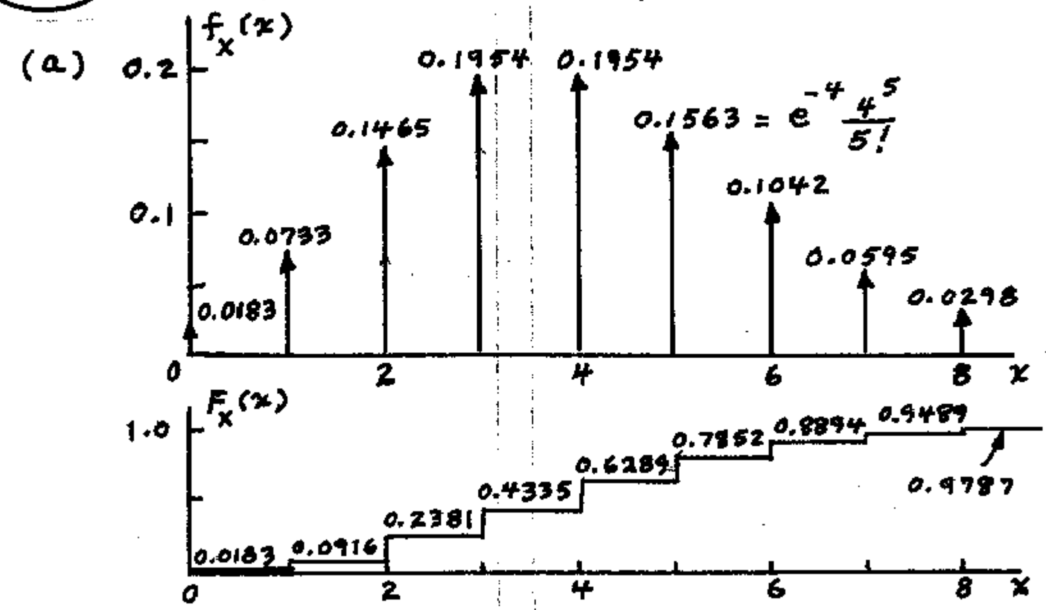
$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{[\ln(x-b) - a_X]/\sigma_X} e^{-\eta^2/2}\,d\eta$$

$$= F\left[\frac{\ln(x-b) - a_X}{\sigma_X}\right], \quad x \geq b$$

$$= 0, \quad x < b.$$

2.5-6. From (2.5-4) and (2.5-5)

(a)



(b) $P\{0 \leq X \leq 5\} = P\{X = 0\} + P\{X = 1\} + P\{X = 2\}$
$+ P\{X = 3\} + P\{X = 4\} + P\{X = 5\} = F_X(5) = 0.7852.$

2.5-7. Here $f_X(x) = e^{-2}\sum_{k=0}^{\infty}(2^k/k!)\,\delta(x-k)$.

(a) $P\{X>3\} = 1 - P\{X \le 3\} = 1 - P\{X=0\} - P\{X=1\} - P\{X=2\}$ $- P\{X=3\} = 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right] = 1 - e^{-2}\left(\frac{19}{3}\right)$ $\approx 0.1429.$

(b) $P\{X=0\} = e^{-2} \approx 0.1353.$

2.5-8. Mode is $x = \sqrt{b/2}$. $P\{X > \sqrt{b/2}\} = 1 - P\{X \le \sqrt{b/2}\}$ $= 1 - F_X(\sqrt{b/2}) = 1 - \left[1 - e^{-(b/2)/b}\right] = e^{-1/2} = 0.6065.$

2.5-9. (a) $P(5 \text{ or more}) = 1 - P(0) - P(1) - P(2) - P(3) - P(4)$ $= 1 - e^{-3}\left[1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}\right] = 1 - \frac{131}{8}e^{-3} = 0.1847.$

(b) $P(0) = e^{-3} = 0.0498$. Average number of weeks per year with no murders $= 52(e^{-3}) = 2.5889$ weeks.

(c) 3 or more murders exceeds the average so $P(3 \text{ or more}) = 1 - P(0) - P(1) - P(2) = 1 - e^{-3}\left[1 + 3 + \frac{3^2}{2}\right]$ $= 1 - \frac{17}{2}e^{-3} = 0.5768$. Average number of weeks per year that number of murders exceeds the average $= 52\left[1 - \frac{17}{2}e^{-3}\right] = 29.9941$ weeks.

2.5-10. $\lambda = 1/\text{MTBF} = 1/200$, $T = 7(24) = 168$ h (1 week). $P(0 \text{ failures in } 1 \text{ wk}) = e^{-\lambda T} = e^{-168/200} = 0.4317.$

2.5-11. $P(\text{no failures in time } T) = e^{-T/200}.$

2.5-12. (a) $P(\text{significant down time}) = P(\text{more than 3 failures in 6 weeks}) = 1 - P(3 \text{ or less failures in 6 weeks}) = 1 - F_X(3) = 1 - P(0 \text{ failure}) - P(1 \text{ failure}) -$

2.5-12 (Continued)

$$P(2 \text{ failures}) - P(3 \text{ failures}) = 1 - e^{-6/4}\left[1 + (6/4) + \frac{(6/4)^2}{2!} + \frac{(6/4)^3}{3!}\right] = 1 - \frac{67}{16}e^{-1.5} = 0.0656 \text{ from } (2.5\text{-}5).$$

(b) Same as (a) except use $T = 8$ weeks: P(significant down time) $= 1 - e^{-8/4}\left[1 + (8/4) + \frac{(8/4)^2}{2!} + \frac{(8/4)^3}{3!}\right]$

$$= 1 - \frac{19}{3}e^{-2} = 0.1429.$$

2.5-13 $f_X(x) = x e^{-x^2/2}$, $P\{\text{false detection}\} = \int_V^{\infty} f_X(x)\,dx = \int_V^{\infty} x e^{-x^2/2}\,dx$, $\xi = -x^2/2$

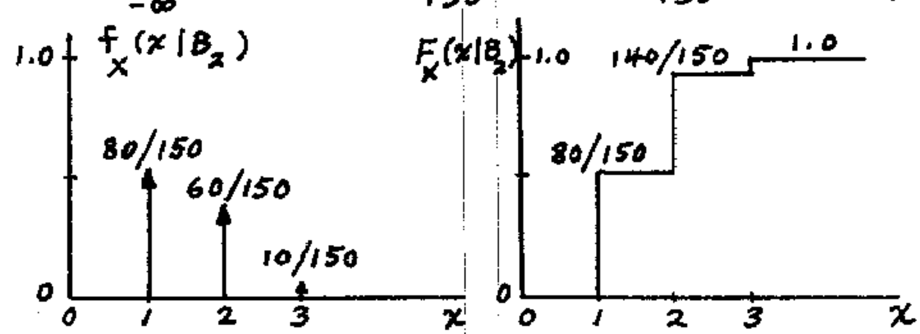
$$= -\int_{-V/2}^{-\infty} e^{\xi}\,d\xi = e^{-V^2/2} \stackrel{\text{must}}{=} 0.001 \quad \text{so} \quad V^2/2 = \ln(1000)$$

and $V = \sqrt{6\ln(10)} = 3.7169$ volts.

2.6-1. $f_X(x|B_2) = \sum_{i=1}^{3} P(X = i \mid B_2)\,\delta(x - i)$

$$= \frac{80}{150}\delta(x-1) + \frac{60}{150}\delta(x-2) + \frac{10}{150}\delta(x-3)$$

$$F_X(x|B_2) = \int_{-\infty}^{x} f_X(\xi|B_2)\,d\xi = \frac{80}{150}u(x-1) + \frac{60}{150}u(x-2) + \frac{10}{150}u(x-3).$$



★**2.6-2.** With $B = \{a < X \le b\}$ (2.6-2) becomes

$$F_X(x \mid a < X \le b) = \frac{P\{X \le x \cap a < X \le b\}}{P\{a < X \le b\}}.$$

(2.6-1.) (Continued)

If $x < a$ then $F_X(x \mid a < X \le b) = 0$ because (1)
$P\{X \le x \cap a < X \le b\} = P(\emptyset) = 0$.

If $a \le x < b$ then $P\{X \le x \cap a < X \le b\} = P\{a < X \le x\}$
$= F_X(x) - F_X(a)$ and $P\{a < X \le b\} = F_X(b) - F_X(a)$ so

$$F_X(x \mid a < X \le b) = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}, \quad a \le x < b \qquad (2)$$

★(2.6-2.) (Continued) If $x \ge b$ then $P\{X \le x \cap a < X \le b\}$
$= P\{a < X \le b\}$ and

$$F_X(x \mid a < X \le b) = \frac{F_X(b) - F_X(a)}{F_X(b) - F_X(a)} = 1.0, \quad x \ge b. \ (3)$$

By combining (1), (2), and (3):

$$F_X(x \mid a < X \le b) = \begin{cases} 0 & x < a \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}, & a \le x < b \\ 1.0, & b \le x. \end{cases} \qquad (4)$$

By differentiating (4)

$$f_X(x \mid a < X \le b) = \begin{cases} 0, & x < a \\ \dfrac{f_X(x)}{F_X(b) - F_X(a)} = \dfrac{f_X(x)}{\int_a^b f_X(x)\,dx}, & a \le x < b \\ 0, & x \ge b. \end{cases}$$

★(2.6-3.) The expressions given in Problem 2.6-2 apply. Specifically, let $a = 20$ and $b = \infty$ so

*2.6-3. (Continued)

$$f_X(x \mid 20 < X) = 0, \qquad x < 20$$
$$= \frac{f_X(x)}{\int_{20}^{\infty} f_X(x)\,dx}, \qquad 20 \le x < \infty. \qquad (1)$$

However, here

$$f_X(x) = \frac{x}{200} e^{-x^2/400}, \qquad 0 \le x$$
$$= 0, \qquad x < 0$$

so

$$\int_{20}^{\infty} f_X(x)\,dx = 1 - \int_{-\infty}^{20} f_X(x)\,dx = 1 - F_X(20) = e^{-(20)^2/400} = e^{-1}. \quad (2)$$

Since the probability of system lifetime being larger than 26 weeks, given that it has survived beyond 20 weeks, is $P\{X>26 \mid X>20\}$, we use (2) with (1) to obtain

$$P\{X>26 \mid X>20\} = \int_{26}^{\infty} f_X(x \mid X>20)\,dx$$
$$= e\int_{26}^{\infty} \frac{x}{200} e^{-x^2/400} dx .$$

Let $\xi = x^2/400$, $d\xi = x\,dx/200$ and use (C-45) to get

$$P\{X>26 \mid X>20\} = e\int_{(26)^2/400}^{\infty} e^{-\xi} d\xi = e\left[-e^{-\xi}\Big|_{(26)^2/400}^{\infty}\right]$$
$$= e^{-0.69} \approx 0.5016 .$$

★ 2.6-4. This is a conditional problem with $B=\{X\le 20\}$ where $b=20$ weeks in (2.6-17). Thus,

$$P\{X\le 10|X\le 20\}=\frac{F_X(10)}{F_X(20)}=\frac{1-e^{-10^2/30}}{1-e^{-20^2/30}}=0.9643.$$

★ 2.6-5. From (2.5-10) $F_X(x)=u(x)[1-e^{-x/13.5}]$.

(a) $P(\text{overflow})=P\{X>40.6\,m\}=1-P\{X\le 40.6\}=1-F_X(40.6)=e^{-40.6/13.5}=0.0494$. (b) $P(\text{power}|\text{no overflow})=\dfrac{P\{X>8.6\cap X\le 40.6\}}{P\{X\le 40.6\}}=1-\dfrac{P\{X\le 8.6\cap X\le 40.6\}}{P\{X\le 40.6\}}$

$=1-[F_X(8.6)/F_X(40.6)]=1-[(1-e^{-8.6/13.5})/(1-e^{-40.6/13.5})]$

$=0.5044$. (c) $P(\text{no power})=P\{X\le 8.6\}=F_X(8.6)$

$=1-e^{-8.6/13.5}=0.4711$.

★ 2.6-6. Without conditions (2.5-9) and (2.5-10) apply with $a=0$ and $b=13.5$:

$$f_X(x)=(1/13.5)\,u(x)\exp(-x/13.5) \qquad (1a)$$

$$F_X(x)=u(x)[1-\exp(-x/13.5)]. \qquad (1b)$$

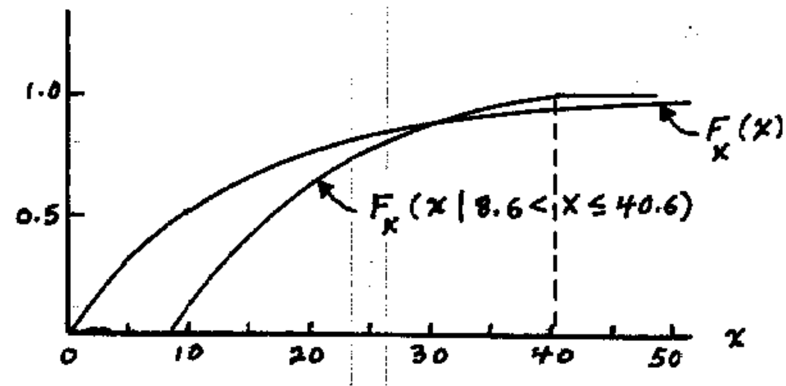
With the conditioning event $B=\{8.6<X\le 40.6\}$ the results of Problem 2.6-2 apply with $a=8.6$ m and $b=40.6$ m:

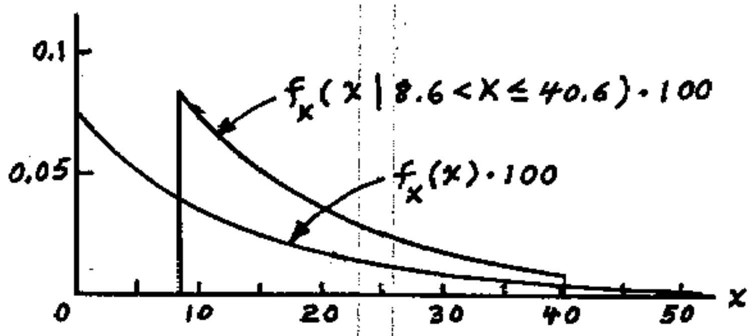
$$f_X(x|8.6<X\le 40.6)=\begin{cases}0, & x<8.6 \text{ and } x>40.6\\ \dfrac{(1/13.5)\,e^{-x/13.5}}{e^{-8.6/13.5}-e^{-40.6/13.5}}, & 8.6\le x\le 40.6\end{cases} \qquad (2a)$$

$$F_X(x|8.6<X\le 40.6)=\begin{cases}0, & x<8.6\\ \dfrac{e^{-8.6/13.5}-e^{-x/13.5}}{e^{-8.6/13.5}-e^{-40.6/13.5}}, & 8.6\le x\le 40.6\\ 1, & x>40.6\end{cases} \qquad (2b)$$

★ 2.6-6. (Continued)

Both (1) and (2) are sketched:





★ 2.6-7. This a problem with a conditioning event $B = \{10 < X \le 50\}$ so Problem 2.6-2 applies with $a = 10$ and $b = 50$. $P(\text{qualified}) = P(X \le 25 \mid 10 < X \le 50) =$

$$\frac{F_X(25) - F_X(10)}{F_X(50) - F_X(10)} = \frac{e^{-10^2/800} - e^{-25^2/800}}{e^{-10^2/800} - e^{-50^2/800}} = \frac{e^{-1/8} - e^{-6.25/8}}{e^{-1/8} - e^{-25/8}}$$

$= 0.5064.$

2.6-8. Here $P(D_1) = 0.30$, $P(X = 0 \mid D = D_1) = 0.1$, $P(X = 100 \mid D = D_1) = 0.2$, $P(X = 1000 \mid D = D_1) = 0.7$; $P(D_2) = 0.45$, $P(X = 0 \mid D = D_2) = 0.5$, $P(X = 100 \mid D = D_2) = 0.35$, $P(X = 1000 \mid D = D_2) = 0.15$; $P(D_3) = 0.25$, $P(X = 0 \mid D = D_3) = 0.8$, $P(X = 100 \mid D = D_3) = 0.15$, $P(X = 1000 \mid D = D_3) = 0.05$.

2.6-8. (Continued) (a) $F_X(x|D=D_1) = 0.1\,u(x) + 0.2\,u(x-100) + 0.7\,u(x-1000)$

$f_X(x) = 0.1\,\delta(x) + 0.2\,\delta(x-100) + 0.7\,\delta(x-1000)$. (b) $f_X(x|D=D_2) = 0.5\,\delta(x) + 0.35\,\delta(x-100) + 0.15\,\delta(x-1000)$. (c) $f_X(x|D=D_3) = 0.8\,\delta(x) + 0.15\,\delta(x-100) + 0.05\,\delta(x-1000)$. (d) $P\{X=\#0\} = P\{X=0|D=D_1\}P\{D_1\} + P\{X=0|D=D_2\}P\{D_2\} + P\{X=0|D=D_3\}P\{D_3\} = 0.1(0.3) + 0.5(0.45) + 0.8(0.25) = 0.455$. Similarly, $P\{X=\#100\} = 0.255$ and $P\{X=\#1000\}$ ~~$= 0.290$. Thus, $f_X(x) = 0.455\,\delta(x) + 0.255\,\delta(x-100) + 0.290\,\delta(x-1000)$.~~

2.6-9. Use results of part (d) of solution to Problem 2.6-8:
(a) $P\{\#0\} = 0.455$, (b) $P\{\#100\} = 0.255$, and (c) $P\{\#1000\} = 0.290$.

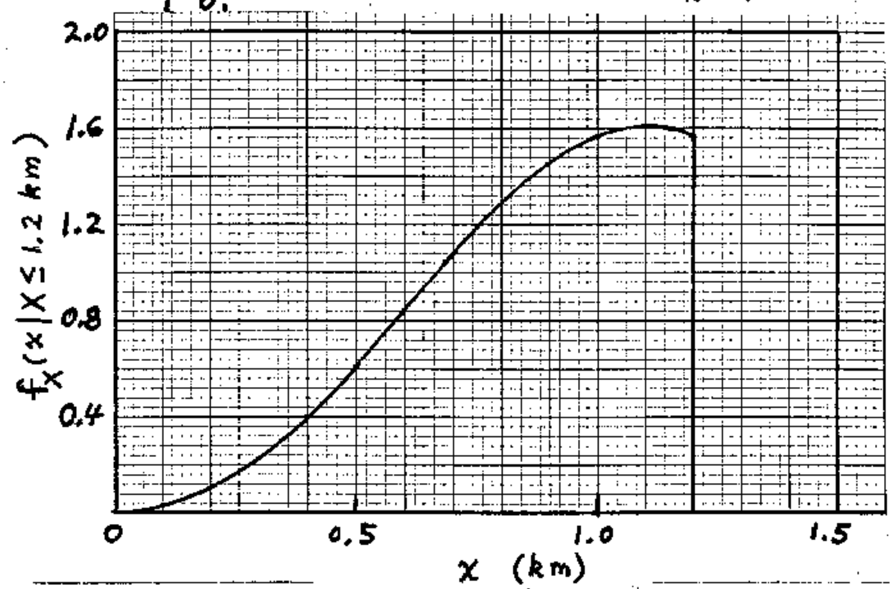
★ **2.6-10.** From (2.6-11) with $b = 1.2$ km so that $F_X(b) = 1 - e^{-(1.2)^3/2} = 0.5785 = 1/1.7285$, we have: $F_X(x|X \le 1.2\text{ km}) = 1.7285[1 - e^{-x^3/2}$, $x < 1.2$ km, and $F_X(x|X \le 1.2\text{ km}) = 0$, $x > 1.2$ km.
$P\{X = 0.6\text{ km} | X \le 1.2\text{ km}\} = 1.7285[1 - e^{-(0.6)^3/2}] = 0.1770$.

★ **2.6-11.** Use (2.6-12). $f_X(x) = -e^{-x^3/2}\left(-\frac{3x^2}{2}\right)u(x) + \delta(x)[1 - e^{-x^3/2}]$
$= \frac{3}{2}x^2 e^{-x^3/2}u(x)$. $f_X(x|X \le 1.2\text{ km}) = \dfrac{(3/2)x^2}{0.5785}e^{-x^3/2}u(x)$, $x < 1.2$ km.

(Over)

★2.6-11. (Continued)

or $f_X(x \mid X \le 1.2\text{ km}) = \begin{cases} 2.5928\, x^2\, e^{-x^3/3} u(x), & x < 1.2\text{ km} \\ 0, & x > 1.2\text{ km} \end{cases}$



CHAPTER 3

(3.1-1.) Use (3.1-4):

$$E[X] = \overline{X} = \sum_{i=1}^{5} x_i P(x_i) = 1.0(0.4) + 4(0.25) + 9(0.15) + 16(0.1) + 25(0.1) = 6.85.$$

(3.1-2.) $E[X] = \sum_{n=1}^{\infty} n P(x_n) = \sum_{n=1}^{\infty} \frac{n}{2^n}$. From Jolley (1961, p. 8) it is known that

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \quad \text{if} \quad x < 1.$$

Thus, if $x = 1/2$: $E[X] = \frac{1/2}{(1-\frac{1}{2})^2} = 2$.

(3.1-3.) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} p^n \delta(x-n)\,dx = \sum_{n=1}^{\infty} p^n$.

This series is known (Jolley, 1961, p. 8) to be given by

$$\sum_{n=1}^{\infty} p^n = \frac{p}{1-p}, \quad p < 1.$$

But $\int_{-\infty}^{\infty} f_X(x)\,dx$ must equal unity so $p/(1-p) = 1$ is necessary. Only $p = 1/2$ satisfies this condition.

(3.1-4.) (a) Average winnings $= E[X] = -1(\frac{1}{8}) + 1(\frac{3}{8}) + 2(\frac{3}{8}) + 3(\frac{1}{8}) = 15/8 = \$1.875$. (b) $P(\text{win}) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$.

3.1-5. $\bar{X} = \int_{-a}^{a} \frac{x\,dx}{\pi\sqrt{a^2-x^2}} = \left[\int_{-a}^{0} + \int_{0}^{a}\right] \frac{x\,dx}{\pi\sqrt{a^2-x^2}}$. Let $u = x^2$, $du = 2x\,dx$ so

$$\bar{X} = \left[\int_{a^2}^{0} + \int_{0}^{a^2}\right] \frac{du}{2\pi\sqrt{a^2-u}} = \frac{1}{2\pi}\left[\int_{0}^{a^2} - \int_{0}^{a^2}\right] \frac{du}{\sqrt{a^2-u}} = 0$$

$$\overline{X^2} = \int_{-a}^{a} \frac{x^2\,dx}{\pi\sqrt{a^2-x^2}} = \frac{1}{\pi}\left[-\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]\Bigg|_{-a}^{a}$$

(Dwight, p. 68) or $\overline{X^2} = a^2/2$.

★**3.1-6.** We use (2.5-11) and (2.5-12) with $a = 0$ and $b = 30$ weeks in (2.6-12) with $b = 20$ to get

$$f_X(x|X \le 20) = \begin{cases} \dfrac{x e^{-x^2/30}}{15\left[1 - e^{-400/30}\right]}, & x < 20 \\ 0, & x \ge 20 \text{ and } x \le 0 \end{cases}$$

From (3.1-8):

$$E[X|X \le 20] = \int_0^{20} \frac{x^2 e^{-x^2/30}}{15\left[1-e^{-40/3}\right]}dx \approx \int_0^{\infty} \frac{x^2 e^{-x^2/30}}{15\left[1-e^{-40/3}\right]}dx$$

$$= \frac{\sqrt{7.5\pi}}{1 - e^{-40/3}} = 4.854 \text{ (from Dwight, p. 230).}$$

3.1-7. $E[g(X)] = E[X^3] = \int_0^{\infty} \frac{1}{2} x^3 e^{-x/2} dx$ ← use (C-48)

$$= \frac{1}{2}\left(\frac{6}{(1/2)^4}\right) = 48.$$

3.1-8. (a) $f_X(x) = 0.35\,\delta(x-1) + 0.25\,\delta(x-5) + 0.20\,\delta(x-10) + 0.15\,\delta(x-25) + 0.05\,\delta(x-50)$. (b) $E[X] = \bar{X} = 1(0.35) + 5(0.25) + 10(0.20) + 25(0.15) + 50(0.05) = 9.85$ ¢

(3.1-9.) (a) $P(R_1) = E_2^2/R_2 = E_1^2 R_2^2/[(R_1+R_2)^2 R_2] = E_1^2 R_2/(R_1+R_2)^2$.

(b) $E[P] = \bar{P} = \int_{-\infty}^{\infty} P(R_1) f_{R_1}(R_1)\, dR_1 = \frac{E_1^2 R_2}{2\Delta R}\int_{R_0-\Delta R}^{R_0+\Delta R} \frac{dR_1}{(R_1+R_2)^2}$ ← use (C-22)

$= E_1^2 R_2/[(R_0+R_2)^2 - \Delta R^2]$. (c) $E[P] = 144(1000)/[2500^2 - 10{,}000]$

$= 23.077(10^{-3})$ W.

★(3.1-10.) (a) Use (3.1-10). $\int_{-\infty}^{15} f_p(p)\,dp = \int_0^{15} \frac{e^{-p/10}}{10}\,dp = 1 - e^{-15/10} \approx 0.777$ mW.

Thus,
$$f_p(p \mid P \le 15\text{ mW}) = \frac{e^{-p/10}}{1-e^{-1.5}}[u(p) - u(p-15)]$$

(b) $E[P \mid P \le 15] = \frac{\int_0^{15} p\, e^{-p/10}\,dp}{10[1-e^{-1.5}]} = \frac{10(1-2.5e^{-1.5})}{1-e^{-1.5}} \approx 5.692$ mW.

(3.1-11.) $E[g(X)] = E[4X^2] = \int_{-\infty}^{\infty} 4x^2 f_X(x)\,dx = 4\int_{-\pi/2}^{\pi/2} \frac{x^2}{2}\cos(x)\,dx$

$= \pi^2 - 8 \approx 1.8696$.

(3.1-12.) Use $\int x^4 \cos(x)\,dx = (4x^3 - 24x)\cos(x) + (x^4 - 12x^2 + 24)\sin(x)$ from Dwight, p. 101. $E[4X^4] = 4\int_{-\pi/2}^{\pi/2} \frac{x^4}{2}\cos(x)\,dx = \frac{\pi^4}{4} - 12\pi^2 + 96$

≈ 1.917.

(3.1-13.) Here $L = 128$, $p(x_i) = 1/128$ so

$$H = \frac{-1}{\ln(2)}\sum_{i=1}^{128} \frac{1}{128}\ln\left(\frac{1}{128}\right) = \frac{-1}{128\ln(2)}[128\ln(2)] = 1$$

bit/symbol.

(3.1-14.) Here $Y = g(X) = e^{-X/5}$ so $E[Y] = E[g(X)]$

$= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = \int_{-5}^{15} e^{-x/5}\frac{1}{15-(-5)}\,dx = \frac{1}{20}\left[-5e^{-x/5}\Big|_{-5}^{15}\right]$

$= \frac{1}{4}(e^1 - e^{-3}) \approx 0.667$.

(3.1-15) Here $Y = g(X) = 5X^2$. Thus, $E[Y] = E[g(X)]$
$= \int_{-\infty}^{\infty} 5x^2 \frac{e^{-x^2/2\sigma_x^2}}{\sqrt{2\pi}\,\sigma_x} dx$. Let $\xi = x/\sigma_x$, $d\xi = dx/\sigma_x$.

$$E[Y] = 5\sigma_x^2 \underbrace{\int_{-\infty}^{\infty} \xi^2 \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} d\xi}_{} , \text{ where } \sigma_x^2 = 9.$$

Variance of zero-mean Gaussian r.v. with variance = 1 so $E[Y] = 45$.

★(3.1-16) It was found in the solution of Problem 2.6-3 that

$$f_X(x|X>20) = \frac{e\,x\,u(x-20)}{200} e^{-x^2/400}.$$

Thus, $E[X|X>20] = \int_{20}^{\infty} \frac{e}{200} x^2 e^{-x^2/400} dx$. Let
$\xi = x/20$, $d\xi = dx/20$ so $E[X|X>20] = 40e\int_1^{\infty} \xi^2 e^{-\xi^2} d\xi$.

This integral is not known in closed form. It is approximated as follows by using a series expansion for $e^{-\xi^2}$. By using (C-52):

$$E[X|X>20] = 40e\left[\frac{\sqrt{\pi}}{4} - \int_0^1 \xi^2 e^{-\xi^2} d\xi\right]$$

$$\int_0^1 \xi^2 e^{-\xi^2} d\xi = \int_0^1 \xi^2 \sum_{n=0}^{\infty} \frac{(-\xi^2)^n}{n!} d\xi = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+3}}{n!\,(2n+3)}\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+3)} = \frac{1}{3} - \frac{1}{5} + \frac{1}{14} - \frac{1}{54} + \frac{1}{264} - \cdots$$

$$\approx 0.1895.$$

$$E[X|X>20] \approx 40e\left[(\sqrt{\pi}/4) - 0.1895\right] \approx 27.58 \text{ weeks.}$$

(3.2-1.) One example (there are many) is a random variable with two values x_1 and $x_2 \neq x_1$ that are equally probable. Here $E[X] = (x_1 + x_2)/2 \neq (x_1$ or $x_2)$.

(3.2-2.) (a) Since $f_X(x)$ is symmetrical about $x = x_0$ its mean value is $E[X] = x_0$.

(b) We calculate

$$\sigma_X^2 = E[(X-\bar{X})^2] = \int_{-\infty}^{\infty} (x - x_0)^2 f_X(x)\, dx .$$

But

$$\begin{aligned} f_X(x) &= 0, & x < x_0 - \alpha \text{ and } x \geq x_0 + \alpha \\ &= \frac{1}{\alpha^2}(x - x_0 + \alpha), & x_0 - \alpha \leq x < x_0 \\ &= \frac{1}{\alpha} - \frac{1}{\alpha^2}(x - x_0), & x_0 \leq x < x_0 + \alpha . \end{aligned}$$

so

$$\sigma_X^2 = \int_{x_0-\alpha}^{x_0} \frac{(x-x_0)^2}{\alpha^2}(x - x_0 + \alpha)\, dx$$

$$+ \int_{x_0}^{x_0+\alpha} \frac{(x-x_0)^2}{\alpha}\left[1 - \frac{1}{\alpha}(x - x_0)\right] dx = \alpha^2/6 .$$

(3.2-3.) $\bar{X} = E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_a^b x \frac{1}{b-a}\, dx = \frac{b+a}{2}$.

Since $\sigma_X^2 = E[(X-\bar{X})^2] = E[X^2 - 2X\bar{X} + \bar{X}^2] = E[X^2] - \bar{X}^2$ we first find $E[X^2]$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_a^b x^2 \frac{dx}{b-a} = \frac{(b+a)^2 - ab}{3} .$$

Thus,

$$\sigma_X^2 = \frac{(b+a)^2}{3} - \frac{ab}{3} - \frac{(b+a)^2}{4} = \frac{(b-a)^2}{12} .$$

3.2-4. (a) Since all positions are equally probable for a fair wheel, the density function of the random variable X = "pointer position" is uniform (see Problem 3.2-3) with $a = 0$ and $b = 100$. Thus, $E[X] = \bar{X} = (a+b)/2 = 100/2 = 50$.

(b) From Problem 3.2-3: $\sigma_X = (b-a)/\sqrt{12} = 100/\sqrt{12} \approx 28.87$.

3.2-5. Here X is discrete with values 1, 2, and 3 having probabilities $P\{X=1\} = P\{-21 \le s \le 18.3\} = [18.3-(-21)]/[49-(-21)] \approx 0.561$, $P\{X=2\} = P\{18.3 < s \le 21.7\} = [21.7-18.3]/[49-(-21)] \approx 0.049$, $P\{X=3\} = P\{21.7 < s \le 49\} = [49-21.7]/[49-(-21)] \approx 0.390$. Thus,

$$f_X(x) = 0.561\,\delta(x-1) + 0.049\,\delta(x-2) + 0.390\,\delta(x-3).$$

(a) $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = P\{X=1\} + 2P\{X=2\} + 3P\{X=3\}$

≈ 1.829.

(b) $E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = P\{X=1\} + (2)^2 P\{X=2\}$

$+ (3)^2 P\{X=3\} \approx 4.267$.

$$\sigma_X^2 = E[X^2] - \bar{X}^2 \approx 4.267 - (1.829)^2 \approx 0.922.$$

★ 3.2-6. $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \int_{-\infty}^{\infty} x\,\delta(x-k)\,dx$

$= \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k}\, k = \sum_{k=1}^{N} \frac{N!\, p^k (1-p)^{N-k}}{(k-1)!\,(N-k)!}$

★3.2-6 (Continued)

$$= Np \sum_{k=1}^{N} \frac{(N-1)!\, p^{k-1} (1-p)^{(N-1)-(k-1)}}{(k-1)!\,[(N-1)-(k-1)]!} .$$

Let $j = k-1$ to get

$$E[X] = Np \sum_{j=0}^{N-1} \binom{N-1}{j} p^j (1-p)^{(N-1)-j}$$

The summation equals $[p+(1-p)]^{N-1} = 1$ by the binomial expansion. Thus, $E[X] = Np$.

$$E[X^2] = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} k^2 = \sum_{k=1}^{N} \frac{N!\, k^2 p^k (1-p)^{N-k}}{k!\,(N-k)!}$$

$$= Np \sum_{k=1}^{N} \frac{(N-1)!\,(k-1+1)\, p^{k-1} (1-p)^{(N-1)-(k-1)}}{(k-1)!\,[(N-1)-(k-1)]!}$$

$$= Np \left\{ \sum_{k=1}^{N} \frac{(N-1)!\, p^{k-1} (1-p)^{(N-1)-(k-1)}}{(k-1)!\,[(N-1)-(k-1)]!} + \sum_{k=1}^{N} \frac{(N-1)(N-2)!\, p\, p^{k-2} (1-p)^{(N-2)-(k-2)}}{(k-2)!\,[(N-2)-(k-2)]!} \right\}$$

$$= Np \left\{ \sum_{j=0}^{N-1} \binom{N-1}{j} p^j (1-p)^{(N-1)-j} + (N-1)p \sum_{j=0}^{N-2} \binom{N-2}{j} p^j (1-p)^{(N-2)-j} \right\}$$

Use binomial expansion again.

$$= Np[1 + (N-1)p].$$

Finally,

$$\sigma_X^2 = E[X^2] - \bar{X}^2 = Np + Np^2(N-1) - (Np)^2$$

$$= Np(1-p).$$

3.2-7. Define R as the random variable "resistance." Here its values and their probabilities are: $R_1 = 180\,\Omega$, $P(R_1) = 1/4$, $R_2 = 470\,\Omega$, $P(R_2) = 1/4$, $R_3 = 1000\,\Omega$, $P(R_3) = 1/4$, $R_4 = 2200\,\Omega$, and $P(R_4) = 1/4$.

(a) $E[R] = \sum_{i=1}^{4} R_i P(R_i) = \frac{180}{4} + \frac{470}{4} + \frac{1000}{4} + \frac{2200}{4} = 962.5\,\Omega.$

(b) $E_2 = 12\left(\frac{962.5}{962.5+820}\right) \approx 6.48$ V.

(c) By using the voltage values found in the solution to Problem 2.1-11 and the fact that these values are equally probable, we have

$$E[E_2] = \frac{1}{4}\left\{12\left(\frac{180}{1000}\right) + 12\left(\frac{470}{1290}\right) + 12\left(\frac{1000}{1820}\right) + 12\left(\frac{2200}{3020}\right)\right\}$$
$$\approx 5.47 \text{ V}.$$

Hence $E[E_2] \neq E_2$ of (b) produced by an average value of resistance. This fact happens because values of E_2 are not linearly related to values of resistance.

3.2-8. (a) $E[X] = a_X$ is true if $E[(X-a_X)] = 0$. Now $\int_{-\infty}^{\infty}(x-a_X)f_X(x)\,dx$, but $(x-a_X)$ is odd about $x=a_X$ while $f_X(x)$ is even about $x=a_X$. The integrand is therefore odd and the integral is zero.

(b) $\text{Var.} = E[(X-a_X)^2] = \int_{-\infty}^{\infty} \frac{(x-a_X)^2}{\sqrt{2\pi}\,\sigma_X} e^{-(x-a_X)^2/2\sigma_X^2}\,dx$

(3.2-8.) (Continued)

Let $\xi = (x-a_X)/\sqrt{2}\,\sigma_X$, $d\xi = dx/\sqrt{2}\,\sigma_X$ so

$$\text{Variance} = \frac{2\sigma_X^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}d\xi = \frac{4\sigma_X^2}{\sqrt{\pi}}\int_0^{\infty}\xi^2 e^{-\xi^2}d\xi .$$

Use (C-52) to finally obtain

$$\text{Variance} = \frac{4\sigma_X^2}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{4} = \sigma_X^2 .$$

(3.2-9.) $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^{\infty} \frac{2(x-a)x}{b} e^{-(x-a)^2/b}\,dx$

$$= 2\int_a^{\infty}\frac{(x-a)^2}{b}e^{-(x-a)^2/b}dx + 2a\int_a^{\infty}(x-a)\,e^{-(x-a)^2/b}\frac{dx}{b} .$$

Let $\xi = (x-a)/\sqrt{b}$, $d\xi = dx/\sqrt{b}$ in the first right-side integral and $\eta = (x-a)^2/b$, $d\eta = 2(x-a)\,dx/b$ in the second.

$$E[X] = 2\sqrt{b}\int_0^{\infty}\xi^2 e^{-\xi^2}d\xi + a\int_0^{\infty}e^{-\eta}d\eta .$$

On using (C-52) and (C-45):

$$E[X] = 2\sqrt{b}\,\frac{\sqrt{\pi}}{4} + a = a + \sqrt{\pi b/4} .$$

$$E[X^2] = \int_a^{\infty} x^2\,\frac{2}{b}(x-a)\,e^{-(x-a)^2/b}dx$$

$$= \int_a^{\infty}\frac{2}{b}(x-a+a)^2(x-a)\,e^{-(x-a)^2/b}dx$$

$$= \frac{2}{b}\int_a^{\infty}(x-a)^3 e^{-(x-a)^2/b}dx \quad \text{Let } \xi = (x-a)^2/b,\ d\xi = \frac{2(x-a)dx}{b}$$

$$+ \frac{4a}{b}\int_a^{\infty}(x-a)^2 e^{-(x-a)^2/b}dx \quad \text{Let } \xi = \frac{x-a}{\sqrt{b}},\ d\xi = \frac{dx}{\sqrt{b}}$$

(3.2-9.) (Continued)

$$+\frac{2a^2}{b}\int_a^\infty (x-a)\, e^{-(x-a)^2/b}dx \leftarrow \text{Let } \xi = \frac{(x-a)^2}{b},\ d\xi = \frac{2(x-a)\,dx}{b}$$

$$= b\int_0^\infty \xi e^{-\xi} d\xi + 4a\sqrt{b}\int_0^\infty \xi^2 e^{-\xi^2} d\xi + a^2\int_0^\infty e^{-\xi} d\xi$$

use (c-46)   use (c-52)   use (c-45)

$$= a^2 + a\sqrt{\pi b} + b$$

Finally, $\sigma_X^2 = E[X^2] - \bar{X}^2 = a^2 + b + a\sqrt{\pi b} - \left[a + \sqrt{\pi b/4}\right]^2$

$$= b - \frac{\pi b}{4} = b(4-\pi)/4.$$

(3.2-10.) Lifetime here is a Rayleigh random variable X defined by (2.5-6) where $a=0$ and $b=400$. By using the result given in Problem 3.2-9:

$$E[X] = 0 + \sqrt{400\pi/4} = 10\sqrt{\pi} \approx 17.72 \text{ weeks.}$$

(3.2-11.) (a) Because $f_X(x)$ has even symmetry about $x = m$, we have $E[X] = m$.

(b) $\sigma_X^2 = \int_{-\infty}^\infty \frac{(x-m)^2}{2b} e^{-|x-m|/b} dx$. Let $\xi = (x-m)/b$,

$d\xi = dx/b$: $\sigma_X^2 = \frac{b^2}{2}\int_{-\infty}^\infty \xi^2 e^{-|\xi|} d\xi = b^2\int_0^\infty \xi^2 e^{-\xi} d\xi$.

Use (c-47):

$$\sigma_X^2 = b^2\left\{e^{-\xi}\left[-\xi^2 - 2\xi - 2\right]\Big|_0^\infty\right\} = 2b^2.$$

(3.2-12) $E[X] = \int_{-\infty}^{\infty} \frac{x(b/\pi)dx}{b^2+(x-a)^2}$. Let $\xi = x-a$, $d\xi = dx$.

$E[X] = \frac{b}{\pi}\int_{-\infty}^{\infty} \frac{\xi\, d\xi}{b^2+\xi^2} + \frac{ab}{\pi}\int_{-\infty}^{\infty}\frac{dx}{b^2+\xi^2}$. The second integral equals a from (C-25) but the first is undefined from (C-26) except in the Cauchy principal value interpretation where it equals zero. Thus, in general $E[X]$ is undefined. Even in the principal value interpretation with $E[X]=a$ we use (C-27) and find that variance is undefined because the integral involved is infinite.

★(3.2-13) (a) From (3.1-4):

$E[X] = \sum_{k=0}^{\infty} \frac{b^k e^{-b} k}{k!} = b\sum_{k=1}^{\infty}\frac{b^{k-1}e^{-b}}{(k-1)!}$. Let $j=k-1$.

$E[X] = b\sum_{j=0}^{\infty}\frac{b^j e^{-b}}{j!}$. From the distribution function, which is $F_X(x) = \sum_{k=0}^{\infty}\frac{b^k e^{-b}}{k!}u(x-k)$, we have $F_X(\infty) = \sum_{k=0}^{\infty}\frac{b^k e^{-b}}{k!} = 1$. Thus, $E[X]=b$.

(b) $E[X^2] = \sum_{k=0}^{\infty}\frac{b^k e^{-b}k^2}{k!}$ from (3.1-7). Expanding:

$E[X^2] = \sum_{k=1}^{\infty}\frac{b^k e^{-b} k}{(k-1)!} = b\sum_{k=1}^{\infty}\frac{b^{k-1}e^{-b}}{(k-1)!}(k-1+1)$

$= b\left\{\sum_{k=1}^{\infty}\frac{b^{k-1}e^{-b}}{(k-1)!}(k-1) + \sum_{k=1}^{\infty}\frac{b^{k-1}e^{-b}}{(k-1)!}\right\}$

$= b\left\{b\sum_{k=2}^{\infty}\frac{b^{k-2}e^{-b}}{(k-2)!} + \sum_{j=0}^{\infty}\frac{b^j e^{-b}}{j!}\right\}$

★ 3.2-13. (Continued)

$$= b^2 \sum_{j=0}^{\infty} \frac{b^j e^{-b}}{j!} + b \sum_{j=0}^{\infty} \frac{b^j e^{-b}}{j!} = b(b+1)$$

Finally,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = b^2 + b - b^2 = b.$$

3.2-14. In general $m_n = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = \frac{1}{b}\int_a^{\infty} x^n e^{-(x-a)/b}\,dx.$

(a) $m_1 = \frac{1}{b}\int_a^{\infty} x e^{-(x-a)/b}\,dx = \frac{1}{b}\int_a^{\infty}(x-a)e^{-(x-a)/b}\,dx$

$+\frac{a}{b}\int_a^{\infty} e^{-(x-a)/b}\,dx.$ Let $\xi = (x-a)/b$, $d\xi = dx/b$.

$$m_1 = b\underbrace{\int_0^{\infty}\xi e^{-\xi}\,d\xi}_{=1 \text{ from } (C-46)} + a\underbrace{\int_0^{\infty} e^{-\xi}\,d\xi}_{=1 \text{ from } (C-45)} = a+b.$$

$$m_2 = \frac{1}{b}\int_a^{\infty} x^2 e^{-(x-a)/b}\,dx = \int_0^{\infty}(b\xi+a)^2 e^{-\xi}\,d\xi$$

$$= \int_0^{\infty}(b^2\xi^2 + 2ab\xi + a^2)e^{-\xi}\,d\xi.$$

Use (C-45), (C-46), and (C-47) to get

$$m_2 = 2b^2 + 2ab + a^2.$$

$$m_3 = \frac{1}{b}\int_a^{\infty} x^3 e^{-(x-a)/b}\,dx = \int_0^{\infty}(b\xi+a)^3 e^{-\xi}\,d\xi$$

$= \int_0^{\infty}(b^3\xi^3 + 3ab^2\xi^2 + 3a^2b\xi + a^3)e^{-\xi}\,d\xi.$ Use (C-45) through (C-48) to get $m_3 = 6b^3 + 6ab^2 + 3a^2b + a^3.$

(b) $$\frac{d\Phi}{d\omega} = e^{j\omega a}\left[\frac{ja}{(1-j\omega b)} + \frac{jb}{(1-j\omega b)^2}\right]$$

$$\frac{d^2\Phi}{d\omega^2} = e^{j\omega a}\left[\frac{-a^2}{(1-j\omega b)} + \frac{-2ab}{(1-j\omega b)^2} + \frac{-2b^2}{(1-j\omega b)^3}\right]$$

(3.2-14.) (Continued)

$$\frac{d^3\Phi}{d\omega^3} = e^{j\omega a}\left[\frac{-j\,a^3}{(1-j\omega b)} + \frac{-j\,3a^2b}{(1-j\omega b)^2} + \frac{-j\,6ab^2}{(1-j\omega b)^3} + \frac{-j\,6b^3}{(1-j\omega b)^4}\right]$$

$$m_1 = (-j)\left.\frac{d\Phi}{d\omega}\right|_{\omega=0} = (a+b)$$

$$m_2 = (-j)^2\left.\frac{d^2\Phi}{d\omega^2}\right|_{\omega=0} = 2b^2 + 2ab + a^2$$

$$m_3 = (-j)^3\left.\frac{d^3\Phi}{d\omega^3}\right|_{\omega=0} = 6b^3 + 6ab^2 + 3a^2b + a^3.$$

(3.2-15.) $m_n = \int_a^b \frac{x^n\,dx}{b-a} = \left.\frac{x^{n+1}}{(n+1)(b-a)}\right|_a^b = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$.

$\mu_n = \int_a^b \frac{(x-\bar{x})^n dx}{(b-a)} = \frac{(b-\bar{x})^{n+1} - (a-\bar{x})^{n+1}}{(n+1)(b-a)}$. But $\bar{x} = m_1$ $= (a+b)/2$ so

$$\mu_n = \frac{\left(\frac{b-a}{2}\right)^{n+1} - \left(\frac{a-b}{2}\right)^{n+1}}{(n+1)(b-a)} = \frac{(b-a)^{n+1} - (a-b)^{n+1}}{(n+1)\,2^{n+1}\,(b-a)}.$$

(3.2-16.) $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = \int_{-\infty}^{x_0} 0\cdot f_X(x)\,dx$ $+ \int_{x_0}^{\infty} 1\cdot f_X(x)\,dx = 1 - \int_{-\infty}^{x_0} f_X(x)\,dx = 1 - F_X(x_0)$.

(3.2-17.) Define $V = E[(X-a)^2] = E[X^2 - 2aX + a^2]$ $= E[X^2] - 2aE[X] + a^2$. $\frac{dV}{da} = -2\bar{X} + 2a = 0$ when $a = \bar{X}$. To show that V is truly minimum:

$\frac{d^2V}{da^2} = 2 > 0$ so a minimum occurs at $a = \bar{X}$.

(3.2-18.) Since $f_X(x) = \sum_{i=1}^{N} P(x_i)\,\delta(x-x_i)$ from (3.1-3), we use (3.2-2) to get

$$m_n = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = \int_{-\infty}^{\infty} x^n \sum_{i=1}^{N} P(x_i)\,\delta(x-x_i)\,dx$$

$$= \sum_{i=1}^{N} P(x_i)\,x_i^n .$$

From (3.2-4):

$$\mu_n = \int_{-\infty}^{\infty} (x-\bar{X})^n f_X(x)\,dx = \int_{-\infty}^{\infty} (x-\bar{X})^n \sum_{i=1}^{N} P(x_i)\,\delta(x-x_i)\,dx$$

$$= \sum_{i=1}^{N} P(x_i)\,(x_i-\bar{X})^n .$$

(3.2-19.) From the binomial expansion

$$\mu_n = E[(X-\bar{X})^n] = E\left[\sum_{k=0}^{n} \binom{n}{k} X^k (-\bar{X})^{n-k}\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-\bar{X})^{n-k} E[X^k] = \sum_{k=0}^{n} \binom{n}{k} (-\bar{X})^{n-k} m_k .$$

(3.2-20.) $m_n' = \int_{-\infty}^{\infty} x^n f_X(x-\alpha)\,dx$. Let $\xi = x-\alpha$, $d\xi = dx$.

$m_n' = \int_{-\infty}^{\infty} (\xi+\alpha)^n f_X(\xi)\,d\xi$. Expand $(x+\alpha)^n$ by the binomial theorem.

$$m_n' = \int_{-\infty}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \xi^k \alpha^{n-k} f_X(\xi)\,d\xi$$

$$= \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \int_{-\infty}^{\infty} \xi^k f_X(\xi)\,d\xi = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} m_k .$$

(3.2-21.) $\overline{Y^n} = E[Y^n] = E[e^{-nX/5}] = \int_{-5}^{15} \frac{e^{-nx/5}}{20}\,dx$
$= \frac{e^{n} - e^{-3n}}{4n}$.

(3.2-22.) Bolts for sale are uniformly distributed on (650, 920) in millimeters of length. (a) From Problem 3-6 $\bar{X} = (a+b)/2 = (650+920)/2 = 785$ mm. (b) Also from Problem 3-6 $\sigma_X^2 = (b-a)^2/12 = (270/\sqrt{12})^2$ so $\sigma_X = 270/\sqrt{12}$ $= 135/\sqrt{3} = 77.942$. (c) $P\{\bar{X} - \sigma_X < X \le \bar{X} + \sigma_X\} =$
$\frac{(\bar{X}+\sigma_X) - (\bar{X}-\sigma_X)}{b-a} = \frac{2(135)/\sqrt{3}}{270} = 1/\sqrt{3} = 0.5774.$
(d) Tolerance $= [(785-760)/760]100 = 3.8295\,\%$.

(3.2-23.) (a) $\bar{X} = \int_{-4}^{4} \frac{\pi}{16} x \cos\left(\frac{\pi x}{8}\right) dx$ ← Let $\alpha = \pi x/8$, $d\alpha = \pi\, dx/8$
$= \frac{4}{\pi}\int_{-\pi/2}^{\pi/2} \alpha \cos(\alpha)\, d\alpha = 0$ from (C-40). (b) $\overline{X^2} =$
$\int_{-4}^{4} \frac{\pi}{16} x^2 \cos\left(\frac{\pi x}{8}\right) dx = \frac{32}{\pi^2}\int_{-\pi/2}^{\pi/2} \alpha^2 \cos(\alpha)\, d\alpha = 16\left(1 - \frac{8}{\pi^2}\right).$
(c) $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = \overline{X^2} = 16\left[1 - (8/\pi^2)\right]$.

(3.2-24.) With the dc signal present it acts as a shift in the density of the random noise. The shift becomes the mean of a gaussian random variable. Thus, $P\{0 < X\} = 0.2514 = 1 - F_X(0) = 1 - F\left(\frac{0-\bar{X}}{\sigma_X}\right)$ and $F(-\bar{X}/\sigma_X) = 1 - 0.2514 = 0.7486$. From Table B-1 this occurs where $-\bar{X}/\sigma_X = 0.67$ so $\bar{X} = -0.67\sigma_X$ $= -0.67(10^{-3}) = -670\,\mu V$.

3.2-25. From Problem 3-12 $\bar{X} = \sqrt{\pi b/4}$ and $\sigma_X^2 = b(1-\frac{\pi}{4})$. From (3.2-6) $\overline{X^2} = \sigma_X^2 + \bar{X}^2 = b(1-\frac{\pi}{4}) - b\frac{\pi}{4} = b$. Next,

$\overline{X^3} = \int_0^\infty \frac{2x^4}{b} e^{-x^2/b} dx \leftarrow$ let $u = x^2/b$, $du = 2x\,dx/b$:

$\overline{X^3} = b^{3/2}\int_0^\infty u^{2-\frac{1}{2}} e^{-u} du = 3\sqrt{\pi}\, b^{3/2}/4$ from Gradshteyn and Ryzhik, p. 317. Finally, $\mu_3 = \overline{(X-\bar{X})^3}$

$= \overline{X^3} - 3\bar{X}\,\overline{X^2} + 2\bar{X}^3 = b^{3/2}\frac{3\sqrt{\pi}}{4} - 3b^{1/2}\frac{\sqrt{\pi}}{2} b + \frac{2\pi\sqrt{\pi}\, b^{3/2}}{8}$

$= \frac{\sqrt{\pi}}{4}(\pi-3)b^{3/2}$ and $\frac{\mu_3}{\sigma_X^3} = \frac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^{3/2}} = 0.6311$.

3.2-26. In general

$$m_n = \int_2^6 x^n \frac{3}{32}(-x^2+8x-12)\,dx$$

$$= \frac{3}{32}\left\{\frac{-6^{n+3}+2^{n+3}}{n+3} + 8\,\frac{6^{n+2}-2^{n+2}}{n+2} - 12\,\frac{6^{n+1}-2^{n+1}}{n+1}\right\}.$$

(a) $m_0 = 1$, (b) $m_1 = 4$, (c) $m_2 = 28(3)/5 = 16.8$.

(d) $\mu_2 = m_2 - m_1^2 = 16.8 - 16 = 0.8$.

3.2-27. (a) $\bar{X} = \int_0^\infty x \frac{x^{(N/2)-1}}{2^{N/2}\Gamma(N/2)} e^{-x/2} dx \leftarrow$ Let $\xi = x/2$, $d\xi = dx/2$

$$\bar{X} = \frac{2}{\Gamma(N/2)}\int_0^\infty \xi^{(\frac{N}{2}+1)-1} e^{-\xi} d\xi = \frac{2\Gamma(\frac{N}{2}+1)}{\Gamma(N/2)} = \frac{2\frac{N}{2}\Gamma(\frac{N}{2})}{\Gamma(N/2)} = N.$$

(b) $\overline{X^2} = \int_0^\infty x^2 \frac{x^{\frac{N}{2}-1} e^{-x/2}}{2^{N/2}\Gamma(N/2)} dx$

$= \frac{4}{\Gamma(N/2)}\int_0^\infty \xi^{(\frac{N}{2}+2)-1} e^{-\xi} d\xi = \frac{4\Gamma(\frac{N}{2}+2)}{\Gamma(N/2)} = \frac{4(\frac{N}{2}+1)\frac{N}{2}\Gamma(\frac{N}{2})}{\Gamma(N/2)}$

$= N(2+N)$. (c) $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 2N + N^2 - N^2 = 2N$.

3.2-28. $\overline{X^n} = \int_0^\infty x^n \frac{x^{\frac{N}{2}-1} e^{-x/2}}{2^{N/2}\Gamma(N/2)} dx \leftarrow$ Let $\xi = x/2$, $d\xi = dx/2$

$$\overline{X^n} = \frac{2^n}{\Gamma(N/2)}\int_0^\infty \xi^{(\frac{N}{2}+n)-1} e^{-\xi} d\xi = \frac{2^n\Gamma(\frac{N}{2}+n)}{\Gamma(N/2)}.$$

3.2-29. (a) $\overline{X} = \int_0^\infty ab\, x^b e^{-ax^b} dx \leftarrow$ Let $\xi = ax^b$, $d\xi = abx^{b-1}dx$

$\overline{X} = \frac{1}{a^{1/b}} \int_0^\infty \xi^{(\frac{1}{b}+1)-1} e^{-\xi} d\xi = \frac{\Gamma(\frac{1}{b}+1)}{a^{1/b}}$.

(b) $\overline{X^2} = \int_0^\infty ab\, x^{b+1} e^{-ax^b} dx = \frac{1}{a^{2/b}} \int_0^\infty \xi^{(\frac{2}{b}+1)-1} e^{-\xi} d\xi$

$= \frac{\Gamma(\frac{2}{b}+1)}{a^{2/b}}$. (c) $\sigma_X^2 = \overline{X^2} - \overline{X}^2 = a^{-2/b}\left\{\Gamma(\frac{2}{b}+1) - [\Gamma(\frac{1}{b}+1)]^2\right\}$.

★**3.2-30.** $\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^\infty e^{j\omega x} \sum_{k=0}^N \binom{N}{k} p^k (1-p)^{N-k}$

$\cdot \delta(x-k)\, dx = \sum_{k=0}^N \binom{N}{k} p^k (1-p)^{N-k} \int_{-\infty}^\infty \delta(x-k) e^{j\omega x} dx$

$= \sum_{k=0}^N \binom{N}{k} (pe^{j\omega})^k (1-p)^{N-k}$. But $(a+b)^N = \sum_{k=0}^N \binom{N}{k}$

$\cdot a^k b^{N-k}$ so $\Phi_X(\omega) = (1-p+pe^{j\omega})^N$.

★**3.2-31.** $\Phi_X(\omega) = \int_{-\infty}^\infty f_X(x) e^{j\omega x} dx = \int_{-\infty}^\infty e^{-b} \sum_{k=0}^\infty \frac{b^k}{k!} \delta(x-k) e^{j\omega x} dx$

$= e^{-b} \sum_{k=0}^\infty \frac{b^k}{k!} \int_{-\infty}^\infty \delta(x-k) e^{j\omega x} dx = e^{-b} \sum_{k=0}^\infty \frac{(be^{j\omega})^k}{k!}$

$= e^{-b+b\exp(j\omega)} = e^{-b(1-e^{j\omega})}$.

★**3.2-32.** Use (3.3-4). $\overline{X} = m_1 = -j \frac{d\Phi_X(\omega)}{d\omega}\Big|_{\omega=0}$

$= -j a^N \left[\frac{-N(-j)}{(a-j\omega)^{N+1}}\right]\Big|_{\omega=0} = \frac{N}{a}$. $\overline{X^2} = m_2 = -\frac{d^2\Phi(\omega)}{d\omega^2}\Big|_{\omega=0}$

$= -jNa^N \left[\frac{-(N+1)(-j)}{(a-j\omega)^{N+2}}\right]\Big|_{\omega=0} = \frac{N(N+1)}{a^2}$.

$\sigma_X^2 = \overline{X^2} - \overline{X}^2 = m_2 - m_1^2 = \frac{N(N+1)}{a^2} - \frac{N^2}{a^2} = N/a^2$.

3.2-33. (a) $E[Y] = E[2X-3] = 2\overline{X} - 3 = 2(-3) - 3 = -9$.

(b) $E[Y^2] = E[(2X-3)^2] = 4\overline{X^2} - 12\overline{X} - 9 = 4(11) - 12(-3) - 9$

$= 89$. (c) $\sigma_Y^2 = \overline{Y^2} - \overline{Y}^2 = 89 - 81 = 8$.

3.2-34. (a) $E[X] = \int_0^1 x \frac{5}{4}(1-x^4)\,dx = \frac{5}{4}\left(\frac{x^2}{2} - \frac{x^6}{6}\right)\Big|_0^1 = 5/12.$

(b) $E[4X+2] = 4\bar{X} + 2 = 4(5/12) + 2 = 11/3.$

(c) $E[X^2] = \int_0^1 x^2 \frac{5}{4}(1-x^4)\,dx = \frac{5}{4}\left(\frac{x^3}{3} - \frac{x^7}{7}\right)\Big|_0^1 = 5/21.$

3.2-35. $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = \int_0^{\infty} \frac{a^b x^{n+b-1}}{\Gamma(b)} e^{-ax}\,dx$ ← $\xi = ax$, $d\xi = a\,dx$

$$= \int_0^{\infty} \frac{a^b \xi^{n+b-1}}{\Gamma(b)\, a^{n+b-1}} e^{-\xi} \frac{d\xi}{a} = \frac{1}{a^n \Gamma(b)} \int_0^{\infty} \xi^{n+b-1} e^{-\xi}\,d\xi = \frac{\Gamma(b+n)}{a^n \Gamma(b)}.$$

$E[X^2] = \frac{\Gamma(b+2)}{a^2 \Gamma(b)} = \frac{(b+1)b}{a^2} = \frac{b^2+b}{a^2}$. Thus, we have

$E[X] = \frac{\Gamma(b+1)}{a\,\Gamma(b)} = \frac{b}{a}$ and $\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{b^2+b}{a^2} - \frac{b^2}{a^2} = \frac{b}{a^2}.$

3.2-36. Use (C-29) and (C-30): $E[X] = \int_{-\infty}^{\infty} \frac{x\,(16/\pi)}{(2^2+x^2)^2}\,dx = 0,$

$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2 (16/\pi)\,dx}{(2^2+x^2)^2} = 1/4, \quad \sigma_X^2 = \overline{X^2} - \bar{X}^2 = \overline{X^2} = 1/4.$

★ 3.3-1. $\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x}\,dx$. But since

$\left|\int g(x)\,dx\right| \le \int |g(x)|\,dx$ for any function $g(x)$, then

$$|\Phi_X(\omega)| = \left|\int_{-\infty}^{\infty} f_X(x) e^{j\omega x}\,dx\right| \le \int_{-\infty}^{\infty} |f_X(x) e^{j\omega x}|\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\,dx = \Phi_X(0) = 1.$$

★ 3.3-2. We use the series $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ two times below. From the definition of $\Phi_X(\omega)$:

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x}\,dx = \int_{-\infty}^{\infty} f_X(x) \sum_{n=0}^{\infty} \frac{(j\omega x)^n}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} m_n = \sum_{n=0}^{\infty} \frac{(j)^n m_n}{n!} \omega^n. \qquad (1)$$

★3.3-2. (Continued)

Also

$$\Phi_X(\omega) = e^{-\sigma_X^2\omega^2/2} = \sum_{k=0}^{\infty}\left(\frac{-\sigma_X^2\omega^2}{2}\right)^k \frac{1}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k\sigma_X^{2k}}{2^k\,k!}\,\omega^{2k}. \quad (2)$$

Next, equate (1) and (2) term-by-term:

$m_n = 0$, n odd. For n even, $k = n/2$, so

$$\frac{(j)^n m_n}{n!} = \frac{\sigma_X^{\,n}(-1)^{n/2}}{2^{n/2}\,(n/2)!}, \quad n \text{ even}$$

or

$$m_n = \frac{n!\,\sigma_X^{\,n}}{2^{n/2}\,(n/2)!} = \frac{n(n-1)\cdots[n-(n/2)+1]}{2^{n/2}}\,\sigma_X^{\,n},$$

n even.

★3.3-3. $M_X(\nu) = \int_{-\infty}^{\infty} f_X(x)\, e^{\nu x}\,dx = \int_{-\infty}^{\infty} f_X(x)\sum_{n=0}^{\infty}\frac{(\nu x)^n}{n!}\,dx$

$$= \sum_{n=0}^{\infty}\frac{\nu^n}{n!}\,m_n. \quad (1)$$

$$M_X(\nu) = e^{\sigma_X^2\nu^2/2} = \sum_{k=0}^{\infty}\left(\frac{\sigma_X^2\nu^2}{2}\right)^k\frac{1}{k!} = \sum_{k=0}^{\infty}\frac{\sigma_X^{2k}}{2^k\,k!}\,\nu^{2k}. \quad (2)$$

By equating (1) and (2) term-by-term:

$$m_n = 0, \quad n \text{ odd}.$$

For n even:

$$\frac{m_n}{n!} = \frac{\sigma_X^{\,n}}{2^{n/2}\,(n/2)!}$$

or

$$m_n = \frac{n!\;\sigma_X^{\,n}}{2^{n/2}(n/2)!} = \frac{n(n-1)\cdots[n-(n/2)+1]}{2^{n/2}}\,\sigma_X^{\,n}, \quad n \text{ even}.$$

★3.3-4. Here $f_X(x) = \sum_{k=0}^{N} \left(\frac{1}{N+1}\right) \delta(x - k\Delta)$ so

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x} dx = \int_{-\infty}^{\infty} e^{j\omega x} \sum_{k=0}^{N} \left(\frac{1}{N+1}\right) \delta(x-k\Delta)\, dx$$

$$= \sum_{k=0}^{N} \left(\frac{1}{N+1}\right) e^{j\omega k\Delta}.$$

From Jolley (1961, pp. 88, 90) the sum is known:

$$\sum_{k=0}^{N} e^{jk\beta} = \frac{\sin\left(\frac{N+1}{2}\beta\right)}{\sin\left(\frac{\beta}{2}\right)} e^{jN\beta/2}.$$

Let $\beta = \omega\Delta$ so

$$\Phi_X(\omega) = \frac{1}{N+1} \frac{\sin[(N+1)\omega\Delta/2]}{\sin(\omega\Delta/2)} e^{jN\omega\Delta/2}.$$

★3.3-5. $m_1 = E[X] = -j \left.\frac{d\Phi_X(\omega)}{d\omega}\right|_{\omega=0} = (-j) \left.\frac{[1+(b\omega)^2] e^{j\omega m} - e^{j\omega m} 2(b\omega)b}{[1+(b\omega)^2]^2}\right|_{\omega=0}$

$= m.$ $\quad m_2 = (-j)^2 \left.\frac{d^2\Phi_X(\omega)}{d\omega^2}\right|_{\omega=0} = \left.\frac{-[1+(b\omega)^2] e^{j\omega m}(jm)^2 + e^{j\omega m} 2(b\omega) b(jm)}{[1+(b\omega)^2]^2}\right|_{\omega=0}$

$+ \left.\frac{[1+(b\omega)^2]^2 [2b^2\omega\, e^{j\omega m}(jm) + e^{j\omega m}(2b^2)] - b^2\omega\, 2\, e^{j\omega m} 2[1+(b\omega)^2] 2(b\omega)b}{[1+(b\omega)^2]^4}\right|_{\omega=0}$

$= 2b^2 + m^2.$ $\quad \sigma_X^2 = m_2 - m_1^2 = 2b^2 + m^2 - m^2 = 2b^2.$

★3.3-6. (a) $\bar{X} = -j \left.\frac{d\Phi_X(\omega)}{d\omega}\right|_{\omega=0} = (-j)\left(-\frac{N}{2}\right) \left.\frac{-j2}{(1-j2\omega)^{\frac{N}{2}+1}}\right|_{\omega=0} = N.$

(b) $\overline{X^2} = (-j)^2 \left.\frac{d^2\Phi_X(\omega)}{d\omega^2}\right|_{\omega=0} = -\frac{d}{d\omega}\left\{\frac{jN}{(1-j2\omega)^{\frac{N}{2}+1}}\right\}\Bigg|_{\omega=0} = \left.\frac{2N\left(\frac{N}{2}+1\right)}{(1-j2\omega)^{\frac{N}{2}+2}}\right|_{\omega=0}$

$= N^2 + 2N.$

★3.3-7. $M_X(\nu) = \int_{-\infty}^{\infty} f_X(x) e^{\nu x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2/2)+\nu x} dx$

$= \sqrt{2\pi}\, \frac{e^{\nu^2/2}}{\sqrt{2\pi}}$ from (C-51). $M_X(\nu) = e^{\nu^2/2}$

$e^{-\nu a} M_X(\nu) = e^{-\nu a + (\nu^2/2)}$

$\frac{d}{d\nu}\left[e^{-\nu a} M_X(\nu)\right] = e^{-\nu a + (\nu^2/2)}\left[-a+\nu\right] = 0$ when $\nu = a$.

Thus, $P\{X \geq a\} \leq e^{-a^2} e^{a^2/2} = e^{-a^2/2}$ (Chernoff's bound)

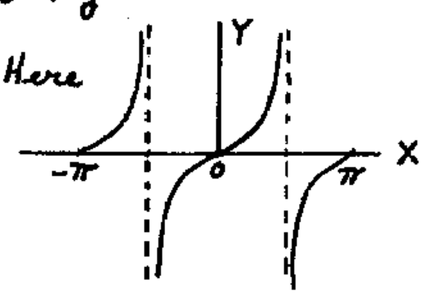
3.4-1. $T(X)$ is monotonically increasing in X so (3.4-6) applies. $X = T^{-1}(Y) = \tan^{-1}(Y/a)$

$$\frac{dT^{-1}(y)}{dy} = \frac{d\tan^{-1}(y/a)}{dy} = \frac{1/a}{1+(y/a)^2}.$$

Since $f_X(x) = 1/\pi$, $-\pi/2 < x < \pi/2$ and is zero of all other x, we have $f_X[T^{-1}(y)] = 1/\pi$ and

$$f_Y(y) = \frac{1}{\pi}\,\frac{1/a}{1+(y/a)^2} = \frac{a/\pi}{a^2+y^2}, \quad -\infty < y < \infty.$$

3.4-2. A sketch is helpful. Here $Y = T(X)$ is not monotonic. We use (3.4-11).



When $y < 0$:

$$-\pi/2 < x < \tan^{-1}(y/a)$$

and $\pi/2 < x < \pi + \tan^{-1}(y/a)$ so

$$F_Y(y) = \int_{-\pi/2}^{\tan^{-1}(y/a)} \frac{dx}{2\pi} + \int_{\pi/2}^{\pi+\tan^{-1}(y/a)} \frac{dx}{2\pi}$$

$$= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(y/a), \quad y < 0. \tag{1}$$

3.4-2. (Continued) When $y \geq 0$:

$$-\pi < x < -\pi + \tan^{-1}(y/a)$$

and $-\pi/2 < x < \tan^{-1}(y/a)$

and $\pi/2 < x < \pi$ so

$$F_Y(y) = \int_{-\pi}^{-\pi+\tan^{-1}(y/a)} \frac{dx}{2\pi} + \int_{-\pi/2}^{\tan^{-1}(y/a)} \frac{dx}{2\pi} + \int_{\pi/2}^{\pi} \frac{dx}{2\pi}$$

$$= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(y/a), \quad y \geq 0. \qquad (2)$$

Combining (1) and (2)

$$F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{y}{a}\right), \quad -\infty < y < \infty.$$

By differentiation

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{a/\pi}{a^2 + y^2}, \quad -\infty < y < \infty.$$

3.4-3. $Y = a/X$, $X = a/Y$. We use (3.4-11).

For $a > 0$: $y < 0$ corresponds to $a/y < x < 0$.

$$F_Y(y) = \int_{a/y}^{0} f_X(x)\,dx = F_X(0) - F_X(a/y), \quad y < 0 \qquad (1)$$

$y \geq 0$ corresponds to $-\infty < x < 0$ and $a/y < x < \infty$.

$$F_Y(y) = \int_{-\infty}^{0} f_X(x)\,dx + \int_{a/y}^{\infty} f_X(x)\,dx = F_X(0) - F_X(-\infty) + F_X(\infty) - F_X(a/y)$$

$$= 1 + F_X(0) - F_X(a/y), \quad y \geq 0. \qquad (2)$$

By differentiation of (1) and (2):

$$f_Y(y) = \frac{a}{y^2} f_X(a/y), \quad -\infty < y < \infty \text{ and } a > 0. \qquad (3)$$

3.4-3. (Continued) For $a < 0$: $y < 0$ corresponds to $0 < x < a/y$.

$$F_Y(y) = \int_0^{a/y} f_X(x)\,dx = F_X(a/y) - F_X(0), \quad y < 0. \tag{4}$$

$y \geq 0$ corresponds to $0 < x < \infty$ and $-\infty < x < a/y$.

$$F_Y(y) = \int_0^{\infty} f_X(x)\,dx + \int_{-\infty}^{a/y} f_X(x)\,dx = F_X(\infty) - F_X(0) + F_X(a/y) - F_X(-\infty)$$

$$= 1 - F_X(0) + F_X(a/y), \quad y \geq 0. \tag{5}$$

By differentiation of (4) and (5):

$$f_Y(y) = \frac{-a}{y^2} f_X(a/y), \; -\infty < y < \infty \text{ and } a < 0. \tag{6}$$

By combining (3) and (6):

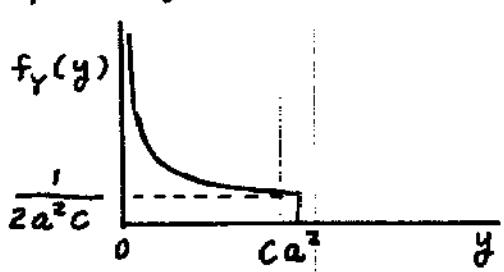
$$f_Y(y) = \frac{|a|}{y^2} f_X(a/y), \; -\infty < y < \infty.$$

3.4-4. Here $f_X(x) = 1/2a$, $-a < x < a$ and is zero for other x. From Example 3.4-2:

$$f_Y(y) = \frac{f_X(\sqrt{y/c}) + f_X(-\sqrt{y/c})}{2\sqrt{cy}}, \quad y \geq 0.$$

Therefore, since $0 < y < ca^2$ when $-a < x < a$, we have

$$f_Y(y) = 1/2a\sqrt{cy}, \quad 0 < y < ca^2.$$



(3.4-5.) Use (3.4-11). There are no values of x that correspond to $\{Y<0\}$ so $F_Y(y)=0$, $y<0$. The event $\{Y=0\}$ corresponds to $\{X \le 0\}$ so $F_Y(0)=0.5$ since $f_X(x)$ has half its area to the left of the origin for a zero-mean Gaussian random variable. For $y \ge 0$ we have $-\infty < x < y/c$ and

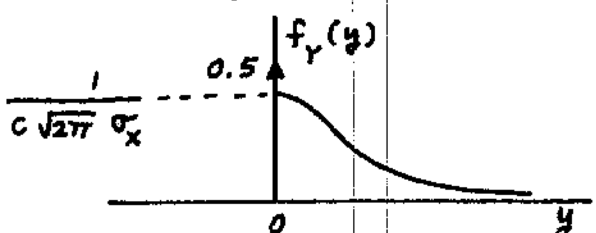
$$F_Y(y) = \int_{-\infty}^{y/c} f_X(x)\,dx = F_X(y/c), \quad y \ge 0.$$

after differentiation:

$$f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{c}u(x) f_X(y/c)$$

where

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-x^2/2\sigma_X^2}.$$

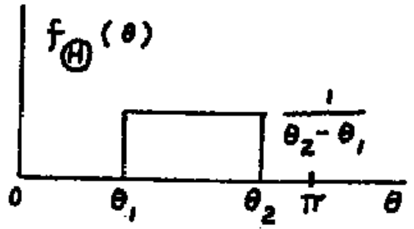


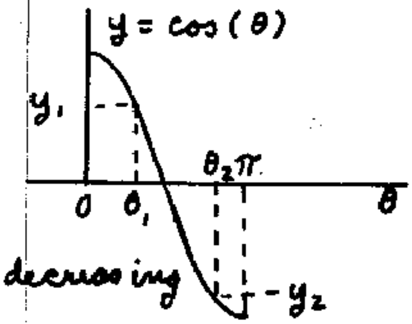
(3.4-6.) Here $f_X(x)$ is given by (2.5-6). Since $a \ge 0$ the transformation is monotonically increasing and (3.4-6) applies: $Y = cX$, $X>0$, $X = T^{-1}(Y) = Y/c$, $Y>0$, $dT^{-1}(y)/dy = 1/c$.

$$f_Y(y) = \frac{2}{cb}\left(\frac{y}{c}-a\right) e^{-\left(\frac{y}{c}-a\right)^2/b}, \quad y \ge ca$$
$$= 0, \qquad y < ca.$$

This function is a Rayleigh density so the effect of transformation is to generate another Rayleigh r.v.

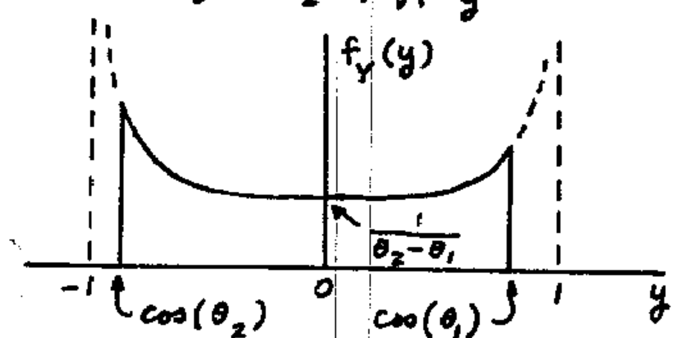
★ 3.4-7. Sketches are helpful.





This is a monotonically decreasing transformation and (3.4-10) applies: $\theta = T^{-1}(y) = \cos^{-1}(y)$, $d\theta/dy = d\cos^{-1}(y)/dy = -1/\sqrt{1-y^2}$. When $\theta_1 < \theta < \theta_2$ we have $y_2 < y < y_1$ where $y_1 = \cos(\theta_1)$ and $y_2 = \cos(\theta_2)$. Thus,

$$f_Y(y) = f_{\Theta}(y)\left|\frac{d\theta}{dy}\right| = \frac{1}{\theta_2-\theta_1}\,\frac{1}{\sqrt{1-y^2}}\ ,\quad \cos(\theta_2)<y<\cos(\theta_1).$$



3.4-8. Y is a discrete random variable. Its values are $y_1 = 3(-4)^3 = -192$, $y_2 = 3(-1)^3 = -3$, $y_3 = 3(2)^3 = 24$, $y_4 = 3(3)^3 = 81$, and $y_5 = 3(4)^3 = 192$. All values occur with probability 1/5 because mapping from X to Y is one-to-one.

(a) Use (2.3-5):

$$f_Y(y) = \tfrac{1}{5}\delta(y+192) + \tfrac{1}{5}\delta(y+3) + \tfrac{1}{5}\delta(y-24) + \tfrac{1}{5}\delta(y-81) + \tfrac{1}{5}\delta(y-192).$$

3.4-8. (Continued)

(b) Use (3.1-4):

$$E[Y] = \frac{1}{5}[-192-3+24+81+192] = 20.4.$$

(c) Use (3.1-7) with $g(X) = X^2$ to first find $E[Y^2]$. $E[Y^2] = \frac{1}{5}[(-192)^2+(-3)^2+(24)^2+(81)^2+(192)^2]$

$$= 16174.8.$$

Thus,

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2 = 16174.8 - (20.8)^2$$

$$= 15758.64$$

$$\sigma_Y \approx 125.53.$$

3.4-9. (a) This problem is an application of Example 3.4-2 with $c = 1/2$. Thus, $f_Y(y) = \frac{e^{-8y}+e^{-8y}}{\sqrt{\pi}\sqrt{y/2}}u(y)$ $= \sqrt{\frac{2}{\pi y}}\,2e^{-8y}u(y)$. (b) $\overline{Y^n} = E[Y^n] = 2\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{y^n e^{-8y}}{\sqrt{y}}dy$. Let $\xi = 8y$, $d\xi = 8\,dy$ so $\overline{Y^n} = \frac{1}{8^n\sqrt{\pi}}\int_0^\infty \xi^{(n+\frac{1}{2})-1}e^{-\xi}d\xi$

$$= \frac{\Gamma(n+\frac{1}{2})}{8^n\sqrt{\pi}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{16^n}.$$

3.4-10. (a) Y is discrete. $P\{Y=-4\} = P\{X<-1\} = F_X(-1)$ $= F\left(\frac{-1-0.6}{0.8}\right) = F(-2) = 1-F(2) = 1-0.9772 = 0.0228.$

$P\{Y=-2\} = P\{-1\le X<0\} = F\left(\frac{0-0.6}{0.8}\right) - F\left(\frac{-1-0.6}{0.8}\right) =$ $-F(0.75)+F(2.0) = 0.9772 - 0.7734 = 0.2038.$

$P\{Y=2\} = P\{0\le X<1\} = F\left(\frac{1-0.6}{0.8}\right) - F\left(\frac{0-0.6}{0.8}\right) = 0.7734$ $-1.0+0.6915 = 0.4649.$ $P\{Y=4\} = P\{1\le X<\infty\}$ $= 1-F\left(\frac{1-0.6}{0.8}\right) = 1-0.6915 = 0.3085.$

(3.4-10.) (Continued) Finally $f_Y(y) = 0.0228\,\delta(y+4)$
$+ 0.2038\,\delta(y+2) + 0.4649\,\delta(y-2) + 0.3085\,\delta(y-4)$.

(b) $\bar{Y} = E[Y] = -4(0.0228) - 2(0.2038) + 2(0.4649) + 4(0.3085)$
$= 1.6650$. Here we used (3.1-4). Similarly, $\overline{Y^2} = E[Y^2]$
$= (-4)^2 0.0228 + (-2)^2 0.2038 + (2)^2 0.4649 + (4)^2 0.3085 =$
7.9756, so $\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = 7.9756 - (1.6650)^2 = 5.2034$.

(3.4-11.) $Y = T(X)$ is monotonically increasing on $(-\pi/2, \pi/2)$ for X. Now $X = \sin^{-1}(Y/a)$ so $\frac{dx}{dy} = 1/\sqrt{a^2 - y^2}$. From (3.4-10): $f_Y(y) = \frac{1}{\pi\sqrt{a^2-y^2}}$ for $-a < Y < a$ and $f_Y(y) = 0$, elsewhere.

(3.4-12.) $Y = b + e^X$ is monotonically increasing so we use (3.4-7). Since $X = \ln(Y-b)$, we have $\frac{dx}{dy} = \frac{1}{Y-b}$ and

$$f_Y(y) = f_X[x = T^{-1}(y)]\frac{dx}{dy} = \frac{e^{-[\ln(y-b) - a_X^2]^2/2\sigma_X^2}}{\sqrt{2\pi\sigma_X^2}\;(y-b)}.$$

(3.4-13.) From a sketch of Y versus X we find that Y varies from -4 to 14. On using $X = 3 \pm \sqrt{(Y+4)/2}$ with (3.4-12) we have

$$f_Y(y) = \frac{d}{dy}\int_{3-\sqrt{(y+4)/2}}^{3+\sqrt{(y+4)/2}} dx/6 = \frac{d}{dy}\left\{\frac{1}{3}\sqrt{\frac{y+4}{2}}\right\} = \frac{1}{6\sqrt{2}\sqrt{y+4}}, -4 \le y \le 14$$

and $f_Y(y) = 0$, elsewhere. (b) $\bar{Y} = \frac{1}{6\sqrt{2}}\int_{-4}^{14}\frac{y\,dy}{\sqrt{y+4}} = 2$.

(c) $\overline{Y^2} = \frac{1}{6\sqrt{2}}\int_{-4}^{14}\frac{y^2\,dy}{\sqrt{y+4}} = 32.8$ so $\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = 32.8 - 4$
$= 28.8$.

(3.4-14) Since $X \geq 0$ the transformation is monotonic and (3.4-6) applies. $x = \sqrt{y/c} = (y/c)^{1/2}$, $\frac{dx}{dy} = \frac{1}{2}\left(\frac{y}{c}\right)^{-1/2}\frac{1}{c} = \frac{1}{2\sqrt{yc}}$

$$f_Y(y) = f_X(T^{-1})\frac{dT^{-1}}{dy} = \frac{u(\sqrt{y/c})}{\sigma_X^2}\sqrt{\frac{y}{c}}\, e^{-y/2c\sigma_X^2}\frac{1}{2\sqrt{yc}} = \frac{u(y)}{2c\sigma_X^2}\, e^{-y/(2c\sigma_X^2)}.$$

(3.4-15) This function is monotonic but is best handled in two regions of X. For $X \geq 0$ where $0 \leq Y < V$:

$$y = V(1 - e^{-x/a}), \quad x = a \ln\left(\frac{V}{V-y}\right) = a \ln\left[\frac{1}{1-(y/V)}\right]$$

$$dx/dy = a/(V-y)$$

$$f_Y(y) = f_X\left\{a \ln\left[\frac{1}{1-(y/V)}\right]\right\}\frac{a}{(V-y)}, \quad 0 \leq y < V.$$

For $X \leq 0$ where $-V < Y \leq 0$:

$$y = -V(1 - e^{x/a}), \quad x = a \ln\left(\frac{V+y}{V}\right), \quad \frac{dx}{dy} = \frac{a}{V+y}.$$

$$f_Y(y) = f_X\left\{a \ln\left(\frac{V+y}{V}\right)\right\}\frac{a}{(V+y)} = f_X\left\{-a \ln\left[\frac{1}{1+(y/V)}\right]\right\}\frac{a}{(V+y)}, \quad -V < y \leq 0.$$

(3.4-16)
$$f_X(x) = \begin{cases} \frac{1}{2b}\, e^{-x/b}, & x \geq 0 \\ \frac{1}{2b}\, e^{x/b}, & x \leq 0 \end{cases}$$

$$f_Y(y) = \frac{a}{(V-y)}\frac{1}{2b}\, e^{-\frac{a}{b}\ln\left(\frac{V}{V-y}\right)}[u(y) - u(y-V)]$$

$$+ \frac{a}{(V+y)}\frac{1}{2b}\, e^{-\frac{a}{b}\ln\left(\frac{V}{V+y}\right)}[u(-y) - u(-y-V)]$$

$$= \frac{(a/2b)}{V - |y|}\, e^{-\frac{a}{b}\ln\left(\frac{V}{V-|y|}\right)}[u(y+V) - u(y-V)]$$

3.4-16. (Continued)

$$= \frac{a}{2b\, V^{a/b}} (V - |y|)^{\frac{a}{b}-1} [u(y+V) - u(y-V)]$$

If $a = b$: $f_Y(y) = \frac{1}{2V}[u(y+V) - u(y-V)]$, Y is uniform on $(-V, V)$.

3.5-1. Here $F_Y(y) = 1 - \exp(-y/b) = x$ for $0 < x < 1$. On solving for y: $(1-x) = -\exp(-y/b)$ or $y = T(x) = -b \ln(1-x)$, $0 < x < 1$.

3.5-2. Here $F_Y(y) = 1 - \exp[-ay^b] = x$, $0 < x < 1$. On solving for y: $(1-x) = \exp(-ay^b)$, so $y = \left[-\frac{1}{a}\ln(1-x)\right]^{1/b}$, $0 < x < 1$.

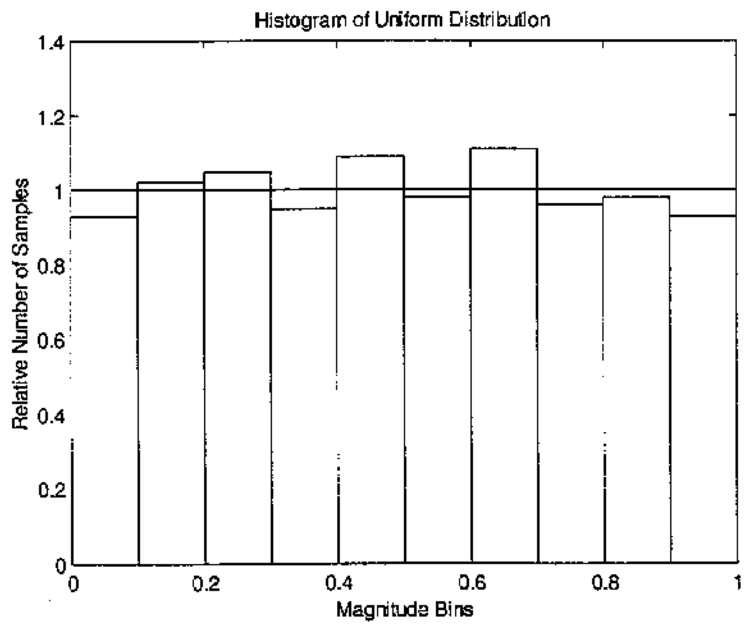
3.5-3. Here $F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(y/b) = x$, $0 < x < 1$. On solving for y: $(x - \frac{1}{2})\pi = \tan^{-1}(y/b)$, so $y = b\tan\left[\frac{\pi}{2}(2x-1)\right]$, $0 < x < 1$.

3.5-4. mean $= 4a^4\int_0^\infty \frac{\xi^2}{(a^2+\xi^2)^3}d\xi = 4a^4\left\{\frac{-\xi}{4(a^2+\xi^2)^2}\right.$
$\left. + \frac{\xi}{8a^2(a^2+\xi^2)} + \frac{1}{8a^3}\tan^{-1}(x/a)\right\}\Big|_0^\infty = \pi a/4$,

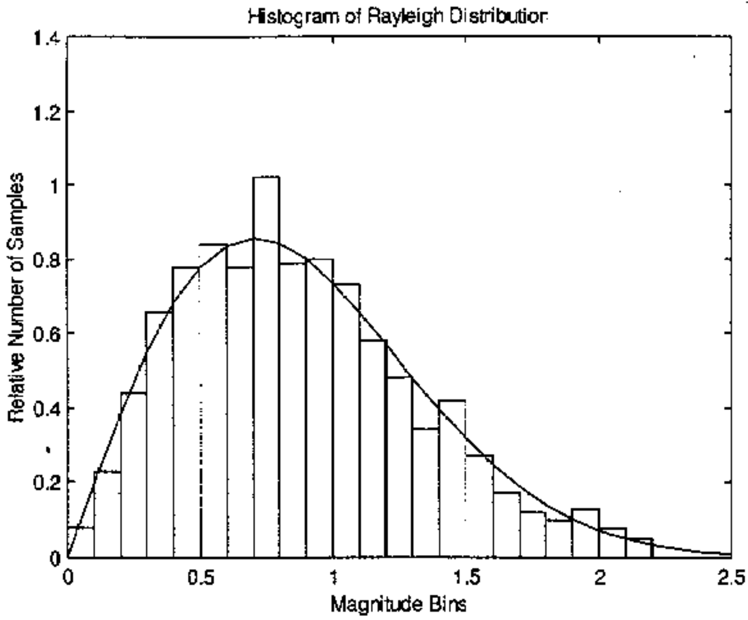
second moment $= 4a^4\int_0^\infty \frac{\xi^3}{(a^2+\xi^2)^3}d\xi = 4a^4\left\{\frac{-1}{2(a^2+\xi^2)}\right.$
$\left. + \frac{a^2}{4(a^2+\xi^2)^2}\right\}\Big|_0^\infty = a^2$, variance = second moment
$-(\text{mean})^2 = a^2 - (\pi^2 a^2/16) = a^2(16-\pi^2)/16$.

$F_Y(y) = \int_0^y f_Y(\xi)d\xi = 4a^4\int_0^y \frac{\xi}{(a^2+\xi^2)^3}d\xi = 4a^4\left\{\frac{-1}{4(a^2+\xi^2)^2}\Big|_0^y\right\}$
$= \left\{1 - \frac{1}{[1+(y/a)^2]^2}\right\}u(y)$. Since we must solve $F_Y(y) = x$ for $y \ge 0$ to get the transformation, we solve and find $y = a\left\{\sqrt{\sqrt{\frac{1}{1-x}} - 1}\right\}$ for $x > 0$.

(3.5-5.) We use the code from Example 3.5-3 except with $N = 1000$. Results are plotted in the following two figures. In the first we see that the results are closer to the ideal 1.0 over the bins from 0 to 1.0 for data on the histogram-representation of the uniform density. (Compare with Figure 3.5-2.)



For the Rayleigh density the computed histogram is shown in the following figure. On comparing with Figure 3.5-3 we see that the new histogram ($N=1000$) is more close to the ideal curve at nearly all bin values ($N=1000$). The bin from 0.7 to 0.8 remains in error by an amount roughly equal to the larger errors for $N=100$. However, 3.5-5 (Continued) significant improvement is generally observed as N is raised from 100 to 1000.



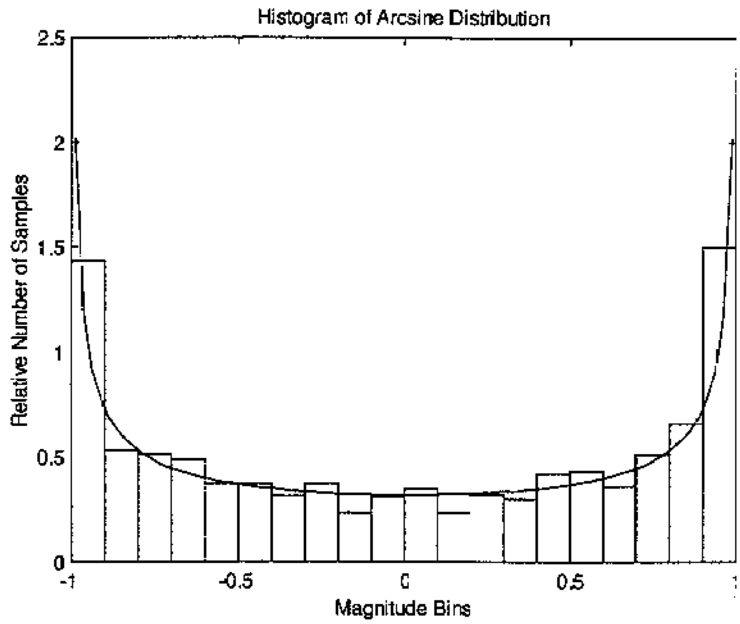
3.5-6 The MATLAB code follows:

```
%%%%%%%%%%%%%%% Problem 3.5-6 %%%%%%%%%%%%%%%%%%

clear

N = 1000; % number of random variables to generate
stp = 0.1; % step size
a = 1; % arcsin parameter

x = rand(1,N);  % uniformly distributed random numbers
y = a*sin(pi*(2*x - 1)/2);    % arcsin distributed random numbers

yabscissa = -a+stp/2:stp:a-stp/2; % abscissa
yabscissa2 = -a+stp/8:stp/4:a-stp/8;

yhist = hist(y,yabscissa); % compute histogram (not normalized)
ytrue = 1./(a*pi*sqrt(1 - (yabscissa2/a).^2)); % compute the analytic values

% plot results
clf
bar(yabscissa,yhist./(N*stp),1,'w')
hold on
plot(yabscissa2,ytrue,'k') % arcsine distribution

xlabel('Magnitude Bins')
ylabel('Relative Number of Samples')
title('Histogram of Arcsine Distribution')
```

3.5-6. (Continued) The computed histogram for the probability density function is shown below. Also shown is the exact density of the arcsine random variable, which is

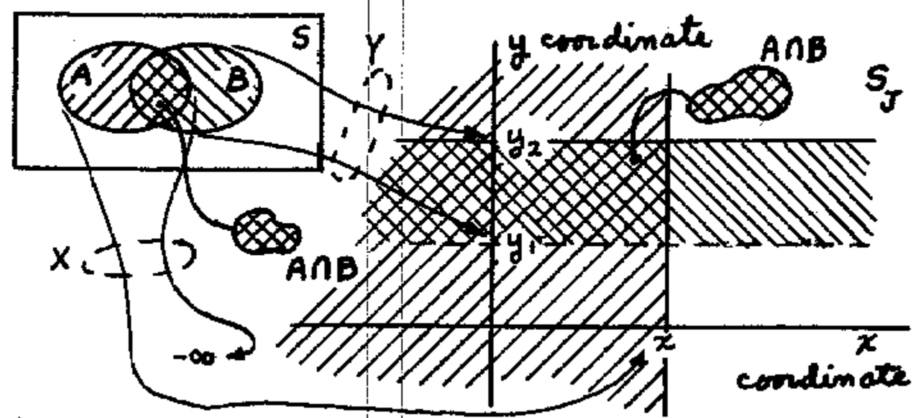
$$f_Y(y) = \frac{\text{rect}(y/2a)}{\pi\sqrt{a^2 - y^2}}$$

from (F-22).



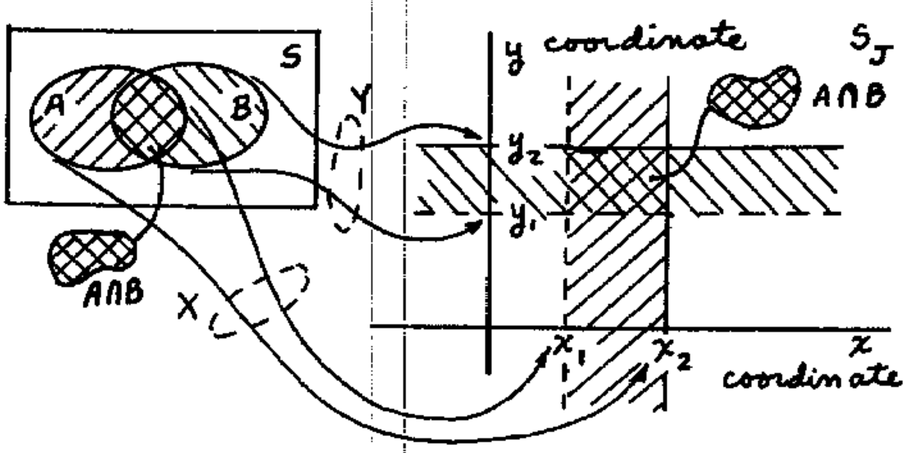
CHAPTER 4

(4.1-1.)

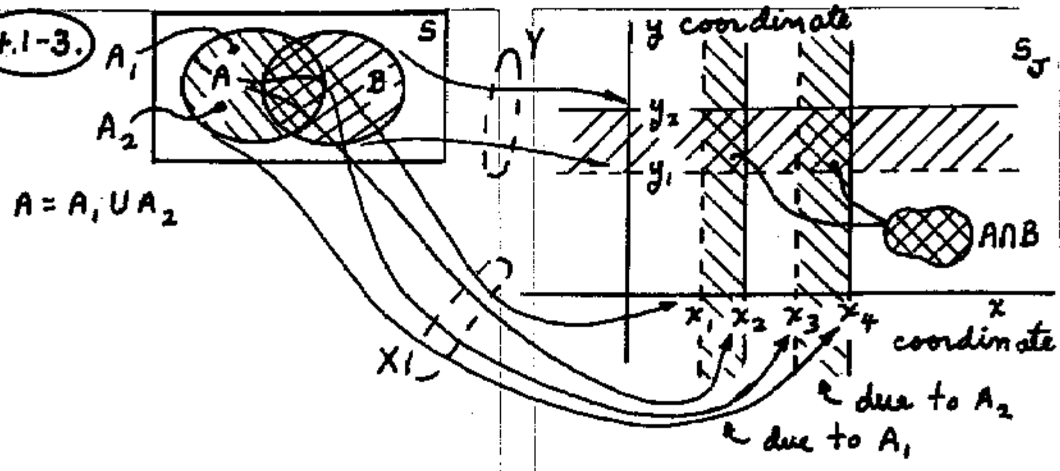


Solid lines included in cross-hatched area but dashed lines are not.

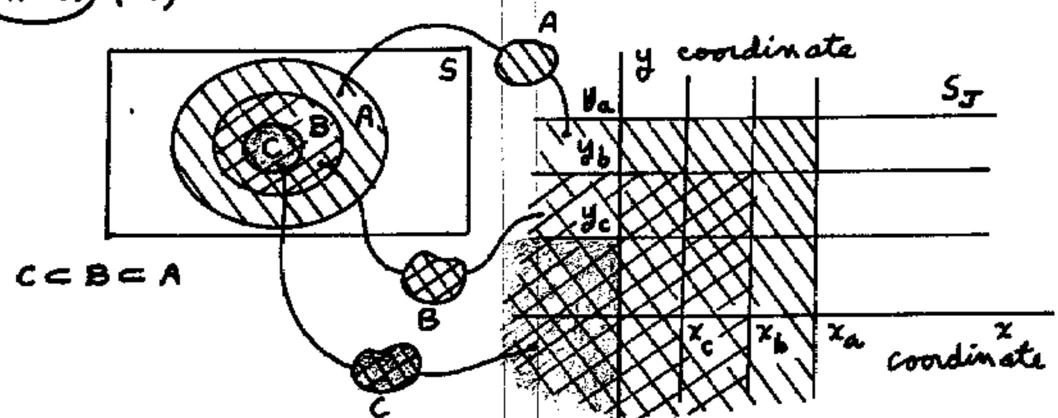
(4.1-2.)



4.1-3.



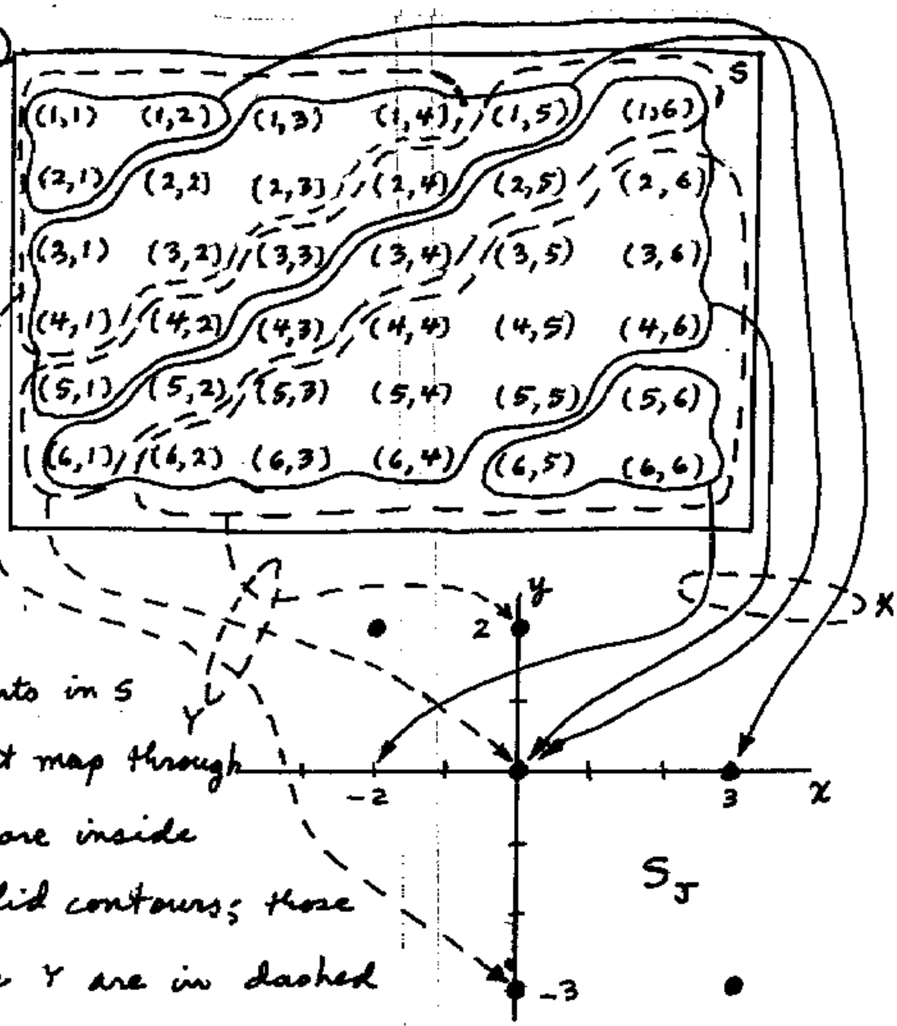
4.1-4. (a)



(b) Here $A \cap B \cap C = C$ since $C \subset B \subset A$.

4.1-5.

(a)



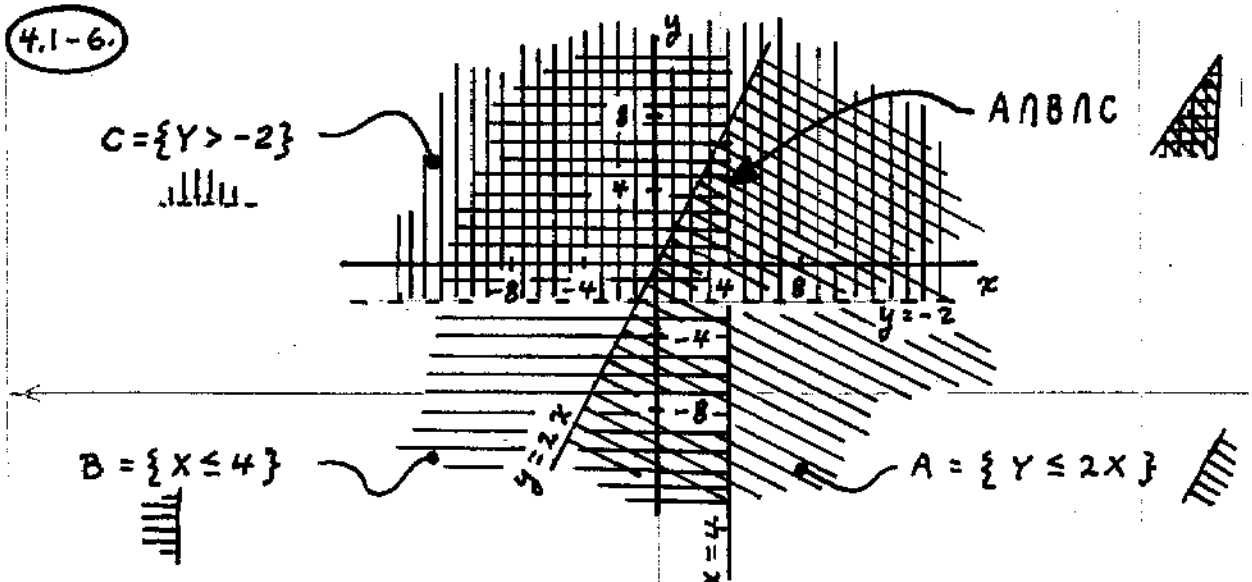
Points in S that map through X are inside solid contours; those for Y are in dashed contours. Heavy dots in S_J are possible outcomes that are intersections of contours in S.

(b) Probabilities of possible outcomes in S_J relate to points in intersections in S:

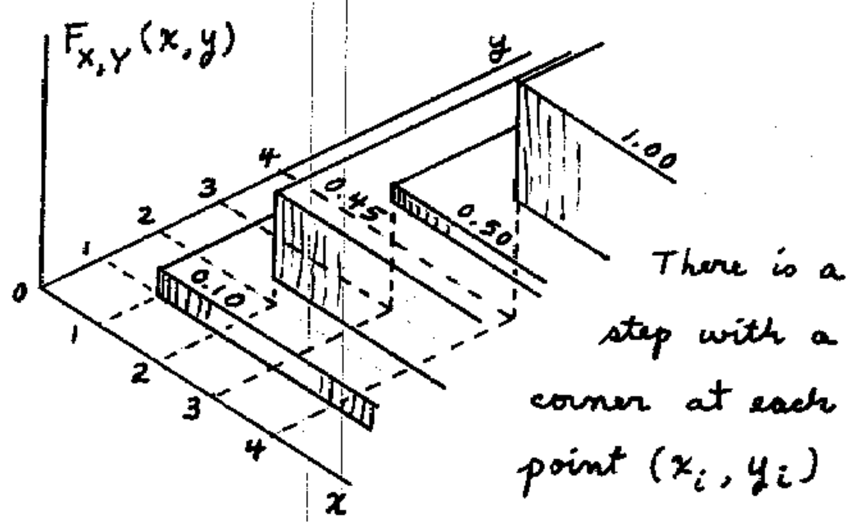
$P(-2, 2) = 3/36$, $P(0, -3) = 3/36$, $P(0, 0) = 6/36$,

$P(0, 2) = 12/36$, $P(3, 0) = 5/36$, $P(3, -3) = 7/36$.

4.1-6.



4.2-1. (a)



$$F_{X,Y}(x,y) = 0.10\, u(x-1)\, u(y-1)$$

$+ 0.35\, u(x-2)\, u(y-2) + 0.05\, u(x-3)\, u(y-3)$

$+ 0.50\, u(x-4)\, u(y-4)$.

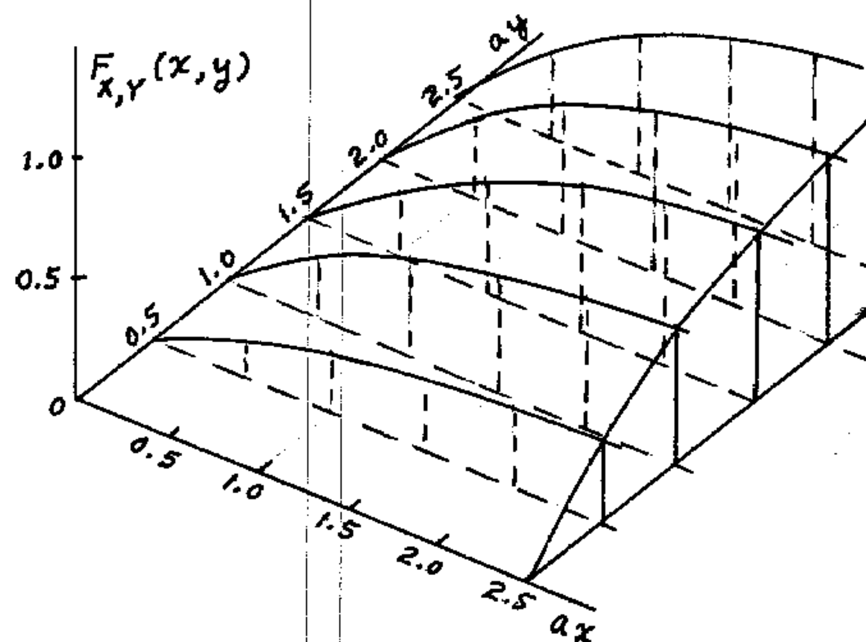
(b) $P\{X \le 2.5, Y \le 6.0\} = F_{X,Y}(2.5, 6.0) = 0.1 + 0.35 = 0.45$. (c) $P\{X \le 3.0\} = F_X(3.0) = F_{X,Y}(3.0, \infty) = 0.1 + 0.35 + 0.05 = 0.5$.

(4.2-2.) As found in Problem 4.2-1:

$$F_{X,Y}(x,y) = 0.10\,u(x-1)\,u(y-1) + 0.35\,u(x-2)\,u(y-2) + 0.05\,u(x-3)\,u(y-3) + 0.50\,u(x-4)\,u(y-4).$$

(4.2-3.) $F_{X,Y}(x,y)$ can be written as

$$F_{X,Y}(x,y) = u(x)\,u(y)\left[1-e^{-ax}\right]\left[1-e^{-ay}\right].$$



(4.2-4.) (a) $P\{X \le 1, Y \le 2\} = F_{X,Y}(1,2) = 1-e^{-a}-e^{-2a}+e^{-3a}$ $= 1-e^{-0.5}-e^{-1.0}+e^{-1.5} \approx 0.249$. (b) $P\{0.5 < X \le 1.5\}$ $= F_X(1.5) - F_X(0.5)$. By use of (4.2-6f) this reduces to $P\{0.5 < X \le 1.5\} = F_{X,Y}(1.5,\infty) - F_{X,Y}(0.5,\infty) = \left[1-e^{-1.5(0.5)}\right]-\left[1-e^{-0.5(0.5)}\right] \approx 0.306.$
(c) $P\{-1.5 < X \le 2,\ 1 < Y \le 3\} = F_{X,Y}(2,3) + F_{X,Y}(-1.5,1) - F_{X,Y}(-1.5,3) - F_{X,Y}(2,1)$ after using (4.2-6e).

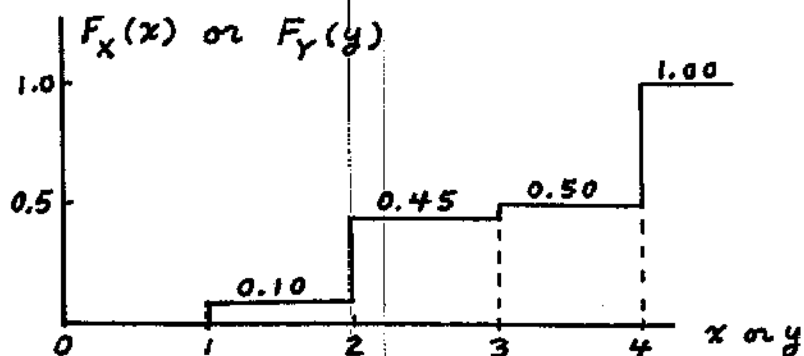
(4.2-4) (Continued)

Thus, $P\{-1.5 < X \le 2, 1 < Y \le 3\} = [1 - e^{-0.5(2)} - e^{-0.5(3)} + e^{-0.5(5)}] + 0 - 0 - [1 - e^{-0.5(2)} - e^{-0.5(1)} + e^{-0.5(3)}]$
≈ 0.242.

(4.2-5) From (4.2-4) $F_{X,Y}(x,y) = 0.10\,u(x-1)u(y-1) + 0.35\,u(x-2)u(y-2) + 0.05\,u(x-3)u(y-3) + 0.50\,u(x-4) \cdot u(y-4)$. From (4.2-6f)

$$F_X(x) = F_{X,Y}(x,\infty) = 0.10\,u(x-1) + 0.35\,u(x-2) + 0.05\,u(x-3) + 0.50\,u(x-4)$$

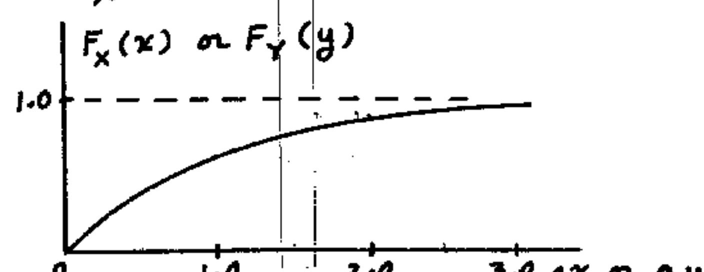
$$F_Y(y) = F_{X,Y}(\infty,y) = 0.10\,u(y-1) + 0.35\,u(y-2) + 0.05\,u(y-3) + 0.50\,u(y-4).$$



(4.2-6.) By using (4.2-6f):

$$F_X(x) = F_{X,Y}(x,\infty) = u(x)[1 - e^{-ax}]$$

$$F_Y(y) = F_{X,Y}(\infty,y) = u(y)[1 - e^{-ay}].$$



(4.2-7.) 1. Because of the unit-step function's behavior $G_{X,Y}(-\infty,-\infty)=0$, $G_{X,Y}(-\infty,y)=0$, $G_{X,Y}(x,-\infty)=0$.

2. Similarly, $G_{X,Y}(\infty,\infty)=1$

3. Since $0 \le \exp[-(x+y)] \le 1$ for all $x>0$ and $y>0$, $0 \le G_{X,Y}(x,y) \le 1$.

4. $dG_{X,Y}(x,y)/dx = e^{-x}e^{-y}$, $x>0$, $y>0$

$dG_{X,Y}(x,y)/dy = e^{-x}e^{-y}$, $x>0$, $y>0$.

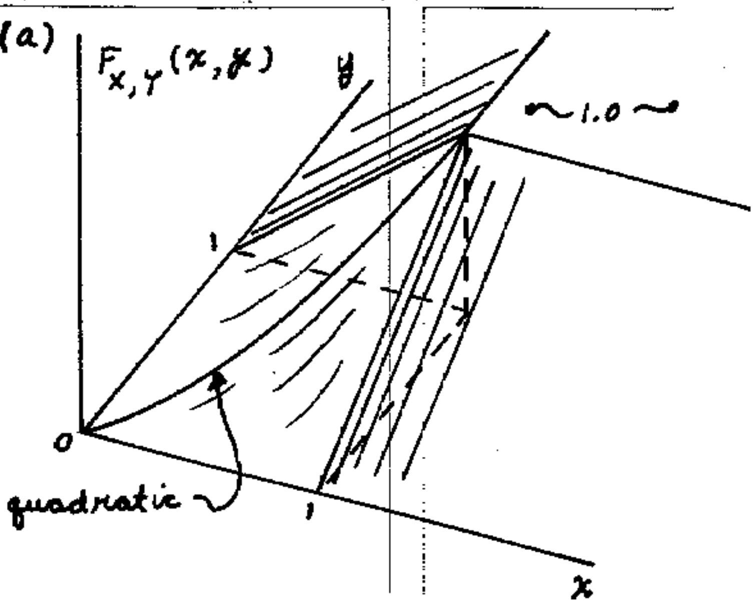
These functions are the slopes of $G_{X,Y}(x,y)$ and are nonnegative for all x and y.

5. Assume $0<x_1<x_2$ and $0<y_1<y_2$:

$P\{x_1<X\le x_2,\ y_1<Y\le y_2\} = [1-e^{-x_1-y_1}]+[1-e^{-x_2-y_2}] - [1-e^{-x_1-y_2}]-[1-e^{-x_2-y_1}] = (e^{-x_1}-e^{-x_2})(e^{-y_2}-e^{-y_1})$.

Now $(e^{-x_1}-e^{-x_2})>0$ and $(e^{-y_2}-e^{-y_1})<0$ so $P\{x_1<X\le x_2,\ y_1<Y\le y_2\}<0$ which is not allowed and property 5 is violated.

(4.2-8.) (a)



(4.2-8.) (Continued)

(b) $F_X(x) = F_{X,Y}(x, \infty) = 0, \quad x < 0$

$= x, \quad 0 \le x < 1$

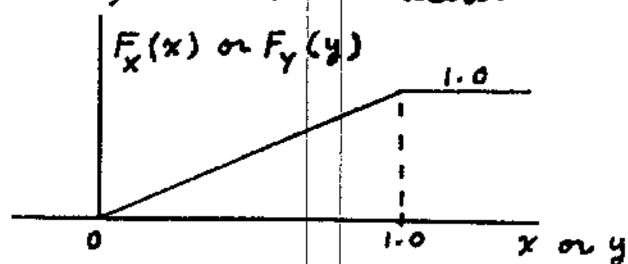
$= 1, \quad 1 \le x.$

$F_Y(y) = F_{X,Y}(\infty, y) = 0, \quad y < 0$

$= y, \quad 0 \le y < 1$

$= 1, \quad 1 \le y.$

Here (4.2-6f) has been used.

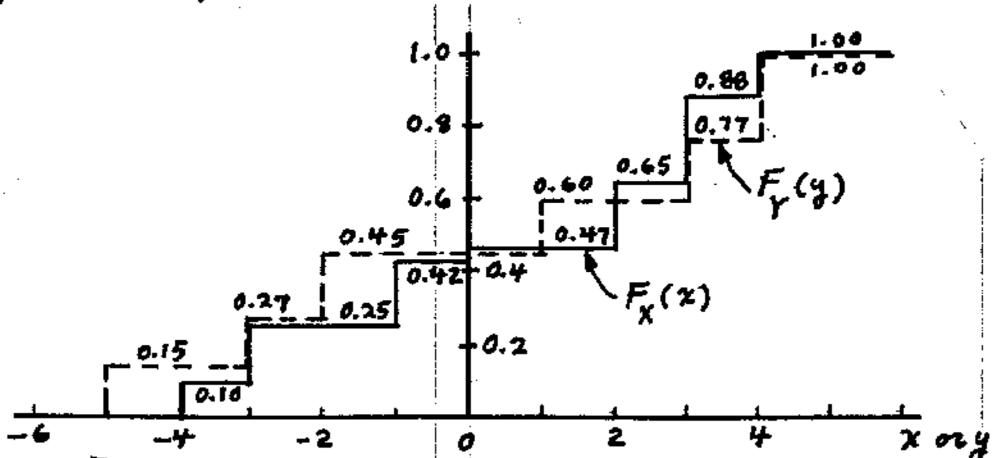


(4.2-9.) Choose any points satisfying $x_1 < y_1 \le x_2 < y_2$. Then $P\{x_1 < X \le x_2, y_1 < Y \le y_2\} = G_{X,Y}(x_1, y_1) + G_{X,Y}(x_2, y_2) - G_{X,Y}(x_1, y_2) - G_{X,Y}(x_2, y_1) = 0 + 0 - 0 - 1 = -1$ which is not allowed so $G_{X,Y}(x,y)$ is not a valid distribution.

(4.2-10.) (a) Use (4.2-6f) to get $F_X(x) = F_{X,Y}(x, \infty) =$
$0.1\,u(x+4) + 0.15\,u(x+3) + 0.17\,u(x+1) + 0.05\,u(x) + 0.18\,u(x-2) + 0.23\,u(x-3) + 0.12\,u(x-4)$ and

$F_Y(y) = F_{X,Y}(\infty, y) = 0.1\,u(y-1) + 0.15\,u(y+5) + 0.17\,u(y-3) + 0.05\,u(y-1) + 0.18\,u(y+2) + 0.23\,u(y-4) + 0.12\,u(y+3) = 0.15\,u(y-5) + 0.12\,u(y+3) + 0.18\,u(y+2) + 0.15\,u(y-1) + 0.17\,u(y-3) + 0.23\,u(y-4).$

4.2-10. (Continued)



(b) $\bar{X} = \sum_{i=1}^{7} P(x_i)x_i = -4(0.1) - 3(0.15) - 1(0.17) + 0(0.05) + 2(0.18) + 3(0.23) + 4(0.12) = 0.51$. $\bar{Y} = -5(0.15) - 3(0.12) - 2(0.18) + 1(0.15) + 3(0.17) + 4(0.23) = 0.11$. (c) $P\{-1 < X \le 4, -3 < Y \le 3\} = 0.05 + 0.18 = 0.23$.

4.2-11. (a)

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 0, & x < 0 \\ \dfrac{5x}{4(x+1)}, & 0 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$

$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 0, & y < 0 \\ 1 + \frac{1}{4}e^{-5y^2} - \frac{5}{4}e^{-y^2}, & y \ge 0 \end{cases}$$

(b) $P\{3 < X \le 5, 1 < Y \le 2\} = F_{X,Y}(5,2) + F_{X,Y}(3,1) - F_{X,Y}(3,2) - F_{X,Y}(5,1) = \left[1 + \frac{1}{4}e^{-5(4)} - \frac{5}{4}e^{-4}\right] + \frac{5}{4}\left[\frac{3+e^{-4}}{4} - e^{-1}\right] - \frac{5}{4}\left[\frac{3+e^{-4(4)}}{4} - e^{-4}\right] - \left[1 + \frac{1}{4}e^{-5} - \frac{5}{4}e^{-1}\right] = 0.004039.$

4.2-12. $F_{X,Y}(x,y) = 0, \quad x < 0$ and/or $y < 0$

$$F_{X,Y}(x,y) = \frac{10}{4}\int_{\alpha=0}^{x}\int_{\beta=0}^{y} \beta^3 e^{-(\alpha+1)\beta^2} d\beta\, d\alpha, \quad 0 \le x \le 4 \text{ and } y > 0,$$

$$= \frac{10}{4}\int_{\beta=0}^{y} \beta^3 e^{-\beta^2}\int_{\alpha=0}^{x} e^{-\beta^2\alpha} d\alpha\, d\beta = \frac{10}{4}\int_{\beta=0}^{y} \beta e^{-\beta^2}\left[1 - e^{-\beta^2 x}\right] d\beta$$

4.2-12. (Continued)

$$= \frac{5}{4}\left\{\frac{x + e^{-(x+1)y^2}}{x+1} - e^{-y^2}\right\}, \quad 0 \le x \le 4 \text{ and } y > 0.$$

$$F_{X,Y}(x,y) = 1 + \frac{1}{4}\left[e^{-5y^2} - 5e^{-y^2}\right], \quad x > 4 \text{ and } y > 0.$$

4.2-13. $F_{X,Y}(\infty,\infty) = a\pi^2 \underset{=}{\text{must}}\ 1$ so $a = 1/\pi^2$.

4.2-14. $F_{X,Y}(\infty,\infty) = a\pi^2 \underset{=}{\text{must}}\ 1$ so again $a = 1/\pi^2$.

4.2-15. (a) For $1 \le y \to \infty$ and any $0 \le x < 1$:

$$F_X(x) = F_{X,Y}(x,\infty) = \frac{27}{26}x\left(1 - \frac{x^2}{27}\right)$$

For $1 \le y \to \infty$ and any $1 \le x$: $F_X(x) = F_{X,Y}(x,\infty) = 1$

For $1 \le y \to \infty$ and any $x < 0$: $F_X(x) = F_{X,Y}(x,\infty) = 0$. Thus,

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{27}{26}x\left(1 - \frac{x^2}{27}\right), & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}$$

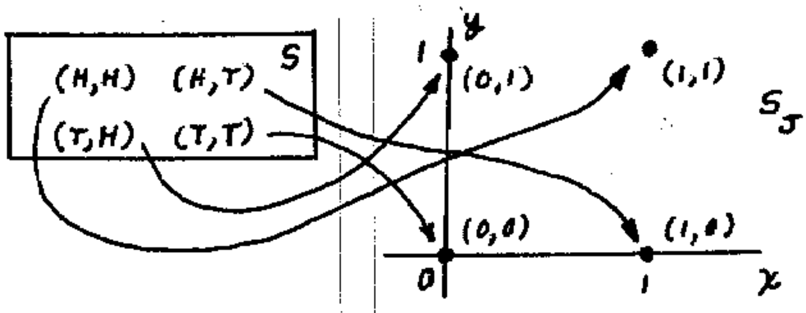
Similarly,

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{27}{26}y\left(1 - y^2/27\right), & 0 \le y < 1 \\ 1, & 1 \le y \end{cases}$$

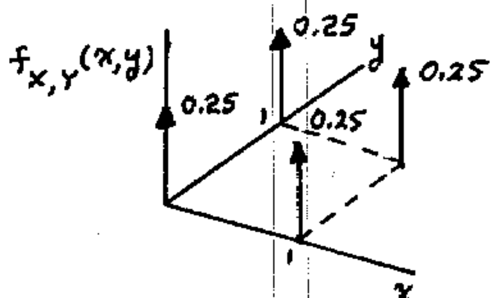
(b) $P\{0 < X \le 0.5,\ 0 < Y \le 0.25\} = F_{X,Y}(0.5, 0.25) + F_{X,Y}(0,0) - F_{X,Y}(0.5,0)$
$- F_{X,Y}(0, 0.25) = \frac{1727}{13{,}312} \approx 0.1297.$

4.3-1. Only four outcomes define the sample space. $S = \{(H,H), (H,T), (T,H), (T,T)\}$. Each outcome has probability $1/4$. The mapping here is:

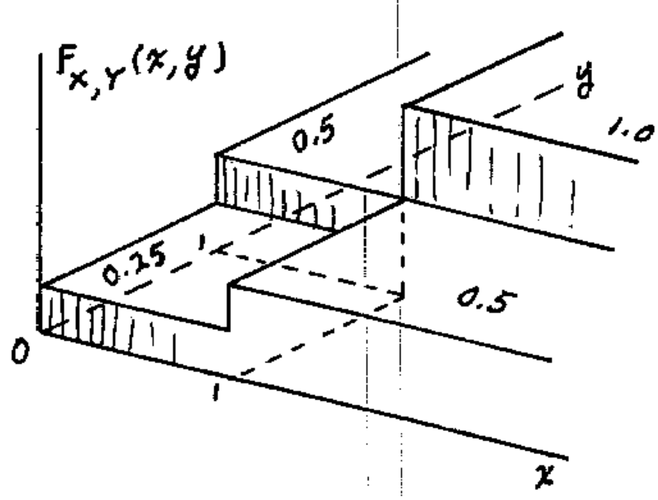
(4.3-1.) (Continued)



(a) $f_{X,Y}(x,y) = \frac{1}{4}\delta(x)\delta(y) + \frac{1}{4}\delta(x)\delta(y-1) + \frac{1}{4}\delta(x-1)\delta(y)$
$+ \frac{1}{4}\delta(x-1)\delta(y-1)$ (1)

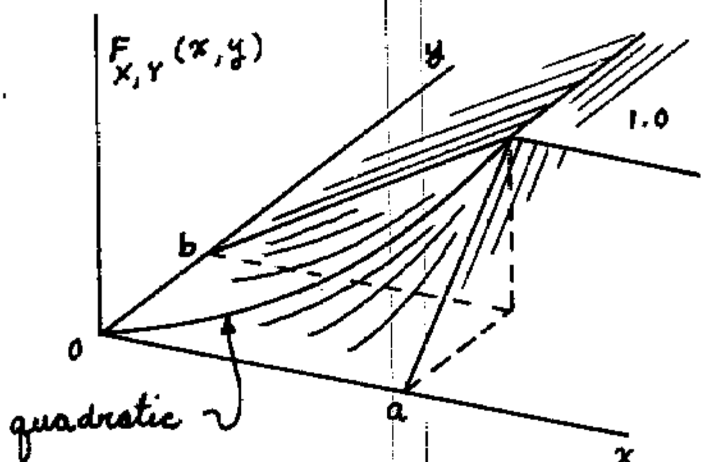


(b) By integration of (1): $F_{X,Y}(x,y) = \frac{1}{4}u(x)u(y)$
$+\frac{1}{4}u(x)u(y-1) + \frac{1}{4}u(x-1)u(y) + \frac{1}{4}u(x-1)u(y-1)$.



(4.3-2.) $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv$

$$\begin{aligned}
&= xy/ab, && 0 \le x < a \text{ and } 0 \le y < b\\
&= x/a, && 0 \le x < a \text{ and } b \le y\\
&= y/b, && a \le x \text{ and } 0 \le y < b\\
&= 1, && a \le x \text{ and } b \le y\\
&= 0, && x < 0 \text{ or } y < 0.
\end{aligned}$$



(4.3-3.) (a) $P\{X+Y \le 3a/4\} = \int_{y=0}^{3a/4}\int_{x=0}^{(3a/4)-y} \frac{dx\,dy}{ab}$

$$= \int_{y=0}^{3a/4} \frac{(3a/4)-y}{ab}\,dy = 9a/32b.$$

(b) $P\{Y \le 2bX/a\} = \int_{y=0}^{b}\int_{x=ay/2b}^{a} \frac{dx\,dy}{ab}$

$$= \int_0^b \frac{a-(ay/2b)}{ab}\,dy = 3/4.$$

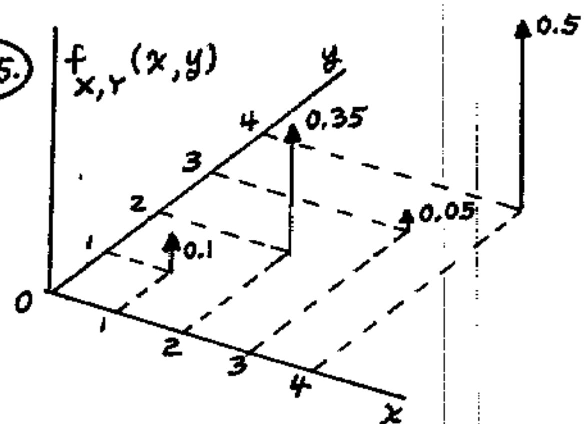
(4.3-4.) $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv = 0$, ($x$ or y) < 0.

$$F_{X,Y}(x,y) = \int_0^{y}\int_0^{x} u\, e^{-u(v+1)}\,du\,dv = \int_0^{x} u\, e^{-u}\int_0^{y} e^{-uv}\,dv\,du$$

$$= \int_0^{x} u\, e^{-u}\left[-\frac{e^{-uy}}{u} + \frac{1}{u}\right]du = -\int_0^{x} e^{-u(y+1)}\,du + \int_0^{x} e^{-u}\,du$$

$$= \frac{e^{-x}\left[e^{-xy} - y - 1\right] + y}{y+1}, \quad x \geq 0 \text{ and } y \geq 0.$$

(4.3-5.)



$$f_{X,Y}(x,y) = 0.1\,\delta(x-1)\delta(y-1) + 0.35\,\delta(x-2)\delta(y-2) + 0.05\,\delta(x-3)\delta(y-3) + 0.50\,\delta(x-4)\delta(y-4).$$

(4.3-6.) By using (4.3-1): $f_{X,Y}(x,y) = \partial^2 F_{X,Y}(x,y)/\partial x \partial y$

$$= \frac{\partial^2}{\partial x \partial y}\left\{u(x)[1-e^{-ax}]\,u(y)[1-e^{-ay}]\right\}$$

$$= \frac{\partial}{\partial x}\left\{u(x)[1-e^{-ax}]\left\langle u(y)\,a\,e^{-ay} + [1-e^{-ay}]\,\delta(y)\right\rangle\right\}$$

$$= \left\{u(x)\,a\,e^{-ax} + \delta(x)\cancelto{0}{[1-e^{-ax}]}\right\}\left\{u(y)\,a\,e^{-ay} + \delta(y)\cancelto{0}{[1-e^{-ay}]}\right\}$$

$$= u(x)\,u(y)\,a^2 e^{-a(x+y)}.$$

Next, $F_X(x) = F_{X,Y}(x,\infty) = u(x)\,[1-e^{-ax}]$

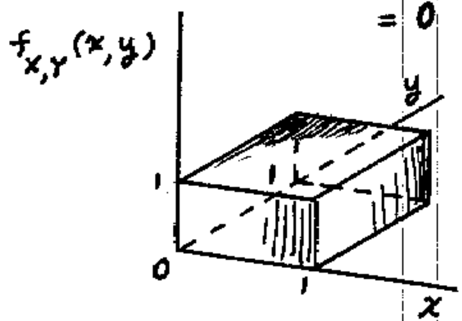
$F_Y(y) = F_{X,Y}(\infty,y) = u(y)\,[1-e^{-ay}]$,

so $f_X(x) = u(x)\,a\,e^{-ax}$

$f_Y(y) = u(y)\,a\,e^{-ay}$.

(4.3-7.) Differentiate $F_{X,Y}(x,y)$ in each region according to (4.3-1): $f_{X,Y}(x,y) = 1$, $0<x<1$ and $0<y<1$

$= 0$, elsewhere.



(4.3-8.) (a) The requirement $f_{X,Y}(x,y) \geq 0$ means that $b>0$ is necessary. Since area must be unity:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_{y=0}^{\infty}\int_{x=0}^{a} b\,e^{-x}e^{-y}\,dx\,dy = b[1-e^{-a}]$$

$= 1$, $\quad b = [1-e^{-a}]^{-1}$.

(b) $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\alpha,\beta)\,d\alpha\,d\beta$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} u(\alpha)\,u(\beta)\,u(a-\alpha)\,\frac{e^{-\alpha}\,e^{-\beta}}{(1-e^{-a})}\,d\alpha\,d\beta$$

This function is zero when $x<0$ or $y<0$. Straightforward solution for the remaining cases produces

$$F_{X,Y}(x,y) = 0 \qquad x<0 \text{ or } y<0$$

$$= \frac{(1-e^{-x})(1-e^{-y})}{(1-e^{-a})}, \quad 0 \leq x < a \text{ and } 0 \leq y$$

$$= 1-e^{-y}, \quad a \leq x \text{ and } 0 \leq y.$$

4.3-9. (a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{\infty} u(x)\,u(a-x)\,b\,e^{-x-y}\,dy$

$$= u(x)\,u(a-x)\,b\,e^{-x}.$$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^a u(y)\,b\,e^{-y}e^{-x}\,dx$

$$= u(y)\,b\,[1-e^{-a}]\,e^{-y}$$

The value of b is $1/[1-e^{-a}]$ from Problem 4.3-8:

(b) $P\{0.5a < X \le 0.75a\} = \int_{0.5a}^{0.75a} f_X(x)\,dx$

$= b\int_{0.5a}^{0.75a} e^{-x}\,dx = b\left[e^{-0.5a} - e^{-0.75a}\right].$

4.3-10. (a) $\int_0^b \int_0^1 3xy\,dx\,dy = \frac{3}{4}b^2 \overset{\text{must}}{=\!=} 1$ so $b = 2/\sqrt{3}$.

(b) $\int_0^1 \int_0^{0.5} bx(1-y)\,dx\,dy = \frac{b}{16} \overset{\text{must}}{=\!=} 1$ so $b = 16$.

(c) $\int_0^2 \int_{-1}^1 b(x^2+4y^2)\,dx\,dy = \frac{68}{3}b \overset{\text{must}}{=\!=} 1$ so $b = 3/68$.

★**4.3-11.** Here $f_{X,Y}(x,y)$ exists (is nonzero) over the area of the xy plane within a circle of radius $\sqrt{b}$ that is centered on the origin.

(a) $\underset{\substack{\text{over circular area}\\ \text{in } xy \text{ plane}}}{\int\int} [(x^2+y^2)/8\pi]\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{b}} \frac{r^2}{8\pi}\,r\,dr\,d\theta = 1$

Here $r = \sqrt{x^2+y^2}$, $r\,dr\,d\theta = dx\,dy$ and $\theta = \tan^{-1}(y/x)$ define a change from rectangular to polar coordinates. The integrals reduce to

$\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{b}} \frac{r^3}{8\pi}\,dr\,d\theta = b^2/16 \overset{\text{must}}{=\!=} 1$ so $b = 4$.

★ 4.3-11. (Continued) (b) The set $\{0.5b < X^2+Y^2 \le 0.8b\}$ corresponds to the annulus defined by $\sqrt{b/2} < r \le \sqrt{0.8b}$ and $0 < \theta \le 2\pi$. Thus,

$$P\{0.5b < X^2+Y^2 \le 0.8b\} = \int_{\theta=0}^{2\pi}\int_{r=\sqrt{0.5b}}^{\sqrt{0.8b}} \frac{r^3}{8\pi}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{32\pi}\Big|_{\sqrt{0.5b}}^{\sqrt{0.8b}} d\theta = 0.39\, b^2/16 = 0.39.$$

★ 4.3-12. We want to find P(bullets inside circle of of diameter 6 cm) = 0.8. This probability is the volume over the circular area in the xy plane with radius R (R = 3 cm here) centered on the origin. In polar coordinates r, θ this volume is

$$0.8 = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} \frac{e^{-r^2/2\sigma^2}}{2\pi\sigma^2}\, r\,dr\,d\theta = \int_{r=0}^{R} \frac{re^{-r^2/2\sigma^2}}{\sigma^2}\,dr$$

Let $\xi = r^2/2\sigma^2$, $d\xi = r\,dr/\sigma^2$ so

$$0.8 = \int_0^{R^2/2\sigma^2} e^{-\xi}\,d\xi = 1 - e^{-R^2/2\sigma^2} \quad \text{or} \quad e^{R^2/2\sigma^2} = 5.$$

Solving for σ: $\sigma = R/\sqrt{2\ln(5)} = 3/\sqrt{2\ln(5)}$ or

$$\sigma \approx 1.672 \text{ cm}.$$

4.3-13. (a) Area must be unity. $\int_{-3}^{3}\int_{-2}^{2} b(x+y)^2\,dx\,dy$

$$= \int_{-3}^{3} b\,\frac{(x+y)^3}{3}\Big|_{-2}^{2} dy = \int_{-3}^{3} \frac{b}{3}\left[(2+y)^3 - (-2+y)^3\right] dy$$

$$= \frac{b}{3}\left\{\frac{(2+y)^4}{4} - \frac{(-2+y)^4}{4}\right\}\Big|_{-3}^{3} = 104\,b \overset{\text{must}}{=} 1. \text{ Thus, } b = 1/104.$$

4.3-13. (Continued) (b) $f_X(x) = \int_{-3}^{3} b(x+y)^2\,dy = \frac{b}{3}(x+y)^3\Big|_{-3}^{3}$

$= \dfrac{(x+3)^3-(x-3)^3}{312}$, $-2<x<2$ (and zero elsewhere).

$f_Y(y) = \int_{-2}^{2} b(x+y)^2\,dx = \dfrac{(2+y)^3-(-2+y)^3}{312}$, $-3<y<3$

(and zero for other values of y).

4.3-14. Test 1: $f_{X,Y}(x,y) \ge 0$ is true if $b>0$.

Test 2: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$ required. Thus,

$$\int_{y=1}^{\infty}\int_{x=2}^{\infty} bxy^2\exp(-2xy)\,dx\,dy = \int_{y=1}^{\infty} by^2\int_{x=2}^{\infty} x e^{-2yx}\,dx\,dy$$

$= \int_1^{\infty} b\left(y+\frac{1}{4}\right)e^{-4y}\,dy = 3be^{-4}/8 \overset{\text{must}}{=} 1$ so $b = \frac{8e^4}{3}$.

4.3-15. $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2+y^2}} \frac{dx}{\pi r^2}$

$$= \begin{cases} \frac{2}{\pi}\frac{\sqrt{r^2-y^2}}{r^2}, & |y|<r \\ 0, & |y|>r \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{dy}{\pi r^2} = \begin{cases} \frac{2}{\pi}\frac{\sqrt{r^2-x^2}}{r^2}, & |x|<r \\ 0, & |x|>r. \end{cases}$$

4.3-16. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$= \frac{10}{4}[u(x)-u(x-4)]\int_0^{\infty} y^3 e^{-(x+1)y^2}\,dy = \frac{10}{8}\frac{[u(x)-u(x-4)]}{(x+1)^2}$.

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{10}{4}u(y)y^3e^{-y^2}\int_0^4 \exp(-y^2x)\,dx$

$= \frac{10}{4}u(y)y\left[e^{-y^2}-e^{-5y^2}\right]$. $F_X(x) = 0$, $x<0$ and

$F_X(x) = \frac{10}{8}\int_0^x \frac{d\xi}{(\xi+1)^2} = \frac{10}{8}\left(\frac{x}{x+1}\right)$, $0\le x\le 4$. Finally,

$F_X(x) = 1$, $x>4$. Repeat the procedures to

4.3-16 (Continued)

obtain $F_Y(y)$. $F_Y(y) = 0$, $y < 0$. $F_Y(y) =$

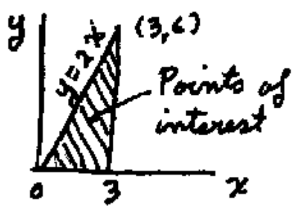
$$\frac{10}{4}\int_0^y \xi\left[e^{-\xi^2}-e^{-5\xi^2}\right]d\xi = 1-e^{-y^2},\quad y \ge 0$$

4.3-17 $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = 2u(x)e^{-x/2}\int_0^{\infty} e^{-4y}\,dy$

$= \frac{1}{2}u(x)e^{-x/2}$. $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = 2u(y)e^{-4y}$

$\int_0^{\infty} e^{-x/2}\,dx = 4u(y)e^{-4y}$.

4.3-18 Use (4.3-5f) for points in x,y plane applicable to the event given.



$$\text{Probability} = \int_{x=0}^{3}\int_{y=0}^{2x} 2e^{-4y-x/2}\,dy\,dx$$

$$= 2\int_{x=0}^{3} e^{-x/2}\left(\frac{1-e^{-8x}}{4}\right)dx$$

$$= \frac{1}{2}\int_0^3\left(e^{-x/2}-e^{-17x/2}\right)dx = \frac{16}{17}-e^{-3/2}+\frac{e^{-51/2}}{17} = 0.7180.$$

4.3-19 $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$

$= 0.1\,u(x)u(y) + 0.12\,u(x-4)u(y) + 0.05\,u(x)u(y-1) + 0.25\,u(x-2)u(y-1)$

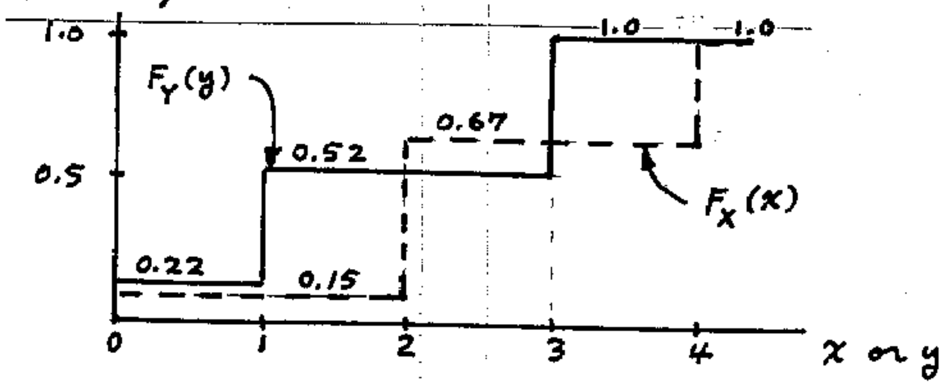
$+ 0.3\,u(x-2)u(y-3) + 0.18\,u(x-4)u(y-3)$.

$F_X(x) = F_{X,Y}(x,\infty) = 0.1\,u(x) + 0.12\,u(x-4) + 0.25\,u(x-2) + 0.05\,u(x)$

$+ 0.18\,u(x-4) + 0.3\,u(x-2) = 0.15\,u(x) + 0.55\,u(x-2) + 0.30\,u(x-4)$.

$F_Y(y) = F_{X,Y}(\infty,y) = 0.22\,u(y) + 0.30\,u(y-1) + 0.48\,u(y-3)$.

(4.3-19) (Continued)



(4.3-20) $$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = a \cdot \frac{1/3}{1+(y/3)^2} \cdot \frac{1/2}{1+(x/2)^2} = \frac{6a}{(9+y^2)(4+x^2)}$$

$$f_X(x) = \frac{d}{dx}\left[F_{X,Y}(x,\infty)\right] = \frac{a\pi/2}{1+(x/2)^2} = \frac{a\,2\pi}{4+x^2}, \quad f_Y(y) = \frac{a\,3\pi}{9+y^2}.$$

(4.3-21) $$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = a \cdot \frac{2(\sqrt{3})^3}{(3+x^2)^2} \cdot \frac{2(\sqrt{5})^3}{(5+y^2)^2}$$

$$f_X(x) = \frac{d}{dx}\left[F_{X,Y}(x,\infty)\right] = a\pi \frac{2(\sqrt{3})^3}{(3+x^2)^2}, \quad f_Y(y) = \frac{d}{dy}\left[F_{X,Y}(\infty,y)\right] = a\pi \frac{2(\sqrt{5})^3}{(5+y^2)^2}.$$

(4.3-22) (a) $$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = \frac{27}{26} \frac{\partial}{\partial y}\left(y - \frac{3x^2y^3}{27}\right)$$

$$= \begin{cases} \frac{27}{26}\left(1 - \frac{x^2y^2}{3}\right), & 0 \le x < 1 \text{ and } 0 \le y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{27}{26}\left(1 - \frac{x^2}{9}\right)$, $0 \le x < 1$ and 0 elsewhere.

Similarly, $f_Y(y) = \frac{27}{26}\left(1 - \frac{y^2}{9}\right)$, $0 \le y < 1$ and 0 elsewhere.

(c) $$P\{Y > 1 - X\} = \int_{x=0}^{1} \int_{\xi=1-x}^{1} f_{X,Y}(x,\xi)\,d\xi\,dx$$

$$= \int_{x=0}^{1} \frac{27}{26}\left[\xi - \frac{x^2\xi^3}{9}\right]\Bigg|_{1-x}^{1} dx = \frac{27}{26}\int_0^1 \left[x - \frac{x^2}{9}(3x - 3x^2 + x^3)\right]dx = \frac{251}{520} \approx 0.4827.$$

(4.3-23) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^a \frac{25}{23ab}\left(\frac{y}{a}\right)\left[1-\left(\frac{x}{b}\right)^4\left(\frac{y}{a}\right)^3\right]dy$

$$= \begin{cases} \frac{5}{46b}\left[5-2\left(\frac{x}{b}\right)^4\right], & -b < x < b \\ 0, & \text{elsewhere.} \end{cases}$$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-b}^{b} \frac{25}{23ab}\left(\frac{y}{a}\right)\left[1-\left(\frac{x}{b}\right)^4\left(\frac{y}{a}\right)^3\right]dx$

$$= \begin{cases} \frac{5}{46a}\left(\frac{y}{a}\right)\left[20-4\left(\frac{y}{a}\right)^3\right], & 0 < y < a \\ 0, & \text{elsewhere.} \end{cases}$$

(4.4-1.) Here $P(y_1) = (2/15) + (3/15) = 5/15$, $P(y_2) = 1/15$, $P(x_1) = (2/15) + (4/15) = 6/15$, $P(x_2) = (3/15) + (1/15) + (5/15) = 9/15$. From (4.4-8):

$$f_X(x|Y=y_1) = \frac{2}{5}\delta(x-x_1) + \frac{3}{5}\delta(x-x_2)$$

$$f_X(x|Y=y_2) = \delta(x-x_2).$$

Repeating the text analysis leading to (4.4-8) gives

$$f_Y(y|X=x_k) = \sum_{j=1}^{M} \frac{P(x_k, y_j)}{P(x_k)}\delta(y-y_j).$$

Thus, $f_Y(y|X=x_1) = \frac{2}{6}\delta(y-y_1) + \frac{4}{6}\delta(y-y_2)$

$$f_Y(y|X=x_2) = \frac{3}{9}\delta(y-y_1) + \frac{1}{9}\delta(y-y_2) + \frac{5}{9}\delta(y-y_3).$$

(4.4-2.) From Example 4.3-2:

$$f_{X,Y}(x,y) = u(x)\,u(y)\,x\,e^{-x(y+1)}$$

$$f_Y(y) = \frac{u(y)}{(y+1)^2}.$$

Use (4.4-12) to get

$$f_X(x|y) = u(x)\,u(y)\,x\,(y+1)^2\,e^{-x(y+1)}.$$

4.4-3. From Example 4.4-2 $f_Y(y|X=x) = u(x)u(y)\,x e^{-xy}$.

Thus, $P\{Y \le 2 | X=1\} = \int_{-\infty}^{2} f_Y(y|X=1)\,dy = \int_0^2 e^{-y}\,dy$

$= 1 - e^{-2} \approx 0.8647.$

4.4-4. From (4.4-12), (4.4-13) and the solution to Problem 4.3-15:

$$f_X(x|Y=y) = \frac{\frac{1}{\pi r^2}}{\frac{2}{\pi}\sqrt{r^2-y^2}} = \begin{cases} \frac{1}{2\sqrt{r^2-y^2}}, & |x| < \sqrt{r^2-y^2} \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_Y(y|X=x) = \frac{\frac{1}{\pi r^2}}{\frac{2}{\pi}\sqrt{r^2-x^2}} = \begin{cases} \frac{1}{2\sqrt{r^2-x^2}}, & |y| < \sqrt{r^2-x^2} \\ 0, & \text{elsewhere.} \end{cases}$$

4.4-5. From (4.4-12), (4.4-13) and the solution to Problem 4.3-16:

$$f_X(x|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{[u(x)-u(x-4)]u(y)y^2 e^{-(x+1)y^2}}{(e^{-y^2}-e^{-5y^2})}$$

$$f_Y(y|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = 2[u(x)-u(x-4)]u(y)(x+1)^2 y^3 e^{-(x+1)y^2}.$$

***4.4-6.** First need to find $f_X(x)$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{T_1}^{\infty} c(y-T_1)e^{-(y-T_1)(x+1)}\,dy\,[u(x)-u(x-4)]$$

$$= \frac{c\,6\,[u(x)-u(x-4)]}{(x+1)^4} \text{ after using (C-48).}$$

(a) From (4.4-13)

$$f_Y(y|X=x) = \frac{1}{6}(x+1)^4[u(x)-u(x-4)](y-T_1)^3 u(y-T_1)e^{-(y-T_1)(x+1)}$$

From (3.1-8): $E[Y|X=x] = \int_{-\infty}^{\infty} y\, f_Y(y|X=x)\,dy$

$$= \frac{(x+1)^4[u(x)-u(x-4)]}{6}\int_{T_1}^{\infty} y(y-T_1)^3 e^{-(y-T_1)(x+1)}\,dy$$

Expand and use (C-48) and an integral from Dwight to get $E[Y|X=x] = \left[T_1 + \frac{4}{x+1}\right][u(x)-u(x-4)]$.

★4.4-6 (Continued)

For a 7:30 A.M. departure $x = 0$ and $E[Y|X=0] = T_1 + 4$. This result is independent of T_0. (b) For a departure at 7:30 A.M. plus T_0 $x = T_0$ and we get $E[Y|X=T_0] = T_1 + [4/(1+T_0)]$ or $T_1 + 2$ if $T_0 = 1$ h.

$$\bar{X} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{T_0} \frac{6c\,x}{(x+1)^4}\,dx \quad \text{use Dwight, p. 23}$$

$$= \frac{T_0^2(T_0+3)}{2[(T_0+1)^3-1]}.$$ For $T_0 = 1$ h $\bar{X} = 2/7$ h.

★4.4-7. Define $B = \{x_a < X \le x_b\}$ with $P(B) \ne 0$ assumed.

$$F_Y(y|x_a < X \le x_b) = \frac{P\{Y \le y \cap x_a < X \le x_b\}}{P\{x_a < X \le x_b\}}$$

$$= \frac{\int_{-\infty}^{y}\int_{x_a}^{x_b} f_{X,Y}(x,\xi)\,dx\,d\xi}{\int_{x_a}^{x_b} f_X(x)\,dx} = \frac{\int_{-\infty}^{y}\int_{x_a}^{x_b} f_{X,Y}(x,\xi)\,dx\,d\xi}{\int_{x_a}^{x_b}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy}$$

On differentiation according to (4.4-2):

$$f_Y(y|x_a < X \le x_b) = \frac{\int_{x_a}^{x_b} f_{X,Y}(x,y)\,dx}{\int_{x_a}^{x_b}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy}.$$

★4.4-8. Define $B = \{y_a < Y \le y_b\}$.

$$P\{X \le x, Y \le y | y_a < Y \le y_b\} = \frac{P\{X \le x, Y \le y \cap y_a < Y \le y_b\}}{P\{y_a < Y \le y_b\}} \quad \text{so}$$

$$F_{X,Y}(x,y|y_a < Y \le y_b) = 0, \quad y \le y_a$$

$$= \frac{\int_{y_a}^{y}\int_{-\infty}^{x} f_{X,Y}(\alpha,\beta)\,d\alpha\,d\beta}{F_Y(y_b) - F_Y(y_a)} = \frac{F_{X,Y}(x,y) - F_{X,Y}(x,y_a)}{F_Y(y_b) - F_Y(y_a)},$$

for $y_a < y \le y_b$

$$= \frac{F_{X,Y}(x,y_b) - F_{X,Y}(x,y_a)}{F_Y(y_b) - F_Y(y_a)}, \quad y > y_b.$$

★ 4.4-8. (Continued)

Next,

$$f_{X,Y}(x,y \mid y_a < Y \le y_b) = \begin{cases} 0, & y \le y_a \\ 0, & y_b < y \\ \dfrac{f_{X,Y}(x,y)}{F_Y(y_b) - F_Y(y_a)}, & y_a < y \le y_b . \end{cases}$$

★ 4.4-9. First, note that (4.4-16) applies if y_a, y_b, y and x are changed to x_a, x_b, x and y, respectively. Thus,

$$f_Y(y \mid x_a < X \le x_b) = \frac{\int_{x_a}^{x_b} f_{X,Y}(x,y)\,dx}{\int_{x_a}^{x_b}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx}.$$

For $x_a = -b < X \le x_b = 0$:

$$\int_{-b}^{0} f_{X,Y}(x,y)\,dx = \int_{-b}^{0} \frac{25}{23ab}\left(\frac{y}{a}\right)\left[1 - \left(\frac{x}{b}\right)^4\left(\frac{y}{a}\right)^3\right]dx = \frac{25}{23a}\left(\frac{y}{a}\right)\left[1 - \frac{1}{5}\left(\frac{y}{a}\right)^3\right],$$

$0 < y \le a$.

$$\int_{-b}^{0}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = \int_{0}^{a} \frac{25}{23a}\left(\frac{y}{a}\right)\left[1 - \frac{1}{5}\left(\frac{y}{a}\right)^3\right]dy = \frac{1}{2}. \quad \text{Thus,}$$

$$f_Y(y \mid -b < X \le 0) = \begin{cases} \dfrac{10}{23a}\left(\dfrac{y}{a}\right)\left[5 - \left(\dfrac{y}{a}\right)^3\right], & 0 < y \le a \\ 0, & \text{elsewhere.} \end{cases}$$

where $a = 5$ km and $b = 25$ km.

A similar development for $0 < X \le b$ gives the same result.

$$f_Y(y \mid 0 < X \le b) = \begin{cases} \dfrac{10}{23a}\left(\dfrac{y}{a}\right)\left[5 - \left(\dfrac{y}{a}\right)^3\right], & 0 < y \le a \\ 0, & \text{elsewhere.} \end{cases}$$

★ 4.4-10. Use (4.4-16) with $y_a = 10$ km and $y_b = 25$ km $= a$.

$$\int_{y_a}^{y_b} f_{X,Y}(x,y)\,dy = \int_{y_a}^{a} \frac{25}{23ab}\left[\left(\frac{y}{a}\right) - \left(\frac{x}{b}\right)^4\left(\frac{y}{a}\right)^4\right] dy$$

$$= \frac{25}{23b}\left\{\frac{1}{2} - \frac{1}{5}\left(\frac{x}{b}\right)^4 - \frac{1}{2}\left(\frac{y_a}{a}\right)^2 + \frac{1}{5}\left(\frac{x}{b}\right)^4\left(\frac{y_a}{a}\right)^5\right\}, \quad \int_{y_a}^{y_b}\int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy =$$

$$= \frac{50}{23}\left\{\frac{1}{2} - \frac{1}{2}\left(\frac{y_a}{a}\right)^2 - \frac{1}{25} + \frac{1}{25}\left(\frac{y_a}{a}\right)^5\right\}. \text{ Thus,}$$

$$f_X(x \mid y_a < Y \le y_b = a) = \frac{\frac{25}{23b}\left\{\frac{1}{2} - \frac{1}{5}\left(\frac{x}{b}\right)^4 - \frac{1}{2}\left(\frac{y_a}{a}\right)^2 + \frac{1}{5}\left(\frac{x}{b}\right)^4\left(\frac{y_a}{a}\right)^5\right\}}{\frac{50}{23}\left\{\frac{1}{2} - \frac{1}{2}\left(\frac{y_a}{a}\right)^2 - \frac{1}{25} + \frac{1}{25}\left(\frac{y_a}{a}\right)^5\right\}}$$

$$= \begin{cases} \dfrac{6{,}562.5 - 3{,}093\,(x/5)^4}{59{,}439}, & -5 < x < 5 \\ 0, & \text{elsewhere.} \end{cases}$$

4.5-1. (a) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{\infty} \frac{e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}}\,dy$$

$$= \frac{e^{-x^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-(y^2-2\rho xy+\rho^2x^2-\rho^2x^2)/2(1-\rho^2)}\,dy$$

$$= \frac{e^{-(x^2-\rho^2x^2)/2(1-\rho^2)}}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty}\frac{e^{-(y-\rho x)^2/2(1-\rho^2)}}{\sqrt{2\pi(1-\rho^2)}}\,dy}_{\text{area of Gaussian density with mean } \rho x \text{ and variance } (1-\rho^2).}$$

$$f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

By simply interchanging x and y in the above analysis we obtain $f_Y(y) = \dfrac{e^{-y^2/2}}{\sqrt{2\pi}}$.

(b) Since $f_X(x)\,f_Y(y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2} \ne f_{X,Y}(x,y)$

4.5-1. (Continued)

X and Y are not independent except when $\rho = 0$.

4.5-2. From (4.4-11):

$$f_X(x|Y=y) = \frac{\dfrac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}}{\dfrac{1}{\sqrt{2\pi}} e^{-y^2/2}}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-(x-\rho y)^2/2(1-\rho^2)}.$$

Note that $f_X(x|Y=y)$ is a Gaussian density with mean ρy and variance $(1-\rho^2)$.

4.5-3. Joint and marginal densities were found in Problem 4.3-6: $f_{X,Y}(x,y) = u(x)u(y)\,a^2 e^{-a(x+y)}$,

$f_X(x) = u(x)\,a\,e^{-ax}$, $f_Y(y) = u(y)\,a\,e^{-ay}$.

(a) $f_X(x|Y=y) = f_{X,Y}(x,y)/f_Y(y)$

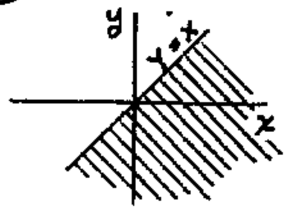
$$= \frac{u(x)u(y)\,a^2 e^{-ax-ay}}{u(y)\,a\,e^{-ay}} = u(x)\,a\,e^{-ax}.$$

$$f_Y(y|X=x) = \frac{u(x)u(y)\,a^2 e^{-ax-ay}}{u(x)\,a\,e^{-ax}} = u(y)\,a\,e^{-ay}.$$

(b) Here $f_X(x|Y=y) = f_X(x)$ and $f_Y(y|X=x) = f_Y(y)$ so X and Y are statistically independent.

4.5-4. The required probability is the volume over the shaded figure's area in the xy plane where $Y \le X$.

(4.5-4.) (Continued)



By integrating strips in the y direction $P\{Y \le X\} = \int_{v=-\infty}^{x} \int_{x=-\infty}^{\infty} f_{X,Y}(x,v)\,dx\,dv = \int_{x=-\infty}^{\infty} f_X(x)\left[\int_{v=-\infty}^{x} f_Y(v)\,dv\right] dx = \int_{-\infty}^{\infty} f_X(x)\,F_Y(x)\,dx.$

By integrating strips along the x direction

$$P\{Y \le X\} = \int_{y=-\infty}^{\infty} \int_{u=y}^{\infty} f_{X,Y}(u,y)\,du\,dy$$

$$= \int_{y=-\infty}^{\infty} f_Y(y)\left[\int_{u=y}^{\infty} f_X(u)\,du\right] dy = \int_{-\infty}^{\infty} f_Y(y)\,F_X(\infty)\,dy$$

$$- \int_{-\infty}^{\infty} f_Y(y)\,F_X(y)\,dy = 1 - \int_{-\infty}^{\infty} f_Y(y)\,F_X(y)\,dy.$$

(4.5-5.) (a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^x \frac{5}{16} x^2 y\,dy$

$= \frac{5}{32} x^4, \quad 0 < x < 2.$ (and zero elsewhere).

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_y^2 \frac{5}{16} x^2 y\,dx = \frac{5}{48} y(8-y^3), \; 0 < y < 2$

(and zero elsewhere).

(b) Since $f_X(x)\,f_Y(y) = \dfrac{25\,x^4 y(8-y^3)}{32\,(48)} \ne f_{X,Y}(x,y)$ then X and Y are not statistically independent.

(4.5-6.) From solution of Problem 4.4-6 $f_X(x) = 6c[u(x) - u(x-T_0)]/(x+1)^4$. Also $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = c\int_0^{T_0} (y-T_1)^3 u(y-T_1) e^{-(y-T_1)(x+1)} dx = cu(y-T_1)(y-T_1)^2 e^{-(y-T_1)}[1 - e^{-(y-T_1)T_0}]$. Clearly $f_X(x) f_Y(y) \neq f_{X,Y}(x,y)$ so X and Y are not statistically independent.

(4.5-7.) From the solution to Problem 4.3-17 we have $f_X(x) f_Y(y) = f_{X,Y}(x,y)$ so X and Y are independent.

(4.5-8.) Because the given joint density is factorable with all factors equal in form and being densities, the X_i, $i=1,2,3,4$ are statistically independent so: (a) $f_{X_1,X_2,X_3}(x_1,x_2,x_3 \mid x_4) = \prod_{i=1}^{3} \exp(-2|x_i|)$

(b) $f_{X_1,X_2}(x_1,x_2 \mid x_3,x_4) = \prod_{i=1}^{2} \exp(-2|x_i|)$

(c) $f_{X_1}(x_1 \mid x_2,x_3,x_4) = \exp(-2|x_1|)$.

(4.5-9.) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-1}^{1} k\cos^2(\pi x y/2)\,dy = k[1 + \frac{\sin(\pi x)}{\pi x}]$ for $-1<x<1$ and 0 for all other x. Similarly, $f_Y(y) = \int_{-1}^{1} k\cos^2(\pi x y/2)\,dx = k[1 + \frac{\sin(\pi y)}{\pi y}]$, $-1<y<1$, and zero for other y. Finally $f_X(x) f_Y(y) \neq f_{X,Y}(x,y)$ so X and Y are not statistically independent.

(4.5-10.) (a) $P\{2 < X \le 4, -1 < Y \le 5\} = \int_{y=0}^{5}\int_{x=2}^{4} \frac{1}{12} e^{-x/4} e^{-y/3} dx\,dy = (1-e^{-5/3})(e^{-1/2}-e^{-1}) \approx 0.1936$. (b) $P\{0 < X < \infty, -\infty < Y \le -2\} = 0$ since Y has no values less than zero.

(4.6-1.) area $= \int_{-\infty}^{\infty} f_Y(y|x)\,dy = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)\,dy}{f_X(x)} = \frac{f_X(x)}{f_X(x)} = 1.$

★(4.6-2.) (a) $P\{0 < R \le 1,\ 0 < \Theta \le \pi/2\} = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} \frac{r}{2\pi} e^{-r^2/2}\,dr\,d\theta$

By use of the variable change $\xi = r^2/2$, $d\xi = r\,dr$:

$P\{0 < R \le 1,\ 0 < \Theta \le \pi/2\} = \int_0^{1/2} \frac{1}{4} e^{-\xi}\,d\xi = \frac{1}{4}(1 - e^{-1/2}) \approx 0.0984.$

(b) Use (4.4-11) after finding $f_\Theta(\theta)$.

$$f_\Theta(\theta) = \int_{r=0}^{\infty} f_{R,\Theta}(r,\theta)\,dr = \frac{u(\theta) - u(\theta - 2\pi)}{2\pi}\int_0^{\infty} r\, e^{-r^2/2}\,dr$$

$$= [u(\theta) - u(\theta - 2\pi)]/2\pi$$

$$f_R(r|\Theta = \pi) = \frac{\frac{1}{2\pi}[u(\pi) - u(\pi - 2\pi)]\,u(r)\,r\,e^{-r^2/2}}{\frac{1}{2\pi}[u(\pi) - u(\pi - 2\pi)]}$$

$$= u(r)\,r\,e^{-r^2/2}.$$

(c) apply (4.4-16).

$$f_R(r|\Theta \le \pi) = \frac{\int_0^{\pi} \frac{u(r)\,r}{2\pi} e^{-r^2/2}\,d\theta}{\int_0^{\pi}\int_0^{\infty} \frac{r e^{-r^2/2}}{2\pi}\,dr\,d\theta} = u(r)\,r\,e^{-r^2/2}.$$

This is the same result as in (b) because R and Θ are actually independent so $f_R(r|\Theta \le \text{any value}) = f_R(r)$.

(4.6-3.) Use (4.6-5). $f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\,f_X(w-y)\,dy$

$$= \int_0^{\infty} b\,e^{-by}\,\frac{1}{a}[u(w-y) - u(w-y-a)]\,dy$$

$= \int_0^{w} \frac{b}{a} e^{-by}\,dy - \int_0^{w-a} \frac{b}{a} e^{-by}\,dy$. These integrals are

(4.6-3.) (Continued)

zero when $w \le 0$. There are two other cases:

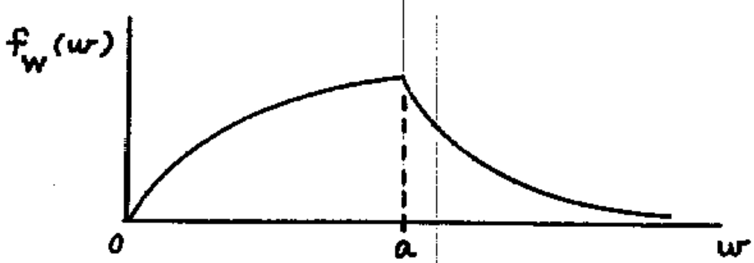
$\underline{0 < w \le a}$ $\quad f_W(w) = \int_0^w \frac{b}{a} e^{-by}\,dy = \frac{1}{a}[1 - e^{-bw}]$.

$\underline{a < w}$ $\quad f_W(w) = \frac{1}{a}[1 - e^{-bw}] - \frac{b}{a}\int_0^{w-a} e^{-by}\,dy$

$= \frac{e^{-bw}}{a}[e^{ba} - 1]$.

Combining results:

$$
\begin{aligned}
f_W(w) &= 0, && w \le 0 \\
&= \frac{1}{a}(1 - e^{-bw}), && 0 < w \le a \\
&= \frac{1}{a} e^{-bw}(e^{ba} - 1), && a < w.
\end{aligned}
$$



(4.6-4.) Apply (4.6-5). $f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\,dy$

$= \int_{-\infty}^{\infty} [0.4\,\delta(y-5) + 0.5\,\delta(y-6) + 0.1\,\delta(y-7)][0.1\,\delta(w-y-1)$

$\quad + 0.2\,\delta(w-y-2) + 0.4\,\delta(w-y-3) + 0.3\,\delta(w-y-4)]\,dy$.

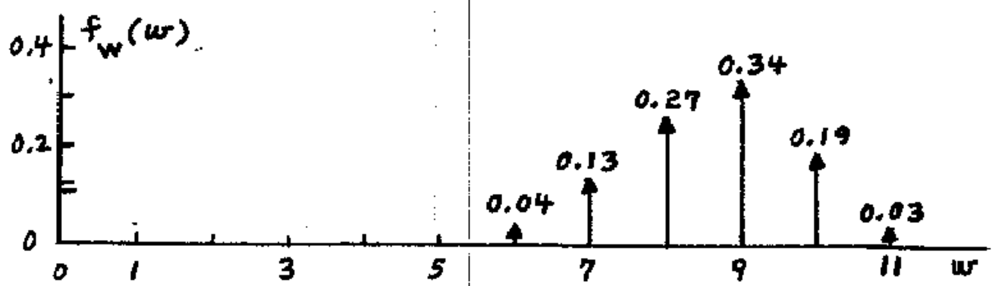
Each integral is of the form $k\int_{-\infty}^{\infty}\delta(y-y_k)\delta(w-y-x_m)\,dy$ where y_k and x_m are numbers and k is a constant. Impulses coincide only when $w = x_m + y_k$ and the integral equals $k\,\delta(w - y_k - x_m)$. Hence,

4.6-4. (Continued)

$$f_W(w) = 0.04\,\delta(w-6) + 0.08\,\delta(w-7) + 0.16\,\delta(w-8)$$
$$+ 0.12\,\delta(w-9) + 0.05\,\delta(w-7) + 0.10\,\delta(w-8)$$
$$+ 0.20\,\delta(w-9) + 0.15\,\delta(w-10) + 0.01\,\delta(w-8)$$
$$+ 0.02\,\delta(w-9) + 0.04\,\delta(w-10) + 0.03\,\delta(w-11)$$

or

$$f_W(w) = 0.04\,\delta(w-6) + 0.13\,\delta(w-7) + 0.27\,\delta(w-8)$$
$$+ 0.34\,\delta(w-9) + 0.19\,\delta(w-10) + 0.03\,\delta(w-11).$$



4.6-5. Here $f_X(x) = \frac{1}{5}[u(x) - u(x-5)]$

$$f_Y(y) = 0.4\,\delta(y-5) + 0.5\,\delta(y-6) + 0.1\,\delta(y-7).$$

Thus,

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy =$$

$$\int_{-\infty}^{\infty} [0.4\,\delta(y-5) + 0.5\,\delta(y-6) + 0.1\,\delta(y-7)]$$

$$\cdot \frac{1}{5}[u(w-y) - u(w-y-5)]\, dy =$$

$$\frac{1}{5}\int_{-\infty}^{w} [0.4\,\delta(y-5) + 0.5\,\delta(y-6) + 0.1\,\delta(y-7)]\, dy$$

$$-\frac{1}{5}\int_{0}^{w-5} [0.4\,\delta(y-5) + 0.5\,\delta(y-6) + 0.1\,\delta(y-7)]\, dy.$$

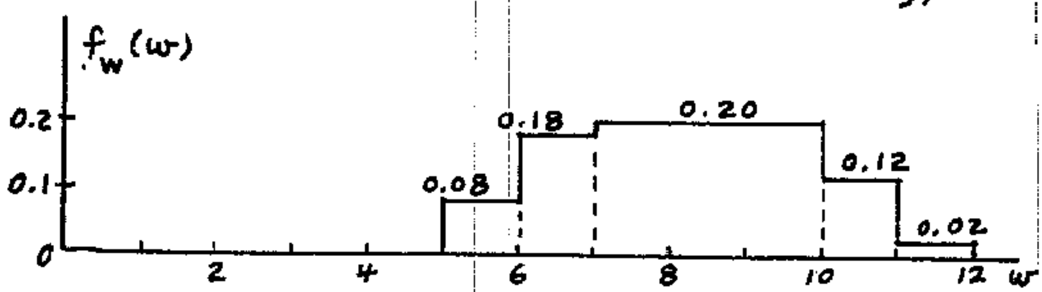
4.6-5. (Continued)

If $w < 5$ $f_W(w) = 0$. If $5 \le w < 10$ only the first integral is nonzero. For $10 \le w$ both integrals are nonzero. By combining cases:

$$f_W(w) = 0, \qquad w < 0$$

$$= \frac{1}{5}[0.4u(w-5) + 0.5u(w-6) + 0.1u(w-7)], \quad 0 \le w < 10$$

$$= \frac{1}{5}[0.4u(w-5) + 0.5u(w-6) + 0.1u(w-7)$$
$$- 0.4u(w-10) - 0.5u(w-11) - 0.1u(w-12)], \quad 10 \le w.$$



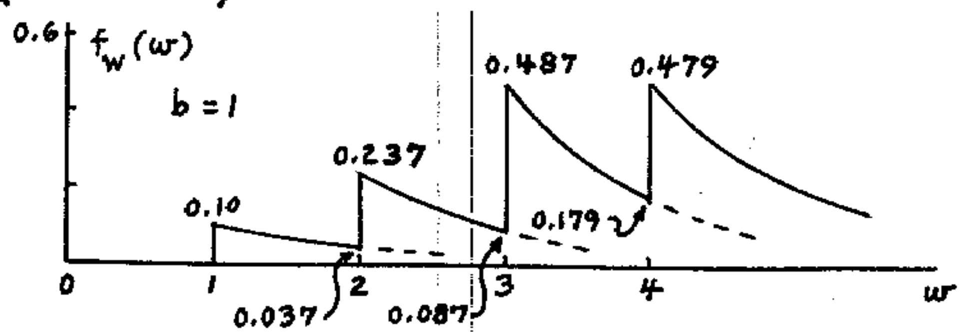
4.6-6. Here $f_X(x) = 0.1\delta(x-1) + 0.2\delta(x-2) + 0.4\delta(x-3)$
$+ 0.3\delta(x-4)$

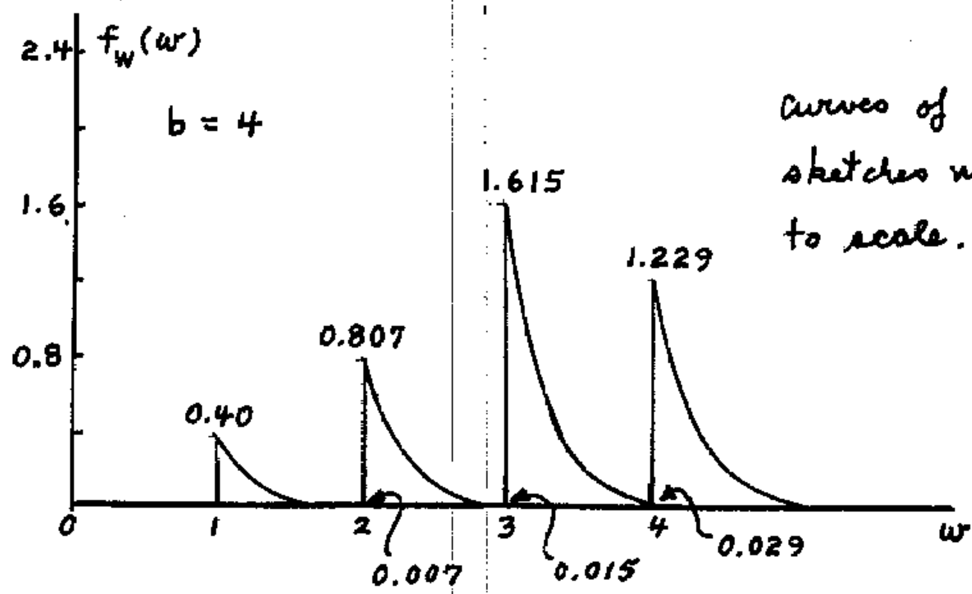
$$f_Y(y) = b\,u(y)\,e^{-by}.$$

Thus, $f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\,dy = b\int_{-\infty}^{\infty} u(y)e^{-by}[0.1\delta(w$
$-y-1) + 0.2\delta(w-y-2) + 0.4\delta(w-y-3) + 0.3\delta(w-y-4)]\,dy$

$$f_W(w) = b\{0.1u(w-1)e^{-b(w-1)} + 0.2u(w-2)e^{-b(w-2)}$$
$$+ 0.4u(w-3)e^{-b(w-3)} + 0.3u(w-4)e^{-b(w-4)}\}.$$

(4.6-6.) (Continued)





Curves of sketches not to scale.

* (4.6-7.) Define $W_1 = X_1 + X_2$. Then $f_{W_1}(w_1) = \int_{-\infty}^{\infty} f_{X_1}(x)$

$\cdot f_{X_2}(w_1 - x)\,dx = \int_{-\infty}^{\infty} \frac{1}{a^2}[u(x) - u(x-a)][u(w_1 - x) - u(w_1$

$- x - a)]\,dx$.

The above integrals easily evaluate after considering where the integrands are nonzero.

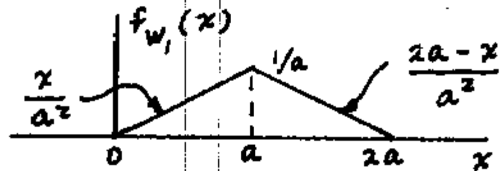
$$
\begin{aligned}
f_{W_1}(w_1) &= 0, && w_1 \le 0 \\
&= w_1/a^2, && 0 < w_1 \le a \\
&= (2a - w_1)/a^2, && a < w_1 \le 2a
\end{aligned}
$$

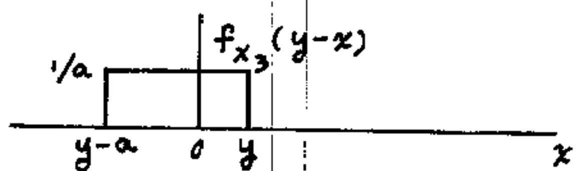
★4.6-7 (Continued)

$= 0, \qquad 2a < w_1$

Next, let $Y = W_1 + X_3$. $f_Y(y) = \int_{-\infty}^{\infty} f_{W_1}(x) f_{X_3}(y-x)\, dx$.

From geometry:





$f_Y(y) = 0, \quad y \le 0$

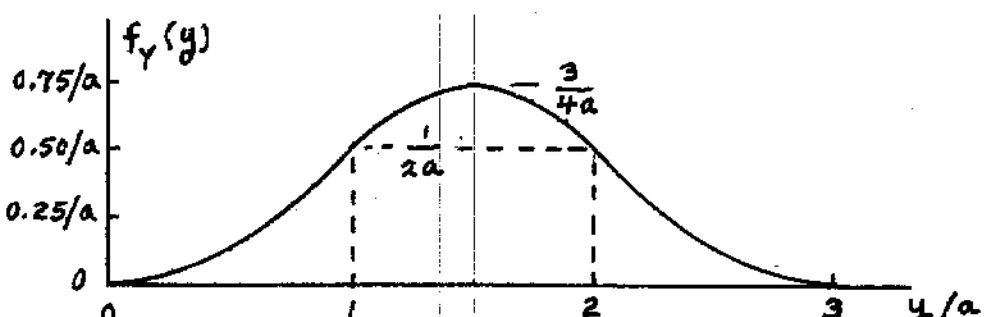
$f_Y(y) = \int_0^y \frac{x}{a^3}\, dx = y^2/2a^3, \quad 0 < y \le a$

$f_Y(y) = \int_{y-a}^{a} \frac{x}{a^3}\, dx + \int_a^y \frac{(2a-x)}{a^3}\, dx = \frac{-2y^2+6ay-3a^2}{2a^3}$

for $a < y \le 2a$

$f_Y(y) = \int_{y-a}^{2a} \frac{(2a-x)}{a^3}\, dx = \frac{(y-3a)^2}{2a^3}, \quad 2a < y \le 3a$

$f_Y(y) = 0, \quad 3a < y$.



4.6-8. $W = X - Y$, $F_W(w) = P\{W \le w\} = P\{X - Y \le w\}$

$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{w+y} f_Y(y) f_X(x)\, dx\, dy = \int_{y=-\infty}^{\infty} f_Y(y) \int_{x=-\infty}^{w+y} f_X(x)\, dx\, dy$$

$f_W(w) = \frac{dF_W(w)}{dw} = \int_{-\infty}^{\infty} f_Y(y) f_X(w+y)\, dy$. This is a <u>correlation</u> <u>integral</u> not a convolution.

4.6-9. (a) $P\{Y \le 8 - \frac{2|X|}{3}\} = \int_{x=-12}^{12} \int_{y=0}^{8-\frac{2|X|}{3}} \frac{1}{12}\left(1 - \frac{|X|}{12}\right)\frac{1}{4} e^{-y/4}\, dy\, dx$

Symmetric in x

This integral easily reduces to

$P\{Y \le 8 - \frac{2|X|}{3}\} = \frac{1}{2}[1 + 3e^{-2}] = 0.7030$. (b) $P\{Y \le 8 + \frac{2|X|}{3}\}$
$= 1 - \frac{1}{2}e^{-2} - \frac{1}{2}e^{-4} = 0.9232$ by a similar procedure. The second probability is larger because $\{Y \le 8 + \frac{2|X|}{3}\}$ has more points.

4.6-10. $f_W(w) = \int_{-\infty}^{\infty} f_X(w-y) f_Y(y)\, dy = \int_{-\infty}^{\infty} 5e^{-5(w-y)} u(w-y) u(y)$
$\cdot 2e^{-2y}\, dy = 10\int_0^w e^{-5w-(2-5)y}\, dy$, $w > 0$
$= \frac{10}{3} u(w)[e^{-2w} - e^{-5w}]$.

★4.6-11. Use (4.6-7) by repeated iteration of (4.6-5).

$f_{X_1+X_2}(w) = \int_{-\infty}^{\infty} a e^{-a(w-\xi)} u(w-\xi)\, a e^{-a\xi} u(\xi)\, d\xi$
$= a^2 e^{-wa} \int_0^{w>0} d\xi = a^2 u(w)\, w\, e^{-aw}$. $f_{X_1+X_2+X_3}(w)$
$= \int_{-\infty}^{\infty} f_X(w-\xi) f_{X_1+X_2}(\xi)\, d\xi = \int_0^{w>0} a e^{-a(w-\xi)} a^2 \xi e^{-a\xi}\, d\xi$
$= a^3 u(w) \frac{w^2}{2} e^{-aw}$. $f_{X_1+X_2+X_3+X_4}(w) = \int_{-\infty}^{\infty} f_X(w-\xi)$
$f_{X_1+X_2+X_3}(\xi)\, d\xi = \frac{a^3}{2}\int_0^{w>0} a e^{-a(w-\xi)} \xi^2 e^{-a\xi}\, d\xi$
$= a^4 \frac{w^3}{2\cdot 3} u(w) e^{-aw}$. Clearly, the form of the density of the sum is the exponential $u(w)e^{-aw}$ modified by a factor that has the form $a \prod_{i=1}^{N-1} \left(\frac{aw}{i}\right)$ so $f_W(w) = \frac{a^N u(w)\, w^{N-1}}{(N-1)!} e^{-aw}$.

★ **4.6-12.** $f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx = \int_{-a}^{a} \frac{3}{2a^3} x^2 \frac{1}{2} \mathrm{rect}\left(\frac{w-x}{\pi}\right) \cos(w-x)\, dx$

$= \frac{3}{4a^3} \int_{-a}^{a} x^2\, \mathrm{rect}\left(\frac{x-w}{\pi}\right) \cos(x-w)\, dx$. Two cases: $w + \frac{\pi}{2} < a$, and

$w + \frac{\pi}{2} > a$ when $w - \frac{\pi}{2} < a$.

★ **4.6-12.** (Continued) For $w + (\pi/2) < a$, or $w < a - (\pi/2)$:

$$f_W(w) = \frac{3}{4a^3} \int_{w-(\pi/2)}^{w+(\pi/2)} x^2 \cos(x-w)\, dx = \frac{3}{2a^3}\left[\left(\frac{\pi^2}{4} - 2\right) + w^2\right], \quad w < a - \frac{\pi}{2}.$$

For $w + (\pi/2) > a$ when $w - \frac{\pi}{2} < a$:

$$f_W(w) = \frac{3}{4a^3} \int_{w-(\pi/2)}^{a} x^2 \cos(w-x)\, dx = \frac{3}{4a^3}\Big\{2a\cos(a-w) + (a^2-2)\sin(a-w)$$

$$+ \left[\left(\frac{\pi}{2} - w\right)^2 - 2\right]\Big\}, \quad a - (\pi/2) < w < a + (\pi/2).$$ By combining these

two cases and noting the symmetry for $w < 0$:

$$f_W(w) = \begin{cases} \frac{3}{2a^3}\left[\left(\frac{\pi^2}{4} - 2\right) + w^2\right], & |w| < a - \frac{\pi}{2} \\ \frac{3}{4a^3}\left\{\left(\frac{\pi}{2} - w\right)^2 - 2 + 2a\cos(a-w) + (a^2-2)\sin(a-w)\right\}, & a - \frac{\pi}{2} < |w| < a + \frac{\pi}{2} \\ 0, & a + \frac{\pi}{2} < |w|. \end{cases}$$

4.6-13. $f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy = \int_{-\infty}^{\infty} \frac{1}{4} u(y-3)\, e^{-(y-3)/4} \frac{1}{2} u(w-y-1)$

$\cdot\, e^{-(w-y-1)/2}\, dy = \frac{1}{2} u(w-4)\left[e^{-(w-4)/4} - e^{-(w-4)/2}\right]$

4.6-14. $f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx = \frac{3}{64} \int_{-2}^{2} (4-x^2)\left[u(w-x+1) - u(w-x-1)\right] dx$

Bracketed term is nonzero only for $w - x + 1 > 0$ and for $x - w - 1 < 0$. Thus, there are two cases that depend on the value of w.

For $w + 1 \le 2$ (or $w \le 1$):

4.6-14 (Continued)

$$f_W(w) = \frac{3}{64}\int_{w-1}^{w+1}(4-x^2)\,dx = \frac{1}{32}(11-3w^2), \quad w \le 1.$$

For $w+1 > 2$ (or $w > 1$) and $w-1 \le 2$ (or $w \le 3$):

$$f_W(w) = \frac{3}{64}\int_{w-1}^{2}(4-x^2)\,dx = \frac{1}{16}\left\{7-3w+\frac{1}{4}(w-1)^3\right\}, \quad 1 < w \le 3.$$

Because X and Y have symmetric densities, $f_W(w)$ will be symmetric and only $w > 0$ need be analyzed. Thus,

$$f_W(w) = \begin{cases} \frac{1}{32}(11-3w^2), & 0 < |w| \le 1 \\ \frac{1}{32}\left\{14-6w+\frac{1}{2}(w-1)^3\right\}, & 1 < |w| \le 3 \\ 0 & \text{elsewhere.} \end{cases}$$

★ **4.7-1.** From (4.6-5) with $c = 3/2a^3$:

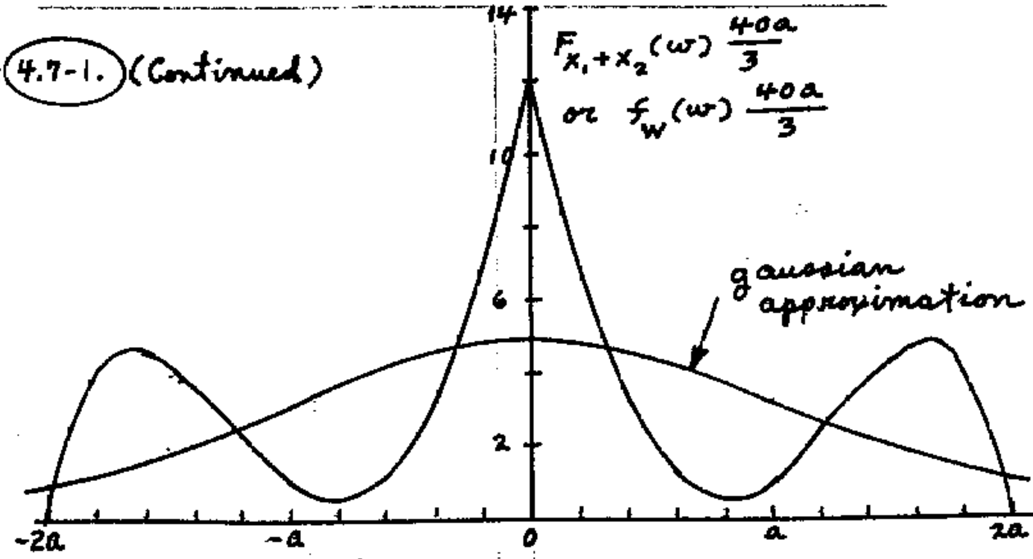
$$f_{X_1+X_2}(w) = \int_{-\infty}^{\infty} c\,\xi^2 \operatorname{rect}\left(\frac{\xi}{2a}\right) c\,(w-\xi)^2 \operatorname{rect}\left(\frac{w-\xi}{2a}\right) d\xi.$$

From a sketch of the rect(·) functions we see the integral $\neq 0$ only for $-2a < w < 2a$. For $0 \le w \le 2a$

$$f_{X_1+X_2}(w) = c^2\int_{w-a}^{a}\xi^2(w-\xi)^2\,d\xi = \frac{3}{40a}\left[12-30\left(\frac{w}{a}\right)+20\left(\frac{w}{a}\right)^2-\left(\frac{w}{a}\right)^5\right].$$

For $-2a \le w \le 0$ we use the fact that $f_{X_1+X_2}(w)$ has even symmetry so $f_{X_1+X_2}(w) = \frac{3}{40a}\left[12+30\left(\frac{w}{a}\right)+20\left(\frac{w}{a}\right)^2+\left(\frac{w}{a}\right)^5\right]$. Next, $\sigma_X^2 = \frac{3}{2a^3}\int_{-a}^{a}x^4\,dx = 3a^2/5$, $\overline{X} = 0$, so the gaussian approximation is $f_W(w) = \dfrac{\exp[-w^2/2(6a^2/5)]}{\sqrt{2\pi(6a^2/5)}}$.

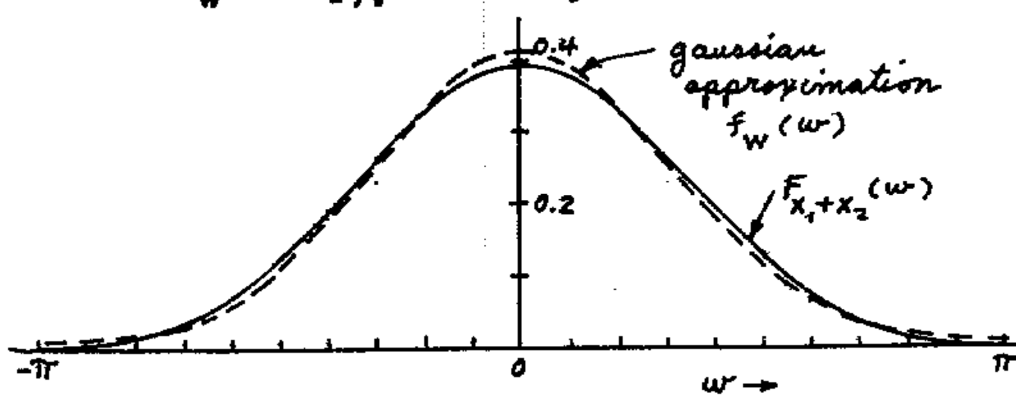
★ 4.7-1. (Continued)



This approximation is not a good one.

★ 4.7-2. Procedures are as in solution of Problem 4.7-1.

$f_{X_1+X_2}(w) = \frac{1}{4}\int_{-\infty}^{\infty} \text{rect}\left(\frac{\xi}{\pi}\right) \text{rect}\left(\frac{w-\xi}{\pi}\right) \cos(\xi)\cos(w-\xi)\, d\xi$ $\neq 0$ only for $-\pi \le w \le \pi$. For $0 \le w \le \pi$: $f_{X_1+X_2}(w) =$ $\frac{1}{4}\int_{w-\frac{\pi}{2}}^{\pi/2} \cos(\xi)\cos(w-\xi)\, d\xi = \frac{1}{8}\left[(\pi - w)\cos(w) + \sin(w)\right]$. From even symmetry $f_{X_1+X_2}(w) = \frac{1}{8}\left[(\pi + w)\right.$ $\left.\cdot \cos(w) - \sin(w)\right]$, $-\pi \le w \le 0$. Next, $\bar{x} = 0$ and $\sigma_x^2 = \frac{1}{2}\int_{-\pi/2}^{\pi/2} x^2 \cos(x)\, dx = \left(\frac{\pi^2}{4} - 2\right)$. The gaussian approximation is $f_W(w) = \left[1/\sqrt{\pi(\pi^2-8)}\right] \exp\left[-w^2/(\pi^2-8)\right]$.



This approximation is a very good one.

★ 4.7-3. $\bar{X} = \bar{X}_1 + \bar{X}_2 + \bar{X}_3 = -1 + 0.6 + 1.8 = 1.4$

$\sigma_X^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 = 2.0 + 1.5 + 0.8 = 4.3$

Thus,

$$f_X(x) \approx \frac{e^{-(x-1.4)^2/[2(4.3)]}}{\sqrt{2\pi(4.3)}} = \frac{e^{-(x-1.4)^2/8.6}}{\sqrt{8.6\pi}}$$

★ 4.7-4. (a) $f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_X(w-x)\,dx = \int_0^a \frac{2x}{a^2}\cdot\frac{2(w-x)}{a^2}\left[u(w-x) - u(w-x-a)\right]dx$

For $0 \le w < a$:

$$f_W(w) = \frac{4}{a^4}\int_0^w x(w-x)\,dx = \frac{2}{3a^4}w^3, \quad 0 \le w < a$$

For $a \le w < 2a$:

$$f_W(w) = \frac{4}{a^4}\int_{w-a}^{a} x(w-x)\,dx = \frac{2}{3a^4}\left\{3a^2\left(w - \frac{2a}{3}\right) - (w-a)^2(w+2a)\right\}, \quad a \le w < 2a$$

Thus,

$$f_W(w) = \begin{cases} \frac{2}{3a}\left(\frac{w}{a}\right)^3, & 0 \le \frac{w}{a} < 1 \\ \frac{2}{3a}\left\{3\left(\frac{w}{a}\right) - 2 - \left[\left(\frac{w}{a}\right) - 1\right]^2\left[\left(\frac{w}{a} + 2\right)\right]\right\}, & 1 \le \frac{w}{a} < 2 \\ 0, & \text{elsewhere} \end{cases} \qquad (1)$$

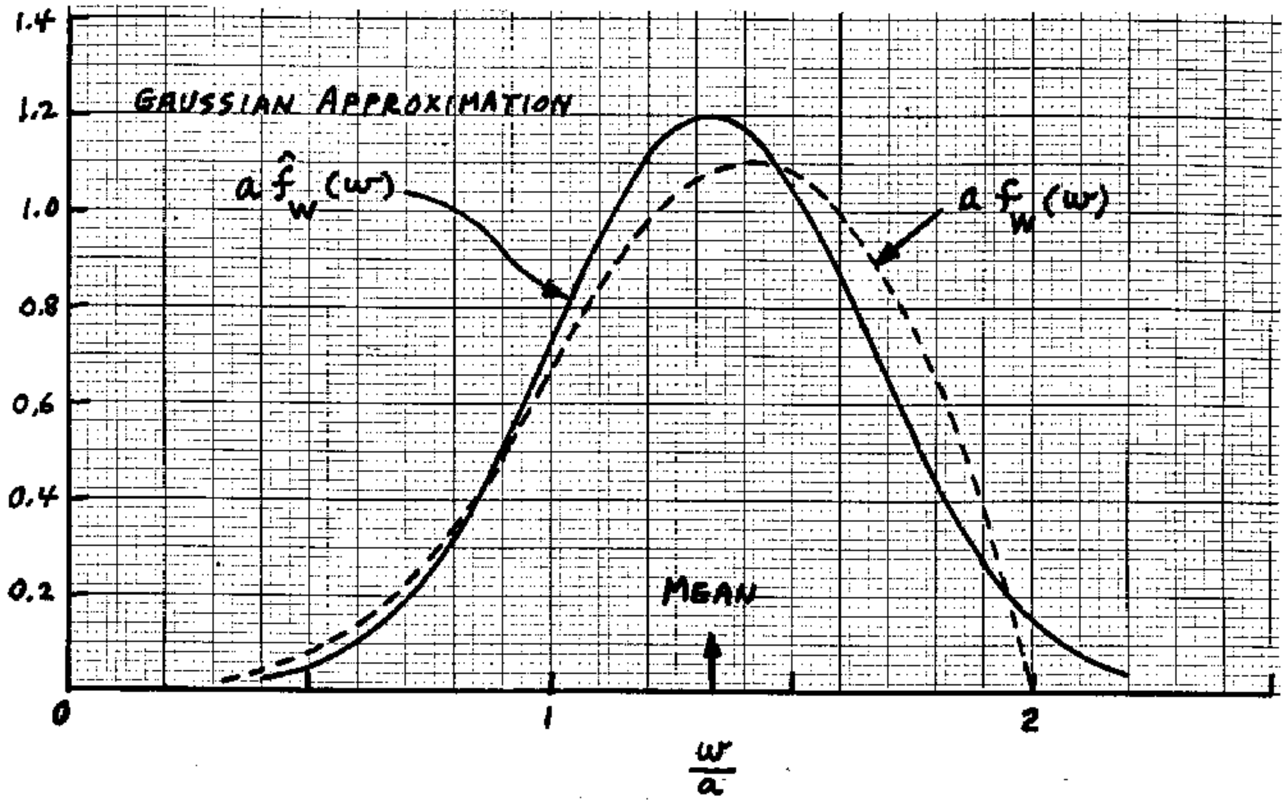
(b) $\bar{X}_i = \int_0^a \frac{2x^2}{a^2}dx = 2a/3$, $\overline{X_i^2} = \int_0^a \frac{2x^3}{a^2}dx = a^2/2$, $\sigma_{X_i}^2 = \overline{X_i^2} - \bar{X}_i^2$
$= (a^2/2) - (2a/3)^2 = a^2/18$, $\bar{W} = 2\bar{X}_i = 4a/3$, $\sigma_W^2 = 2\sigma_{X_i}^2 = a^2/9$.

Gaussian approximation: $\hat{f}_W(w) = \dfrac{e^{-(w-\frac{4a}{3})^2/2(a^2/9)}}{\sqrt{2\pi(a^2/9)}}$ or

$$a\hat{f}_W(w) = \frac{3}{\sqrt{2\pi}}\exp\left[-\left(\frac{w}{a} - \frac{4}{3}\right)^2\frac{9}{2}\right]. \qquad (2)$$

(c) Plots of $a f_W(w)$, given by (1), and $a\hat{f}_W(w)$, given by (2), are shown below.

★ 4.7-4. (Continued)



★ 4.7-5. (a) $f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_W(w-y)\,dy = \int_0^{\infty} b e^{-by} \frac{2}{a^2}(w-y)[u(w-y)-u(w-y-a)]\,dy$

For $0 \le w \le a$:

$$f_W(w) = \frac{2b}{a^2}\int_0^{w}(w-y)e^{-by}dy = \frac{2}{a}\left\{\frac{1}{ab}e^{-bw} + \left(\frac{w}{a} - \frac{1}{ab}\right)\right\}, \quad 0 \le w \le a$$

For $a < w < \infty$:

$$f_W(w) = \frac{2b}{a^2}\int_{w-a}^{w}(w-y)e^{-by}dy = \frac{2e^{-bw}}{a}\left\{\frac{1}{ab} + e^{ab}\left(1 - \frac{1}{ab}\right)\right\}, \quad a < w < \infty$$

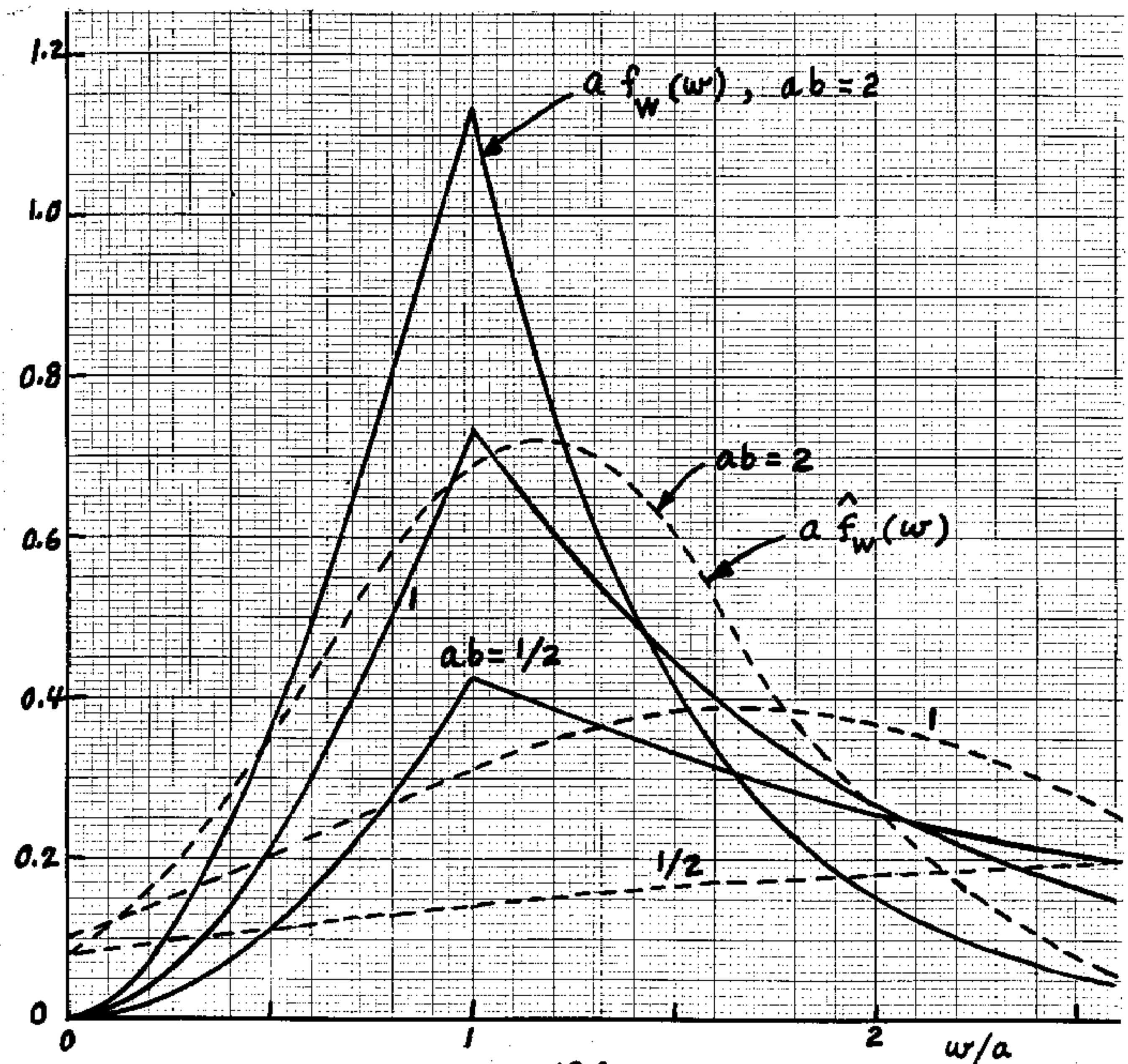
Thus,

$$f_W(w) = \begin{cases} \frac{2}{a}\left\{\frac{1}{ab}e^{-ab(w/a)} + \left(\frac{w}{a} - \frac{1}{ab}\right)\right\}, & 0 \le \frac{w}{a} \le 1 \\ \frac{2}{a}\left\{\frac{1}{ab} + e^{ab}\left(1 - \frac{1}{ab}\right)\right\} e^{-ab(w/a)}, & 1 < \frac{w}{a} < \infty \\ 0, & \text{elsewhere} \end{cases}$$

★ 4.7-5. (Continued) (b) From the solution to Prob. 4.7-4, $\bar{X} = 2a/3$, $\sigma_X^2 = a^2/18$. From (F-47) and (F-48), $\bar{Y} = 1/b$ and $\sigma_Y^2 = 1/b^2$.
Thus, $\bar{W} = \bar{X} + \bar{Y} = \frac{2a}{3} + \frac{1}{b}$, $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 = \frac{a^2}{18} + \frac{1}{b^2}$, and

$$\hat{f}_W(w) = \frac{1}{a\sqrt{2\pi\left(\frac{1}{18} + \frac{1}{a^2b^2}\right)}} \exp\left\{-\frac{\left[\frac{w}{a} - \left(\frac{2}{3} + \frac{1}{ab}\right)\right]^2}{2\left(\frac{1}{18} + \frac{1}{a^2b^2}\right)}\right\}$$

(c) Approximation is clearly better (from plots below) as ab is larger.



CHAPTER 5

5.1-1. From (5.1-1): $E[(XY)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2y^2 f_{X,Y}(x,y)\,dx\,dy$

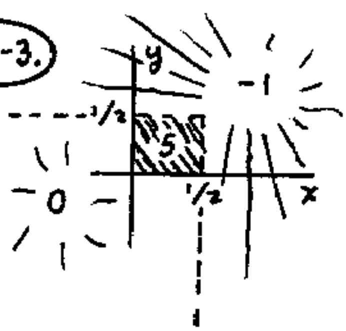
$$= \int_{y=0}^{4}\int_{x=0}^{6} \frac{x^2y^2}{24}\,dx\,dy = 64.$$

5.1-2. By using (5.1-2): $E[X_1^{n_1}X_2^{n_2}X_3^{n_3}X_4^{n_4}]$

$$= \int_{x_4=0}^{d}\int_0^{c}\int_0^{b}\int_{x_1=0}^{a} \frac{x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}}{a\,b\,c\,d}\,dx_1\,dx_2\,dx_3\,dx_4$$

$$= \frac{a^{n_1+1}\,b^{n_2+1}\,c^{n_3+1}\,d^{n_4+1}}{(n_1+1)(n_2+1)(n_3+1)(n_4+1)\,a\,b\,c\,d}.$$

5.1-3.



By using (5.1-1):

$$E[g(X,Y)] = \int_{y=0}^{1/2}\int_0^{1/2} 5(16)e^{-4x-4y}\,dx\,dy - \int_{y=0}^{\infty}\int_{1/2}^{\infty} 16e^{-4x-4y}\,dx\,dy$$

$$-\int_{y=1/2}^{\infty}\int_0^{1/2} 16e^{-4x-4y}\,dx\,dy = 6e^{-4} - 12e^{-2} + 5 \approx 3.486.$$

5.1-4. From (5.1-1): $E[e^{-2(X^2+Y^2)}] = \int_0^{\infty}\int_0^{\infty} e^{-2(x^2+y^2)}$

$$\cdot 16e^{-4(x+y)}\,dx\,dy = 16\int_0^{\infty} e^{-2x^2-4x}\,dx\int_0^{\infty} e^{-2y^2-4y}\,dy$$

$$= 16e^4\int_0^{\infty} e^{-2(x+1)^2}\,dx\int_0^{\infty} e^{-2(y+1)^2}\,dy.$$

5.1-4. (Continued) Let $\xi = 2(x+1)$, $d\xi = 2dx$ in the first integral and write $E[e^{-2(X^2+Y^2)}] =$

$$16e^4\left[\int_2^\infty e^{-\xi^2/2}d\xi \frac{1}{2}\right]^2 = 8\pi e^4\left[\int_2^\infty \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}d\xi\right]^2.$$

Next, use (2.4-3) and Appendix B to obtain

$$E[e^{-2(X^2+Y^2)}] = 8\pi e^4[1-F(2)]^2 = 8\pi e^4[1-0.9772]^2 \approx 0.713.$$

5.1-5 Here $\bar{X}_1 = 3$, $\bar{X}_2 = 6$, $\bar{X}_3 = -2$.

(a) $E[X_1+3X_2+4X_3] = \bar{X}_1+3\bar{X}_2+4\bar{X}_3 = 3+18-8 = 13.$

(b) $E[X_1X_2X_3] = \bar{X}_1\bar{X}_2\bar{X}_3$ from the fact that X_1, X_2 and X_3 are independent. Thus, $E[X_1X_2X_3] = -36.$

(c) $E[-2X_1X_2 - 3X_1X_3 + 4X_2X_3] = -2\overline{X_1X_2} - 3\overline{X_1X_3} + 4\overline{X_2X_3} = -2\bar{X}_1\bar{X}_2 - 3\bar{X}_1\bar{X}_3 + 4\bar{X}_2\bar{X}_3 = -36+18-48 = -66.$ (d) $E[X_1+X_2+X_3] = \bar{X}_1+\bar{X}_2+\bar{X}_3 = 7.$

5.1-6. From (5.1-1): $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)\frac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2}dx\,dy$

$$= \int_{-\infty}^{\infty}\frac{x^2e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}dx\int_{-\infty}^{\infty}\frac{e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}dy + \int_{-\infty}^{\infty}\frac{y^2e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}dy\int_{-\infty}^{\infty}\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}dx$$

Both double integrals are of the same form. The second factors equal 1 because they are areas of a Gaussian density. The first factors equal σ^2 because they are second moments of a Gaussian density with zero mean and variance σ^2.

Thus $E[g(X,Y)] = E[(X^2+Y^2)] = 2\sigma^2.$

(5.1-7.) Here $\bar{X}=2$, $\bar{Y}=4$, $\overline{X^2}=E[X^2]=8$, $\overline{Y^2}=E[Y^2]=25$.

(a) $E[W]=E[3X-Y]=3\bar{X}-\bar{Y}=6-4=2$.

(b) $E[W^2]=E[(3X-Y)^2]=E[9X^2-6XY+Y^2]=9\overline{X^2}$
$-6\overline{XY}+\overline{Y^2}=9(8)-6\bar{X}\bar{Y}+25=97-6(2)4=49$.

(c) $\sigma_W^2=E[(W-\bar{W})^2]=E[W^2-2W\bar{W}+\bar{W}^2]=\overline{W^2}-\bar{W}^2$
$=49-4=45$.

(5.1-8.) Here $\bar{X}=1$, $\bar{Y}=2$, $\sigma_X^2=4$, $\sigma_Y^2=1$, $\rho_{XY}=0.4$.

(a) $\bar{V}=E[V]=E[-X+2Y]=-\bar{X}+2\bar{Y}=-1+4=3$.
$\bar{W}=E[W]=E[X+3Y]=\bar{X}+3\bar{Y}=1+6=7$.

(b) $\sigma_V^2=E[(V-\bar{V})^2]=E[(-X+2Y+\bar{X}-2\bar{Y})^2]$
$=E[\{-(X-\bar{X})+2(Y-\bar{Y})\}^2]=\overline{(X-\bar{X})^2}-4\overline{(X-\bar{X})(Y-\bar{Y})}$
$+4\overline{(Y-\bar{Y})^2}=\sigma_X^2-4C_{XY}+4\sigma_Y^2$. But $\rho_{XY}=\dfrac{C_{XY}}{\sigma_X\sigma_Y}$
$=0.4$, so $C_{XY}=0.4\sigma_X\sigma_Y=0.8$ and
$\sigma_V^2=4-4(0.8)+4=4.8$. Next, $\sigma_W^2=E[(W-\bar{W})^2]$
$=E[\{(X-\bar{X})+3(Y-\bar{Y})\}^2]=\sigma_X^2+6C_{XY}+9\sigma_Y^2=17.8$.

(c) $R_{VW}=E[VW]=E[(-X+2Y)(X+3Y)]=-\overline{X^2}-\overline{XY}+6\overline{Y^2}$
$=-[\sigma_X^2+\bar{X}^2]-R_{XY}+6[\sigma_Y^2+\bar{Y}^2]=-[4+1]-[C_{XY}$
$+\bar{X}\bar{Y}]+6[1+4]=-5-[0.8+2]+6(5)=22.2$.

(d) $\rho_{VW}=C_{VW}/\sigma_V\sigma_W=(R_{VW}-\bar{V}\bar{W})/\sigma_V\sigma_W$
$=(22.2-21)/\sqrt{4.8(17.8)}\approx 0.1298$.

(5.1-9.) $\rho = E[(X-\bar{X})(Y-\bar{Y})]/\sigma_X\sigma_Y$.

(a) Now $\bar{Y} = E[Y] = a\bar{X} + b$, $(Y-\bar{Y}) = a(X-\bar{X})$

$$\sigma_Y^2 = E[(Y-\bar{Y})^2] = a^2 E[(X-\bar{X})^2] = a^2\sigma_X^2$$

$$\sigma_Y = |a|\sigma_X.$$

Thus, $\rho = E[(X-\bar{X})a(X-\bar{X})]/\sigma_X\sigma_Y = a\,\sigma_X^2/\sigma_X\sigma_Y$

$$= a\,\sigma_X^2/|a|\sigma_X^2 = a/|a| = \left.\begin{matrix} 1 \text{ if } a>0 \\ -1 \text{ if } a<0 \end{matrix}\right\} \text{any } b.$$

(b) $C_{XY} = E[(X-\bar{X})(Y-\bar{Y})] = E[(X-\bar{X})a(X-\bar{X})] = a\sigma_X^2$.

★(5.1-10.) Since $E[\{(X-\bar{X})\alpha + (Y-\bar{Y})\}^2] \geq 0$ for any real number α, expand to get $E[\alpha^2(X-\bar{X})^2 + 2\alpha(X-\bar{X})(Y-\bar{Y}) + (Y-\bar{Y})^2] = \alpha^2\sigma_X^2 + 2\alpha\mu_{11} + \sigma_Y^2 \geq 0$. To remain nonnegative this quadratic form in α must not have any real zeros. Thus, the discriminant must be negative or zero to force all roots to be imaginary (or complex). That is, we require $(2\mu_{11})^2 - 4\sigma_X^2\sigma_Y^2 \leq 0$ so

$$\frac{\mu_{11}^2}{\sigma_X^2\sigma_Y^2} \leq 1 \quad \text{or} \quad \frac{|\mu_{11}|}{\sigma_X\sigma_Y} \leq 1.$$

But $\sigma_X = \sqrt{\mu_{20}}$ and $\sigma_Y = \sqrt{\mu_{02}}$ so $|\rho| = \dfrac{|\mu_{11}|}{\sqrt{\mu_{02}\mu_{20}}} \leq 1$.

(5.1-11.) Here $f_{X,Y}(x,y) = u(x)u(y)\,16\exp[-4(x+y)]$. From (5.1-5) the second-order moments are found.

$$m_{20} = \int_0^\infty\int_0^\infty x^2 16 e^{-4x-4y}\,dx\,dy = 16\int_0^\infty x^2 e^{-4x}dx\int_0^\infty e^{-4y}dy$$

$$= 1/8 \text{ from (C-45) and (C-47).}$$

5.1-11. (Continued) In the same way we find m_{02}.

$$m_{02} = \int_0^\infty \int_0^\infty y^2 16\, e^{-4y-4x}\, dx\, dy = 1/8.$$

$$m_{11} = \int_0^\infty \int_0^\infty xy\, 16\, e^{-4y-4x}\, dx\, dy = 1/16 \text{ from use of (C-46).}$$

For central moments first find means.

$m_{10} = \bar{X} = \int_0^\infty \int_0^\infty x\, 16\, e^{-4x-4y}\, dx\, dy = 1/4$ from (C-45) and (C-46). $m_{01} = \bar{Y} = \int_0^\infty \int_0^\infty y\, 16\, e^{-4x-4y}\, dx\, dy = 1/4.$

Thus,

$$\mu_{20} = \sigma_X^2 = m_{20} - (m_{10})^2 = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$$

$$\mu_{02} = \sigma_Y^2 = m_{02} - (m_{01})^2 = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$$

$$\mu_{11} = R_{XY} - \bar{X}\,\bar{Y} \;[\text{from } (5.1\text{-}14)] = m_{11} - m_{10}\, m_{01}$$

$$= \frac{1}{16} - \frac{1}{16} = 0.$$

5.1-12. (a) $m_{20} = \int_{-3}^{3} \int_{-1}^{1} \frac{x^2(x+y)^2}{40}\, dx\, dy = \frac{9}{25} = 0.36.$

$$m_{02} = \int_{-3}^{3} \int_{-1}^{1} \frac{y^2(x+y)^2}{40}\, dx\, dy = \frac{129}{25} = 5.16.$$

$$m_{11} = \int_{-3}^{3} \int_{-1}^{1} \frac{xy(x+y)^2}{40}\, dx\, dy = \frac{6}{10} = 0.6.$$

(b) $\sigma_X^2 = m_{20} - (m_{10})^2, \quad \sigma_Y^2 = m_{02} - (m_{01})^2$

$m_{10} = \int_{-3}^{3} \int_{-1}^{1} \frac{x(x+y)^2}{40}\, dx\, dy = 0, \quad m_{01} = \int_{-3}^{3} \int_{-1}^{1} \frac{y(x+y)^2}{40}\, dx\, dy = 0.$

$\sigma_X^2 = m_{20} = 0.36, \quad \sigma_Y^2 = m_{02} = 5.16.$

(c) Use (5.1-17): $\rho_{XY} = E[(X-\bar{X})(Y-\bar{Y})]/\sigma_X\sigma_Y = C_{XY}/\sigma_X\sigma_Y =$ $[R_{XY} - \bar{X}\,\bar{Y}]/\sigma_X\sigma_Y = [m_{11} - m_{10}m_{01}]/\sqrt{m_{20}m_{01}} \approx 0.440.$

(5.1-13.) $m_{03} = \int_{-3}^{3}\int_{-1}^{1} \frac{y^3(x+y)^2}{40}\,dx\,dy = 0.$

$m_{30} = \int_{-3}^{3}\int_{-1}^{1} \frac{x^3(x+y)^2}{40}\,dx\,dy = 0.$

$m_{12} = \int_{-3}^{3}\int_{-1}^{1} \frac{xy^2(x+y)^2}{40}\,dx\,dy = 0.$

$m_{21} = \int_{-3}^{3}\int_{-1}^{1} \frac{x^2y(x+y)^2}{40}\,dx\,dy = 0.$

(5.1-14.) From (4.3-2) $f_{X,Y}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i,y_j)\,\delta(x-x_i)\,\delta(y-y_j).$

(a) Use (5.1-5). $m_{nk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k \sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i,y_j)$

$\cdot\,\delta(x-x_i)\,\delta(y-y_j)\,dx\,dy = \sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i,y_j)\,x_i^n\,y_j^k\,.$

(b) Use (5.1-10). $\mu_{nk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})^n (y-\bar{Y})^k$

$\cdot \sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i,y_j)\,\delta(x-x_i)\,\delta(y-y_j)\,dx\,dy$

$= \sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i,y_j)\,(x_i-\bar{X})^n\,(y_j-\bar{Y})^k\,.$

(5.1-15.) (a) Use (5.1-6). $R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f_{X,Y}(x,y)$

$dx\,dy = 0.15(-1)(0) + 0.1(0)(0) + 0.1(0)(2) + 0.4(1)(-2)$

$+ 0.2(1)(1) + 0.05(1)(3) = -0.45.$

(b) From (5.1-14) $C_{XY} = E[(X-\bar{X})(Y-\bar{Y})] = R_{XY} - \bar{X}\bar{Y}.$

First find $\bar{X}$ and $\bar{Y}$ from (5.1-5).

$\bar{X} = 0.15(-1) + 0.1(0) + 0.1(0) + 0.4(1) + 0.2(1) + 0.05(1)$

$= 0.5.$

(5.1-15) (Continued) $\overline{Y} = 0.15(0) + 0.1(0) + 0.1(2) + 0.4(-2) + 0.2(1) + 0.05(3) = -0.25.$

Thus, $C_{XY} = -0.45 - (-0.125) = -0.325.$

(c) $\rho_{XY} = C_{XY}/\sigma_X \sigma_Y$. We must find σ_X and σ_Y.

$$\overline{X^2} = 0.15(-1)^2 + 0.1(0)^2 + 0.1(0)^2 + 0.4(1)^2 + 0.2(1)^2 + 0.05(1)^2 = 0.80.$$

$$\overline{Y^2} = 0.15(0)^2 + 0.1(0)^2 + 0.1(2)^2 + 0.4(-2)^2 + 0.2(1)^2 + 0.05(3)^2 = 2.65.$$

$$\sigma_X^2 = \overline{X^2} - (\overline{X})^2 = 0.8 - (0.5)^2 = 0.55$$

$$\sigma_Y^2 = \overline{Y^2} - (\overline{Y})^2 = 2.65 - (-0.25)^2 = 2.5875.$$

$$\rho_{XY} = \frac{-0.325}{\sqrt{0.55(2.5875)}} \approx -0.272.$$

(d) $C_{XY} \neq 0$ so X and Y are not uncorrelated. $R_{XY} \neq 0$ so X and Y are not orthogonal.

(5.1-16) From (5.1-6) $R_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$

$= 0.4(-\alpha)(2) + 0.3(\alpha)(2) + 0.1(\alpha)(\alpha) + 0.2(1)(1)$

$= 0.1\alpha^2 - 0.2\alpha + 0.2$. Now for a minimum: $dR_{XY}/d\alpha = 0.2\alpha - 0.2 = 0$ when $\alpha = 1$. To check that a minimum does occur $d^2R_{XY}/d\alpha^2\big|_{\alpha=1}$ must be positive: $d^2R_{XY}/d\alpha^2 = 0.2 > 0$. At $\alpha = 1$ $R_{XY} = 0.1 - 0.2 + 0.2 = 0.1$. X and Y are not orthogonal because $R_{XY} \neq 0$.

5.1-17. Use (5.1-14). $C_{XY} = R_{XY} - \bar{X}\bar{Y}$. We first find $\bar{X}, \bar{Y}$ and R_{XY}. From (5.1-5) $\bar{X} = m_{10} = 0.3(\alpha) + 0.5(-\alpha) + 0.2(-2) = -0.2\alpha - 0.4$, $\bar{Y} = m_{01} = 0.3(\alpha) + 0.5(4) + 0.2(-2) = 0.3\alpha + 1.6$, $R_{XY} = m_{11} = 0.3(\alpha)(\alpha) + 0.5(-\alpha)(4) + 0.2(-2)(-2) = 0.3\alpha^2 - 2\alpha + 0.8$.

Thus, $C_{XY} = 0.3\alpha^2 - 2\alpha + 0.8 - (-0.2\alpha - 0.4)(0.3\alpha + 1.6)$

$$= 0.36\alpha^2 - 1.56\alpha + 1.44$$

$\dfrac{dC_{XY}}{d\alpha} = 0.72\alpha - 1.56 = 0$ when $\alpha = \dfrac{1.56}{0.72} \approx 2.17$.

$\dfrac{d^2C_{XY}}{d\alpha^2} = 0.72 > 0$ so $\alpha \approx 2.17$ is a minimum.

$C_{XY(\text{min.})} = 0.36\left(\dfrac{1.56}{0.72}\right)^2 - 1.56\left(\dfrac{1.56}{0.72}\right) + 1.44 = -0.25.$

X and Y are not uncorrelated because $C_{XY(\text{min.})} \neq 0$.

5.1-18. (a) $R_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy = \int_0^3\int_0^2 \dfrac{x^2y^2}{9}\,dx\,dy = 8/3$. $E[X] = \int_0^3\int_0^2 \dfrac{x^2y}{9}\,dx\,dy = 4/3$, $E[Y] = \int_0^3\int_0^2 \dfrac{xy^2}{9}\,dx\,dy = 2$. Thus, since $R_{XY} = 8/3 = E[X]E[Y] = 2(4/3) = 8/3$ we have X and Y uncorrelated from (5.1-7).

(b) From marginal densities $f_X(x) = \int_0^3 \dfrac{xy}{9}\,dy = \dfrac{x}{2}$, $0 < x < 2$ and $f_Y(y) = \int_0^2 \dfrac{xy}{9}\,dx = \dfrac{2y}{9}$, $0 < y < 3$, we have $f_X(x)\,f_Y(y) = xy/9$, $0 < x < 2$ and $0 < y < 3$ (and zero elsewhere). Thus, $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$ and X and Y are statistically independent.

(5.1-19.) (a) Use (5.1-5). $m_{10} = E[X] = \int_0^3\int_0^2 x \frac{2}{43}(x+0.5y)^2 dx\, dy$

$= 57/43 \approx 1.326$. $m_{01} = E[Y] = \int_0^3\int_0^2 y \frac{2}{43}(x+0.5y)^2 dx\, dy$

$= 321/172 \approx 1.866$. $m_{20} = E[X^2] = \int_0^3\int_0^2 x^2 \frac{2}{43}(x+0.5y)^2 dx\, dy$

$= 432/215 \approx 2.009$. $m_{02} = E[Y^2] = \int_0^3\int_0^2 y^2 \frac{2}{43}(x+0.5y)^2 dx\, dy$

$= 888/215 \approx 4.130$. $m_{11} = E[XY] = \int_0^3\int_0^2 xy \frac{2}{43}(x+0.5y)^2 dx\, dy$

$= 417/172 \approx 2.424$.

(b) From (5.1-14) $C_{XY} = R_{XY} - E[X]E[Y] = m_{11} - m_{10}m_{01}$

$= \frac{417}{172} - \frac{57}{43}\left(\frac{321}{172}\right) = \frac{-366}{(43)^2 4} \approx -0.0495$.

(c) X and Y are not uncorrelated because $C_{XY} \neq 0$.

(5.1-20.) $R_{VW} = E[VW] = E[(X+aY)(X-aY)] = E[X^2 - a^2Y^2]$

$= E[X^2] - a^2E[Y^2] = 0$ if $a^2 = E[X^2]/E[Y^2]$ or

$|a| = \sqrt{m_{20}/m_{02}}$. The sign of a is unimportant.

★(5.1-21.) V and W are linear transformations of Gaussian random variables X and Y so they are also Gaussian from (5.5-15). Thus, V and W are independent if uncorrelated. Since $\bar{V} = \bar{X} + a\bar{Y}$ $\bar{W} = \bar{X} - a\bar{Y}$, then $C_{VW} = E[(V-\bar{V})(W-\bar{W})] = E[\{X + aY - \bar{X} - a\bar{Y}\}\{X - aY - \bar{X} + a\bar{Y}\}] = E[\{(X-\bar{X}) + a(Y-\bar{Y})\} \cdot \{(X-\bar{X}) - a(Y-\bar{Y})\}] = E[(X-\bar{X})^2 - a^2(Y-\bar{Y})^2] = \sigma_X^2 - a^2\sigma_Y^2$
$= 0$ if $a^2 = \sigma_X^2/\sigma_Y^2$.

★5.1-22. (a) $E[Y] = E[X_1 - 2X_2 + 3X_3] = \bar{X}_1 - 2\bar{X}_2 + 3\bar{X}_3 = 1 - 2(-3) + 3(1.5) = 11.5$. (b) $\sigma_Y^2 = E[(Y-\bar{Y})^2] = E[\{(X_1-\bar{X}_1) - 2(X_2-\bar{X}_2)^2 + 3(X_3-\bar{X}_3)\}^2] = E[(X_1-\bar{X}_1)^2 + 4(X_2-\bar{X}_2)^2 + 9(X_3-\bar{X}_3)^2 - 4(X_1-\bar{X}_1)(X_2-\bar{X}_2) + 6(X_1-\bar{X}_1)(X_3-\bar{X}_3) - 12(X_2-\bar{X}_2)(X_3-\bar{X}_3)] = \sigma_{X_1}^2 + 4\sigma_{X_2}^2 + 9\sigma_{X_3}^2 - 4C_{X_1X_2} + 6C_{X_1X_3} - 12C_{X_2X_3}$. But $C_{X_iX_j} = 0$ for $i \neq j$ because X_1, X_2, and X_3 are uncorrelated. Thus, $\sigma_Y^2 = \sigma_{X_1}^2 + 4\sigma_{X_2}^2 + 9\sigma_{X_3}^2 = E[X_1^2] - (\bar{X}_1)^2 + 4E[X_2^2] - 4(\bar{X}_2)^2 + 9E[X_3^2] - 9(\bar{X}_3)^2 = 2.5 - (1)^2 + 4(11) - 4(-3)^2 + 9(3.5) - 9(1.5)^2 = 20.75$.

5.1-23. (a) $E[W] = \bar{W} = E[(aX+3Y)^2] = E[a^2X^2 + 6aXY + 9Y^2] = a^2E[X^2] + 6aR_{XY} + 9E[Y^2]$. Since X and Y are zero mean variables $E[X^2] = \sigma_X^2 = 4$ and $E[Y^2] = \sigma_Y^2 = 16$ while $R_{XY} = \rho\sigma_X\sigma_Y = -0.5(2)(4) = -4$. Thus, $\bar{W} = 4a^2 - 24a + 144$. By differentiation of $\bar{W}$: $d\bar{W}/da = 8a - 24 = 0$ when $a = 3$. (This is a minimum because $d^2\bar{W}/da^2 = 8 > 0$.) (b) $\bar{W}_{min} = 4(3)^2 - 24(3) + 144 = 108$.

5.1-24. $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$

$$= \int_{x=-r}^{r}\int_{y=-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{x^2+y^2}{\pi r^2}\,dx\,dy = \frac{2}{\pi r^2}\int_{-r}^{r}\left[x^2\sqrt{r^2-x^2} + \frac{(r^2-x^2)^{3/2}}{3}\right]dx = r^2/2.$$

★ 5.1-25. (a) From Problem 4.4-8 we have

$$E\{g(X,Y)|y_a<Y\le y_b\} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy}.$$

Let $y_a = y - \Delta y$, $y_b = y + \Delta y$ where $\Delta y \to 0$. (b)

$$E\{g(X,Y)|Y=y\} = \lim_{\Delta y\to 0} \frac{\int_{y-\Delta y}^{y+\Delta y}\int_{-\infty}^{\infty} g(x,\xi) f_{X,Y}(x,\xi)\,dx\,d\xi}{\int_{y-\Delta y}^{y+\Delta y} f_Y(\xi)\,d\xi}$$

$$= \lim_{\Delta y\to 0} \frac{\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,2\Delta y}{f_Y(y)\,2\Delta y} = \frac{\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

$$= \int_{-\infty}^{\infty} g(x,y) f_X(x|y)\,dx \quad \text{from (4.4-12).}$$

5.1-26. (a) $\rho = \frac{C_{XY}}{\sigma_X\sigma_Y}$ so $C_{XY} = \rho\sigma_X\sigma_Y = -\frac{2}{3}\sqrt{6}\sqrt{9} = -2\sqrt{6}$

(b) $C_{XY} = R_{XY} - \bar{X}\bar{Y}$ so $R_{XY} = C_{XY} + \bar{X}\bar{Y} = -2\sqrt{6} + 2 = 2(1-\sqrt{6})$

(c) $m_{20} = \overline{X^2} = \sigma_X^2 + \bar{X}^2 = 6 + 1 = 7$

$m_{02} = \overline{Y^2} = \sigma_Y^2 + \bar{Y}^2 = 9 + 4 = 13.$

5.1-27. (a) $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$. $\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = \frac{19}{2} - 4 = \frac{11}{2}$. $R_{XY} = \overline{XY} = C_{XY} + \bar{X}\bar{Y} = -\frac{1}{2\sqrt{3}} + \frac{1}{2}(2) = \frac{1}{2}(2 - \frac{1}{\sqrt{3}})$.

$\rho = C_{XY}/\sigma_X\sigma_Y = \frac{-1/2\sqrt{3}}{(3/2)(\sqrt{11/2})} = \frac{-1}{3}\sqrt{\frac{2}{33}}$. (b) $\bar{W} = \overline{(X+3Y)^2 + 2X + 3} = 3 + 2\bar{X} + \overline{X^2} + 6\overline{XY} + 9\overline{Y^2} = \frac{5}{2} + 6\left(\frac{1}{2}\right)\left(2 + \frac{1}{\sqrt{3}}\right) + 9\,\frac{19}{2} + 2\left(\frac{1}{2}\right) + 3 = 98 - \sqrt{3}.$

5.1-28 (a) $R_{XY} = \bar{X}\,\bar{Y} = 3/4$. (b) $R_{XW} = E[X(X-2Y+1)] = \overline{X^2} - 2\overline{XY} + \bar{X} = 4 - 2(0.75) + 0.75 = 3.25$. (c) $R_{YW} = E[Y(X-2Y+1)] = \overline{YX} - 2\overline{Y^2} + \bar{Y} = 0.75 - 2(5) + 1 = -8.25$. (d) $C_{XY} = R_{XY} - \bar{X}\,\bar{Y} = \bar{X}\,\bar{Y} - \bar{X}\,\bar{Y} = 0$. (e) Yes, X and Y are uncorrelated because $C_{XY} = 0$.

5.1-29 Because of independence $\mu_{22} = E[(X-\bar{X})^2(Y-\bar{Y})^2] = E[(X-\bar{X})^2]E[(Y-\bar{Y})^2] = \sigma_X^2\sigma_Y^2 = \mu_{20}\mu_{02}$. Now

5.1-29. (Continued) $m_{11} = \overline{XY} = \bar{X}\,\bar{Y} = -6$ so $\bar{Y} = -6/\bar{X} = -6/m_{10} = -6/2 = -3$, $\mu_{20} = \sigma_X^2 = \overline{X^2} - \bar{X}^2 = m_{20} - m_{10}^2 = 14 - 4 = 10$, $\mu_{02} = \sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = m_{02} - m_{01}^2 = 12 - (-3)^2 = 3$. Thus, $\mu_{22} = 10(3) = 30$

5.1-30. $m_{nk} = \int_0^1\int_0^1 x^n y^k x(y + \frac{3}{2})\,dx\,dy$

$= \int_0^1 y^k(y+\frac{3}{2})\,dy \int_0^1 x^{n+1}dx = \left[\frac{1}{k+2} + \frac{3/2}{k+1}\right]\frac{1}{(n+2)}$

$$= \frac{5k+8}{2(k+2)(k+1)(n+2)}.$$

5.1-31. From Problem 5.1-30's solution $\bar{X} = 2/3$ and $\bar{Y} = 13/24$. $\mu_{nk} = \int_0^1\int_0^1 (x-\bar{X})^n(y-\bar{Y})^k x(y+\frac{3}{2})\,dx\,dy$

$= \int_0^1 (x-\frac{2}{3})^n x\,dx \int_0^1 (y - \frac{13}{24})^k(y+\frac{3}{2})\,dy$. These two integrals readily reduce to give

$$\mu_{nk} = \left[\frac{3n+5+(-2)^{n+2}}{3^{n+2}(n+1)(n+2)}\right]\left[\frac{(11)^{k+1}(60k+109) - (-13)^{k+1}(36k+85)}{24^{k+2}(k+1)(k+2)}\right].$$

5.1-32 $m_{10} = E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy = 0.1(0) + 0.12(4) + 0.05(0) + 0.25(2) + 0.3(2) + 0.18(4) = 2.30.$ $m_{01} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy = E[Y] = 0.1(0) + 0.12(0) + 0.05(1) + 0.25(1) + 0.3(3) + 0.18(3) = 1.74.$

$m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy = 0.1(0)0 + 0.12(4)0 + 0.05(0)1 + 0.25(2)1 + 0.3(2)3 + 0.18(4)3 = 4.46.$ $m_{20} = E[X^2] = 0.1(0^2) + 0.12(4^2) + 0.05(0^2) + 0.25(2^2) + 0.3(2^2) + 0.18(4^2) = 7.00.$

$m_{02} = E[Y^2] = 0.1(0^2) + 0.12(0^2) + 0.05(1^2) + 0.25(1^2) + 0.3(3^2) + 0.18(3^2) = 4.62.$ X and Y are not orthogonal, since $R_{XY} \neq 0$. X and Y are not uncorrelated, since $R_{XY} \neq \bar{X}\bar{Y} = m_{10}m_{01}$.

5.1-33 $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 9 - 4 = 5.$ $\bar{Y} = \overline{-1.5X + 2} = -1.5\bar{X} + 2 = -1.5(-2) + 2 = 5.$ $\overline{Y^2} = \overline{[(-1.5X+2)^2]} = \frac{9}{4}\overline{X^2} - 6\bar{X} + 4 = \frac{9}{4}(9) - 6(-2) + 4 = \frac{145}{4}.$

$\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = \frac{145}{4} - 25 = \frac{45}{4}.$ $R_{XY} = \overline{XY} = \overline{X(-\frac{3}{2}X + 2)} = -\frac{3}{2}\overline{X^2} + 2\bar{X} = -\frac{3}{2}(9) + 2(-2) = \frac{-35}{2}.$

5.1-34. $\bar{W} = \overline{2X+Y} = 2\bar{X} + \bar{Y} = 0 - 1 = -1,$ $\bar{U} = \overline{-X-3Y} = -\bar{X} - 3\bar{Y} = 3.$

$\overline{W^2} = \overline{(2X+Y)^2} = 4\overline{X^2} + 4\overline{XY} + \overline{Y^2} = 4(2) + 4(-2) + 4 = 4,$ $\overline{U^2} = \overline{(-X-3Y)^2} = \overline{X^2} + 6\overline{XY} + 9\overline{Y^2} = 2 + 6(-2) + 9(4) = 26,$ $R_{WU} = \overline{WU} = \overline{(2X+Y)(-X-3Y)} = -2\overline{X^2} - 7\overline{XY} - 3\overline{Y^2} = -2,$ $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 2 - 0 = 2,$ $\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = 4 - 1 = 3.$

5.1-35. $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 4 - 1 = 3,$ $\sigma_Y^2 = \frac{11}{4} - \frac{1}{4} = \frac{5}{2},$ $R_{XY} = \overline{XY} = \bar{X}\bar{Y} = -1/2,$ $C_{XY} = \overline{(X-\bar{X})(Y-\bar{Y})} = \overline{(X-\bar{X})}\,\overline{(Y-\bar{Y})} = 0,$ $\bar{W} = 3\overline{X^2} + 2\bar{Y} + 1 = 12,$ $R_{WY} = \overline{WY} = 3\overline{X^2Y} + 2\overline{Y^2} + \bar{Y} = 3(4)(-\frac{1}{2}) + 2(\frac{11}{4}) + (-\frac{1}{2}) = -1.$

(5.1-36.) $R_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} xy\,k\cos^2(\pi xy/2)\,dx\,dy$
$= k\int_{-1}^{1} x\int_{-1}^{1} y\left[\frac{1}{2}+\frac{1}{2}\cos(\pi xy)\right]dy\,dx = 0$, $E[X] = \int_{-1}^{1}\int_{-1}^{1} xk\cos^2(\pi xy/2)\,dx\,dy$
$= 0$. Similarly, $E[Y]=0$. Thus, $R_{XY}=0$ so X and Y are uncorrelated. Since $E[X]$ and $E[Y]$ are zero, then $\rho=0$ also.

(5.1-37.) $E[(aX-Y)^2] = a^2\overline{X^2} - 2a\overline{XY} + \overline{Y^2} \geq 0$. To be non-negative the discriminant of the quadratic in the variable a must be negative or zero. Thus, $(-2\overline{XY})^2 - 4(\overline{X^2}\,\overline{Y^2}) \leq 0$ is necessary. Thus, $(\overline{XY})^2 \leq \overline{X^2}\,\overline{Y^2}$ or $[E(XY)]^2 \leq E(X^2)E(Y^2)$.

(5.1-38.) $\overline{(X+Y)^2} = \overline{X^2} + \overline{Y^2} + 2\overline{XY}$, $\left(\sqrt{\overline{X^2}} + \sqrt{\overline{Y^2}}\right)^2 = \overline{X^2} + \overline{Y^2} + 2\sqrt{\overline{X^2}}\sqrt{\overline{Y^2}}$. Combining these two results: $\overline{(X+Y)^2} = \left(\sqrt{\overline{X^2}} + \sqrt{\overline{Y^2}}\right)^2 - 2\sqrt{\overline{X^2}}\sqrt{\overline{Y^2}} + 2\overline{XY}$. Now use the cosine inequality in the last term $\overline{(X+Y)^2} \leq \left(\sqrt{\overline{X^2}} + \sqrt{\overline{Y^2}}\right)^2 - 2\sqrt{\overline{X^2}}\sqrt{\overline{Y^2}} + 2\sqrt{\overline{X^2}}\sqrt{\overline{Y^2}}$ or $\sqrt{\overline{(X+Y)^2}} \leq \sqrt{\overline{X^2}} + \sqrt{\overline{Y^2}}$.

★(5.2-1.) Use (5.2-2). $\Phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\cdot e^{j\omega_1 x + j\omega_2 y}\,dx\,dy = \int_0^{\infty}\int_0^{\infty} 16\, e^{-4x-4y+j\omega_1 x + j\omega_2 y}\,dx\,dy$
$= 4\int_0^{\infty} e^{(j\omega_1-4)x}\,dx\; 4\int_0^{\infty} e^{j(\omega_2-4)y}\,dy = \frac{16}{(4-j\omega_1)(4-j\omega_2)}\cdot\left(\begin{matrix}\text{from}\\ (C-45)\end{matrix}\right)$

★ **5.2-2.** From (5.2-7) $\Phi_{X_1,\cdots,X_N}(\omega_1,\cdots,\omega_N) =$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1,\cdots,X_N}(x_1,\cdots,x_N)\, e^{j\omega_1 x_1+\cdots+j\omega_N x_N}\,dx_1\cdots dx_N.$$

But for independent random variables

$$f_{X_1,\cdots,X_N}(x_1,\cdots,x_N) = f_{X_1}(x_1)\cdots f_{X_N}(x_N)$$

so

$$\Phi_{X_1,\cdots,X_N}(\omega_1,\cdots,\omega_N) = \prod_{i=1}^{N}\int_{-\infty}^{\infty} f_{X_i}(x_i)\,e^{j\omega_i x_i}\,dx_i$$

$$= \prod_{i=1}^{N}\Phi_{X_i}(\omega_i).$$

★ **5.2-3.** $\left|\Phi_{X_1,\cdots,X_N}(\omega_1,\cdots,\omega_N)\right| = \left| E\left[e^{j\sum_{i=1}^{N}\omega_i X_i}\right]\right|$

$$= \left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1,\cdots,X_N}(x_1,\cdots,x_N)\,e^{j\sum_{i=1}^{N}\omega_i x_i}\,dx_1\cdots dx_N\right|$$

$$\le \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|f_{X_1,\cdots,X_N}(x_1,\cdots,x_N)\,e^{j\sum_{i=1}^{N}\omega_i x_i}\right|dx_1\cdots dx_N$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1,\cdots,X_N}(x_1,\cdots,x_N)\,dx_1\cdots dx_N = \Phi_{X_1,\cdots,X_N}(0,\cdots,0) = 1.$$

★ **5.2-4.**

$$\Phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_X^2}-\frac{2\rho xy}{\sigma_X\sigma_Y}+\frac{y^2}{\sigma_Y^2}\right]}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\,e^{j\omega_1 x+j\omega_2 y}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_X^2}\right]+j\omega_1 x}}{2\pi\sigma_X\sqrt{1-\rho^2}}\underbrace{\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{y^2}{\sigma_Y^2}-\frac{2\rho xy}{\sigma_X\sigma_Y}\right]+j\omega_2 y}}{\sigma_Y}\,dy}_{\text{Define to be } I_y}\,dx$$

Let $\xi = y/\sigma_Y$, $d\xi = dy/\sigma_Y$

★ 5.2-4. (Continued)

$$I_y = \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\xi^2 - \frac{2\rho x \xi}{\sigma_X} - j2(1-\rho^2)\omega_2 \sigma_Y \xi\right]} d\xi$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\xi^2 - 2\{j(1-\rho^2)\omega_2\sigma_Y + \frac{\rho x}{\sigma_X}\}\xi\right]} d\xi$$

Let $p = 1/2(1-\rho^2)$, $q = [j(1-\rho^2)\omega_2\sigma_Y + \frac{\rho x}{\sigma_X}]/(1-\rho^2)$.

$$I_y = \int_{-\infty}^{\infty} e^{-p\xi^2 + q\xi} d\xi = \text{a known integral (C-51)}$$

$$= \sqrt{\pi/p}\; e^{+q^2/4p} = \sqrt{2\pi(1-\rho^2)}\; e^{\frac{[j(1-\rho^2)\omega_2\sigma_Y + (\rho x/\sigma_X)]^2}{2(1-\rho^2)}}$$

Thus, on substituting I_y into the expression for $\Phi_{X,Y}(\omega_1,\omega_2)$ and letting $\xi = x/\sigma_X$, $d\xi = dx/\sigma_X$:

$$\Phi_{X,Y}(\omega_1,\omega_2) = \frac{e^{-\frac{1}{2}(1-\rho^2)\omega_2^2\sigma_Y^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2} + j(\rho\omega_2\sigma_Y + \omega_1\sigma_X)\xi} d\xi$$

$$= e^{-\frac{1}{2}[\sigma_X^2\omega_1^2 + 2\rho\sigma_X\sigma_Y\omega_1\omega_2 + \omega_2^2\sigma_Y^2]}$$ after using (C-51) again.

★ 5.2-5. $\Phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, e^{j\omega_1 x + j\omega_2 y} dx\, dy$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{x}{\pi}\right)\mathrm{rect}\left(\frac{x+y}{\pi}\right)\cos(x+y)\, e^{j\omega_1 x + j\omega_2 y} dx\, dy$$

$$= \frac{1}{2\pi}\int_{x=-\pi/2}^{\pi/2}\int_{y=-\frac{\pi}{2}-x}^{\pi/2-x} \cos(x+y)\, e^{j\omega_1 x + j\omega_2 y} dx\, dy$$

$$= \frac{1}{4\pi}\int_{x=-\pi/2}^{\pi/2}\left\{e^{j(\omega_1+1)x}\int_{-\pi/2-x}^{\pi/2-x} e^{j(\omega_2+1)y} dy\right.$$

$$\left. + e^{j(\omega_1-1)x}\int_{-\pi/2-x}^{\pi/2-x} e^{j(\omega_2-1)y} dy\right\} dx$$

These integrals are readily reduced to obtain

★ 5.2-5. (Continued)

$$\Phi_{X,Y}(\omega_1,\omega_2) = \frac{\cos(\omega_2\pi/2)}{(1-\omega_2^2)}\, Sa\left[(\omega_1-\omega_2)\pi/2\right].$$

also

$$\Phi_X(\omega_1) = \Phi_{X,Y}(\omega_1,0) = Sa[\omega_1\pi/2]$$

$$\Phi_Y(\omega_2) = \Phi_{X,Y}(0,\omega_2) = \frac{\cos(\omega_2\pi/2)}{(1-\omega_2^2)}\, Sa(\omega_2\pi/2)$$

$$= \frac{Sa(\omega_2\pi)}{1-\omega_2^2}.$$

★ 5.2-6. First, differentiate to obtain

$$\frac{\partial\Phi}{\partial\omega_1} = \frac{1}{(1-j2\omega_2)^{N/2}}\left[\frac{jN}{(1-j2\omega_1)^{(N/2)+1}}\right]$$

$$\frac{\partial\Phi}{\partial\omega_2} = \frac{1}{(1-j2\omega_1)^{N/2}}\left[\frac{jN}{(1-j2\omega_2)^{(N/2)+1}}\right]$$

$$\frac{\partial^2\Phi}{\partial\omega_1^2} = \frac{1}{(1-j2\omega_2)^{N/2}}\left[\frac{-N(N+2)}{(1-j2\omega_1)^{(N/2)+2}}\right]$$

$$\frac{\partial^2\Phi}{\partial\omega_2^2} = \frac{1}{(1-j2\omega_1)^{N/2}}\left[\frac{-N(N+2)}{(1-j2\omega_2)^{(N/2)+2}}\right]$$

(a) $m_{20} = (-j)^2\frac{\partial^2\Phi}{\partial\omega_1^2}\Big|_{\substack{\omega_1=0\\ \omega_2=0}} = N(N+2)$, $m_{02} = (-j)^2\frac{\partial^2\Phi}{\partial\omega_2^2}\Big|_{\substack{\omega_1=0\\ \omega_2=0}}$

$= N(N+2)$, $R_{X_1X_2} = m_{11} = (-j)^2\frac{\partial^2\Phi}{\partial\omega_1\partial\omega_2}\Big|_{\substack{\omega_1=0\\ \omega_2=0}} =$

$$\left[\frac{jN(-j)}{(1-j2\omega_1)^{(N/2)+1}}\right]\left[\frac{jN(-j)}{(1-j2\omega_2)^{(N/2)+1}}\right]_{\substack{\omega_1=0\\ \omega_2=0}} = +N^2$$

(b) $m_{10} = -j\frac{\partial\Phi}{\partial\omega_1}\Big|_{\substack{\omega_1=0\\ \omega_2=0}} = N$, $m_{01} = m_{10} = N$, (c) $\rho = \frac{C_{X_1X_2}}{\sigma_{X_1}\sigma_{X_2}}$

$$= \frac{R_{X_1X_2} - \bar{X}_1\bar{X}_2}{\sigma_X\sigma_Y} = \frac{m_{11} - m_{10}m_{01}}{\sqrt{m_{20}m_{02}}} = \frac{N^2-N^2}{\sqrt{4N^2}} = 0$$

★ 5.2-7. $\Phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, e^{j\omega_1 x + j\omega_2 y}\,dx\,dy =$

★5.2-7. (Continued) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{m=0}^{M}\sum_{n=0}^{N} P(x_m, y_n)\,\delta(x-x_m)$

$\cdot\,\delta(y-y_n)\, e^{j\omega_1 x + j\omega_2 y}\, dx\, dy$ where $P(x_m, y_n) = \frac{1}{(M+1)(N+1)}$.

Thus, $\Phi_{X,Y}(\omega_1,\omega_2) = \frac{1}{(N+1)}\sum_{n=0}^{N} e^{jn\omega_2 d}\,\frac{1}{(M+1)}\sum_{m=0}^{M} e^{jm\omega_1 b}$

$$= \frac{e^{jM\omega_1 b/2 + jN\omega_2 d/2}}{(M+1)(N+1)}\;\frac{\sin[(M+1)\omega_1 b/2]}{\sin(\omega_1 b/2)}\;\frac{\sin[(N+1)\omega_2 d/2]}{\sin(\omega_2 d/2)}$$

where (C-60) has been used.

★5.2-8. From Problem 3.2-31 $\Phi_{X_k}(\omega) = \exp[b_k(e^{j\omega}-1)]$.

From Problem 5.2-2 with $\omega_1 = \omega_2 = \cdots = \omega_K = \omega$

$$\Phi_X(\omega) = E\left[e^{j\omega\sum_{k=1}^{K} X_k}\right] = \prod_{k=1}^{K} E\left[e^{j\omega X_k}\right] = \prod_{k=1}^{K}\Phi_{X_k}(\omega)$$

$= e^{\left(\sum_{k=1}^{K} b_k\right)(e^{j\omega}-1)}$. Let $b = \sum_{k=1}^{K} b_k$ so

$\Phi_X(\omega) = \exp\{b(e^{j\omega}-1)\}$ which is the characteristic function of a Poisson random variable with mean and variance b.

★5.2-9. From (5.2-7) with $\omega_i = \omega$, all i, and $X = \sum_{i=1}^{N} X_i$

$$\Phi_{X_1,\cdots,X_N}(\omega,\cdots,\omega) = E\left[e^{j\omega X}\right] = \Phi_X(\omega)$$

From Problem 5-25:

$$\Phi_{X_1,\cdots,X_N}(\omega,\cdots,\omega) = \prod_{i=1}^{N}\Phi_{X_i}(\omega) = \prod_{i=1}^{N}\exp[b_i(e^{j\omega}-1)] = \exp\left[\left(\sum_{i=1}^{N} b_i\right)(e^{j\omega}-1)\right]$$

when (F-21) is used. On equating these two expressions,

$$\Phi_X(\omega) = \exp[b(e^{j\omega}-1)] \qquad (1)$$

where

$$b = \sum_{i=1}^{N} b_i \qquad (2)$$

★ 5.2-9. (Continued) Since (1) is the characteristic function of a Poisson random variable, X must be Poisson and its mean is given by (2).

★ 5.2-10. From Example 3.3-1 with $a=0$ and b replaced by $1/a$:

$$\Phi_{X_i}(\omega) = \frac{a}{a-j\omega}.$$

From (5.2-7) with $\omega_i = \omega$, all i, and $X = \sum_{i=1}^{N} X_i$:

$$\Phi_{X_1,\cdots,X_N}(\omega,\cdots,\omega) = E\left[e^{j\omega_i \sum_{i=1}^{N} X_i}\right] = E\left[e^{j\omega X}\right] = \Phi_X(\omega).$$

From Problem 5.2-2:

$$\Phi_{X_1,\cdots,X_N}(\omega,\cdots,\omega) = \prod_{i=1}^{N} \Phi_{X_i}(\omega) = \prod_{i=1}^{N} \frac{a}{a-j\omega} = \left(\frac{a}{a-j\omega}\right)^N.$$

This result is the characteristic function of an Erlang random variable [see (F-40)-(F-44)] so X is Erlang; its mean and variance are: $E[X] = N/a = Nb$, $\sigma_X^2 = N/a^2 = Nb^2$.

★ 5.2-11. From (F-39) with $N=1$: $\Phi_{X_i}(\omega) = 1/(1-j2\omega)^{1/2}$. From (5.2-7) with $\omega_i = \omega$, all i, and $X = \sum_{i=1}^{N} X_i$:

$$\Phi_{X_1,\cdots,X_N}(\omega,\cdots,\omega) = E\left[e^{j\omega \sum_{i=1}^{N} X_i}\right] = E\left[e^{j\omega X}\right] = \Phi_X(\omega). \quad (1)$$

From Problem 5.2-2:

$$\Phi_{X_1,\cdots,X_N}(\omega,\cdots,\omega) = \prod_{i=1}^{N} \Phi_{X_i}(\omega) = \prod_{i=1}^{N} \left(\frac{1}{1-j2\omega}\right)^{1/2} = \left(\frac{1}{1-j2\omega}\right)^{N/2}. \quad (2)$$

On equating (1) and (2), and recognizing that (2) is the characteristic function of a chi-square random variable with N degrees of freedom, proves the result desired.

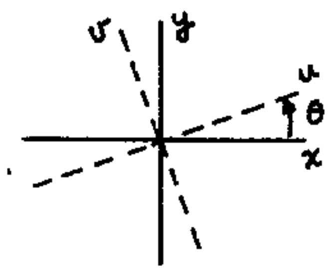
★ 5.3-1. Here $\sigma_X^2 = 3$, $\sigma_Y^2 = 4$, $\rho = -1/4$, $\rho^2 = 1/16$, $(1-\rho^2) = 15/16$.

(a) From (5.3-1):

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{12}\sqrt{15/16}}\, e^{-\frac{8}{15}\left[\frac{x^2}{3} - \frac{2(-1/4)xy}{\sqrt{12}} + \frac{y^2}{4}\right]}$$

$$= \frac{1}{3\pi\sqrt{5}}\, e^{-\frac{2}{45}\left[4x^2+\sqrt{3}\,xy+3y^2\right]}$$

(b)



$$u = x\cos(\theta) + y\sin(\theta) \qquad (1)$$

$$v = -x\sin(\theta) + y\cos(\theta) \qquad (2)$$

or

$$x = u\cos(\theta) - v\sin(\theta) = T_1^{-1}(u,v)$$

$$y = u\sin(\theta) + v\cos(\theta) = T_2^{-1}(u,v)$$

$$|J| = \left|\begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix}\right| = \left|\cos^2(\theta) + \sin^2(\theta)\right| = 1$$

From (5.3-11): $\theta = \frac{1}{2}\tan^{-1}\left[\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2}\right] = \frac{1}{2}\tan^{-1}\left[\frac{-2(\frac{1}{4})\sqrt{12}}{3-4}\right]$

$$= \frac{1}{2}\tan^{-1}\left[\sqrt{3}\right] = \pi/6.$$

We desire to substitute into (5.3-1) using (5.4-6). Quantities required are: $\cos(\theta) = \sqrt{3}/2$, $\sin(\theta) = 1/2$,

$x^2 = u^2\cos^2(\theta) + v^2\sin^2(\theta) - 2uv\cos(\theta)\sin(\theta)$

$= \frac{3}{4}u^2 + \frac{1}{4}v^2 - \frac{\sqrt{3}}{2}uv$,

$y^2 = \frac{1}{4}u^2 + \frac{3}{4}v^2 + \frac{\sqrt{3}}{2}uv$, $\quad xy = u^2\cos(\theta)\sin(\theta)$

$-uv\left[\sin^2(\theta) - \cos^2(\theta)\right] - v^2\sin(\theta)\cos(\theta)$

$= \frac{\sqrt{3}}{4}u^2 - \frac{\sqrt{3}}{4}v^2 + \frac{1}{2}uv$. On final substitution after reducing the algebra:

★ 5.3-1. (Continued)

$$f_{U,V}(u,v) = \frac{1}{3\pi\sqrt{5}} e^{-(u^2/5)-(v^2/9)} \quad (3)$$

Now since $E[U] = 0$ from (1), then $\sigma_U^2 = E[U^2] = E[X^2\cos^2(\theta) + Y^2\sin^2(\theta) + 2XY\cos(\theta)\sin(\theta)] = \sigma_X^2\frac{3}{4} + \sigma_Y^2\frac{1}{4} + R_{XY}\frac{\sqrt{3}}{2} = \frac{9}{4} + 1 + \frac{\sqrt{3}}{2}\rho\sigma_X\sigma_Y = \frac{13}{4} - \frac{3}{4} = 2.5.$
Similarly, from (2) $\sigma_V^2 = 4.5$. Thus, (3) can be written in the form

$$f_{U,V}(u,v) = \frac{e^{-u^2/2\sigma_U^2}}{\sqrt{2\pi\sigma_U^2}} \frac{e^{-v^2/2\sigma_V^2}}{\sqrt{2\pi\sigma_V^2}}$$

showing that U and V are statistically independent.

★ 5.3-2. Use (5.3-1) and the form of (2.4-1) in either (4.4-12) or (4.4-13).

$$f_X(x|Y=y) = \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right]\right\}}{\frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left\{-\frac{(y-\bar{Y})^2}{2\sigma_Y^2}\right\}}$$

On algebraic reduction:

$$f_X(x|Y=y) = \frac{1}{\sqrt{2\pi}\,\sigma_X\sqrt{1-\rho^2}} \exp\left\{-\frac{\left[x-\left\langle\bar{X}+\frac{\rho\sigma_X(y-\bar{Y})}{\sigma_Y}\right\rangle\right]^2}{2\sigma_X^2(1-\rho^2)}\right\} \quad (1)$$

Similarly,

$$f_Y(y|X=x) = \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right]\right\}}{\frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left\{-\frac{(x-\bar{X})^2}{2\sigma_X^2}\right\}}$$

★ 5.3-2. (Continued)

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{\left[y - \left\langle \bar{Y} + \frac{\rho(x-\bar{X})\sigma_Y}{\sigma_X}\right\rangle\right]^2}{2\sigma_Y^2(1-\rho^2)}\right\} \qquad (2)$$

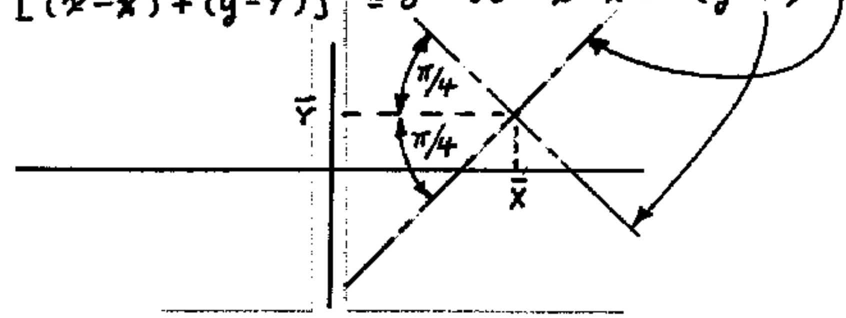
Eq. (1) is Gaussian with mean $\bar{X} + [\rho\sigma_X(y-\bar{Y})/\sigma_Y]$ and variance $\sigma_X^2(1-\rho^2)$. Eq. (2) is Gaussian with mean $\bar{Y} + [\rho\sigma_Y(x-\bar{X})/\sigma_X]$ and variance $\sigma_Y^2(1-\rho^2)$.

5.3-3. $f_{X,Y}(x,y) = \dfrac{e^{-\frac{1}{2\sigma^2(1-\rho^2)}\left[(x-\bar{X})^2 - 2\rho(x-\bar{X})(y-\bar{Y}) + (y-\bar{Y})^2\right]}}{2\pi\sigma^2\sqrt{1-\rho^2}}$

The locus of maxima occur where the exponent is zero: $(x-\bar{X})^2 - 2\rho(x-\bar{X})(y-\bar{Y}) + (y-\bar{Y})^2 = 0$.

$\rho = +1: \quad [(x-\bar{X}) - (y-\bar{Y})]^2 = 0 \quad$ or $\quad x - \bar{X} = y - \bar{Y}$

$\rho = -1: \quad [(x-\bar{X}) + (y-\bar{Y})]^2 = 0 \quad$ or $\quad x - \bar{X} = -(y-\bar{Y})$



5.3-4. Use (5.3-11): $\rho = \dfrac{\sigma_X^2 - \sigma_Y^2}{2\sigma_X\sigma_Y}\tan(2\theta) = \dfrac{9-4}{2\sqrt{9}\sqrt{4}}\tan\left(\dfrac{-\pi}{4}\right) = \dfrac{-5}{12}$.

★ 5.3-5. Use (5.3-17): $\theta = \frac{1}{2}\tan^{-1}\left[\dfrac{2\rho\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2}\right] = \frac{1}{2}\tan^{-1}(-\infty)$

$= -\frac{\pi}{4}$ so $[T] = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$.

5.3-6. (a) $C_{XY} = \overline{(X-\bar{X})(Y-\bar{Y})} = R_{XY} - \bar{X}\bar{Y} = -1.724 - (-1.0)\,1.5 = -0.224.$

(b) $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 1.16 - 1 = 0.16$, $\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = 2.89 - 1.5^2 = 0.64$,

$\rho = C_{XY}/\sigma_X\sigma_Y = -0.224/\sqrt{0.16(0.64)} = -0.70.$ (c) $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2}\right)$

$= \frac{1}{2}\tan^{-1}\left(\frac{2(-0.7)0.4(0.8)}{0.16 - 0.64}\right) = 0.3755$ rad.

5.3-7. $\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{R_{XY} - \bar{X}\bar{Y}}{\sigma_X\sigma_Y}$, so $\bar{Y} = \frac{R_{XY} - \rho\sigma_X\sigma_Y}{\bar{X}} = \frac{81.476 - 0.82(1.5)(1.2)}{10}$

$= 8.0$ m.

5.3-8. $\tan(2\theta) = \tan(\pi/3) = \sqrt{3} = \frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2} = \frac{(1.9/2)\sigma_Y}{(1.9)^2 - \sigma_Y^2}$, or

$\sigma_Y^2 + \frac{1.9}{2\sqrt{3}}\sigma_Y - (1.9)^2 = 0$. This occurs when $\sigma_Y = \frac{1.9}{4\sqrt{3}}(-1 \pm \sqrt{49})$.

Only positive sign leads to positive σ_Y so $\sigma_Y = (1.9/4\sqrt{3})(-1+7)$

$= 11.4/4\sqrt{3} \approx 1.6454.$

★ **5.3-9.** $\cancel{(a)\ \overline{X_1^2} = \sigma_{X_1}^2 + \bar{X}_1^2 = 9 + 4 = 13.}$ $\cancel{(b)\ \overline{X_2^2} = \sigma_{X_2}^2 + \bar{X}_2^2 = 4 + 1 = 5.}$

(c) $\rho_{X_1X_2} = C_{X_1X_2}/(\sigma_{X_1}\sigma_{X_2}) = -3/2(3) = -1/2$. Next, $[C_Y] = [T][C_X][T]^t$

$= \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 19 & 3 \\ 3 & 36 \end{bmatrix} = \begin{bmatrix} \sigma_{Y_1}^2 & C_{Y_1Y_2} \\ C_{Y_1Y_2} & \sigma_{Y_2}^2 \end{bmatrix}$, so: (d) $\sigma_{Y_1}^2 = 19$,

(e) $\sigma_{Y_2}^2 = 36$, and (f) $C_{Y_1Y_2} = 3$.

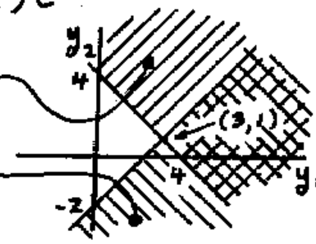
★ **5.4-1.** (a) From Example 5.4-3 with $a = 1$, $b = 1$, $c = 1$ and $d = -1$:

$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{|-2|} f_{X_1,X_2}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right)$

$= \frac{1}{6} u\left(\frac{y_1+y_2-4}{2}\right) u\left(\frac{y_1-y_2-2}{2}\right)(y_1-y_2)(y_1^2-y_2^2)e^{4-\frac{1}{2}(y_1^2-y_2^2)}$

(b) $u\left(\frac{y_1+y_2-4}{2}\right) > 0$ only for $y_2 > 4 - y_1$

$u\left(\frac{y_1-y_2-2}{2}\right) > 0$ only for $y_2 < y_1 - 2$



Applicable points shown crosshatched.

★ 5.4-2. Equation (5.5-17) applies.

$$[C_Y]=\begin{bmatrix}4 & -1 & -2\\ 2 & 2 & 1\\ -3 & -1 & 3\end{bmatrix}\begin{bmatrix}3 & 1.8 & 1.1\\ 1.8 & 3 & 1.8\\ 1.1 & 1.8 & 3\end{bmatrix}\begin{bmatrix}4 & 2 & -3\\ -1 & 2 & -1\\ -2 & 1 & 3\end{bmatrix}=\begin{bmatrix}38.2 & 13.8 & -34.8\\ 13.8 & 53.0 & -17.1\\ -34.8 & -17.1 & 37.2\end{bmatrix}$$

★ 5.4-3.

$$f_{X_1,X_2}(x_1,x_2)=\frac{1}{2\pi\sigma^2}e^{-x_1^2/(2\sigma^2)-x_2^2/(2\sigma^2)}$$

From (5) of Example 5.4-2

$$f_Y(y)=\int_{-y}^{y}\frac{y}{\sqrt{y^2-x_2^2}}\left\{\frac{1}{2\pi\sigma^2}e^{-(y_2^2-x_2^2)/(2\sigma^2)-x_2^2/(2\sigma^2)}\right.$$

$$\left.+\frac{1}{2\pi\sigma^2}e^{-(y^2-x_2^2)/(2\sigma^2)-x_2^2/(2\sigma^2)}\right\}dx_2$$

$$=\frac{y}{\pi\sigma^2}e^{-y^2/(2\sigma^2)}\int_{-y}^{y}\frac{dx_2}{\sqrt{y^2-x_2^2}}$$

But $\int_0^y\frac{d\xi}{\sqrt{a^2-\xi^2}}=\sin^{-1}\left(\frac{\xi}{a}\right)\Big|_0^y$ so

$$f_Y(y)=\frac{y}{\sigma^2}e^{-y^2/(2\sigma^2)}$$

★ 5.5-1. Here $[T]=\begin{bmatrix}5 & 2 & -1\\ -1 & 3 & 1\\ 2 & -1 & 2\end{bmatrix}$.

(a) From (5.5-17):

$$[C_Y]=\begin{bmatrix}5 & 2 & -1\\ -1 & 3 & 1\\ 2 & -1 & 2\end{bmatrix}\begin{bmatrix}4 & 2.05 & 1.05\\ 2.05 & 4 & 2.05\\ 1.05 & 2.05 & 4\end{bmatrix}\begin{bmatrix}5 & -1 & 2\\ 2 & 3 & -1\\ -1 & 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}142.30 & 30.90 & 40.60\\ 30.90 & 41.90 & 12.60\\ 40.60 & 12.60 & 28.00\end{bmatrix} \qquad (1)$$

★5.5-1 (Continued)

(b) $|[C_Y]| = (142.3)41.9(28.0) + 30.9(12.6)40.6(2)$
$- (40.6)^2 41.9 - (30.9)^2 28.0 - (12.6)^2 142.3$
$= 80168.256.$

$$|[C_Y]^{-1}|^{1/2} = 1/\sqrt{80168.256} \approx 3.532(10^{-3}).$$

By straightforward inversion of (1):

$$[C_Y]^{-1} \approx \begin{bmatrix} 1.265(10^{-2}) & -4.411(10^{-3}) & -1.636(10^{-2}) \\ -4.411(10^{-3}) & 2.914(10^{-2}) & -6.716(10^{-3}) \\ -1.636(10^{-2}) & -6.716(10^{-3}) & 6.246(10^{-2}) \end{bmatrix}$$

Equation (5.5-15) becomes

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) \approx \frac{3.532(10^{-3})}{(2\pi)^{3/2}} e^{-\frac{1}{2}[y_1\ y_2\ y_3][C_Y]^{-1}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}.$$

★5.5-2 (a) $[\bar{Y}] = [T][\bar{X}] = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$

(b) $[C_Y] = [T][C_X][T]^t$ from (5.5-17). On reduction:

$$[C_Y] = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 5 & -2/\sqrt{5} \\ -2/\sqrt{5} & 4 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(6 - \frac{2}{\sqrt{5}}\right) & \left(\frac{9}{2} - \frac{5}{2\sqrt{5}}\right) \\ \left(\frac{9}{2} - \frac{5}{2\sqrt{5}}\right) & \left(\frac{21}{4} - \frac{2}{\sqrt{5}}\right) \end{bmatrix}.$$

(c) $C_{Y_1Y_2} = \rho\sigma_{Y_1}\sigma_{Y_2} = \rho\left(6 - \frac{2}{\sqrt{5}}\right)^{1/2} \cdot \left(\frac{21}{4} - \frac{2}{\sqrt{5}}\right)^{1/2} = \left(\frac{9}{2} - \frac{5}{2\sqrt{5}}\right)$ so $\rho = \frac{9\sqrt{5} - 5}{\sqrt{(6\sqrt{5}-2)(21\sqrt{5}-8)}} = 0.717.$

★ 5.6-1. $\left.\begin{array}{l} Y_1 = \sqrt{-2\ln(X_1)}\cos(2\pi X_2) \\ Y_2 = \sqrt{-2\ln(X_1)}\sin(2\pi X_2) \end{array}\right\}$ $\tan(2\pi X_2) = Y_2/Y_1$

$X_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{Y_2}{Y_1}\right) = \frac{1}{2\pi}\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{Y_1}{Y_2}\right)\right\}$

$\left.\begin{array}{l} Y_1^2 = -2\ln(X_1)\cos^2(2\pi X_2) \\ Y_2^2 = -2\ln(X_1)\sin^2(2\pi X_2) \end{array}\right\}$ $Y_1^2 + Y_2^2 = -2\ln(X_1)$

$X_1 = \exp\left[-\frac{1}{2}(Y_1^2 + Y_2^2)\right]$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} -Y_1\exp[-(Y_1^2+Y_2^2)/2] & -Y_2\exp[-(Y_1^2+Y_2^2)] \\ \frac{-(1/Y_2)}{2\pi\left[1+\left(\frac{Y_1}{Y_2}\right)^2\right]} & \frac{(1/Y_1)}{2\pi\left[1+\left(\frac{Y_2}{Y_1}\right)^2\right]} \end{vmatrix}$$

$|J| = \frac{1}{2\pi}e^{-\frac{1}{2}(Y_1^2+Y_2^2)}$. since $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2) = 1$, $0<x_1<1$ and $0<x_2<1$, then (5.4-6) gives (5.6-2).

★ 5.6-2. Here $(Y_1 - \bar{Y}_1)^2 = -2\sigma_{Y_1}^2\ln(X_1)\cos^2(2\pi X_2)$ (1)

$(Y_2 - \bar{Y}_2)^2 = -2\sigma_{Y_2}^2\ln(X_1)\sin^2(2\pi X_2)$. (2)

By dividing (2) by (1), and adding (1) and (2) and solving for X_1:

$\left(\frac{Y_2-\bar{Y}_2}{Y_1-\bar{Y}_1}\right)\frac{\sigma_{Y_1}}{\sigma_{Y_2}} = \tan(2\pi X_2)$ or $X_2 = \frac{1}{2\pi}\left\{\frac{\pi}{2} - \tan^{-1}\left[\left(\frac{Y_1-\bar{Y}_1}{Y_2-\bar{Y}_2}\right)\frac{\sigma_{Y_2}}{\sigma_{Y_1}}\right]\right\}$

$X_1 = \exp\left\{-\frac{1}{2\sigma_{Y_1}^2}(Y_1-\bar{Y}_1)^2 - \frac{1}{2\sigma_{Y_2}^2}(Y_2-\bar{Y}_2)^2\right\}$.

Next,

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} X_1\left[\frac{-(Y_1-\bar{Y}_1)}{\sigma_{Y_1}^2}\right] & X_1\left[\frac{-(Y_2-\bar{Y}_2)}{\sigma_{Y_2}^2}\right] \\ \frac{-\frac{\sigma_{Y_2}}{\sigma_{Y_1}}\left(\frac{1}{Y_2-\bar{Y}_2}\right)}{2\pi\left[1+\left(\frac{Y_1-\bar{Y}_1}{Y_2-\bar{Y}_2}\right)^2\frac{\sigma_{Y_2}^2}{\sigma_{Y_1}^2}\right]} & \frac{\frac{\sigma_{Y_1}}{\sigma_{Y_2}}\left(\frac{1}{Y_1-\bar{Y}_1}\right)}{2\pi\left[1+\left(\frac{Y_2-\bar{Y}_2}{Y_1-\bar{Y}_1}\right)^2\frac{\sigma_{Y_1}^2}{\sigma_{Y_2}^2}\right]} \end{vmatrix}$$

$$= \frac{-1}{2\pi\sigma_{Y_1}\sigma_{Y_2}}e^{-\frac{1}{2\sigma_{Y_1}^2}(Y_1-\bar{Y}_1)^2 - \frac{1}{2\sigma_{Y_2}^2}(Y_2-\bar{Y}_2)^2}.$$

★ (5.6-2.) (Continued)

From (5.4-8) $f_{Y_1,Y_2}(y_1,y_2) = |J|$, so the required joint density is that of gaussian random variables that are statistically independent and have the prescribed means and variances.

★ (5.6-3.) Use (5.6-7) to obtain W_1 and W_2 and then apply (5.6-1) to get in terms of X_1 and X_2. First, $W_1 = \bar{W}_1 + \sigma_{W_1} Y_1$ and $W_2 = \bar{W}_2 + \rho_W \sigma_{W_2} Y_1 + \sigma_{W_2}\sqrt{1-\rho_W^2}\, Y_2$. Next, $W_1 = \bar{W}_1 + \sigma_{W_1}\sqrt{-2\ln(X_1)}\cos(2\pi X_2)$ and $W_2 = \bar{W}_2 + \rho_W \sigma_{W_2}\sqrt{-2\ln(X_1)}\cos(2\pi X_2) + \sigma_{W_2}\sqrt{1-\rho_W^2}\sqrt{-2\ln(X_1)}\sin(2\pi X_2)$.

★ (5.6-4.) For R and Θ (5.6-8) and (5.6-9) apply. For W_1 and W_2 we apply (5.6-7) with $\rho_W = 0$ and $\sigma_{W_1} = \sigma_{W_2} = \sigma_W$. For Y_1 and Y_2 we apply (5.6-1):

$$R = \sqrt{W_1^2 + W_2^2} = \left\{ [\bar{W}_1 + \sigma Y_1]^2 + [\bar{W}_2 + \sigma Y_2]^2 \right\}^{1/2}$$

$$= \left\{ \left[\bar{W}_1 + \sigma\sqrt{-2\ln(X_1)}\cos(2\pi X_2)\right]^2 + \left[\bar{W}_2 + \sigma\sqrt{-2\ln(X_1)}\sin(2\pi X_2)\right]^2 \right\}^{1/2}$$

$$= \left\{ \bar{W}_1^2 + \bar{W}_2^2 - 2\sigma^2\ln(X_1) + 2\sigma\sqrt{-2\ln(X_1)}\left[\bar{W}_1\cos(2\pi X_2) + \bar{W}_2\sin(2\pi X_2)\right] \right\}^{1/2}$$

$$= \left\{ A_0^2 - 2\sigma^2\ln(X_1) + 2\sigma\sqrt{-2\ln(X_1)}\left[A_0\cos(\theta_0)\cos(2\pi X_2) + A_0\sin(\theta_0)\sin(2\pi X_2)\right] \right\}^{1/2}$$

$$R = \left\{ A_0^2 - 2\sigma^2\ln(X_1) + 2\sigma\sqrt{-2\ln(X_1)}\, A_0\cos(2\pi X_2 - \theta_0) \right\}^{1/2}$$

$$\Theta = \tan^{-1}\left\{ \frac{\bar{W}_2 + \sigma Y_2}{\bar{W}_1 + \sigma Y_1} \right\} = \tan^{-1}\left\{ \frac{\bar{W}_2 + \sigma\sqrt{-2\ln(X_1)}\sin(2\pi X_2)}{\bar{W}_1 + \sigma\sqrt{-2\ln(X_1)}\cos(2\pi X_2)} \right\}$$

$$\Theta = \tan^{-1}\left\{ \frac{A_0\sin(\theta_0) + \sigma\sqrt{-2\ln(X_1)}\sin(2\pi X_2)}{A_0\cos(\theta_0) + \sigma\sqrt{-2\ln(X_1)}\cos(2\pi X_2)} \right\}$$

★ (5.6-5.) We use the MATLAB code of Example 5.6-1 except with $N=1000$. Results are tabulated below.

	Mean		Standard Deviation		Corr. Coef.
	m_1	m_2	σ_1	σ_2	ρ
True values	0	0	2	3	-0.4
Estimated (N=1000)	0.03	-0.07	2.01	2.99	-0.42
Percent Error	-	-	0.5%	-0.3%	5.0%

When compared to results of Table 5.6-1 we find significant reduction of all errors when $N=1000$ values are used.

(5.7-1.) $E\left[\widehat{X^2_N}\right] = E\left[\frac{1}{N}\sum_{n=1}^{N} X_n^2\right] = \frac{1}{N}\sum_{n=1}^{N} E[X_n^2]$. But $\overline{X_n^2} = \overline{X^2}$, all n, so $E\left[\widehat{X^2_N}\right] = \overline{X^2}$. This is the definition of an unbiased estimator. For a biased estimator the equality is not true.

(5.7-2.) variance of $\widehat{X^2_N} = E\left\{\left[\widehat{X^2_N} - \overline{X^2}\right]^2\right\}$ since the mean of $\widehat{X^2_N} = \overline{X^2}$ from results of Problem 5.7-1. Hence,

variance of $\widehat{X^2_N} = E\left\{(\widehat{X^2_N})^2 - 2\,\overline{X^2}\,\widehat{X^2_N} + (\overline{X^2})^2\right\} = -(\overline{X^2})^2$

$+E\left\{\frac{1}{N}\sum_n X_n^2 \frac{1}{N}\sum_m X_m^2\right\} = -(\overline{X^2})^2 + \frac{1}{N^2}\sum_n\sum_m \overline{X_n^2 X_m^2}$. But

since $\overline{X_m^2 X_n^2} = \overline{X^4}, \quad n=m$

$\qquad\qquad = (\overline{X^2})^2, \quad n \neq m$

then

variance of $\widehat{X^2_N} = -(\overline{X^2})^2 + \frac{1}{N^2}\left\{N(\overline{X^4}) + (N^2-N)(\overline{X^2})^2\right\}$

$= \frac{1}{N}\left[\overline{X^4} - (\overline{X^2})^2\right] = \frac{1}{N}[\text{variance of } X^2] = \frac{1}{N}[\text{variance of power}] \to 0$ as $N \to \infty$.

5.7-3. $E[\widehat{\sigma_X^2}] = E\left[\frac{1}{N}\sum_{n=1}^{N}(X_n - \hat{\hat{X}}_N)^2\right]$. But $E(\hat{\hat{X}}_N) = \bar{X}$ from (5.7-3). Also, since $\hat{\hat{X}}_N = (1/N)\sum_n X_n$, we have

$$E[\widehat{\sigma_X^2}] = \frac{1}{N}\sum_n E[X_n^2 - 2\hat{\hat{X}}_N X_n + (\hat{\hat{X}}_N)^2] = \frac{1}{N}\sum_n \left\{\overline{X^2} - \frac{2}{N}\sum_m\sum_n \overline{X_n X_m} + \frac{1}{N^2}\sum_m\sum_k \overline{X_m X_k}\right.$$

But $\overline{X_n X_m} = \overline{X_n^2} = \overline{X^2}$, $m = n$

$= \bar{X}_n \bar{X}_m = (\bar{X})^2$, $m \neq n$

Thus, we reduce to get

$$E[\widehat{\sigma_X^2}] = \left(\frac{N-1}{N}\right)[\overline{X^2} - (\bar{X})^2] = \sigma_X^2 - \overbrace{(\sigma_X^2/N)}^{\text{bias}}$$ and bias $\to 0$ as $N \to \infty$.

[M] **5.7-4.** We use the MATLAB code of Example 5.7-3 except with $N = 1000$. Tabulated results are given below.

	Mean	2nd Moment	Variance
True values	1.57	4.00	1.53
Estimated (N=1000)	1.55	3.87	1.47
Percent Error	-1.3%	-3.3%	-3.9%

When these results are compared with those of Table 5.7-1 for $N = 250$ we find significant reduction of errors.

★5.8-1. (a) $E[Z] = E[\cos(X)] + jE[\sin(Y)]$

$$E[\cos(X)] = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos(x)\frac{1}{(2\pi)^2}\,dx\,dy = 0.$$

$$E[\sin(Y)] = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sin(y)\frac{1}{(2\pi)^2}\,dx\,dy = 0.$$

Thus, $E[Z] = 0$.

(b) $\sigma_Z^2 = E[|Z|^2]$ since $E[Z]=0$.

$$\sigma_Z^2 = E[\cos^2(X) + \sin^2(Y)] = E[\cos^2(X)] + E[\sin^2(Y)].$$

$$E[\cos^2(X)] = E\left[\tfrac{1}{2} + \tfrac{1}{2}\cos(2X)\right] = \tfrac{1}{2} + \tfrac{1}{2}E[\cos(2X)]$$

$$= \frac{1}{2} + \frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos(2x)\frac{1}{(2\pi)^2}\,dx\,dy = \frac{1}{2}$$

$$E[\sin^2(Y)] = E\left[\tfrac{1}{2} - \tfrac{1}{2}\cos(2Y)\right] = \tfrac{1}{2} - \tfrac{1}{2}E[\cos(2Y)]$$

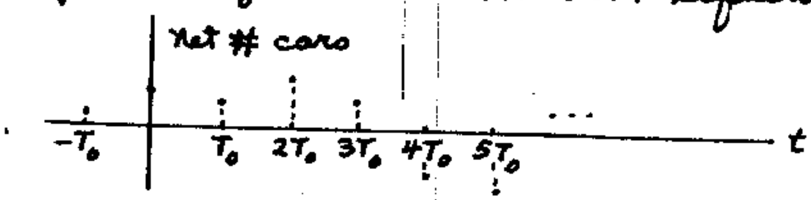
$$= \frac{1}{2} - \frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos(2Y)\frac{1}{(2\pi)^2}\,dx\,dy = \frac{1}{2}$$

Thus, $\sigma_Z^2 = 1$.

★5.8-2. Define $Z_1 = X_1 + jY_1$, $Z_2 = X_2 + jY_2$. Then $R_{Z_1Z_2} = E[Z_1^* Z_2] = E[(X_1 - jY_1)(X_2 + jY_2)] = \overline{X_1X_2} + j\,\overline{X_1Y_2} - j\,\overline{X_2Y_1} + \overline{Y_1Y_2} = 4 + j0 - j0 + 6 = 10$. (b) Since $E[Z_1^* Z_2] = R_{Z_1Z_2} = 10 \neq E[Z_1^*]E[Z_2] = 0$ we find that Z_1 and Z_2 are not statistically independent.

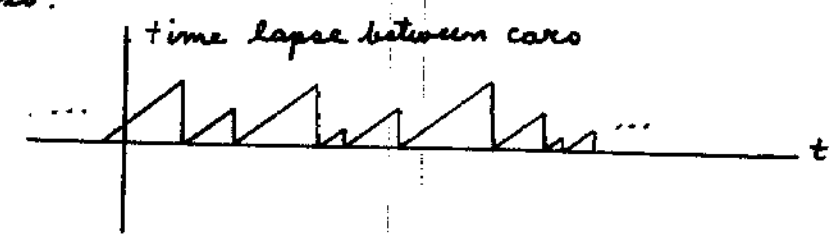
CHAPTER 6

(6.1-1.) 1. The net number of cars passing in one direction (reverse direction produces a negative number in the chosen direction) during each unit of time T_0: Discrete random sequence



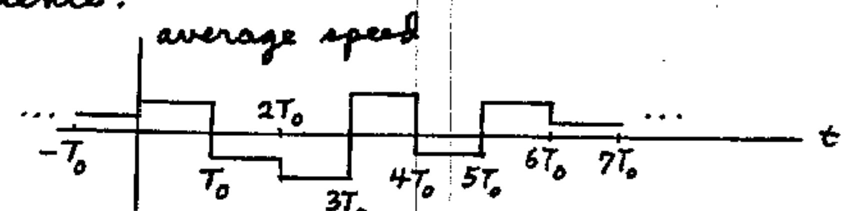
2. The net number of cars on a cumulative basis that has passed in a chosen direction at <u>any</u> time: Discrete random process.

3. The cumulating time between cars passing (direction unimportant): Continuous random process.



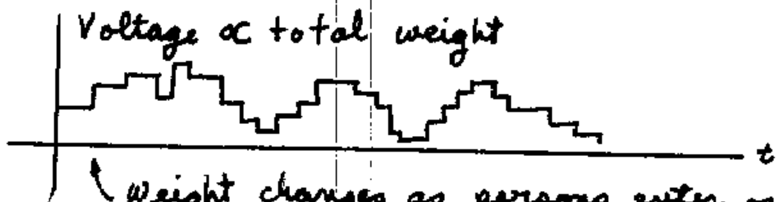
4. The average velocity (speed with direction)

(6.1-1.) (Continued) of all cars that pass during each interval of time T_0: Continuous random sequence.



Many other processes can also be defined.

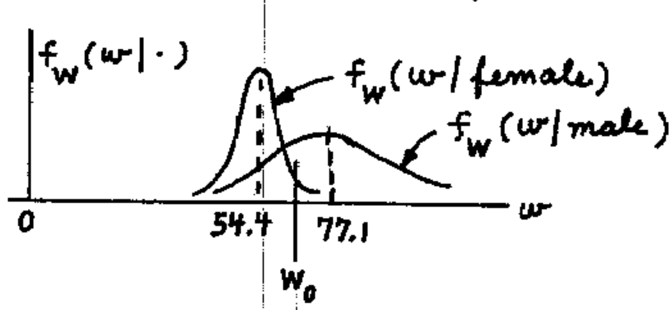
(6.1-2.) (a)



(b) The underlying experiment can be taken as "Chose a 10 m sidewalk section for measurements."

(c) This is a continuous random process.

★ (6.1-3.)



(a) $P\{W \le W_0 | \text{female}\} = \int_{-\infty}^{W_0} f_W(w|\text{female})\, dw$

$= P\{W > W_0 | \text{male}\} = 1 - \int_{-\infty}^{W_0} f_W(w|\text{male})\, dw$

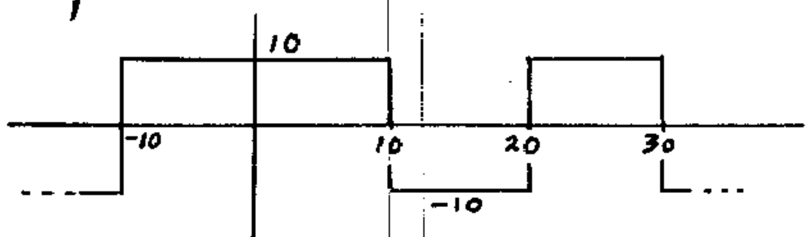
Since both densities are Gaussian, we use (2.4-7) and (2.4-3) to get

★6.1-3. (Continued) $F\left(\frac{W_0-54.4}{6.8}\right) = 1 - F\left(\frac{W_0-77.1}{11.3}\right)$

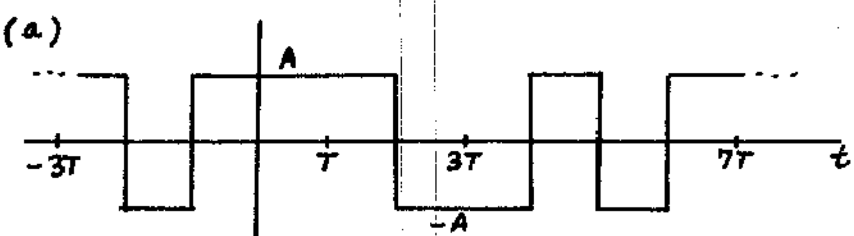
$= F\left(\frac{77.1-W_0}{11.3}\right)$ and therefore $\frac{W_0-54.4}{6.8} = \frac{77.1-W_0}{11.3}$

or $W_0 = 62.93$. (b) We decide weight is that of a male if it exceeds 62.93 kg. We decide it is a female if weight does not exceed 62.93 kg. W_0 acts as a decision boundary that splits the range of possible weights into two regions "male" and "female."

(c)



6.1-4. (a)



(b) This is a discrete random process.

(c) It is not deterministic.

6.1-5. (a) yes, (b) yes, (c) yes, (d) no, (e) yes, (f) no, and (g) no.

6.1-6. (a) Each sample function is a constant with time. (b) $X(t)$ is a continuous random process. (c) The process is deterministic.

(6.1-7.) (a) Each sample function is a linear function with time that passes through the origin with a random slope. (b) $X(t)$ is a continuous random process. (c) The process is deterministic.

(6.2-1.) (a) Yes. (b) $f_X(x) = 0.6\,\delta(x-1) + 0.3\,\delta(x-2) + 0.1\,\delta(x-3)$, since a random process for any one time instant is just a random variable.

(6.2-2.) $E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)\,dx_1\,dx_2$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1+\Delta, t_2+\Delta)\,dx_1\,dx_2$. Now let arbitrary Δ equal $-t_1$: $E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; 0, t_2-t_1)\,dx_1\,dx_2$. Next let $\tau = t_2 - t_1$, $t_1 = t$: $E[X(t)X(t+\tau)] = R_{XX}(t, t+\tau)$ $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; \tau)\,dx_1\,dx_2 = R_{XX}(\tau)$.

★(6.2-3.) (a) and (b) Let ϵ have values e. Now $P\{X \le x \mid \epsilon = e\} = F_X(x \mid \epsilon = e)$ and for any ϵ must be zero for $x < 0$ because $x(t)$ is never negative. The event $\{X \le 0\}$ is satisfied whenever $x(t)$ is zero. This happens during the fraction of time $(T-2t_0)/T$. Hence, $F_X(x \mid \epsilon = e) = [(T-2t_0)/T]\,u(x)$, $x < 0$. For $0 \le x < B$ the additional time interval or fraction of time where $X \le x$ becomes $2t_0 x/BT$. Thus

★6.2-3. (Continued)

$$F_X(x \mid \epsilon = e) = \left(\frac{T-2t_0}{T}\right) u(x) + \frac{2t_0 x}{BT}, \quad 0 \le x < B$$
$$= 1, \qquad B \le x$$
$$= 0, \qquad x < 0.$$

By differentiation:

$$f_X(x \mid \epsilon = e) = \left(\frac{T-2t_0}{T}\right)\delta(x) + \frac{2t_0}{BT}, \quad 0 \le x < B.$$
$$= 0, \qquad \text{elsewhere.}$$

Next, $f_{X,\epsilon}(x, e) = f_X(x \mid \epsilon = e) f_\epsilon(e)$

$$= \left(\frac{T-2t_0}{T^2}\right)\delta(x) + \frac{2t_0}{BT^2}, \quad 0 \le x < B \text{ and } 0 < e < T.$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,\epsilon}(x, e)\,de = \left(\frac{T-2t_0}{T}\right)\delta(x) + \frac{2t_0}{BT}, \quad 0 \le x < B.$$
$$= 0 \quad \text{for other values of } x.$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi = \left(\frac{T-2t_0}{T}\right)u(x) + \frac{2t_0 x}{BT}, \quad 0 \le x < B$$
$$= 0, \qquad x < 0$$
$$= 1, \qquad B \le x.$$

(c) $E[X(t)] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x\left(\frac{T-2t_0}{T}\right)\delta(x)\,dx$

$+ \int_0^B \frac{2t_0 x}{BT}\,dx = \frac{t_0 B}{T}$, $\quad E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$= \int_0^B \frac{2t_0 x^2}{BT}\,dx = \frac{2t_0 B^2}{3T}$. $\quad \sigma_X^2 = E[X^2(t)] - \{E[X(t)]\}^2$

$$= \frac{2t_0 B^2}{3T} - \frac{t_0^2 B^2}{T^2} = \frac{t_0 B^2}{T}\left[\frac{2}{3} - \frac{t_0}{T}\right].$$

★6.2-4. (a) and (b) The procedures are identical to Problem 6.2-3. Here $F_X(x \mid \epsilon = e) = P\{X \le x \mid \epsilon = e\} = [(T-t_0)/T]$ $u(x) + (t_0/T)\,u(x-A)$ because X can have only

★ 6.2-4. (Continued)

values of zero and A. Thus, $f_X(x|\epsilon=e) = [(T-t_0)/T]\,\delta(x) + (t_0/T)\,\delta(x-A)$. Next,

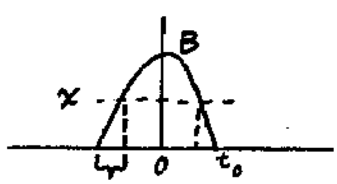
$$f_X(x) = \int_{-\infty}^{\infty} f_X(x|\epsilon=e) f_\epsilon(e)\,de = [(T-t_0)/T]\,\delta(x) + (t_0/T)\,\delta(x-A).$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi = \left(\frac{T-t_0}{T}\right)u(x) + \frac{t_0}{T}\,u(x-A).$$

(c) $E[X(t)] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = t_0 A/T.$

$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = t_0 A^2/T$. $\sigma_X^2 = E[X^2(t)] - \{E[X(t)]\}^2 = \frac{t_0 A^2}{T}\left(\frac{T-t_0}{T}\right)$.

★ 6.2-5. (a) and (b) The procedures are identical to those of Problem 6.2-3. Hence $F_X(x|\epsilon=e) = P\{X \le x|\epsilon=e\} = [(T-2t_0)/T]\,u(x)$, $x<0$.



$t_0 - \frac{2t_0}{\pi}\cos^{-1}\left(\frac{x}{B}\right)$

For $0 \le x < B$ the figure is helpful. The additional fraction of time that $X \le x$ is $\frac{2}{T}\left\{t_0 - \frac{2t_0}{\pi}\cos^{-1}\left(\frac{x}{B}\right)\right\}$.

Hence,

$$F_X(x|\epsilon=e) = \left(\frac{T-2t_0}{T}\right)u(x) + \frac{2t_0}{T}\left[1-\frac{2}{\pi}\cos^{-1}\left(\frac{x}{B}\right)\right]u(x)$$

for $0 \le x < B$ and 1 for $B \le x$. By differentiation

$$f_X(x|\epsilon=e) = \left[1-\frac{4t_0}{\pi T}\cos^{-1}\left(\frac{x}{B}\right)\right]\delta(x)$$

★ 6.2-5. (Continued)

$$+ \frac{4t_0}{\pi T}\frac{u(x)}{\sqrt{B^2 - x^2}}, \qquad x < B$$

$$= 0, \qquad x \geq B.$$

$$f_X(x) = \int_{-\infty}^{\infty} f_X(x|\epsilon = e) f_\epsilon(e)\, de =$$

$$= \left[1 - \frac{2t_0}{T}\right]\delta(x) + \frac{4t_0}{\pi T}\frac{u(x)}{\sqrt{B^2 - x^2}}, \qquad x < B$$

$$= 0, \qquad x \geq B.$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi = \left(1 - \frac{2t_0}{T}\right) u(x)$$

$+\int_0^x \frac{4t_0}{\pi T}\frac{d\xi}{\sqrt{B^2 - \xi^2}}$. This integral is known (Dwight, 1961, p. 67) so that

$$F_X(x) = \left(1 - \frac{2t_0}{T}\right) u(x) + \frac{4t_0}{\pi T}\sin^{-1}\left(\frac{x}{B}\right) u(x), \qquad x < B$$

$$= 1, \qquad x \geq B.$$

(c) $E[X(t)] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{B} x\left[1 - \frac{2t_0}{T}\right]\delta(x)\, dx$

$+ \frac{4t_0}{\pi T}\int_0^B \frac{x\, dx}{\sqrt{B^2 - x^2}}$. The integral is tabulated (Dwight, 1961, p. 68) and we find $E[X(t)] = \frac{4t_0|B|}{\pi T}$.

$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_{-\infty}^{B} x^2\left[1 - \frac{2t_0}{T}\right]\delta(x)\, dx$

$+ \frac{4t_0}{\pi T}\int_0^B \frac{x^2\, dx}{\sqrt{B^2 - \xi^2}} = \frac{t_0 B^2}{T}$ after using another integral from Dwight (1961, p.68). Finally,

$$\sigma_X^2 = E[X^2(t)] - \{E[X(t)]\}^2 = \frac{t_0 B^2}{T} - \frac{16 t_0^2 B^2}{\pi^2 T^2} = \frac{t_0 B^2}{T}\left[1 - \frac{16 t_0}{\pi^2 T}\right].$$

(6.2-6.) (a) $E[X(t)] = E[X_0] + E[V]t$. But X_0 and V are uniformly distributed so $\bar{X}_0 = (X_{02}+X_{01})/2$ and $\bar{V} = (V_2+V_1)/2$ and $E[X(t)] = \frac{X_{02}+X_{01}}{2} + \frac{V_2+V_1}{2}t$.

(b) $R_{XX}(t,t+\tau) = E[\{X_0+Vt\}\{X_0+V(t+\tau)\}] = \overline{X_0^2} + \overline{X_0V}(t+\tau) + \overline{X_0V}\,t + \overline{V^2}\,t(t+\tau) = \overline{X_0^2} + \bar{X}_0\bar{V}(2t+\tau) + \overline{V^2}(t^2+t\tau)$.

But since $\overline{X_0^2} = (X_{02}^3 - X_{01}^3)/3(X_{02}-X_{01}) = (X_{02}^2 + X_{02}X_{01} + X_{01}^2)/3$ and $\overline{V^2} = (V_2^2+V_2V_1+V_1^2)/3$ because of uniform distributions,

$$R_{XX}(t,t+\tau) = \frac{(X_{01}^2+X_{01}X_{02}+X_{02}^2)}{3} + \frac{(X_{01}+X_{02})(V_1+V_2)}{4}(2t+\tau) + \frac{(V_1^2+V_1V_2+V_2^2)}{3}(t^2+\tau t).$$

(c) $C_{XX}(t,t+\tau) = E[\{(X_0-\bar{X}_0)+(V-\bar{V})t\}\{(X_0-\bar{X}_0)+(V-\bar{V})(t+\tau)\}] = \sigma_{X_0}^2 + C_{X_0V}(2t+\tau) + \sigma_V^2(t^2+\tau t)$. But $C_{X_0V}=0$ and $\sigma_{X_0}^2 = \frac{(X_{02}-X_{01})^2}{12}$ and $\sigma_V^2 = \frac{(V_2-V_1)^2}{12}$ so

$$C_{XX}(t,t+\tau) = \frac{(X_{02}-X_{01})^2}{12} + \frac{(V_2-V_1)^2}{12}(t^2+\tau t).$$

(d) $X(t)$ is not stationary in any sense.

★(6.2-7.) Since V is uniform $Y=Vt$ is uniform on $[V_1t, V_2t]$. (a) $f_X(x;t) = \int_{-\infty}^{\infty} f_{X_0}(x-y)f_Y(y)\,dy$ is true because $X = X_0 + Vt = X_0 + Y$. Thus, $f_X(x;t) =$

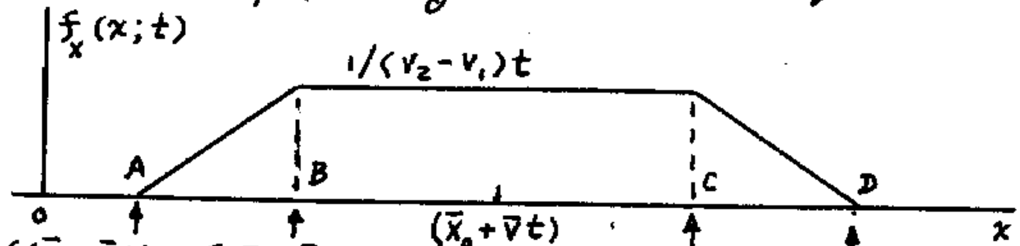
$$\int_{-\infty}^{\infty} \frac{1}{(X_{02}-X_{01})}\operatorname{rect}\left[\frac{\xi-\bar{X}_0}{X_{02}-X_{01}}\right]\frac{1}{(V_2-V_1)t}\operatorname{rect}\left[\frac{x-\xi-\bar{V}t}{(V_2-V_1)t}\right]d\xi.$$

There are two cases of interest. Case 1:

Case 1: $(V_2-V_1)t > (X_{02}-X_{01})$. Here the rect functions are centered on each other when $x = \bar{X}_0 + \bar{V}t$. The integral with full overlap equals $1/(V_2-V_1)t$. For

$\star$ 6.2-7. (Continued)

other x we graphically evaluate the integral as:



A: $\left\{(\bar{X}_0+\bar{v}t) - \frac{(X_{02}-X_{01})}{2} - \frac{(V_2-V_1)t}{2}\right\}$

B: $\left\{(\bar{X}_0+\bar{v}t) + \frac{(X_{02}-X_{01})}{2} - \frac{(V_2-V_1)t}{2}\right\}$

C: $\left\{(\bar{X}_0+\bar{v}t) - \frac{(X_{02}-X_{01})}{2} + \frac{(V_2-V_1)t}{2}\right\}$

D: $\left\{(\bar{X}_0+\bar{v}t) + \frac{(X_{02}-X_{01})}{2} + \frac{(V_2-V_1)t}{2}\right\}$

Case 2: Same as case 1 except interchange X_{01} and X_{02} with V_1t and V_2t, respectively. In this case we have $(V_2-V_1)t \leq (X_{02}-X_{01})$. The points A, B, C, and D are now $A = (\bar{X}_0+\bar{v}t) - \frac{(X_{02}-X_{01})}{2} - \frac{(V_2-V_1)t}{2} = X_{01}+V_1t$

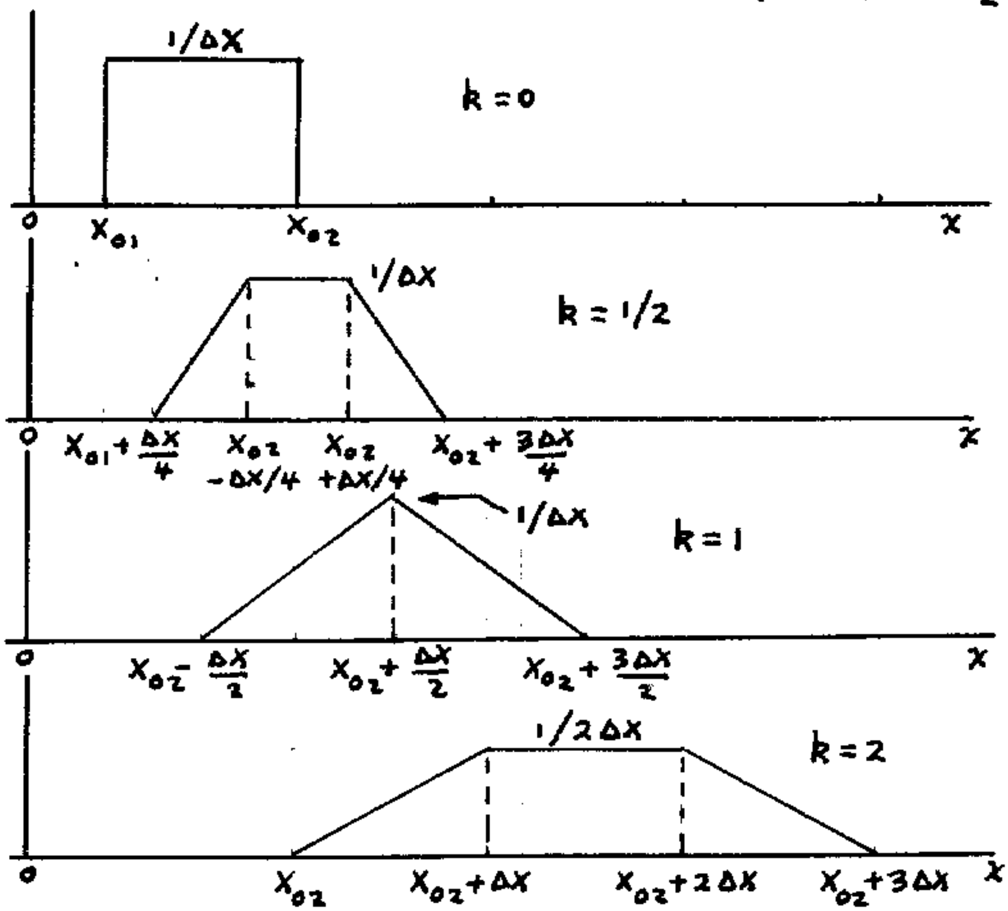
$B = (\bar{X}_0+\bar{v}t) - \frac{(X_{02}-X_{01})}{2} + \frac{(V_2-V_1)t}{2} = X_{01}+V_2t$

$C = (\bar{X}_0+\bar{v}t) + \frac{(X_{02}-X_{01})}{2} - \frac{(V_2-V_1)t}{2} = X_{02}+V_1t$

$D = (\bar{X}_0+\bar{v}t) + \frac{(X_{02}-X_{01})}{2} + \frac{(V_2-V_1)t}{2} = X_{02}+V_2t.$

(over)

★6.2-7. (Continued) The amplitude of $f_X(x;t)$ at $\bar{X}_0+\bar{V}t$ is now $1/(X_{02}-X_{01})$. (b) Here $\frac{(V_2-V_1)t}{X_{02}-X_{01}} = k\begin{cases} >1, & \text{case } 1 \\ \le 1, & \text{case } 2 \end{cases}$ so $k=2$ is case 1 while $k=0, \frac{1}{2}$, and 1 are case 2. Finally, we evaluate our results for the special case where $V_2 = 3V_1$ and where we define $\Delta X = \frac{X_{02}-X_{01}}{2}$.



6.2-8. (a) $\overline{\epsilon^2} = E\left[\{X(t_1+\tau) - \hat{X}(t_1+\tau)\}^2\right] = \overline{X^2(t_1+\tau)} + \overline{\hat{X}^2(t_1+\tau)}$
$-2\,\overline{X(t_1+\tau)\hat{X}(t_1+\tau)} = R_{XX}(0) + \overline{[\alpha X(t_1)+\beta]^2}$
$-2\,\overline{[\alpha X(t_1)X(t_1+\tau) + \beta X(t_1+\tau)]} = R_{XX}(0) + \alpha^2 R_{XX}(0) + \beta^2$
$+2\alpha\beta\overline{X(t_1)} - 2\alpha R_{XX}(\tau) - 2\beta\overline{X(t+\tau)} = R_{XX}(0)(1+\alpha^2) + \beta^2$
$+2\beta(\alpha-1)\bar{X} - 2\alpha R_{XX}(\tau)$. Differentiate to minimize.

(6.2-8.) (Continued) $\dfrac{\partial \overline{\varepsilon^2}}{\partial \beta} = 2\beta + 2(\alpha - 1)\bar{X} = 0$

$\dfrac{\partial \overline{\varepsilon^2}}{\partial \alpha} = 2\alpha R_{XX}(0) + 2\beta\bar{X} - 2R_{XX}(\tau) = 0$. On solving for α and β we have

$$\alpha = \frac{R_{XX}(\tau) - \bar{X}^2}{R_{XX}(0) - \bar{X}^2}, \quad \beta = \frac{\bar{X}\left[R_{XX}(0) - R_{XX}(\tau)\right]}{R_{XX}(0) - \bar{X}^2}$$

(b)

$$\overline{\varepsilon^2}_{min} = R_{XX}(0)\left[1 + \left\{\frac{R_{XX}(\tau) - \bar{X}^2}{R_{XX}(0) - \bar{X}^2}\right\}^2\right] + \bar{X}^2\left[\frac{R_{XX}(0) - R_{XX}(\tau)}{R_{XX}(0) - \bar{X}^2}\right]^2$$

$$+2\bar{X}^2\left[\frac{R_{XX}(0) - R_{XX}(\tau)}{R_{XX}(0) - \bar{X}^2}\right]\left[\frac{R_{XX}(\tau) - \bar{X}^2}{R_{XX}(0) - \bar{X}^2} - 1\right] - 2R_{XX}(\tau)\left[\frac{R_{XX}(\tau) - \bar{X}^2}{R_{XX}(0) - \bar{X}^2}\right]$$

$$= \left[R_{XX}(0) - R_{XX}(\tau)\right]\left[1 + \frac{R_{XX}(\tau) - \bar{X}^2}{R_{XX}(0) - \bar{X}^2}\right]$$

An alternate form uses $C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$ to get

$$\overline{\varepsilon^2}_{min} = \frac{C_{XX}^2(0) - C_{XX}^2(\tau)}{C_{XX}(0)}.$$

(6.2-9) $\mathcal{R}_{XX}(t, t+\tau) = \lim_{T\to\infty} \dfrac{1}{2T}\displaystyle\int_{-T}^{T} A\cos(\omega_0 t + \theta) A\cos(\omega_0 t + \omega_0\tau + \theta) \cdot dt$

$$= \frac{A^2}{2} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[\cos(\omega_0\tau) + \cos(2\omega_0 t + 2\theta + \omega_0\tau)\right] dt$$

$$= \frac{A^2}{2}\left[\cos(\omega_0\tau) + \lim_{T\to\infty} \frac{\sin(2\omega_0 T + 2\theta + \omega_0\tau) - \sin(-2\omega_0 T + 2\theta + \omega_0\tau)}{4\omega_0 T}\right]$$

The second term approaches 0 as $T\to\infty$ so

$\mathcal{R}_{XX}(\tau) = \dfrac{A^2}{2}\cos(\omega_0\tau)$. $\quad \bar{x} = \lim_{T\to\infty}\dfrac{1}{2T}\displaystyle\int_{-T}^{T} A\cos(\omega_0 t + \theta)\, dt = 0.$

On comparing with $\bar{X}$ and $R_{XX}(t, t+\tau)$ we see the means are the same but time averaging has removed the time-dependent term in $R_{XX}(t, t+\tau)$.

(6.2-10) (a) From the given function, which has no periodic component, we use (6.3-8) and (6.3-7) to

6.2-10. (Continued) get $|\bar{X}|^2 = 18$ so $\bar{X} = \pm\sqrt{18}$. (b) No.

(c) $P_{XX} = R_{XX}(0) = 18 + (10/6) = 118/6 = 59/3$ W.

6.2-11. (a) First assume $\tau > 0$. $R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$

$= 1 \cdot P\{X(t)$ and $X(t+\tau)$ have the same sign$\}$

$- 1 \cdot P\{X(t)$ and $X(t+\tau)$ have opposite signs$\}$

$= P\{$even number of transitions$\} - P\{$odd number of transitions$\}$ $= e^{-\lambda\tau}\left[\sum_{\substack{k=0\\ k\ \text{even}}}^{\infty} \frac{(\lambda\tau)^k}{k!} - \sum_{\substack{k=0\\ k\ \text{odd}}}^{\infty} \frac{(\lambda\tau)^k}{k!}\right]$

$= e^{-\lambda\tau}\sum_{k=0}^{\infty}\frac{(-\lambda\tau)^k}{k!} = e^{-2\lambda\tau} = R_{XX}(\tau)$. Because $R_{XX}(\tau)$ must be even we have $R_{XX}(\tau) = e^{-2\lambda|\tau|}$.

(b) For any $t > t_0$, t_0 arbitrary, $P\{X(t) = 1\} =$

$P\{X(t_0) = 1\}\,P\{$even number of transitions from t_0 to $t\}$

$+ P\{X(t_0) = -1\}\,P\{$odd number of transitions from t_0 to $t\}$

$= \frac{1}{2}[P\{$even number$\} + P\{$odd number$\}] = \frac{1}{2} = P\{X(t) = -1\}$.

(c) $E[X(t)] = +1 \cdot P\{X(t) = 1\} + (-1)P\{X(t) = -1\} = 0$. (d) Since the mean is constant and the autocorrelation function is independent of t, $X(t)$ is wide-sense stationary.

6.2-12. (a) $X(t_1)$ and $X(t_2)$ can only have values of 0 and 1. There are 4 possibilities (0,0), (0,1), (1,0), and (1,1) for which $X(t_1)X(t_2) = 0, 0, 0$, and 1, respectively. $E[X(t_1)X(t_2)]$ is the average of a discrete random variable: $R_{XX}(t_1, t_2) = 0\,P\{X(t_1) = 0, X(t_2) = 0\}$

$+ 0\,P\{X(t_1) = 0, X(t_2) = 1\} + 0\,P\{X(t_1) = 1, X(t_2) = 0\}$

$+ 1\,P\{X(t_1) = 1, X(t_2) = 1\} = P\{X(t_1) = 1, X(t_2) = 1\}$.

Now $X(t_1) = 1$ and $X(t_2) = 1$ when an even number of

(6.2-12.) (Continued) transitions between t_1 and t_2. If $\tau = t_2 - t_1$ then $P\{X(t_1)=1, X(t_2)=1\} = P\{X(t_2)=1 | X(t_1)=1\} \cdot P\{X(t_1)=1\} = \frac{1}{2} P\{X(t_2)=1 | X(t_1)=1\} = \frac{1}{2} P\{k \text{ even over } \tau\}$

$$= \frac{1}{2} \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} = \frac{1}{2} e^{-\lambda\tau} \Big\{ \frac{1}{2} \sum_{k \text{ even}} (\,) + \frac{1}{2} \sum_{k \text{ odd}} (\,) + \frac{1}{2} \sum_{k \text{ even}} (\,) - \frac{1}{2} \sum_{k \text{ odd}} (\,) \Big\}$$

$$= \frac{1}{2} e^{-\lambda\tau} \Big\{ \frac{1}{2} \sum_{k=0}^{\infty} \frac{(\lambda\tau)^k}{k!} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-\lambda\tau)^k}{k!} \Big\} = \frac{1}{2} e^{-\lambda\tau} \Big\{ \frac{1}{2} e^{\lambda\tau} + \frac{1}{2} e^{-\lambda\tau} \Big\}$$

$= \frac{1}{4} + \frac{1}{4} e^{-2\lambda\tau}$. Since $R_{XX}(\tau)$ is even in τ we have $R_{XX}(\tau) = \frac{1}{4} + \frac{1}{4} e^{-2\lambda|\tau|}$. (b) A proof identical to Problem 6-44 shows $P\{X(t)=1\} = P\{X(t)=0\} = 1/2$.

(c) $E[X(t)] = 0\, P\{X(t)=0\} + 1\, P\{X(t)=1\} = 1/2$.

(d) $\bar{X}$ = constant and $R_{XX}(\tau)$ is not a function of t so $X(t)$ is wide-sense stationary.

(6.2-13.) (a) False, (b) true, (c) false, (d) true, (e) false, and (f) false.

(6.2-14.) (a) Since sample functions are constants, $X(t+\tau)$ is the same as $X(t)$, so the density does not change with time; $X(t)$ is first-order stationary. (b) Also $R_{XX}(t, t+\tau) = E[A^2] = \int_0^1 a^2 da = 1/3$ so it does not depend on t, and $X(t)$ is wide-sense stationary. (c) Results of (a) do not change, while results of (b) only differ in the value of $E[A^2]$. Hence $X(t)$ remains wide-sense stationary.

(6.2-15.) Let $f_A(a)$ be the density of A. First-Order Case: Since sample functions are constants, the density does not change with time and $f_X(x_1;t_1) = f_A(x_1)$, any $x_1; t_1$. Second-Order Case: Since $X(t_2) = X(t_1)$, then $f_X(x_2;t_2 | x_1;t_1) = \delta(x_2 - x_1)$. But, from (4.4-11), $f_X(x_1,x_2;t_1,t_2) = f_X(x_2;t_2 | x_1;t_1) f_X(x_1;t_1)$ $= f_A(x_1)\delta(x_2 - x_1)$.

(6.2-16.) $\bar{X} = E[X(t)] = E[A] = \int_0^1 a\,da = 1/2$.

$R_{XX}(t,t+\tau) = E[X(t)X(t+\tau)] = E[A^2] = \int_0^1 a^2\,da = 1/3$.

(6.2-17.) No; since $E[X(t)] = E[A] = \bar{A}$, $C_{XX}(t,t+\tau) =$ variance of $A = \sigma_A^2$, $C_{XX}(\tau)$ does not $\to 0$ as $\tau \to \infty$. Integral of (6.2-35) equals $\sigma_A^2 \neq 0$.

(6.2-18.) Use (6.2-37) with $C_{NN}(\tau) = R_{NN}(\tau)$ since $E[N(t)] = 0$.

$$\sigma_{X_T}^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\tau|}{2T}\right)\frac{\mathcal{N}_0}{2}\delta(\tau)\,d\tau = \lim_{T\to\infty}\frac{1}{2T}\left\{\int_{-2T}^{2T}\frac{\mathcal{N}_0}{2}\delta(\tau)\,d\tau - \int_{-2T}^{2T}\frac{|\tau|}{2T}\frac{\mathcal{N}_0}{2}\delta(\tau)\,d\tau\right\}.$$

The second term is zero since $|\tau|\delta(\tau) = 0$ at $\tau = 0$. Thus, $\sigma_{X_T}^2 = \lim_{T\to\infty}\frac{1}{2T}\frac{\mathcal{N}_0}{2} = 0$ and $N(t)$ is mean-ergodic.

(6.2-19.) For vertical strips of integration over the area required in Figure 6.2-1 we write

$$\sigma_{X_T}^2 = \lim_{T\to\infty}\frac{1}{4T^2}\left\{\int_{\tau=-2T}^{0}\int_{t=-T-\tau}^{T} C_{XX}(\tau)\,dt\,d\tau + \int_{\tau=0}^{2T}\int_{t=-T}^{T-\tau} C_{XX}(\tau)\,dt\,d\tau\right\} = \lim_{T\to\infty}\frac{1}{4T^2}[I_1 + I_2].$$

6.2-19. (Continued) We solve the integrals I_1 and I_2:

$$I_1 = \int_{-2T}^{0} [T-(-T-\tau)]\, C_{XX}(\tau)\, d\tau = 2T\int_{-2T}^{0}\left(1-\frac{|\tau|}{2T}\right) C_{XX}(\tau)\, d\tau$$

$$I_2 = \int_{0}^{2T} [T-\tau-(-T)]\, C_{XX}(\tau)\, d\tau = 2T\int_{0}^{2T}\left(1-\frac{\tau}{2T}\right) C_{XX}(\tau)\, d\tau\ ,\ \text{so}$$

$$\sigma^2_{X_T} = \lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right) C_{XX}(\tau)\, d\tau .$$

6.3-1. (a) $R_{YY}(t,t+\tau) = E[X^2(t)X^2(t+\tau)]$

$$= E[A^2\sin^2(\omega_0 t+\theta)\, A^2\sin^2(\omega_0 t+\theta+\omega_0\tau)]$$

$$= \frac{A^4}{4} E[\{1-\cos(2\omega_0 t+2\theta)\}\{1-\cos(2\omega_0 t+2\theta+2\omega_0\tau)\}]$$

$$= \frac{A^4}{4} E[1-\cos(2\omega_0 t+2\theta)-\cos(2\omega_0 t+2\theta+2\omega_0\tau)$$
$$+\cos(2\omega_0 t+2\theta)\cos(2\omega_0 t+2\theta+2\omega_0\tau)]$$

Since Θ is uniform on $(-\pi,\pi)$ the middle two terms become zero when (3.1-6) is used to take the expected value. Thus,

$$R_{YY}(t,t+\tau) = \frac{A^4}{4} E[1+\tfrac{1}{2}\cos(2\omega_0\tau)+\tfrac{1}{2}\cos(4\omega_0 t+4\theta+2\omega_0\tau)] = \frac{A^4}{4}[1+\tfrac{1}{2}\cos(2\omega_0\tau)] = R_{YY}(\tau).$$

(b) $R_{XY}(t,t+\tau) = E[A\sin(\omega_0 t+\theta)\, A^2\sin^2(\omega_0 t+\theta+\omega_0\tau)]$

$$= \frac{A^3}{2} E[\sin(\omega_0 t+\theta)\{1-\cos(2\omega_0 t+2\theta+2\omega_0\tau)\}]$$

$$= (A^3/2)\, E[\sin(\omega_0 t+\theta)+\tfrac{1}{2}\sin(\omega_0 t+\theta+2\omega_0\tau)$$
$$-\tfrac{1}{2}\sin(3\omega_0 t+3\theta+2\omega_0\tau)] = 0 \text{ after}$$

(3.1-6) is used. (c) For $X(t)$: $E[X(t)] = E[A\sin(\omega_0 t+\theta)] = 0$, $R_{XX}(t,t+\tau) = E[A\sin(\omega_0 t+\theta)A\sin(\omega_0 t+\theta+\omega_0\tau)] = \frac{A^2}{2}E[\cos(\omega_0\tau)-\cos(2\omega_0 t+2\theta+\omega_0\tau)]$

6.3-1. (Continued)

$= \frac{A^2}{2}\cos(\omega_0\tau)$ after (3.1-6) is used. Thus, $X(t)$ is wide-sense stationary. For $Y(t)$: $E[Y(t)] = E[X^2(t)] = R_{XX}(0) = A^2/2 =$ constant. Thus, $Y(t)$ is wide-sense stationary, and, since $R_{XY}(t,t+\tau)$ depends only on τ and not t (equals zero above) $X(t)$ and $Y(t)$ are jointly wide-sense also. (d) Yes.

6.3-2. (a) $E[Y(t)] = E[X(t)\cos(\omega_0 t+\theta)] = E_X[X(t)] \cdot \int_{-\pi}^{\pi}\cos(\omega_0 t+\theta)\frac{1}{2\pi}\,d\theta = 0$ where $E_X[\cdot]$ represents expectation with respect to X only. (b) $R_{YY}(t,t+\tau)$

$= E[X(t)\cos(\omega_0 t+\theta)X(t+\tau)\cos(\omega_0 t+\theta+\omega_0\tau)]$

$= R_{XX}(\tau)\frac{1}{2}E[\cos(\omega_0\tau)+\cos(2\omega_0 t+2\theta+\omega_0\tau)]$

$= \frac{1}{2}R_{XX}(\tau)\cos(\omega_0\tau) = R_{YY}(\tau)$.

(c) Yes, $Y(t)$ is wide-sense stationary because $E[Y(t)] = 0$ is constant while $R_{YY}(t,t+\tau) = R_{YY}(\tau)$ does not depend on t.

6.3-3. $E[X(t)] = E[A\cos(\omega_0 t)+B\sin(\omega_0 t)] = E(A)\cos(\omega_0 t) + E[B]\sin(\omega_0 t) = 0$. $E[X(t)X(t+\tau)] = R_{XX}(t,t+\tau)$

$= E[A\cos(\omega_0 t)A\cos(\omega_0 t+\omega_0\tau)$

$+ A\cos(\omega_0 t)B\sin(\omega_0 t+\omega_0\tau)$

$+ B\sin(\omega_0 t)A\cos(\omega_0 t+\omega_0\tau)$

(6.3-3.) (Continued)

$+ B \sin(\omega_0 t) B \sin(\omega_0 t + \omega_0 \tau)]$

$= E[A^2] \cos(\omega_0 t) \cos(\omega_0 t + \omega_0 \tau) + E[B^2] \sin(\omega_0 t) \sin(\omega_0 t + \omega_0 \tau)$

$= \sigma^2 \cos(\omega_0 \tau) = R_{XX}(\tau)$. Thus, $X(t)$ is wide-sense stationary. To show $X(t)$ is not strictly stationary we prove $E[X^3(t)]$ depends on t.

$E[X^3(t)] = E[A^3 \cos^3(\omega_0 t) + 3A^2 B \cos^2(\omega_0 t) \sin(\omega_0 t) + 3AB^2 \cos(\omega_0 t) \sin^2(\omega_0 t) + B^3 \sin^3(\omega_0 t)]$. With no knowledge about third-order moments we conclude that $E[X^3(t)]$ depends on t in general.

(6.3-4.) (a) $E[Y] = E\left[\int_0^2 X(t)\,dt\right] = \int_0^2 E[X(t)]\,dt = 3\int_0^2 dt = 6$. (b) $E[Y^2] = E\left[\int_0^2 X(t)\,dt \int_0^2 X(u)\,du\right] = \int_0^2 \int_0^2 E[X(t)X(u)]\,du\,dt = \int_0^2 \int_0^2 R_{XX}(t-u)\,dt\,du = \int_0^2 \int_0^2 [9 + 2e^{-|t-u|}]\,dt\,du$

$= 36 + 2\int_0^2 \int_0^2 e^{-|t-u|}\,dt\,du = 4(10 + e^{-2})$ after using (C-45). $\sigma_Y^2 = E[Y^2] - (E[Y])^2 = 4(1 + e^{-2}) \approx 4.541$.

(6.3-5.) (a) For $t = 0$: $X(0) = A$ so $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_A} \exp[-x^2/2\sigma_A^2]$.

For $t = 1$: $X(1) = A\cos(\pi) = -A$ so again $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_A} \exp[-x^2/2\sigma_A^2]$. (b) $E[X(t)] = E[A] \cdot \cos(\pi t) = 0$ because $E[A] = 0$. $E[X^2(t)] = E[A^2] \cdot \cos^2(\pi t) = \sigma_A^2 \cos^2(\pi t)$. Since $E[X^2(t)]$ is time-dependent $X(t)$ is not even wide-sense stationary.

6.3-6. (a) $E[X(t)] = A\,P(A) + (-A)\,P(-A) = \frac{A}{2} - \frac{A}{2} = 0$.

(b) Here $R_{XX}(t_1, t_2) = A^2$ if both t_1 and t_2 are in the same interval $(n-1)T < t_1, t_2 < nT$, $n = 0, \pm 1, \pm 2, \cdots$ and $R_{XX}(t_1, t_2) = 0$ otherwise. Hence $R_{XX}(0.5T, 0.7T) = A^2$, $R_{XX}(0.2T, 1.2T) = 0$.

6.3-7. Let $x_1 = 2$, $x_2 = 2\cos(t)$ and $x_3 = 3\sin(t)$. Then $f_X(x) = \frac{1}{3}\delta(x - x_1) + \frac{1}{3}\delta(x - x_2) + \frac{1}{3}\delta(x - x_3)$ and $E[X(t)] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x\left[\frac{1}{3}\delta(x - x_1) + \frac{1}{3}\delta(x - x_2) + \frac{1}{3}\delta(x - x_3)\right]dx = \frac{1}{3}[2 + 2\cos(t) + 3\sin(t)]$. The mean value is time dependent so $X(t)$ is not stationary in any sense.

6.3-8. (a) $R_{W_1W_1}(t, t+\tau) = E[\{X(t) + Y(t)\}\{X(t+\tau) + Y(t+\tau)\}]$ $= R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(t, t+\tau) + R_{YX}(t, t+\tau)$. But $X(t)$ and $Y(t)$ are independent so $R_{XY}(t, t+\tau) = E[X(t)]\,E[Y(t+\tau)] = 0$ and $R_{YX}(t, t+\tau) = 0$. Thus,

(over)

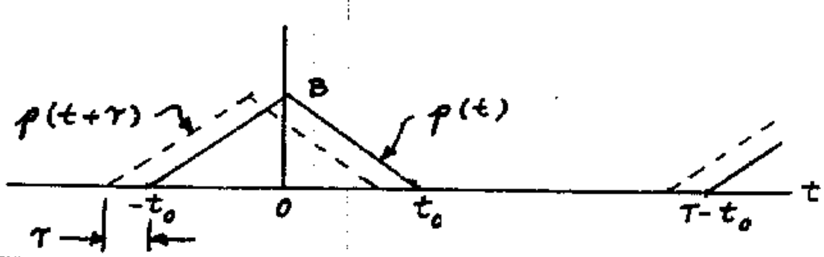
(6.3-8.) (Continued) $R_{W_1W_1}(t,t+\tau) = R_{XX}(\tau) + R_{YY}(\tau)$ $= \cos(2\pi\tau) + e^{-|\tau|}$. (b) $R_{W_2W_2}(t,t+\tau) = E[\{X(t)-Y(t)\}\{X(t+\tau)-Y(t+\tau)\}] = R_{XX}(\tau) + R_{YY}(\tau) = R_{W_2W_2}(\tau)$ $= \cos(2\pi\tau) + e^{-|\tau|}$. Cross terms are zero because $X(t)$ and $Y(t)$ are independent with zero means. (c) $R_{W_1W_2}(t,t+\tau) = E[\{X(t)+Y(t)\}\{X(t+\tau) - Y(t+\tau)\}] = R_{XX}(\tau) - R_{YY}(\tau) = e^{-|\tau|} - \cos(2\pi\tau)$.

(6.3-9.) $X(t)X(t+\tau) = p(t+\epsilon)\,p(t+\epsilon+\tau)$. The expectation is with respect to ϵ (having values e) where

$$f_\epsilon(e) = \frac{1}{T}, \quad 0 < e < T$$
$$= 0, \quad \text{elsewhere}.$$

Thus, $E[X(t)X(t+\tau)] = \int_{-\infty}^{\infty} p(t+e)\,p(t+e+\tau)\,f_\epsilon(e)\,de$ $= \int_0^T p(t+e)\,p(t+e+\tau)\,\frac{1}{T}\,de$. Let $\xi = t+e$, $d\xi = de$ so $E[X(t)X(t+\tau)] = \frac{1}{T}\int_t^{t+T} p(\xi)p(\xi+\tau)\,d\xi$. Now since $p(t)$ is periodic with period T, the integral is the same for any t. Letting $t=0$: $E[X(t)X(t+\tau)] = \frac{1}{T}\int_0^T p(\xi)p(\xi+\tau)\,d\xi = R_{XX}(\tau)$.

★(6.3-10.) (a)



*(6.3-10.) (Continued) Two cases are of interest. First, let $0 \le \tau \le t_0$: In the central period we have

$p(t)p(t+\tau) = 0$ for $t < -t_0$ and $t > t_0 - \tau$.

$$= \left(\frac{B}{t_0}\right)^2 (t_0+t)(t_0+t+\tau), \quad -t_0 < t < -\tau$$

$$= \left(\frac{B}{t_0}\right)^2 (t_0+t)(t_0-t-\tau), \quad -\tau < t < 0$$

$$= \left(\frac{B}{t_0}\right)^2 (t_0-t)(t_0-t-\tau), \quad 0 < t < t_0-\tau.$$

$$R_{XX}(\tau) = \frac{1}{T}\left(\frac{B}{t_0}\right)^2 \left\{ \int_{-t_0}^{-\tau} [(t_0+t)^2 + \tau(t_0+t)]\,dt \right.$$

$$+ \int_{-\tau}^{0} (t_0+t)(t_0-t-\tau)\,dt$$

$$\left. + \int_0^{t_0-\tau} (t_0-t)(t_0-t-\tau)\,dt \right\}$$

$$= \frac{1}{T}\left(\frac{B}{t_0}\right)^2 \frac{1}{6}\left[4(t_0-\tau)^3 + 6\tau(t_0-\tau)^2 + 6t_0\tau(t_0-\tau) + \tau^3\right] \quad (1)$$

for $0 \le \tau \le t_0$. Next, let $t_0 < \tau \le 2t_0$:

$$p(t)p(t+\tau) = \left(\frac{B}{t_0}\right)^2 (t_0+t)(t_0-t-\tau), \quad -t_0 < t < t_0-\tau$$

$$= 0, \quad \text{elsewhere.}$$

$$R_{XX}(\tau) = \frac{1}{T}\left(\frac{B}{t_0}\right)^2 \int_{-t_0}^{t_0-\tau} (t_0+t)(t_0-t-\tau)\,dt$$

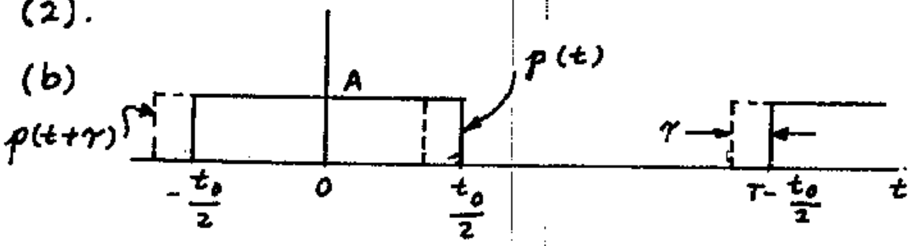
$$= \frac{1}{T}\left(\frac{B}{t_0}\right)^2 \frac{1}{6}\left[-2(t_0-\tau)^3 + 6(t_0-\tau)^2\left(t_0-\frac{\tau}{2}\right) + t_0^2(4t_0-3\tau)\right] \quad (2)$$

for $t_0 < \tau < 2t_0$.

$R_{XX}(\tau)$ is the combination of (1) and (2) for $0 < \tau < 2t_0$. It also gives $R_{XX}(\tau)$ for $-2t_0 < \tau < 0$ because $R_{XX}(\tau)$ must be an even function of τ. Finally, $R_{XX}(\tau)$ is periodic because $p(t)$ is and

*(6.3-10.) (Continued) (1) and (2) displaced in τ by nT, $n = 0, \pm 1, \pm 2, \cdots$ completes the description of $R_{XX}(\tau)$. Note that the condition $4t_0 \le T$ prevents the displaced "replicas" of (1) and (2) from overlapping the portion given by (1) and (2).

(b)



For $0 \le \tau \le t_0$: $R_{YY}(\tau) = \frac{1}{T}\int_{-t_0/2}^{(t_0/2)-\tau} A^2 dt = \frac{A^2}{T}(t_0 - \tau)$.

Because $R_{YY}(\tau)$ is even:

$$R_{YY}(\tau) = \frac{A^2}{T}(t_0 - |\tau|), \quad -t_0 \le \tau \le t_0$$
$$= 0, \qquad \text{elsewhere} \quad (3)$$

for the central period and replicas of (3) exist at nT, $n = 0, \pm 1, \pm 2, \cdots$. They do not overlap since $2t_0 \le T$ was assumed.

(6.3-11.) $R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = E[p_1(t+\epsilon)p_2(t+\epsilon+\tau)]$

$= \int_{-\infty}^{\infty} p_1(t+e)p_2(t+\tau+e) f_\epsilon(e)\, de = \frac{1}{T}\int_0^T p_1(t+e)p_2(t+e+\tau) de.$

Let $\xi = t + e$, $d\xi = de$ so

$$R_{XY}(t, t+\tau) = \frac{1}{T}\int_t^{t+T} p_1(\xi)\, p_2(\xi+\tau)\, d\xi$$

which has to be true for any t because $p_1(\xi)$

6.3-11. (Continued) and $f_2(\xi)$ are periodic. By letting $t=0$:

$$R_{XY}(t,t+\tau) = \frac{1}{T}\int_0^T f_1(\xi) f_2(\xi+\tau)\,d\xi = R_{XY}(\tau).$$

6.3-12. (a) $E[\{X(t+\tau) \pm X(t)\}^2] = E[X^2(t+\tau) + X^2(t) \pm 2X(t)X(t+\tau)] = 2R_{XX}(0) \pm 2R_{XX}(\tau) \geq 0$. Now if $R_{XX}(\tau) > 0$ for some τ we use the negative sign and write $R_{XX}(0) - |R_{XX}(\tau)| \geq 0$. If $R_{XX}(\tau) < 0$ for some τ we use the positive sign and write $R_{XX}(0) - |R_{XX}(\tau)| \geq 0$. Thus, in any case $R_{XX}(0) \geq |R_{XX}(\tau)|$ which proves (6.3-4).
(b) $R_{XX}(-\tau) = E[X(t)X(t-\tau)]$. Let $\xi = t-\tau$ so $R_{XX}(-\tau) = E[X(\xi+\tau)X(\xi)] = R_{XX}(\tau)$ which proves (6.3-5).

6.3-13. Since $R_{XX}(\tau) = E[X(t)X(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$

we see that large τ means large separation between values of $X(t)$ in time. Now because $X(t)$ is ergodic we expect any sample function over time to take on all the statistically allowable values of $X(t)$. For large enough τ we expect these values to fluctuate with time in a manner unrelated to values a large time (τ) away. In other words we expect $X(t)$ and $X(t+\tau)$ to

(6.3-13.) (Continued) be approximately independent so that $E[X(t)X(t+\tau)] \to E[X(t)]E[X(t+\tau)] = 0$ as $\tau \to \infty$ if $X(t)$ has no mean value.

(6.3-14.) (a) From (6.3-7): $E[X(t)] = \bar{X} = \sqrt{20}$.
(b) From (6.3-6): $E[X^2(t)] = \overline{X^2} = R_{XX}(0) = 50.$
(c) From (3.2-6): $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 50 - 20 = 30.$

(6.3-15.) (a) $E[Y(t)] = E[X(t) - X(t+\tau)] = E[X(t)] - E[X(t+\tau)] = \bar{X} - \bar{X} = 0$ because $X(t)$ is wide-sense stationary. (b) Since $\bar{Y} = 0$, $\sigma_Y^2 = E[Y^2(t)] = E[\{X(t) - X(t+\tau)\}^2] = E[X^2(t) - 2X(t)X(t+\tau) + X^2(t+\tau)] = 2R_{XX}(0) - 2R_{XX}(\tau)$. (c) $E[Y(t)] = E[X(t) + X(t+\tau)] = 2\bar{X}$. $\sigma_Y^2 = E[Y^2(t)] - (2\bar{X})^2$. $E[Y^2(t)] = E[X^2(t) + 2X(t)X(t+\tau) + X^2(t+\tau)] = 2R_{XX}(0) + 2R_{XX}(\tau)$.
$\sigma_Y^2 = 2R_{XX}(0) + 2R_{XX}(\tau) - 4\bar{X}^2$. But since $\sigma_X^2 = R_{XX}(0) - \bar{X}^2$ we have $\sigma_Y^2 = 4\sigma_X^2 - 2[R_{XX}(0) - R_{XX}(\tau)]$. These results are significantly different than those of (a) and (b) because of the presence of a mean value due to the change in sign.

(6.3-16.) Here $\bar{X} = 0$, $\bar{Y} = 0$, $\sigma_X^2 = R_{XX}(0) = 5$, $\sigma_Y^2 = R_{YY}(0) = 10$.
(a) Function does not have even symmetry. (b) Function does not have even symmetry. (c) Function does not satisfy $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)R_{YY}(0)} = \sqrt{50}$.

6.3-16. (Continued) (d) Function is negative for $\tau = 0$. (e) Function does not satisfy $R_{YY}(0) = 10$. (f) Function has a constant component which is not allowed because $Y(t)$ has zero mean.

6.3-17. (a) $R_{WW}(t, t+\tau) = E[\{X(t)+Y(t)\}\{X(t+\tau)+Y(t+\tau)\}]$

$= R_{XX}(t, t+\tau) + R_{XY}(t, t+\tau) + R_{YX}(t, t+\tau) + R_{YY}(t, t+\tau)$.

(b) Here (6.3-29) applies: $R_{WW}(t, t+\tau) =$

$$R_{XX}(t, t+\tau) + R_{YY}(t, t+\tau) + E[X(t)]E[Y(t+\tau)] + E[Y(t)]E[X(t+\tau)].$$

(c) For zero means: $R_{WW}(t, t+\tau) = R_{XX}(t, t+\tau) + R_{YY}(t, t+\tau)$.

Note that the processes were not specified to be stationary.

6.3-18. $E[\{Y(t+\tau) + \alpha X(t)\}^2] = E[Y^2(t+\tau) + 2\alpha X(t) Y(t+\tau) + \alpha^2 X^2(t)] = R_{YY}(0) + 2\alpha R_{XY}(\tau) + \alpha^2 R_{XX}(0)$

$= a\alpha^2 + b\alpha + c \geq 0$. For this quadratic form to be nonnegative with α the roots cannot be real (except as a double real root). Roots will be imaginary or complex if

$$b^2 - 4ac = 4R_{XY}^2(\tau) - 4R_{XX}(0)R_{YY}(0) \leq 0$$

or

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) R_{YY}(0)}\,.$$

6.3-19 (a) $E[\dot{X}(t)] = E\left[\lim_{\epsilon \to 0} \frac{X(t+\epsilon) - X(t)}{\epsilon}\right]$

$$= \lim_{\epsilon \to 0} \frac{E[X(t+\epsilon)] - E[X(t)]}{\epsilon} = \lim_{\epsilon \to 0} \frac{\bar{X} - \bar{X}}{\epsilon} = 0.$$

(b) $E[X(t)\dot{X}(t+\tau)] = R_{X\dot{X}}(\tau) = E\left[\lim_{\epsilon \to 0}\left\{\frac{X(t)X(t+\tau+\epsilon)}{\epsilon}\right.\right.$

$$\left.\left. - \frac{X(t)X(t+\tau)}{\epsilon}\right\}\right] = \lim_{\epsilon \to 0} \frac{R_{XX}(\tau+\epsilon) - R_{XX}(\tau)}{\epsilon} = \frac{dR_{XX}(\tau)}{d\tau}.$$

(c) $R_{\dot{X}\dot{X}}(\tau) = E[\dot{X}(t)\dot{X}(t+\tau)] = E\left[\lim_{\epsilon \to 0} \frac{X(t+\epsilon) - X(t)}{\epsilon}\dot{X}(t+\tau)\right]$

$$= \lim_{\epsilon \to 0} \frac{R_{X\dot{X}}(\tau-\epsilon) - R_{X\dot{X}}(\tau)}{\epsilon} = -\frac{dR_{X\dot{X}}(\tau)}{d\tau} = -\frac{d^2R_{XX}(\tau)}{d\tau^2}$$

after the result of part (b) is used.

6.3-20. (a) No — has a periodic component. (b) Yes.
(c) No — not even in τ, total power incorrect, (d) No — depends on t, not even in τ, average power is ∞.

6.3-21. $R_{XX}(t, t+\tau) = \frac{1}{T}\int_0^T p(\xi)p(\xi+\tau)\,d\xi$

$$= \frac{A^2}{T}\int_0^T \cos^2\left(\frac{2\pi\xi}{T}\right)\cos^2\left(\frac{2\pi\xi + 2\pi\tau}{T}\right)d\xi$$

$$= \frac{A^2}{T}\int_0^T \left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{4\pi\xi}{T}\right)\right]\left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{4\pi\xi}{T} + \frac{4\pi\tau}{T}\right)\right]d\xi$$

$$= \frac{A^2}{8}\left[2 + \cos\left(\frac{4\pi\tau}{T}\right)\right].$$

6.3-22. (a) $E[X(t)] = E[At] = t\int_0^1 a\,dt = t/2$. (b) $R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = E[At\,A(t+\tau)] = E[A^2](t^2+\tau t) = \int_0^1 a^2\,da\,(t^2+\tau t) = \frac{1}{3}(t^2+\tau t)$. (c) $X(t)$ is not stationary in any sense.

6.3-23. (a) $E[W(t)] = E[X^2(t)\cos^2(\omega_0 t + \Theta)] = \frac{E[X^2(t)]}{2}\left\{1 + \int_{-\pi}^{\pi}\cos(2\omega_0 t + 2\theta)\frac{d\theta}{2\pi}\right.$ $= \frac{1}{2}E[X^2(t)]$. (b) $R_{WW}(t, t+\tau) = E\{X^2(t)\cos^2(\omega_0 t + \Theta)X^2(t+\tau)\cos^2(\omega_0 t + \omega_0\tau + \Theta)\}$ $= \frac{1}{4}E[X^2(t)X^2(t+\tau)]\left[\frac{1}{2} + \cos^2(\omega_0\tau)\right]$ which is a function of τ only because of second-order stationarity of $X(t)$. (c) $W(t)$ is wide-sense stationary since $X(t)$ has a constant mean and autocorrelation that is not a function of t.

6.3-24. (a) Function of t not allowed. (b) Odd autocorrelation not allowed. Autocorrelation maximum is not where $\tau = 0$. (c) Maximum is at $\tau = 2$ and not $\tau = 0$. Autocorrelation must be even function of τ. (d) $R_{XX}(t,t)$ cannot be negative. (e) Odd function of τ not allowed. Maximum is not at $\tau = 0$, as required.

6.3-25. (a) Odd function of τ not allowed. (b) $R_{YY}(t,t) \neq 4$, as required. (c) Not an even function of τ, as required. (d) Not an even function of τ, as required. (e) Function cannot depend on t.

6.3-26. (a) $R_{Y_1Y_2}(t, t+\tau) = E\{X(t)\cos(\omega_0 t)Y(t+\tau)\cos(\omega_0 t + \omega_0\tau + \Theta)\}$ $= \frac{1}{2}R_{XY}(\tau)[\cos(\omega_0\tau + \Theta) + \cos(2\omega_0 t + \omega_0\tau + \Theta)]$. There is no constant value of Θ that will make both terms zero for all τ. (c) after the expectation in part (a) form the expectation with respect to Θ. $R_{Y_1Y_2}(t, t+\tau) = \frac{1}{2}R_{XY}(\tau)\int_{\theta_1}^{\theta_2}[\cos(\omega_0\tau + \theta) + \cos(2\omega_0 t + \omega_0\tau + \theta)]\frac{d\theta}{(\theta_2 - \theta_1)} = 0$ if $\theta_2 = \theta_1 + 2\pi$. Thus, Θ uniform on $(\theta_1, \theta_1 + 2\pi)$ makes $Y_1(t)$ and $Y_2(t)$ orthogonal.

6.3-27. $R_{XY}(\tau)$ must satisfy (6.3-17). The largest $|R_{XY}(\tau)|$ is found: $\frac{dR_{XY}(\tau)}{d\tau} = Ke^{-\tau^2}(\pi)\cos(\pi\tau) + K\sin(\pi\tau)e^{-\tau^2}(-2\tau) = 0$ when $\tan(\pi\tau) = \pi/(2\tau)$. By trial-and-error this condition occurs when $\tau = 0.41734$. At this value of τ $R_{XX}(0.41734) = Ke^{-0.41734^2}\sin(0.41734\pi) = 0.8120K \overset{\text{must}}{\leq} \sqrt{4(6)} = \sqrt{24} \approx 4.8990$ so $K \leq 6.0334$. With this value of K, (6.3-18) is also satisfied.

6.4-1. For a gaussian random variable to have probability of 0.9606 of being in an interval centered on its mean the interval width is 4.12σ from Appendix B. Thus $|X-\bar{X}| = 2.06\sigma$. But since $|X-\bar{X}|$ must be 0.1 V here, then $\sigma = 0.1/2.06$. However, $(\sigma_{\hat{\bar{X}}})^2 = \sigma_X^2/N$ so we require $\sigma_X^2/N \leq \sigma^2 = 10^{-2}/(2.06)^2$ or $\sigma_X^2 \leq 10^{-2}N/(2.06)^2 = 1/(2.06)^2 = 0.2356$

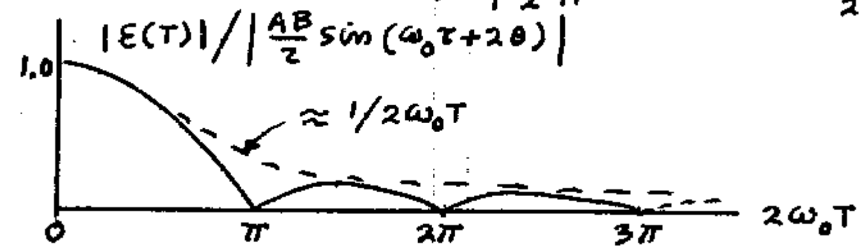
6.4-2. $\overline{X(t)} = s(t) + \overline{N(t)} = s(t) + \bar{N}$. $R_{XX}(t,t+\tau) = E[\{s(t)+N(t)\}\{s(t+\tau)+N(t+\tau)\}] = s(t)s(t+\tau) + s(t)\bar{N} + \bar{N}s(t+\tau) + R_{NN}(\tau)$. $C_{XX}(t,t+\tau) = E[\{X(t) - \overline{X(t)}\}\cdot\{X(t+\tau) - \overline{X(t+\tau)}\}] = E[\{N(t)-\bar{N}\}\{N(t+\tau)-\bar{N}\}] = R_{NN}(\tau) - \bar{N}^2$. $X(t)$ is not stationary in any sense because the mean is time dependent; however, the autocorrelation function is independent of t.

(6.4-3.) (a) $R_{XY}(t,t+\tau) = E[AB\cos(\omega_0 t + \Theta)\sin(\omega_0 t + \Theta + \omega_0\tau)] = \frac{AB}{2}E[\sin(\omega_0\tau) + \sin(2\omega_0 t + 2\Theta + \omega_0\tau)]$

$= \frac{AB}{2}\sin(\omega_0\tau) = R_{XY}(\tau)$. This is not a function of t so $X(t)$ and $Y(t)$ are jointly wide-sense stationary. (b) $R_0(2T) = \frac{1}{2T}\int_{-T}^{T} AB\cos(\omega_0 t + \theta)\sin(\omega_0 t + \theta + \omega_0\tau)\,dt = \frac{AB}{2}\cdot\frac{1}{2T}\int_{-T}^{T}[\sin(\omega_0\tau) + \sin(2\omega_0 t + 2\theta + \omega_0\tau)]\,dt$

$$= \underbrace{\frac{AB}{2}\sin(\omega_0\tau)}_{R_{XY}(\tau)} + \underbrace{\frac{AB}{2}\frac{\sin(\omega_0\tau+2\theta)\sin(2\omega_0 T)}{2\omega_0 T}}_{\varepsilon(T)}$$

(c) $|\varepsilon(T)| = \left|\frac{AB}{2}\right|\left|\sin(\omega_0\tau+2\theta)\frac{\sin(2\omega_0 T)}{2\omega_0 T}\right|$



For $|\varepsilon(T)| \le \left|\frac{AB}{2}\right|\left|\frac{\sin(2\omega_0 T)}{2\omega_0 T}\right| \le \left|\frac{AB}{2}\right|\frac{1}{2\omega_0 T} \le 0.01\left|\frac{AB}{2}\right|$ we require $T \ge 50/\omega_0$.

(6.4-4.) (a) By the same procedures as Example 6.2-1 we find $R_{XX}(t,t+\tau) = \frac{A^2}{2}\cos(\omega_0\tau) = R_{XX}(\tau)$

$R_{YY}(t,t+\tau) = \frac{B^2}{2}\cos(\omega_1\tau) = R_{YY}(\tau)$

also $E[Y(t)] = B\int_0^{2\pi}\cos(\omega_0 t + \theta)\frac{d\theta}{2\pi} = 0$, $E[X(t)] = 0$ by the same procedure. Next, $R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)]$

$= E[AB\cos(\omega_0 t + \Theta)\cos(\omega_1 t + \omega_1\tau + \Phi)]$

$= \frac{AB}{2}E[\cos\{(\omega_1-\omega_0)t + \omega_1\tau + \Phi - \Theta\} + \cos\{(\omega_1+\omega_0)t + \omega_1\tau + \Phi + \Theta\}]$

$= 0$ which is not a function of t so $X(t)$ and $Y(t)$ are jointly wide-sense stationary. (b) If $\Phi = \Theta$:

6.4-4. (Continued)

$R_{XY}(t, t+\tau) = \frac{AB}{2}\cos[(\omega_1 - \omega_0)t + \omega_1\tau]$ which is not a function of t only if $\omega_1 = \omega_0$.

6.4-5. $t_k - t_i = (2k-2) - (2i-2) = 2(k-i)$

$$R_{XX}(t_k - t_i) = \begin{cases} 13[1 - |k-i|/3], & |k-i| \le 3 \\ 0, & |k-i| > 3. \end{cases}$$

$$C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - \bar{X}_i \bar{X}_k = R_{XX}(t_k - t_i) - \bar{X}_i \bar{X}_k$$

$$= \begin{cases} 13[1 - |k-i|/3], & |k-i| \le 3 \\ 0, & |k-i| > 3. \end{cases}$$

The covariance matrix of the $X_i = X(t_i)$ is

$$[C_X] = \begin{bmatrix} 13 & (\frac{26}{3}-0) & (\frac{13}{3}-0) & 0 & 0 \\ (\frac{26}{3}-0) & 13 & (\frac{26}{3}-0) & (\frac{13}{3}-0) & 0 \\ (\frac{13}{3}-0) & (\frac{26}{3}-0) & 13 & (\frac{26}{3}-0) & (\frac{13}{3}-0) \\ 0 & (\frac{13}{3}-0) & (\frac{26}{3}-0) & 13 & (\frac{26}{3}-0) \\ 0 & 0 & (\frac{13}{3}-0) & (\frac{26}{3}-0) & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 26 & 13 & 0 & 0 \\ 26 & 39 & 26 & 13 & 0 \\ 13 & 26 & 39 & 26 & 13 \\ 0 & 13 & 26 & 39 & 26 \\ 0 & 0 & 13 & 26 & 39 \end{bmatrix} \frac{1}{3}.$$

6.4-6. From (6.4-2): $R_0(2T) = \frac{1}{2T}\int_{-T}^{T} x(t)\cos(\omega_0 t + \theta)\, x(t+\tau)\cos(\omega_0 t + \omega_0\tau + \theta)\, dt = \frac{1}{2}\cos(\omega_0\tau)\frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt + \frac{1}{2}\frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau) \cdot \cos(2\omega_0 t + 2\theta + \omega_0\tau)\,dt$. The second integral is approximately zero because the cosine term cycles rapidly relative to $x(t)x(t+\tau)$. Thus, $R_0(2T) \approx \frac{1}{2}R_{XX}(\tau)\cos(\omega_0\tau)$.

(6.5-1.) Define $X_1 = X(t)$, $X_2 = X(t+1)$, $X_3 = X(t+2)$, $X_4 = X(t+3)$. The covariance matrix of the X_i, $i = 1, 2, 3, 4$ will be a 4×4 matrix with elements $C_{ij} = E[X_i X_j] = E[X(t+i-1)X(t+j-1)] = R_{XX}(j-i)$. These steps are allowed because $X(t)$ is zero-mean from (6.3-7). Thus, $C_{ij} = R_{XX}(j-i) = 6\exp[-|j-i|/2]$ so

$$[C] = 6\begin{bmatrix} 1 & e^{-1/2} & e^{-1} & e^{-3/2} \\ e^{-1/2} & 1 & e^{-1/2} & e^{-1} \\ e^{-1} & e^{-1/2} & 1 & e^{-1/2} \\ e^{-3/2} & e^{-1} & e^{-1/2} & 1 \end{bmatrix}$$

where $[C]$ denotes the desired covariance matrix.

(6.5-2.) Procedures are identical to those in Problem 6-29. Now $C_{ij} = R_{XX}(j-i) = 6\,\dfrac{\sin[\pi(j-i)]}{\pi(j-i)} = 6,\ i=j$
$= 0,\ i \neq j.$

Thus,

$$[C] = 6\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(6.5-3) From (6.3-7) the autocorrelation function will now have a constant term $\bar{X}^2 = 4$ added. However, from (6.5-4) this constant is removed from

(6.5-3.) (Continued)

$C_{XX}(t_i, t_k)$ so no net change occurs in the elements of $[C_X]$. This matrix $[x-\bar{x}]$ of (6.5-1) does change so the joint density is different.

(6.6-1.) $P(\text{miss 1 or more aircraft}) = 1 - P(\text{miss } 0) = 1 - P(0 \text{ arrive})$

$= 1 - \frac{(\lambda t)^0}{0!} e^{-\lambda t} = 1 - e^{-\lambda t} = 1 - e^{-\frac{12}{60}(2)} = 1 - e^{-0.4} = 0.3297.$

(6.6-2.) $P(0 \text{ calls}) = \frac{\left[\frac{75}{60} \cdot 5\right]^0}{0!} \exp\left(-\frac{75}{60} \cdot 5\right) = e^{-75(5)/60}$

$P(1 \text{ call}) = \frac{\left[\frac{75}{60} \cdot 5\right]^1}{1!} \exp\left(-\frac{75}{60} \cdot 5\right) = \frac{75(5)}{60} e^{-75(5)/60}$

Similarly, $P(2 \text{ calls}) = \left[\frac{75(5)}{60}\right]^2 \frac{1}{2} e^{-75(5)/60}$, $P(3 \text{ calls}) = \left[\frac{75(5)}{60}\right]^3 \frac{1}{6} e^{-75(5)/60}$.

$P(\text{more than 3 calls}) = 1 - P(0 \text{ calls}) - P(1 \text{ call}) - P(2 \text{ calls}) - P(3 \text{ calls})$

$= 1 - e^{-75(5)/60}\left[1 + \frac{75(5)}{60} + \left(\frac{75(5)}{60}\right)^2 \frac{1}{2} + \left(\frac{75(5)}{60}\right)^3 \frac{1}{6}\right] \approx 0.8697.$

(6.6-3.) $P(\text{waiting line}) = P(\geq 3 \text{ arrive}) = 1 - P(0 \text{ arrive}) - P(1 \text{ arrives}) - P(2 \text{ arrive})$. (a) $\lambda = 2/\text{minute}$. $P(\text{line}) = 1 - e^{-2}\left\{\frac{[2(1)]^0}{0!} + \frac{[2(1)]^1}{1!} + \frac{[2(1)]^2}{2!}\right\} = 1 - 5e^{-2} \approx 0.3233$. (b) $\lambda = 1/\text{minute}$. $P(\text{line}) = 1 - e^{-1}\left\{\frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2!}\right\} = 1 - 2.5e^{-1} \approx 0.0803$. (c) $\lambda = 1/2\ /\text{minute}$. $P(\text{line}) =$

$$1 - e^{-1/2}\left\{\frac{(1/2)^0}{0!} + \frac{(1/2)^1}{1!} + \frac{(1/2)^2}{2!}\right\} = 1 - \frac{13}{8} e^{-1/2} \approx 0.0144.$$

(6.6-4.) Desired probability is that of the joint occurrence of the two events $A = \{X(30) = 1\}$ and $B = \{X(60) - X(30) = 0\}$. Since

$P(A) = P\{X(30) = 1\} = \frac{\left[\frac{2}{30} \cdot 30\right]^1}{1!} \exp\left[-\frac{2}{30} \cdot 30\right] = 2e^{-2}$

6.6-4. (Continued)

$$P(B) = P\{X(60)-X(30)=0\} = \frac{\left[\frac{2}{30}(60-30)\right]}{0!}\exp\left[-\frac{2}{30}(60-30)\right] = e^{-2}$$

then, from independence of intervals $P(A,B) = P(A)P(B) = 2e^{-4}$ ≈ 0.0366.

6.6-5. $R_{XX}(t,t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1,x_2)\,dx_1 dx_2$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=k_1}^{\infty} P(k_1,k_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\,\delta(x_1-k_1)\delta(x_2-k_2)\,dx_1 dx_2$$

$= \sum_{k_1=0}^{\infty}\sum_{k_2=k_1}^{\infty} k_1 k_2 P(k_1,k_2)$. Next, use (6.6-7) to get $R_{XX}(t,t+\tau)$

$$= \sum_{k_1=0}^{\infty}\frac{k_1(\lambda t_1)^{k_1}e^{-\lambda t_1}}{k_1!}\sum_{k_2=k_1}^{\infty}\frac{k_2[\lambda(t_2-t_1)]^{k_2-k_1}}{(k_2-k_1)!}e^{-\lambda(t_2-t_1)} \leftarrow m = k_2-k_1$$

$$= \sum_{k_1=0}^{\infty}\frac{k_1(\lambda t_1)^{k_1}e^{-\lambda t_1}}{k_1!}\left\{\underbrace{\sum_{m=0}^{\infty}\frac{m[\lambda(t_2-t_1)]^m e^{-\lambda(t_2-t_1)}}{m!}}_{=\lambda(t_2-t_1)\text{ from (6.6-3)}} + k_1\underbrace{\sum_{m=0}^{\infty}\frac{[\lambda(t_2-t_1)]^m e^{-\lambda(t_2-t_1)}}{m!}}_{=1}\right\}$$

$$= \lambda(t_2-t_1)\underbrace{\sum_{k_1=0}^{\infty}\frac{k_1(\lambda t_1)^{k_1}e^{-\lambda t_1}}{k_1!}}_{=\lambda t_1\text{ from (6.6-3)}} + \underbrace{\sum_{k_1=0}^{\infty}\frac{k_1^2(\lambda t_1)^{k_1}e^{-\lambda t_1}}{k_1!}}_{=\lambda t_1+\lambda^2 t_2^2\text{ from (6.6-4)}} = \lambda t_1(1+\lambda t_2)$$

By repeating the above procedure the case for $t_1 > t_2$ can be proven.

6.6-6. For $t_2 > t_1$: $C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - E[X(t_1)]E[X(t_2)] = \lambda t_1 + \lambda^2 t_1 t_2 - \lambda t_1 \lambda t_2 = \lambda t_1$, $t_1 < t_2$. Similarly, for $t_1 > t_2$: $C_{XX}(t_1,t_2) = \lambda t_2$, $t_1 > t_2$.

★6.7-1. $E[|Z(t)|^2] = E[\{X(t)+jY(t)\}\{X(t)-jY(t)\}]$

$= E[X^2(t)+Y^2(t)] = E[X^2(t)] + E[Y^2(t)]$

$= R_{XX}(0) + R_{YY}(0).$

★6.7-2. (a) $R_{Z_1Z_2}(t,t+\tau) = E[Z_1^*(t)Z_2(t+\tau)]$

$= E[\{X_1(t)-jY_1(t)\}\{X_2(t+\tau)-jY_2(t+\tau)\}]$

$= R_{X_1X_2}(t,t+\tau) - R_{Y_1Y_2}(t,t+\tau) - jR_{X_1Y_2}(t,t+\tau) - jR_{Y_1X_2}(t,t+\tau).$

(b) For uncorrelated processes (6.3-29) applies.

$R_{Z_1Z_2}(t,t+\tau) = E[X_1(t)X_2(t+\tau) - Y_1(t)Y_2(t+\tau)$
$\qquad -jX_1(t)Y_2(t+\tau) - jY_1(t)X_2(t+\tau)]$

$= E[X_1(t)]E[X_2(t+\tau)] - E[Y_1(t)]E[Y_2(t+\tau)]$
$\qquad -jE[X_1(t)]E[Y_2(t+\tau)] - jE[Y_1(t)]E[X_2(t+\tau)]$

$= E[Z_1^*(t)]E[Z_2(t+\tau)].$

(c) For uncorrelated processes with zero means

$$R_{Z_1Z_2}(t,t+\tau) = 0.$$

★6.7-3. $E[|W|^2] = E\left[\int_a^{a+T} Z^*(t)\,dt \int_a^{a+T} Z(u)\,du\right]$

$= \int_a^{a+T}\int_a^{a+T} R_{ZZ}(u-t)\,du\,dt.$ Let $x = t-a$,

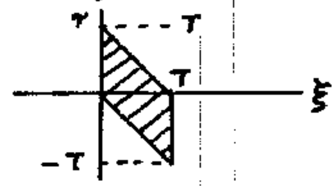
$dx = dt$, $y = u-a$, $dy = du$. Then $E[|W|^2]$

$= \int_0^T\int_0^T R_{ZZ}(y-x)\,dy\,dx$. Next, let

$$\tau = y-x, \quad d\tau = dy$$
$$\xi = x, \quad d\xi = dx$$

★ (6.7-3.) (Continued)

The region of integration is hatched in the figure.



Now $E[|W|^2] =$

$$\int_{\tau=-T}^{0} \int_{\xi=-\tau}^{T} R_{zz}(\tau)\, d\xi\, d\tau + \int_{\tau=0}^{T} \int_{\xi=0}^{T-\tau} R_{zz}(\tau)\, d\xi\, d\tau$$

$$= \int_{-T}^{0} (T+\tau) R_{zz}(\tau)\, d\tau + \int_{0}^{T} (T-\tau) R_{zz}(\tau)\, d\tau$$

$$= \int_{-T}^{T} (T-|\tau|) R_{zz}(\tau)\, d\tau .$$

★ (6.7-4.) Since the z_i are zero-mean and uncorrelated $E[z_i^* z_k] = 0$, $i \neq k$ and $= \sigma_{z_i}^2$, $i = k$. (a) $E[z(t)] = \sum_{i=1}^{N} E[z_i]\, e^{j\omega_i t} = 0$. (b) $R_{zz}(t, t+\tau) =$ $E[z^*(t) z(t+\tau)] = E\left[\sum_{i=1} z_i^* e^{-j\omega_i t} \sum_{k=1}^{N} z_k\, e^{j\omega_k (t+\tau)}\right]$
$= \sum_{i=1}^{N} \sum_{k=1}^{N} E[z_i^* z_k]\, e^{j(\omega_k - \omega_i)t + j\omega_k \tau} = \sum_{i=1}^{N} \sigma_{z_i}^2\, e^{j\omega_i \tau} = R_{zz}(\tau)$.
$z(t)$ is wide-sense stationary since the mean is constant and the autocorrelation function does not depend on t.

★ 6.7-5 (a) $E[Z(t)] = \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} e^{j\omega t}\frac{d\omega}{2\Delta\omega} = e^{j\omega_0 t}\frac{\sin(\Delta\omega t)}{\Delta\omega t}$

(b) $R_{ZZ}(t,t+\tau) = E[e^{-j\Omega_1 t + j\Omega_1 t + j\Omega_1 \tau}] = E[e^{j\Omega_1\tau}]$

$= e^{j\omega_0\tau}\frac{\sin(\Delta\omega\tau)}{\Delta\omega\tau} = R_{ZZ}(\tau).$ ← Same as (a) except τ instead of t.

(c) $Z(t)$ is not wide-sense stationary because the mean depends on time.

★ 6.7-6. (a) $E[Z(t)] = \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}\left[e^{j\omega t} + e^{-j\omega t}\right]\frac{d\omega}{2\pi} =$

$e^{j\omega_0 t}\frac{\sin(\Delta\omega t)}{\Delta\omega t} + e^{-j\omega_0 t}\frac{\sin(\Delta\omega t)}{\Delta\omega t} = 2\cos(\omega_0 t)\frac{\sin(\Delta\omega t)}{\Delta\omega t}$

(b) $R_{ZZ}(t,t+\tau) = E\left[2\cos(\Omega_1 t)\, 2\cos(\Omega_1 t + \Omega_1\tau)\right]$

$= 2E\{\cos(\Omega_1\tau) + \cos[\Omega_1(2t+\tau)]\}$

$= 2\cos(\omega_0\tau)\frac{\sin(\Delta\omega\tau)}{\Delta\omega\tau} + 2\frac{\sin[2\Delta\omega(2t+\tau)]}{2\Delta\omega(2t+\tau)}$. (c) $Z(t)$ is not wide-sense stationary because both the mean and autocorrelation functions depend on time.

★ 6.7-7. $R_{ZZ}(t,t+\tau) = E\Big[\{X(t)\cos(\omega_0 t) - jY(t)\sin(\omega_0 t)\}$

$\cdot\{X(t+\tau)\cos(\omega_0 t + \omega_0\tau) + jY(t+\tau)\sin(\omega_0 t + \omega_0\tau)\}\Big]$

$= E\Big[X(t)X(t+\tau)\cos(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)$

$+ jX(t)Y(t+\tau)\cos(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)$

$- jY(t)X(t+\tau)\sin(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)$

$+ Y(t)Y(t+\tau)\sin(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)\Big]$

$= R(\tau)\cos(\omega_0\tau) + j\bar{X}\bar{Y}\sin(\omega_0\tau)$. $E[Z(t)] = \bar{X}\cos(\omega_0 t) + j\bar{Y}\sin(\omega_0 t)$. Thus, $Z(t)$ is not wide-sense stationary because the mean is a function of t.

CHAPTER 7

(7.1-1.) (a) $E[X(t)] = \int_{-\infty}^{\infty} x f_{\Theta}(\theta)\,d\theta = \int_0^{\pi} A\cos(\omega_0 t + \theta)\frac{1}{\pi}\,d\theta$

$= \frac{A}{\pi}\left[\sin(\omega_0 t + \pi) - \sin(\omega_0 t)\right] = \frac{-2A}{\pi}\sin(\omega_0 t)$. This mean value is time dependent so $X(t)$ is not wide-sense stationary. (b) $E[X^2(t)] = E[A^2 \cdot \cos^2(\omega_0 t + \Theta)] = \frac{A^2}{2}E[1 + \cos(2\omega_0 t + 2\Theta)]$

$= \frac{A^2}{2} + \frac{A^2}{2}\int_0^{\pi}\cos(2\omega_0 t + 2\theta)\frac{1}{\pi}\,d\theta = A^2/2$. Thus,

$P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{A^2}{2}\,dt = A^2/2$. (c) From Example 7.1-2 $X_T(\omega) = ATe^{j\Theta}Sa[(\omega-\omega_0)T] + ATe^{-j\Theta} \cdot Sa[(\omega+\omega_0)T]$ so $|X_T(\omega)|^2 = X_T(\omega)X_T^*(\omega)$

$= A^2T^2\left\{Sa^2[(\omega-\omega_0)T] + Sa^2[(\omega+\omega_0)T] + (e^{-j2\Theta} + e^{j2\Theta}) \cdot Sa[(\omega-\omega_0)T]\,Sa[(\omega+\omega_0)T]\right\}$ and $E[|X_T(\omega)|^2]$

$= A^2T^2\left\{Sa^2[(\omega-\omega_0)T] + Sa^2[(\omega+\omega_0)T] + Sa[(\omega-\omega_0)T] \cdot Sa[(\omega+\omega_0)T]\frac{2}{\pi}\int_0^{\pi}\cancelto{0}{\cos(2\theta)}\,d\theta\right.$. Thus, the power spectrum is the limit as $T\to\infty$ of

$$\frac{E[|X_T(\omega)|^2]}{2T} = \frac{A^2\pi}{2}\left\{\frac{T}{\pi}Sa^2[(\omega-\omega_0)T] + \frac{T}{\pi}Sa^2[(\omega+\omega_0)T]\right\}.$$

Since this result is identical to that of Example 7.1-2, the power spectrum is the same,

$$\mathcal{S}_{XX}(\omega) = \frac{A^2\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right],$$

7.1-1. (Continued) and the power is the same, $P_{XX} = A^2/2$. The two power calculations do agree.

7.1-2. (a) $E[X(t)] = 0$ if $t<0$ and $E[X(t)] = -(2A/\pi)\sin(\omega_0 t)$ if $t>0$ (same as Problem 7.1-1). $X(t)$ is not wide-sense stationary. (b) $E[X^2(t)] = 0$ if $t<0$ and $E[X^2(t)] = A^2/2$ if $t>0$ (same as Problem 7.1-1). Now: $P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2} u(t)\,dt = A^2/4$. (c) $x_T(t) = u(t) A\cos(\omega_0 t + \Theta)$, $-T<t<T$ and zero for other t. $X_T(\omega)$ = Fourier transform of $x_T(t) = \int_0^T A\cos(\omega_0 t+\Theta)e^{-j\omega t}dt$. Expanding $\cos(\alpha) = \frac{1}{2}(e^{j\alpha}+e^{-j\alpha})$ and using (C-45), we have

$$X_T(\omega) = \frac{AT}{2}\Big\{ e^{j\Theta - j(\omega-\omega_0)T/2}\, Sa[(\omega-\omega_0)T/2] + e^{-j\Theta - j(\omega+\omega_0)T/2}\, Sa[(\omega+\omega_0)T/2]\Big\},$$

$$|X_T(\omega)|^2 = \frac{A^2T^2}{4}\Big\{ Sa^2[(\omega-\omega_0)T/2] + Sa^2[(\omega+\omega_0)T/2] + 2\cos[\omega_0 T + 2\Theta]\, Sa[(\omega-\omega_0)T/2]\, Sa[(\omega+\omega_0)T/2]\Big\}.$$

The third term is zero when the expected value is taken because $\cos(\omega_0 T + 2\theta)$ is integrated over a full period. By using the fact that

$$\lim_{T/2\to\infty} \frac{T/2}{\pi} Sa^2[(\omega\pm\omega_0)T/2] = \delta(\omega\pm\omega_0) \quad (1)$$

we have

$$\mathcal{S}_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T} = \frac{A^2\pi}{4}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

From (7.1-12) $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\,d\omega = A^2/4$ which

7.1-2. (Continued) agrees with P_{XX} found in (b).

★7.1-3. (a) $E[X(t)] = 0$, $t < 0$. $E[X(t)] = \int_{-\infty}^{\infty} x f_{\Theta}(\theta)\,d\theta$

$= \int_0^{\pi/2} u(t) A \cos(\omega_0 t + \theta) \frac{d\theta}{\pi/2} = \frac{2A}{\pi} u(t)[\cos(\omega_0 t) - \sin(\omega_0 t)]$,

$t > 0$. $X(t)$ is not wide-sense stationary because $E[X(t)]$ is time-dependent. (b) $E[X^2(t)] = 0$, $t < 0$. $E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_{\Theta}(\theta)\,d\theta = \int_0^{\pi/2} u(t) A^2 \cos^2(\omega_0 t + \theta) \frac{d\theta}{\pi/2} = u(t)\frac{A^2}{2}\left[1 - \frac{2}{\pi}\sin(2\omega_0 t)\right]$, $t > 0$. Thus,

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t)\frac{A^2}{2}\left[1 - \frac{2}{\pi}\sin(2\omega_0 t)\right] dt = A^2/4.$$

(c) $|X_T(\omega)|^2$ is the same as in Problem 7.1-2. Now since $E[2\cos(\omega_0 T + 2\Theta)] = 2\int_0^{\pi/2} \cos(\omega_0 T + 2\theta)\frac{d\theta}{\pi/2}$

$= \frac{2}{\pi}[-\sin(\omega_0 T) - \sin(\omega_0 T)] = \frac{-4}{\pi}\sin(\omega_0 T)$, we have

$$\frac{E[|X_T(\omega)|^2]}{2T} = \frac{A^2\pi}{4}\left\{\frac{T}{2\pi} Sa^2[(\omega-\omega_0)T/2] + \frac{T}{2\pi} Sa^2[(\omega+\omega_0)T/2]\right\}$$
$$- \frac{A^2 T}{2\pi}\sin(\omega_0 T)]\, Sa[(\omega-\omega_0)T/2]$$
$$\cdot\, Sa[(\omega+\omega_0)T/2].$$

As $T\to\infty$ as a limit the two $Sa(\cdot)$ factors in the third term have values at ω_0 and $-\omega_0$ only so the product becomes an impulse at either ω_0 or $-\omega_0$ multiplied by zero (due to opposing factor). The third term is therefore zero. The power spectrum then becomes

★7.1-3. (Continued) $\mathcal{S}_{XX}(\omega) = \frac{A^2\pi}{4}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

where (1) in the solution to Problem 7.1-2 has been used. Finally, $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega = A^2/4$ which agrees with P_{XX} of part (b).

7.1-4. (a) $E[X(t)] = \int_{-\infty}^{\infty} x\, f_{\Theta}(\theta)\,d\theta = \int_0^{\pi} A\sin(\omega_0 t + \theta)\frac{1}{\pi}\,d\theta$ $= (2A/\pi)\cos(\omega_0 t)$. $X(t)$ is not wide-sense stationary because $E[X(t)]$ is time dependent.

(b) $E[X^2(t)] = E[A^2\sin^2(\omega_0 t + \Theta)] = \int_0^{\pi} A^2\sin^2(\omega_0 t+\theta)\frac{d\theta}{\pi}$ $= A^2/2$. $P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{A^2}{2}\,dt = A^2/2$. (c) $X_T(\omega)$ $= \int_{-T}^{T} A\sin(\omega_0 t + \Theta)\,e^{-j\omega t}\,dt$. By expanding $2\cos(\alpha)$ $= e^{j\alpha} + e^{-j\alpha}$ and using (C-45) we obtain

$$X_T(\omega) = \frac{AT}{j}\{e^{j\Theta}Sa[(\omega-\omega_0)T] - e^{-j\Theta}Sa[(\omega+\omega_0)T]\},$$

$$|X_T(\omega)|^2 = A^2T^2\{Sa^2[(\omega-\omega_0)T] + Sa^2[(\omega+\omega_0)T]$$
$$-2\cos(2\Theta)\,Sa[(\omega-\omega_0)T]\,Sa[(\omega+\omega_0)T]\}.$$

As in Problem 7.1-1 the third term is zero when the expected value is taken. The remaining terms are identical to those in Problem 7.1-1 so $\mathcal{S}_{XX}(\omega) = \frac{A^2\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$ and $P_{XX} = A^2/2$ as in Problem 7.1-1. P_{XX} does agree with that found in part (b).

★7.1-5. Write $X(t) = A^2\cos^2(\omega_0 t + \Theta) = \frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega_0 t + 2\Theta)$. (a) $E[X(t)] = \int_{-\infty}^{\infty} x\, f_\Theta(\theta)\,d\theta = \int_0^{\pi}\left[\frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega_0 t + 2\theta)\right]\frac{1}{\pi}\,d\theta = A^2/2$ (constant in time).

$E[X(t)X(t+\tau)] = R_{XX}(t, t+\tau) = E\left[\frac{A^2}{2}\{1+\cos(2\omega_0 t + 2\Theta)\}\cdot\frac{A^2}{2}\{1+\cos(2\omega_0 t + 2\Theta + 2\omega_0\tau)\}\right] = \frac{A^4}{4}\left[1 + \frac{1}{2}\cos(2\omega_0\tau)\right]$.

Since $E[X(t)]$ is constant in t and the autocorrelation function does not depend on t, $X(t)$ is wide-sense stationary. (b) $E[X^2(t)] = R_{XX}(0) = \frac{3A^4}{8}$. Since this is constant with time $A\{E[X^2(t)]\} = E[X^2(t)]$ and $P_{XX} = 3A^4/8$. (c) $X_T(\omega) = \int_{-T}^{T}\frac{A^2}{2}[1+\cos(2\omega_0 t + 2\Theta)]e^{-j\omega t}dt$. Expand $2\cos(\alpha) = e^{j\alpha} + e^{-j\alpha}$ and use (C-45) to get

$$X_T(\omega) = A^2 T\,Sa(\omega T) + \frac{A^2 T}{2}\left\{e^{j2\Theta}Sa[(\omega - 2\omega_0)T] + e^{-j2\Theta}Sa[(\omega + 2\omega_0)T]\right\}.$$

By forming $|X_T(\omega)|^2 = X_T(\omega)X_T^*(\omega)$ and taking the expected value with respect to Θ we get

$$\frac{E[|X_T(\omega)|^2]}{2T} = \frac{A^4\pi}{2}\left[\frac{T}{\pi}Sa(\omega T)\right] + \frac{A^4\pi}{8}\left\{\frac{T}{\pi}Sa^2[(\omega - 2\omega_0)T] + \frac{T}{\pi}Sa^2[(\omega+2\omega_0)T]\right\}.$$

Next, since $\lim_{T\to\infty}\frac{T}{\pi}Sa^2(\alpha T) = \delta(\alpha)$, then

$$\mathcal{S}_{XX}(\omega) = \lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T} = \frac{A^4\pi}{2}\left\{\delta(\omega) + \frac{1}{4}\delta(\omega - 2\omega_0) + \frac{1}{4}\delta(\omega + 2\omega_0)\right\}$$

$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega = \frac{3A^4}{8}$ and both powers agree.

7.1-6. (a) Here $E[A]=0$, $E[B]=0$, $E[(A-\bar{A})^2]=E[A^2]$ $=\sigma^2$, $E[(B-\bar{B})^2]=E[B^2]=\sigma^2$ (same variance as B], and $E[(A-\bar{A})(B-\bar{B})]=E[AB]=0$. Then,

$E[X(t)]=E[A]\cos(\omega_0 t)+E[B]\sin(\omega_0 t)=0$,

$R_{XX}(t,t+\tau)=E[X(t)X(t+\tau)]=E[\{A\cos(\omega_0 t)+B\sin(\omega_0 t)\}\{A\cos(\omega_0 t+\omega_0\tau)+B\sin(\omega_0 t+\omega_0\tau)\}]$

$=E[A^2]\cos(\omega_0 t)\cos(\omega_0 t+\omega_0\tau)+E[AB]\cos(\omega_0 t)$

$\cdot\sin(\omega_0 t+\omega_0\tau)+E[B^2]\sin(\omega_0 t)\sin(\omega_0 t+\omega_0\tau)$

$+E[AB]\sin(\omega_0 t)\cos(\omega_0 t+\omega_0\tau)=\sigma^2\cos(\omega_0\tau)$

$=R_{XX}(\tau)$. $X(t)$ is wide-sense stationary because the mean is a constant with t and the auto-correlation function is independent of t.

(b) From part (a): $R_{XX}(\tau)=\sigma^2\cos(\omega_0\tau)$.

(c) Since $R_{XX}(\tau)$ is identical in form to that in Example 7.2-1, we have by inspection

$$\mathcal{S}_{XX}(\omega)=\sigma^2\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

7.1-7. The limiting forms are impulses if amplitude $\to\infty$ as duration $\to 0$ while area is constant at unity, all when $T\to\infty$. Clearly amplitudes and durations are satisfied in (a) and (b). Check areas:

(a) $$\int_{-\infty}^{\infty} T e^{-\pi\alpha^2 T^2}\,d\alpha=\int_{-\infty}^{\infty}\frac{e^{-\alpha^2/2(1/2\pi T^2)}}{\sqrt{2\pi(1/2\pi T^2)}}\,d\alpha=1,$$

because this is the area of a Gaussian density

7.1-7. (Continued) having a mean of zero and a variance of $1/2\pi T^2$. (b) area $= \int_{-\infty}^{\infty} \frac{T}{2} e^{-T|\alpha|} d\alpha$ $= T\int_0^{\infty} e^{-T\alpha} d\alpha = 1$ from (C-45).

7.1-8. By applying the explanation in Problem 7.1-7 we only need check the areas of the functions.

(a) area $= \int_{-\infty}^{\infty} \frac{T}{\pi} \frac{\sin(T\alpha)}{T\alpha} d\alpha$. Let $\xi = \alpha T$, $d\xi = d\alpha T$,

$$\text{area} = \frac{2}{\pi}\int_0^{\infty} Sa(\xi)\, d\xi = 1 \quad \text{from (C-53)}.$$

(b) area $= \int_{-1/T}^{1/T} T[1-|\alpha|T]\, d\alpha = 2T\int_0^{1/T} (1-\alpha T)\, d\alpha = 1$

7.1-9. Let $x_T(t)$ be any Fourier transformable signal. Let $X_T(\omega)$ be its transform. $X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt$.

$X_T(-\omega) = \int_{-\infty}^{\infty} x_T(t) e^{j\omega t} dt = X_T^*(\omega)$ for real $x_T(t)$.

Thus, $X_T^*(-\omega) = X_T(\omega)$ so, for real $X(t)$,

$$\mathcal{S}_{XX}(-\omega) = \lim_{T\to\infty} \frac{E[X_T^*(-\omega) X_T(-\omega)]}{2T}$$

$$= \lim_{T\to\infty} \frac{E[X_T(\omega) X_T^*(\omega)]}{2T} = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T} = \mathcal{S}_{XX}(\omega).$$

7.1-10. Define $\dot{x}_T(t) = \frac{dx(t)}{dt}$, $-T < t < T$
$= 0$, elsewhere.

Then $\dot{x}_T(t) = \lim_{\epsilon\to 0} \frac{x(t+\epsilon) - x(t)}{\epsilon}$, $-T < (t \text{ and } t+\epsilon) < T$,
$= 0$, elsewhere.

Use (D-6) to Fourier transform $\dot{x}_T(t)$ to obtain

7.1-10. (Continued) its transform denoted $\dot{X}_T(\omega)$:

$$\dot{X}_T(\omega) = \mathcal{F}\{\dot{x}_T(t)\} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[X_T(\omega)e^{j\omega\epsilon} - X_T(\omega)\right]$$

$$= X_T(\omega)\lim_{\epsilon\to 0} e^{j\omega\epsilon/2} j\omega \frac{\sin(\omega\epsilon/2)}{(\omega\epsilon/2)} = j\omega X_T(\omega)$$

where $x_T(t) \longleftrightarrow X_T(\omega)$. Next,

$$\mathcal{S}_{\dot{X}\dot{X}}(\omega) = \lim_{T\to\infty}\frac{E[|\dot{X}_T(\omega)|^2]}{2T} = \lim_{T\to\infty}\frac{E[|j\omega X_T(\omega)|^2]}{2T}$$

$$= \omega^2\lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T} = \omega^2\mathcal{S}_{XX}(\omega).$$

7.1-11. Define $x_T(t)$ and $X_T(\omega)$ by (7.1-3) and (7.1-5), respectively. Define $Y_T(t) = X_T(t)\cos(\omega_0 t + \Theta)$ and

$$Y_T(\omega) = \int_{-T}^{T} x_T(t)\cos(\omega_0 t + \Theta)e^{-j\omega t}dt$$

$$= \int_{-T}^{T}\frac{1}{2}\left[x_T(t)e^{j\omega_0 t + j\Theta} + x_T(t)e^{-j\omega_0 t - j\Theta}\right]e^{-j\omega t}dt$$

$$= \frac{e^{j\Theta}}{2}X_T(\omega-\omega_0) + \frac{e^{-j\Theta}}{2}X_T(\omega+\omega_0) \quad \text{from (D-7).}$$

Next,

$$4|Y_T(\omega)|^2 = |X_T(\omega-\omega_0)|^2 + |X_T(\omega+\omega_0)|^2$$
$$+ e^{j2\Theta}X_T(\omega-\omega_0)X_T^*(\omega+\omega_0)$$
$$+ e^{-j2\Theta}X_T^*(\omega-\omega_0)X_T(\omega+\omega_0) \qquad (1)$$

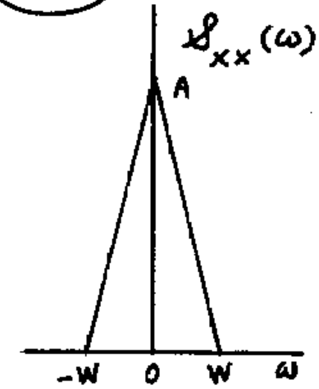
First, take the expected value of (1) with respect to Θ. Two terms involve:

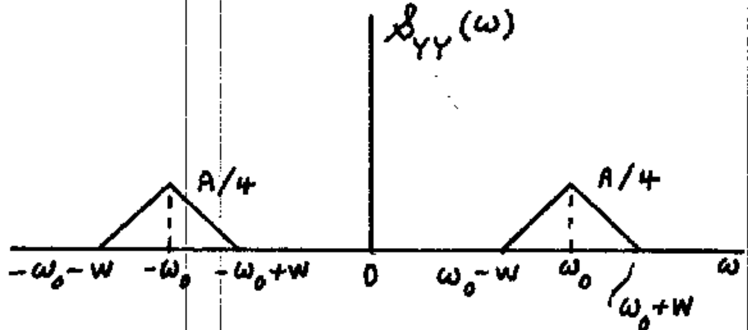
$$E\left[e^{\pm j2\Theta}\right] = \int_0^{2\pi}[\cos(\theta) \pm j\sin(\theta)]\frac{1}{2\pi}d\theta = 0.$$

Next, take expected value of (1) with respect to $X(t)$. Then

$$\mathcal{S}_{YY}(\omega) = \lim_{T\to\infty}\frac{E[|Y_T(\omega)|^2]}{2T} = \lim_{T\to\infty}\frac{E[|X_T(\omega-\omega_0)|^2] + E[|X_T(\omega+\omega_0)|^2]}{2T(4)}$$

(7.1-11.) (Continued) or $\mathcal{S}_{YY}(\omega) = \frac{1}{4}\left[\mathcal{S}_{XX}(\omega-\omega_0) + \mathcal{S}_{XX}(\omega+\omega_0)\right]$.





(7.1-12.) (a) Valid. (b) No — not an even function of ω. (c) No — not everywhere nonnegative. (d) No — not real.

(7.1-13.) (a) No — not everywhere nonnegative. (b) Valid. (c) No — not even in ω. (d) No — not always real.

(7.1-14.) $R_{XX}(t,t+\tau) = E[X(t)X(t+\tau)] = E\left[\sum_{i=1}^{N}\alpha_i X_i(t) \cdot \sum_{j=1}^{N}\alpha_j X_j(t+\tau)\right] = \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j E[X_i(t)X_j(t+\tau)]$. Since processes are orthogonal $E[X_i(t)X_j(t+\tau)] = 0$, $i \neq j$ and $E[X_i(t)X_j(t+\tau)] = R_{X_iX_i}(\tau)$, $i = j$, so

$$R_{XX}(t,t+\tau) = \sum_{i=1}^{N}\alpha_i^2 R_{X_iX_i}(\tau) = R_{XX}(\tau)$$

By using (7.1-20):

$$\mathcal{S}_{XX}(\omega) = \sum_{i=1}^{N}\alpha_i^2 \mathcal{S}_{X_iX_i}(\omega).$$

7.1-15. $R_{XX}(t, t+\tau) = E[A\cos(\Omega t + \Theta) A\cos(\Omega t + \Omega\tau + \Theta)]$

$= (A^2/2) E[\cos(\Omega\tau) + \cos(2\Omega t + \Omega\tau + 2\Theta)]$. Take expectation with respect to Θ first.

$$R_{XX}(t,t+\tau) = \frac{A^2}{2} E_\Omega\left[\cos(\Omega\tau) + \int_0^{2\pi} \cos(2\Omega t + \Omega\tau + 2\theta)\frac{1}{2\pi}d\theta\right]$$

$$= \frac{A^2}{2} E_\Omega[\cos(\Omega\tau)] = R_{XX}(\tau).$$

Next, take expected value with respect to Ω.

$$R_{XX}(\tau) = \int_\Omega \frac{A^2}{2}\cos(\eta\tau) f_\Omega(\eta)\, d\eta.$$

Fourier transform to get the power spectrum.

$$\mathcal{S}_{XX}(\omega) = \int_{-\infty}^{\infty}\left\{\int_\Omega \frac{A^2}{4}\left(e^{j\eta\tau} + e^{-j\eta\tau}\right) f_\Omega(\eta)d\eta\right\} e^{-j\omega\tau}\, d\tau$$

$$= \int_\Omega \frac{A^2}{4}\left\{\underbrace{\int_{-\infty}^{\infty}\left(e^{j\eta\tau} + e^{-j\eta\tau}\right)e^{-j\omega\tau}\, d\tau}_{2\pi[\delta(\omega-\eta)+\delta(\omega+\eta)] \text{ from pair 9 of Appendix E.}}\right\} f_\Omega(\eta)\, d\eta$$

$$= \int_\Omega \frac{A^2\pi}{2}[\delta(\omega-\eta) + \delta(\omega+\eta)] f_\Omega(\eta)\, d\eta$$

$$= \frac{A^2\pi}{2}[f_\Omega(\omega) + f_\Omega(-\omega)],$$

where (A-2) has been used.

7.1-16. $R_{YY}(t,t+\tau) = E[Y(t)Y(t+\tau)] = E[\{A+BX(t)\}\{A+BX(t+\tau)\}]$

$= E[A^2 + ABX(t) + ABX(t+\tau) + B^2X(t)X(t+\tau)] = A^2 + 2AB\bar{X} + B^2R_{XX}(\tau) = R_{YY}(\tau)$. From (7.1-20) and pair 2 of Appendix E:

$$\mathcal{S}_{YY}(\omega) = (A^2 + 2AB\bar{X})2\pi\delta(\omega) + B^2\mathcal{S}_{XX}(\omega).$$

(7.1-17) We calculate (7.1-22) using (C-25) and (C-27):

$$\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\,d\omega = \int_{-KW}^{KW} \frac{P\,d\omega}{1+(\omega/W)^2} = PW\tan^{-1}\left(\frac{\omega}{W}\right)\Big|_{-KW}^{KW}$$

$$= 2PW\tan^{-1}(K).$$

$$\int_{-\infty}^{\infty} \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega = \int_{-KW}^{KW} \frac{P\omega^2 d\omega}{1+(\omega/W)^2}$$

$$= PW^2\left\{\omega - W\tan^{-1}\left(\frac{\omega}{W}\right)\right\}\Big|_{-KW}^{KW} = 2PW^2\{KW - W\tan^{-1}(K)\}$$

$$W_{rms}^2 = \frac{2PW^3[K-\tan^{-1}(K)]}{2PW\tan^{-1}(K)} = W^2\left[\frac{K}{\tan^{-1}(K)} - 1\right].$$

as $K \to \infty$, $W_{rms} \to \infty$, which says that the power spectrum has an infinite rms bandwidth.

(7.1-18.) $\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\,d\omega = \int_{-W}^{W} P\cos\left(\frac{\pi\omega}{2W}\right)d\omega = \frac{4WP}{\pi}$.

$\int_{-\infty}^{\infty} \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega = \int_{-W}^{W} P\omega^2\cos\left(\frac{\pi\omega}{2W}\right)d\omega$. Let $x = \frac{\pi\omega}{2W}$

$dx = (\pi/2W)\,d\omega$. $\int_{-\infty}^{\infty} \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega = P\left(\frac{2W}{\pi}\right)^3\int_{-\pi/2}^{\pi/2} x^2\cos(x)\,dx$

$= P\left(\frac{2W}{\pi}\right)^3\left(\frac{\pi^2-8}{2}\right)$ after use of (C-41).

$$W_{rms}^2 = \frac{P\left(\frac{2W}{\pi}\right)^3\left(\frac{\pi^2-8}{2}\right)}{2P\left(\frac{2W}{\pi}\right)} = W^2\left(1-\frac{8}{\pi^2}\right) \approx 0.189W^2$$

(7.1-19.) (a) $\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\,d\omega = P\int_{-W}^{W} d\omega = 2WP$. $\int_{-\infty}^{\infty} \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega$ $= P\int_{-W}^{W} \omega^2 d\omega = 2PW^3/3$. $W_{rms}^2 = \frac{2PW^3/3}{2PW} = W^2/3$.

(b) $\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\,d\omega = \int_{-W}^{W} P\left[1-\frac{|\omega|}{W}\right]d\omega = 2P\int_{0}^{W}\left[1-\frac{\omega}{W}\right]d\omega$

$= PW$. $\int_{-\infty}^{\infty} \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega = 2P\int_{0}^{W} \omega^2\left[1-\frac{\omega}{W}\right]d\omega = \frac{PW^3}{6}$

(7.1-19.) (Continued) $W_{rms}^2 = \dfrac{PW^3/6}{PW} = W^2/6.$

★(7.1-20.) $\displaystyle\int_0^\infty \mathcal{S}_{XX}(\omega)\,d\omega = \frac{1}{2}\int_{-\infty}^\infty \mathcal{S}_{XX}(\omega)\,d\omega = \frac{P}{2}\int_{-\infty}^\infty \frac{d\omega}{\left[1+\left(\frac{\omega+\alpha}{W}\right)^2\right]^2}$

$\displaystyle + \frac{P}{2}\int_{-\infty}^\infty \frac{d\omega}{\left[1+\left(\frac{\omega-\alpha}{W}\right)^2\right]^2}$ Let $\xi = \dfrac{\omega-\alpha}{W}$, $d\xi = d\omega/W$. Let $\xi = \dfrac{\omega+\alpha}{W}$, $d\xi = d\omega/W$.

$\displaystyle = \frac{PW}{2}\int_{-\infty}^\infty \frac{d\xi}{(1+\xi^2)^2} + \frac{PW}{2}\int_{-\infty}^\infty \frac{d\xi}{(1+\xi^2)^2} = \pi PW/2$ from (C-28). $\displaystyle\int_0^\infty \omega\,\mathcal{S}_{XX}(\omega)\,d\omega = P\int_0^\infty \frac{\omega\,d\omega}{\left[1+\left(\frac{\omega+\alpha}{W}\right)^2\right]^2}$

$\displaystyle + P\int_0^\infty \frac{\omega\,d\omega}{\left[1+\left(\frac{\omega-\alpha}{W}\right)^2\right]^2} = WP\int_{\alpha/W}^\infty \frac{(W\xi-\alpha)\,d\xi}{(1+\xi^2)^2} + WP\int_{-\alpha/W}^\infty \frac{(W\xi+\alpha)\,d\xi}{(1+\xi^2)^2}$

$= PW^2\left[1+\frac{\alpha}{W}\tan^{-1}\left(\frac{\alpha}{W}\right)\right]$ after using (C-28) and (C-29). $\displaystyle\int_{-\infty}^\infty \omega^2\,\mathcal{S}_{XX}(\omega)\,d\omega = P\int_{-\infty}^\infty \frac{\omega^2\,d\omega}{\left[1+\left(\frac{\omega+\alpha}{W}\right)^2\right]^2}$

$\displaystyle + P\int_{-\infty}^\infty \frac{\omega^2\,d\omega}{\left[1+\left(\frac{\omega-\alpha}{W}\right)^2\right]^2} = PW\int_{-\infty}^\infty \frac{(W\xi-\alpha)^2}{(1+\xi^2)^2}d\xi + PW\int_{-\infty}^\infty \frac{(W\xi+\alpha)^2 d\xi}{(1+\xi^2)^2}$

$= PW^3\pi\left(1+\frac{\alpha^2}{W^2}\right)$ after using (C-28), (C-29), and (C-30). From (7.1-23):

$$\bar{\omega}_0 = \frac{PW^2\left[1+\frac{\alpha}{W}\tan^{-1}\left(\frac{\alpha}{W}\right)\right]}{\pi PW/2} = \frac{2W}{\pi}\left[1+\frac{\alpha}{W}\tan^{-1}\left(\frac{\alpha}{W}\right)\right].$$

From Problem 7-35 $W_{rms}^2 = 4\left[\overline{W^2} - \bar{\omega}_0^2\right]$ where $\overline{W^2}$ is given by the right side of (7.1-22):

$$W_{rms}^2 = 4\left\{\frac{\pi PW^3\left(1+\frac{\alpha^2}{W^2}\right)}{\pi PW} - \frac{4W^2}{\pi^2}\left[1+\frac{\alpha}{W}\tan^{-1}\left(\frac{\alpha}{W}\right)\right]^2\right\}$$

$$= 4\left\{W^2+\alpha^2 - \frac{4W^2}{\pi^2}\left[1+\frac{\alpha}{W}\tan^{-1}\left(\frac{\alpha}{W}\right)\right]^2\right\}.$$

7.1-21. From (7.1-24): $W_{rms}^2 \int_0^\infty \mathcal{S}_{XX}(\omega)\,d\omega =$

$$4\int_0^\infty (\omega^2 - 2\bar{\omega}_0\omega + \bar{\omega}_0^2)\mathcal{S}_{XX}(\omega)\,d\omega$$

$$= 4\left\{\int_0^\infty \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega - 2\bar{\omega}_0\int_0^\infty \omega\,\mathcal{S}_{XX}(\omega)\,d\omega + \bar{\omega}_0^2\int_0^\infty \mathcal{S}_{XX}(\omega)\,d\omega\right\}.$$

$$W_{rms}^2 = 4\left\{\frac{\int_0^\infty \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega}{\int_0^\infty \mathcal{S}_{XX}(\omega)\,d\omega} - 2\bar{\omega}_0^2 + \bar{\omega}_0^2\right\}$$ where

(7.1-23) has been used. Since $\mathcal{S}_{XX}(\omega)$ is even in ω:

$W_{rms}^2 = 4[\overline{W^2} - \bar{\omega}_0^2]$ where

$$\overline{W^2} = \frac{\int_{-\infty}^\infty \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega}{\int_{-\infty}^\infty \mathcal{S}_{XX}(\omega)\,d\omega}.$$

7.1-22. (a) Use pairs 2 and 20 of Appendix E to get $\mathcal{S}_{XX}(\omega) = 6\pi\delta(\omega) + \sqrt{\pi}\,e^{-\omega^2/16}$. (b) Power $= R_{XX}(0) = 5\,W$.

(c) Power in $(\frac{-1}{\sqrt{2}} \le \omega \le \frac{1}{\sqrt{2}}) = \frac{1}{2\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} [6\pi\delta(\omega) + \sqrt{\pi}\,e^{-\omega^2/16}]\,d\omega$

$$= 3 + 2\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{e^{-\omega^2/2(8)}}{\sqrt{2\pi 8}}\,d\omega = 3 + 2\left\{F\left(\frac{1/\sqrt{2}}{2\sqrt{2}}\right) - F\left(\frac{-1/\sqrt{2}}{2\sqrt{2}}\right)\right\}$$

$$= 3 + 2\left[F(1/4) - 1 + F(1/4)\right] = 3 + 2[2(0.5987 - 1] = 3.3948.$$

Fractional power $= \frac{3.3948}{5} = 0.6790$ (or 67.9%).

7.1-23. (a) No – not real, (b) No – not real, can be negative, (c) Yes, (d) No – not even in ω, can be negative, (e) Yes, and (f) Yes.

7.1-24. From sketches of the derivatives we find that both can be negative and therefore cannot be valid power spectrums.

(7.1-25.) (a) From the solution to Problem 7.1-15

$R_{XX}(t,t+\tau) = (A^2/2) E_{\Omega_0}[\cos(\Omega_0\tau) + \cos(2\Omega_0 t + \Omega_0\tau + 2\Theta)]$

On time averaging the second term is zero, so

$R_{XX}(\tau) = (A^2/2) E_{\Omega_0}[\cos(\Omega_0\tau)]$ which is the same as in Problem 7-15; therefore $\mathcal{S}_{XX}(\omega)$ is the same as before.

(b) $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi A^2}{2}\left[f_{\Omega_0}(\omega) + f_{\Omega_0}(-\omega)\right]d\omega = \frac{A^2}{2}$.

This result is independent of the form of $f_{\Omega_0}(\omega)$.

(7.1-26.) Use (C-31) and (C-32): $W_{rms}^2 = \int_{-\infty}^{\infty}\frac{\omega^2\,d\omega}{[1+(\omega/W)^2]^3}\Big/\int_{-\infty}^{\infty}\frac{d\omega}{[1+(\omega/W)^2]^3} = \frac{W^3\pi/8}{W\,3\pi/8} = W^2/3$.

(7.1-27.) Use (C-32) and (C-33): $W_{rms}^2 = \int_{-\infty}^{\infty}\frac{\omega^4\,d\omega}{[1+(\omega/W)^2]^3}\Big/\int_{-\infty}^{\infty}\frac{\omega^2\,d\omega}{[1+(\omega/W)^2]^3} = \frac{W^5\,3\pi/8}{W^3\,\pi/8} = 3W^2$.

(7.1-28.) Use (C-34) and (C-35): $W_{rms}^2 = \int_{-\infty}^{\infty}\frac{\omega^2\,d\omega}{[1+(\omega/W)^2]^4}\Big/\int_{-\infty}^{\infty}\frac{d\omega}{[1+(\omega/W)^2]^4} = \frac{W^3\pi/16}{W\,5\pi/16} = \frac{W^2}{5}$.

(7.1-29.) Use (C-35) and (C-36): $W_{rms}^2 = \int_{-\infty}^{\infty}\frac{\omega^4\,d\omega}{[1+(\omega/W)^2]^4}\Big/\int_{-\infty}^{\infty}\frac{\omega^2\,d\omega}{[1+(\omega/W)^2]^4} = \frac{W^5\pi/16}{W^3\pi/16} = W^2$.

★(7.1-30.) From Dwight, pp. 30-31,

$$\int_{-\infty}^{\infty}\frac{x^2\,dx}{(1+x^2)^N} = \frac{1}{2(N-1)}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{N-1}}$$

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^N} = \frac{2N-3}{2(N-1)}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{N-1}}.\quad \text{Thus,}$$

$$W_{rms}^2 = \int_{-\infty}^{\infty}\frac{\omega^2\,d\omega}{[1+(\omega/W)^2]^N}\Big/\int_{-\infty}^{\infty}\frac{d\omega}{[1+(\omega/W)^2]^N} = \frac{W^3}{2(N-1)}\cdot\frac{2(N-1)}{W\,(2N-3)}$$

★ 7.1-30 (Continued) $= W^2/(2N-3)$.

★ 7.1-31. From Gradshteyn and Ryzhik, p. 65:

$$\int_{-\infty}^{\infty}\frac{x^4\,dx}{(1+x^2)^N} = \frac{3}{(2N-5)}\int_{-\infty}^{\infty}\frac{x^2\,dx}{(1+x^2)^N}.$$ Thus,

$$W_{rms}^2 = \int_{-\infty}^{\infty}\frac{\omega^4 d\omega}{[1+(\omega/W)^2]^N}\Big/\int_{-\infty}^{\infty}\frac{\omega^2 d\omega}{[1+(\omega/W)^2]^N} = \frac{W^2 3}{2N-5}.$$

7.1-32. (a) $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-6}^{6}\left[4-\frac{\omega^2}{9}\right]d\omega = \frac{16}{\pi}$.

(b) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2\mathcal{S}_{XX}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-6}^{6}\left[4\omega^2+\frac{\omega^4}{9}\right]d\omega = \frac{4^2 6^2}{5\pi}$.

$W_{rms}^2 = \frac{4^2 6^2 \pi}{5\pi\ 16} = 36/5$. (c) We inverse Fourier transform $\mathcal{S}_{XX}(\omega)$. For the first term use pair 6, Appendix E: $\frac{24}{\pi}\mathrm{Sa}(6\tau) \longleftrightarrow 4\,\mathrm{rect}(\frac{\omega}{12})$. For the second term:

$$\frac{1}{2\pi}\int_{-6}^{6}\frac{\omega^2}{9}e^{j\omega\tau}d\omega = \frac{1}{18\pi}e^{j\omega\tau}\left[\frac{\omega^2}{j\tau}-\frac{2\omega}{-\tau^2}+\frac{2}{-j\tau^3}\right]$$

evaluated from -6 to 6 $= \frac{-4}{18\pi}\left\{\left(\frac{1}{\tau^2}-18\right)\frac{\sin(6\tau)}{\tau}-\frac{6\cos(6\tau)}{\tau^2}\right\}$

Finally by adding the two terms:

$$R_{XX}(\tau) = \frac{4}{3\pi\tau^2}\left\{\mathrm{Sa}(6\tau) - \cos(6\tau)\right\}.$$

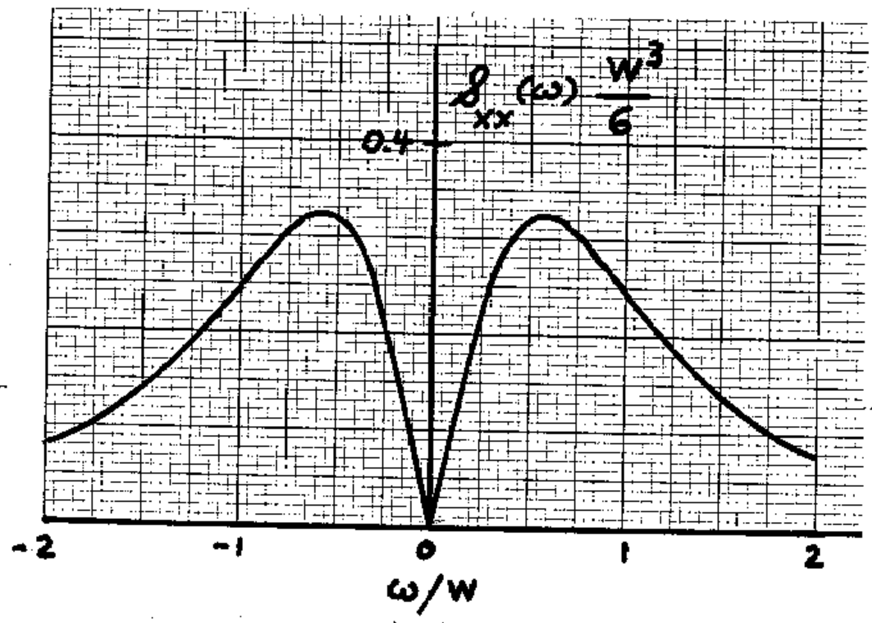
7.1-33. $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,e^{j\omega\tau}d\omega$ so by differentiation

$$\left.\frac{d^2R_{XX}(\tau)}{d\tau^2}\right|_{\tau=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-\omega^2\mathcal{S}_{XX}(\omega)\right]e^{j\omega\tau}d\omega\Big|_{\tau=0}$$

$= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\omega^2\mathcal{S}_{XX}(\omega)\,d\omega$. $W_{rms}^2 = \frac{\int_{\infty}\omega^2\mathcal{S}_{XX}(\omega)\,d\omega}{\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega}$

$$= \frac{-2\pi\left.\frac{d^2R_{XX}(\tau)}{d\tau^2}\right|_{\tau=0}}{2\pi R_{XX}(0)} = \frac{-1}{R_{XX}(0)}\left.\frac{d^2R_{XX}(\tau)}{d\tau^2}\right|_{\tau=0}.$$

(7.1-34.) (a)



(b) For $\omega > 0$: $\dfrac{d\mathcal{S}_{XX}(\omega)}{d\omega} = \dfrac{(W^2+\omega^2)^2 6 - 6\omega\, 2(W^2+\omega^2)\, 2\omega}{(W^2+\omega^2)^4} = 0$

when $W^2 + \omega^2 - 4\omega^2 = W^2 - 3\omega^2 = 0$, or $\omega = \omega_{max} = W/\sqrt{3}$.

(7.1-35.) $\int_0^\infty \mathcal{S}_{XX}(\omega)\,d\omega = 6\int_0^\infty \dfrac{\omega\,d\omega}{(W^2+\omega^2)^2} = \dfrac{3}{W^2}$ from (C-29).

$\int_0^\infty \omega\,\mathcal{S}_{XX}(\omega)\,d\omega = 6\int_0^\infty \dfrac{\omega^2\,d\omega}{(W^2+\omega^2)^2} = \dfrac{3\pi}{2W}$ from (C-30).

Thus,

$$\bar{\omega}_0 = \frac{3\pi/(2W)}{3/W^2} = \frac{\pi W}{2}.$$

Next,

$\int_0^\infty \omega^2\,\mathcal{S}_{XX}(\omega)\,d\omega = 6\int_0^\infty \dfrac{\omega^3\,d\omega}{(W^2+\omega^2)^2} = \infty$ from Dwight, p. 31.

Thus,

$$W_{rms}^2 = \frac{4\int_0^\infty (\omega-\bar{\omega}_0)^2 \mathcal{S}_{XX}(\omega)\,d\omega}{\int_0^\infty \mathcal{S}_{XX}(\omega)\,d\omega} = \frac{4\int_0^\infty \omega^2 \mathcal{S}_{XX}(\omega)\,d\omega}{\int_0^\infty \mathcal{S}_{XX}(\omega)\,d\omega} - 8\bar{\omega}_0^2 + 4\bar{\omega}_0^2 = \infty.$$

(7.1-36.) $\int_0^\infty \mathcal{S}_{XX}(\omega)\,d\omega = 6\int_0^\infty \dfrac{\omega\,d\omega}{(W^2+\omega^2)^3} = \dfrac{3}{2W^4}$ from Dwight, p. 31.

$\int_0^\infty \omega\,\mathcal{S}_{XX}(\omega)\,d\omega = 6\int_0^\infty \dfrac{\omega^2\,d\omega}{(W^2+\omega^2)^3} = \dfrac{3\pi}{8W^3}$ from (C-32).

7.1-36. (Continued) $\int_0^\infty \omega^2 S_{XX}(\omega)\,d\omega = 6\int_0^\infty \frac{\omega^3\,d\omega}{(W^2+\omega^2)^3} = \frac{3}{2W^2}$ from Dwight, p. 32.

Thus, $\bar{\omega}_0 = \dfrac{3\pi/(8W^3)}{3/(2W^4)} = \dfrac{\pi W}{4}$

From the solution to Problem 7.1-35: $W_{rms} = \sqrt{\dfrac{4(3/2W^2)}{3/2W^4} - 4\left(\dfrac{\pi W}{4}\right)^2}$

$= 2W\sqrt{1-(\pi/4)^2}$.

7.1-37. (a) No. $S_{XX}(\omega)$ cannot be negative for any ω. (b) No. $S_{XX}(\omega)$ ~~must be even in ω. (c) Yes. (d) No. $S_{XX}(\omega)$ must be even and~~ nonnegative.

7.1-38. $\int_0^\infty S_{XX}(\omega)\,d\omega = \int_{\omega_0-W}^{\omega_0+2W}\left[3 - \frac{\omega-\omega_0+W}{W}\right]d\omega = \frac{9W}{2}$.

$\int_0^\infty \omega S_{XX}(\omega)\,d\omega = \int_{\omega_0-W}^{\omega_0+2W}\omega\left[3 - \frac{\omega-\omega_0+W}{W}\right]d\omega = \frac{9W\omega_0}{2}$.

$\int_0^\infty \omega^2 S_{XX}(\omega)\,d\omega = \int_{\omega_0-W}^{\omega_0+2W}\omega^2\left[3 - \frac{\omega-\omega_0+W}{W}\right]d\omega = \frac{9W}{2}\left(\omega_0^2 + \frac{W^2}{2}\right)$.

From (7.1-23) and (7.1-24): $\bar{\omega}_0 = (9W\omega_0/2)/(9W/2) = \omega_0$.

$$W_{rms}^2 = \frac{4\int_0^\infty \omega^2 S_{XX}(\omega)\,d\omega}{\int_0^\infty S_{XX}(\omega)\,d\omega} - 4\bar{\omega}_0^2 = \frac{4\left(\frac{9W}{2}\right)\left(\omega_0^2+\frac{W^2}{2}\right)}{\left(\frac{9W}{2}\right)} - 4\omega_0^2 = 2W^2.$$

Thus, $W_{rms} = \sqrt{2}\,W$.

7.2-1. From (C-16): $R_{XX}(\tau) = P\cos^4(\omega_0\tau) = P\{\frac{3}{8} + \frac{1}{2}\cos(2\omega_0\tau) + \frac{1}{8}\cos(4\omega_0\tau)\}$. From pairs 2 and 11 of Appendix E:

$$\mathcal{S}_{XX}(\omega) = \frac{\pi P}{2}\Big\{\frac{3}{2}\delta(\omega) + \delta(\omega-2\omega_0) + \delta(\omega+2\omega_0) + \frac{1}{4}\delta(\omega-4\omega_0) + \frac{1}{4}\delta(\omega+4\omega_0)\Big\}.$$

From (7.1-12) and (A-2):

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega = \frac{1}{2\pi}\,\frac{\pi P}{2}\Big\{\frac{3}{2}+1+1+\frac{1}{4}+\frac{1}{4}\Big\} = P.$$

7.2-2. Use (7.1-12). $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{6\omega^2\,d\omega}{1+\omega^4}$. From (C-38): $P_{XX} = \frac{6}{\pi}\Big\{-\frac{1}{4\sqrt{2}}\ln\Big(\frac{\omega^2+\sqrt{2}\,\omega+1}{\omega^2-\sqrt{2}\,\omega+1}\Big) + \frac{1}{2\sqrt{2}}\tan^{-1}\Big(\frac{\sqrt{2}\,\omega}{1-\omega^2}\Big)\Big\}\Big|_0^{\infty}$

$$= \frac{6}{\pi}\Big\{0 + \frac{1}{2\sqrt{2}}\pi\Big\} = 3/\sqrt{2}.$$

7.2-3. Use (C-32): $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{6\omega^2\,d\omega}{(1+\omega^2)^3}$

$$= \frac{6}{\pi}\Big\{\frac{-\omega}{4(1+\omega^2)^2} + \frac{\omega}{8(1+\omega^2)} + \frac{1}{8}\tan^{-1}(\omega)\Big\}\Big|_0^{\infty} = \frac{3}{8}.$$

7.2-4. Use (C-35): $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{6\omega^2\,d\omega}{(1+\omega^2)^4}$

$$= \frac{6}{\pi}\Big\{\frac{-\omega/6}{(1+\omega^2)^3} + \frac{\omega}{24(1+\omega^2)^2} + \frac{\omega}{16(1+\omega^2)} + \frac{1}{16}\tan^{-1}(\omega)\Big\}\Big|_0^{\infty} = \frac{3}{16}.$$

7.2-5. $\mathcal{S}_{XX}(\omega) = \int_{-\infty}^{\infty}R_{XX}(\tau)e^{-j\omega\tau}d\tau$ from (7.2-11). But $R_{XX}(\tau) = \bar{X}^2 + C_{XX}(\tau)$ from (6.3-25) so, using pair 2 of Appendix E, we get

$$\mathcal{S}_{XX}(\omega) = 2\pi\bar{X}^2\delta(\omega) + \int_{-\infty}^{\infty}C_{XX}(\tau)e^{-j\omega\tau}d\tau.$$

(7.2-6.) From (7.1-20) and pair 20 of Appendix E:

$$\mathcal{S}_{XX}(\omega) = \int_{-\infty}^{\infty} P\, e^{-\tau^2/2a^2}\, e^{-j\omega\tau}\, d\tau = \sqrt{2\pi}\, a\, P\, e^{-a^2\omega^2/2}.$$

(7.2-7.) Here $R_{XX}(\tau) = P\,\mathrm{tri}(2\tau/T)$ where $\mathrm{tri}(\cdot)$ is defined by (E-4). From pair 7 of Appendix E:

$$\mathrm{tri}(t/\tau) \longleftrightarrow \tau\, Sa^2(\omega\tau/2)$$

so $R_{XX}(\tau) = P\,\mathrm{tri}(2\tau/T) \longleftrightarrow \mathcal{S}_{XX}(\omega) = \frac{TP}{2} Sa^2(\omega T/4)$.

★(7.2-8.) Use the notation of (E-4) to write

$R_{XX}(\tau) = \sum_{n=-\infty}^{\infty} P\,\mathrm{tri}\left(\frac{\tau - nT}{T/2}\right)$. But $P\,\mathrm{tri}\left(\frac{\tau}{T/2}\right)$ has the Fourier transform $(TP/2)\,Sa^2(\omega T/4)$ from pair 7 of Appendix E. From (D-6) $P\,\mathrm{tri}\left(\frac{\tau - nT}{T/2}\right) \longleftrightarrow$ $(TP/2)\,Sa^2(\omega T/4)\, e^{-jn\omega T}$ and therefore

$$\mathcal{S}_{XX}(\omega) = \sum_{n=-\infty}^{\infty} \frac{TP}{2} Sa^2(\omega T/4)\, e^{-jn\omega T}$$

$$= \frac{TP}{2} Sa^2(\omega T/4) \sum_{n=-\infty}^{\infty} e^{-jn\omega T}.$$

Next, use

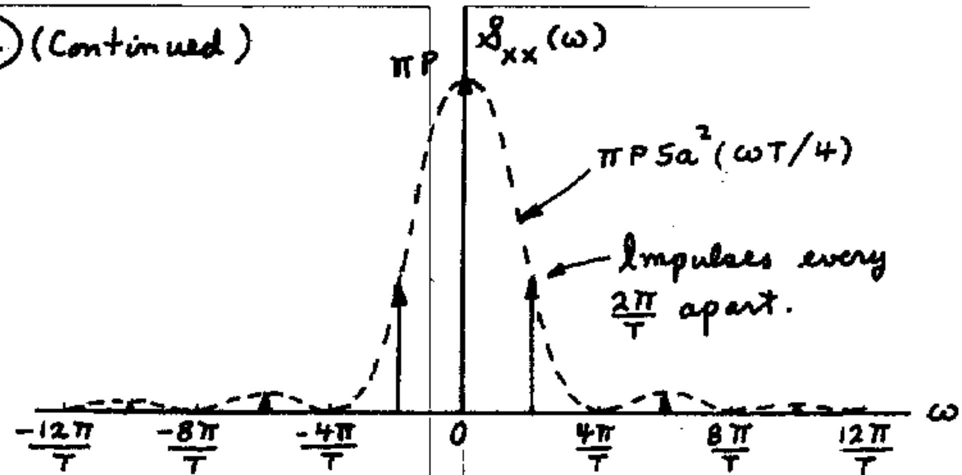
$$\sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{T}\right) = \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} e^{-jn\omega T} \qquad (1)$$

(See problem D-8.) to obtain

$$\mathcal{S}_{XX}(\omega) = \pi P\, Sa^2(\omega T/4) \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{n2\pi}{T}\right). \qquad (2)$$

A sketch of this power spectrum follows.

★7.2-8. (Continued)



7.2-9. $R_{XX}(\tau) = E[X(t)X(t+\tau)] = \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j E[X_i(t)X_j(t+\tau)]$.

Because of independence $E[X_i(t)X_j(t+\tau)] = R_{X_iX_i}(\tau)$, $i=j$ and $E[X_i(t)X_j(t+\tau)] = E[X_i(t)]E[X_j(t+\tau)] = 0$, $i \neq j$ (zero means). Thus,

$$R_{XX}(\tau) = \sum_{i=1}^{N}\alpha_i^2 R_{X_iX_i}(\tau)$$

and

$$\mathcal{S}_{XX}(\omega) = \sum_{i=1}^{N}\alpha_i^2 \mathcal{S}_{X_iX_i}(\omega).$$

For nonzero means $\bar{X}_i \neq 0$, $i=1,2,\cdots,N$:

$$R_{XX}(\tau) = \sum_{i=1}^{N}\alpha_i^2 R_{X_iX_i}(\tau) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\alpha_i\alpha_j\bar{X}_i\bar{X}_j \quad \text{and so}$$

$$\mathcal{S}_{XX}(\omega) = \sum_{i=1}^{N}\alpha_i^2\mathcal{S}_{X_iX_i}(\omega) + \left\{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}2\pi\alpha_i\alpha_j\bar{X}_i\bar{X}_j\right\}\delta(\omega)$$

where pair 2 of Appendix E has been used.

7.2-10. Use pair 19 of Appendix E and (D-7) and (D-5): $A e^{-\alpha|\tau|} \longleftrightarrow \dfrac{2\alpha A}{\alpha^2+\omega^2}$.

But if $f(t) \longleftrightarrow F(\omega)$ then $f(t)\exp(\pm j\omega_0 t) \longleftrightarrow F(\omega \pm \omega_0)$.

Thus,

$$R_{XX}(\tau) = A e^{-\alpha|\tau|}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] \longleftrightarrow \frac{\alpha A}{\alpha^2+(\omega-\omega_0)^2} + \frac{\alpha A}{\alpha^2+(\omega+\omega_0)^2}$$

and
$$\mathcal{S}_{XX}(\omega) = \frac{\alpha A}{\alpha^2+(\omega-\omega_0)^2} + \frac{\alpha A}{\alpha^2+(\omega+\omega_0)^2}.$$

7.2-11. (a) From (7.1-17): $\mathcal{S}_{\dot{X}\dot{X}}(\omega) = \dfrac{6\omega^4}{(1+\omega^2)^3}$.

(b) From (C-33): $P_{\dot{X}\dot{X}} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{6\omega^4\, d\omega}{(1+\omega^2)^3}$

$$= \frac{3}{\pi}\left\{\frac{\omega}{4(1+\omega^2)^2} - \frac{5\omega}{8(1+\omega^2)} + \frac{3}{8}\tan^{-1}(\omega)\right\}\Bigg|_{-\infty}^{\infty} = 9/8.$$

7.2-12. (a) From (7.1-17): $\mathcal{S}_{\dot{X}\dot{X}}(\omega) = \dfrac{6\omega^4}{(1+\omega^2)^4}$.

(b) From (C-36): $P_{\dot{X}\dot{X}} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{6\omega^4\, d\omega}{(1+\omega^2)^4}$

$$= \frac{3}{\pi}\left\{\frac{\omega}{6(1+\omega^2)^3} - \frac{7\omega}{24(1+\omega^2)^2} + \frac{\omega}{16(1+\omega^2)} + \frac{1}{16}\tan^{-1}(\omega)\right\}\Bigg|_{-\infty}^{\infty} = \frac{3}{16}.$$

7.2-13. $R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)]$

$$= E\left[\int_{-\infty}^{\infty} h(\xi)X(t-\xi)\,d\xi \int_{-\infty}^{\infty} h(\eta)X(t+\tau-\eta)\,d\eta\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\xi-\eta)\,h(\xi)h(\eta)\,d\xi\, d\eta = R_{YY}(\tau).$$

Fourier transform $R_{YY}(\tau)$ to obtain $\mathcal{S}_{YY}(\omega)$.

$$\mathcal{S}_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

(7.2-13.) (Continued)

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(\tau+\xi-\eta) h(\xi) h(\eta) \, d\xi \, d\eta \right\} e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} h(\xi) \int_{-\infty}^{\infty} h(\eta) \left\{ \int_{-\infty}^{\infty} R_{XX}(\tau+\xi-\eta) e^{-j\omega\tau} d\tau \right\} d\eta \, d\xi .$$

Let $u = \tau + \xi - \eta$, $du = d\tau$:

$$\mathcal{S}_{YY}(\omega) = \underbrace{\int_{-\infty}^{\infty} h(\xi) e^{j\omega\xi} d\xi}_{H^*(\omega)} \underbrace{\int_{-\infty}^{\infty} h(\eta) e^{-j\omega\eta} d\eta}_{H(\omega)} \underbrace{\int_{-\infty}^{\infty} R_{XX}(u) e^{-j\omega u} du}_{\mathcal{S}_{XX}(\omega)}$$

Thus,

$$\mathcal{S}_{YY}(\omega) = \mathcal{S}_{XX}(\omega) \left| H(\omega) \right|^2 .$$

(7.2-14.) (a) Average power in a sinusoid of peak amplitude A is $A^2/2$ (in $1\,\Omega$ per our normalization convention). For noise power use (7.1-12).

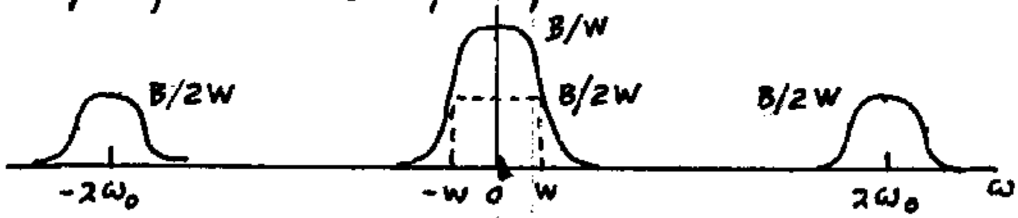
$$P_{NN} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{W^2 d\omega}{W^2 + \omega^2} = \frac{W^2}{2\pi} \left\{ \frac{1}{W} \tan^{-1}\left(\frac{\omega}{W}\right) \Big|_{-\infty}^{\infty} \right\} = W/2$$

from (C-25). Define (S/N) as signal-to-noise ratio. Then $(S/N) = \frac{A^2}{W}$. (b) (S/N) is maximum when $W \to 0$. The consequence of choosing $W \to 0$ is that noise disappears because $\mathcal{S}_{NN}(\omega)$ becomes a vanishingly narrow function around $\omega = 0$.

7.2-15. (a) Expand $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1}{2} + \frac{1}{2}e^{j2x} + \frac{1}{2}e^{-j2x}$

and use (D-7) and pair 19 of Appendix E to get

$$\mathcal{S}_{XX}(\omega) = \underbrace{\frac{BW}{W^2+\omega^2}}_{\text{lowpass part}} + \underbrace{\frac{BW/2}{W^2+(\omega-2\omega_0)^2} + \frac{BW/2}{W^2+(\omega+2\omega_0)^2}}_{\text{bandpass part}}$$



(a) With $\omega_0 \gg W$ we have, from (C-25),

$$P_{lowpass} = \frac{BW}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{W^2+\omega^2} = \frac{B}{2}, \quad P_{bandpass} = \frac{BW}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{W^2+(\omega-2\omega_0)^2} + \frac{BW}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{W^2+(\omega+2\omega_0)^2} = \frac{B}{2}.$$

Note that bandpass and lowpass components each possess half of the total power.

★**7.2-16.** We use expansions of $\cos^N(x)$ given on pp. 25-26 of Gradshteyn and Ryzhik along with pair 19 and (D-7). (a) N is odd ≥ 1.

$$\cos^N(x) = \frac{1}{2^{N-1}}\sum_{k=0}^{(N-1)/2}\binom{N}{k}\cos[(N-2k)x]$$

$$= \frac{1}{2^{N-1}}\sum_{k=0}^{(N-1)/2}\binom{N}{k}\frac{1}{2}\left[e^{j(N-2k)x} + e^{-j(N-2k)x}\right]$$

Thus, we easily obtain

$$\mathcal{S}_{XX}(\omega) = \frac{1}{2^{N-1}}\sum_{k=0}^{(N-1)/2}\binom{N}{k}\left\{\frac{BW}{W^2+[\omega-(N-2k)\omega_0]^2} + \frac{BW}{W^2+[\omega+(N-2k)\omega_0]^2}\right\}$$

★ 7.2-16 (Continued)

Because $N = N, N-2, \cdots, 3, 1$ only, terms in $\mathcal{S}_{XX}(\omega)$ are all bandpass and occur at $\pm$(odd multiples of ω_0).

(b) $$\cos^N(x) = \frac{1}{2^N}\left\{\binom{N}{N/2} + 2\sum_{k=0}^{\frac{N}{2}-1}\binom{N}{k}\cos[(N-2k)x]\right\}$$

for N even ≥ 2. Thus, by the preceding procedures

$$\mathcal{S}_{XX}(\omega) = \underbrace{\frac{\frac{1}{2^{N-1}}\binom{N}{N/2}BW}{W^2+\omega^2}}_{\text{baseband or lowpass term}} + \sum_{k=0}^{\frac{N}{2}-1}\left\{\frac{\frac{1}{2^{N-1}}\binom{N}{k}BW}{W^2+[\omega-(N-2k)\omega_0]^2} + \frac{\frac{1}{2^{N-1}}\binom{N}{k}BW}{W^2+[\omega+(N-2k)\omega_0]^2}\right\}$$

Terms in the summation are bandpass and are located at $\pm$(even multiples of ω_0).

★ 7.2-17. (a) $R_{YY}(t,t+\tau) = E[Y(t)Y(t+\tau)] = E[\underset{\uparrow t_1}{X(t)}\cdot \underset{\uparrow t_2}{X(t-T)}\,\underset{\uparrow t_3}{X(t+\tau)}\,\underset{\uparrow t_4}{X(t+\tau-T)}] = R_{XX}(t_2-t_1)R_{XX}(t_4-t_3)$

$+ R_{XX}(t_3-t_1)R_{XX}(t_4-t_2)$

$+ R_{XX}(t_4-t_1)R_{XX}(t_3-t_2) - 2\bar{X}^4 = R_{XX}(-T)R_{XX}(-T) + R_{XX}(\tau)R_{XX}(\tau) + R_{XX}(\tau-T)R_{XX}(\tau+T) - 2\bar{X}^4$ so

$$R_{YY}(\tau) = [R_{XX}^2(T) - \bar{X}^4] + R_{XX}^2(\tau) + R_{XX}(\tau-T)R_{XX}(\tau+T). \quad (1)$$

(b) We use pair 2 of Appendix E to transform the first term of (1)

$$[R_{XX}^2(T) - 2\bar{X}^4] \longleftrightarrow 2\pi[R_{XX}^2(T)-\bar{X}^4]\delta(\omega).$$

The second term of (1) transforms using (D-17)

★ 7.2-17. (Continued)

$$R_{XX}^2(\tau) \longleftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\xi)\mathcal{S}_{XX}(\omega-\xi)\,d\xi .$$

The third term transforms by using (D-6) on each factor first and then applying (D-17)

$$R_{XX}(\tau-T)R_{XX}(\tau+T) \longleftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\xi)\mathcal{S}_{XX}(\omega-\xi)\,e^{j(\omega-2\xi)T}d\xi.$$

On combining terms and observing that terms in ω that have odd symmetry are zero, we get $\mathcal{S}_{YY}(\omega) = 2\pi[R_{XX}^2(T) - 2\bar{X}^4]\,\delta(\omega)$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\xi)\mathcal{S}_{XX}(\omega-\xi)\{1+\cos[(\omega-2\xi)T]\}\,d\xi.$$

7.2-18. (a) No, because $R_{XX}(t,t+\tau)$ depends on t. (b) $R_{XX}(\tau) = A[R_{XX}(t,t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} 12\,e^{-4\tau^2}\cos^2(24t)\,dt$ $= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}6\,e^{-4\tau^2}[1+\cos(48t)]\,dt = 6\,e^{-4\tau^2}$. (c) $\mathcal{S}_{XX}(\omega) = \mathcal{F}\{R_{XX}(\tau)\}$ $= \mathcal{F}\{6\,e^{-4\tau^2}\} = 3\sqrt{\pi}\,e^{-\omega^2/16}$ from pair 20, Table E-1.

7.2-19. $R_{YY}(t,t+\tau) = E\{[X(t)-X(t-a)][X(t+\tau)-X(t+\tau-a)]\}$ $= 2R_{XX}(\tau) - R_{XX}(\tau-a) - R_{XX}(\tau+a)$. $\mathcal{S}_{YY}(\omega) = 2\mathcal{S}_{XX}(\omega) - \mathcal{S}_{XX}(\omega)e^{-j\omega a}$ $-\mathcal{S}_{XX}(\omega)e^{j\omega a} = 4\,\mathcal{S}_{XX}(\omega)\sin^2(\omega a/2)$.

7.2-20. $\mathcal{S}_{XX}(\omega) = \dfrac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)} = \dfrac{A}{16+\omega^2} + \dfrac{B}{9+\omega^2} = \dfrac{(9A+16B)+(A+B)\omega^2}{(16+\omega^2)(9+\omega^2)}$.

On solving term-by-term equations for A and B, we have $A=5$, $B=7$. From pair 19, Table E-1:

7.2-20. (Continued)

$$R_{XX}(\tau) = \frac{5}{8}e^{-4|\tau|} + \frac{7}{6}e^{-3|\tau|} \longleftrightarrow \mathcal{S}_{XX}(\omega) = \frac{5}{16+\omega^2} + \frac{7}{9+\omega^2}$$

7.2-21. From pair 19, Table E-1, we can write:

$$R_{XX}(\tau) = R(\tau) \star R(\tau) \longleftrightarrow \left(\frac{\sqrt{8}}{9+\omega^2}\right)^2 = \mathcal{S}_{XX}(\omega) \quad , \quad R(\tau) = \frac{\sqrt{8}}{6}e^{-3|\tau|}.$$

Thus,

$$R_{XX}(\tau) = R(\tau) \star R(\tau) = \int_{-\infty}^{\infty} \frac{\sqrt{8}}{6}e^{-3|\xi|}\frac{\sqrt{8}}{6}e^{-3|\tau-\xi|}d\xi.$$

For $\tau \geq 0$:

$$R_{XX}(\tau) = \int_{-\infty}^{0} \frac{8}{36}e^{3\xi}e^{-3(\tau-\xi)}d\xi + \int_{0}^{\tau} \frac{8}{36}e^{-3\xi}e^{-3(\tau-\xi)}d\xi + \int_{\tau}^{\infty} \frac{8}{36}e^{-3\xi}e^{3(\tau-\xi)}d\xi$$

$$= \frac{8}{36}e^{-3\tau}\left(\frac{1}{3}+\tau\right), \quad \tau \geq 0.$$

Since $R_{XX}(-\tau) = R_{XX}(\tau)$, then $R_{XX}(\tau) = \frac{2}{9}e^{-3|\tau|}\left(\frac{1}{3}+|\tau|\right)$.

★ 7.2-22. Use properties of Fourier transforms. From pair 19 and (D-7):

$$e^{-\alpha|\tau|} \longleftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$$

$$e^{-\alpha|\tau|}e^{\pm j\alpha\tau} \longleftrightarrow \frac{2\alpha}{\alpha^2+(\omega\mp\alpha)^2}$$

Thus,

$$\cos(\alpha\tau)e^{-\alpha|\tau|} = e^{-\alpha\tau}\frac{1}{2}\left(e^{j\alpha\tau}+e^{-j\alpha\tau}\right) \longleftrightarrow \frac{\alpha}{\alpha^2+(\omega-\alpha)^2} + \frac{\alpha}{\alpha^2+(\omega+\alpha)^2}$$

$$\sin(\alpha|\tau|)e^{-\alpha|\tau|} = u(\tau)\sin(\alpha\tau)e^{-\alpha\tau} + u(-\tau)e^{\alpha\tau}\sin(-\alpha\tau)$$

From pair 15:

$$\sin(\alpha|\tau|)e^{-\alpha|\tau|} \longleftrightarrow \frac{-(\omega-\alpha)}{\alpha^2+(\omega-\alpha)^2} + \frac{(\omega+\alpha)}{\alpha^2+(\omega+\alpha)^2}.$$

Finally,

$$R_{XX}(\tau) \longleftrightarrow \frac{\alpha-(\omega-\alpha)}{\alpha^2+(\omega-\alpha)^2} + \frac{\alpha+(\omega+\alpha)}{\alpha^2+(\omega+\alpha)^2} = \frac{8\alpha^3}{4\alpha^4+\omega^4} = \mathcal{S}_{XX}(\omega).$$

★7.3-1. (a) $E[W(t)] = E[X(t)\cos(\omega_0 t) + Y(t)\sin(\omega_0 t)]$ $= \bar{X}\cos(\omega_0 t) + \bar{Y}\sin(\omega_0 t)$ = constant with t only if $\bar{X}=0$ and $\bar{Y}=0$. $R_{WW}(t,t+\tau) = E[\{X(t)\cos(\omega_0 t) + Y(t)\sin(\omega_0 t)\}\{X(t+\tau)\cos(\omega_0 t+\omega_0\tau) + Y(t+\tau)\sin(\omega_0 t + \omega_0\tau)\}] = R_{XX}(\tau)\cos(\omega_0 t)\cos(\omega_0 t+\omega_0\tau)$

$$+ R_{YY}(\tau)\sin(\omega_0 t)\sin(\omega_0 t+\omega_0\tau)$$
$$+ R_{XY}(\tau)\cos(\omega_0 t)\sin(\omega_0 t+\omega_0\tau)$$
$$+ R_{YX}(\tau)\sin(\omega_0 t)\cos(\omega_0 t+\omega_0\tau)$$

$= [R_{XX}(\tau)+R_{YY}(\tau)]\frac{1}{2}\cos(\omega_0\tau) + [R_{XY}(\tau)-R_{YX}(\tau)]\frac{1}{2}\sin(\omega_0\tau)$
$+ [R_{XX}(\tau)-R_{YY}(\tau)]\frac{1}{2}\cos(2\omega_0 t+\omega_0\tau)$
$+ [R_{XY}(\tau)+R_{YX}(\tau)]\frac{1}{2}\sin(2\omega_0 t+\omega_0\tau)$. This function is independent of t only if $R_{XX}(\tau) = R_{YY}(\tau)$ and

$R_{YX}(\tau) = -R_{XY}(\tau)$. Since $R_{YX}(\tau) = R_{XY}(-\tau)$ [See (6.3-16)] this last condition means $R_{XY}(\tau)$ must be an odd function of τ. Thus, for $W(t)$ to be wide-sense stationary we require

$$E[X(t)]=0,\quad E[Y(t)]=0$$
$$R_{XX}(\tau) = R_{YY}(\tau)$$
$$R_{XY}(\tau) = -R_{YX}(\tau) = -R_{XY}(-\tau).$$

(b) $R_{WW}(\tau) = R_{XX}(\tau)\cos(\omega_0\tau) + R_{XY}(\tau)\sin(\omega_0\tau)$.

If $R_{XX}(\tau) \leftrightarrow \mathcal{S}_{XX}(\omega)$, $R_{XY}(\tau) \leftrightarrow \mathcal{S}_{XY}(\omega)$, then from Fourier transform property (D-7)

$$R_{XX}(\tau)\cos(\omega_0\tau) \leftrightarrow \frac{1}{2}[\mathcal{S}_{XX}(\omega-\omega_0) + \mathcal{S}_{XX}(\omega+\omega_0)]$$

*7.3-1. (Continued)

$$R_{XY}(\tau)\sin(\omega_0\tau) \leftrightarrow \frac{1}{2j}\left[\mathcal{S}_{XY}(\omega-\omega_0) - \mathcal{S}_{XY}(\omega+\omega_0)\right]$$

and

$$\mathcal{S}_{WW}(\omega) = \frac{1}{2}\Big\{\mathcal{S}_{XX}(\omega-\omega_0) + \mathcal{S}_{XX}(\omega+\omega_0) - j\,\mathcal{S}_{XY}(\omega-\omega_0) + j\,\mathcal{S}_{XY}(\omega+\omega_0)\Big\}.$$

(c) For uncorrelated $X(t)$ and $Y(t)$ $R_{XY}(\tau) = \bar{X}\bar{Y} = 0$, so $\mathcal{S}_{XY}(\omega) = 0$. Then

$$\mathcal{S}_{WW}(\omega) = \frac{1}{2}\left[\mathcal{S}_{XX}(\omega-\omega_0) + \mathcal{S}_{XX}(\omega+\omega_0)\right].$$

7.3-2. (a) $R_{WW}(t,t+\tau) = E[\{AX(t)+BY(t)\}\{AX(t+\tau) + BY(t+\tau)\}] = A^2R_{XX}(\tau) + B^2R_{YY}(\tau) + AB\,R_{XY}(\tau) + AB\,R_{YX}(\tau)$. $\mathcal{S}_{WW}(\omega) = A^2\mathcal{S}_{XX}(\omega) + B^2\mathcal{S}_{YY}(\omega) + AB\,\mathcal{S}_{XY}(\omega) + AB\,\mathcal{S}_{YX}(\omega)$.

(b) For uncorrelated $X(t)$ and $Y(t)$: $R_{XY}(\tau) = \bar{X}\bar{Y}$,

$R_{YX}(\tau) = \bar{X}\bar{Y}$ so

$$\mathcal{S}_{WW}(\omega) = A^2\mathcal{S}_{XX}(\omega) + B^2\mathcal{S}_{YY}(\omega) + 4\pi\bar{X}\bar{Y}AB\,\delta(\omega)$$

from pair 2 of Appendix E.

(c) $R_{XW}(t,t+\tau) = E[X(t)\{AX(t+\tau) + BY(t+\tau)\}] = AR_{XX}(\tau) + BR_{XY}(\tau)$. $\mathcal{S}_{XW}(\omega) = A\,\mathcal{S}_{XX}(\omega) + B\,\mathcal{S}_{XY}(\omega)$.

$R_{YW}(t,t+\tau) = E[Y(t)\{AX(t+\tau) + BY(t+\tau)\}] = A\,R_{YX}(\tau) + B\,R_{YY}(\tau)$. $\mathcal{S}_{YW}(\omega) = A\,\mathcal{S}_{YX}(\omega) + B\,\mathcal{S}_{YY}(\omega)$.

★7.3-3. $X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt = \int_{-T}^{T} \frac{A}{2}\left[e^{j\omega_0 t + j\Theta} + e^{-j\omega_0 t - j\Theta}\right]$

$\cdot e^{-j\omega t} dt = AT e^{j\Theta} Sa[(\omega-\omega_0)T] + AT e^{-j\Theta} Sa[(\omega+\omega_0)T]$.

$Y_T(\omega) = \int_{-T}^{T} Y(t) e^{-j\omega t} dt = \int_{-T}^{T} \frac{W(t)}{2}\left[e^{j\omega_0 t + j\Theta} + e^{-j\omega_0 t - j\Theta}\right]$

$\cdot e^{-j\omega t} dt = \frac{e^{j\Theta}}{2}\int_{-T}^{T} W(t) e^{-j(\omega-\omega_0)t} dt + \frac{e^{-j\Theta}}{2}\int_{-T}^{T} W(t)$

$\cdot e^{-j(\omega+\omega_0)t} dt$. Combining these results:

$$E[X_T^*(\omega) Y_T(\omega)] = E\Big[\frac{AT}{2} Sa[(\omega-\omega_0)T]\int_{-T}^{T} W(t) e^{-j(\omega-\omega_0)t} dt$$

$$+ \frac{AT}{2} e^{-j2\Theta} Sa[(\omega-\omega_0)T]\int_{-T}^{T} W(t) e^{-j(\omega+\omega_0)t} dt$$

$$+ \frac{AT}{2} e^{j2\Theta} Sa[(\omega+\omega_0)T]\int_{-T}^{T} W(t) e^{-j(\omega-\omega_0)t} dt$$

$$+ \frac{AT}{2} Sa[(\omega+\omega_0)T]\int_{-T}^{T} W(t) e^{-j(\omega+\omega_0)t} dt\Big]$$

$$= \frac{A T \overline{W}}{2}\Big\{2T Sa^2[(\omega-\omega_0)T] + 2T Sa^2[(\omega+\omega_0)T]$$

$$+ 2T E[e^{-j2\Theta}] Sa[(\omega-\omega_0)T] Sa[(\omega+\omega_0)T]$$

$$+ 2T E[e^{+j2\Theta}] Sa[(\omega-\omega_0)T] Sa[(\omega+\omega_0)T]\Big\}.$$

$$\lim_{T\to\infty} \frac{E[X_T^*(\omega) Y_T(\omega)]}{2T} = \frac{A\overline{W}}{2}\Big\{\lim_{T\to\infty} T\, Sa^2[(\omega-\omega_0)T]$$

$$+ \lim_{T\to\infty} T\, Sa^2[(\omega+\omega_0)T]$$

$$+ E[e^{j2\Theta} + e^{-j2\Theta}] \lim_{T\to\infty} T\, Sa[(\omega-\omega_0)T] \underbrace{\lim_{T\to\infty} Sa[(\omega+\omega_0)T]\Big\}}_{=0 \text{ except } =1 \text{ at } -\omega_0.}$$

From Example 7.1-2 and Problem 7.1-8:

$$\lim_{T\to\infty} \frac{T}{\pi} Sa^2[\alpha T] = \delta(\alpha)$$

$$\lim_{T\to\infty} \frac{T}{\pi} Sa[\alpha T] = \delta(\alpha)$$

★ 7.3-3. (Continued)

so

$$\mathcal{S}_{XY}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T} = \frac{\pi A\bar{W}}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

This result is independent of the density of Θ because the term involving $E[e^{j2\Theta}+e^{-j2\Theta}]$ never has to be evaluated.

7.3-4. Use (7.3-12) and (7.3-14):

$$R_{XY}(\omega) = Re[\mathcal{S}_{XY}(\omega)] = \frac{1}{2}[\mathcal{S}_{XY}(\omega)+\mathcal{S}_{XY}^*(\omega)]$$

$$= \frac{1}{2}\lim_{T\to\infty}\frac{1}{2T}E\left[X_T^*(\omega)Y_T(\omega)+X_T(\omega)Y_T^*(\omega)\right] \quad (1)$$

$$R_{YX}(\omega) = Re[\mathcal{S}_{YX}(\omega)] = \frac{1}{2}[\mathcal{S}_{YX}(\omega)+\mathcal{S}_{YX}^*(\omega)]$$

$$= \frac{1}{2}\lim_{T\to\infty}\frac{1}{2T}E\left[Y_T^*(\omega)X_T(\omega)+Y_T(\omega)X_T^*(\omega)\right]. \quad (2)$$

Thus, from (1) and (2), $R_{XY}(\omega) = R_{YX}(\omega)$. Next,

$$I_{XY}(\omega) = Im[\mathcal{S}_{XY}(\omega)] = \frac{1}{2j}[\mathcal{S}_{XY}(\omega)-\mathcal{S}_{XY}^*(\omega)]$$

$$= \frac{1}{2j}\lim_{T\to\infty}\frac{1}{2T}E\left[X_T^*(\omega)Y_T(\omega)-X_T(\omega)Y_T^*(\omega)\right] \quad (3)$$

$$I_{YX}(\omega) = Im[\mathcal{S}_{YX}(\omega)] = \frac{1}{2j}[\mathcal{S}_{YX}(\omega)-\mathcal{S}_{YX}^*(\omega)]$$

$$= \frac{1}{2j}\lim_{T\to\infty}\frac{1}{2T}E\left[Y_T^*(\omega)X_T(\omega)-Y_T(\omega)X_T^*(\omega)\right]. \quad (4)$$

Thus, from (3) and (4), $I_{XY}(\omega) = -I_{YX}(\omega)$. Next, since $X(t)$ and $Y(t)$ are real processes it is easy to show that $X_T^*(\omega) = X_T(-\omega)$ and $Y_T^*(\omega) = Y_T(-\omega)$. // Proof for $X_T(\omega)$: $X_T(\omega) = \int_{-T}^{T} x(t) e^{-j\omega t}dt$. $X_T^*(\omega) = \int_{-T}^{T} x(t)e^{j\omega t}dt = X_T(-\omega)$ if

(7.3-4.) (Continued) $x(t)$ is real. A similar proof holds for $Y_T(\omega)$. // Equations (2) and (4) become:

$$R_{YX}(-\omega) = \frac{1}{2}\lim_{T\to\infty}\frac{1}{2T}E\left[Y_T^*(-\omega)X_T(-\omega) + Y_T(-\omega)X_T^*(-\omega)\right]$$

$$= \frac{1}{2}\lim_{T\to\infty}\frac{1}{2T}E\left[Y_T(\omega)X_T^*(\omega) + Y_T^*(\omega)X_T(\omega)\right] = R_{YX}(\omega)$$

$$I_{YX}(-\omega) = \frac{1}{2j}\lim_{T\to\infty}\frac{1}{2T}E\left[Y_T^*(-\omega)X_T(-\omega) - Y_T(-\omega)X_T^*(-\omega)\right]$$

$$= \frac{1}{2j}\lim_{T\to\infty}\frac{1}{2T}E\left[Y_T(\omega)X_T^*(\omega) - Y_T^*(\omega)X_T(\omega)\right] = -I_{YX}(\omega).$$

By combining all the above:

$$R_{XY}(\omega) = R_{YX}(\omega) = R_{YX}(-\omega)$$

$$I_{XY}(\omega) = -I_{YX}(\omega) = I_{YX}(-\omega).$$

(7.3-5.) $\mathcal{S}_{XY}(\omega) = R_{XY}(\omega) + jI_{XY}(\omega) = R_{YX}(-\omega) + jI_{YX}(-\omega)$

$$= \mathcal{S}_{YX}(-\omega)$$

$$\mathcal{S}_{XY}(\omega) = R_{XY}(\omega) + jI_{XY}(\omega) = R_{YX}(\omega) - jI_{YX}(\omega)$$

$$= \mathcal{S}_{YX}^*(\omega)$$

Thus,

$$\mathcal{S}_{XY}(\omega) = \mathcal{S}_{YX}(-\omega) = \mathcal{S}_{YX}^*(\omega).$$

(7.3-6.) For orthogonal $X(t)$ and $Y(t)$ $R_{XY}(\tau) = 0$ and $R_{YX}(\tau) = 0$ so the Fourier transforms are zero: $\mathcal{S}_{XY}(\omega) = 0$, $\mathcal{S}_{YX}(\omega) = 0$. For uncorrelated $X(t)$ and $Y(t)$ $R_{XY}(\tau) = \bar{X}\,\bar{Y} = R_{YX}(\tau)$. By using pair 2 of Appendix E the Fourier transforms are $\mathcal{S}_{XY}(\omega) = 2\pi\bar{X}\,\bar{Y}\,\delta(\omega) = \mathcal{S}_{YX}(\omega)$.

7.3-7. By the same procedures used in Problem 7.3-7 we get $R_{XY}(t,t+\tau) = E[X(t)X(t+\tau)X(t+\tau-T)]$ with $t_1 = t$, $t_2 = t+\tau$, $t_3 = t+\tau-T$

$$= \bar{X} R_{XX}(t_3 - t_2) + \bar{X} R_{XX}(t_1 - t_3) + \bar{X} R_{XX}(t_2 - t_1) - 2\bar{X}^3$$

$$\therefore R_{XY}(\tau) = \bar{X}[R_{XX}(\tau - T) + R_{XX}(-T) + R_{XX}(\tau)] - 2\bar{X}^3. \qquad (1)$$

By using the time-shifting property of Fourier transforms the transform of (1) follows:

$$\mathcal{S}_{XY}(\omega) = 2\pi[\bar{X}R_{XX}(T) - 2\bar{X}^3]\delta(\omega) + 2\bar{X}\mathcal{S}_{XX}(\omega)\cos(\omega T/2)\, e^{-j\omega T/2}.$$

7.3-8. (a) yes, (b) no, cannot be imaginary, (c) no, cannot be odd, (d) no, cannot be negative, (e) no, cannot depend on τ, (f) no, imaginary part must be odd, (g) yes.

7.3-9. (a) $R_{WW}(t,t+\tau) = E[X(t)Y(t)X(t+\tau)Y(t+\tau)]$ $= R_{XX}(\tau)R_{YY}(\tau)$. Use (D-17) to get

$$\mathcal{S}_{WW}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\mathcal{S}_{YY}(\omega - \xi)\,d\xi.$$

(b) $R_{XW}(t,t+\tau) = E[X(t)X(t+\tau)Y(t+\tau)] = R_{XX}(\tau)\bar{Y}$

$\mathcal{S}_{WX}(\omega) = \bar{Y}\mathcal{S}_{XX}(\omega)$. (c) $R_{WX}(t,t+\tau) = E[X(t)Y(t)X(t+\tau)]$ $= R_{XX}(\tau)\bar{Y} = R_{XW}(t,t+\tau)$. $\mathcal{S}_{WX}(\omega) = \bar{Y}\mathcal{S}_{XX}(\omega) = \mathcal{S}_{XW}(\omega)$.

(d) Form product of explicit functions given:

$$R_{WW}(t,t+\tau) = R_{WW}(\tau) = \frac{W_1 W_2}{\pi^2} Sa(W_1\tau)\, Sa(W_2\tau).$$

From pair 5, appendix E, $\mathcal{S}_{XX}(\omega) = \text{rect}(\omega/2W_1)$ and $\mathcal{S}_{YY}(\omega) = \text{rect}(\omega/2W_2)$. We graphically form the

(7.3-9.) (Continued)

convolution required to obtain $\mathcal{S}_{WW}(\omega)$ in part (a). The result is

$$\mathcal{S}_{WW}(\omega) = \begin{cases} (W_1+W_2+\omega)/2\pi, & -(W_2+W_1) \le \omega \le -(W_2-W_1) \\ 2W_1/2\pi & -(W_2-W_1) \le \omega \le (W_2-W_1) \\ (W_1+W_2-\omega)/2\pi, & (W_2-W_1) \le \omega \le (W_2+W_1) \end{cases}$$

and zero for all other ω.

(7.3-10.) We use (D-11) and pair 20 of Appendix E to get $\mathcal{S}_{XY}(\omega) = \frac{P\sqrt{\pi}}{W^2}\left[W - j\frac{\omega}{2W}\right]e^{-(\omega/2W)^2}$

This function satisfies all three properties (7.3-16) through (7.3-18) so his fears are not valid.

(7.3-11.) (a) From (D-10) $\mathcal{S}_{XY}(\omega) = j\omega\mathcal{S}_{XX}(\omega)$. From (7.3-16) $\mathcal{S}_{YX}(\omega) = \mathcal{S}_{XY}(-\omega) = -j\omega\mathcal{S}_{XX}(\omega)$. (b) $\mathcal{S}_{XY}(\omega)$ is imaginary with odd symmetry.

(7.3-12.) (a) From (6.3-16) $R_{YX}(\tau) = R_{XY}(-\tau) = Bu(-\tau)e^{W\tau}$.

(b) $\mathcal{S}_{XY}(\omega) = \mathcal{F}\{R_{XY}(\tau)\} = \int_{-\infty}^{\infty} Bu(\tau)e^{-W\tau}e^{-j\omega\tau}d\tau$

$= B\int_0^{\infty} e^{-(W+j\omega)\tau}d\tau = B/(W+j\omega)$. From (7.3-16)

$\mathcal{S}_{YX}(\omega) = \mathcal{S}_{XY}(-\omega) = B/(W-j\omega)$.

(7.3-13.) (a) Use (6.3-16): $R_{YX}(\tau) = R_{XY}(-\tau) = -Bu(-\tau)\tau e^{W\tau}$.

(b) Use pair 16, Appendix E, to get $\mathcal{S}_{XY}(\omega) = B/(W+j\omega)^2$.

From (7.3-16) $\mathcal{S}_{YX}(\omega) = \mathcal{S}_{XY}(-\omega) = B/(W-j\omega)^2$.

(7.3-14.) $R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega) H(\omega) e^{j\omega\tau} d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(\alpha) e^{-j\omega\alpha} d\alpha \int_{-\infty}^{\infty} h(\beta) e^{-j\omega\beta} d\beta\, e^{j\omega\tau} d\omega$

$= \int_{-\infty}^{\infty} R_{XX}(\alpha) \int_{-\infty}^{\infty} h(\beta) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau-\beta-\alpha)} d\omega\, d\beta\, d\alpha$

$= \int_{-\infty}^{\infty} R_{XX}(\alpha) \int_{-\infty}^{\infty} h(\beta)\, \delta(\alpha+\beta-\tau)\, d\beta\, d\alpha$ or

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\beta)\, h(\beta)\, d\beta$$

From (6.3-16)

$$R_{YX}(\tau) = R_{XY}(-\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\beta)\, h(-\beta)\, d\beta .$$

(7.3-15.) $\mathcal{F}\left\{\frac{u(-\tau)}{a+b} e^{b\tau}\right\} = \int_{-\infty}^{0} \frac{e^{(b-j\omega)\tau}}{a+b} d\tau = \frac{1}{(a+b)(b-j\omega)} \triangleq F_1(\omega)$

$\mathcal{F}\left\{\frac{u(\tau) e^{-b\tau}}{a-b}\right\} = \int_{0}^{\infty} \frac{e^{-(b+j\omega)\tau}}{a-b} d\tau = \frac{1}{(a-b)(b+j\omega)} \triangleq F_2(\omega)$

$\mathcal{F}\left\{\frac{u(\tau)(-2b) e^{-a\tau}}{a^2-b^2}\right\} = \frac{-2b}{a^2-b^2}\int_{0}^{\infty} e^{-(a+j\omega)\tau} d\tau = \frac{-2b}{a^2-b^2}\cdot\frac{1}{(a+j\omega)} \triangleq F_3(\omega)$

$\mathcal{S}_{XX}(\omega) = F_1(\omega) + F_2(\omega) + F_3(\omega) = \frac{2b}{(b^2+\omega^2)(a+j\omega)}$.

★ (7.3-16.) Write $\mathcal{S}_{XY}(\omega) = G_1(\omega) G_2(\omega)$ where $G_1(\omega) = \frac{6}{9+\omega^2}$, $G_2(\omega) = \frac{1}{(3+j\omega)^2}$.

From Table E-1:

$g_1(\tau) = e^{-3|\tau|} \longleftrightarrow G_1(\omega)$

$g_2(\tau) = u(\tau)\tau e^{-3\tau} \longleftrightarrow G_2(\omega)$.

From (D-16):

$R_{XY}(\tau) = \int_{-\infty}^{\infty} g_1(\xi) g_2(\tau-\xi) d\xi = \int_{-\infty}^{\tau} g_1(\xi) g_2(\tau-\xi) e^{-3(\tau-\xi)} d\xi$.

For $\tau < 0$:

$R_{XY}(\tau) = \int_{-\infty}^{\tau} e^{3\xi}(\tau-\xi) e^{-3\tau+3\xi} d\xi = \frac{1}{36} e^{3\tau}, \quad \tau < 0.$

★ 7.3-16. (Continued)

For $\tau \geq 0$:

$$R_{XY}(\tau) = \int_{-\infty}^{0} e^{3\xi}(\tau-\xi)\, e^{-3\tau+3\xi}\, d\xi + \int_{0}^{\tau} e^{-3\xi}(\tau-\xi)\, e^{-3\tau+3\xi}\, d\xi$$

$$= \left(\frac{9\tau-1}{36}\right) e^{-3\tau}, \quad \tau \geq 0.$$

Thus,

$$R_{XY}(\tau) = \frac{e^{-3|\tau|}}{36}\left[(9\tau-1)\,u(\tau) + u(-\tau)\right].$$

★ 7.4-1. (a) $R_{XY}(t,t+\tau) = E[A\cos(\omega_0 t + \Theta) W(t+\tau) \cos(\omega_0 t$
$+ \omega_0\tau + \Theta)] = \frac{A}{2} E[W(t+\tau)\{\cos(\omega_0\tau) + \cos(2\omega_0 t + \omega_0\tau + 2\Theta)\}]$

$$= \frac{A\overline{W}}{2}\{\cos(\omega_0\tau) + E[\cos(2\Theta)]\cos(2\omega_0 t + \omega_0\tau) - E[\sin(2\Theta)]\sin(2\omega_0 t + \omega_0\tau)\}.$$

(b) $A[R_{XY}(t,t+\tau)] = \frac{A\overline{W}}{2}\{\cos(\omega_0\tau)$

$$+ E[\cos(2\Theta)] \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos(2\omega_0 t + \omega_0\tau)\, dt$$

$$- E[\sin(2\Theta)] \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin(2\omega_0 t + \omega_0\tau)\, dt \}$$

$$= (A\overline{W}/2)\cos(\omega_0\tau) = R_{XY}(\tau).$$

Now since

$$\cos(\omega_0\tau) \longleftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

from pair 11 of Appendix E, then

$$\mathcal{S}_{XY}(\omega) = \frac{\pi A\overline{W}}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)].$$

7.5-1. $\mathcal{S}_{X_sX_s}(\omega) = \int_{-\infty}^{\infty} R_{X_sX_s}(\tau)\, e^{-j\omega\tau}\, d\tau$

$$= \int_{-\infty}^{\infty} \frac{1}{T_p} R_{XX}(\tau) \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{\tau - nT_s}{T_p}\right) e^{-j\omega\tau}\, d\tau$$

7.5-1. (Continued)

$$= \frac{1}{T_p} \sum_{n=-\infty}^{\infty} \int_{nT_s-(T_p/2)}^{nT_s+(T_p/2)} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$\approx \frac{1}{T_p} \sum_{n=-\infty}^{\infty} R_{XX}(nT_s) \int_{nT_s-(T_p/2)}^{nT_s+(T_p/2)} e^{-j\omega\tau} d\tau$$

$$= \sum_{n=-\infty}^{\infty} R_{XX}(nT_s)\, e^{-jn\omega T_s} \frac{(e^{j\omega T_p/2} - e^{-j\omega T_p/2})}{2j\,(\omega T_p/2)}$$

$$= \sum_{n=-\infty}^{\infty} R_{XX}(nT_s)\, Sa(\omega T_p/2)\, e^{-jn\omega T_s}.$$ As $T_p \to 0$ $Sa(\omega T_p/2) \to 1$ for all ω, so

$$\mathcal{S}_{X_sX_s}(\omega) = \sum_{n=-\infty}^{\infty} R_{XX}(nT_s)\, e^{-jn\omega T_s} \quad \text{as } T_p \to 0.$$

7.5-2. $\sum_n \delta(t-nT_s)$ is periodic with period T_s so it has a Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t}, \quad \omega_s = 2\pi/T_s.$$

Here

$$C_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} f(t) e^{-jn\omega_s t} dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \sum_m \delta(t-mT_s) e^{-jn\omega_s t} dt$$

$= \frac{1}{T_s}$, all n, because only $\delta(t)$ falls in the range of integration. Thus,

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}.$$

7.5-3. $R_{X_sX_s}(\tau) = R_{XX}(\tau) \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s \tau}$

$$\mathcal{S}_{X_sX_s}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{jn\omega_s\tau} e^{-j\omega\tau} d\tau$$

7.5-3. (Continued) $$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j(\omega - n\omega_s)\tau} d\tau}_{\mathcal{S}_{XX}(\omega - n\omega_s)}$$

Thus, $\mathcal{S}_{X_sX_s}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \mathcal{S}_{XX}(\omega - n\omega_s)$.

7.5-4. $\mathcal{S}_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{XX}(nT_s) Sa\left[\frac{\omega_s}{2}(\tau - nT_s)\right] e^{-j\omega\tau} d\tau$$

$$= \sum_{n=-\infty}^{\infty} R_{XX}(nT_s) \mathcal{F}\left\{Sa\left[\frac{\omega_s}{2}(\tau - nT_s)\right]\right\} \qquad (1)$$

From pair 6 of Table E-1:

$$Sa(W\tau) \longleftrightarrow \frac{\pi}{W} rect\left(\frac{\omega}{2W}\right)$$

$$\therefore Sa\left(\frac{\omega_s}{2}\tau\right) \longleftrightarrow \frac{2\pi}{\omega_s} rect\left(\frac{\omega}{2\omega_s}\right) = T_s \, rect\left(\frac{\omega}{\omega_s}\right)$$

But, from the time shifting property of Fourier transforms: $Sa\left[\frac{\omega_s}{2}(\tau - nT_s)\right] \longleftrightarrow T_s \, rect(\omega/\omega_s) \exp(-jn\omega T_s)$ so (1) becomes

$$\mathcal{S}_{XX}(\omega) = T_s \sum_{n=-\infty}^{\infty} R_{XX}(nT_s) rect\left(\frac{\omega}{\omega_s}\right) e^{-jn\omega T_s}$$

$$= T_s \, rect\left(\frac{\omega}{\omega_s}\right) \underbrace{\sum_{n=-\infty}^{\infty} R_{XX}(nT_s) e^{-jn\omega T_s}}_{\mathcal{S}_{X_sX_s}(\omega)}.$$

7.5-5. From (7.5-2)

$$\mathcal{S}_{N_sN_s}(\omega) = \sum_{n=-\infty}^{\infty} R_{NN}(nT_s) e^{-jn\omega T_s} = \begin{cases} \sigma_N^2 e^{j0} = \sigma_N^2, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

(7.5-6) $R_{XX}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{S}_{X_sX_s}(e^{j\Omega})\, e^{jn\Omega}\, d\Omega$

$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} A_0 \cos(\Omega)\, e^{jn\Omega}\, d\Omega$ (since $\text{rect}(\Omega/\pi) = 0$ for $|\Omega| > \pi/2$). Expand $\cos(\Omega)$:

$$R_{XX}[n] = \frac{A_0}{2\pi}\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[e^{j(1+n)\Omega} + e^{j(-1+n)\Omega}\right] d\Omega$$

These integrals easily reduce to give

$$R_{XX}[n] = \frac{A_0}{\pi}\,\frac{\cos(n\pi/2)}{(1-n^2)} = \begin{cases} A_0\pi/8, & n = 0 \\ 0, & n \\ \dfrac{A_0(-1)^{n/2}}{\pi(1-n^2)}, & |n| \geq 2 \text{ and even.} \end{cases}$$

(7.5-7) $R_{YY}[n, n+k] = E\{Y[n]\,Y[n+k]\} = E\langle\{N[n] + b_1 N[n-1]\}$ $\{N[n+k] + b_1 N[n+k-1]\}\rangle = \overline{N[n]\,N[n+k]} + b_1\overline{N[n]N[n+k-1]}$ $+ b_1\overline{N[n-1]\,N[n+k]} + b_1^2\,\overline{N[n-1]\,N[n+k-1]} = R_{NN}[k] +$ $b_1 R_{NN}[k-1] + b_1 R_{NN}[k+1] + b_1^2 R_{NN}[k]$

$R_{NN}[k-1]$: σ_N^2, $k = 1$; 0, $k \neq 1$. $R_{NN}[k+1]$: σ_N^2, $k = -1$; 0, $k \neq -1$. $R_{NN}[k]$: σ_N^2, $k = 0$; 0, $k \neq 0$.

$$= \left\{\begin{array}{ll} (1+b_1^2)\sigma_N^2, & k = 0 \\ b_1\sigma_N^2, & k = +1 \text{ and } -1 \\ 0, & \text{all other } k \end{array}\right\} = R_{YY}[k]$$

(7.5-8) $\mathcal{S}_{Y_sY_s}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{YY}[n]\, e^{-jn\Omega}$

$= (1+b_1^2)\sigma_N^2 + b_1\sigma_N^2\left(e^{j\Omega} + e^{-j\Omega}\right) = (1+b_1^2)\sigma_N^2 +$ $2b_1\sigma_N^2\cos(\Omega)$, where we use the results of Problem 7.5-7 to substitute for $R_{YY}[k]$.

(7.5-9) This problem is the same as Problem 7.5-7 except the subscript 1 there is replaced here by m. By inspection we have

$$R_{YY}[n,n+k] = \begin{cases} (1+b_m^2)\,\sigma_N^2, & k=0 \\ b_m\,\sigma_N^2, & k=+m \text{ and } -m \\ 0, & \text{all other } k \end{cases}$$

From (7.5-11): $\mathcal{S}_{Y_sY_s}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{YY}[n]\,e^{-jn\Omega}$

$$= (1+b_m^2)\sigma_N^2 + b_m\sigma_N^2\left(e^{jm\Omega} + e^{-jm\Omega}\right)$$

$$= (1+b_m^2)\sigma_N^2 + 2\,b_m\,\sigma_N^2\cos(m\Omega)$$

(7.5-10) $R_{YY}[k] = E\{Y[n]Y[n+k]\} = E\langle\{N[n]+b_1N[n-1]+ b_2N[n-2]\}\{N[n+k]+b_1N[n+k-1]+b_2N[n+k-2]\}\rangle$

$$= R_{NN}[k] + b_1R_{NN}[k-1] + b_2R_{NN}[k-2] + b_1^2R_{NN}[k] + b_1b_2R_{NN}[k-1] + b_1R_{NN}[k+1] + b_2R_{NN}[k+2] + b_2^2R_{NN}[k] + b_1b_2R_{NN}[k+1]$$

$$= \begin{cases} (1+b_1^2+b_2^2)\,\sigma_N^2, & k=0 \\ (b_1+b_1b_2)\,\sigma_N^2, & k=+1 \text{ and } -1 \\ b_2\,\sigma_N^2, & k=+2 \text{ and } -2 \\ 0, & \text{all other } k \end{cases}$$

$$\mathcal{S}_{YY}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{YY}[n]\,e^{-jn\Omega} = (1+b_1^2+b_2^2)\,\sigma_N^2$$

$$+\, 2b_1(1+b_2)\sigma_N^2\cos(\Omega) + 2b_2\sigma_N^2\cos(2\Omega)$$

(7.5-11) $E\{W[n]\} = E\{X[n]\} + E\{N[n]\} = 0$, all n.

$R_{WW}[n,n+k] = E\{\langle X[n]+N[n]\rangle\langle X[n+k]+N[n+k]\rangle\}$ $X[n] = X$

(7.5-11.) (Continued) $= E(X^2) + E\{XN[n+k]\} + E\{XN[n]\}$
$+ E\{N[n]N[n+k]\} = \sigma_X^2 + R_{NN}[k]$. But $R_{NN}[k] = \sigma_N^2$ when $k=0$ and zero when $k \neq 0$, so

$$\mathcal{S}_{WW}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{WW}[n]\, e^{-jn\Omega} = \sigma_N^2 + \sum_{n=-\infty}^{\infty} \sigma_X^2\, e^{-jn\Omega}$$

$$= \sigma_N^2 + \sigma_X^2\, 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - n2\pi)$$

(7.5-12.) $$\mathcal{S}_{X_sX_s}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n]\, e^{-jn\Omega} = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{-jn\Omega}$$

$$= \sum_{n=-N}^{N} e^{-jn\Omega} - \frac{1}{N}\sum_{n=1}^{N} n e^{-jn\Omega} + \frac{1}{N}\sum_{n=-N}^{-1} n\, e^{-jn\Omega}$$

Use (C-62) on the first sum and let $m = -n$ in the third:

$$\mathcal{S}_{X_sX_s}(e^{j\Omega}) = \frac{\sin[(2N+1)\Omega/2]}{\sin(\Omega/2)} - \frac{1}{N}\sum_{n=1}^{N} n\, 2\cos(n\Omega) \qquad (1)$$

The sum evaluates using sum number 429, pp. 80-81 of Jolley:

$$\sum_{n=1}^{N} n\cos(n\Omega) = \frac{(N+1)\sin[(2N+1)\Omega/2]}{2\sin(\Omega/2)} - \frac{1-\cos[(N+1)\Omega]}{4\sin^2(\Omega/2)} \quad (2)$$

On using (2) in (1) and reducing some trigonometric algebra we find:

$$\mathcal{S}_{X_sX_s}(e^{j\Omega}) = \frac{1}{N}\left\{\frac{\sin(N\Omega/2)}{\sin(\Omega/2)}\right\}^2$$

(7.5-13.) Here $Y[n] = X[n] + \frac{1}{2}X[n-1] + \frac{1}{4}X[n-2]$, so

$$R_{YY}[k] = E\left\langle \{X[n] + \tfrac{1}{2}X[n-1] + \tfrac{1}{4}X[n-2]\}\{X[n+k] + \tfrac{1}{2}X[n+k-1] + \tfrac{1}{4}X[n+k-2]\}\right\rangle = E\langle X[n]X[n+k] + \tfrac{1}{2}X[n]X[n+k-1]$$

(7.5-13) (Continued) $+\frac{1}{4}X[n]X[n+k-2]+\frac{1}{2}X[n-1]X[n+k]$
$+\frac{1}{4}X[n-1]X[n+k-1]+\frac{1}{8}X[n-1]X[n+k-2]+\frac{1}{4}X[n-2]X[n+k]$
$+\frac{1}{8}X[n-2]X[n+k-1]+\frac{1}{16}X[n-2]X[n+k-2]\rangle$

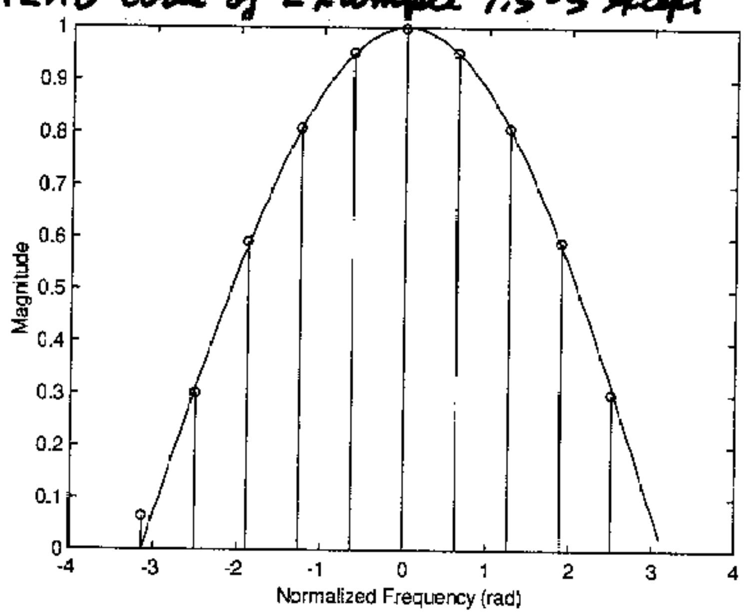
$$= \frac{21}{16}R_{XX}[k]+\frac{5}{8}\langle R_{XX}[k-1]+R_{XX}[k+1]\rangle+\frac{1}{4}\langle R_{XX}[k-2]+R_{XX}[k+2]\rangle$$

$$= \begin{cases} (21/16)\,\sigma_X^2, & k=0 \\ (5/8)\,\sigma_X^2, & k=+1 \text{ and } -1 \\ (1/4)\,\sigma_X^2, & k=+2 \text{ and } -2 \\ 0, & \text{all other } k. \end{cases}$$

(7.5-14) $\mathcal{S}_{XX}(e^{j\Omega})=\sum_{k=-\infty}^{\infty} R_{XX}[k]\,e^{-jk\Omega}$ For a finite length record where $R_{XX}[k]$ is nonzero only for $-3 \le k \le 3$:

$$\mathcal{S}_{XX}(e^{j\Omega})=\sum_{k=-3}^{3} R_{XX}[k]e^{-jk\Omega}=\frac{21}{16}\sigma_X^2+\frac{5}{8}\sigma_X^2\left(e^{-j\Omega}+e^{j\Omega}\right)$$
$$+\frac{1}{4}\sigma_X^2\left(e^{-j2\Omega}+e^{j2\Omega}\right)=\frac{\sigma_X^2}{16}\left[21+20\cos(\Omega)+8\cos(2\Omega)\right]$$

M (7.5-15) Use the MATLAB code of Example 7.5-5 except with N=10. As shown (right), we have twice as many points, all but one of which, are quite accurate (solid line is exact power spectrum).



(7.5-16) We form the inverse Fourier transform:

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-W_x}^{W_x} k\cos^2\left(\frac{\pi\omega}{2W_x}\right) e^{j\omega\tau}\,d\omega$$

$$= \frac{K}{8\pi}\int_{-W_x}^{W_x}\left(e^{j\pi\omega/2W_x} + e^{-j\pi\omega/2W_x}\right)^2 e^{j\omega\tau}\,d\omega$$

$$= \frac{K}{8\pi}\int_{-W_x}^{W_x}\left[e^{j\left(\frac{\pi}{W_x}+\tau\right)\omega} + 2e^{j\omega\tau} + e^{-j\left(\frac{\pi}{W_x}-\tau\right)\omega}\right]d\omega$$

$$= \frac{K}{8\pi}\left\{\frac{2\sin\left[\left(\frac{\pi}{W_x}+\tau\right)W_x\right]}{\left(\frac{\pi}{W_x}+\tau\right)} + \frac{4\sin(W_x\tau)}{\tau} + \frac{2\sin\left[\left(\frac{\pi}{W_x}-\tau\right)W_x\right]}{\left(\frac{\pi}{W_x}-\tau\right)}\right\}$$

$$= \frac{KW_x}{4\pi}\left\{2\,Sa(W_x\tau) + Sa(W_x\tau-\pi) + Sa(W_x\tau+\pi)\right\}$$

Expansion of the sampling functions and use of some trigonometric identities prove the second form given.

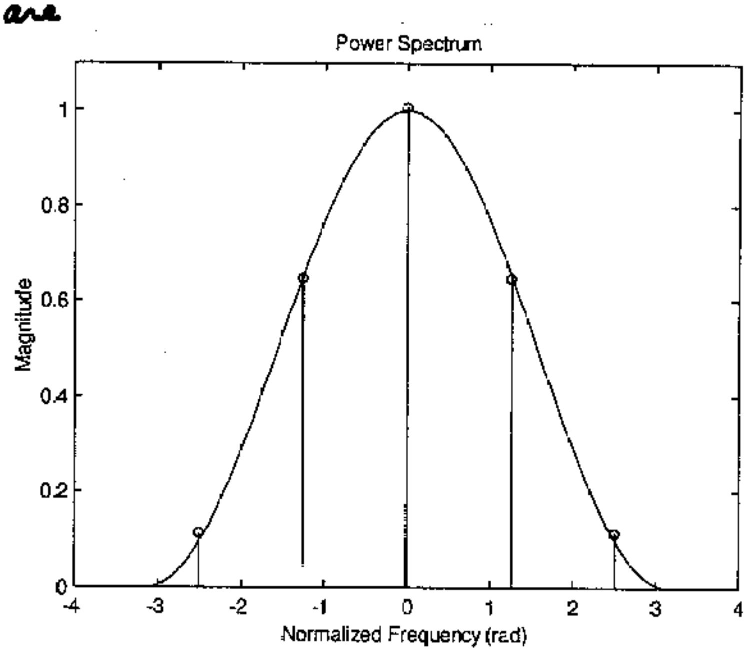
[M] (7.5-17) The MATLAB code is shown below.

```
%%%%%%%%%%%%%%% Problem 7.5-17 %%%%%%%%%%%%%%%%%

clear

N = 5; % number of samples
k = 1;
Ts = 1; % sample period

wx = pi/Ts;

if rem(N,2) == 0
   w = -wx : 2*wx/N : wx - 2*wx/N; % frequency vector
else
   w = -wx + wx/N : 2*wx/N : wx;
end

tau = -0.5*N:(0.5*N-1);

sinc1 = zeros(1,length(tau)); % initialize
sinc2 = zeros(1,length(tau));
sinc3 = zeros(1,length(tau));

f = find(pi*tau ~= 0); % avoid sin(0)/0
sinc1(f) = sin(pi*tau(f))./(pi*tau(f));
f = find(pi*tau == 0); % sin(0)/0 = 1
sinc1(f) = ones(1,length(f));
```

(7.5-17) (Continued) The code is continued below.

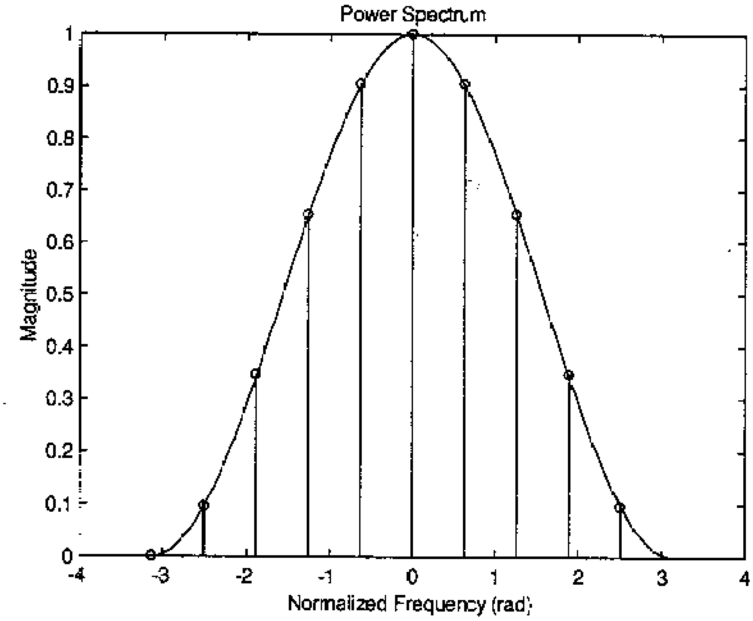
```
f = find(pi*tau - pi ~= 0);
sinc2(f) = sin(pi*tau(f) - pi)./(pi*tau(f) - pi);
f = find(pi*tau - pi == 0);
sinc2(f) = ones(1,length(f));

f = find(pi*tau + pi ~= 0);
sinc3(f) = sin(pi*tau(f) + pi)./(pi*tau(f) + pi);
f = find(pi*tau + pi == 0);
sinc3(f) = ones(1,length(f));

Rtrue = Ts*k*wx/(4*pi)*(2*sinc1 + sinc2 + sinc3); % true autocorrelation

w2 = -wx : 2*wx/128 : wx - 2*wx/128; % frequency vector
Strue = k*cos(pi*w2/(2*wx)).^2; % true power spectrum
Sxx = abs(fftshift(fft(Rtrue,N))); % estimated power spectrum

clf
plot(w2,Strue,'k')
hold
stem(w,Sxx,'k')

xlabel('Normalized Frequency (rad)')
ylabel('Magnitude')
title('Power Spectrum')
```

The calculated results are the stem plots for 5 points (right) and 10 points (below). All points are very accurate (compare with exact curve shown by the solid line) with the results for 10 points being slightly more accurate.



Problem 7.5-17 (N=5)

(7.5-17.) (Continued)



Problem 7.5-17 (N=10)

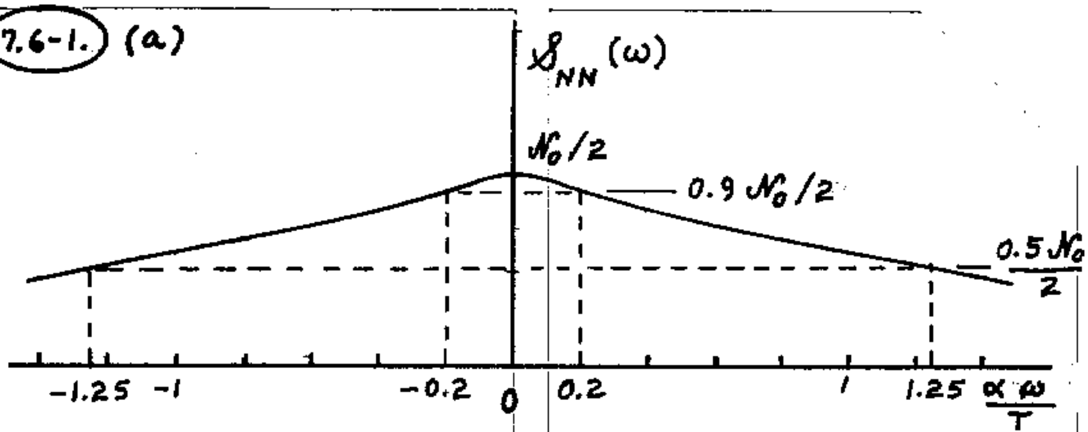
(7.5-18.) $\Omega = \omega T_s$, $d\Omega = T_s d\omega$

$$R_{xx}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{S}_{x_s x_s}(e^{j\Omega})\, e^{jn\Omega}\, d\Omega$$ Use (7.5-10) and change variables.

$$= \frac{1}{2\pi}\int_{-\pi/T_s}^{\pi/T_s} \mathcal{S}_{xx}(\omega)\, e^{jn\omega T_s}\, d\omega\, T_s$$

$$= \frac{T_s}{2\pi}\int_{-\pi/T_s}^{\pi/T_s} \mathcal{S}_{xx}(\omega)\, e^{jn\omega T_s}\, d\omega$$

7.6-1. (a)



(b) $|\omega| \le 1.25\,T/\alpha = 1.25(4.2)/7.64(10^{-12})$

$\approx 687.2\,(10^9)$ $\quad$ (109.4 GHz)

7.6-2. $R_{NN}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{S}_{NN}(\omega)\, e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-W}^{W} \frac{P\pi}{W}\, e^{j\omega\tau} d\omega$

$$= \frac{P}{2W}\,\frac{e^{j\omega\tau}}{j\tau}\Big|_{-W}^{W} = P\,\frac{\sin(W\tau)}{W\tau} = P\,Sa[W\tau].$$

7.6-3. $R_{NN}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{S}_{NN}(\omega)\, e^{j\omega\tau} d\omega$

$$= \frac{1}{2\pi}\int_{-\omega_0-\frac{W}{2}}^{-\omega_0+\frac{W}{2}} \frac{\pi P}{W}\, e^{j\omega\tau} d\omega + \frac{1}{2\pi}\int_{\omega_0-\frac{W}{2}}^{\omega_0+\frac{W}{2}} \frac{\pi P}{W}\, e^{j\omega\tau} d\omega$$

$$= \frac{P}{2W}\,\frac{e^{j\omega\tau}}{j\tau}\Big|_{-\omega_0-\frac{W}{2}}^{-\omega_0+\frac{W}{2}} + \frac{P}{2W}\,\frac{e^{j\omega\tau}}{j\tau}\Big|_{\omega_0-\frac{W}{2}}^{\omega_0+\frac{W}{2}}$$

$$= \frac{P}{2}\, e^{-j\omega_0\tau}\,\frac{\sin(W\tau/2)}{W\tau/2} + \frac{P}{2}\, e^{j\omega_0\tau}\,\frac{\sin(W\tau/2)}{W\tau/2}$$

$$= P\cos(\omega_0\tau)\,\frac{\sin(W\tau/2)}{W\tau/2}.$$

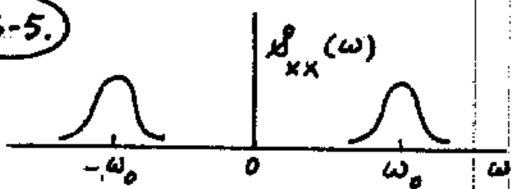
7.6-4. The power in the actual process is

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{S}_{XX}(\omega)\, d\omega.$$

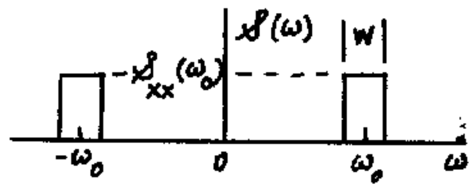
7.6-4. (Continued) The power in the bandlimited white noise is $P_{XX} = \frac{1}{2\pi}\int_{-W}^{W} \mathcal{S}_{XX}(0)\, d\omega = \frac{\mathcal{S}_{XX}(0)}{2\pi}\int_{-W}^{W} d\omega$ $= \mathcal{S}_{XX}(0)\,\frac{W}{\pi}$. Equating these two expressions:

$$W = \frac{\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\, d\omega}{2\,\mathcal{S}_{XX}(0)} = \frac{\int_{0}^{\infty} \mathcal{S}_{XX}(\omega)\, d\omega}{\mathcal{S}_{XX}(0)}.$$

7.6-5.



actual

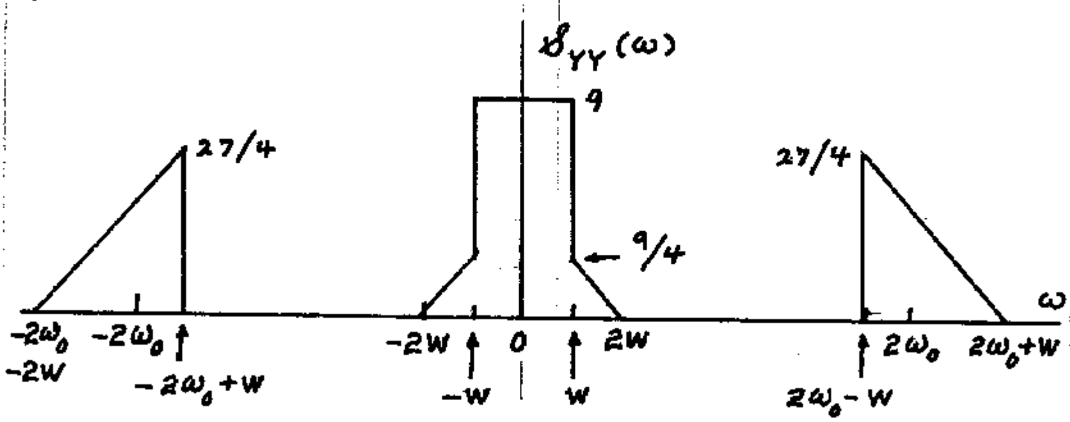


Bandlimited Band-pass

$$\frac{1}{2\pi}\int_{0}^{\infty} \mathcal{S}_{XX}(\omega)\, d\omega = \frac{1}{2\pi}\int_{\omega_0 - \frac{W}{2}}^{\omega_0 + \frac{W}{2}} \mathcal{S}_{XX}(\omega_0)\, d\omega = \mathcal{S}_{XX}(\omega_0)\,\frac{W}{2\pi}.$$

$$W = \frac{\int_{0}^{\infty} \mathcal{S}_{XX}(\omega)\, d\omega}{\mathcal{S}_{XX}(\omega_0)}.$$

7.6-6. Shift the given power spectrum and add components as indicated in (7.6-12):

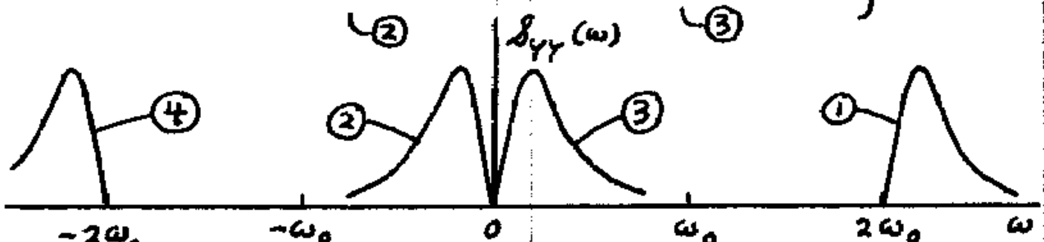


7.6-7. $R_{YY}(t,t+\tau) = E[A_0 X(t)\cos(\omega_0 t + \Theta) A_0 X(t+\tau)\cos(\omega_0 t + \omega_0\tau + \Theta)] = \frac{A_0^2}{2} R_{XX}(\tau)\, E_{\Theta}[\cos(\omega_0\tau) + \cos(2\omega_0 t + \omega_0\tau + 2\Theta)] = (A_0^2/2) R_{XX}(\tau)\cos(\omega_0\tau) = R_{YY}(\tau)$. Because this result is the same as in the text, the power spectrum is the same

$$\mathcal{S}_{YY}(\omega) = \frac{A_0^2}{4}\left[\mathcal{S}_{XX}(\omega-\omega_0) + \mathcal{S}_{XX}(\omega+\omega_0)\right].$$

7.6-8. (a) From (7.6-12):

$$\mathcal{S}_{YY}(\omega) = \frac{P}{4}\Big\{\underbrace{u(\omega-2\omega_0)(\omega-2\omega_0)e^{-(\omega-2\omega_0)^2/b}}_{①} \underbrace{- u(-\omega-2\omega_0)(\omega+2\omega_0)e^{-(\omega+2\omega_0)^2/b}}_{④} \underbrace{- u(-\omega)\,\omega\, e^{-\omega^2/b}}_{②} + \underbrace{u(\omega)\,\omega\, e^{-\omega^2/b}}_{③}\Big\}$$



(b) $P_{XX} = \frac{2}{2\pi}\int_{\omega_0}^{\infty} P(\omega-\omega_0)\, e^{-(\omega-\omega_0)^2/b}\, d\omega = Pb/2\pi$

$P_{YY} = \frac{1}{2\pi}\,\frac{P}{4}\,4\int_0^{\infty}\omega\, e^{-\omega^2/b}\, d\omega = Pb/4\pi$

7.6-9. $e^x \approx 1 + x + \frac{x^2}{2}$ so $e^{\alpha|\omega|T} \approx 1 + \alpha|\omega|T + \frac{(\alpha|\omega|T)^2}{2}$ and

$$\frac{\mathcal{N}_0}{2}\,\frac{\alpha|\omega|T}{e^{\alpha|\omega|T}-1} \approx \frac{\mathcal{N}_0}{2}\,\frac{\alpha|\omega|T}{\left[1+\alpha|\omega|T + \frac{(\alpha|\omega|T)^2}{2}\right]-1} = \frac{\mathcal{N}_0/2}{1+\frac{\alpha|\omega|T}{2}} = \frac{\mathcal{N}_0}{4} \text{ when}$$

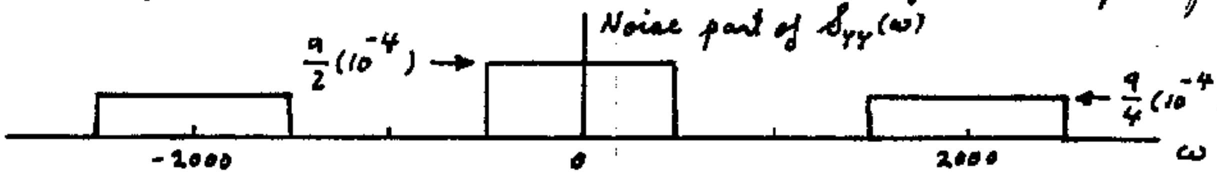
$\alpha|\omega|T/2 = 1$ or $|\omega| = 2/\alpha T = 130.89(10^9)$ rad/s. Thus,

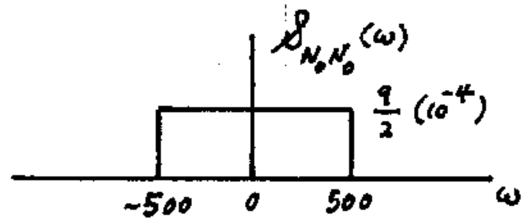
$$\text{average power} = \frac{1}{2\pi}\int_{-130.89(10^9)}^{130.89(10^9)} (\mathcal{N}_0/2)\, d\omega = 229.15(10^{-10}).$$

7.6-10. (a) Signal part of $Y(t) = 2.3\cos(1000t)\, 3\cos(1000t)$

$= 3.45[1 + \cos(2000t)]$. $\quad V_0 =$ dc term $= 3.45$ V.

(b) Here $A_0 = 3$ V. From (7.6-5) and the action of the lowpass filter:





(c) $E[N_0^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{N_0N_0}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-500}^{500}\frac{9}{2}(10^{-4})\,d\omega = 71.62\,(10^{-3})$.

$V_0^2/E[N_0^2(t)] = (3.45)^2/7.62(10^{-3}) = 166.19$.

★ **7.7-1.** $R_{ZZ}(t, t+\tau) = E[Z^*(t)Z(t+\tau)] = E[A^* e^{-j\Omega t}$

$\cdot A e^{j\Omega(t+\tau)}] = |A|^2 E[e^{j\Omega\tau}] = |A|^2\int_{-\infty}^{\infty} f_\Omega(\omega)e^{j\omega\tau}\,d\omega$

The integral is 2π times the inverse Fourier transform of $f_\Omega(\omega)$ and is a function of τ (not t) that is the autocorrelation function of $Z(t)$. Therefore $2\pi|A|^2 f_\Omega(\omega)$ must be the power spectrum of $Z(t)$:

$$R_{ZZ}(\tau) \longleftrightarrow \mathcal{S}_{ZZ}(\omega) = 2\pi|A|^2 f_\Omega(\omega).$$

★7.7-2. From the solution to Problem 6.7-4

$R_{ZZ}(\tau) = \sum_{i=1}^{N} \sigma_{Z_i}^2 \, e^{j\omega_i \tau}$. But since $e^{j\omega_i \tau} \leftrightarrow 2\pi\delta(\omega-\omega_i)$

we get $\mathcal{S}_{ZZ}(\omega) = 2\pi \sum_{i=1}^{N} \sigma_{Z_i}^2 \delta(\omega-\omega_i)$.

★7.7-3. (a) From pairs 2 and 19 of Appendix E:

$R_{XX}(\tau) = (1/\pi) + 5e^{-10|\tau|}$, $R_{YY}(\tau) = 4e^{-2|\tau|}$. Thus,

$R_{ZZ}(t,t+\tau) = E[Z^*(t)Z(t+\tau)] = E\{[X(t) - jY(t)]e^{-j\omega_0 t}$

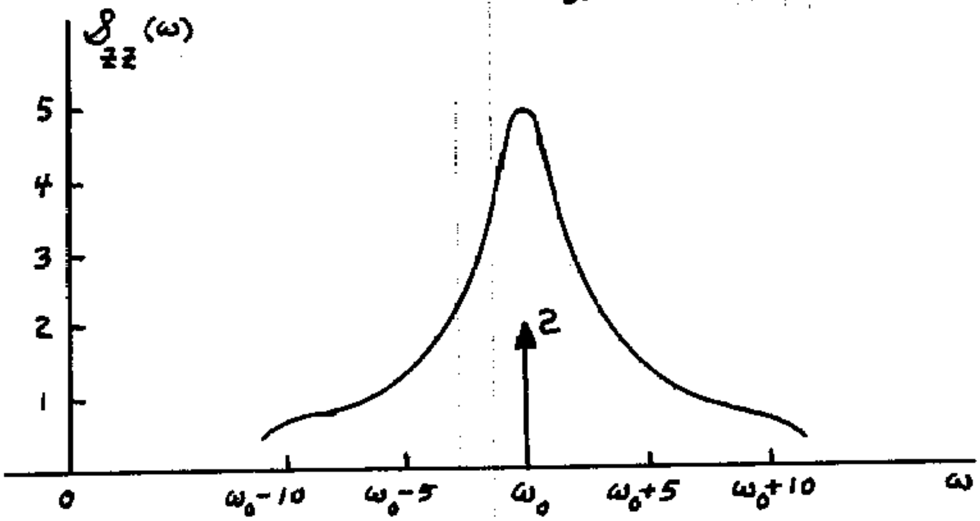
$\cdot [X(t+\tau) + jY(t+\tau)]e^{j\omega_0 t + j\omega_0 \tau}\} = e^{j\omega_0\tau}[R_{XX}(\tau)$

$+ R_{YY}(\tau) + j\bar{X}\bar{Y} - j\bar{Y}\bar{X}] = e^{j\omega_0\tau}[R_{XX}(\tau) + R_{YY}(\tau)]$, or

$R_{ZZ}(\tau) = e^{j\omega_0\tau}\left[\frac{1}{\pi} + 5e^{-10|\tau|} + 4e^{-2|\tau|}\right]$

(b) From use of (D-7):

$$\mathcal{S}_{ZZ}(\omega) = 2\delta(\omega-\omega_0) + \frac{100}{100+(\omega-\omega_0)^2} + \frac{16}{4+(\omega-\omega_0)^2}$$



CHAPTER 8

8.1-1. From (8.1-10) $y(t) = \int_{-\infty}^{\infty} h(\xi)\, x(t-\xi)\, d\xi$

$$= \int_{-\infty}^{\infty} W\, u(\xi)\, e^{-W\xi}\, u(t-\xi)\, e^{-\alpha(t-\xi)}\, d\xi.$$

Since $u(\xi) = 1$ for $\xi > 0$ and $u(t-\xi) = 1$ for $\xi < t$:

$$y(t) = 0, \quad t < 0$$

$$= \int_0^t W e^{-\alpha t} e^{-(W-\alpha)\xi}\, d\xi = \frac{W}{W-\alpha}\left(e^{-\alpha t} - e^{-Wt}\right)$$

for $t > 0$ from (C-45). Thus,

$$y(t) = \frac{W}{W-\alpha}\, u(t)\left[e^{-\alpha t} - e^{-Wt}\right].$$

8.1-2. From pair 15 of Appendix E:

$$x(t) = u(t)\, e^{-\alpha t} \longleftrightarrow \frac{1}{\alpha + j\omega} = X(\omega)$$

$$h(t) = W\, u(t)\, e^{-Wt} \longleftrightarrow \frac{W}{W + j\omega} = H(\omega).$$

Hence, from (8.1-11),

$$Y(\omega) = X(\omega)\, H(\omega) = \frac{W}{(\alpha + j\omega)(W + j\omega)}.$$

8.1-3. (a) Write $x(t) = A\,u(t) - A\,u(t-T)$. The output is $y(t) = \int_{-\infty}^{\infty} x(\xi)\, h(t-\xi)\, d\xi = \int_{-\infty}^{\infty} A\, u(\xi)\, W\, u(t-\xi)\, e^{-W(t-\xi)}\, d\xi$

$$- \int_{-\infty}^{\infty} A\, u(\xi - T)\, W\, u(t-\xi)\, e^{-W(t-\xi)}\, d\xi.$$

The first right-side integrand is zero except

(8.1-3.) (Continued) when $0<\xi<t$ while the second integrand is zero except when $T<\xi<t$. Therefore $y(t)=0$ for $t<0$. For $0<t<T$:

$$y(t) = AW\int_0^t e^{-W(t-\xi)}d\xi = A\,u(t)\,(1-e^{-Wt}).$$

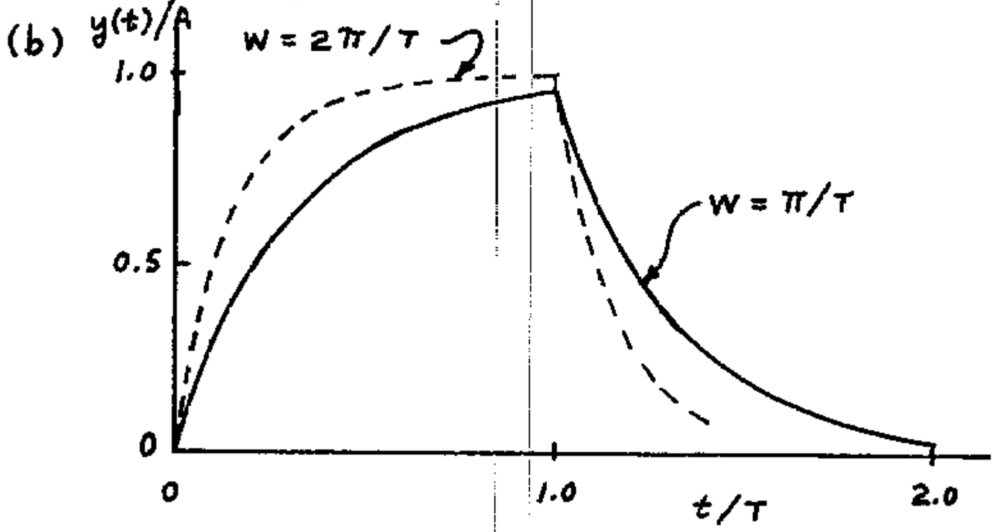
For $t>T$: $y(t) = A\,u(t)(1-e^{-Wt}) - AW\int_T^t e^{-W(t-\xi)}d\xi$

$$= A\,u(t)(1-e^{-Wt}) - A\left[1-e^{-W(t-T)}\right]u(t-T).$$

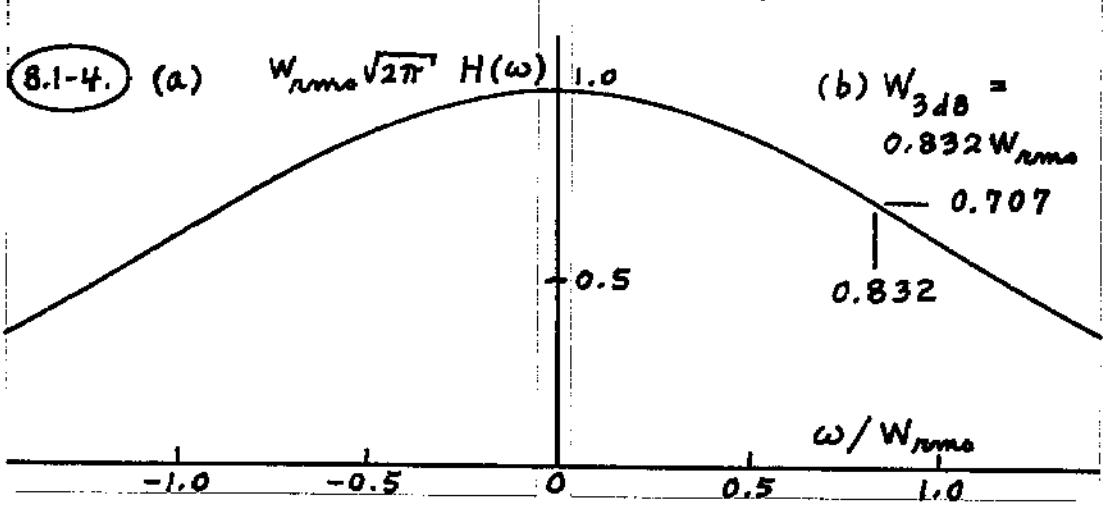
Thus, for any t:

$$y(t) = A\left\{u(t)\left[1-e^{-Wt}\right] - u(t-T)\left[1-e^{-W(t-T)}\right]\right\}.$$

(b)



(8.1-4.) (a)



8.1-5.

$$x(t) \longrightarrow \boxed{\text{Network } 1} \xrightarrow{\;y(t)\;} \boxed{\text{Network } 2} \longrightarrow w(t)$$

$$X(\omega) \qquad H_1(\omega) \qquad Y(\omega) \qquad H_2(\omega) \qquad W(\omega)$$

(a) From (8.1-11) $W(\omega) = H_2(\omega)Y(\omega)$ and $Y(\omega) = H_1(\omega)X(\omega)$ so $W(\omega) = H_2(\omega)H_1(\omega)X(\omega) = H(\omega)X(\omega)$ where $H(\omega) = H_2(\omega)H_1(\omega)$. (1)

(b) By repeated application of (1):

$$H(\omega) = H_N(\omega)\cdots H_1(\omega) = \prod_{n=1}^{N} H_n(\omega).$$

★ **8.1-6.** From the solution to Problem 8.1-1 the first network's output is

$$y_1(t) = \frac{W}{W-\alpha}\,u(t)\left[e^{-\alpha t} - e^{-Wt}\right].$$

The output of the second network is

$$y_2(t) = \int_{-\infty}^{\infty} y_1(\xi)h_2(t-\xi)\,d\xi$$

$$= \int_{-\infty}^{\infty} \frac{W}{W-\alpha}\,u(\xi)\left[e^{-\alpha\xi} - e^{-W\xi}\right] W u(t-\xi)\,e^{-W(t-\xi)}\,d\xi$$

$$= \frac{W^2}{W-\alpha}\left\{e^{-Wt}\int_0^t e^{-(\alpha-W)\xi}\,d\xi - e^{-Wt}\int_0^t d\xi\right\}$$

$$= \left(\frac{W}{W-\alpha}\right)^2 u(t)\left\{e^{-\alpha t} - e^{-Wt}\left[1+(W-\alpha)t\right]\right\}.$$

8.1-7. By writing $x(t) = A\,u(t) - A\,u(t-T)$ we have

$$y(t) = \int_{-\infty}^{\infty} h(\xi)x(t-\xi)\,d\xi = \int_{-\infty}^{\infty} u(\xi)\,\xi^3 e^{-\xi^2} A\,u(t-\xi)\,d\xi$$

$$- \int_{-\infty}^{\infty} u(\xi)\,\xi^3 e^{-\xi^2} A\,u(t-T-\xi)\,d\xi$$

$$= A\int_0^t \xi^3 e^{-\xi^2}\,d\xi - A\int_0^{t-T} \xi^3 e^{-\xi^2}\,d\xi.$$

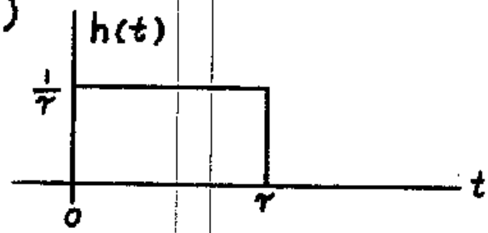
(8.1-7.) (Continued) By letting $x = \xi^2$, $dx = 2\xi\, d\xi$ in these integrals and using (C-46) they reduce to
$$y(t) = \frac{A}{2}\left\{[1+(t-T)^2]e^{-(t-T)^2} - (1+t^2)e^{-t^2}\right\}, \quad T<t$$
$$= \frac{A}{2}\left\{1-(1+t^2)e^{-t^2}\right\}, \quad 0<t<T.$$

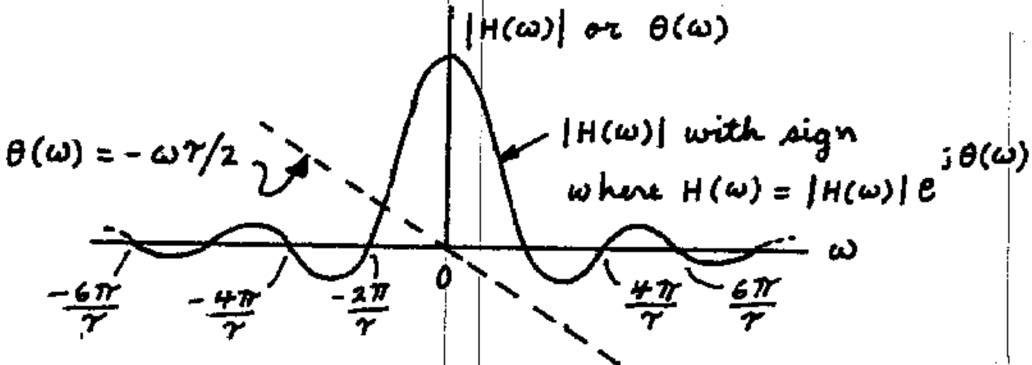
(8.1-8.) Again write $x(t) = Au(t) - Au(t-T)$. Then
$$y(t) = \int_{-\infty}^{\infty} h(\xi)x(t-\xi)\,d\xi = \int_{-\infty}^{\infty} u(\xi)\xi^3 e^{-\xi} A u(t-\xi)\,d\xi$$
$$- \int_{-\infty}^{\infty} u(\xi)\xi^3 e^{-\xi} A u(t-T-\xi)\,d\xi$$
$$= A\int_0^t \xi^3 e^{-\xi}\,d\xi - A\int_0^{t-T} \xi^3 e^{-\xi}\,d\xi.$$
By using (C-48) these integrals reduce to
$$y(t) = A\{6 - e^{-t}(6+6t+3t^2+t^3)\}, \quad 0<t\le T$$
$$= A\{e^{-(t-T)}[6+6(t-T)+3(t-T)^2+(t-T)^3]$$
$$- e^{-t}[6+6t+3t^2+t^3]\}, \quad T<t.$$

(8.1-9.) (a) One path passes the impulse straight through to produce $\frac{1}{\tau}u(t)$ at the integrator's output. The same happens due to the second path except the response is delayed by τ and changed in sign: $-\frac{1}{\tau}u(t-\tau)$. Thus, $h(t) = \frac{1}{\tau}[u(t) - u(t-\tau)]$. (b) $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$
$$= \int_0^{\tau} \frac{1}{\tau}e^{-j\omega t}dt = e^{-j\omega\tau/2}\frac{\sin(\omega\tau/2)}{\omega\tau/2} \text{ from (C-45).}$$

8.1-9. (Continued)

(c)

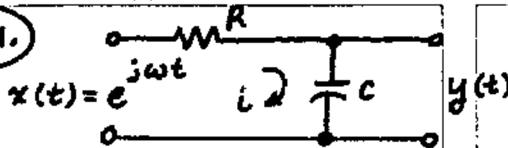




8.1-10. Here $x(t) = e^{j\omega t}$. The direct path response is $\frac{1}{\tau}\int_{-\infty}^{t} e^{j\omega\xi}\,d\xi$. The delay path response is $\frac{-1}{\tau}\int_{-\infty}^{t} e^{j\omega(\eta-\tau)}\,d\eta$. Let $\eta - \tau = \xi$, $d\xi = d\eta$ here to obtain $\frac{-1}{\tau}\int_{-\infty}^{t-\tau} e^{j\omega\xi}\,d\xi$. The output is the sum of responses: $y(t) = L[e^{j\omega t}] = \frac{1}{\tau}\int_{-\infty}^{t} e^{j\omega\xi}\,d\xi$ $-\frac{1}{\tau}\int_{-\infty}^{t-\tau} e^{j\omega\xi}\,d\xi = \frac{1}{\tau}\int_{t-\tau}^{t} e^{j\omega\xi}\,d\xi = e^{j\omega t}\, e^{-j\omega\tau/2}$ $\cdot \frac{\sin(\omega\tau/2)}{\omega\tau/2}$. Now since $y(t) = H(\omega)\,x(t)$ we obtain

$$H(\omega) = \frac{y(t)}{x(t)} = \frac{L[e^{j\omega t}]}{e^{j\omega t}} = e^{-j\omega\tau/2}\,\frac{\sin(\omega\tau/2)}{\omega\tau/2}.$$

8.1-11.



$$x(t) = iR + y(t) \qquad (1)$$

$$i = C\,\frac{dy(t)}{dt} \qquad (2)$$

$$y(t) = H(\omega)\,x(t) = H(\omega)\,e^{j\omega t} \qquad (3)$$

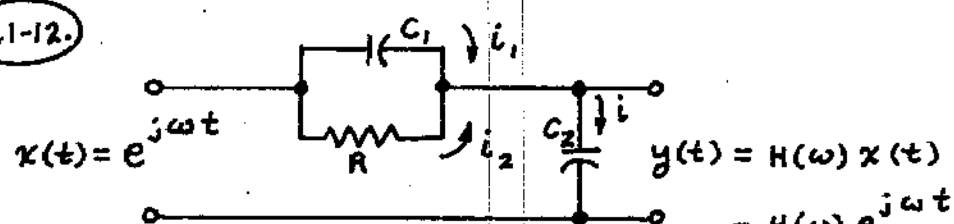
$$dy(t)/dt = H(\omega)\,j\omega\, e^{j\omega t} = j\omega H(\omega)\,x(t) \qquad (4)$$

Use (4) with (2) to obtain i then substitute i along with (3) in (1):

$$x(t) = RCj\omega H(\omega)\,x(t) + H(\omega)\,x(t)$$

so $$H(\omega) = 1/(1+j\omega RC).$$

8.1-12.



$$\frac{dy(t)}{dt} = H(\omega)\,j\omega\, e^{j\omega t} = j\omega H(\omega)\,x(t) \qquad (1)$$

$$x(t) - y(t) = [1 - H(\omega)]\,x(t) \qquad (2)$$

$$\frac{d}{dt}[x(t) - y(t)] = j\omega\,[1 - H(\omega)]\,x(t) \qquad (3)$$

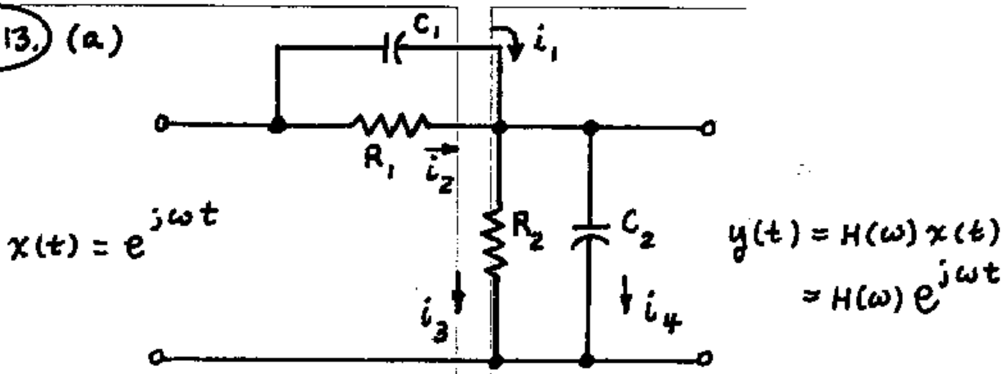
Since $i = C_2\,\dfrac{dy(t)}{dt} = i_1 + i_2 = C_1\,\dfrac{d}{dt}[x(t)-y(t)] + \dfrac{x(t)-y(t)}{R}$, then on substitution of (1)–(3):

$$j\omega C_2 H(\omega)\,x(t) = j\omega C_1\,[1 - H(\omega)]\,x(t) + \frac{1}{R}[1-H(\omega)]\,x(t).$$

Solve for $H(\omega)$:

$$H(\omega) = \frac{1 + j\omega RC_1}{1 + j\omega R(C_1 + C_2)}.$$

★ 8.1-13. (a)



$$i_1 + i_2 = C_1 \frac{d}{dt}[x(t) - y(t)] + \frac{1}{R_1}[x(t) - y(t)] = i_3 + i_4$$

$$= \frac{1}{R_2} y(t) + C_2 \frac{dy(t)}{dt}.$$

Substitute $x(t)$ and $y(t)$ from above:

$$j\omega C_1 [1 - H(\omega)] e^{j\omega t} + \frac{1}{R_1}[1 - H(\omega)] e^{j\omega t} = \frac{H(\omega)}{R_2} e^{j\omega t} + j\omega C_2 H(\omega) e^{j\omega t}.$$

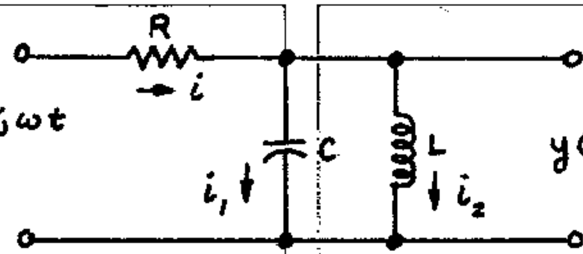
Thus, $H(\omega)$ can be found and put in the form

$$H(\omega) = \left(\frac{R_2}{R_1 + R_2}\right) \frac{1 + j\omega R_1 C_1}{1 + j\omega R_1 C_1 \left(\dfrac{1 + \frac{C_2}{C_1}}{1 + \frac{R_1}{R_2}}\right)}.$$

(b) If $C_2 / C_1 \gg R_1 / R_2$ the network is approximately a lowpass filter out to the denominator's "break" angular frequency $\left(1 + \frac{R_1}{R_2}\right) / \left(1 + \frac{C_2}{C_1}\right) R_1 C_1$.

(c) If $C_2 / C_1 = R_1 / R_2$ then $H(\omega) = R_2 / (R_1 + R_2)$ which is the transfer function of a purely resistive attenuator at all frequencies.

8.1-14. (a)



$x(t) = e^{j\omega t}$ $\qquad$ $y(t) = H(\omega) e^{j\omega t}$

$$i = \frac{x(t) - y(t)}{R} = i_1 + i_2 = C\,\frac{dy(t)}{dt} + \frac{1}{L}\int y(t)\,dt.$$

After substituting $x(t)$ and $y(t)$ above:

$$\frac{1}{R}[1 - H(\omega)]\, e^{j\omega t} = j\omega C\, H(\omega)\, e^{j\omega t} + \frac{H(\omega)}{j\omega L}\, e^{j\omega t}$$

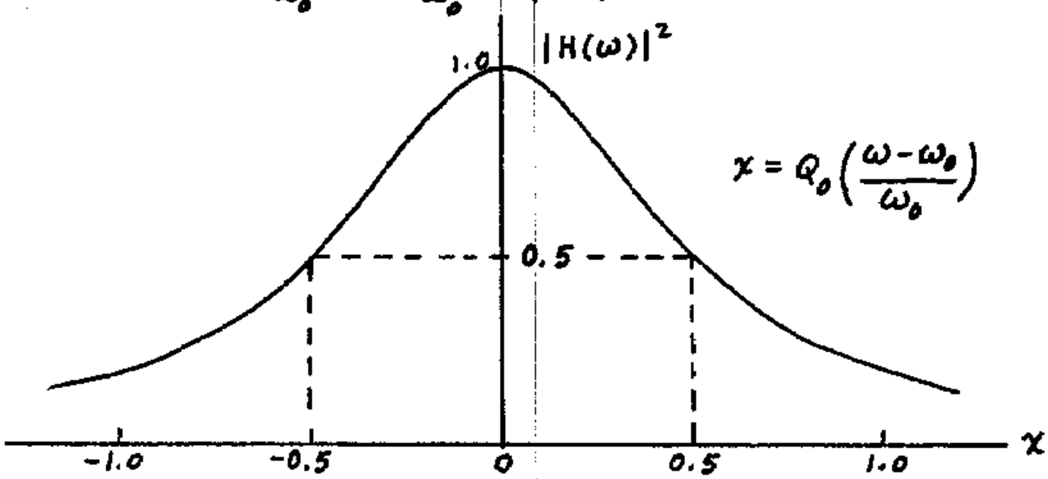
$$H(\omega) = \frac{j\omega L/R}{(1-\omega^2 LC) + j(\omega L/R)}.$$

(b)

$$H(\omega) = \frac{j\dfrac{\omega}{\omega_0}\dfrac{\omega_0 L}{R}}{\left(\dfrac{\omega_0^2 - \omega^2}{\omega_0^2}\right) + j\dfrac{\omega}{\omega_0}\dfrac{\omega_0 L}{R}} = \frac{j\dfrac{\omega}{\omega_0}\dfrac{1}{Q_0}}{\dfrac{(\omega_0-\omega)(\omega_0+\omega)}{\omega_0^2} + j\dfrac{\omega}{\omega_0}\dfrac{1}{Q_0}}.$$

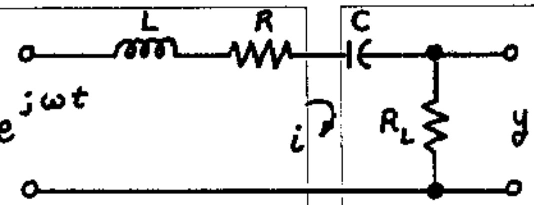
Since Q_0 is large ($Q_0 \geq 10$) then $\omega \approx \omega_0$ in the region of major response so

$$|H(\omega)|^2 = \frac{1}{1 + \dfrac{(\omega-\omega_0)^2}{\omega_0^2}\dfrac{(\omega+\omega_0)^2}{\omega_0^2}\left(\dfrac{\omega_0}{\omega}\right)^2 Q_0^2} \approx \frac{1}{1+4x^2}.$$



★ 8.1-15. (a)

$x(t) = e^{j\omega t}$



$y(t) = H(\omega)\, e^{j\omega t}$

$$x(t) = e^{j\omega t} = L\,\frac{di(t)}{dt} + iR + \frac{1}{C}\int i\,dt + y(t) \qquad (1)$$

But since
$$i = y(t)/R_L = \frac{H(\omega)}{R_L}\, e^{j\omega t} \qquad (2)$$

$$\frac{di(t)}{dt} = j\omega\,\frac{H(\omega)}{R_L}\, e^{j\omega t} \qquad (3)$$

$$\int i\,dt = \frac{H(\omega)}{j\omega R_L}\, e^{j\omega t}, \qquad (4)$$

we substitute (2) – (4) into (1) to find

$$H(\omega) = \frac{j\omega R_L C}{\left(1-\omega^2 LC\right) + j\omega(R+R_L)C}.$$

(b)
$$H(\omega) = \frac{j\dfrac{\omega}{\omega_0}\,\omega_0(R+R_L)C\,\dfrac{R_L}{R+R_L}}{\dfrac{(\omega_0-\omega)(\omega_0+\omega)}{\omega_0^2} + j\dfrac{\omega}{\omega_0}\,\omega_0(R+R_L)C} \approx \frac{R_L/(R+R_L)}{1+j\left(\dfrac{\omega-\omega_0}{\omega_0}\right)2Q_0}$$

(c) Energy in impulse response is $E = \int_{-\infty}^{\infty} \frac{h^2(t)}{R_L}\,dt.$

From Parseval's theorem $E = \frac{1}{2\pi R_L}\int_{-\infty}^{\infty} |H(\omega)|^2\,d\omega$

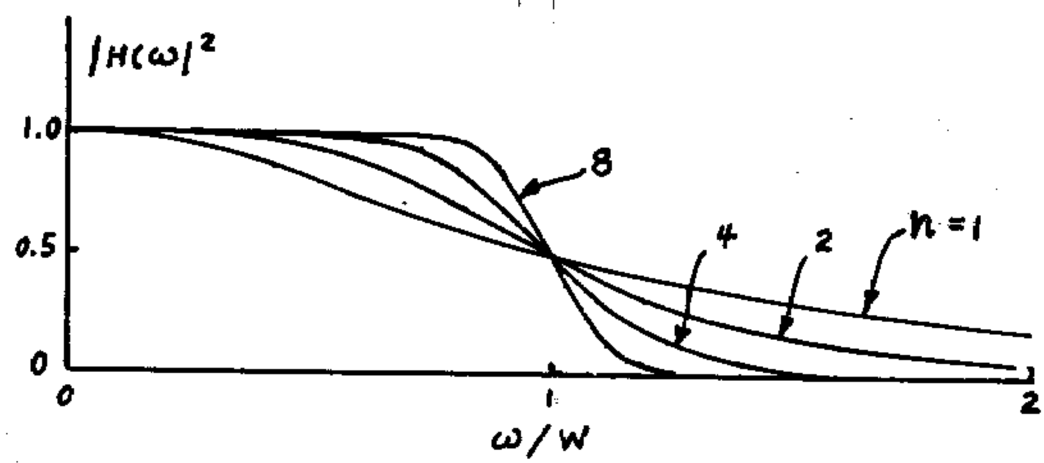
$$\approx \frac{1}{\pi R_L}\int_0^{\infty} \frac{\left(\frac{R_L}{R+R_L}\right)^2 d\omega}{1+\left(\frac{\omega-\omega_0}{\omega_0}\right)^2 4Q_0^2}.$$ Let $\xi = (\omega-\omega_0)2Q_0/\omega_0$,

$d\xi = 2Q_0\,d\omega/\omega_0$ and use (C-25) to obtain

$$E = \frac{R_L\,\omega_0^2\,(R+R_L)C}{2\pi(R+R_L)^2}\left[\frac{\pi}{2} + \tan^{-1}(2Q_0)\right].$$ But $Q_0 \gg 1$

so $$E \approx \frac{\omega_0^2\, C\, R_L}{2(R+R_L)}.$$

(8.1-16.)



As $n \to \infty$ $|H(\omega)|^2$ approaches an ideal lowpass filter characteristic, i.e., it becomes rectangular.

(8.1-17.) (a) Not realizable because it is not causal. Not stable. To see this let $x(t) = A$. Then

$$y(t) = \int_{-\infty}^{\infty} h(\xi)\, x(t-\xi)\, d\xi = A\int_{-\infty}^{\infty} u(\xi+3)\, d\xi$$

$$= A\int_{-3}^{\infty} d\xi = \infty .$$

(b) Realizable and stable. (c) Not realizable because it is not causal. Not stable because it grows without bound. (d) Realizable because it is causal. Stable because it satisfies (8.1-16).

(8.1-18.) If $x(t) = 0$ for $t < t_0$ then write (8.1-10) as $y(t) = \int_{-\infty}^{t-t_0} h(\xi)\, x(t-\xi)\, d\xi$. Now if $y(t) = 0$ for $t < t_0$ is required then $y(t) = 0$ for $t - t_0 < 0$. The integral must therefore evaluate to zero

8.1-18. (Continued) whenever $-\infty<\xi<t-t_0<0$. This result is assured for arbitrary $x(t)$ only if $h(\xi)=0$ for $\xi<0$. Hence $h(t)=0$, $t<0$, is necessary for $y(t)=0$, $t<t_0$, when $x(t)=0$, $t<t_0$.

8.1-19. If $|x(t)|<M$, a constant, then

$$y(t)=\int_{-\infty}^{\infty} h(\xi)x(t-\xi)\,d\xi \text{ in general, and}$$

$$|y(t)|=\left|\int_{-\infty}^{\infty} h(\xi)x(t-\xi)\,d\xi\right| \le \int_{-\infty}^{\infty} |h(\xi)x(t-\xi)|\,d\xi$$

$$\le M\int_{-\infty}^{\infty}|h(\xi)|\,d\xi = MI < \infty \quad \text{if}$$

$$I=\int_{-\infty}^{\infty}|h(\xi)|\,d\xi<\infty.$$

8.1-20. Linear: $y_n(t)=\int_{-\infty}^{t} x_n(\xi)\,d\xi$. $y(t)=\int_{-\infty}^{t}\sum_{n=1}^{N}\alpha_n \cdot x_n(\xi)\,d\xi = \sum_{n=1}^{N}\alpha_n\int_{-\infty}^{t}x_n(\xi)\,d\xi = \sum_{n=1}^{N}\alpha_n y_n(t)$.

Time-Invariant: Let $x(t)=\delta(t-t_0)$. Then $y(t)=\int_{-\infty}^{t}\delta(\xi-t_0)\,d\xi = u(t-t_0)$. Since this response shifts with t_0 as the input impulse the system is time invariant.

Causal: If $x(t)=0$, $t<0$, then $y(t)=\int_{-\infty}^{t}x(\xi)\,d\xi = \int_{0}^{t}x(\xi)\,d\xi$ which is nonzero only if $t>0$ so the system is causal.

8.1-21. (a) $z_i = \dfrac{100\left(50 + \frac{100(50)}{100+50}\right)}{100 + 50 + \frac{100(50)}{100+150}} = \dfrac{500}{11}\,\Omega$.

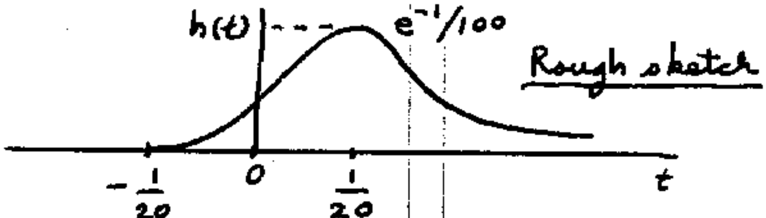
(b) $Z_o = \dfrac{100(100)}{100+100} = 50\,\Omega$. (c) $G_a = \dfrac{\overline{e_o^2}(\text{no load})\, R_s}{\overline{e_s^2}\qquad R_o} =$

$\dfrac{\overline{e_o^2}(\text{no load})}{\overline{e_s^2}}\, 2$, $\dfrac{\overline{e_o^2}(\text{no load})}{\overline{e_i^2}} = \left(\dfrac{100}{150}\right)^2$, $\dfrac{\overline{e_i^2}}{\overline{e_s^2}} = \left(\dfrac{500/11}{100+\frac{500}{11}}\right)^2$,

$G_a = 2\left(\dfrac{100}{150}\right)^2\left(\dfrac{500}{1100+500}\right)^2 = \dfrac{25}{144}$. (d) No. The network with source is matched to the load but the network with load is not matched to the source.

8.1-22. (a) From pair 17, Appendix E, and (D-6):

$h(t) = u(t+\frac{1}{20})(t+\frac{1}{20})^2 \exp[-20(t+\frac{1}{20})]$.



(b) No. Because it is not causal. (c) $I = \int_{-1/20}^{\infty} (t+\frac{1}{20})^2 \exp[-20(t+\frac{1}{20})]\,dt = \int_0^\infty x^2 e^{-20x}\,dx = 2/(20^3)$ from (C-47). Thus, $I < \infty$ and the network is stable.

★**8.1-23.** For one network use pair 15, Appendix E, $h_1(t) = u(t)e^{-at}$. For two networks $h_2(t) = \int_{-\infty}^{\infty} h_1(\xi)h_1(t-\xi)\,d\xi = \int_{-\infty}^{\infty} u(\xi)e^{-a\xi}u(t-\xi)e^{-a(t-\xi)}\,d\xi$ $= e^{-at}\int_0^t dt\; u(t) = u(t)\,t\,e^{-at}$. For three networks $h_3(t) = \int_{-\infty}^{\infty} h_2(\xi)h_1(t-\xi)\,d\xi = u(t)e^{-at}\int_0^t \xi\,d\xi = u(t)\frac{t^2}{2}e^{-at}$.

For four networks $h_4(t) = \int_{-\infty}^{\infty} h_3(\xi)h_1(t-\xi)\,d\xi =$

★ 8.1-23. (Continued) $u(t)\,e^{-at}\,\frac{1}{2}\int_0^t \xi^2 d\xi = u(t)\frac{t^3}{2\cdot 3}e^{-at}$.

By continuing the expansion it is clear that

$h_N(t) = u(t)\frac{t^{N-1}}{(N-1)!}\,e^{-at}$.

8.1-24. $y(t) = \frac{1}{T}\int_{-\infty}^{t}[x(\xi) - x(\xi - T)]d\xi = \frac{1}{T}\int_{-\infty}^{t}x(\xi)d\xi - \frac{1}{T}\int_{-\infty}^{t}x(\xi - T)d\xi$

$= \frac{1}{T}\int_{-\infty}^{t}x(\xi)d\xi - \frac{1}{T}\int_{-\infty}^{t-T}x(\alpha)d\alpha = \frac{1}{T}\int_{t-T}^{t}x(\xi)d\xi$.

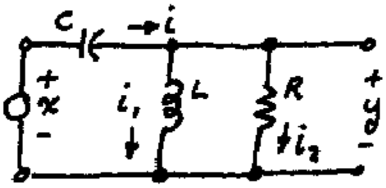
8.1-25. If $x(t)$ is applied to the given filter:

$y(t) = \int_{-\infty}^{\infty}x(\xi)h(t-\xi)d\xi = \int_{-\infty}^{\infty}x(\xi)\frac{1}{T}[u(t-\xi) - u(t-\xi-T)]d\xi$

$= \frac{1}{T}\int_{-\infty}^{\infty}x(\xi)[u(t-\xi) - u(t-\xi-T)]d\xi = \frac{1}{T}\int_{t-T}^{t}x(\xi)d\xi$. From pair 5 and (D-6):

$\frac{1}{T}[u(t) - u(t-T)] = \frac{1}{T}\,\mathrm{rect}\left(\frac{t - T/2}{T}\right) \longleftrightarrow H(\omega) = Sa\left(\frac{\omega T}{2}\right)e^{-j\omega T/2}$.

8.1-26.



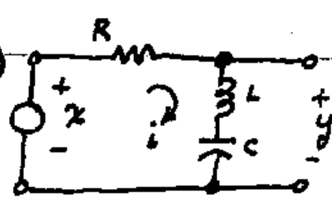
$y = L\frac{di_1}{dt} = Lj\omega i_1$, $i_1 = y/(j\omega L)$

$y = i_2 R$, $i_2 = y/R$

$x - y = \frac{1}{C}\int i\,dt = \frac{i_1 + i_2}{j\omega C} = \frac{1}{j\omega C}\left[\frac{y}{j\omega L} + \frac{y}{R}\right]$, $x = y\left[1 + \frac{1}{j\omega C}\left(\frac{1}{j\omega L} + \frac{1}{R}\right)\right]$

$H(\omega) = \frac{y}{x} = \frac{1}{1 + \frac{1}{j\omega C}\left(\frac{1}{j\omega L} + \frac{1}{R}\right)} = \frac{-\omega^2 RLC}{R - \omega^2 LC + j\omega L}$.

8.1-27.



$y = L\frac{di}{dt} + \frac{1}{C}\int i\,dt = j\omega L\,i + \frac{1}{j\omega C}i$

$x - y = iR = R\frac{y}{j\omega L + \frac{1}{j\omega C}}$, $x = y\left[1 + \frac{j\omega RC}{1 + j\omega L j\omega C}\right]$

$H(\omega) = \frac{y}{x} = \frac{1 + j\omega L j\omega C}{j\omega RC + 1 + j\omega L j\omega C} = \frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega RC}$.

8.2-1. $Y(t) = \int_{-\infty}^{\infty} h(\xi) X(t-\xi)\, d\xi = \int_{-\infty}^{\infty} W u(\xi) e^{-W\xi} A \sin[\omega_0 t - \omega_0 \xi$ $+ \Theta]\, d\xi = WA \int_0^{\infty} e^{-W\xi} \left[\frac{e^{j\omega_0 t - j\omega_0 \xi + j\Theta}}{2j} - \frac{e^{-j\omega_0 t + j\omega_0 \xi - j\Theta}}{2j} \right] d\xi$

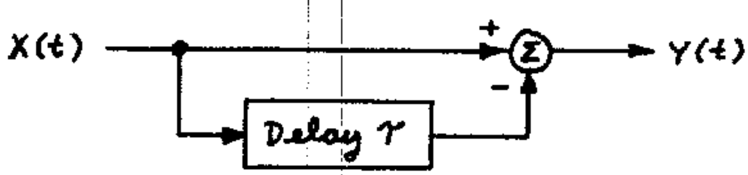
$= \frac{WA}{W^2 + \omega_0^2} \left\{ W \sin(\omega_0 t + \Theta) - \omega_0 \cos(\omega_0 t + \Theta) \right\}$

after (C-45) is used.

8.2-2. $Y(t) = \int_{-\infty}^{\infty} h(\xi) X(t-\xi)\, d\xi = \int_{-\infty}^{\infty} u(\xi) \xi e^{-\xi} A \sin[\omega_0 t$ $- \omega_0 \xi + \Theta]\, d\xi = A \int_0^{\infty} \xi e^{-\xi} \left[\frac{e^{j\omega_0 t - j\omega_0 \xi + j\Theta}}{2j} \right.$ $\left. - \frac{e^{-j\omega_0 t + j\omega_0 \xi - j\Theta}}{2j} \right] d\xi$. By using (C-46):

$Y(t) = \frac{A}{(1+\omega_0^2)^2} \left\{ (1-\omega_0^2) \sin(\omega_0 t + \Theta) - 2\omega_0 \cos(\omega_0 t + \Theta) \right\}.$

8.2-3. (a)



(b) The impulse response is $h(t) = \delta(t) - \delta(t-\tau)$. But since $\delta(t) \leftrightarrow 1$ and $\delta(t-\tau) \leftrightarrow e^{-j\omega\tau}$, then

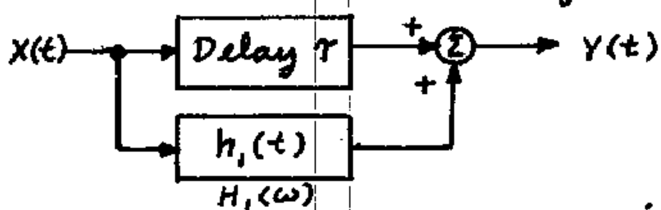
$$h(t) \leftrightarrow H(\omega) = 1 - e^{-j\omega\tau} = 2j\, e^{-j\omega\tau/2} \sin(\omega\tau/2).$$

8.2-4. $Y(t) = X(t-\tau) + \int_{t_1}^{t_2} X(t-\xi)\, d\xi$.

(a) The first term is the output of a delay line. Write the second as

$$\int_{t_1}^{t_2} X(t-\xi)\, d\xi = \int_{-\infty}^{\infty} [u(\xi - t_1) - u(\xi - t_2)\, X(t-\xi)\, d\xi.$$

8.2-4. (Continued) This function represents the output of a network with impulse response $h_1(t) = u(t-t_1) - u(t-t_2)$. The block diagram is:



(b) For the delay: $h(t) = \delta(t-\tau) \leftrightarrow e^{-j\omega\tau}$ from pair 10 of Appendix E. For the second network:

$$H_1(\omega) = \int_{-\infty}^{\infty} h_1(t)\, e^{-j\omega t}\, dt = \int_{t_1}^{t_2} e^{-j\omega t}\, dt$$

$$= 2\, e^{-j\omega(t_1+t_2)/2}\; \frac{\sin[\omega(t_2-t_1)/2]}{\omega}.$$

For both:

$$H(\omega) = e^{-j\omega\tau} + (t_2-t_1)\, e^{-j\omega(t_2+t_1)/2}\, \frac{\sin[\omega(t_2-t_1)/2]}{\omega(t_2-t_1)/2}.$$

8.2-5. From (6.3-7) the mean value of $X(t)$ is $\bar{X} = \pm|A| = +A$ (A given as positive). From (8.2-4): $E[Y(t)] = A\int_{-\infty}^{\infty} h(\xi)\,d\xi = A\int_0^{\infty} e^{-Wt}\,dt = \frac{A}{W}$ from (C-45).

8.2-6. Again the mean value of $X(t)$ is $\bar{X} = +A$ from (6.3-7). From (8.2-4) and (C-46):

$$E[Y(t)] = \bar{Y} = \bar{X}\int_{-\infty}^{\infty} h(t)\,dt = A\int_0^{\infty} t\, e^{-Wt}\,dt = \frac{A}{W^2}.$$

8.2-7. From (6.3-7) and the fact that A is positive: $\bar{X} = A$. From (8.2-4) and (C-49):

$$E[Y(t)] = \bar{Y} = \bar{X}\int_{-\infty}^{\infty} h(t)\,dt = A\int_{0}^{\infty} e^{-Wt}\sin(\omega_0 t)\,dt$$

$$= \frac{A\,\omega_0}{\omega_0^2 + W^2}.$$

8.2-8. Here $(\mathcal{N}_0/2)\,\delta(\tau) \longleftrightarrow (\mathcal{N}_0/2) = 5$ W/Hz so $R_{NN}(\tau) = 5\,\delta(\tau)$. From (8.2-7):

$$\overline{Y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 5\,\delta(\xi_1 - \xi_2)\,h(\xi_1)\,h(\xi_2)\,d\xi_1\,d\xi_2$$

$$= 5\int_{-\infty}^{\infty} h^2(\xi_2)\,d\xi_2 = 5\int_{0}^{\infty} e^{-2W\xi_2}\,d\xi_2 = \frac{2.5}{W}$$

from (C-45).

8.2-9. From Problem 8.2-8 $R_{NN}(\tau) = 5\,\delta(\tau)$, From (8.2-7): $\overline{Y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 5\,\delta(\xi_1 - \xi_2)\,h(\xi_1)\,h(\xi_2)\,d\xi_1\,d\xi_2$

$$= 5\int_{-\infty}^{\infty} h^2(\xi_2)\,d\xi_2 = 5\int_{0}^{\infty} \xi_2^2\, e^{-2W\xi_2}\,d\xi_2 = \frac{1.25}{W^3}$$

from using (C-47).

8.2-10. From Problem 8.2-8 $R_{NN}(\tau) = 5\,\delta(\tau)$. From (8.2-7): $\overline{Y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 5\,\delta(\xi_1 - \xi_2)\,h(\xi_1)\,h(\xi_2)\,d\xi_1\,d\xi_2$

$$= 5\int_{-\infty}^{\infty} h^2(\xi_2)\,d\xi_2 = 5\int_{0}^{\infty} e^{-2W\xi_2}\sin^2(\omega_0\xi_2)\,d\xi_2$$

$$= 5\int_{0}^{\infty} e^{-2W\xi_2}\left[\tfrac{1}{2} - \tfrac{1}{2}\cos(2\omega_0\xi_2)\right]d\xi_2 = \frac{1.25\,\omega_0^2/W}{\omega_0^2 + W^2}$$

from use of (C-45) and (C-50).

(8.2-11.) (a) Since the system is linear $Y(t) = Y_1(t) + Y_2(t)$ so $\bar{Y} = \bar{Y}_1 + \bar{Y}_2 = E[Y(t)]$. From (8.2-4):
$\bar{Y} = (\bar{X}_1 + \bar{X}_2)\int_{-\infty}^{\infty} h(t)\,dt$. (b) $R_{YY}(t, t+\tau) =$

$E[Y(t)Y(t+\tau)] = E[\{Y_1(t) + Y_2(t)\}\{Y_1(t+\tau) + Y_2(t+\tau)\}]$
$= R_{Y_1Y_1}(t, t+\tau) + R_{Y_1Y_2}(t, t+\tau) + R_{Y_2Y_1}(t, t+\tau)$

$+ R_{Y_2Y_2}(t, t+\tau)$. Now since responses of a linear time-invariant network to wide-sense stationary input processes are also wide-sense stationary, and since the cross-correlations above are functions of τ and not t from Problem 8.2-12 we have $Y_1(t)$ and $Y_2(t)$ jointly wide-sense and $R_{YY}(t, t+\tau) = R_{Y_1Y_1}(\tau) + R_{Y_1Y_2}(\tau) + R_{Y_2Y_1}(\tau) + R_{Y_2Y_2}(\tau)$
$= R_{YY}(\tau)$. In terms of $h(t)$ and inputs we use (8.2-9):

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{X_1X_1}(\tau+\xi_1-\xi_2)\,h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{X_1X_2}(\tau+\xi_1-\xi_2)\,h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{X_2X_1}(\tau+\xi_1-\xi_2)\,h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{X_2X_2}(\tau+\xi_1-\xi_2)\,h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2.$$

(8.2-12.) $R_{Y_1Y_2}(t, t+\tau) = E[Y_1(t)Y_2(t+\tau)]$
$$= E\left[\int_{-\infty}^{\infty} X_1(t-\xi_1)h(\xi_1)\,d\xi_1 \int_{-\infty}^{\infty} X_2(t+\tau-\xi_2)h(\xi_2)\,d\xi_2\right]$$

8.2-12. (Continued)

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[X_1(t-\xi_1)X_2(t+\tau-\xi_2)]\, h(\xi_1) h(\xi_2)\, d\xi_1\, d\xi_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{X_1X_2}(\tau+\xi_1-\xi_2)\, h(\xi_1) h(\xi_2)\, d\xi_1\, d\xi_2 .$$

8.2-13. $Y_1(t) = \int_{-\infty}^{\infty} X_1(t-\xi_1)\, h_1(\xi_1)\, d\xi_1$

$Y_2(t) = \int_{-\infty}^{\infty} X_2(t-\xi_2)\, h_2(\xi_2)\, d\xi_2$

$$R_{Y_1Y_2}(t, t+\tau) = E[Y_1(t)\, Y_2(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} X_1(t-\xi_1) h_1(\xi_1)\, d\xi_1 \int_{-\infty}^{\infty} X_2(t+\tau-\xi_2) h_2(\xi_2)\, d\xi_2\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{X_1X_2}(\tau+\xi_1-\xi_2)\, h_1(\xi_1) h_2(\xi_2)\, d\xi_1\, d\xi_2$$

$$= R_{Y_1Y_2}(\tau) .$$

8.2-14. $W(t) = \int_{-\infty}^{\infty} X(t-\xi_1)\, h_1(\xi_1)\, d\xi_1$

$Y(t) = \int_{-\infty}^{\infty} W(t-\xi_2)\, h_2(\xi_2)\, d\xi_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\xi_1-\xi_2)\, h_1(\xi_1) h_2(\xi_2)\, d\xi_1$ $d\xi_2$. $R_{WY}(t, t+\tau) = E[W(t)\, Y(t+\tau)]$

$$= E\left[\int_{-\infty}^{\infty} X(t-\xi_3) h_1(\xi_3)\, d\xi_3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t+\tau-\xi_1-\xi_2)\, h_1(\xi_1) h_2(\xi_2)\, d\xi_1\, d\xi_2\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau-\xi_1-\xi_2+\xi_3)\, h_1(\xi_1) h_2(\xi_2)\, h_1(\xi_3)\, d\xi_1\, d\xi_2\, d\xi_3 = R_{WY}(\tau).$$

(8.2-15.) From the solution to Problem 8.2-14 with $h_1(t) = h_2(t) = h(t)$: $Y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$ so

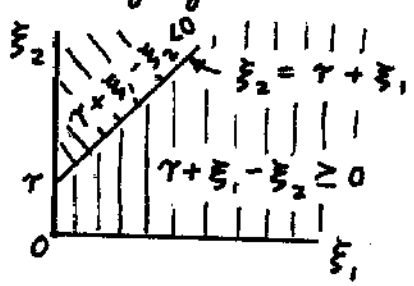
$$E[Y(t)] = \bar{X}\int_{-\infty}^{\infty} h(\xi_1)d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)d\xi_2 = \bar{X}\left[\int_{-\infty}^{\infty} h(\xi)d\xi\right]^2$$

$$= \bar{X}\left[\int_0^{\infty} \xi e^{-W\xi}d\xi\right]^2 = \frac{\bar{X}}{W^4} = \frac{2}{81} \quad \text{from (C-46).}$$

(8.2-16.) Use (8.2-9): $R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P e^{-\alpha|\tau+\xi_1-\xi_2|}$

$$\cdot W^2 u(\xi_1)e^{-W\xi_1}u(\xi_2)e^{-W\xi_2}d\xi_1 d\xi_2$$

$$= W^2 P\int_0^{\infty}\int_0^{\infty} e^{-\alpha|\tau+\xi_1-\xi_2|}e^{-W(\xi_1+\xi_2)}d\xi_1 d\xi_2.$$



The sketch is helpful. Assume $\tau \geq 0$.

$$R_{YY}(\tau) = W^2P\int_{\xi_2=\tau}^{\infty}\int_{\xi_1=0}^{\xi_2-\tau} e^{\alpha(\tau+\xi_1-\xi_2)-W(\xi_1+\xi_2)}d\xi_1 d\xi_2$$

$$+W^2P\int_{\xi_2=0}^{\tau}\int_{\xi_1=0}^{\infty} e^{-\alpha(\tau+\xi_1-\xi_2)-W(\xi_1+\xi_2)}d\xi_1 d\xi_2$$

$$+W^2P\int_{\xi_2=\tau}^{\tau+\xi_1}\int_{\xi_1=0}^{\infty} e^{-\alpha(\tau+\xi_1-\xi_2)-W(\xi_1+\xi_2)}d\xi_1 d\xi_2$$

$$= \frac{WP}{\alpha^2-W^2}\left[\alpha e^{-W\tau} - We^{-\alpha\tau}\right] \quad \text{from (C-45).}$$

8.2-16. (Continued) Now since $R_{YY}(\tau)$ must be an even function of τ we have:

$$R_{YY}(\tau) = \frac{WP}{\alpha^2 - W^2}\left[\alpha e^{-W|\tau|} - We^{-\alpha|\tau|}\right].$$

8.2-17. Use (8.2-12). $R_{XY}(\tau) = \int_{-\infty}^{\infty} P e^{-\alpha|\tau-\xi|} W u(\xi) \cdot e^{-W\xi} d\xi = WP\int_0^{\infty} e^{-\alpha|\tau-\xi| - W\xi} d\xi$. Assume first $\tau \geq 0$. $R_{XY}(\tau) = WP\left\{\int_0^{\tau} e^{-\alpha(\tau-\xi) - W\xi} d\xi + \int_{\tau}^{\infty} e^{\alpha(\tau-\xi)} \cdot e^{-W\xi} d\xi\right\} = \frac{WP}{\alpha^2 - W^2}\left[2\alpha e^{-W\tau} - (\alpha+W)e^{-\alpha\tau}\right]$, $\tau \geq 0$, from (C-45). Next assume $\tau < 0$.

$R_{XY}(\tau) = WP\left\{\int_0^{\infty} e^{\alpha\tau - \alpha\xi - W\xi} d\xi\right\} = \frac{WP}{\alpha^2 - W^2}(\alpha - W)e^{\alpha\tau}$, $\tau < 0$. By combining results:

$$R_{XY}(\tau) = \frac{WP}{\alpha^2 - W^2}\left[2\alpha e^{-W\tau} - (\alpha+W)e^{-\alpha\tau}\right], \quad \tau \geq 0$$

$$= \frac{WP}{\alpha^2 - W^2}\left[(\alpha - W)e^{\alpha\tau}\right], \quad \tau < 0.$$

(8.2-18.) On use of (C-46): $y(t) = \int_{-\infty}^{\infty} h(\xi)\, x(t-\xi)\, d\xi$

$= W^2 \int_0^{\infty} \xi\, e^{-W\xi} u(t-\xi)\, e^{-\alpha(t-\xi)} d\xi = W^2 u(t) e^{-\alpha t} \int_0^t \xi$

$\cdot\, e^{-(W-\alpha)\xi} d\xi = \frac{W^2 u(t)}{(W-\alpha)^2} \left\{ e^{-\alpha t} - [1+(W-\alpha)t]\, e^{-Wt} \right\}.$

(8.2-19.) Use (C-47): $y(t) = \int_{-\infty}^{\infty} h(\xi)\, x(t-\xi)\, d\xi =$

$W^3 \int_0^{\infty} \xi^2 e^{-W\xi} u(t-\xi)\, e^{-\alpha(t-\xi)} d\xi = W^3 u(t) e^{-\alpha t} \int_0^t \xi^2 e^{-(W-\alpha)\xi} d\xi$

$= \frac{2W^3 u(t)}{(W-\alpha)^3} \left\{ e^{-\alpha t} - [1+(W-\alpha)t + \frac{(W-\alpha)^2 t^2}{2}]\, e^{-Wt} \right\}.$

(8.2-20.) From (8.2-4): $\bar{Y} = \bar{X} \int_{-\infty}^{\infty} h(t)dt = 2\int_0^{\infty} 3t^2 e^{-8t} dt = 3/128$

Here we have also used (C-47).

(8.2-21.) $\overline{Y^2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 5\,\delta(\xi_1 - \xi_2)\, h(\xi_1) h(\xi_2)\, d\xi_1\, d\xi_2$

$= 5\int_{-\infty}^{\infty} h^2(\xi) d\xi = 5\int_0^{\infty} 9\, \xi^4 e^{-16\xi} d\xi = 135/2(16^4)$ from

an integral of Dwight, p. 134.

(8.2-22.) From Example 8.2-2:

$R_{XY}(\tau) = \frac{\mathcal{N}_0}{2} h(\tau) = \frac{\mathcal{N}_0}{2} u(\tau)\, W\tau\, e^{-W\tau}$

$R_{YX}(\tau) = R_{XY}(-\tau) = -\frac{\mathcal{N}_0}{2} u(-\tau)\, W\tau\, e^{W\tau}.$

(8.2-23.) From Example 8.2-2:

$R_{XY}(\tau) = \frac{\mathcal{N}_0}{2} h(\tau) = \frac{\mathcal{N}_0}{2} u(\tau)\, W\tau\, e^{-W\tau} \sin(\omega_0 \tau)$

$R_{YX}(\tau) = R_{XY}(-\tau) = \frac{\mathcal{N}_0}{2} u(-\tau)\, W\tau\, e^{W\tau} \sin(\omega_0 \tau).$

8.2-24. (a) Use (8.2-16): $R_{YY}(\tau) = \int_{-\infty}^{\infty} R_{XY}(\tau+\xi)h(\xi)\,d\xi$ $= e^{-b\tau}\int_{-\infty}^{\infty} u(\xi)u(\xi+\tau)(\tau\xi+\xi^2)\,e^{-2b\xi}\,d\xi$. There are two cases of interest; $\tau \geq 0$ and $\tau < 0$. Because $R_{YY}(\tau)$ is an even function we solve only the case $\tau \geq 0$.

$R_{YY}(\tau) = e^{-b\tau}\int_0^{\infty}(\tau\xi+\xi^2)\,e^{-2b\xi}\,d\xi = \dfrac{(1+b\tau)}{4b^3}e^{-b\tau}$.

For any τ: $R_{YY}(\tau) = \dfrac{(1+b|\tau|)}{4b^3}e^{-b|\tau|}$. (b) Power in $Y(t) = R_{YY}(0) = 1/4b^3$.

8.2-25. (a) Use (8.2-16) assuming $\tau \geq 0$:

$R_{YY}(\tau) = e^{-b\tau}\int_0^{\infty}(\tau^2\xi^2+2\tau\xi^3+\xi^4)\,e^{-2b\xi}\,d\xi$

$= \dfrac{1}{4b^5}(3+3b\tau+b^2\tau^2)\,e^{-b\tau}$, $\tau \geq 0$. Here we used an integral from Dwight, p. 134. Since $R_{YY}(\tau)$ must be even we have $R_{YY}(\tau) = \dfrac{1}{4b^5}(3+3b|\tau|+b^2\tau^2)\,e^{-b|\tau|}$.

(b) Power in $Y(t) = R_{YY}(0) = 3/4b^5$.

8.2-26. (a) $X(t) \longrightarrow \boxed{h(t)} \xrightarrow{Y_1(t)} \boxed{h(t)} \longrightarrow Y(t)$

$Y_1(t) = \int_{-\infty}^{\infty} X(t-\xi_1)h(\xi_1)\,d\xi_1$, $Y(t) = \int_{-\infty}^{\infty} Y_1(t-\xi_2)h(\xi_2)\,d\xi_2$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\xi_2-\xi_1)h(\xi_1)\,d\xi_1\, h(\xi_2)\,d\xi_2$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\xi_1-\xi_2)h(\xi_1)h(\xi_2)\,d\xi_1 d\xi_2$

$= 9\int_0^{\infty}\int_0^{\infty} X(t-\xi_1-\xi_2)\,\xi_1\xi_2\,e^{-4(\xi_1+\xi_2)}\,d\xi_1 d\xi_2$.

(b) $E[Y(t)] = 9\bar{X}\left[\int_0^{\infty}\xi\,e^{-4\xi}\,d\xi\right]^2 = 27/128$.

8.2-27. $R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2 \leftarrow \begin{array}{l}\alpha = \tau+\xi_1-\xi_2\\ d\alpha = -d\xi_2\end{array}$

$$= \int_{-\infty}^{\infty} R_{XX}(\alpha)\int_{-\infty}^{\infty} h(\xi_1)h(\tau-\alpha+\xi_1)\,d\xi_1\,d\alpha = \int_{-\infty}^{\infty} R_{XX}(\alpha)\mathcal{R}_{hh}(\tau-\alpha)\,d\alpha$$

$$= R_{XX}(\tau) * \mathcal{R}_{hh}(\tau).$$

8.2-28. $\mathcal{R}_{hh}(\xi) = \int_{-\infty}^{\infty} h(t)h(t+\xi)\,d\xi = \int_{-\infty}^{\infty}\frac{1}{T^2}[u(t)-u(t-T)][u(t+\xi)-u(t+\xi-T)]\,dt$

Two cases: For $0 \le \xi \le T$

$$\left.\begin{array}{l}\mathcal{R}_{hh}(\xi) = \frac{1}{T^2}\int_0^{T-\xi} dt = \frac{T-\xi}{T^2}, \quad 0\le\xi\le T\\ \text{For } -T\le\xi<0\\ \mathcal{R}_{hh}(\xi) = \frac{1}{T^2}\int_{-\xi}^{T} d\xi = \frac{T+\xi}{T^2}, \quad -T\le\xi<0\end{array}\right\} \mathcal{R}_{hh}(\xi) = \frac{1}{T}\left(1-\frac{|\xi|}{T}\right), -T\le\xi\le T. \text{ (and zero elsewhere)}$$

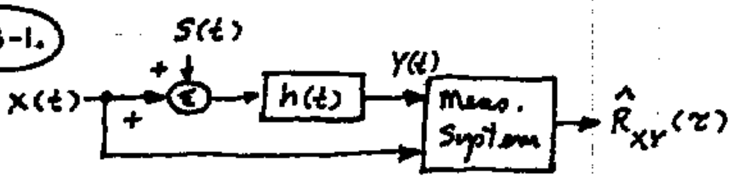
Thus, for $-T \le \xi \le T$: $\quad \xi = \tau-\alpha,\ d\xi = -d\alpha$

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\alpha)\frac{1}{T}\left(1-\frac{|\tau-\alpha|}{T}\right)d\alpha = \int_{-\infty}^{\infty} R_{XX}(\tau-\xi)\frac{1}{T}\left(1-\frac{|\xi|}{T}\right)d\xi$$

or

$$R_{YY}(\tau) = \int_{-T}^{T} R_{XX}(\tau-\xi)\frac{1}{T}\left(1-\frac{|\xi|}{T}\right)d\xi$$

8.3-1.



$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E\left\{X(t)\int_{-\infty}^{\infty}[X(t+\tau-\xi)+S(t+\tau-\xi)]h(\xi)d\xi\right\}$$

$$= \int_{-\infty}^{\infty}[R_{XX}(\tau-\xi)+R_{XS}(\tau-\xi)]h(\xi)d\xi = \int_{-\infty}^{\infty}\frac{\mathcal{N}_0}{2}\delta(\tau-\xi)h(\xi)d\xi +$$

$\int_{-\infty}^{\infty} R_{XS}(\tau-\xi)h(\xi)d\xi \leftarrow$ Second term $=0$ if $R_{XX}(\tau-\xi)=0$ due to ~~orthogonality. Thus, $R_{XY}(\tau) = (\mathcal{N}_0/2)h(\tau)$ and (8.3-4)~~ applies because this result is just (8.2-3) again.

(8.4-1.) From Example 8.1-1 $|H(\omega)|^2 = 1/[1+(\omega/W)^2]$ where $W = R/L$. (a) For the output noise, defined as $N(t)$: $\mathcal{S}_{NN}(\omega) = (\mathcal{N}_0/2)/[1+(\omega/W)^2]$.

For the input signal $X(t) = A\sin(\omega_0 t + \Theta)$:

$$R_{XX}(t,t+\tau) = E[A\sin(\omega_0 t+\Theta)\,A\sin(\omega_0 t+\omega_0\tau+\Theta)]$$

$$= \frac{A^2}{2}E[\cos(\omega_0\tau) - \cos(2\omega_0 t+\omega_0\tau+2\Theta)]$$

$$= \frac{A^2}{2}\left\{\cos(\omega_0\tau) - \int_{-\pi}^{\pi}\cos(2\omega_0 t+\omega_0\tau+2\theta)\frac{1}{2\pi}d\theta\right\}$$

$$= \frac{A^2}{2}\cos(\omega_0\tau) = R_{XX}(\tau).$$ The Fourier transform of $R_{XX}(\tau)$ is given by pair 11 of Appendix E:

$$\mathcal{S}_{XX}(\omega) = \frac{A^2\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

By defining $S(t)$ to be the output signal, then

$$\mathcal{S}_{SS}(\omega) = \frac{A^2\pi/2}{1+(\omega_0/W)^2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

from (8.4-5). (b) Output signal power S_0 is

$$S_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{SS}(\omega)\,d\omega = \frac{A^2/2}{1+(\omega_0/W)^2}.$$

Output noise power N_0 is

$$N_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{NN}(\omega)\,d\omega = \frac{\mathcal{N}_0}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{1+(\omega/W)^2}$$

$= \mathcal{N}_0 W/4$ from (C-25). Thus, $(S_0/N_0) = \dfrac{2A^2/\mathcal{N}_0 W}{1+(\omega_0/W)^2}$.

(c) $$\frac{d(S_0/N_0)}{dW} = \frac{2A^2}{\mathcal{N}_0}\left\{\frac{(W^2+\omega_0^2)-W(2W)}{(W^2+\omega_0^2)^2}\right\} = 0$$ when

$W = \pm\omega_0$. Since W cannot be negative $W = \omega_0$.

8.4-1. (Continued)

a sketch proves this corresponds to a maximum and not a minimum.

8.4-2. Since $X_1(t)$ and $X_2(t)$ are wide-sense stationary:

$$R_{XX}(\tau) = E[\{X_1(t)+X_2(t)\}\{X_1(t+\tau)+X_2(t+\tau)\}]$$
$$= R_{X_1X_1}(\tau) + R_{X_1X_2}(\tau) + R_{X_2X_1}(\tau) + R_{X_2X_2}(\tau) \quad \text{and}$$

$\mathcal{S}_{XX}(\omega) = \mathcal{S}_{X_1X_1}(\omega) + \mathcal{S}_{X_1X_2}(\omega) + \mathcal{S}_{X_2X_1}(\omega) + \mathcal{S}_{X_2X_2}(\omega)$. From (8.4-5):

$$\mathcal{S}_{YY}(\omega) = |H(\omega)|^2[\mathcal{S}_{X_1X_1}(\omega) + \mathcal{S}_{X_1X_2}(\omega) + \mathcal{S}_{X_2X_1}(\omega) + \mathcal{S}_{X_2X_2}(\omega)].$$

8.4-3. Since (6.3-14) applies, $R_{XY}(\tau) = R_{YX}(\tau) = \bar{X}\bar{Y}$. As applied here: $R_{X_1X_2}(\tau) = R_{X_2X_1}(\tau) = \bar{X}_1\bar{X}_2$. Thus, $\mathcal{S}_{X_1X_2}(\omega) = \mathcal{S}_{X_2X_1}(\omega) = 2\pi\bar{X}_1\bar{X}_2\,\delta(\omega)$ from pair 2 of Appendix E. Finally, from Problem 8.4-2,

$$\mathcal{S}_{YY}(\omega) = |H(\omega)|^2[\mathcal{S}_{X_1X_1}(\omega) + \mathcal{S}_{X_2X_2}(\omega) + 4\pi\bar{X}\bar{Y}\,\delta(\omega)].$$

8.4-4. Now $\mathcal{S}_{YY}(\omega) = \dfrac{\mathcal{N}_0/2}{[1+(\omega L/R)^2]^2}$ and

$$P_{YY} = \frac{\mathcal{N}_0/2}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{[1+(\omega L/R)^2]^2} = \mathcal{N}_0 R/8L \text{ from (C-28).}$$

8.4-5. From (8.2-12): $R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\xi)h(\xi)\,d\xi$ so

$\mathcal{S}_{XY}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau-\xi)h(\xi)\,d\xi\, e^{-j\omega\tau}\,d\tau$. Let $\eta = \tau-\xi$,

(8.4-5.) (Continued)

$d\eta = d\tau$. $\mathcal{S}_{XY}(\omega) = \underbrace{\int_{-\infty}^{\infty} R_{XX}(\eta)\, e^{-j\omega\eta} d\eta}_{\mathcal{S}_{XX}(\omega)} \underbrace{\int_{-\infty}^{\infty} h(\xi)\, e^{-j\omega\xi} d\xi}_{H(\omega)}$.

From (8.2-14) and the same procedure:

$\mathcal{S}_{YX}(\omega) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} R_{XX}(\tau - \xi)\, h(-\xi)\, d\xi \right\} e^{-j\omega\tau} d\tau$. Let

$\eta = \tau - \xi$, $d\eta = d\tau$, $\mu = -\xi$, $d\mu = -d\xi$:

$$\mathcal{S}_{YX}(\omega) = \underbrace{\int_{-\infty}^{\infty} R_{XX}(\eta)\, e^{-j\omega\eta} d\eta}_{\mathcal{S}_{XX}(\omega)} \underbrace{\int_{-\infty}^{\infty} h(\mu)\, e^{j\omega\mu} d\mu}_{H(-\omega)} .$$

(8.4-6.) (a) From (8.4-7): $\mathcal{S}_{X\dot{X}}(\omega) = \mathcal{S}_{XX}(\omega)\, j\omega$. By inverse Fourier transformation:

$$R_{X\dot{X}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{X\dot{X}}(\omega)\, e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega) \left[j\omega\, e^{j\omega\tau} \right] d\tau$$

$$= \frac{d}{d\tau}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\, e^{j\omega\tau} d\omega \right] = \frac{d\, R_{XX}(\tau)}{d\tau} .$$

(b) From (8.4-1): $\mathcal{S}_{\dot{X}\dot{X}}(\omega) = \mathcal{S}_{XX}(\omega)\, \omega^2$. Thus,

$$R_{\dot{X}\dot{X}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\; \omega^2\, e^{j\omega\tau} d\omega$$

$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\,(j\omega)^2\, e^{j\omega\tau} d\omega = \frac{-d^2}{d\tau^2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega) e^{j\omega\tau} d\tau \right]$$

$$= \frac{-d^2 R_{XX}(\tau)}{d\tau^2} .$$

(8.4-7.) $Y(t) = \frac{1}{2T}\int_{-T+t}^{T+t} X(\xi)\,d\xi$. From (8.2-1)

$Y(t) = \int_{-\infty}^{\infty} X(\xi)h(t-\xi)\,d\xi$ is equivalent if

$h(t-\xi) = \frac{1}{2T}$, $t-T<\xi<t+T$ and $h(t-\xi)=0$ otherwise. In other words, if $X(t)$ passes through a network for which $h(t) = 1/2T$, $-T<t<T$ and $h(t)=0$ elsewhere we have equivalence. From Fourier transformation: $H(\omega) = \int_{-T}^{T} \frac{1}{2T} e^{-j\omega t}\,dt$

$= \frac{\sin(\omega T)}{\omega T}$ from (C-45). From (8.4-1)

$$\mathcal{S}_{YY}(\omega) = \mathcal{S}_{XX}(\omega)\left[\frac{\sin(\omega T)}{\omega T}\right]^2.$$

(8.4-8.) (a) From pair 19, Appendix E,

$$\mathcal{S}_{XX}(\omega) = \frac{16}{16+\omega^2}.$$

(b) $H(\omega) = \dfrac{1/j\omega C_2}{\dfrac{1}{j\omega C_2} + \dfrac{R(1/j\omega C_1)}{R+(1/j\omega C_1)}}$

$= \frac{1+j\omega RC_1}{1+j\omega R(C_1+C_2)} = \frac{1+j6\omega}{1+j9\omega}$. $|H(\omega)|^2 = H(\omega)H^*(\omega)$

$= \frac{1+\omega^2R^2C_1^2}{1+\omega^2R^2(C_1+C_2)^2} = \frac{1+36\omega^2}{1+81\omega^2}$. (c) From (8.4-5):

$$\mathcal{S}_{YY}(\omega) = \mathcal{S}_{XX}(\omega)|H(\omega)|^2 = \frac{16\,(1+36\omega^2)}{(16+\omega^2)(1+81\omega^2)}.$$

(8.4-9.) (a) From pair 19, Appendix E, $H(\omega) = \frac{32}{16+\omega^2}$.

(b) $\bar{Y} = \bar{X}\int_{-\infty}^{\infty} h(\xi)\,d\xi = 5\int_{-\infty}^{\infty} 4e^{-4|\xi|}\,d\xi = 40\int_0^{\infty} e^{-4\xi}\,d\xi = 10.$

(c) $\mathcal{S}_{YY}(\omega) = \mathcal{S}_{XX}(\omega)|H(\omega)|^2 = 50\pi\delta(\omega)|H(0)|^2 + \frac{12\,(32^2)}{(4+\omega^2)(16+\omega^2)}$

(8.4-9.) (Continued)

or $\mathcal{S}_{YY}(\omega) = 200\pi\delta(\omega) + \dfrac{12{,}288.}{(4+\omega^2)(16+\omega^2)}$.

(8.4-10.) (a) $P_{YY} = R_{YY}(0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 10^{-2}\delta(\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$

$= 10^{-2}\int_{-\infty}^{\infty} h^2(\xi)d\xi = 10^{-2}(9)\int_0^{\infty}\xi^2 e^{-8\xi}d\xi = \dfrac{9}{256(10^2)}$.

(b) Use pairs 1 and 16, appendix E: $\mathcal{S}_{YY}(\omega) =$

$\mathcal{S}_{XX}(\omega)|H(\omega)|^2 = 10^{-2}\dfrac{9}{(16+\omega^2)^2}$.

(8.4-11) $P_{YY} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)|H(\omega)|^2 d\omega = \dfrac{1}{2\pi}\int_{-W}^{W} 6(10^{-6})d\omega = \dfrac{6(10^{-6})W}{\pi}$

$= 15$ so $W = 5\pi(10^6)/2$.

(8.4-12.) $P_{YY} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)|H(\omega)|^2 d\omega = \dfrac{1}{2\pi}\int_{-4}^{4}\dfrac{50}{\sqrt{8\pi}}e^{-\omega^2/8}\,8\,d\omega$

$= \dfrac{200}{\pi}\int_{-4}^{4}\dfrac{e^{-\omega^2/2(4)}}{\sqrt{2\pi(4)}}d\omega$. Use (B-1) and (B-2) to

get $P_{YY} = (200/\pi)[F(2) - F(-2)] = (200/\pi)[2F(2) - 1]$

$= 190.88/\pi$.

(8.4-13.) (a) $10e^{-|\tau|} \longleftrightarrow \dfrac{20}{1+\omega^2}$. $\mathcal{S}_{\substack{signal\\out}}(\omega) = \dfrac{20}{1+\omega^2}\cdot\dfrac{4}{(1+\omega^2)^3} = \dfrac{80}{(1+\omega^2)^4}$.

$P_{signal\ out} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\dfrac{80\,d\omega}{(1+\omega^2)^4} = 12.5$ from (C-34). (b) $\mathcal{S}_{\substack{noise\\out}}(\omega) =$

$\dfrac{\mathcal{N}_0}{2}\dfrac{4}{(1+\omega^2)^3} = \dfrac{2\mathcal{N}_0}{(1+\omega^2)^3}$. $P_{noise\ out} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\dfrac{2\mathcal{N}_0\,d\omega}{(1+\omega^2)^3} = \dfrac{3}{4}\left(\dfrac{\mathcal{N}_0}{2}\right)$ from

(C-31). (c) $P_{signal\ out}/P_{noise\ out} = 50{,}000/3 = 16{,}667$.

(8.4-14.) (a) $P_{XX} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)d\omega = \dfrac{3}{2\pi}\int_{-\infty}^{\infty}\dfrac{d\omega}{49+\omega^2} = \dfrac{3}{14} \approx 0.2143$.

(b) $h_2(t) = u(t)t^2e^{-7t} \longleftrightarrow \dfrac{2}{(7+j\omega)^3} = H_2(\omega)$ from pair 17.

$\mathcal{S}_{YY}(\omega) = \mathcal{S}_{XX}(\omega)|H_1(\omega)H_2(\omega)|^2 = \dfrac{12\omega^2}{(49+\omega^2)^4}$. (c) $P_{YY} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\dfrac{12\omega^2\,d\omega}{(49+\omega^2)^4}$

$= \dfrac{3}{134{,}456} \approx 2.231(10^{-5})$ from (C-35).

(8.5-1.) For the actual network $P_{YY}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 \cdot \frac{\mathcal{N}_0}{2}\, d\omega = \frac{\mathcal{N}_0}{2\pi}\int_0^{\infty} |H(\omega)|^2 d\omega$ since $|H(\omega)|^2$ is an even function of ω for any real network. For an idealized network with mid-band gain $|H(\omega_0)|^2$, bandwidth W_N: $P_{YY} = \frac{1}{2\pi}\int_{-\omega_0 - \frac{W_N}{2}}^{-\omega_0 + \frac{W_N}{2}} \frac{\mathcal{N}_0}{2} \cdot |H(\omega_0)|^2 d\omega + \frac{1}{2\pi}\int_{\omega_0 - \frac{W_N}{2}}^{\omega_0 + \frac{W_N}{2}} \frac{\mathcal{N}_0}{2} |H(\omega_0)|^2 d\omega = \frac{\mathcal{N}_0 W_N |H(\omega_0)|^2}{2\pi}$.

By equating powers: $W_N = \dfrac{\int_0^{\infty} |H(\omega)|^2 d\omega}{|H(\omega_0)|^2}$.

(8.5-2.) $|H(\omega)|^2 = 1 + (\omega/W_H)^2$, $-W_X < \omega < W_X$ (and zero for other ω). $\mathcal{S}_{YY}(\omega) = \mathcal{S}_{XX}(\omega)\left[1 + \left(\frac{\omega}{W_H}\right)^2\right]$, $-W_X < \omega < W_X$ (and zero elsewhere). Hence,

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{YY}(\omega)\, d\omega = \frac{1}{2\pi}\int_{-W_X}^{W_X} \left\{ \mathcal{S}_{XX}(\omega) + \frac{\omega^2}{W_H^2} \mathcal{S}_{XX}(\omega) \right\} d\omega$$

$$= \frac{1}{2\pi}\int_{-W_X}^{W_X} \mathcal{S}_{XX}(\omega)\, d\omega + \frac{1}{W_H^2}\,\frac{1}{2\pi}\int_{-W_X}^{W_X} \omega^2 \mathcal{S}_{XX}(\omega)\, d\omega$$

$= P_{XX}\left[1 + (W_{rms}/W_H)^2\right]$ from (7.1-22) and (7.1-12). As $W_X \to \infty$ there is no affect on W_{rms} or P_{XX}; it just means $\mathcal{S}_{XX}(\omega)$ is nonzero for all ω. The main affect is on the network. As $W_X \to \infty$ $H(\omega) = 1 + (j\omega/W_H)$ becomes increasingly difficult to approximate in practice.

(8.5-3) From (8.5-5) with $H(0)=1$: $W_N = \int_0^\infty |H(\omega)|^2 d\omega$

$= \int_0^\infty \frac{d\omega}{1+(\omega/W)^4} = \pi W/2\sqrt{2}$ after using (C-37).

(8.5-4.) From (8.5-5) with $H(0)=1$: $W_N = \int_0^\infty |H(\omega)|^2 d\omega$

$= \int_0^\infty \frac{d\omega}{[1+(\omega/W)^2]^2} = \pi W/4$ after using (C-28).

(8.5-5.) From (8.5-5) with $H(0)=1$: $W_N = \int_0^\infty |H(\omega)|^2 d\omega$

$= \int_0^\infty \frac{d\omega}{[1+(\omega/W)^2]^3} = 3\pi W/16$ after using (C-31).

(8.5-6.) From (8.5-4): $P_{YY} = \frac{\mathcal{N}_0 |H(0)|^2 W_N}{2\pi} = 0.1$ so

$\mathcal{N}_0 = \frac{0.1(2\pi)}{4(2\pi)2(10^6)} = 10^{-7}/8 = 1.25(10^{-8})$ W/Hz.

(8.5-7.) Use (8.5-7): $P_{YY} = \mathcal{N}_0 |H(0)|^2 W_N/2\pi$ so

$\mathcal{N}_0 = 0.5/64(12)10^6 = 6.51(10^{-10})$ W/Hz.

(8.5-8.) (a) Use (8.5-5): $W_N = \frac{\int_0^\infty \frac{16}{(4^4+\omega^4)} d\omega}{(16/4^4)} = 256\int_0^\infty \frac{d\omega}{4^4+\omega^4}$

$= \pi\sqrt{2}$ from Dwight, p. 213. (b) Use (8.5-7):

$P_{YY} = \mathcal{N}_0 |H(0)|^2 W_N/2\pi = 2(6)10^{-3}\frac{16}{256}\frac{1}{\sqrt{2}} = \frac{3(10^{-3})}{4\sqrt{2}}$.

8.5-9. (a) $d|H(\omega)|^2/d\omega\Big|_{\omega=\omega_0} = \dfrac{[1+(\omega/W)^2]2\omega - 4\omega^2[1+(\omega/W)^2]^3 2(\omega/W)(1/W)}{[1+(\omega/W)^2]^8}\Bigg|_{\omega=\omega_0}$

$= 0$ when $[1+(\omega_0/W)^2] - 4(\omega_0/W)^2 = 0$ or $\omega_0 = W/\sqrt{3}$. (b) $|H(\omega_0)|^2$

$= \dfrac{W^2/3}{(1+\frac{1}{3})^4} = \dfrac{27W^2}{16^2}$. From (8.5-6): $W_N = \dfrac{16^2}{27W^2}\displaystyle\int_0^\infty \frac{\omega^2 d\omega}{[1+(\omega/W)^2]^4}$

$= \dfrac{8\pi W}{27}$ from (C-35).

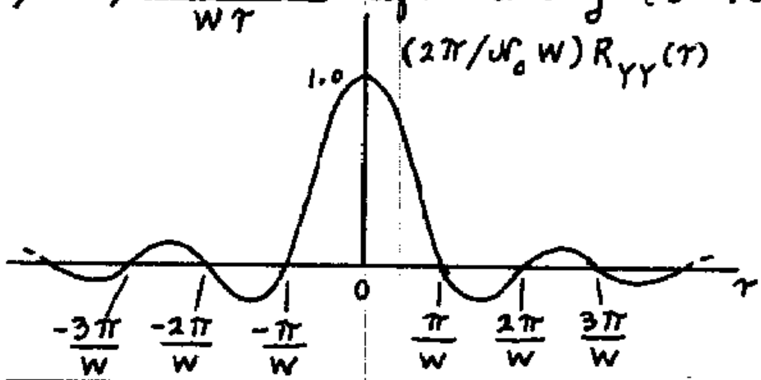
8.5-10. (a) $d|H(\omega)|^2/d\omega\Big|_{\omega=\omega_0} = \dfrac{[1+(\omega/W)^2]^4 3\omega^3 - 4\omega^4[1+(\omega/W)^2]^3 2(\omega/W)(1/W)}{[1+(\omega/W)^2]^8}\Bigg|_{\omega=\omega_0}$

$= 0$ when $\omega_0 = W\sqrt{3/5}$. (b) $|H(\omega_0)|^2 = \dfrac{W^4 9/25}{(1+3/5)^4} = \dfrac{9(25)W^4}{8^4}$.

From (8.5-6): $W_N = \dfrac{8^4 W^8}{9(25)W^4}\displaystyle\int_0^\infty \frac{\omega^4 d\omega}{(W^2+\omega^2)^4} = \frac{128\pi W}{225}$ from (C-36).

8.6-1. (a) Because $H(0)=1$ for an ideal lowpass filter (8.4-1) gives $\mathcal{S}_{YY}(\omega) = \mathcal{N}_0/2$, $-W<\omega<W$ and zero for other ω. By inverse Fourier transformation: $R_{YY}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-W}^{W}\frac{\mathcal{N}_0}{2}e^{j\omega\tau}d\omega$

$= (\mathcal{N}_0 W/2\pi)\dfrac{\sin(W\tau)}{W\tau}$ after using (C-45).



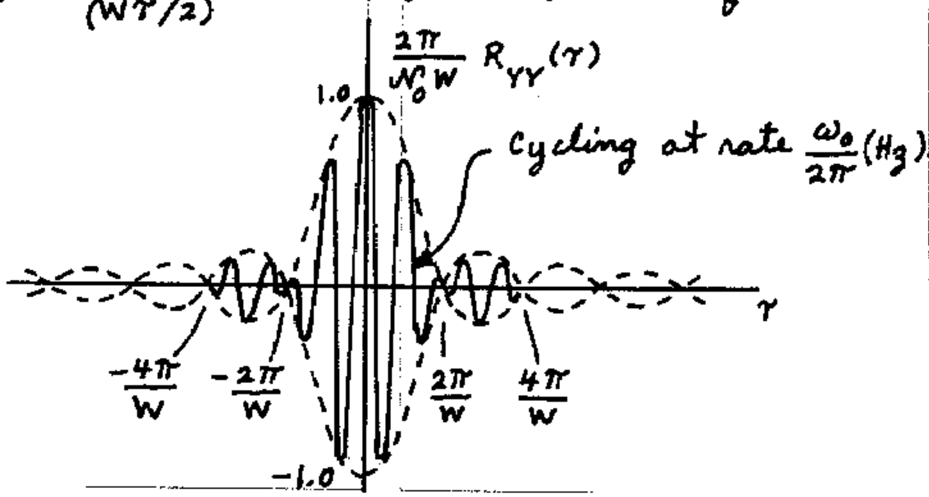
8.6-1. (Continued) Let the kth random variable correspond to the sample at time t_k. Another random variable, for a sample separated from the kth one by $\tau_n = n\pi/W$, occurs at time t_{n+k}. The separation is $\tau_n = t_{n+k} - t_k = (n+k)\frac{\pi}{W} - k\frac{\pi}{W} = n\frac{\pi}{W}$, $n = 0, \pm1, \pm2, \cdots$.

Clearly, from the sketch $R_{YY}(\tau_n) = 0$ for all n except $n = 0$ (no separation, both variables the same). Thus, all random variables are orthogonal.

8.6-2. We apply (8.4-1) to get $\mathcal{S}_{YY}(\omega) = \mathcal{N}_0/2$, $-(W/2) - \omega_0 < \omega < -\omega_0 + (W/2)$ and $-(W/2) + \omega_0 < \omega < \omega_0 + (W/2)$ (and zero elsewhere).

(a) $$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\omega_0-\frac{W}{2}}^{-\omega_0+\frac{W}{2}} \frac{\mathcal{N}_0}{2} e^{j\omega\tau} d\omega + \frac{1}{2\pi}\int_{\omega_0-\frac{W}{2}}^{\omega_0+\frac{W}{2}} \frac{\mathcal{N}_0}{2} e^{j\omega\tau} d\tau$$

$$= \frac{\mathcal{N}_0 W}{2\pi} \frac{\sin(W\tau/2)}{(W\tau/2)} \cos(\omega_0\tau),$$ after using (C-45).



8.6-2. (Continued) (b) As in Problem 8.6-1 the random variables are orthogonal because $R_{YY}(\tau_n) = 0$ at separation times $\tau_n = n2\pi/W$, $n = \pm 1, \pm 2, \pm 3, \cdots$.

8.6-3. (a) From (8.6-10) and (8.6-11): $P_{NN} = E[N^2(t)] = E[X^2(t)] = R_{XX}(0) = \frac{1}{\pi}\int_0^\infty \mathcal{S}_{NN}(\omega)\cos[(\omega-\omega_0)\tau]\,d\omega$

$$(\text{for } \tau = 0) = \frac{1}{\pi}\int_{\omega_0-\frac{W}{2}}^{\omega_0+\frac{W}{2}} P\cos[\pi(\omega-\omega_0)/W]\,d\omega = \frac{2PW}{\pi^2}.$$

(b) Use (8.6-16). $\mathcal{S}_{XX}(\omega) = L_p\{P\cos[\pi(\omega-2\omega_0)/W]$

$\cdot\operatorname{rect}[(\omega-2\omega_0)/W] + 2P\cos[\pi\omega/W]\operatorname{rect}(\omega/W)$

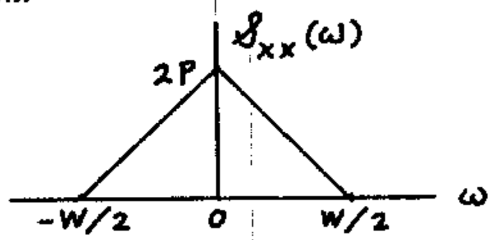
$+ P\cos[\pi(\omega+2\omega_0)/W]\operatorname{rect}[(\omega+2\omega_0)/W]\}$

$= 2P\operatorname{rect}(\omega/W)\cos(\pi\omega/W)$.

(c) From (8.6-13) $R_{XY}(\tau) = 0$ because $\mathcal{S}_{NN}(\omega)$ is even about $\omega = \omega_0$.

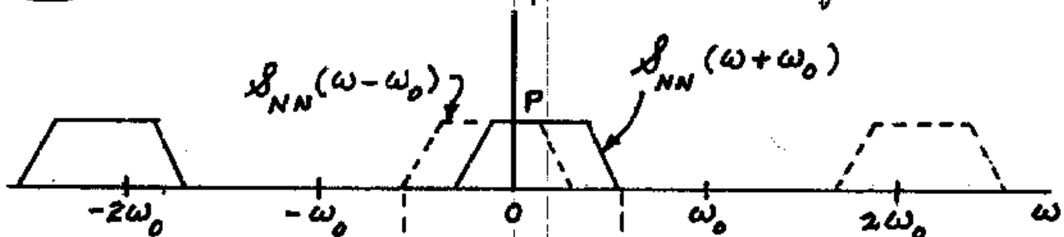
(d) Yes because $R_{XY}(\tau) = 0$.

8.6-4. (a) $\mathcal{S}_{XX}(\omega)$ is given by (8.6-16); it is the sum of two components which are the lowpass portions of $\mathcal{S}_{XX}(\omega-\omega_0)$ and $\mathcal{S}_{XX}(\omega+\omega_0)$:

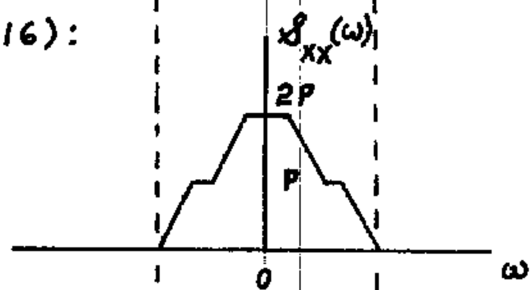


(8.6-4.) (Continued) (b) Because the lowpass components of the shifted power spectrums $\mathcal{S}_{NN}(\omega+\omega_0)$ and $\mathcal{S}_{NN}(\omega-\omega_0)$ are identical, their difference required by (8.6-18) is zero. Therefore, $\mathcal{S}_{XY}(\omega) = 0$ and a sketch is not relevant.

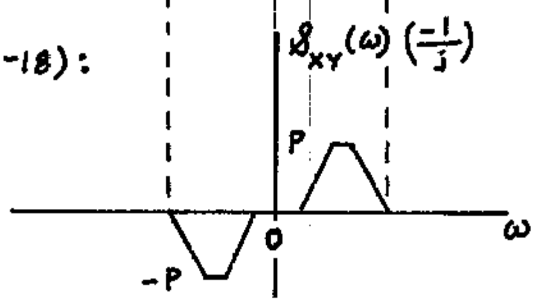
(8.6-5.) The needed shifted spectrums are first sketched.



(a) Use (8.6-16):



(b) Use (8.6-18):



*(8.6-6.) $R_{NN}(t,t+\tau) = E[N(t)N(t+\tau)] = E[\{X(t)\cos(\omega_0 t) - Y(t)\sin(\omega_0 t)\}\{X(t+\tau)\cos(\omega_0 t+\omega_0\tau) - Y(t+\tau)\cos(\omega_0 t + \omega_0\tau)\}] = E[X(t)X(t+\tau)\cos(\omega_0 t)\cos(\omega_0 t+\omega_0\tau)$

★8.6-6. (Continued) $-X(t)Y(t+\tau)\cos(\omega_0 t)\sin(\omega_0 t+\omega_0\tau)$

$-Y(t)X(t+\tau)\sin(\omega_0 t)\cos(\omega_0 t+\omega_0\tau) + Y(t)Y(t+\tau)$

$\cdot\sin(\omega_0 t)\sin(\omega_0 t+\omega_0\tau)] = R_{XX}(\tau)\frac{1}{2}\{\cos(2\omega_0 t+\omega_0\tau)$

$+\cos(\omega_0\tau)\} - R_{XY}(\tau)\frac{1}{2}\{\sin(2\omega_0 t+\omega_0\tau)+\sin(\omega_0\tau)\}$

$-R_{YX}(\tau)\frac{1}{2}\{\sin(2\omega_0 t+\omega_0\tau)-\sin(\omega_0\tau)\} + R_{YY}(\tau)\cdot$

$\cdot\frac{1}{2}\{-\cos(2\omega_0 t+\omega_0\tau)+\cos(\omega_0\tau)\}$

$$= [R_{XX}(\tau)+R_{YY}(\tau)]\frac{1}{2}\cos(\omega_0\tau)$$
$$+[R_{XX}(\tau)-R_{YY}(\tau)]\frac{1}{2}\cos(2\omega_0 t+\omega_0\tau)$$
$$-[R_{XY}(\tau)-R_{YX}(\tau)]\frac{1}{2}\sin(\omega_0\tau)$$
$$-[R_{XY}(\tau)+R_{YX}(\tau)]\frac{1}{2}\sin(2\omega_0 t+\omega_0\tau).$$

★8.6-7. (a) From (8.6-16): $\mathcal{S}_{XX}(\omega) = L_p[\mathcal{S}_{NN}(\omega-\omega_0)+\mathcal{S}_{NN}(\omega+\omega_0)]$

$$= L_p\{B[u(\omega-2\omega_0+W_1)-u(\omega-2\omega_0-W_2)]\exp[-a(\omega-2\omega_0+W_1)] \quad ①$$
$$+B[u(-\omega+W_1)-u(-\omega-W_2)]\exp[-a(-\omega+W_1)] \quad ②$$
$$+B[u(\omega+W_1)-u(\omega-W_2)]\exp[-a(\omega+W_1)] \quad ③$$
$$+B[u(-\omega-2\omega_0+W_1)-u(-\omega-2\omega_0-W_2)]\exp[-a(-\omega-2\omega_0+W_1)]\} \quad ④$$
$$= B[u(-\omega+W_1)-u(-\omega-W_2)]\exp[-a(-\omega+W_1)]$$
$$+B[u(\omega+W_1)-u(\omega-W_2)]\exp[-a(\omega+W_1)]$$

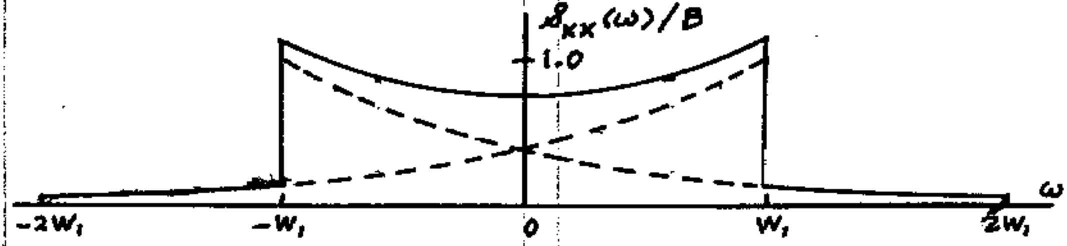
(b) From (8.6-18): $\mathcal{S}_{XY}(\omega) = jL_p\{\mathcal{S}_{NN}(\omega-\omega_0)-\mathcal{S}_{NN}(\omega+\omega_0)]$

$$= jL_p\{① + ② - ③ - ④\} = j\{② - ③\}$$
$$= jB[u(-\omega+W_1)-u(-\omega-W_2)]\exp[-a(-\omega+W_1)]$$
$$-jB[u(\omega+W_1)-u(\omega-W_2)]\exp[-a(\omega+W_1)].$$

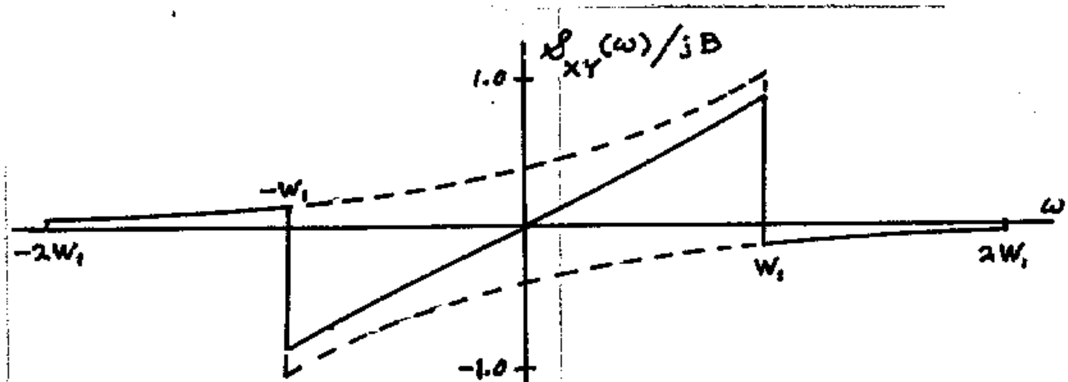
(c) $W_1 = W_2/2$, $a = 1/W_1$:

★8.6-7 (Continued)

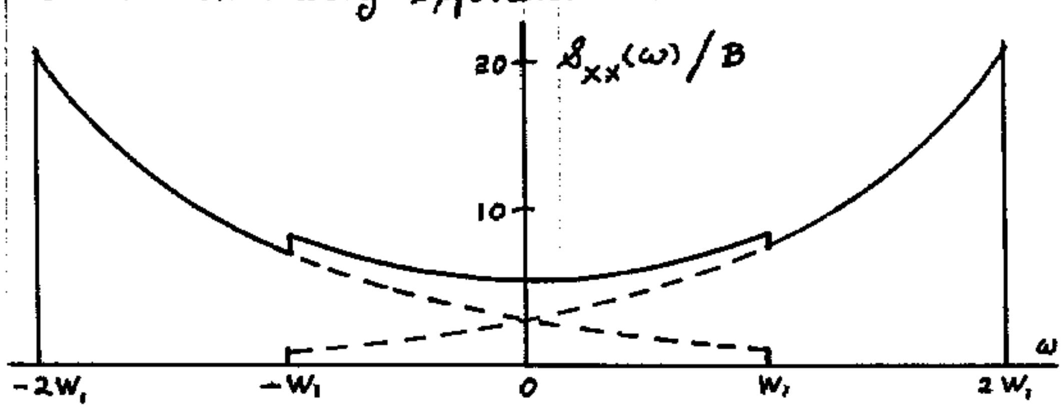
$$\mathcal{S}_{XX}(\omega) = B\,[u(-\omega+W_1) - u(-\omega-2W_1)]\exp[-(-\omega+W_1)/W_1]$$
$$+ B\,[u(\omega+W_1) - u(\omega-2W_1)]\exp[-(\omega+W_1)/W_1].$$



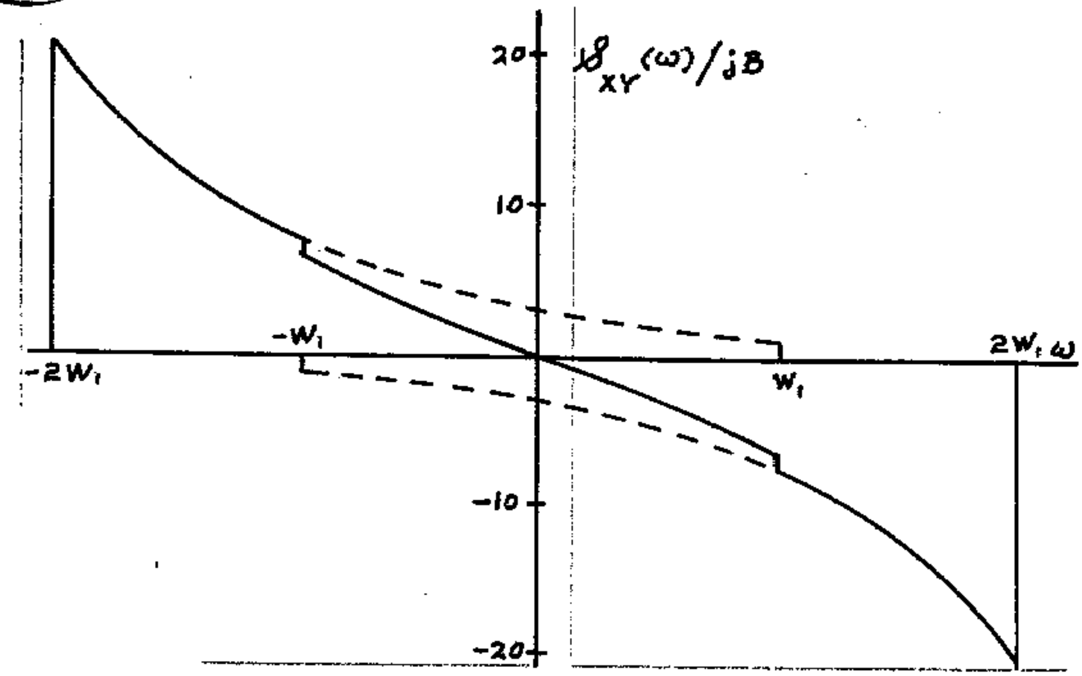
$$\mathcal{S}_{XY}(\omega) = jB\,[u(-\omega+W_1) - u(-\omega-2W_1)]\exp[-(-\omega+W_1)/W_1]$$
$$-jB\,[u(\omega+W_1) - u(\omega-2W_1)]\exp[-(\omega+W_1)/W_1]$$



(d) $W_1 = W_2/2$, $a = -1/W_1$. These conditions produce the same components as in (c) except instead of decreasing we have increasing exponentials.



*(8.6-7.) (Continued)



(8.6-8.) From (8.6-11) using (C-50):

$$R_{XX}(\tau) = \frac{1}{\pi}\int_0^{\infty} \mathcal{S}_{NN}(\omega)\cos[(\omega-\omega_0)\tau]\,d\omega$$

$$= \frac{1}{\pi}\int_{\omega_0-W_1}^{\omega_0+W_2} e^{-a(\omega-\omega_0+W_1)}\cos[(\omega-\omega_0)\tau]\,d\omega$$

$$= \frac{1}{\pi(a^2+\tau^2)}\Big\{[a\cos(W_1\tau)+\tau\sin(W_1\tau)]$$
$$-[a\cos(W_2\tau)-\tau\sin(W_2\tau)]\,e^{-a(W_1+W_2)}\Big\}.$$

From (8.6-13) using (C-49):

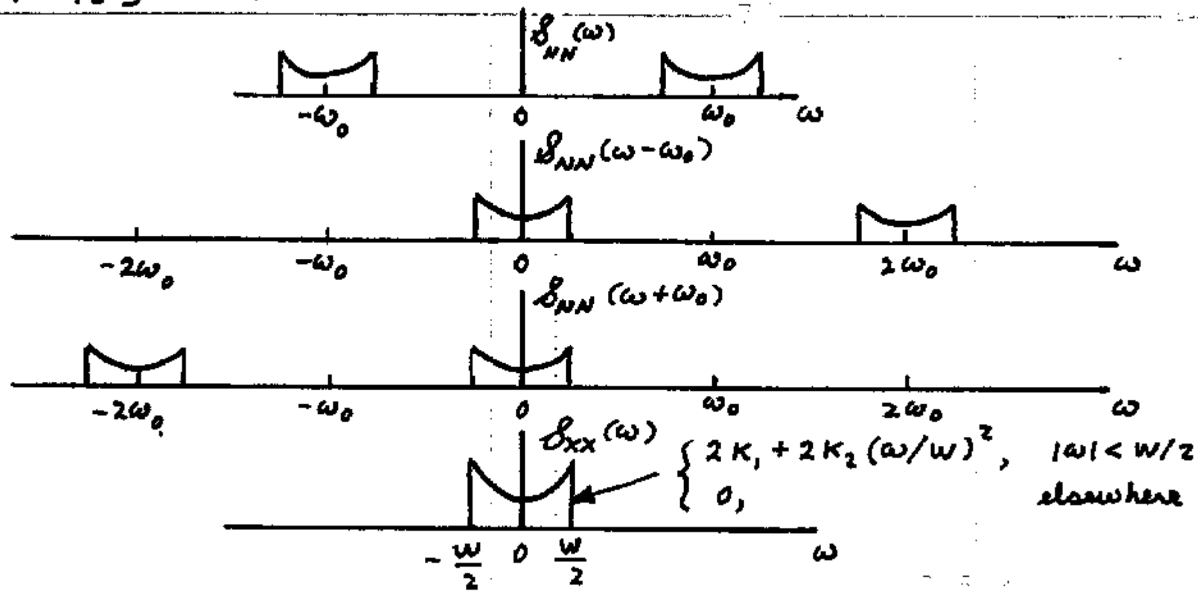
$$R_{XY}(\tau) = \frac{1}{\pi}\int_0^{\infty} \mathcal{S}_{NN}(\omega)\sin[(\omega-\omega_0)\tau]\,d\omega$$

$$= \frac{1}{\pi}\int_{\omega_0-W_1}^{\omega_0+W_2} e^{-a(\omega-\omega_0+W_1)}\sin[(\omega-\omega_0)\tau]\,d\omega$$

$$= \frac{1}{\pi(a^2+\tau^2)}\Big\{[-a\sin(W_1\tau)+\tau\cos(W_1\tau)]$$
$$-[a\sin(W_2\tau)+\tau\cos(W_2\tau)]\,e^{-a(W_1+W_2)}\Big\}.$$

(8.6-9.) (a) Power $= E[N^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{NN}(\omega)\,d\omega = \frac{1}{\pi}\int_{\omega_0-(W/2)}^{\omega_0+(W/2)} [K_1 + \frac{K_2}{W^2}(\omega-\omega_0)^2]d\omega$

$= \frac{W}{\pi}[K_1 + \frac{K_2}{12}]$. (b) From (8.6-16):



(c) From (8.6-13): $R_{XY}(\tau) = \frac{1}{\pi}\int_0^{\infty}\mathcal{S}_{NN}(\omega)\sin[(\omega-\omega_0)\tau]\,d\omega = 0$ because the integrand is odd about $\omega = \omega_0$. (d) Yes, because $R_{XY}(\tau) = 0$.

(8.6-10.) $R_{XX}(\tau) = \frac{1}{\pi}\int_0^{\infty}\mathcal{S}_{NN}(\omega)\cos[(\omega-\omega_0)\tau]\,d\omega$

$= \frac{1}{\pi}\int_{\omega_0-(W/2)}^{\omega_0+(W/2)} [K_1 + K_2(\frac{\omega-\omega_0}{W})^2]\cos[(\omega-\omega_0)\tau]\,d\omega$ $\xi = (\omega-\omega_0)\tau$, $d\xi = d\omega\,\tau$

$= \frac{1}{\pi\tau}\int_{-W\tau/2}^{W\tau/2} [K_1 + \frac{K_2\xi^2}{W^2\tau^2}]\cos(\xi)\,d\xi$ use (C-41)

$= \frac{K_1 W}{\pi} Sa\left(\frac{W\tau}{2}\right) + \frac{K_2}{\pi W\tau^2}\left\{2\cos\left(\frac{W\tau}{2}\right) + \left(\frac{W^2\tau^2}{4} - 2\right) Sa\left(\frac{W\tau}{2}\right)\right\}$.

Other forms are also possible.

8.7-1. Since $\frac{1}{T_p}\,\text{rect}\left(\frac{t}{T_p}\right) \longleftrightarrow Sa(\omega T_p/2)$

$$\frac{1}{T_p}\,\text{rect}\left(\frac{t-nT_o}{T_p}\right) \longleftrightarrow Sa(\omega T_p/2)\,e^{-jn\omega T_o}$$

$$\frac{1}{T_p}\,\text{rect}\left(\frac{t-nT_o+T_o}{T_p}\right) \longleftrightarrow Sa(\omega T_p/2)\,e^{-j\omega(nT_o-T_o)}$$

the transform of (8.7-2) becomes

$$P(\omega)=\sum_{n=-\infty}^{\infty} Sa(\omega T_p/2)\,e^{-j\omega(nT_o-T_o)}$$

$$=\frac{2\pi}{T_o}e^{j\omega T_o}Sa(\omega T_p/2)\,\underbrace{\frac{1}{\omega_o}\sum_{n=-\infty}^{\infty}e^{-jn\omega T_o}}$$

But this $=\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_o)$ as given in Problem D-8, so (8.7-3) is proved.

8.7-2 $G_s(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\xi)\,\frac{2\pi}{T_s}\,e^{j(\omega-\xi)T_o}\,Sa[(\omega-\xi)T_p/2]$

$$\cdot\sum_{n=-\infty}^{\infty}\delta(\omega-\xi-n\omega_s)\,d\xi$$

$$=\frac{1}{T_s}\sum_{n=-\infty}^{\infty}Sa[n\omega_s T_p/2]\,e^{jn\omega_s T_o}\,G(\omega-n\omega_s)$$

★ 8.7-3. $g_s(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G_s(\omega)\,e^{j\omega t}\,d\omega$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{T_s}\sum_{n=-\infty}^{\infty}G(\omega-n\omega_s)\,e^{jn\omega_s T_o}\,e^{j\omega t}\,d\omega$$

$$=\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\omega-n\omega_s)\,e^{j(\omega-n\omega_s)t}\,d\omega}_{\text{Let }\xi=\omega-n\omega_s,\ d\xi=d\omega}\,e^{jn\omega_s t+jn\omega_s T_o}$$

$=\frac{1}{T_s}\,g(t)\sum_{n=-\infty}^{\infty}e^{jn\omega_s(t+T_o)}$. Next, use the result of

★8.7-3. (Continued) Problem D-8 on, it can easily be shown that $\pi \sum_{n=-\infty}^{\infty} \frac{2}{\omega_s} \delta(t - nT_s + T_0) = \sum_{n=-\infty}^{\infty} e^{jn\omega_s(t+T_0)}$. On substituting above we have

$$g(t) = \frac{1}{T_s} g(t) \frac{2\pi}{\omega_s} \sum_{n=-\infty}^{\infty} \delta(t - nT_s + T_0)$$

$$= \sum_{n=-\infty}^{\infty} g(nT_s - T_0)\,\delta(t - nT_s + T_0)$$

8.7-4. From (8.7-20) and (8.7-21)

$$X_I(t) = A(t)\cos[\omega_d t + \Theta(t)]$$

$$X_Q(t) = A(t)\sin[\omega_d t + \Theta(t)]$$

From Fig. 8.7-6 and (8.7-19):

$$Y(t) = X_I(t)\cos(\omega_0 t) - X_Q(t)\sin(\omega_0 t)$$

$$= A(t)\cos[\omega_d t + \Theta(t)]\cos(\omega_0 t)$$

$$- A(t)\sin[\omega_d t + \Theta(t)]\sin(\omega_0 t)$$

$$= \frac{1}{2}A(t)\{\cos[\omega_0 t - \omega_d t - \Theta(t)] + \cos[\omega_0 t + \omega_d t + \Theta(t)]\}$$

$$- \frac{1}{2}A(t)\{\cos[\omega_0 t - \omega_d t - \Theta(t)] - \cos[\omega_0 t + \omega_d t + \Theta(t)]\}$$

$$= A(t)\cos[(\omega_0 + \omega_d)t + \Theta(t)] = X(t)$$

8.7-5. Choose time origin so sample times are $(1+2k)T_s/2$. This choice causes samples to be zero for all k except $k=0$ and $k=-1$ if $T_s = \pi/W_X$, since $\cos(W_X t) \rightarrow \cos\{\frac{\pi}{T_s}(\frac{1}{2}+k)T_s\} = \cos[\pi(1+2k)/2] = -\sin(k\pi) = 0$ for all k. But, the denominator is also zero when $k=0,-1$. From L'Hospital's rule for $k=0$ and -1 we have $x(0)=1$

(8.7-5.) (Continued) and $x(-\frac{T_s}{2}) = 1$. Thus,

$$x(t) = Sa\left[W_x\left(t - \frac{T_s}{2}\right)\right] + Sa\left[W_x\left(t + \frac{T_s}{2}\right)\right]$$

(8.7-6.) From (8.7-9) with a sample rate $\omega_s = 2W_x$, only the sample at $t = 0$ is nonzero because $x(kT_s) = Sa(W_x kT_s) = Sa(\frac{\pi}{T_s} kT_s) = Sa(k\pi) = 0$ all k except $k = 0$.

(8.7-7.) (a) Minimum bandwidth must at least equal the signal's spectral extent: Bandwidth $\geq 2\pi(17.5)10^3$ rad/s $= W_x$. (b) Minimum rate is the Nyquist rate: $\omega_s > 2W_x = 2(2\pi)17.5(10^3) = 2\pi(35)10^3$ rad/s.

(8.7-8.) The signal $x^2(t)$ is $x(t)$ multiplied by itself. The Fourier transform of the product of two waveforms is the convolution of the two Fourier transforms of the two signals times $1/2\pi$. This convolution <u>doubles</u> the spectral extent of the spectrum of $x^2(t)$ compared to that of $x(t)$. It is still band-limited, however, and therefore has a sampling-theory representation.

(8.7-9.) Minimum rate $= 2(W_x/2\pi)$ Hz $= 2(20\text{ kHz}) = 40$ kHz. 4 times minimum rate $= 4(40) = 160$ kHz.

(8.7-10.) $R_{X_sX_s}(\tau) = \sum_{n=-\infty}^{\infty} R_{XX}(nT_s)\delta(\tau - nT_s) = R_{XX}(\tau)\sum_{n=-\infty}^{\infty}\delta(\tau - nT_s)$

Use (7.5-3) to get $R_{X_sX_s}(\tau) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} R_{XX}(\tau)\exp(jn\omega_s\tau)$. Fourier transform using the frequency shifting property of transforms to get $\mathcal{S}_{X_sX_s}(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\mathcal{S}_{XX}(\omega - n\omega_s)$.

8.7-11. Consider the replicas for $n=0$ and $n=1$ (treat aliasing from farther-removed pairs as negligible). Halfway between the two corresponds to $\omega_s/2$. We require $[1+(\omega/W)^2]^{-2} = 0.05$ when $\omega = \omega_s/2$. Solving for ω_s we get $\omega_s = 2W\sqrt{\sqrt{20}-1} \approx 3.7267W$ rad/s.

8.7-12. Consider the replicas for $n=0$ and $n=1$ (treat aliasing from farther-removed pairs as negligible). Halfway between the two corresponds to $\omega_s/2$. We require $[1+(\omega/W)^2]^{-4} = 0.05$ when $\omega = \omega_s/2$. Solving for ω_s we get $\omega_s = 2W\sqrt{20^{1/4}-1} \approx 2.1116\,W$ rad/s.

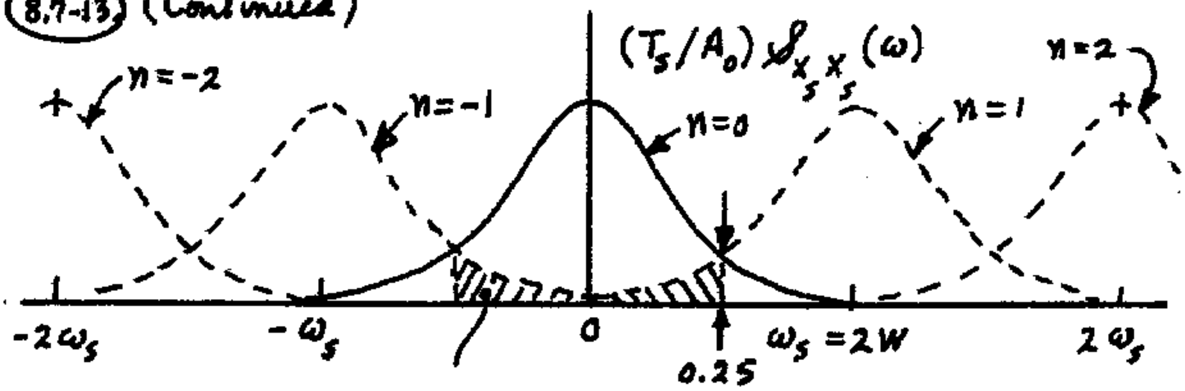
8.7-13. $\mathcal{S}_{X_sX_s}(\omega) = \dfrac{A_0}{T_s}\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{[1+\{(\omega-n\omega_s)/W\}^2]^2}$. Replicas are separated by $2W$. For the term for $n=0$ replicas cause aliasing for region $-W = -\omega_s/2 < \omega < \omega_s/2 = W$. Contributions at the regions edges are:

$$\frac{A_0}{T_s}\frac{1}{[1+(1)^2]^2} = \frac{A_0}{T_s}\frac{1}{4}, \quad \text{for } \omega = W \text{ and } n=1$$

$$\frac{A_0}{T_s}\frac{1}{[1+(-3)^2]^2} = \frac{A_0}{T_s}\frac{1}{100}, \quad \text{for } \omega = W \text{ and } n=2$$

If 1% is considered negligible, the aliasing is mainly caused by only the adjacent replicas where $n=1$ and $n=-1$. A sketch is shown below.

(8.7-13) (Continued)



aliased power for $n = \pm 1$ in region of main response when $n = 0$.

(8.7-14) (a) Take the Fourier transform of $g(t)$.

$$G(\omega) = \frac{KW_x}{8\pi}\Big\{3\,\mathcal{F}\big\langle Sa\big(W_x t - \tfrac{\pi}{2}\big) + Sa\big(W_x t + \tfrac{\pi}{2}\big)\big\rangle + \mathcal{F}\big\langle Sa\big(W_x t - \tfrac{3\pi}{2}\big) + Sa\big(W_x t + \tfrac{3\pi}{2}\big)\big\rangle\Big\}$$

But
$$Sa(W_x t) \longleftrightarrow \frac{\pi}{W_x}\,\mathrm{rect}(\omega/2W_x)$$
$$f(t \pm t_0) \longleftrightarrow F(\omega)\exp(\pm j\omega t_0)$$

so
$$Sa\Big[W_x\Big(t \pm \frac{n\pi}{2W_x}\Big)\Big] \longleftrightarrow \frac{\pi}{W_x}\,\mathrm{rect}\Big(\frac{\omega}{2W_x}\Big)e^{\pm j\omega(n\pi/2W_x)}$$

and

$$G(\omega) = \frac{KW_x}{8\pi}\Big[\frac{\pi}{W_x}\,\mathrm{rect}\Big(\frac{\omega}{2W_x}\Big)\Big]\Big[3e^{-j\omega\pi/2W_x} + 3e^{j\omega\pi/2W_x} + e^{-j\omega 3\pi/2W_x} + e^{j\omega 3\pi/2W_x}\Big]$$

$$= \frac{K}{8}\,\mathrm{rect}\Big(\frac{\omega}{2W_x}\Big)\Big[6\cos\Big(\frac{\pi\omega}{2W_x}\Big) + 2\cos\Big(\frac{3\pi\omega}{2W_x}\Big)\Big]$$ use (C-14)

$$= K\,\mathrm{rect}\Big(\frac{\omega}{2W_x}\Big)\cos^3\Big(\frac{\pi\omega}{2W_x}\Big)$$

(b)
$$g\Big(t - \frac{T_s}{2}\Big) = \frac{K}{8T_s}\Big\{3\,Sa\Big[\omega_s\Big(t - \frac{T_s}{2} - \frac{T_s}{2}\Big)/2\Big] + 3\,Sa\Big[\omega_s\Big(t - \frac{T_s}{2} + \frac{T_s}{2}\Big)/2\Big]$$

8.7-14. (Continued) $+ Sa[\omega_s(t - \frac{T_s}{2} - \frac{3T_s}{2})/2]$

$+ Sa[\omega_s(t - \frac{T_s}{2} + \frac{3T_s}{2})/2]\}$, so

$$g[t-(T_s/2)] = \frac{K}{8T_s}\{3\,Sa[\omega_s(t-T_s)/2] + 3\,Sa(\omega_s t/2) + Sa[\omega_s(t-2T_s)/2] + Sa[\omega_s(t+T_s)/2]\} \qquad (1)$$

(c) This form is that of (8.7-9) with all samples equal to zero except the four for $n = -1, 0, 1,$ and 2, where the sample values are $g(-T_s) = K/8T_s$, $g(0) = 3K/8T_s$, $g(T_s) = 3K/8T_s$, and $g(2T_s) = K/8T_s$.

(d) Time shift (1) to an earlier time by $+T_s/2$ and write

$$g(t) = \frac{K}{8T_s}\{3\,Sa[\omega_s(t-\frac{T_s}{2})/2] + 3\,Sa[\omega_s(t+\frac{T_s}{2})/2] + Sa[\omega_s(t-\frac{3T_s}{2})/2] + Sa[\omega_s(t+\frac{3T_s}{2})/2]\}$$

$$= \frac{K}{8T_s}\{3\,Sa[\omega_s(t+T_0-T_s)/2] + 3\,Sa[\omega_s(t+T_0)/2] + Sa[\omega_s(t+T_0-2T_s)/2] + Sa[\omega_s(t+T_0+T_s)/2]\}$$

$$= \sum_{n=-1}^{2} g(nT_s - T_0)\,Sa[\omega_s(t+T_0-nT_s)/2], \quad \text{where}$$

$g(-T_s-T_0) = g(-3T_s/2) = K/8T_s$ (for $n=-1$), $g(0T_s-T_0) = g(-T_s/2) = 3K/8T_s$ (for $n=0$), $g(T_s-T_0) = g(T_s/2) = 3K/8T_s$ (for $n=1$), and $g(2T_s-T_0) = g(3T_s/2) = K/8T_s$ (for $n=2$).

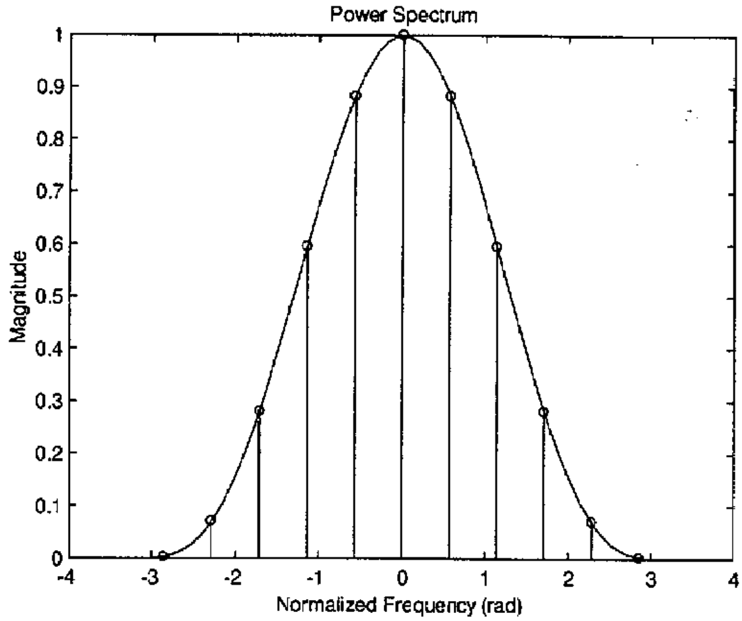
(e) This form, which is (8.7-8) with all samples zero except for the above four, shows that only four samples are needed to specify $g(t)$.

(8.7-15) The required MATLAB code is shown below.

```
%%%%%%%%%%%%%%% Problem 8.7-15 %%%%%%%%%%%%%%%%%

clear

N = 11;
M = (N-1)/2;

Wx = pi;
Ts = pi/Wx; % sample period
K = 1;

n = -M:M;

if rem(N,2) == 0
   w = -pi : 2*pi/N : pi - 2*pi/N; % frequency vector
else
   w = -pi + pi/N : 2*pi/N : pi;
end

gain = [3 3 1 1];

tmp = zeros(1,N);
Rxx = zeros(1,N);
arg = Wx*[n-pi/(2*Wx); n+pi/(2*Wx); n-3*pi/(2*Wx); n+3*pi/(2*Wx)];

for k = 1:4
   f = find(arg(k,:) ~= 0); % avoid sin(0)/0
   tmp(f) = sin(arg(k,f))./(arg(k,f));
   f = find(arg(k,:) == 0); % sin(0)/0 = 1
   tmp(f) = ones(1,length(f));

   Rxx = Rxx + K*Wx/(8*pi)*gain(k)*tmp; % autocorrelation function
end

w2 = -pi : 2*pi/51 : pi - 2*pi/51; % frequency vector
Strue = K*cos(pi*w2/(2*Wx)).^3; % true power spectrum
Sxx = abs(fftshift(fft(Rxx))); % estimated power spectrum

clf
plot(w2,Strue/K,'k')
hold
stem(w,Ts*Sxx/K,'k')

xlabel('Normalized Frequency (rad)')
ylabel('Magnitude')
title('Power Spectrum')
```

The simulation-generated power spectrum is shown as the 11-point stem plot in the figure below. The normalized frequency is $\omega/(\omega_s/2)$. For comparison the true curve

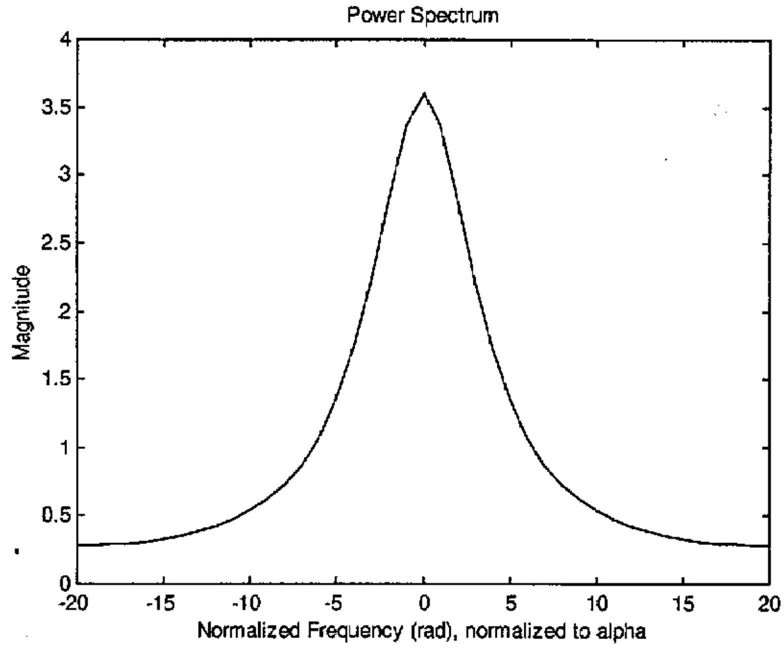
(8.7-15.) (Continued) is shown by the solid-line plot.



(8.7-16.) From Fig. 8.7-5a $X(t)$ and $Y(t)$ replace $X_I(t)$ and $X_Q(t)$. Thus, $X(t)$ can be sampled at a rate of $2(1.6)10^6 = 3.2$ M samples per second and $Y(t)$ at a rate of $2(3.2)10^6 = 6.4$ M samples per second. Two samples of $Y(t)$ must be taken for each one of $X(t)$.

(8.7-17.) Use the MATLAB code of Example 8.7-2 except with $N = 20$. A plot of $(2/\alpha)\mathcal{S}_{X_s X_s}(\omega_k)$ is shown below as a curve of linear line segments connecting calculated points. When compared with Fig. 8.7-8 for $N = 10$ points we see that the same overall function results, but the "curve" is smoother for $N = 20$. Clearly, increasing N to a much larger value will give the curve precisely.

8.7-17. (Continued)



8.8-1. $\sum_{n=-\infty}^{\infty} \delta[n] e^{-jn\Omega} = e^{-jn\Omega}\Big|_{n=0} = 1$

8.8-2. $DTFT\{\delta[n-r]\} = \sum_{n=-\infty}^{\infty} \delta[n-r]\, e^{-jn\Omega}$, But $\delta[n-r]$ $= 1$ for $n = r$ and equals 0 for $n \neq r$. Thus,

$DTFT\{\delta[n-r]\} = e^{-jr\Omega}$ so $\delta[n-r] \longleftrightarrow e^{-jr\Omega}$

8.8-3. $DTFT\{h[n-r]\} = \sum_{n=-\infty}^{\infty} h[n-r]\, e^{-jn\Omega}$ Let $m = n-r$

$$= \sum_{m=-\infty}^{\infty} h[m]\, e^{-j(m+r)\Omega} = e^{-jr\Omega} \sum_{m=-\infty}^{\infty} h[m]\, e^{-jm\Omega}$$

$$= e^{-jr\Omega} H(e^{j\Omega})$$

8.8-4. The system is BIBO stable if (8.8-10) is true.
$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} e^{-2\alpha n}$. Use (C-62) with $z = e^{-2\alpha}$,

(8.8-4.) (Continued) $N_1 = 0$ and $N_2 \to \infty$: We get

$$\sum_{n=0}^{\infty} e^{-2\alpha n} = \frac{1}{1-e^{-2\alpha}} < \infty,$$ so the system is BIBO stable.

(8.8-5.) $$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} = \sum_{n=0}^{\infty} e^{-(\alpha+j\Omega)n}$$

Use (C-62) with $N_1 = 0$, $N_2 \to \infty$ and $z = e^{-(\alpha+j\Omega)}$.

$$H(e^{j\Omega}) = [1 - e^{-(\alpha+j\Omega)}]^{-1}$$

(8.8-6.) First, the power spectrum. $\mathcal{S}_{XX}(e^{j\Omega}) =$

$$\sum_{n=-\infty}^{\infty} e^{-W_x|n| - jn\Omega} = \sum_{n=-\infty}^{0} e^{-(-W_x+j\Omega)n} - 1 + \sum_{n=0}^{\infty} e^{-(W_x+j\Omega)n}$$

Use (C-62) with $N_1 = 0$, $N_2 \to \infty$, $z = \exp(-W_x + j\Omega)$ and $z = \exp(-W_x - j\Omega)$.

$$\mathcal{S}_{xx}(e^{j\Omega}) = -1 + \frac{1}{1-e^{-(W_x+j\Omega)}} + \frac{1}{1-e^{-(W_x-j\Omega)}}$$

On reducing some algebra, we have

$$\mathcal{S}_{XX}(e^{j\Omega}) = \frac{A}{1 - B\cos(\Omega)}$$ where $A = \dfrac{e^{W_x} + e^{-W_x}}{e^{W_x} - e^{-W_x}}$, $B = \dfrac{2}{e^{W_x} - e^{-W_x}}$.

We note that this power spectrum is not band-limited since it is nonzero at $\Omega = +\pi$ and $-\pi$. It can be shown that $A > 0$ and $B > 0$ if $W_x > 0$. For $\Omega = \omega T_s = 0$, $\mathcal{S}_{XX}(e^{j0}) = \dfrac{A}{1-B}$. For $\Omega = \dfrac{\omega_s}{2} T_s = \pi$, $\mathcal{S}_{XX}(e^{j\pi}) = \dfrac{A}{1+B}$.

The ratio $\mathcal{S}_{XX}(e^{j\pi}) / \mathcal{S}_{XX}(e^{j0}) = \dfrac{1-B}{1+B}$ is required to be 0.05. Trial and error calculations give $W_x = 2.21052$.

*8.8-7. (a) $|H(e^{j\Omega})|^2 = |b_0 + b_1 e^{-j\Omega}|^2 = [b_0 + b_1\cos(\Omega)]^2 + [b_1 \sin(\Omega)]^2 = (b_0^2 + b_1^2) + 2b_0 b_1 \cos(\Omega)$. $|H(e^{j0})|^2 = (b_0+b_1)^2$, $|H(e^{j\pi/2})|^2 = (b_0^2+b_1^2)$, $|H(e^{j\pi})|^2 = (b_0 - b_1)^2$.

(b) $|H_f(e^{j0})|^2 = 1 \overset{\text{must}}{=} (b_0 + b_1)^2$, so $b_0 = 1 - b_1$.

$|H_f(e^{j\pi/2})|^2 = \left|\frac{1}{1+j(\pi/2\Omega_0)}\right|^2 = \frac{1}{1+(\pi/2\Omega_0)^2} \overset{\text{must}}{=} b_0^2 + b_1^2$
$= (1-b_1)^2 + b_1^2 = 1 - 2b_1 + 2b_1^2$. Values of b_1 that are solutions of this quadratic equation are:

$$b_1 = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1-(\pi/2\Omega_0)^2}{1+(\pi/2\Omega_0)^2}}.$$ Now b_0 is real if b_1 is real. But b_1 is real if the radical is real which requires $1-(\pi/2\Omega_0)^2 \geq 0$, or $\Omega_0 \geq \pi/2$. (c) For $\Omega_0 = \pi/2$: $b_1 = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1-1}{1+1}} = \frac{1}{2}$, $b_0 = 1 - b_1 = \frac{1}{2}$, $|H_f(e^{j\pi})|^2 = \frac{1}{1+(\pi/2[\pi/2])^2}$
$= 1/5$ while $|H(e^{j\pi})|^2 = (b_0 - b_1)^2 = (\frac{1}{2}-\frac{1}{2})^2 = 0$. Thus, $|H_f(e^{j\pi})|^2$ is in error by $1/5$ which is 20% of the peak value.

8.8-8. (a) From (8.8-16) all $a_r = 0$ and all $b_r = 0$ except $b_0 = 1$, $b_1 = 2$, and $b_2 = 1/2$. Since all $a_r = 0$ this is an FIR system with $N = 0$ and $M = 2$. (b) From (8.8-11) $Y[n] = b_0 X[n] + b_1 X[n-1] + b_2 X[n-2]$, so $R_{YY}[k] = E\{Y[n]Y[n+k]\} = E\{(b_0 X[n] + b_1 X[n-1] + b_2 X[n-2])(b_0 X[n+k] + b_1 X[n+k-1] + b_2 X[n+k-2])\}$. After some algebraic reduction

$$R_{YY}[k] = (b_0^2+b_1^2+b_2^2)R_{XX}[k] + (b_0+b_2)b_1 R_{XX}[k-1] + b_0 b_2 R_{XX}[k-2] + (b_0+b_2)b_1 R_{XX}[k+1] + b_0 b_2 R_{XX}[k+2]$$

8.8-8. (Continued) After values of the constants are inserted:

$$R_{YY}[k] = \frac{21}{4} a^{|k|} + 3\left(a^{|k-1|} + a^{|k+1|}\right) + \frac{1}{2}\left(a^{|k-2|} + a^{|k+2|}\right)$$

8.8-9. (a) Here $Y[n] = \sum_{i=0}^{3} b_i X[n-i]$, so

$R_{YY}[k] = E\left\{\sum_{i=0}^{3}\sum_{j=0}^{3} b_i b_j X[n-i]X[n+k-j]\right\}$. Now there are 16 terms having factors of the form $E\{X[n-i]X[n+k-j]\} = R_{XX}[k+i-j]$. On grouping these terms we find that

$$R_{YY}[k] = (b_0^2+b_1^2+b_2^2+b_3^2)R_{XX}[k] + (b_0b_1+b_1b_2+b_2b_3)(R_{XX}[k-1] + R_{XX}[k+1]) + (b_0b_2+b_1b_3)(R_{XX}[k-2]+R_{XX}[k+2]) + b_0b_3\left(R_{XX}[k-3]+R_{XX}[k+3]\right)$$

But $R_{XX}[k] = \sigma_X^2$ when $k=0$ and equals 0 when $k \neq 0$. Thus,

$$R_{YY}[k] = \begin{cases} (b_0^2+b_1^2+b_2^2+b_3^2)\sigma_X^2, & k=0 \\ (b_0b_1+b_1b_2+b_2b_3)\sigma_X^2, & k=+1 \text{ and } -1 \\ (b_0b_2+b_1b_3)\sigma_X^2, & k=+2 \text{ and } -2 \\ b_0b_3\sigma_X^2, & k=+3 \text{ and } -3 \\ 0, & \text{all other } k \end{cases} \quad (1)$$

(b) From (8.8-13) $\mathcal{S}_{X_sX_s}(\omega) = \sum_{k=-\infty}^{\infty} R_{YY}[k]e^{-jn\omega T_s}$ On substitution of (1) and reduction of some algebra, we have

$$\begin{aligned}\mathcal{S}_{X_sX_s}(\omega) = {} & (b_0^2+b_1^2+b_2^2+b_3^2)\sigma_X^2 \\ & +2(b_0b_1+b_1b_2+b_2b_3)\sigma_X^2\cos(\omega T_s) \\ & +2\ (b_0b_2+b_1b_3)\sigma_X^2\cos(2\omega T_s) \\ & +2\ (b_0b_3)\sigma_X^2\cos(3\omega T_s)\end{aligned}$$

(8.8-10) From (8.8-22) for any DT system $R_{YY}[n] = \sum_{m=-\infty}^{\infty} h[m] \sum_{p=-\infty}^{\infty} h[p] R_{XX}[n+m-p]$, but $R_{XX}[n+m-p] = 0$ except for $p = n+m$ so $R_{YY}[n] = \sum_{m=-\infty}^{\infty} h[m] \sum_{p=-\infty}^{\infty} h[p]\sigma_X^2 \delta[n+m-p]$

$= \sum_{m=-\infty}^{\infty} h[m]h[n+m]\sigma_X^2$ which was to be proved.

(8.8-11.) $R_{YY}[n] = \sigma_X^2 \sum_{m=-\infty}^{\infty} h[m]h[m+n]$, but $h[m] = b_m$ for $m = 0, 1, \cdots, M$ and is 0 for $m < 0$ and $m > M$ from (8.8-12)

so
$$R_{YY}[n] = \sigma_X^2 \sum_{m=0}^{M-n} b_m b_{m+n}, \quad 0 \le n \le M$$
$$= \sigma_X^2 \sum_{m=-n}^{M} b_m b_{m+n}, \quad -M \le n \le 0$$

In the last sum let $k = -n = |n|$, so $k \ge 0$ for $n \le 0$ and $\sigma_X^2 \sum_{m=k}^{M} b_m b_{m-k}$. Next let $r = m-k$ so $\sigma_X^2 \sum_{r=0}^{M-k} b_{r+k} b_r$

$= \sigma_X^2 \sum_{m=0}^{M-k} b_m b_{m+k} = \sigma_X^2 \sum_{m=0}^{M-|n|} b_m b_{m+|n|}, \quad -M \le n \le 0.$ replace $m = r$

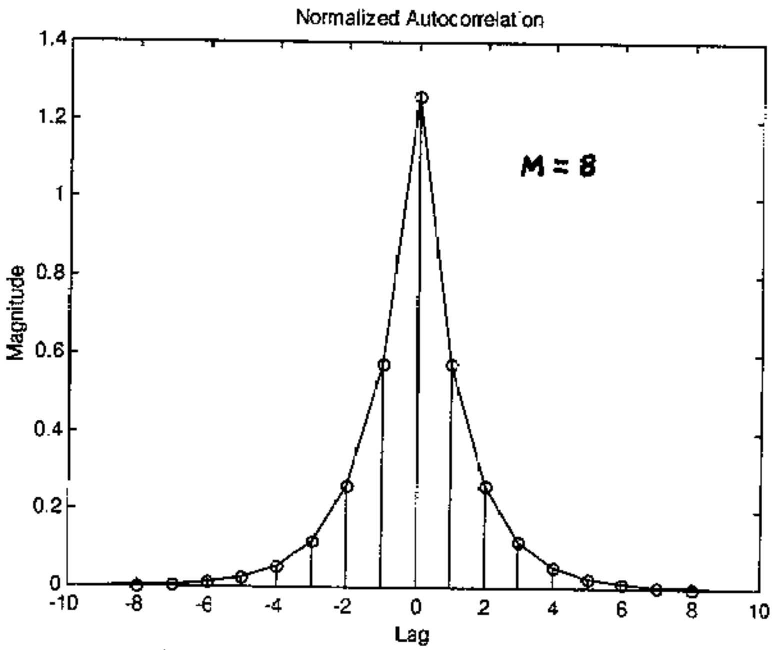
Thus,
$$R_{YY}[n] = \sigma_X^2 \sum_{m=0}^{M-|n|} b_m b_{m+|n|}, \quad 0 \le |n| \le M.$$

M (8.8-12.) Our procedure is to first calculate b_n/α from the given function for $n = 0, 1, \cdots, M$. Next, we use MATLAB to compute $R_{YY}[n]/(\alpha^2\sigma_X^2)$ from the expression given in Problem 8.8-11. Results are shown as stem plots for $-M \le n \le M$ where here we calculate two cases: (a) $M = 8$ and (b) $M = 2$.

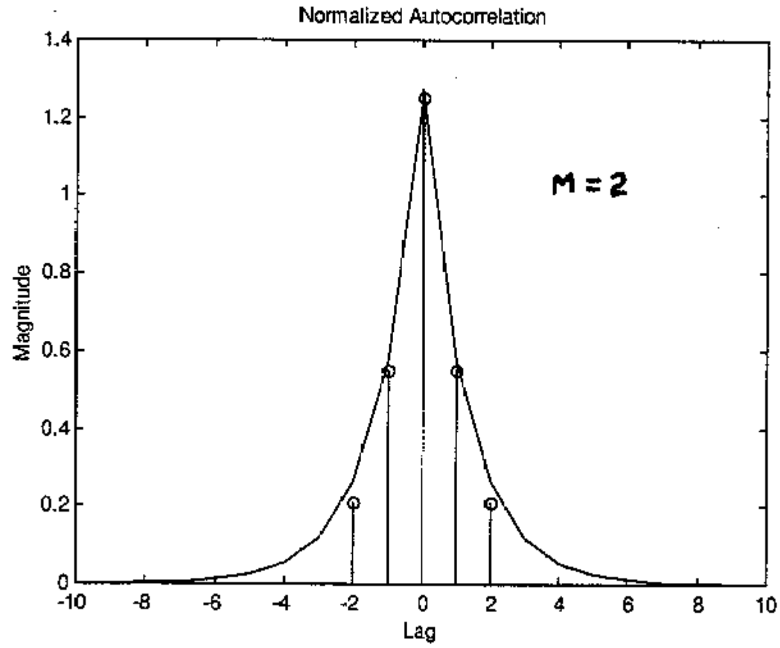
8.8-12. (Continued) The MATLAB code used is given below.

```
%%%%%%%%%%%%%%%% Problem 8.8-12 %%%%%%%%%%%%%%%%%%%

clear

M = 2;   % M should be even
alpha = 1;
Ts = pi/4;
xvar = 1;

n = 0:M;
b = exp(-alpha*n*Ts); % b(n)/alpha

Ryy = zeros(1,M+1); % initialize
for k = -M:M
   ndx1 = max([1 1+k]):min([M+1+k M+1]);
   ndx2 = max([1 1-k]):min([M+1-k M+1]);
   Ryy(k+M+1) = sum(b(ndx1).*b(ndx2));   % auto-correlation normalized by (alpha*sigma)^2
end

L = 128;
k = -L:L;
tmp = (1-exp(-2*alpha*(L-abs(k)+1)*Ts))./(1-exp(-2*alpha*Ts));
Rtrue = exp(-alpha*abs(k)*Ts).*tmp; % true value of normalized auto-correlation

clf
stem(-M:M,Ryy,'k')
hold
plot(-10:10,Rtrue(L+1-10:L+1+10),'k')

xlabel('Lag')
ylabel('Magnitude')
title('Normalized Autocorrelation')
```

Results are shown in the figure at the right as a stem plot when $M = 8$. Also shown is the exact result for $M \to \infty$ as found from the expression given in Problem 8.8-10.



(8.8-12) (Continued) The corresponding results for $M=2$ are shown in the following figure. From the two figures it is clear that M should be a large number to obtain high definition of the true autocorrelation.



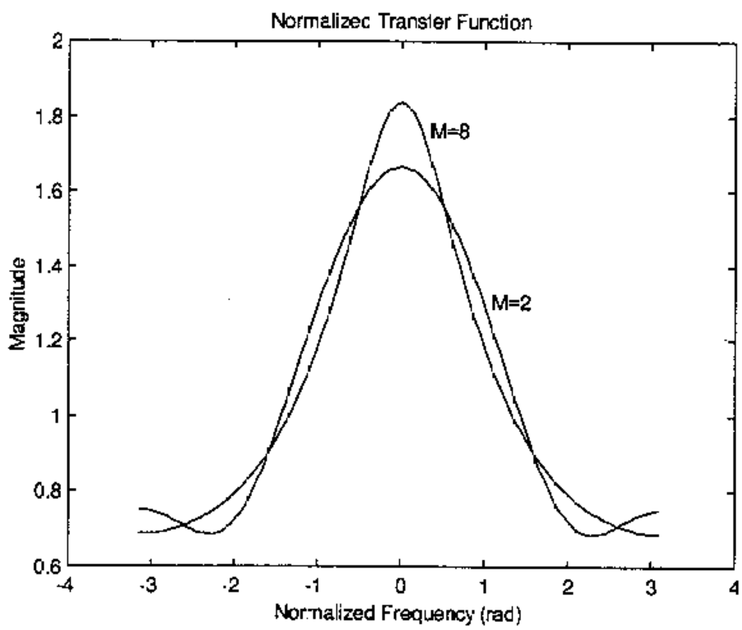
(8.8-13) We use the expression of Problem 8.8-12 to calculate b_n/α. From (8.8-15) we have

$$\left|\frac{H(e^{j\Omega})}{\alpha}\right| = \left|\sum_{n=-M}^{M} \frac{h[n]}{\alpha} e^{-jn\Omega}\right| = \left|\sum_{n=-M}^{M} e^{-(\alpha T_s + j\Omega)n}\right|$$

MATLAB was used to find this result by using its fast Fourier transform operation. The code used is shown on the next page along with computed results for $M=2$ and $M=8$ when $\alpha T_s = \pi/4$. We see that for the larger value of M the computed result is much closer to the single-pole lowpass function of Problem 8.8-12.

M 8.8-13. (Continued)

```
%%%%%%%%%%%%%%% Problem 8.8-13 %%%%%%%%%%%%%%%%%%

clear

M = 8;   % M should be even
alpha = 1;
Ts = pi/4;
xvar = 1;

n = 0:M;
h = exp(-alpha*n*Ts); % h(n)/alpha

N = 128;
H = abs(fftshift(fft(h,N))); % estimated transfer function magnitude

if rem(N,2) == 0
   w = -pi : 2*pi/N : pi - 2*pi/N; % frequency vector
else
   w = -pi + pi/N : 2*pi/N : pi;
end

plot(w,H,'k')

xlabel('Normalized Frequency (rad)')
ylabel('Magnitude')
title('Normalized Transfer Function')
```

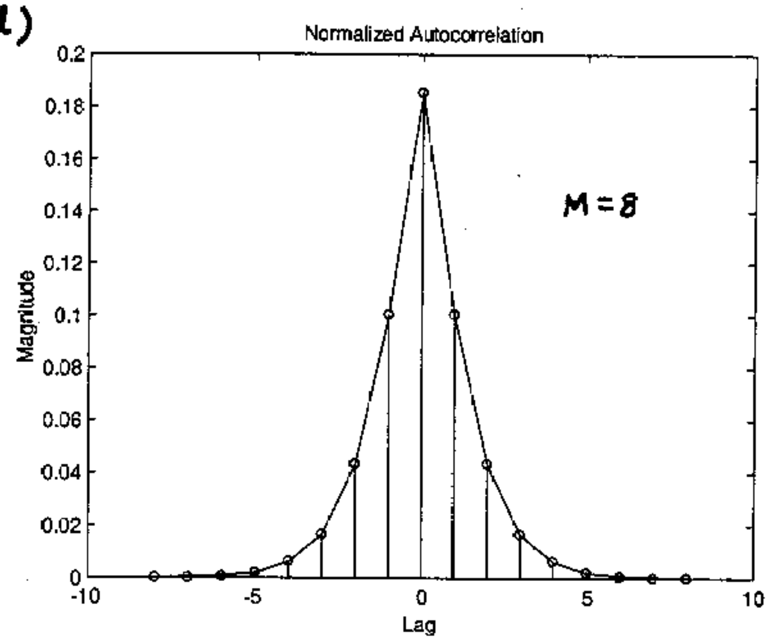


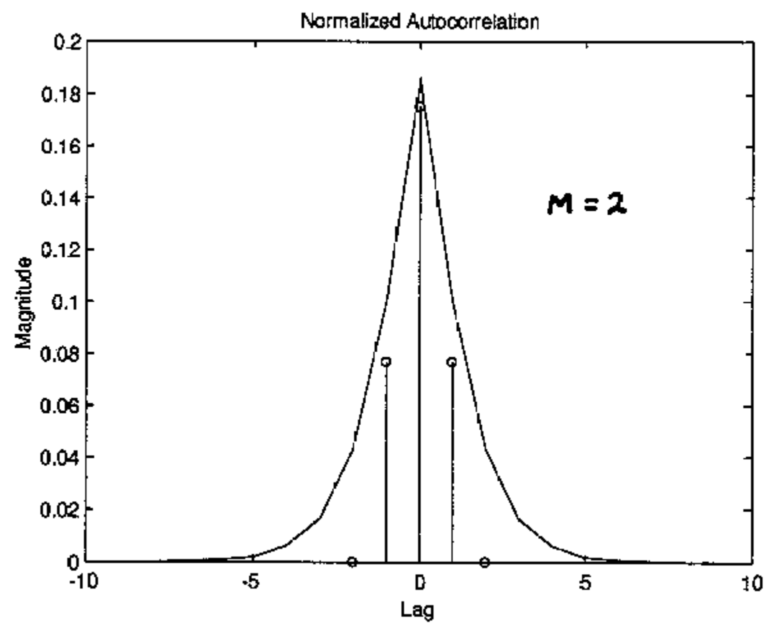
M **8.8-14** The MATLAB code is given below.

```
%%%%%%%%%%%%%%% Problem 8.8-14 %%%%%%%%%%%%%%%%%%

clear

M = 8;  % M should be even
alpha = 1;
Ts = pi/(4*sqrt((sqrt(2)-1)));
xvar = 1;

n = 0:M;
b = alpha*n*Ts.*exp(-alpha*n*Ts); % b(n)/alpha

Ryy = zeros(1,M+1); % initialize
for k = -M:M
   ndx1 = max([1 1+k]):min([M+1+k M+1]);
   ndx2 = max([1 1-k]):min([M+1-k M+1]);
   % auto-correlation normalized by (alpha*sigma)^2
   Ryy(k+M+1) = sum(b(ndx1).*b(ndx2));
end

L = 128;
k = -L:L;

N = L - abs(k);
a = exp(-2*alpha*Ts);

tmp = ((N+1).*(N-1).*a.^N - N.^2.*a.^(N-1));
tmp1 = ((1-a).*tmp + 2*(1-N.*a.^N+(N-1).*a.^(N+1)))./((1-a).^3);
tmp2 = (abs(k)+1)./a.*(1-N.*a.^N+(N-1).*a.^(N+1))./((1-a).^2);

% true value of normalized auto-correlation
Rtrue = a.^2.*(alpha*Ts)^2*exp(-alpha*abs(k)*Ts).*(tmp1+tmp2);

clf
stem(-M:M,Ryy,'k')
hold
plot(-10:10,Rtrue(L+1-10:L+1+10),'k')

xlabel('Lag')
ylabel('Magnitude')
title('Normalized Autocorrelation')
```

The calculated results are shown below for $M=8$ and $M=2$. It is clear that $M=2$ is too small to give adequate definition of the true result shown by the solid line. Calculated results are given as the stem plots.

8.8-14. (Continued)

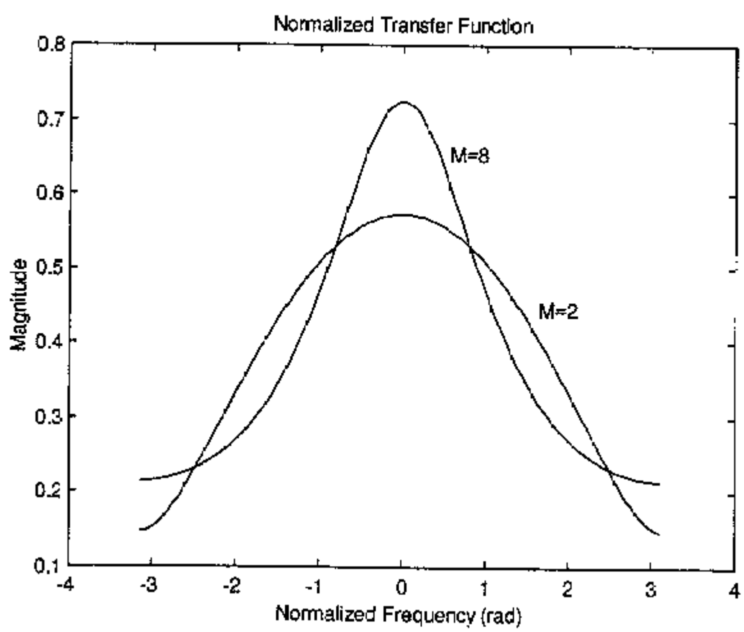




8.8-15. The MATLAB code and computed results for M = 8 and M = 2 are given below. It is found that larger M produces more accurate results.

8.8-15 (Continued)

```
%%%%%%%%%%%%%%% Problem 8.8-15 %%%%%%%%%%%%%%%%%%

clear

M = 8;  % M should be even
alpha = 1;
Ts = pi/(4*sqrt((sqrt(2)-1)));
xvar = 1;

n = 0:M;
h = alpha*n*Ts.*exp(-alpha*n*Ts); % h(n)/alpha

N = 128;
H = abs(fftshift(fft(h,N))); % estimated transfer function magnitude

if rem(N,2) == 0
   w = -pi : 2*pi/N : pi - 2*pi/N; % frequency vector
else
   w = -pi + pi/N : 2*pi/N : pi;
end

plot(w,H,'k')

xlabel('Normalized Frequency (rad)')
ylabel('Magnitude')
title('Normalized Transfer Function')
```

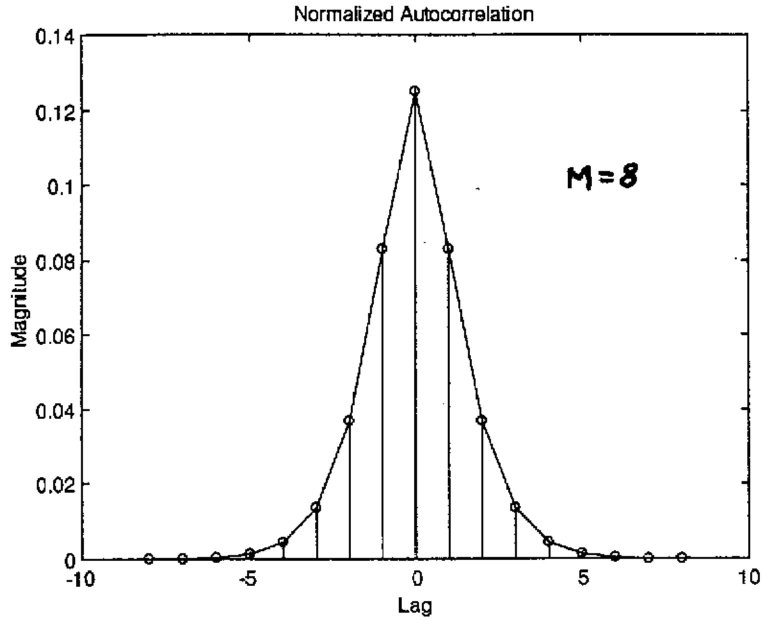


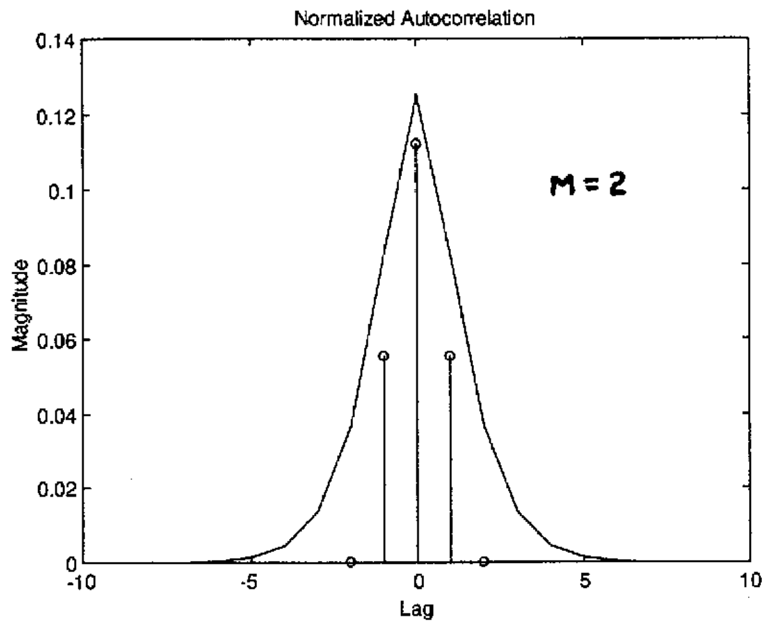
8.8-16 The MATLAB code is given below.

```
%%%%%%%%%%%%%%% Problem 8.8-16 %%%%%%%%%%%%%%%%%

clear

M = 8;  % M should be even
alpha = 1;
Ts = pi/(4*sqrt((2^0.333-1)));
xvar = 1;

n = 0:M;
b = 0.5*alpha^2*(n*Ts).^2.*exp(-alpha*n*Ts); % b(n)/alpha

Ryy = zeros(1,M+1); % initialize
for k = -M:M
   ndx1 = max([1 1+k]):min([M+1+k M+1]);
   ndx2 = max([1 1-k]):min([M+1-k M+1]);
   % auto-correlation normalized by (alpha*sigma)^2
   Ryy(k+M+1) = sum(b(ndx1).*b(ndx2));
end

M2 = 128;
n2 = 0:M2;
b2 = 0.5*alpha^2*(n2*Ts).^2.*exp(-alpha*n2*Ts); % b(n)/alpha

Rtrue = zeros(1,M+1); % initialize
for k = -M2:M2
   ndx1 = max([1 1+k]):min([M2+1+k M2+1]);
   ndx2 = max([1 1-k]):min([M2+1-k M2+1]);
   % auto-correlation normalized by (alpha*sigma)^2
   Rtrue(k+M2+1) = sum(b2(ndx1).*b2(ndx2));
end

clf
stem(-M:M,Ryy,'k')
hold
plot(-10:10,Rtrue(M2+1-10:M2+1+10),'k')

xlabel('Lag')
ylabel('Magnitude')
title('Normalized Autocorrelation')
```

The computed results are shown below as the stem plots (for $M=8$ and $M=2$), along with the exact results. Clearly larger M is necessary to give reasonable definition of the true function.

8.8-16. (Continued)

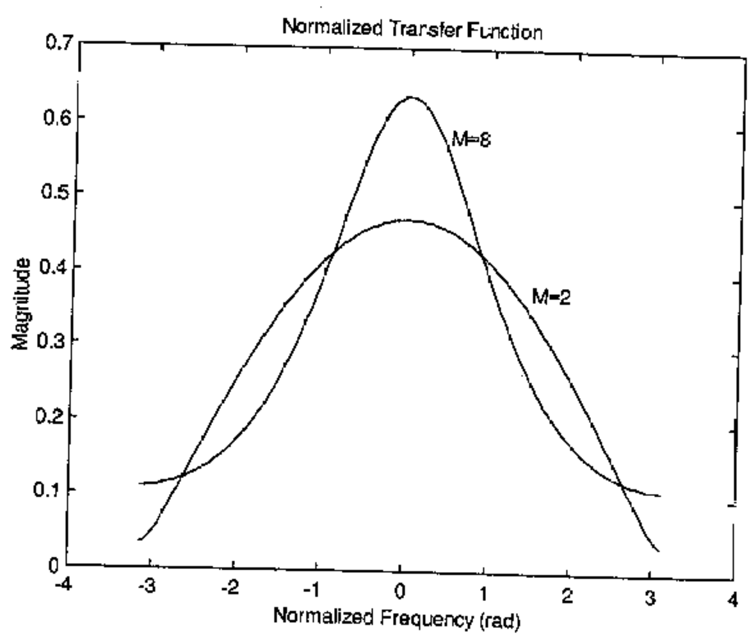




8.8-17. The MATLAB code and computed results are given below for M = 8 and M = 2. As with other systems the best results are obtained for larger M.

8.8-17. (Continued)

```
%%%%%%%%%%%%%%% Problem 8.8-17 %%%%%%%%%%%%%%%%%

clear

M = 8;  % M should be even
alpha = 1;
Ts = pi/(4*sqrt((2^0.333-1)));
xvar = 1;

n = 0:M;
h = 0.5*alpha^2*(n*Ts).^2.*exp(-alpha*n*Ts); % h(n)/alpha

N = 128;
H = abs(fftshift(fft(h,N))); % estimated transfer function magnitude

if rem(N,2) == 0
   w = -pi : 2*pi/N : pi - 2*pi/N; % frequency vector
else
   w = -pi + pi/N : 2*pi/N : pi;
end

plot(w,H,'k')

xlabel('Normalized Frequency (rad)')
ylabel('Magnitude')
title('Normalized Transfer Function')
```



8.8-18 From (8.8-6) $Y[n] = \sum_{m=-\infty}^{\infty} X[m]\,h[n-m] = \sum_{m=0}^{\infty} a^m h[n-m]$

But $h[n-m]=0$ for $n-m<0$ or $m>n$ so

$$Y[n] = \sum_{m=0}^{n} a^m e^{-\alpha(n-m)} \text{ for } n\geq 0 = u[n]\,e^{-\alpha n}\sum_{m=0}^{n} a^m e^{\alpha m}$$

$$= u[n]\,e^{-\alpha n}\sum_{m=0}^{n}(ae^{\alpha})^m.$$ Reduce by using (C-62)

with $N_1=0$, $N_2=n$ and $w=ae^{\alpha}$:

$$Y[n] = u[n]\,e^{-\alpha n}\,\frac{1+(ae^{\alpha})^{n+1}}{1-(ae^{\alpha})} = u[n]\,\frac{a^{n+1}+e^{-(n+1)\alpha}}{a-e^{-\alpha}}$$

8.8-19 A necessary and sufficient condition for BIBO stability is (8.8-10): $\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=0}^{\infty} e^{-\alpha n}$ since $\alpha>0$ and $n>0$. Thus, from (C-62) with $N_1=0$, $N_2=\infty$ and $w=e^{-\alpha}$, we get $\sum_{n=-\infty}^{\infty}|h[n]| = \frac{1}{1-e^{-\alpha}}<\infty$ so $h[n]$ is BIBO stable.

8.8-20 From (8.8-10) $\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=-\infty}^{\infty}\left|u(nT_s)\alpha^2 nT_s\,e^{-\alpha nT_s}\right|$

$= \alpha^2 T_s\sum_{n=1}^{\infty} n\,e^{-\alpha T_s n}$. But, from Dwight, Eq. (7), p.2:

$$\sum_{n=1}^{\infty} n\,x^{n-1} = \frac{1}{x}\sum_{n=1}^{\infty} n\,x^n = \frac{1}{(1-x)^2}, \quad 0<x<1.$$ Thus,

since $\alpha T_s>0$:

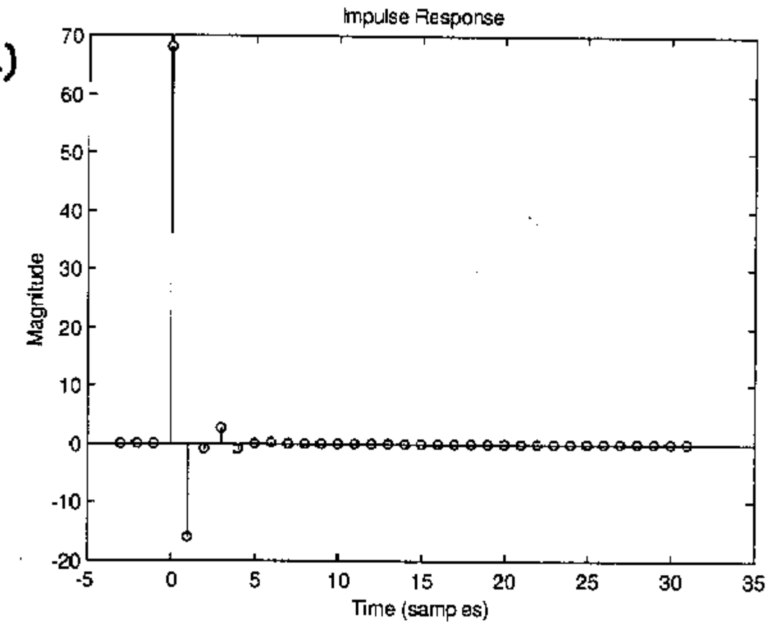
$$\sum_{n=-\infty}^{\infty}|h[n]| = \alpha^2 T_s\,\frac{e^{-\alpha T_s}}{(1-e^{-\alpha T_s})^2}<\infty,$$ then $h[n]$ is stable.

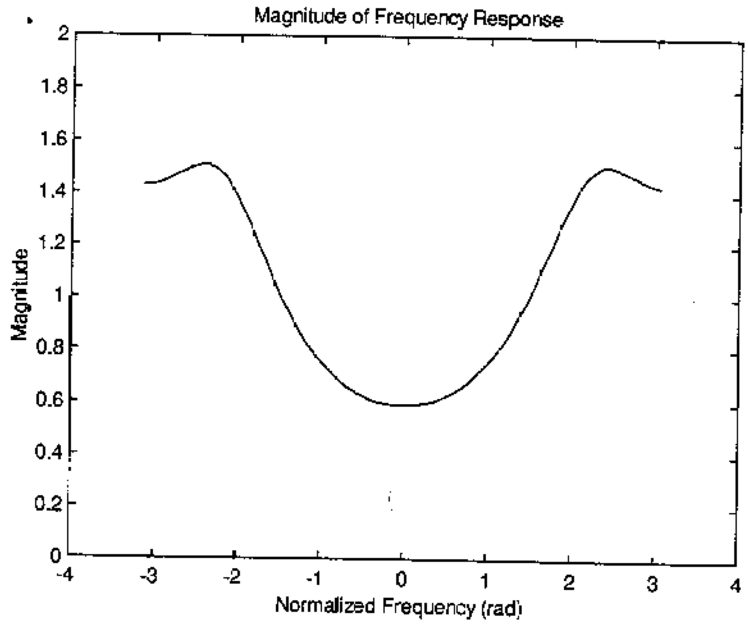
8.8-21. Here $H(e^{j\Omega}) = \dfrac{1}{1+0.5e^{-j\Omega}+0.2e^{-j2\Omega}}$. We use $N=32$ values of $\Omega_k = k\pi/32$ and the FFT of MATLAB to generate 32 values of $h[n]$. The MATLAB code is:

```
%%%%%%%%%%%%%%%% Problem 8.8-21 %%%%%%%%%%%%%%%%%%

clear

N = 32;
N2 = N*2;

if rem(N2,2) == 0
   w = -pi : 2*pi/N2 : pi - 2*pi/N2; % frequency vector
else
   w = -pi + pi/N2 : 2*pi/N2 : pi;
end

H = 1./abs(1 + 0.5*exp(-j*w) + 0.2*exp(-j*2*w)); % magnitude frequency response
h = real(fft(fftshift(H))); % impulse response

clf
stem(-3:N-1,[0 0 0 h(1:N)],'k')
hold on
plot([-5 35],[0 0],'k')

xlabel('Time (samples)')
ylabel('Magnitude')
title('Impulse Response')

figure
plot(w,H,'k')
axis([-4 4 0 2])

xlabel('Normalized Frequency (rad)')
ylabel('Magnitude')
title('Magnitude of Frequency Response')
```

The impulse response $h[n]$ is shown below as a stem plot. Also shown is a plot of $|H(e^{j\Omega})|$; it reveals that our choice of coefficients has produced a highpass type of filter. The dc response of the filter is nonzero, as readily found from the equation given above when $\Omega = 0$.

8.8-21. (Continued)

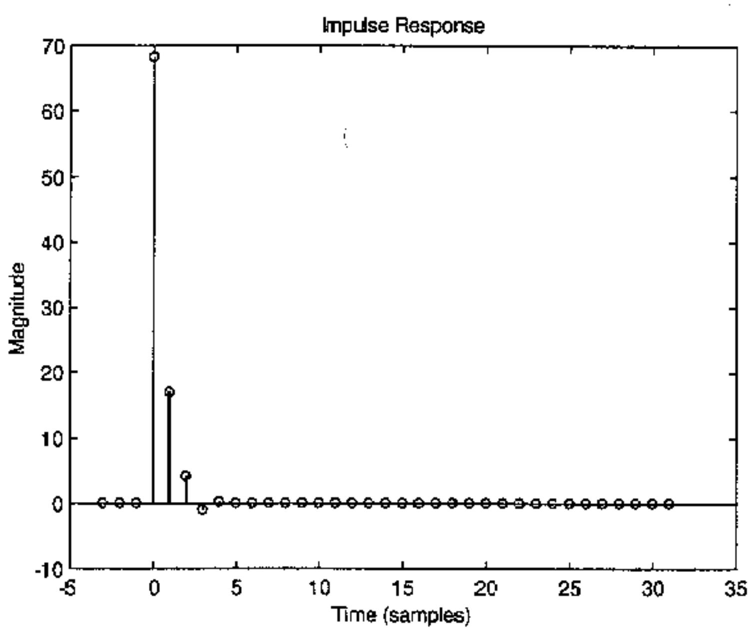




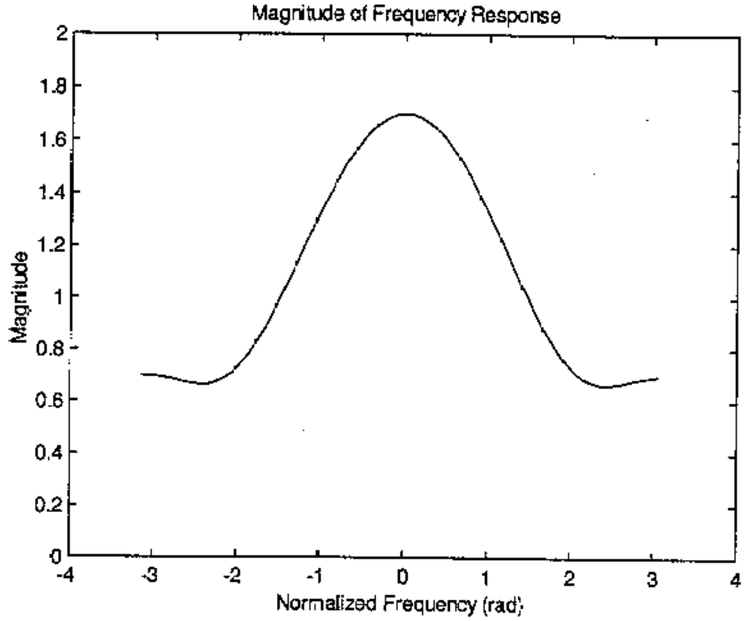
8.8-22. Here $H(e^{j\Omega}) = 1 + 0.5e^{-j\Omega} + 0.2e^{-j2\Omega}$. The MATLAB code is shown below. Also shown is a plot of $|H(e^{j\Omega})|$ which clearly shows that this choice of coefficients leads to a lowpass type of system, but its higher-frequency response is not significantly reduced relative to

M (8.8-22) (Continued) the maximum at $\Omega = 0$.

```
%%%%%%%%%%%%%%%%% Problem 8.8-22 %%%%%%%%%%%%%%%%%%%

clear

N = 32;
N2 = N*2;

if rem(N2,2) == 0
   w = -pi : 2*pi/N2 : pi - 2*pi/N2; % frequency vector
else
   w = -pi + pi/N2 : 2*pi/N2 : pi;
end

H = abs(1 + 0.5*exp(-j*w) + 0.2*exp(-j*2*w)); % magnitude frequency response
h = real(fft(fftshift(H))); % impulse response

clf
stem(-3:N-1,[0 0 0 h(1:N)],'k')
hold on
plot([-5,35],[0 0],'k')

xlabel('Time (samples)')
ylabel('Magnitude')
title('Impulse Response')

figure
plot(w,H,'k')
axis([-4 4 0 2])

xlabel('Normalized Frequency (rad)')
ylabel('Magnitude')
title('Magnitude of Frequency Response')
```



8.8-22. (Continued)



8.9-1. (a) $V = \frac{1}{2T}\int_{-T}^{T} n(t-\tau_T)n(t-\tau_R)\,dt$. Since T is large $V \approx \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} n(t-\tau_T)n(t-\tau_R)\,dt$

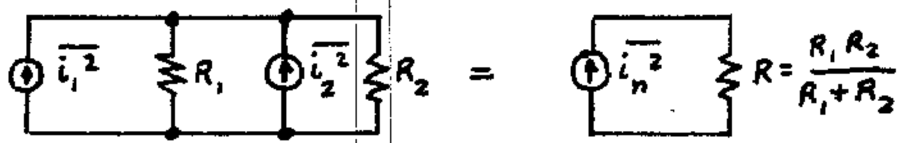
$$= A[n(t-\tau_T)n(t-\tau_R)] = E[N(t-\tau_T)N(t-\tau_R)]$$

since $N(t)$ is ergodic. Thus, $V \approx R_{NN}(\tau_R - \tau_T)$.

(b) Because $|R_{NN}(\tau)| \le R_{NN}(0)$ for any autocorrelation function, V will be maximum if $\tau_R = \tau_T$.

(c) From part (b) V is maximum when receiver delay τ_T equals target delay τ_R. By slowly varying τ_T until V is maximum R can then be found from $R = v\tau_T/2$, τ_T being that for which V is maximum.

8.9-2. Use the current form of equivalent circuit.

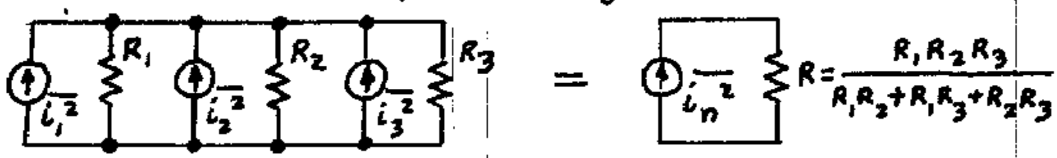


(a) $\overline{i_n^2} = \overline{i_1^2} + \overline{i_2^2} = \dfrac{2kT_1\,d\omega}{\pi R_1} + \dfrac{2kT_2\,d\omega}{\pi R_2}$ where

$\overline{i_n^2} = \dfrac{2kT_s\,d\omega}{\pi R}$. Thus, $T_s = \left(\dfrac{T_1}{R_1} + \dfrac{T_2}{R_2}\right)R = \dfrac{T_1 R_2 + T_2 R_1}{R_1 + R_2}$.

(b) If $T_1 = T_2 = T$ then $T_s = T$.

8.9-3. Use the current form of equivalent circuit.

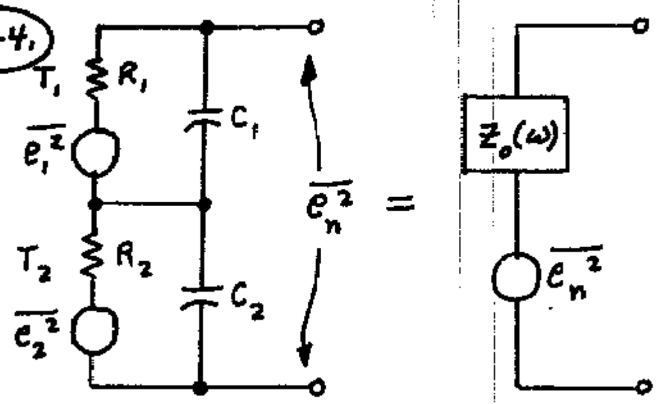


(a) $\overline{i_n^2} = \dfrac{2k}{\pi}\left(\dfrac{T_s}{R}\right)d\omega = \overline{i_1^2} + \overline{i_2^2} + \overline{i_3^2} = \dfrac{2k}{\pi}\left[\dfrac{T_1}{R_1} + \dfrac{T_2}{R_2} + \dfrac{T_3}{R_3}\right]d\omega$.

Thus, $T_s = R\left(\dfrac{T_1}{R_1} + \dfrac{T_2}{R_2} + \dfrac{T_3}{R_3}\right) = \dfrac{T_1 R_2 R_3 + T_2 R_1 R_3 + T_3 R_1 R_2}{R_1 R_3 + R_1 R_2 + R_2 R_3}$.

(b) If $T_1 = T_2 = T_3 = T$ then $T_s = T$.

8.9-4.



$Z_o(\omega) = R_o(\omega) + jX_o(\omega)$

$$z_1 = \frac{R_1\left(\frac{1}{j\omega C_1}\right)}{R_1 + \frac{1}{j\omega C_1}} = \frac{R_1}{1 + j\omega R_1 C_1} = \frac{R_1}{1 + \omega^2 R_1^2 C_1^2} - j\,\frac{\omega R_1^2 C_1}{1 + \omega^2 R_1^2 C_1^2}$$

8.9-4. (Continued) $Z_2 = \dfrac{R_2}{1+\omega^2 R_2^2 C_2^2} - j\,\dfrac{\omega R_2^2 C_2}{1+\omega^2 R_2^2 C_2^2}$.

$$Z_o(\omega) = Z_1 + Z_2 = \underbrace{\left[\frac{R_1}{1+\omega^2 R_1^2 C_1^2} + \frac{R_2}{1+\omega^2 R_2^2 C_2^2}\right]}_{R_o(\omega)} - j\underbrace{\left[\frac{\omega R_1^2 C_1}{1+\omega^2 R_1^2 C_1^2} + \frac{\omega R_2^2 C_2}{1+\omega^2 R_2^2 C_2^2}\right]}_{-X_o(\omega)}$$

Let $\overline{e_{n1}^2}$ and $\overline{e_{n2}^2}$ be the mean-squared voltages across capacitors C_1 and C_2, respectively due to resistor sources $\overline{e_1^2}$ and $\overline{e_2^2}$, respectively.

$$\overline{e_{n1}^2} = \overline{e_1^2}\left|\frac{1}{1+j\omega R_1 C_1}\right|^2 = \frac{2kT_1R_1\,d\omega/\pi}{1+\omega^2 R_1^2 C_1^2}$$

$$\overline{e_{n2}^2} = \overline{e_2^2}\left|\frac{1}{1+j\omega R_2 C_2}\right|^2 = \frac{2kT_2R_2\,d\omega/\pi}{1+\omega^2 R_2^2 C_2^2}$$

Thus, $\overline{e_n^2} = 2k\left[T_s R_o(\omega)\right]\dfrac{d\omega}{\pi} = 2k\left[\dfrac{T_1R_1}{1+\omega^2R_1^2C_1^2} + \dfrac{T_2R_2}{1+\omega^2R_2^2C_2^2}\right]\dfrac{d\omega}{\pi}$

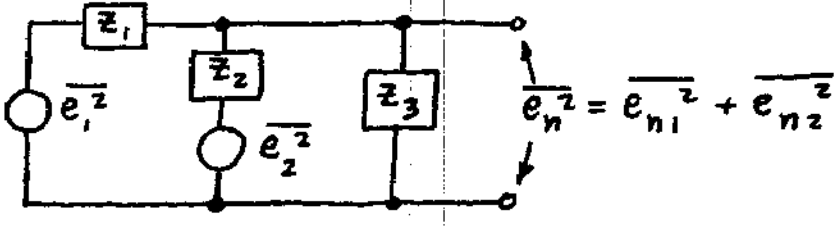
$= \overline{e_{n1}^2} + \overline{e_{n2}^2}$. Solving this equation for T_s gives

$$T_s = \frac{T_1R_1(1+\omega^2R_2^2C_2^2) + T_2R_2(1+\omega^2R_1^2C_1^2)}{R_1(1+\omega^2R_2^2C_2^2) + R_2(1+\omega^2R_1^2C_1^2)}.$$

Yes. If C_2 is chosen so $R_2C_2 = R_1C_1$ then

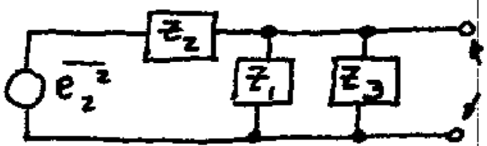
$$T_s = \frac{T_1R_1 + T_2R_2}{R_1+R_2} \quad \text{(independent of } \omega\text{)}.$$

*8.9-5. The equivalent circuit has the form:



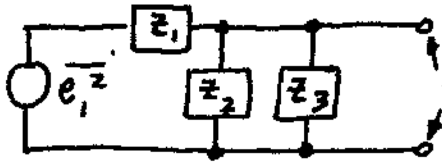
*(8.9-5.) (Continued) $Z_1 = R_1 + j\omega L$, $Z_2 = R_2$, $Z_3 = 1/j\omega C$.

Here $\overline{e_{n1}^2}$ and $\overline{e_{n2}^2}$ are the output contributions from each of the two resistors.



$$\overline{e_{n2}^2} = \overline{e_2^2}\left|\frac{\dfrac{Z_1 Z_3}{Z_1+Z_3}}{Z_2 + \dfrac{Z_1 Z_3}{Z_1+Z_3}}\right|^2$$

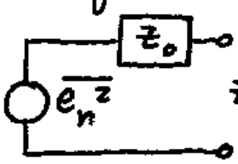
$$= \overline{e_2^2}\left|\frac{Z_1 Z_3}{Z_2(Z_1+Z_3)+Z_1 Z_3}\right|^2$$



$$\overline{e_{n1}^2} = \overline{e_1^2}\left|\frac{\dfrac{Z_2 Z_3}{Z_2+Z_3}}{Z_1 + \dfrac{Z_2 Z_3}{Z_2+Z_3}}\right|^2$$

$$= \overline{e_1^2}\left|\frac{Z_2 Z_3}{Z_1(Z_2+Z_3)+Z_2 Z_3}\right|^2$$

The equivalent circuit is:



$$Z_0 = \frac{1}{\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}} = \frac{Z_1 Z_2 Z_3}{Z_2 Z_3 + Z_1 Z_3 + Z_1 Z_2}$$

$$R_o(\omega) = Re[Z_o] = Re\left[\frac{Z_1 Z_2 Z_3\,(Z_2 Z_3 + Z_1 Z_3 + Z_1 Z_2)^*}{|Z_2 Z_3 + Z_1 Z_3 + Z_1 Z_2|^2}\right]$$

We require

$$\overline{e_1^2}\,\frac{|Z_2 Z_3|^2}{|Z_2 Z_3 + Z_1 Z_3 + Z_1 Z_2|^2} + \overline{e_2^2}\,\frac{|Z_1 Z_3|^2}{|Z_2 Z_3 + Z_1 Z_3 + Z_1 Z_2|^2}$$

$$\overset{\text{must}}{=\!=} 2k\left\{T_o\, Re[Z_o]\right\}\frac{d\omega}{\pi}.$$ From use of (8.9-1)

this equation reduces to:

*8.9-5, (Continued)

$$\frac{T_1R_1\,|Z_2Z_3|^2+T_2R_2\,|Z_1Z_3|^2}{|Z_2Z_3+Z_1Z_3+Z_1Z_2|^2}=\frac{T_s\,\mathrm{Re}\left[Z_1Z_2Z_3\,(Z_2Z_3+Z_1Z_3+Z_1Z_2)^*\right]}{|Z_2Z_3+Z_1Z_3+Z_1Z_2|^2}$$

or
$$T_s=\frac{T_1R_1\,|Z_2Z_3|^2+T_2R_2\,|Z_1Z_3|^2}{\mathrm{Re}\left[Z_1Z_2Z_3\,(Z_2Z_3+Z_1Z_3+Z_1Z_2)^*\right]}\,. \qquad (1)$$

On substitution of $|Z_1Z_3|^2=\dfrac{R_1^2+\omega^2L^2}{\omega^2C^2}$,

$|Z_2Z_3|^2=\dfrac{R_2^2}{\omega^2C^2}$, $Z_1Z_2Z_3=\dfrac{R_2(R_1+j\omega L)}{j\omega C}$,

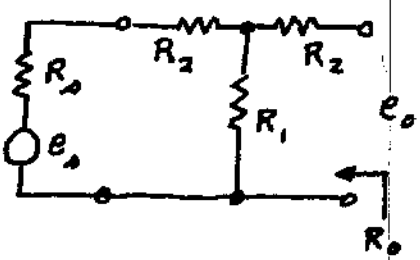
$(Z_2Z_3+Z_1Z_3+Z_1Z_2)^*=\dfrac{-R_2}{j\omega C}-\dfrac{R_1-j\omega L}{j\omega C}+R_2(R_1-j\omega L)$,

and $Z_1Z_2Z_3(Z_2Z_3+Z_1Z_3+Z_1Z_2)^*=\dfrac{-R_2^2(R_1+j\omega L)}{-\omega^2C^2}$

$-\dfrac{R_2(R_1^2+\omega^2L^2)}{-\omega^2C^2}+\dfrac{R_2^2(R_1^2+\omega^2L^2)}{j\omega C}$ into (1)

we get
$$T_s=\frac{T_1R_1R_2^2+T_2R_2(R_1^2+\omega^2L^2)}{R_1R_2^2+R_2(R_1^2+\omega^2L^2)}\,.$$

8.9-6



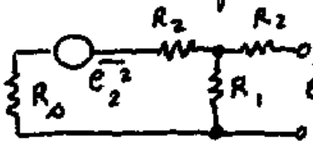
$$e_o=\frac{e_s\,R_1}{R_1+R_2+R_s}$$

$$\overline{e_o^2}=\frac{\overline{e_s^2}\;R_1^2}{(R_1+R_2+R_s)^2}$$

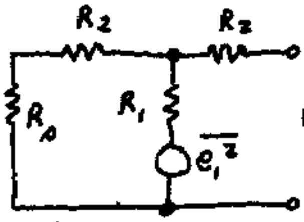
$R_o=R_2+\dfrac{R_1(R_2+R_s)}{R_1+R_2+R_s}$. Now $G_a=\dfrac{dN_{ao}}{dN_{as}}=\dfrac{\overline{e_o^2}/4R_o}{\overline{e_s^2}/4R_s}$

$$=\frac{R_1^2}{(R_1+R_2+R_s)^2}\cdot\frac{R_s}{R_2+\dfrac{R_1(R_2+R_s)}{R_1+R_2+R_s}}=\frac{R_1^2R_s}{(R_1+R_2+R_s)\left[R_1R_2+(R_1+R_2)(R_2+R_s)\right]}.$$

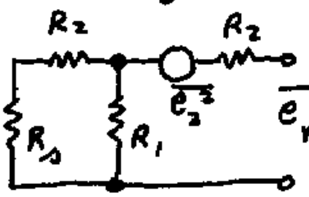
(8.9-7.) Let $\overline{e_{ns}^2}$, $\overline{e_{n1}^2}$, $\overline{e_{n2}^2}$ and $\overline{e_{n3}^2}$ be the contributions to the output mean-squared voltage $\overline{e_n^2}$ from resistors R_s, R_1, R_2 (left) and R_2 (right). Then the following apply:



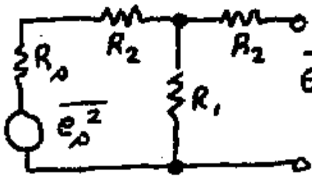
$$\overline{e_{n2}^2} = \frac{\overline{e_2^2}\; R_1^2}{(R_1+R_2+R_s)^2} = 2k\frac{d\omega}{\pi}\,\frac{T_L R_2 R_1^2}{(R_1+R_2+R_s)^2}$$



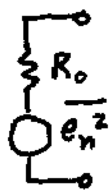
$$\overline{e_{n1}^2} = \frac{\overline{e_1^2}\,(R_2+R_s)^2}{(R_1+R_2+R_s)^2} = 2k\frac{d\omega}{\pi}\,\frac{T_L R_1 (R_2+R_s)^2}{(R_1+R_2+R_s)^2}$$



$$\overline{e_{n3}^2} = \overline{e_2^2} = 2k\frac{d\omega}{\pi}\,T_L R_2$$



$$\overline{e_{ns}^2} = \frac{\overline{e_s^2}\; R_1^2}{(R_1+R_2+R_s)^2} = 2k\frac{d\omega}{\pi}\,\frac{T_s R_s R_1^2}{(R_1+R_2+R_s)^2}$$



$$\overline{e_n^2} = 2k\frac{d\omega}{\pi}\,T_{out} R_o\,, \quad R_o = R_2 + \frac{R_1(R_2+R_s)}{R_1+R_2+R_s}.$$

From (8.10-3):

$$\overline{e_n^2} = 2k\frac{d\omega}{\pi} T_{out} R_o = \frac{R_o}{R_s} G_a\, 2k\frac{d\omega}{\pi}(T_s+T_e)R_s = 2k\frac{d\omega}{\pi}(T_s + T_e) G_a R_o.$$

But also $\overline{e_n^2} = \overline{e_{ns}^2} + \overline{e_{n1}^2} + \overline{e_{n2}^2} + \overline{e_{n3}^2} = 2k\frac{d\omega}{\pi}\left[\frac{T_s R_s R_1^2 + T_L R_1 (R_2+R_s)^2 + T_L R_2 R_1^2}{(R_1+R_2+R_s)^2} + T_L R_2\right].$

8.9-7. (Continued) On equating the above two expressions for $\overline{e_n^2}$:

$$(T_e+T_s)G_a R_o = \frac{T_s R_s R_1^2}{(R_1+R_2+R_s)^2} + \frac{T_L R_1(R_2+R_s)^2 + T_L R_2 R_1^2 + T_L R_2 (R_1+R_2+R_s)^2}{(R_1+R_2+R_s)^2}.$$

However, from problem 8-62: $G_a R_o = \dfrac{R_s R_1^2}{(R_1+R_2+R_s)^2}$ so

$$T_e = T_L\left[\frac{(R_2+R_s)^2}{R_s R_1} + \frac{R_2}{R_s} + \frac{R_2(R_1+R_2+R_s)^2}{R_s R_1^2}\right].$$

8.9-8. From problem 8.9-6 the available power gain of stage one is

$$G_1 = \frac{R_1^2 R_s}{(R_1+R_2+R_s)[R_2(R_1+R_2+R_s)+R_1(R_2+R_s)]}.$$

That of stage two is identical except R_s is replaced by R_o, the output impedance of the first network, where

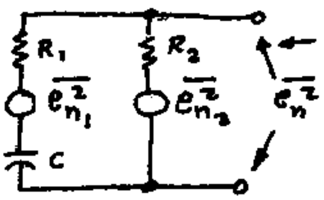
$$R_o = R_2 + \frac{R_1(R_2+R_s)}{(R_1+R_2+R_s)}.$$

Thus, $G_2 = \dfrac{R_1^2 R_o}{(R_1+R_2+R_o)[R_2(R_1+R_2+R_o)+R_1(R_2+R_o)]}$

$$G_a = G_1 G_2 = \frac{R_1^4 R_s}{(R_1+R_2+R_s)^2(R_1+R_2+R_o)[R_2(R_1+R_2+R_o)+R_1(R_2+R_o)]}.$$

With $R_1 = 5\Omega$, $R_2 = 3\Omega$ and $R_s = 7\Omega$ we find $R_o = 95/15\ \Omega$, $G_1 = 7/3(19)$, $G_2 = 25(95)15/215(1345)$ and $G_a = G_1 G_2 = \dfrac{35(95)}{19(43)(269)} = \dfrac{3325}{219773} \approx 0.0153$ or -18.2 dB.

(8.9-9.)



$$Z_0 = R_0 + jX_0 = \frac{1}{\frac{1}{R_2} + \frac{1}{R_1 + \frac{1}{j\omega C}}} = \frac{R_2(1 + j\omega R_1 C)}{1 + j\omega(R_1 + R_2)C}$$

$$\overline{e_n^2} = \overline{e_{n_1}^2}\left|\frac{R_2}{R_1 + R_2 + \frac{1}{j\omega C}}\right|^2 + \left|\frac{R_1 + \frac{1}{j\omega C}}{R_1 + R_2 + \frac{1}{j\omega C}}\right|^2 \overline{e_{n_2}^2}$$

$$= \frac{2k\,d\omega}{\pi}\left[\frac{T_1 R_1 \omega^2 R_2^2 C^2 + T_2 R_2(1 + \omega^2 R_1^2 C^2)}{1 + \omega^2 (R_1 + R_2)^2 C^2}\right]$$

$$\overset{\text{must}}{=} \frac{2kT_s\,d\omega}{\pi} = \frac{2k\,d\omega}{\pi}\, T_s\, \frac{[R_2 + \omega^2 R_1 R_2 (R_1 + R_2) C^2]}{1 + \omega^2 (R_1 + R_2)^2 C^2}, \text{ so}$$

$$T_s = \frac{T_2 + \omega^2 R_1 C^2 (T_1 R_2 + T_2 R_1)}{1 + \omega^2 R_1 C^2 (R_1 + R_2)}.$$

(8.9-10.) (a) $T_s = \dfrac{T_1 \bar{R}_1 + T_2 \bar{R}_2}{\bar{R}_1 + \bar{R}_2} = \dfrac{250(1250) + 330(2450)}{1250 + 2450} = 302.973\text{K}.$

(b) $\bar{T}_s = \dfrac{1}{(500)^2}\displaystyle\int_{R_1=1000}^{1500}\int_{R_2=2200}^{2700} \frac{250R_1 + 330R_2}{R_1 + R_2}\, dR_2\, dR_1$

(8.9-10.) (Continued) Solve the inner integral first:

$$\int_{2200}^{2700}\left[\frac{250R_1\,dx}{R_1 + x} + \frac{330x\,dx}{R_1 + x}\right]$$ ← use (c-21) and integral 91.1 of Dwight, p.23

$$= 250R_1\left\{\ln|R_1 + x|\Big|_{2200}^{2700}\right\} + 330\left\{R_1 + x - R_1 \ln|R_1 + x|\Big|_{2200}^{2700}\right\}$$

$$= 250R_1 \ln|R_1 + 2700| - 250R_1 \ln|R_1 + 2200| + 330(500)$$

$$-330R_1 \ln|R_1 + 2700| + 330R_1 \ln|R_1 + 2200|$$

$$= -80R_1 \ln|R_1 + 2700| + 80R_1 \ln|R_1 + 2200| + 330(500).$$

Next, evaluate the outer integral using integral 620.1, p 144, of Dwight:

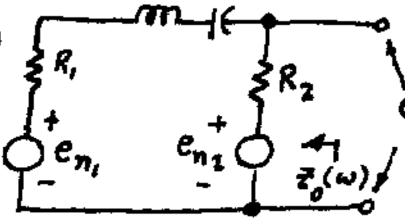
$$\bar{T}_s = 330 + \frac{80}{500^2}\left\{\frac{(R_1^2 - 2200^2)}{2}\ln(R_1 + 2200) + \frac{2200R_1}{2} - \frac{R_1^2}{4}\right\}$$

(8.9-10.) (Continued)

$$-\frac{80}{500^2}\left\{\frac{(R_1^2-2700^2)}{2}\ln(R_1+2700)+\frac{2700R_1}{2}-\frac{R_1^2}{4}\right\}\Bigg|_{1000}^{1500}$$

$= 330 - 26.9873 = 303.0127\ \text{K}.$

(8.9-11.)



$$e_n = e_{n_1}\frac{R_2}{R_1+R_2+j\omega L+\frac{1}{j\omega C}} + e_{n_2} - \frac{e_{n_2}R_2}{R_1+R_2+j\omega L+\frac{1}{j\omega C}}$$

$$= \frac{e_{n_1} j\omega R_2 C + e_{n_2}[1-\omega^2 LC] + e_{n_2} j\omega R_1 C}{[1-\omega^2 LC] + j\omega(R_1+R_2)C}$$

$$\overline{|e_n|^2} = \frac{\overline{e_{n_2}^2}\,[1-\omega^2 LC]^2 + \overline{(e_{n_1}R_2+e_{n_2}R_1)^2}\,\omega^2C^2}{[1-\omega^2LC]^2+\omega^2(R_1+R_2)^2C^2}$$, but $\overline{e_{n_1}e_{n_2}}=0$ so we get

$$\overline{|e_n|^2} = 2k\frac{d\omega}{\pi}\,\frac{T_2R_2[1-\omega^2LC]^2 + R_1R_2\omega^2C^2(T_2R_1+T_1R_2)}{[1-\omega^2LC]^2+\omega^2(R_1+R_2)^2C^2} \overset{\text{must}}{=} 2k\frac{d\omega}{\pi}T_s R_0(\omega).$$ But

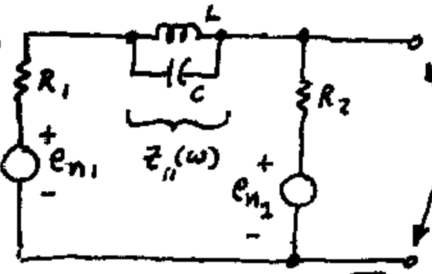
$$Z_0(\omega) = \frac{R_2\left(R_1+j\omega L+\frac{1}{j\omega C}\right)}{R_1+R_2+j\omega L+\frac{1}{j\omega C}} = \frac{R_2\{[1-\omega^2LC]+j\omega R_1C\}}{[1-\omega^2LC]+j\omega(R_1+R_2)C}$$ so

$$R_0(\omega) = \frac{R_2\{[1-\omega^2LC]^2+\omega^2R_1(R_1+R_2)C^2\}}{[1-\omega^2LC]^2+\omega^2(R_1+R_2)^2C^2}$$ and

$$T_s = \frac{T_2R_2[1-\omega^2LC]^2+\omega^2C^2R_1R_2(T_2R_1+T_1R_2)}{R_2[1-\omega^2LC]^2+\omega^2}.$$ For $\omega = 1/\sqrt{LC}$ we get

$$T_s = \frac{T_1R_2+T_2R_1}{R_1+R_2}.$$ At $\omega = \pm\infty$: $T_s = T_2$.

(8.9-12.)



$$e_n = \frac{e_{n_1}R_2}{R_1+R_2+\frac{j\omega L}{1-\omega^2LC}} + e_{n_2} - \frac{e_{n_2}R_2}{R_1+R_2+\frac{j\omega L}{1-\omega^2LC}}$$

$$= \frac{(e_{n_1}R_2+e_{n_2}R_1)(1-\omega^2LC)+e_{n_2}j\omega L}{(R_1+R_2)(1-\omega^2LC)+j\omega L}$$

(8.9-12.) (Continued)

$$Z_{11}(\omega) = \frac{j\omega L\left(\frac{1}{j\omega C}\right)}{j\omega L + \frac{1}{j\omega C}} = \frac{j\omega L}{1-\omega^2 LC}$$

$$\overline{|e_n|^2} = \frac{\overline{(e_{n1}R_2 + e_{n2}R_1)^2}(1-\omega^2 LC)^2 + \overline{e_{n2}^{\,2}}\,\omega^2 L^2}{(R_1+R_2)^2(1-\omega^2 LC)^2 + \omega^2 L^2}.$$ But $\overline{e_{n1}e_{n2}} = 0$ so

$$\overline{|e_n|^2} = 2k\frac{d\omega}{\pi}\,\frac{[1-\omega^2 LC]^2[T_1R_1R_2^2 + T_2R_2R_1^2] + T_2R_2\omega^2L^2}{(R_1+R_2)^2(1-\omega^2LC)^2 + \omega^2L^2} \overset{\text{must}}{=\!=\!=} 2k\frac{d\omega}{\pi}T_o R_o(\omega)$$

$$\text{But, } Z_o(\omega) = \frac{R_2\left(R_1 + \frac{j\omega L}{1-\omega^2LC}\right)}{R_1+R_2+\frac{j\omega L}{1-\omega^2LC}} = \frac{R_2\{R_1(1-\omega^2LC) + j\omega L\}}{(R_1+R_2)(1-\omega^2LC) + j\omega L}$$

so $$R_o(\omega) = \frac{R_2\{R_1(1-\omega^2LC)^2(R_1+R_2) + \omega^2L^2\}}{(R_1+R_2)^2(1-\omega^2LC)^2 + \omega^2L^2}$$ and

$$T_o = \frac{\omega^2L^2T_2R_2 + (1-\omega^2LC)^2R_1R_2(T_1R_2 + T_2R_1)}{\omega^2L^2R_2 + (1-\omega^2LC)^2R_1R_2(R_1+R_2)}.$$ For $\omega = \frac{1}{\sqrt{LC}}$: $T_o = T_2$.

For $\omega = \pm\infty$: $T_o = \dfrac{T_1R_2 + T_2R_1}{R_1+R_2}$.

(8.10-1.) Let $e_{om}(t)$, $m = 1, 2, \cdots, M$, be the open-circuit output voltage of stage m when driven by all previous stages. Similarly, let R_{om} be the output resistance of stage m when driven by all previous stages. Then,

$$G_1 = \frac{R_s\,\overline{e_{o1}^2(t)}}{R_{o1}\,\overline{e_s^2(t)}}, \quad G_2 = \frac{R_{o1}\,\overline{e_{o2}^2(t)}}{R_{o2}\,\overline{e_{o1}^2(t)}}$$

from (8.10-3). For two stages in cascade we apply (8.10-3) again

(8.10-1.) (Continued)

$$G_a = \frac{R_s \overline{e_{o2}^2(t)}}{R_{o2} \overline{e_s^2(t)}} = \frac{R_s \overline{e_{o1}^2(t)}}{R_{o1} \overline{e_s^2(t)}} \cdot \frac{R_{o1} \overline{e_{o2}^2(t)}}{R_{o2} \overline{e_{o1}^2(t)}} = G_1 G_2 .$$

For three stages:

$$G_a = \frac{R_s \overline{e_{o3}^2(t)}}{R_{o3} \overline{e_s^2(t)}} = \frac{R_s \overline{e_{o1}^2(t)}}{R_{o1} \overline{e_s^2(t)}} \cdot \frac{R_{o1} \overline{e_{o2}^2(t)}}{R_{o2} \overline{e_{o1}^2(t)}} \cdot \frac{R_{o2} \overline{e_{o3}^2(t)}}{R_{o3} \overline{e_{o2}^2(t)}} = G_1 G_2 G_3 .$$

Extention to M terms clearly gives

$$G_a = \prod_{m=1}^{M} G_m .$$

(8.10-2.) Apply (8.10-7) using the fact that $F_{op\,m}$ is given by (8.10-11) to satisfy $T_{em} = T_{s(m-1)} \cdot (F_{op\,m} - 1)$, $T_{s0} = T_s$.

$$T_e = T_{e1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1 G_2} + \cdots + \frac{T_{eM}}{G_1 G_2 \cdots G_{M-1}}$$

$$= T_s(F_{op} - 1) = T_s(F_{op1} - 1) + \frac{T_{s1}(F_{op2} - 1)}{G_1} + \cdots + \frac{T_{s(M-1)}(F_{opM} - 1)}{G_1 G_2 \cdots G_{M-1}} .$$

Solve for F_{op}:

$$F_{op} = F_{op1} + \frac{T_{s1}(F_{op2} - 1)}{T_s G_1} + \frac{T_{s2}(F_{op3} - 1)}{T_s G_1 G_2} + \cdots + \frac{T_{s(M-1)}(F_{opM} - 1)}{T_s G_1 G_2 \cdots G_{M-1}} .$$

(8.10-3.) (a) From (8.10-10) with $T_0 = 290\,K$:

$$T_e = T_0(F_0 - 1) = 290\,(6.31 - 1) = 1539.9\,K .$$

(b) From (8.10-11): $F_{op} = 1 + \frac{T_e}{T_s} = 1 + \frac{1539.9}{180} = 9.555$ (or 9.8 dB).

8.10-4 Apply (8.10-10) to stage m : $T_{em} = T_0(F_{om}-1)$, where F_{om} is the standard spot noise figure of stage m when driven by a standard source with impedance equal to that of all m-1 previous stages. From (8.10-7):

$$T_e = T_0(F_0-1) = T_0(F_{01}-1) + \frac{T_0(F_{02}-1)}{G_1} + \cdots + \frac{T_0(F_{0M}-1)}{G_1G_2\cdots G_{M-1}}$$

or $F_0 = F_{01} + \dfrac{F_{02}-1}{G_1} + \dfrac{F_{03}-1}{G_1G_2} + \cdots + \dfrac{F_{0M}-1}{G_1G_2\cdots G_{M-1}}$.

8.10-5 From (8.10-7): $T_e = 250 = T_{e1} + \dfrac{T_{e2}}{G_1} + \dfrac{T_{e3}}{G_1G_2}$

$= 200 + \dfrac{450}{G_1} + \dfrac{1000}{5G_1}$. Solve for $G_1 = 13$.

8.10-6 (a) Substitute T_e from (8.10-11) into (8.10-10):

$F_0 = 1 + \frac{T_0}{T_0}(F_{op}-1) = 1 + \frac{225}{290}(10-1) \approx 7.98$ (9.0 dB).

(b) Here $L \approx 2.089$ (3.2 dB) and $T_L = 290$ K. From (8.10-7), using (8.9-19) for the attenuator and (8.10-10) for the amplifier:

$T_e = T_{e1} + \dfrac{T_{e2}}{G_1} = T_L(L-1) + \dfrac{T_0(F_0-1)}{1/L} = 290[1.089 + 6.98$

$\cdot (2.089)] \approx 4544.4$ K. Next, use (8.10-11):

$F_{op} = 1 + \dfrac{4544.4}{225} = 21.2$ (13.3 dB). (c) From

(8.10-10): $F_0 = 1 + \dfrac{4544.4}{290} = 16.67$ (or 12.2 dB).

8.10-7. (a) For the first manufacturer's receiver $F_{op} = 10$ (or 10 dB) when $T_s = 130$ K. From (8.10-11): $T_{e_1} = 130(10-1) = 1170$ K. For the other receiver $F_o = 3.98$ (or 6.0 dB). From (8.10-10): $T_{e_2} = 290 \cdot (3.98-1) = 864.2$ K. (b) The second is the better unit because $T_{e2} < T_{e1}$.

8.10-8. (a) Use (8.10-7): $G_2 = \dfrac{T_{e3}}{G_1\left(T_e - T_{e_1} - \dfrac{T_{e2}}{G_1}\right)} = \dfrac{600}{10\left(190 - 150 - \frac{350}{10}\right)} = 12.$ (b) $F_o = 1 + \dfrac{T_e}{T_0} = 1 + \dfrac{190}{290} = 1.655$ (or 2.19 dB).

(c) $F_{op} = 1 + \dfrac{T_e}{T_s} = 1 + (190/50) = 4.8$ (or 6.81 dB).

8.10-9. (a) $T_e = T_{e_1} + \dfrac{T_{e2}}{G_1} + \dfrac{T_{e3}}{G_1 G_2} = 40 + \dfrac{100}{8} + \dfrac{280}{8(6)} = \dfrac{175}{3} = 58.3333$ K.

(b) Total available output power $dN_{ao} = k(T_s + T_e)\dfrac{d\omega}{2\pi} G_1 G_2 G_3$.

(1) Fraction due to source $= \dfrac{kT_s \frac{d\omega}{2\pi} G_1 G_2 G_3}{k(T_s + T_e)\frac{d\omega}{2\pi} G_1 G_2 G_3} = \dfrac{T_s}{T_s + T_e}$

$= \dfrac{30}{30 + (175/3)} = 0.3396.$ (2) dN_{ao}(Network 1) $= kT_{e1}\dfrac{d\omega}{2\pi} G_1 G_2 G_3$

Fraction due to network 1 $= \dfrac{T_{e1}}{T_s + T_e} = \dfrac{40}{265/3} = \dfrac{120}{265} = 0.4528.$

(3) dN_{ao} (Network 2) $= kT_{e2}(d\omega/2\pi)G_2G_3$. Fraction due to network 2 $= \dfrac{T_{e2}}{G_1(T_s + T_e)} = \dfrac{15}{106} = 0.1415.$

(4) dN_{ao} (Network 3) $= kT_{e3}\dfrac{d\omega}{2\pi}G_3$. Fraction due to network 3 $= \dfrac{T_{e3}}{G_1 G_2 (T_e + T_s)} = \dfrac{7}{106} = 0.0660.$

8.10-10. Output noise power $= dN_{ao} = \dfrac{k[T_a + T_L(L-1)]}{L}\dfrac{d\omega}{2\pi} \underset{\text{must}}{=} kT_s\dfrac{d\omega}{2\pi}$, T_s = new source's temperature. Thus,

$T_s = \dfrac{T_a + T_L(L-1)}{L} = \dfrac{90 + 270(1.9-1)}{1.9} = 175.26$ K.

(8.10-11.) $T_e = T_{e_1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1 G_2} = T_{e_1}\left(1 + \frac{1}{G} + \frac{1}{G^2}\right)$, $(T_{e_1} - T_e)G^2 + T_{e_1}G + T_{e_1} = 0$

$$G = \frac{-T_{e_1} \pm \sqrt{T_{e_1}^2 - 4(T_{e_1} - T_e)T_{e_1}}}{2(T_{e_1} - T_e)} = \begin{cases} 5 & \leftarrow \text{answer.} \\ -5/6 & \leftarrow \text{negative so not allowed.} \end{cases}$$

(8.10-12.) Sequence — T_e

ABC: $110 + \frac{120}{4} + \frac{150}{4\cdot 6} = 146.25$

ACB: $110 + \frac{150}{4} + \frac{120}{4\cdot 12} = 150.00$

BAC: $120 + \frac{110}{6} + \frac{150}{6\cdot 4} = 144.583$ ← Best sequence.

BCA: $120 + \frac{150}{6} + \frac{110}{6\cdot 12} = 146.528$

CAB: $150 + \frac{110}{12} + \frac{120}{12\cdot 4} = 161.6667$

CBA: $150 + \frac{120}{12} + \frac{110}{12\cdot 6} = 161.528$

(8.10-13.) $T_e = T_{e_1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1 G_2} + \frac{T_{e4}}{G_1 G_2 G_3} + \frac{T_{e5}}{G_1 G_2 G_3 G_4}$

$$= 75 + \frac{1.75(75)}{(\frac{1}{2})} + \frac{(1.75)^2 75}{(\frac{1}{2})1.75(\frac{1}{2})} + \frac{(1.75)^3 75}{(\frac{1}{2})1.75(\frac{1}{2})(1.75)^2(\frac{1}{2})} + \frac{(1.75)^4 75}{(\frac{1}{2})1.75(\frac{1}{2})(1.75)^2(\frac{1}{2})(1.75)^3(\frac{1}{2})}$$

$= 1{,}854.34$ K.

★ (8.10-14.) $G_i = K^{i-1}G_1$, $i = 1,2,3,4,5$, and $T_{e_i} = K^{i-1}T_{e_1}$, $i = 1,2,3,4,5$.

$$T_e = T_{e_1} + \frac{K T_{e_1}}{G_1} + \frac{K^2 T_{e_1}}{K G_1^2} + \frac{K^3 T_{e_1}}{K^3 G_1^3} + \frac{K^4 T_{e_1}}{K^6 G_1^4} = T_{e_1}\left\{1 + \frac{K}{G_1} + \frac{K}{G_1^2} + \frac{1}{G_1^3} + \frac{1}{K^2 G_1^4}\right\}$$

$$= \frac{T_{e_1}}{G_1^4}\left\{\frac{K^3(G_1^3 + G_1^2) + K^2(G_1^4 + G_1) + 1}{K^2}\right\}.$$

$$\frac{\partial T_{e_1}}{\partial K} = \frac{T_{e_1}}{G_1^4}\left\{\frac{K^2[3K^2(G_1^3 + G_1^2) + 2K(G_1^4 + G_1)] - 2K[K^3(G_1^3 + G_1^2) + K^2(G_1^4 + G_1) + 1]}{K^4}\right\} = 0$$

when $K^3(G_1^3 + G_1^2) - 2 = 0$ or $K = \frac{1}{G_1}\left[\frac{2G_1}{1 + G_1}\right]^{1/3}$. Thus,

$$T_{e(min)} = T_{e_1}\left\{1 + \left(\frac{1}{G_1} + \frac{1}{G_1^2}\right)\frac{1}{G_1}\left(\frac{2G_1}{1 + G_1}\right)^{1/3} + \frac{1}{G_1^3} + G_1^2\left(\frac{1 + G_1}{2G_1}\right)^{2/3}\frac{1}{G_1^4}\right\}$$

$$= \frac{T_{e_1}}{G_1^3}\left\{1 + G_1^3 + \left(\frac{1 + G_1}{2G_1}\right)^{2/3} 3G_1\right\}$$

(8.10-15) $F_{op} = 1 + \frac{T_e}{T_a} \le 1.8$ so $T_e \le (1.8-1)T_a = 0.8(160) = 128$ K. But $F_o = 1 + (T_e/T_o) \le 1 + (128/290) = 1.4414$ (or 1.59 dB).

(8.10-16.) (a) Case 1: $F_o = F_{o1} + \frac{F_{o2}-1}{G_{a1}} = 1.6 + \frac{0.4}{12} = 1.6333$

Case 2: $F_o = F_{o2} + \frac{F_{o1}-1}{G_{a2}} = 1.4 + \frac{0.6}{8} = 1.4750$ ← Best case.

Use unit 2 driven by antenna. (b) $F_o = 1.4750$.

(8.10-17.) (a) $T_e = 140 = T_{e1} + \frac{T_{e2}}{G_{a1}} = T_{e1} + \frac{600}{15}$, $T_{e1} = 100$ K.

(b) $dN_{ao} = 4.14(10^{-16}) = k\bar{T}_{sys}\frac{d\omega}{2\pi}G_a = 1.38(10^{-23})10^3 G_{a2}(140+80)15$, so $G_{a2} = 9.0909$. (c) $F_{op} = 1 + (T_e/T_a) = 1 + (140/80) = 2.75$.

(d) $F_o = 1 + (140/290) = 43/29 = 1.4828$.

(8.11-1.) Use (8.11-12): $\bar{T}_e = T_o(\bar{F}_o - 1) \le 290(1.7-1) = 203$ K.

(8.11-2.) Here $\bar{F}_o \approx 1.585$ (or 2.0 dB) and $\bar{F}_{op} \approx 4.467$ (or 6.5 dB). From (8.11-14): $\bar{T}_a = \frac{T_o(\bar{F}_o - 1)}{\bar{F}_{op} - 1}$

$= 290(1.585-1)/(4.467-1) \approx 48.93$ K.

(8.11-3.) Here $T_a = 60$ K, $L \approx 1.738$ (2.4 dB), $T_L = 275$K, and $\bar{T}_{sys} = 820$ K. (a) From (8.11-24):

$\bar{T}_R = [\bar{T}_{sys} - T_a - T_L(L-1)]/L = \frac{820 - 60 - 275(1.738-1)}{1.738}$

≈ 320.5 K.

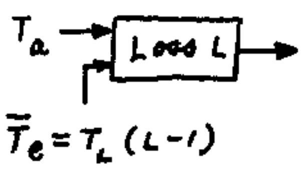
8.11-3. (Continued) (b) $\bar{F}_{op} = 1 + (\bar{T}_e/\bar{T}_o) = 1 + \dfrac{\bar{T}_{sys} - T_a}{T_a}$

$= 1 + [(820-60)/60] \approx 13.67$ (or 11.4 dB).

(c) Here $G_R(\omega_0) = 10^{11}$ (or 110 dB) and $W_N = 2\pi(10^7)$.

$$N_{ao} = k\bar{T}_{sys}\, G_R(\omega_0)\, W_N/(2\pi L) \approx 1.38(10^{-23})\, 820\,(10^{11})$$
$$\cdot 10^7/1.738 \approx 651.1\,(10^{-5}) \approx 6.51 \text{ mW}.$$

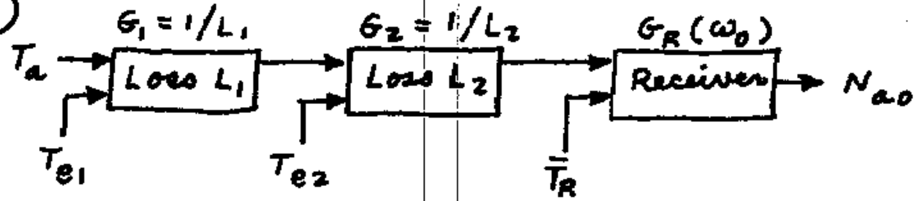
8.11-4.



$$dN_{ao} = k[T_a + T_L(L-1)]\frac{d\omega}{2\pi}$$
$$= k\,T_o\,\frac{d\omega}{2\pi}$$

Thus, $T_o = T_a + T_L(L-1) = 60 + 275(1.738-1) \approx 263$ K.

8.11-5.



Here $T_{e1} = T_1(L_1-1)$, $T_{e2} = T_2(L_2-1)$ from (8.11-19).

For the actual system:

$$N_{ao} = kT_a\frac{G_R(\omega_0)}{L_1L_2}\frac{W_N}{2\pi} + kT_1(L_1-1)\frac{G_R(\omega_0)}{L_1L_2}\frac{W_N}{2\pi}$$
$$+ kT_2(L_2-1)\frac{G_R(\omega_0)}{L_2}\frac{W_N}{2\pi} + k\bar{T}_R\,G_R(\omega_0)\frac{W_N}{2\pi}.$$

For an equivalent system with available gain $G_R(\omega_0)/L_1L_2$: $N_{ao} = k\bar{T}_{sys}\dfrac{G_R(\omega_0)}{L_1L_2}\dfrac{W_N}{2\pi}$. By equating the two equations for N_{ao} we get

$$\bar{T}_{sys} = T_a + T_1(L_1-1) + T_2L_1(L_2-1) + \bar{T}_RL_1L_2.$$

8.11-6. (a) $\bar{T}_e = \bar{T}_s(\bar{F}_{op} - 1) = 60(5-1) = 240$ K.

(b) $\bar{F}_o = 1 + (\bar{T}_e/290) = 1 + (240/290) = 1.8276$ (or 2.62 dB).

(c) $\bar{F}_{op} = 1 + (\bar{T}_e/\bar{T}_s) = 1 + (240/30) = 9.0$ (or 9.54 dB).

8.11-7. $\bar{F}_{op} = 1.8$ with $\bar{T}_s = 80$ K. $\bar{F}_{op} = 1 + \dfrac{\bar{T}_e}{\bar{T}_s} = 1.25$. But $\bar{T}_e = \bar{T}_s(\bar{F}_{op} - 1) = 80(1.8 - 1) = 64$ K so for the second source $\bar{T}_s = \dfrac{\bar{T}_e}{(\bar{F}_{op} - 1)} = \dfrac{64}{0.25} = 256$ K.

8.11-8. (a) $\bar{T}_e(\text{unit } 1) = T_0(\bar{F}_{o1} - 1) = 290(2.98) = 864.2$ K.

(b) $\bar{T}_e(\text{unit } 2) = 290(1.82) = 527.8$ K. (c) Excess power $(\text{unit } 1) = k\bar{T}_e G_{a1}(W_N/2\pi) = 1.38(10^{-23})864.2(2)10^6(10^6) = 2.3852(10^{-8})$ W. (d) Excess power (unit 2) $= 1.38(10^{-23}) \cdot 527.8(10^6)2.9(10^6) = 2.1126(10^{-8})$ W. (e) Unit 2 is the best because excess noise power is most important and unit 2 is best. (f) When source noise dominates $(\bar{T}_s \gg \bar{T}_e)$ and available power gains are the same, bandwidth becomes the deciding factor. Choose unit 1 (smallest bandwidth) since it gives the smallest total noise.

★**8.11-9.** (a) $dN_{aos} = G_a\, dN_{as}$, $dN_{as} = kT_s \dfrac{d\omega}{2\pi}$ (one-sided). Thus, dN_{aos} (two-sided) $= G_a k \dfrac{T_s}{2}\dfrac{d\omega}{2\pi}$

$$= \frac{k\frac{T_s}{2}\frac{d\omega}{2\pi}10^{36}}{(10^6+\omega^2)^4} = \mathcal{S}_{out}(\omega)\,df \quad \text{and therefore}$$

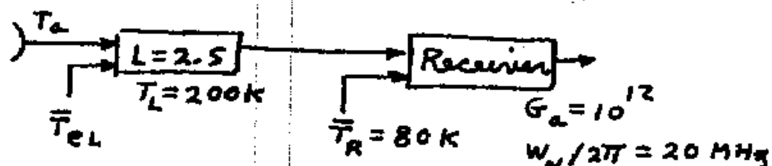
$$\mathcal{S}_{out}(\omega) = \frac{(kT_s/2)10^{36}}{(10^6+\omega^2)^4} = \frac{1.38(10^{-23})(75/2)10^{36}}{(10^6+\omega^2)^4}$$

$$= \frac{5.175(10^{14})}{(10^6+\omega^2)^4}.$$ (b) $N_{aos} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \mathcal{S}_{out}(\omega)\,d\omega =$

★ 8.11-9. (Continued)

$$\frac{5.175(10^{14})}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{(10^6+\omega^2)^4} = \frac{5.175(10^{-7})5}{32} = 8.0859(10^{-8})\,W.$$

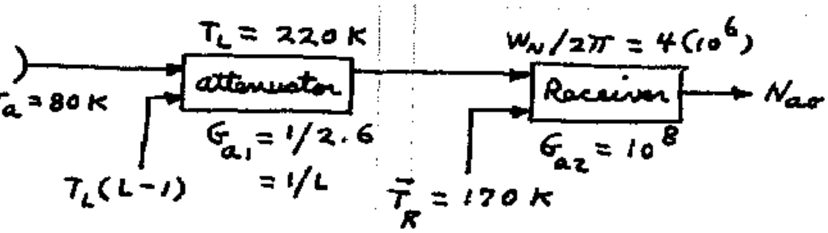
8.11-10.



(a) $\bar{T}_{eL} = T_L(L-1) = 200(1.5) = 300K$, from (8.11-19). (b) From (8.11-24): $\bar{T}_{sys} = T_a + T_L(L-1) + \bar{T}_R L =$ $120 + 300 + 80(2.5) = 620$ K. (c) From (8.11-12): $\bar{F}_o = 1 + \frac{\bar{T}_R}{T_o} = 1 + \frac{80}{290} = 1.2759$ (or 1.06 dB). (d) Use (8.11-23): $N_{ao} = k\bar{T}_{sys}G_a W_N/2\pi L = 1.38(10^{-23})620$ $\cdot(10^{12})20(10^6)/2.5 = 68.45(10^{-3})$ W. (e) From (8.11-25): $\bar{T}_e$ (attenuator-receiver) $= T_L(L-1) + \bar{T}_R L$ $= 300 + 80(2.5) = 500$ K. (f) $\bar{F}_{op} = 1 + \frac{T_L(L-1)+\bar{T}_R L}{T_a}$ $= 1 + \frac{500}{120} = 5.1667$ (or 7.13 dB).

8.11-11. (a) From (8.11-24): $\bar{T}_R = \frac{\bar{T}_{sys} - T_a - T_L(L-1)}{L}$ $= \frac{500-120-75(1.5-1)}{1.5} = 228.33$ K. (b) Since $\bar{T}_{sys} =$ $T_a + \bar{T}_e$, then $\bar{F}_{op} = 1 + \frac{\bar{T}_e}{T_a} = 1 + \frac{\bar{T}_{sys}-T_a}{T_a} = \frac{\bar{T}_{sys}}{T_a} = \frac{500}{120} =$ 4.1667 (or 6.20 dB). (c) $N_{ao} = k\bar{T}_{sys}G_a W_N/2\pi =$ $1.38(10^{-23})500(20)10^6(10^{12})/1.5 = 0.092$ W.

8.11-12.



(a) $\bar{T}_{sys} = T_a + T_L(L-1) + \bar{T}_R L = 80 + 220(1.6) + 170(2.6) = 874K.$

(8.11-12) (Continued)

(b) $N_{ao} = k\bar{T}_{sys}(W_N/2\pi)\mathcal{G}_{a1}\mathcal{G}_{a2} = 1.38(10^{-23})874(4)10^6(10^8)/2.6$
$= 1.8556(10^{-6})$ W. (c) Total power $= k[T_a + T_L(L-1)]$
$(W_N/2\pi)\mathcal{G}_{a1} = 1.38(10^{-23})[80+352]4(10^6)/2.6 = 9.17(10^{-15})$ W.
Power due to antenna $= kT_a(W_N/2\pi)\mathcal{G}_{a1} = 1.38(10^{-23})$
$80(4)10^6/2.6 = 1.6985(10^{-15})$ W. Percentage due to antenna alone $= (1.6985/9.17)100 = 18.52\%$.

(d) $\bar{F}_{op} = \bar{T}_{sys}/T_a = 874/80 = 10.925$ (or 10.38 dB).

(8.11-13) (a) $\int_0^\infty \mathcal{G}_a(\omega)d\omega = \int_0^\infty \frac{64\,d\omega}{(10^2+\omega^2)} = 3.2\pi$ from (C-25).

$\int_0^\infty T_s(\omega)\mathcal{G}_a(\omega)d\omega = \int_0^\infty \frac{8000(64)\ d\omega}{(10^2+\omega^2)^2} = 128\pi$ from (C-28). From (8.11-10):

$\bar{T}_s = \frac{128\pi}{3.2\pi} = 40$ K. (b) $W_N = \frac{100}{64}\int_0^\infty \frac{64\,d\omega}{10^2+\omega^2} = 5\pi\ \frac{\text{rad}}{\text{s}}$ from (8.11-17) with $\omega_0 = 0$.

(8.11-13) (Continued) (c) From (8.11-18) with $\omega_0 = 0$ and $\bar{T}_e = 0$:

$N_{aos} = k\bar{T}_s \frac{W_N}{2\pi}\mathcal{G}_a(\omega_0) = 1.38(10^{-23})40\frac{5\pi}{2\pi}\left(\frac{64}{100}\right) = 8.832(10^{-22})$ W.

(8.11-14) (a) $d\mathcal{G}_a/d\omega\big|_{\omega=\omega_0} = \frac{[W^2+\omega^2]^3\,2K\omega - K\omega^2\,3[W^2+\omega^2]^2\,2\omega}{[W^2+\omega^2]^6}\Big|_{\omega=\omega_0} = 0$

when $(W^2+\omega_0^2) - 3\omega_0^2 = 0$ or $\omega_0 = \pm W/\sqrt{2}$.

(b) $\mathcal{G}_a(\omega_0) = \frac{K(W^2/2)}{[W^2+\frac{W^2}{2}]^3} = \frac{4K}{27W^4}$. $\int_0^\infty \mathcal{G}_a(\omega)d\omega = K\int_0^\infty \frac{\omega^2 d\omega}{[W^2+\omega^2]^3} = \frac{K\pi}{16W^3}$ from (C-32). From (8.11-17): $W_N = 27\pi W/64$.

8.11-15. (a) $dG_a/d\omega|_{\omega=\omega_0} = \dfrac{(W^2+\omega^2)^4 4K\omega^3 - K\omega^4 4(W^2+\omega^2)^2 2\omega}{(W^2+\omega^2)^8}\Big|_{\omega=\omega_0} = 0$

when $(W^2+\omega^2) - 2\omega_0^2 = 0$ or $\omega_0 = \pm W$. (b) $G_a(\omega_0) = \dfrac{KW^4}{(W^2+W^2)^4} = \dfrac{K}{16W^4}$.

$\int_0^\infty G_a(\omega)\,d\omega = K\int_0^\infty \dfrac{\omega^4\,d\omega}{(W^2+\omega^2)^4} = \dfrac{K\pi/2}{16W^3}$ from (C-36). From (8.11-17):

$W_N = \pi W/2$.

8.11-16. From the solution of Problem 8.11-14: $\int_0^\infty G_a(\omega)d\omega = \dfrac{K\pi}{16W^3}$. Next,

$\int_0^\infty T_e(\omega)G_a(\omega)\,d\omega = \int_0^\infty \dfrac{[50 + (\frac{4\omega}{W})^2]\,K\omega^2 d\omega}{[W^2+\omega^2]^3} = \dfrac{K\pi}{W^3}\left(3+\dfrac{25}{8}\right)$. From

(8.11-11): $\bar{T}_e = \dfrac{K\pi}{W^3}\left(3+\dfrac{25}{8}\right)\Big/\left(\dfrac{K\pi}{16W^3}\right) = 98$ K.

8.11-17. (a) $\bar{T}_{sys} = T_a + T_L(L-1) + \bar{T}_R L$, $\bar{T}_R = [558 - 130 - 200(0.6)]/1.6 = 192.5$K.

(b) $N_{ao} = k\bar{T}_{sys}\dfrac{W_N}{2\pi}\dfrac{G_R(\omega_0)}{L} = 1.38(10^{-23})\,558(8)10^6\,\dfrac{5(10^9)}{1.6} = 19.251(10^{-5})$.

(c) $S_{ao} = 55(10^{-12})\dfrac{G_R(\omega_0)}{L} = 55(10^{-12})\dfrac{5(10^9)}{1.6} = 171.875(10^{-3})$ W.

(d) $S_{ao}/N_{ao} = 171.875(10^{-3})/192.51(10^{-6}) = 892.811$ (or 29.51 dB).

(e) $\bar{F}_{0\,(receiver)} = 1 + (\bar{T}_R/T_0) = 1 + (192.5/290) = 1.6638$ (or 2.21 dB).

(f) $\bar{T}_{e\,(loss)} = T_L(L-1) = 200(0.6) = 120$ K.

8.11-18. (a) $N_{ao} = k\bar{T}_R\dfrac{W_N}{2\pi}G_R(\omega_0) + kT_{e_2}\dfrac{W_N}{2\pi}G_R(\omega_0)\dfrac{1}{L_2} + kT_{e_1}\dfrac{W_N}{2\pi}G_R(\omega_0)\dfrac{1}{L_1L_2}$

$+ kT_a\dfrac{W_N}{2\pi}\dfrac{G_R(\omega_0)}{L_1L_2} = k(T_a + T_{e_1} + T_{e_2}L_1 + \bar{T}_R L_1L_2)\dfrac{W_N}{2\pi}\dfrac{G_R(\omega_0)}{L_1L_2}$

$= k[T_a + T_{L_1}(L_1-1) + T_{L_2}(L_2-1)L_1 + \bar{T}_R L_1L_2]\dfrac{W_N}{2\pi}\dfrac{G_R(\omega_0)}{L_1L_2} = 0.01875$ W.

(b) $N_{ai} = N_{ao}\dfrac{L_1L_2}{G_R(\omega_0)} = 0.0570(10^{-12})$, Signal power at antenna's

output $= 5.6995(10^{-11})$ W. (c) $\bar{T}_{sys} = T_a + T_{L_1}(L_1-1) + T_{L_2}(L_2-1)L_1 + \bar{T}_R L_1L_2$

$= 120 + 70(0.6) + 250(0.9)1.6 + 100(1.6)1.9 = 826$ K.

8.11-19. (a) $T_{L1} = T_L(L-1) = 75(1.9-1) = 67.5\ K$, (b) $T_{L2} = 290(1.4-1) = 116\ K$, (c) $\bar{T}_e = T_{e1} + \frac{T_{e2}}{G_{a1}} + \frac{T_{e3}}{G_{a1}G_{a2}} + \frac{\bar{T}_e}{G_{a1}G_{a2}G_{a3}} = 7{,}789.1\ K$,
(d) $\bar{T}_{sys} = 60 + \bar{T}_e = 7{,}849.1\ K$, (e) $\bar{F}_{op} = 1 + (\bar{T}_e/T_a) = 1 + \frac{7{,}789.1}{60} = 130.8183$, (f) Power $= k\,\bar{T}_{sys}\,\frac{W_N}{2\pi}\,\frac{G_a(\text{mixer})\,G_a(\omega_0)}{L_1 L_2}$

$$= 1.38(10^{-23})\,7.8491(10^3)\,10^6\,\frac{0.5(10^7)}{1.9\,(1.4)} = 2.036\,(10^{-7})\ W.$$

CHAPTER 9

(9.1-1.) (a) From (9.1-13):

$$H_{opt}(\omega) = \frac{A\tau}{2\pi C}\left[\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right]^2 \frac{(W_2^2+\omega^2)}{W_2} e^{-j\omega t_0}.$$

(b) Write $H_{opt}(\omega)$ in the form

$$H_{opt}(\omega) = \frac{A}{2\pi C W_2}\left\{W_2^2\tau\left[\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right]^2 + \frac{4}{\tau}\left[\frac{1}{2} - \frac{1}{2}\cos(\omega\tau)\right]\right\}$$

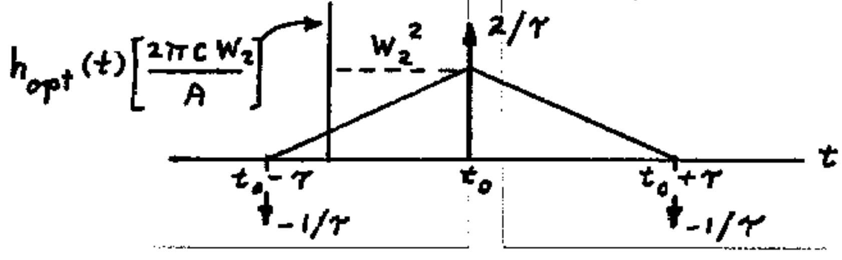
$$\cdot\, e^{-j\omega t_0}.$$ Now since

$$W_2^2\,\mathrm{tri}\left(\frac{t}{\tau}\right) = \begin{cases} W_2^2(\tau+t)/\tau, & -\tau<t<0 \\ W_2^2(\tau-t)/\tau, & 0<t<\tau \\ 0, & \text{elsewhere}\end{cases} \leftrightarrow W_2^2\tau\left[\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right]^2$$

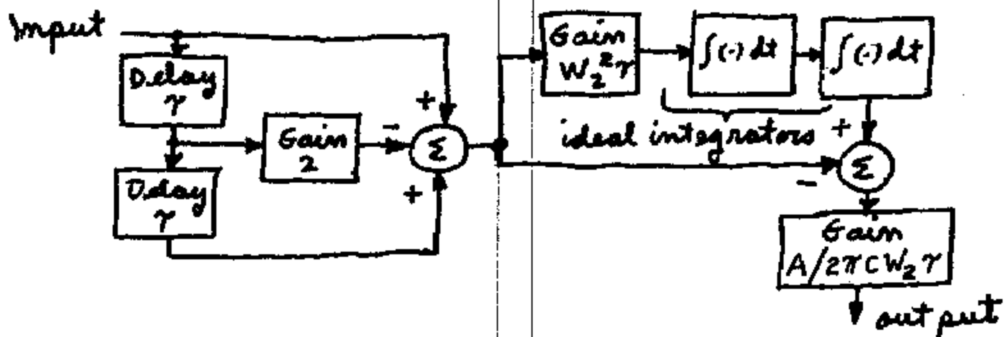
from (E-4) and pair 7 of Appendix E, and since $\frac{2}{\tau}\delta(\tau) \leftrightarrow \frac{2}{\tau}$ and $\frac{1}{\tau}[\delta(t-\tau) + \delta(t+\tau)] \leftrightarrow \frac{2}{\tau}\cos(\omega\tau)$ from pairs 1 and 10 of Appendix E, we get

$$h_{opt}(t) = \frac{A}{2\pi C W_2}\left\{W_2^2\,\mathrm{tri}\left(\frac{t-t_0}{\tau}\right) + \frac{2}{\tau}\delta(t-t_0) - \frac{1}{\tau}\left[\delta(t-t_0-\tau) + \delta(t-t_0+\tau)\right]\right\}$$

from inverse Fourier transformation of $H_{opt}(\omega)$.



(9.1-1.) (Continued) (c) The filter is causal for $t_0 \ge \tau$. (d) One form of network is defined by:



Because gain of a matched filter is arbitrary, the gain $A/2\pi C W_2\tau$ can be eliminated.

(9.1-2.) From pair 15 of Appendix E: $u(t)\,e^{-Wt} \leftrightarrow 1/(W+j\omega)$. Thus, $u(t)\left[e^{-W_2 t} - e^{-\alpha W_2 t}\right] \leftrightarrow$ $\frac{1}{W_2+j\omega} - \frac{1}{\alpha W_2+j\omega} = X(\omega)$. (a) From (9.1-13):

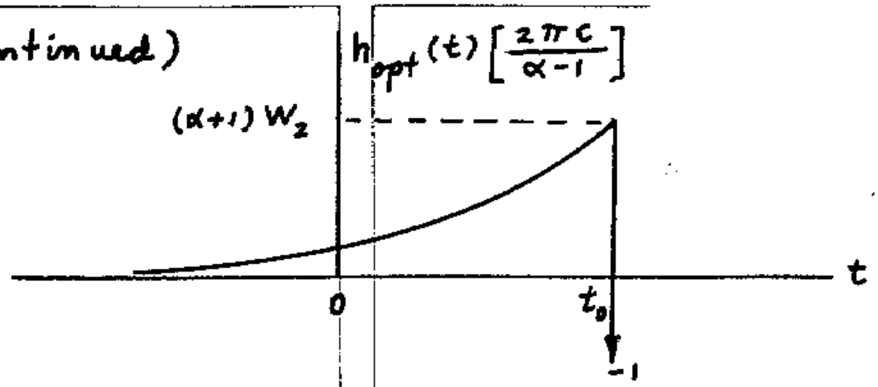
$$H_{opt}(\omega) = \frac{1}{2\pi C}\left[\frac{1}{W_2-j\omega} - \frac{1}{\alpha W_2 - j\omega}\right]\frac{(W_2^2+\omega^2)}{W_2}\, e^{-j\omega t_0}$$

$$= \frac{\alpha-1}{2\pi C}\cdot\frac{W_2+j\omega}{\alpha W_2 - j\omega}\, e^{-j\omega t_0}.$$

(b) Write $H_{opt}(\omega) = \frac{\alpha-1}{2\pi C}\left[-1 + \frac{(\alpha+1)W_2}{\alpha W_2 - j\omega}\right] e^{-j\omega t_0}$. Since $-\delta(t-t_0) \leftrightarrow -e^{-j\omega t_0}$ and $u(t)\,e^{-\alpha W_2 t} \leftrightarrow \frac{1}{\alpha W_2 + j\omega}$, from pairs 10 and 15 of Appendix E, and it can be shown that $f(-t) \leftrightarrow F(-\omega)$ for arbitrary $f(t) \leftrightarrow F(\omega)$, then $u(-t)\,e^{\alpha W_2 t} \leftrightarrow \frac{1}{\alpha W_2 - j\omega}$ and

$$h_{opt}(t) = \frac{\alpha-1}{2\pi C}\left[-\delta(t-t_0) + (\alpha+1)W_2\, u(-t+t_0)\, e^{\alpha W_2 (t-t_0)}\right].$$

(9.1-2.) (Continued)



(c) There is no value of t_0 that makes the filter causal. As $t_0 \to \infty$ $h_{opt}(t)$ for $t<0$ approaches zero.

(9.1-3.) Since $f(-t) \leftrightarrow F(-\omega)$ for any signal $f(t)$ having a transform $F(\omega)$, we have $u(-t)\,e^{W_2 t} \leftrightarrow 1/(W_2 - j\omega)$ and $u(-t)\,e^{\alpha W_2 t} \leftrightarrow 1/(\alpha W_2 - j\omega)$ so

$$x(t) \leftrightarrow X(\omega) = \frac{1}{W_2 - j\omega} - \frac{1}{\alpha W_2 - j\omega} = \frac{(\alpha-1)\,W_2}{(W_2-j\omega)(\alpha W_2 - j\omega)}.$$

(a) From (9.1-13) and the above:

$$H_{opt}(\omega) = \frac{1}{2\pi c}\left[\frac{(\alpha-1)\,W_2}{(W_2+j\omega)(\alpha W_2 + j\omega)}\right]\frac{(W_2^2+\omega^2)}{W_2}\,e^{-j\omega t_0}$$

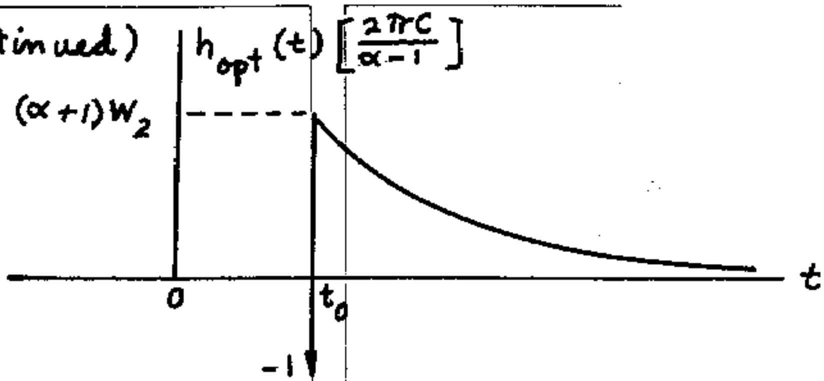
$$= \frac{(\alpha-1)}{\pi 2c}\cdot\frac{W_2 - j\omega}{\alpha W_2 + j\omega}\,e^{-j\omega t_0}.$$

(b) Write as $H_{opt}(\omega) = \dfrac{\alpha-1}{2\pi c}\left[-1 + \dfrac{(\alpha+1)\,W_2}{\alpha W_2 + j\omega}\right]e^{-j\omega t_0}$.

Since $-\delta(t-t_0) \leftrightarrow -e^{-j\omega t_0}$ we have

$$h_{opt}(t) = \frac{\alpha-1}{2\pi c}\left[-\delta(t-t_0) + (\alpha+1)\,W_2\,u(t-t_0)\,e^{-\alpha W_2 (t-t_0)}\right].$$

(9.1-3) (Continued)



(c) Any $t_0 > 0$ will make the filter causal.

*(9.1-4) $H_{opt}(\omega) = \int_{-\infty}^{\infty} h_{opt}(\xi)\, e^{-j\omega\xi}\, d\xi$. By substitution of (9.1-13):

$$\frac{K X^*(\omega)}{\mathcal{S}_{NN}(\omega)} e^{-j\omega t_0} = \int_{-\infty}^{\infty} h_{opt}(\xi)\, e^{-j\omega\xi}\, d\xi \qquad (1)$$

where $K = 1/2\pi C$ is arbitrary. Let $K = 1$. By rewriting (1) and inverse Fourier transforming both sides we get

$$\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega(t_0-t)}\, d\omega}_{x^*(t_0-t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_{opt}(\xi) \underbrace{\int_{-\infty}^{\infty} \mathcal{S}_{NN}(\omega)\, e^{j\omega(t-\xi)}\, d\omega}_{2\pi R_{NN}(t-\xi)}\, d\xi.$$

Thus,

$$x^*(t_0-t) = \int_{-\infty}^{\infty} h_{opt}(\xi)\, R_{NN}(t-\xi)\, d\xi.$$

(9.1-5) Define spectrums by $x(t) \leftrightarrow X(\omega)$, $x_1(t) \leftrightarrow X_1(\omega)$. Thus, from (9.1-14)

$$H_2(\omega) = K X_1^*(\omega)\, e^{-j\omega t_0}$$

because $H_2(\omega)$ is matched to $x_1(t)$ in white noise.

(9.1-5.) (Continued) Next,

$$x_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) H_1(\omega) H_2(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) H_1(\omega) K X_1^*(\omega) e^{j\omega(t-t_0)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) H_1(\omega) K X^*(\omega) H_1^*(\omega) e^{j\omega(t-t_0)} d\omega$$

$$= \frac{K}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \frac{1}{\mathcal{S}_{NN}(\omega)} e^{j\omega(t-t_0)} d\omega .$$

$$[x_0(t_0)]^2 = \frac{K^2}{(2\pi)^2}\left[\int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{\mathcal{S}_{NN}(\omega)} d\omega\right]^2 = \hat{S}_o . \qquad (1)$$

For output noise power:

$$N_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{NN}(\omega) |H_1(\omega)|^2 |H_2(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H_2(\omega)|^2 d\omega$$

$$= \frac{K^2}{2\pi}\int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{\mathcal{S}_{NN}(\omega)} d\omega . \qquad (2)$$

From (1) and (2): $\left(\frac{\hat{S}_o}{N_o}\right) = \frac{[x_0(t_0)]^2}{N_o} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{\mathcal{S}_{NN}(\omega)} d\omega$

which is the maximum value $(\hat{S}_o/N_o)$ can have from (9.1-12). Therefore, the system is a matched filter to $x(t)$ in nonwhite noise.

(9.1-6.) Since $x(t) = A\,\text{rect}[(t + \tau_0 - \frac{\tau}{2})/\tau]$, where rect$(t)$ is defined by (E-2), we use pair 5 of Appendix E and (D-6) to write

$$x(t) \longleftrightarrow X(\omega) = A\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2} e^{-j\omega(\frac{\tau}{2} - \tau_0)} .$$

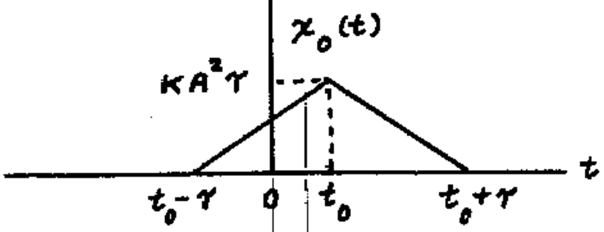
9.1-6. (Continued) Therefore,

$$X_0(\omega) = X(\omega)H_{opt}(\omega) = KA^2\tau^2\left[\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right]^2 e^{-j\omega t_0}.$$

From pair 7 of Appendix E and (D-6):

$$x_0(t) = KA^2\tau\, \mathrm{tri}\left(\frac{t-t_0}{\tau}\right)$$

where tri(t) is defined by (E-4).



9.1-7 $N_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{N}_0}{2}|H_{opt}(\omega)|^2 d\omega = \frac{\mathcal{N}_0 K^2 A^2 \tau^2}{4\pi}\int_{-\infty}^{\infty} Sa^2(\omega\tau/2)\,d\omega$

where $Sa(x)$ is defined by (E-3). By using (C-54) we obtain $N_0 = \mathcal{N}_0 K^2 A^2 \tau/2$. For output signal we use (9.1-16): $x_0(t) = \int_{-\infty}^{\infty} x(t-\xi)h_{opt}(\xi)\,d\xi$ $= K\int_{-\infty}^{\infty} x(t-\xi)x(t_0-\xi)\,d\xi$. Thus, $x_0(t_0) = K\int_{-\infty}^{\infty}$ $\cdot\, x^2(t_0-\xi)\,d\xi = KA^2\tau$ (graphically solve for area). Finally,

$$\left(\frac{\hat{S}_o}{N_o}\right)_{t=t_0} = \frac{[x_0(t_0)]^2}{N_0} = \frac{2A^2\tau}{\mathcal{N}_0}.$$

9.1-8. Here $\mathcal{S}_{NN}(\omega) = \mathcal{N}_0/2$. From (9.1-12): $\left(\frac{\hat{S}_o}{N_o}\right) \le$ $\frac{1}{2\pi\mathcal{N}_0/2}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \frac{2}{\mathcal{N}_0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega\right\}$. The braced factor is the energy E in $x(t)$ from Parseval's

9.1-8. (Continued) theorem, so

$$\left(\frac{\hat{S}_o}{N_o}\right) \le \frac{2E}{\mathcal{N}_0} = \frac{2}{\mathcal{N}_0}\int_{-\infty}^{\infty} |x(t)|^2 dt = \left(\frac{\hat{S}_o}{N_o}\right)_{max}.$$

9.1-9. $$\left(\frac{\hat{S}_o}{N_o}\right)_{max} = \frac{2}{\mathcal{N}_0}\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{2A^2}{\mathcal{N}_0}\int_{-\tau/2}^{\tau/2} \cos^2\left(\frac{\pi t}{\tau}\right) dt$$

$$= \frac{A^2}{\mathcal{N}_0}\int_{-\tau/2}^{\tau/2}\left[1 + \cos\left(\frac{2\pi t}{\tau}\right)\right] dt = A^2\tau/\mathcal{N}_0.$$

9.1-10. First, Fourier transform $x(t)$: $x(t) \leftrightarrow$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = A\int_{-\tau/2}^{\tau/2} \cos\left(\frac{\pi t}{\tau}\right) e^{-j\omega t} dt$$

$$= \frac{A}{2}\int_{-\tau/2}^{\tau/2}\left\{ e^{j\left(\frac{\pi}{\tau}-\omega\right)t} + e^{-j\left(\frac{\pi}{\tau}+\omega\right)t}\right\} dt$$

$$= \frac{\pi A \tau}{2}\,\frac{\cos(\omega\tau/2)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{\omega\tau}{2}\right)^2}$$ from (C-45). From

(9.1-14): $$H_{opt}(\omega) = \frac{K\pi A\tau}{2}\,\frac{\cos(\omega\tau/2)}{(\pi/2)^2 - (\omega\tau/2)^2}\, e^{-j\omega t_0}.$$

9.1-11. From (8.1-10) and (9.1-15):

$$x_0(t) = \int_{-\infty}^{\infty} x(t-\eta)\, h_{opt}(\eta)\, d\eta = K\int_{-\infty}^{\infty} x(t-\eta)\, x^*(t_0-\eta)\, d\eta$$

Let $\xi = t_0 - \eta$, $d\xi = -d\eta$.

$$x_0(t) = K\int_{-\infty}^{\infty} x^*(\xi)\, x(\xi + t - t_0)\, d\xi.$$

(9.1-12.) $|x_o(t)| = \left| K\int_{-\infty}^{\infty} x^*(\xi)\, x(\xi + t - t_o)\, d\xi \right|$. From Schwarz's inequality:

$$|x_o(t)| \le |K| \left\{ \int_{-\infty}^{\infty} |x^*(\xi_1)|^2 d\xi_1 \int_{-\infty}^{\infty} |x(\xi_2 + t - t_o)|^2 d\xi_2 \right\}^{1/2}$$

$$= |K| \left\{ \int_{-\infty}^{\infty} |x(\xi)|^2 d\xi \int_{-\infty}^{\infty} |x(\eta)|^2 d\eta \right\}^{1/2}$$

$$= |K| \int_{-\infty}^{\infty} |x(\xi)|^2 d\xi .$$

Thus,

$$|x_o(t)| = |K| \left| \int_{-\infty}^{\infty} x^*(\xi)\, x(\xi + t - t_o)\, d\xi \right| \le |K| \int_{-\infty}^{\infty} |x(\xi)|^2 d\xi .$$

The middle term realizes the maximum, or right term, value when $t = t_o$. Hence,

$$|x_o(t_o)| = |K| \int_{-\infty}^{\infty} |x(\xi)|^2 d\xi = |x_o(t)|_{max} .$$

(9.1-13.) $X(\omega) = \int_{-\tau_o}^{-\tau_o+\tau} A e^{-j\omega t} dt = A\tau e^{j\omega\tau_o} \dfrac{1 - e^{-j\omega\tau}}{2j(\omega\tau/2)}$

$= A\tau e^{j\omega\tau_o - j(\omega\tau/2)} \dfrac{\sin(\omega\tau/2)}{\omega\tau/2}$ from (C-45).

From (9.1-14):

$$H_{opt}(\omega) = KA\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2} e^{-j\omega[t_o + \tau_o - (\tau/2)]} .$$

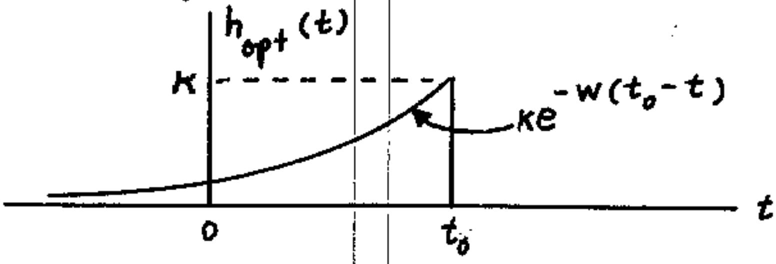
(9.1-14.) (a) From pair 15 of Appendix E and (9.1-14):

$$x(t) = u(t)e^{-Wt} \longleftrightarrow X(\omega) = \frac{1}{W + j\omega}$$

$$H_{opt}(\omega) = \frac{K}{W - j\omega} e^{-j\omega t_o} .$$

(9.1-14.) (Continued) (b) From (9.1-16):

$$h_{opt}(t) = K x(t_0 - t) = K u(t_0 - t) e^{-W(t_0 - t)}.$$



(c) No value of t_0 will make the filter causal.

(d) From Problem 9.1-8 and (C-45):

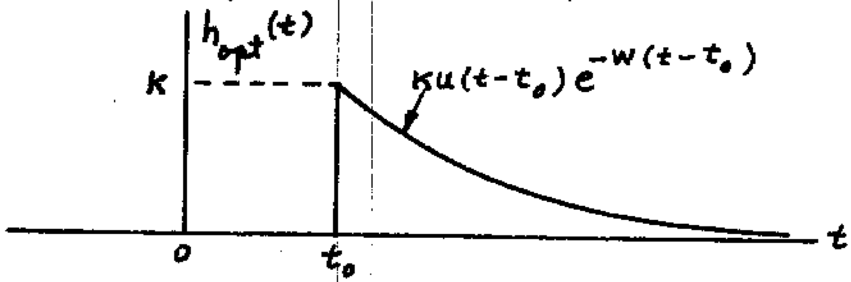
$$\left(\frac{\hat{S}_o}{N_o}\right)_{max} = \frac{2}{\mathcal{N}_0}\int_0^{\infty} e^{-2Wt}\,dt = \frac{1}{\mathcal{N}_0 W}.$$

(9.1-15.) (a) $X(\omega) = \int_{-\infty}^{0} e^{Wt} e^{-j\omega t}\,dt = \left.\frac{e^{(W-j\omega)t}}{W-j\omega}\right|_{-\infty}^{0} = \frac{1}{W-j\omega}$

after using (C-45). From (9.1-14):

$$H_{opt}(\omega) = K\,\frac{e^{-j\omega t_0}}{W+j\omega}.$$

(b) From (9.1-16): $h_{opt}(t) = K x(t_0 - t) = K u(t - t_0) e^{-W(t-t_0)}$.



(c) Yes, any $t_0 \geq 0$ will make the filter causal.

(d) Use the result of Problem 9.1-8:

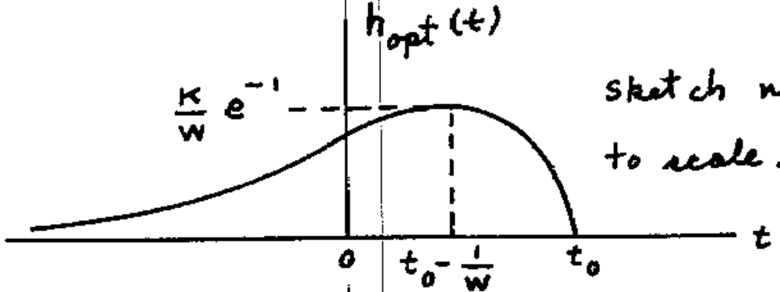
$$\left(\frac{\hat{S}_o}{N_o}\right)_{max} = \frac{2}{\mathcal{N}_0}\int_{-\infty}^{0} e^{2Wt}\,dt = \frac{1}{\mathcal{N}_0 W} \quad \text{from (C-45).}$$

9.1-16. (a) From pair 16 of Appendix E: $x(t) = u(t)\,t \cdot e^{-Wt} \leftrightarrow X(\omega) = \dfrac{1}{(W+j\omega)^2}$. From (9.1-14):

$$H_{opt}(\omega) = K\,\frac{e^{-j\omega t_0}}{(W-j\omega)^2}.$$

(b) Use (9.1-16) to get

$$h_{opt}(t) = K x(t_0 - t) = K u(t_0 - t)[t_0 - t]\,e^{-W(t_0-t)}.$$



(c) No value of t_0 will make the filter causal.

(d) From the result of Problem 9.1-8:

$$\left(\frac{\hat{S}_0}{N_0}\right)_{max} = \frac{2}{\mathcal{N}_0}\int_0^{\infty} t^2 e^{-2Wt}\,dt = \frac{1}{2\mathcal{N}_0 W^3}$$ after using (C-47).

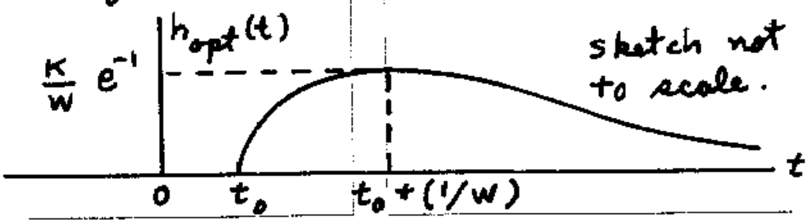
9.1-17. (a) $X(\omega) = \int_{-\infty}^{\infty} -u(-t)\,t\,e^{Wt}e^{-j\omega t}\,dt$. Let $\eta = -t$, $d\eta = -dt$: $X(\omega) = \int_0^{\infty} \eta\, e^{-W\eta + j\omega\eta}\,d\eta = \dfrac{1}{(W-j\omega)^2}$

after using (C-46). From (9.1-14):

$$H_{opt}(\omega) = K\,\frac{e^{-j\omega t_0}}{(W+j\omega)^2}.$$

(b) Use (9.1-16):

$$h_{opt}(t) = K x(t_0 - t) = K u(t - t_0)[t - t_0]\,e^{-W(t-t_0)}.$$



9.1-17. (Continued) (c) Yes, any $t_0 \geq 0$ will make the filter causal. (d) From the result of Problem 9.1-8 and (C-47): $\left(\frac{\hat{S}_0}{\mathcal{N}_0}\right)_{max} = \frac{2}{\mathcal{N}_0}\int_{-\infty}^{0} t^2 e^{2Wt}\,dt = \frac{1}{2\mathcal{N}_0 W^3}$.

9.1-18. $x_0(T) = \int_0^T x^2(t)\,dt$ so $\hat{S}_0 = x_0^2(T) = \left[\int_0^T x^2(\xi)\,d\xi\right]^2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega\right]^2$ from Parseval's theorem.

For noise: $N_0(T) = \int_0^T N(\xi)\,x(\xi)\,d\xi$, $N_0 = E[N_0^2(T)]$

$$= E\left[\int_0^T\int_0^T N(\xi)\,x(\xi)\,N(\eta)\,x(\eta)\,d\xi\,d\eta\right]$$

$$= \int_0^T\int_0^T R_{NN}(\eta-\xi)\,x(\xi)\,x(\eta)\,d\xi\,d\eta$$

$$= \int_0^T\int_0^T \frac{\mathcal{N}_0}{2}\,\delta(\eta-\xi)\,x(\xi)\,x(\eta)\,d\xi\,d\eta = \frac{\mathcal{N}_0}{2}\int_0^T x^2(\xi)\,d\xi .$$

$$\left(\frac{\hat{S}_0}{N_0}\right)_{time\ T} = \frac{\left[\int_0^T x^2(\xi)\,d\xi\right]^2}{\frac{\mathcal{N}_0}{2}\int_0^T x^2(\xi)\,d\xi} = \frac{2}{\mathcal{N}_0}\int_0^T x^2(\xi)\,d\xi = \left(\frac{\hat{S}_0}{N_0}\right)_{max} \quad (1)$$

from the result of Problem 9.1-8 and the fact that $x(t)$ is nonzero only for $0 < t < T$.

Alternatively, using the above spectral result in (1),

$$\left(\frac{\hat{S}_0}{N_0}\right)_{time\ T} = \frac{2}{\mathcal{N}_0}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \left(\frac{\hat{S}_0}{N_0}\right)_{max}$$ from

(9.1-12) for white noise. Since the receiver produces the maximum signal-to-noise ratio of a matched filter (at time T) it must therefore be the equivalent of a matched filter.

9.1-19. From pair 20 of Appendix E:

$$e^{-t^2/2\sigma^2} \longleftrightarrow \sigma\sqrt{2\pi}\, e^{-\sigma^2\omega^2/2}.$$

Identify $\alpha = 1/2\sigma^2$ so $A e^{-\alpha t^2} \longleftrightarrow \sqrt{\frac{\pi}{\alpha}}\, A e^{-\omega^2/4\alpha} =$ $X(\omega)$ and

$$H_{opt}(\omega) = K\sqrt{\frac{\pi}{\alpha}}\, A\, e^{-\frac{\omega^2}{4\alpha} - j\omega t_0}$$

from (9.1-14).

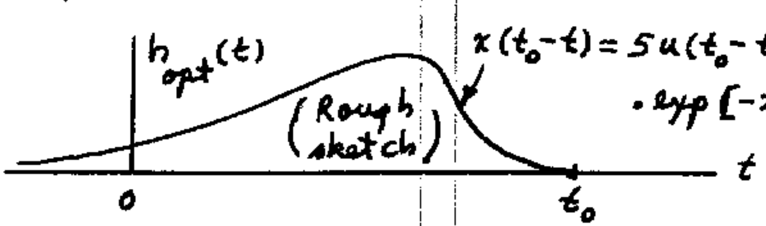
9.1-20. (a) From pair 17, Appendix E, $X(\omega) = \dfrac{10}{(2+j\omega)^3}$.

Thus, $H_{opt}(\omega) = \dfrac{1}{2\pi C}\dfrac{X^*(\omega)}{\mathcal{S}_{NN}(\omega)}e^{-j\omega t_0} = \dfrac{K e^{-j\omega t_0}}{(2-j\omega)^3}$, K = an arbitrary real constant. (b) Use (C-31) to get

$$(\hat{S}_0/N_0) = \frac{1}{2\pi(\mathcal{N}_0/2)}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \frac{50}{\pi}\int_{-\infty}^{\infty}\frac{100\, d\omega}{(4+\omega^2)^3}$$

$= 7500/2^7 = 58.5938$ (or 17.68 dB).

(c)



$x(t_0-t) = 5u(t_0-t)(t_0-t)^2 \cdot \exp[-2(t_0-t)]$

(d) No, because of its response prior to $t=0$.

9.1-21. From pair 17 of Appendix E:

$$x(t) = u(t)t^2 e^{-Wt} \longleftrightarrow X(\omega) = \frac{2}{(W+j\omega)^3}$$

(a) $H_{opt}(\omega) = \dfrac{1}{2\pi C}\dfrac{X^*(\omega)}{\mathcal{S}_{NN}(\omega)}e^{-j\omega t_0} = \dfrac{1}{2\pi C}\dfrac{2(W_N^2+\omega^2)}{P(W-j\omega)^3}e^{-j\omega t_0}$

or $H_{opt}(\omega) = \dfrac{1}{\pi C P}\dfrac{(W_N^2+\omega^2)}{(W-j\omega)^3}e^{-j\omega t_0}$.

(b) From (D-15) and pair 17:

$$W_N^2 u(-t)t^2 e^{Wt} \longleftrightarrow \frac{2W_N^2}{(W-j\omega)^3} \qquad (1)$$

9.1-21. (Continued)

From (D-15) and pair 17:

$$-\frac{d^2}{dt^2}[u(-t)\,t^2 e^{wt}] = -(2+4wt+w^2t^2)\,u(-t)\,e^{wt}$$

$$\longleftrightarrow \frac{2\omega^2}{(W-j\omega)^3}. \qquad (2)$$

By combining (1) and (2) and using (D-6):

$$h_{opt}(t) = \frac{1}{2\pi CP}\,u(-t+t_0)\left[(W_N^2-W^2)(t-t_0)^2-4W(t-t_0)-2\right]e^{W(t-t_0)}.$$

(c) Use (C-31) and (C-32) to evaluate

$$\left(\frac{\hat{S}_o}{N_o}\right)_{max} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|X(\omega)|^2}{\mathcal{S}_{NN}(\omega)}\,d\omega = \frac{2}{\pi P}\int_{-\infty}^{\infty}\frac{(W_N^2+\omega^2)\,d\omega}{(W^2+\omega^2)^3}$$

$$= \frac{1}{4PW^3}\left[1+3\left(\frac{W_N}{W}\right)^2\right].$$

9.1-22. (a) $x_o(t) = \int_{-\infty}^{\infty} x(\xi)\,h(t-\xi)\,d\xi = \int_{-\infty}^{\infty} A\,\mathrm{rect}\left(\frac{\xi}{\tau}\right) W u(t-\xi)\cdot\exp[-W(t-\xi)]\,d\xi$. Because of the unit-step and rect functions $x_o(t)$ must be evaluated for two cases: $-\tau/2 < t < \tau/2$ and $\tau/2 < t$. For the former case $x_o(t) = AW\int_{-\tau/2}^{t} e^{-W(t-\xi)}\,d\xi = A[1-e^{-W(t+\tau/2)}]$, $-\tau/2 < t < \tau/2$. In the latter case $x_o(t) = AW\int_{-\tau/2}^{\tau/2} e^{-W(t-\xi)}\,d\xi = Ae^{-Wt}[e^{W\tau/2}-e^{-W\tau/2}]$, $\tau/2 < t$. Next, $E[N_o^2(t)] = \frac{\mathcal{N}_0}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2\,d\omega = \frac{\mathcal{N}_0}{4\pi}\int_{-\infty}^{\infty}\frac{W^2\,d\omega}{W^2+\omega^2} = \mathcal{N}_0 W/4$, so

$$\left(\frac{\hat{S}_o}{N_o}\right) = \frac{4A^2}{\mathcal{N}_0 W}\left[1-e^{-W(t+\frac{\tau}{2})}\right]^2, \quad -\tau/2 < t < \tau/2$$

$$= \frac{4A^2}{\mathcal{N}_0 W}\left[e^{W\tau/2}-e^{-W\tau/2}\right]^2 e^{-2Wt}, \quad \tau/2 < t.$$

As a function of t $(\hat{S}_o/N_o)$ is maximum when

(9.1-22.) (Continued)

$t = t_0 = \tau/2$. At the maximum we have

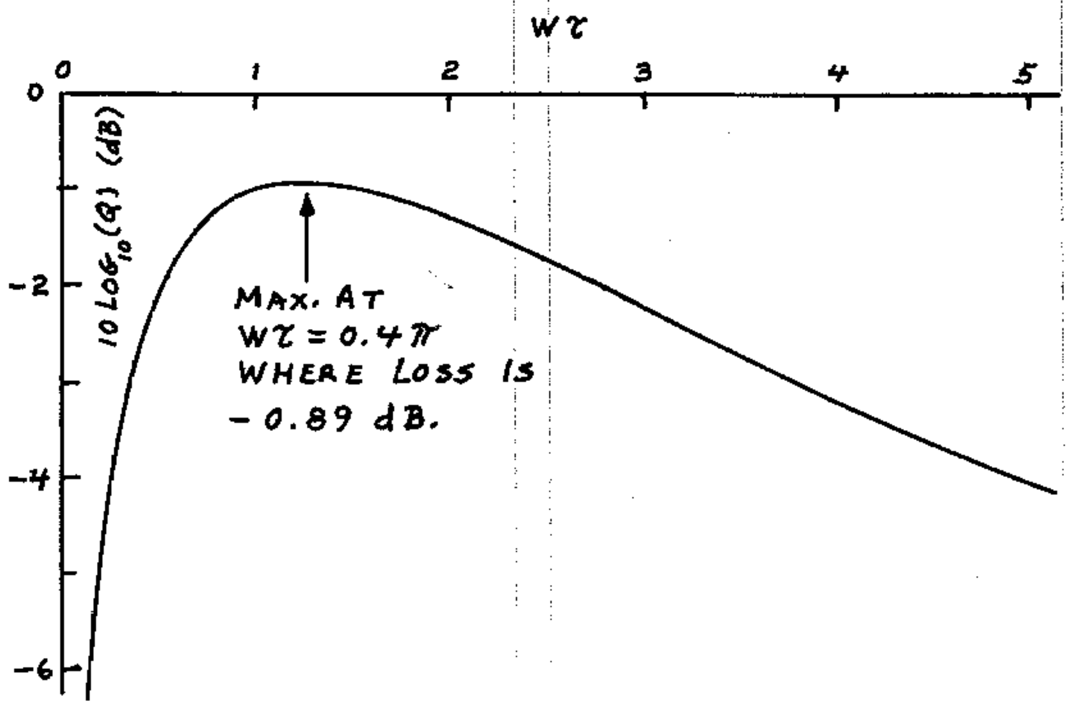
$$\left(\frac{\hat{S}_0}{N_0}\right)_{t_0=\tau/2} = \frac{4A^2}{\mathcal{N}_0^2 W}\left[1-e^{-W\tau}\right]^2 . \qquad (1)$$

(b) Maximize (1) by choosing W through differentiation. $\frac{d}{dW}(\hat{S}_0/N_0)_{t=\tau/2} = \frac{4A^2}{\mathcal{N}_0 W^2}\left\{2W[1-e^{-W\tau}](\tau e^{-W\tau}) - [1-e^{-W\tau}]^2\right\} = 0$ when $W = 0.4\pi/\tau$.

(c) For a matched filter $(S_0/N_0)_{max} = 2E/\mathcal{N}_0$ (Problem 9.1-8) $= 2A^2\tau/\mathcal{N}_0$. The ratio of these results is

$$\frac{(\hat{S}_0/N_0)_{t=\tau/2}}{(\hat{S}_0/N_0)_{max}} = \frac{2}{W\tau}\left(1 - e^{-W\tau}\right)^2 = Q$$

This function is less than unity and is the loss in the system relative to a matched filter. It is plotted below.



★ 9.1-23. Here $h(t) = W^2 u(t)\, t\, e^{-Wt} \leftrightarrow H(\omega) = \dfrac{W^2}{(W+j\omega)^2}$.

(a) $x_o(t) = \int_{-\infty}^{\infty} x(\xi) h(t-\xi)\, d\xi$

$= AW^2 e^{-Wt} \int_{-\infty}^{\infty} \text{rect}\left(\frac{\xi}{\tau}\right) u(t-\xi)\left[t e^{W\xi} - \xi e^{W\xi}\right] d\xi$

After reducing the algebra involved we obtain

$$x_o(t) = A\left\{1 - \left[1 + W\left(t+\frac{\tau}{2}\right)\right] e^{-W\left(t+\frac{\tau}{2}\right)}\right\}, \quad -\tau/2 < t < \tau/2$$

$$= A\left\{\left[1 + W\left(t-\frac{\tau}{2}\right)\right] e^{-W(t-\tau/2)} - \left[1 + W\left(t+\frac{\tau}{2}\right)\right] e^{-W\left(t+\frac{\tau}{2}\right)}\right\}, \quad \tau/2 < t.$$

Next, $E[N_o^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{N}_0}{2} |H(\omega)|^2 d\omega = \frac{\mathcal{N}_0}{4\pi}\int_{-\infty}^{\infty} \frac{W^4\, d\omega}{(W^2+\omega^2)^2} = \frac{\mathcal{N}_0 W}{8}$.

Since $E[N_o^2(t)]$ is not a function of t $(\hat{S}_o/N_o)$ is maximum with t when $|x_o(t)|$ is maximum. Calculation shows $|x_o(t)|$ is monotonically increasing for $-\tau/2 < t < \tau/2$ so the maximum must occur for $\tau/2 < t$. Calculation shows $x_o(t)$ is positive for $\tau/2 < t$ so $|x_o(t)|$ is maximum when $x_o(t)$ is maximum. By differentiation of $x_o(t)$ for $\tau/2 < t$ we find the derivative is zero when

$t_o = t = \dfrac{\tau}{2} + \dfrac{\tau}{e^{W\tau} - 1}$. The second derivative is negative showing that a true maximum occurs.

(b) For t_o found in (a) the signal-to-noise ratio is

$$(\hat{S}_o/N_o)_{t=t_o} = \frac{|x_o(t_o)|^2}{\mathcal{N}_0 W/8} = \frac{8A^2}{\mathcal{N}_0 W}\left[(1-e^{-W\tau})^2 e^{-2\frac{W\tau}{e^{W\tau}-1}}\right].$$

For a true matched filter $(\hat{S}_o/N_o)_{max} = \frac{2}{\mathcal{N}_0}\int_{-\infty}^{\infty} |x(t)|^2 dt$

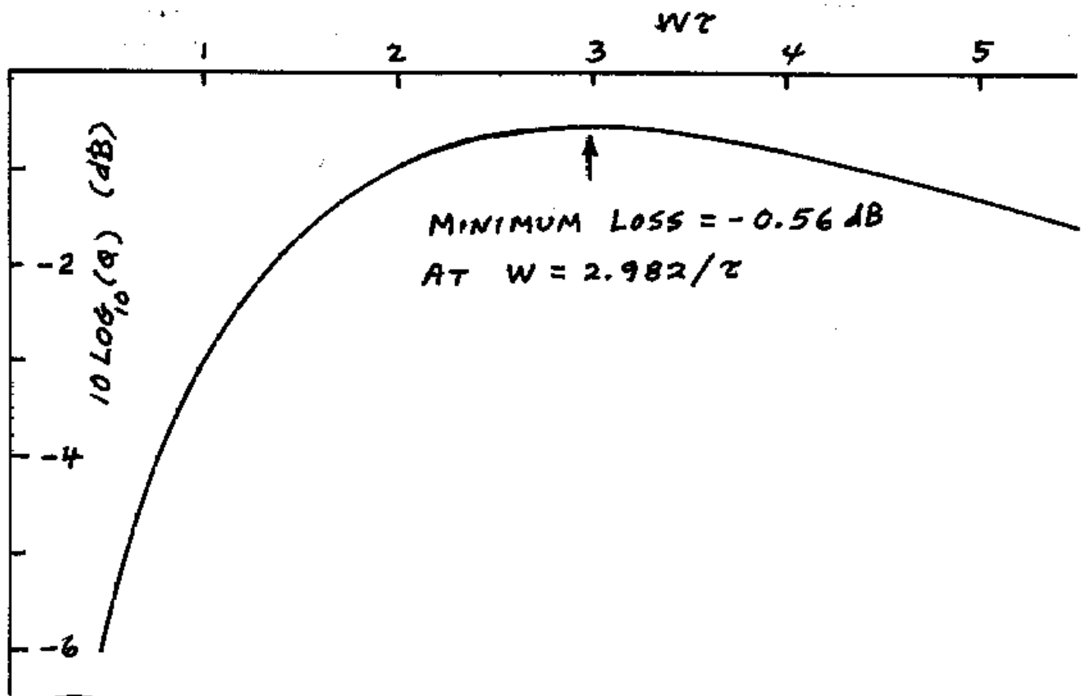
$= 2A^2\tau/\mathcal{N}_0$ so

★ 9.1-23. (Continued)

$$\frac{(\hat{S}_0/N_0)_{t=t_0}}{(\hat{S}_0/N_0)_{max}} = \frac{4}{W\tau}(1-e^{-W\tau})^2 e^{-\frac{2W\tau}{e^{W\tau}-1}} \triangleq Q$$

This function [actually $10\log_{10}(Q)$] is plotted below. Minimum loss (best performance) occurs for $W = 2.982/\tau$. (c) The minimum loss relative to a matched filter is found to be 0.56 dB which occurs at

$$t_0 = \frac{\tau}{2}\left[1 + \frac{2}{e^{2.982}-1}\right] = 1.1068\,\tau/2 = 0.5534\tau.$$



9.1-24. For the given waveform we find its spectrum

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = A\int_{-\tau}^{\tau}\left[1-\left(\frac{t}{\tau}\right)^2\right]e^{-j\omega t}dt$$

$$= \frac{4A\tau}{\omega^2\tau^2}\left[\frac{\sin(\omega\tau)}{\omega\tau} - \cos(\omega\tau)\right].$$

(a) From (9.1-15) $h_{opt}(t) = KA\,\mathrm{rect}\left[\frac{t_0-t}{2\tau}\right]\left[1-\left(\frac{t_0-t}{\tau}\right)^2\right]$

(9.1-24.) (Continued)

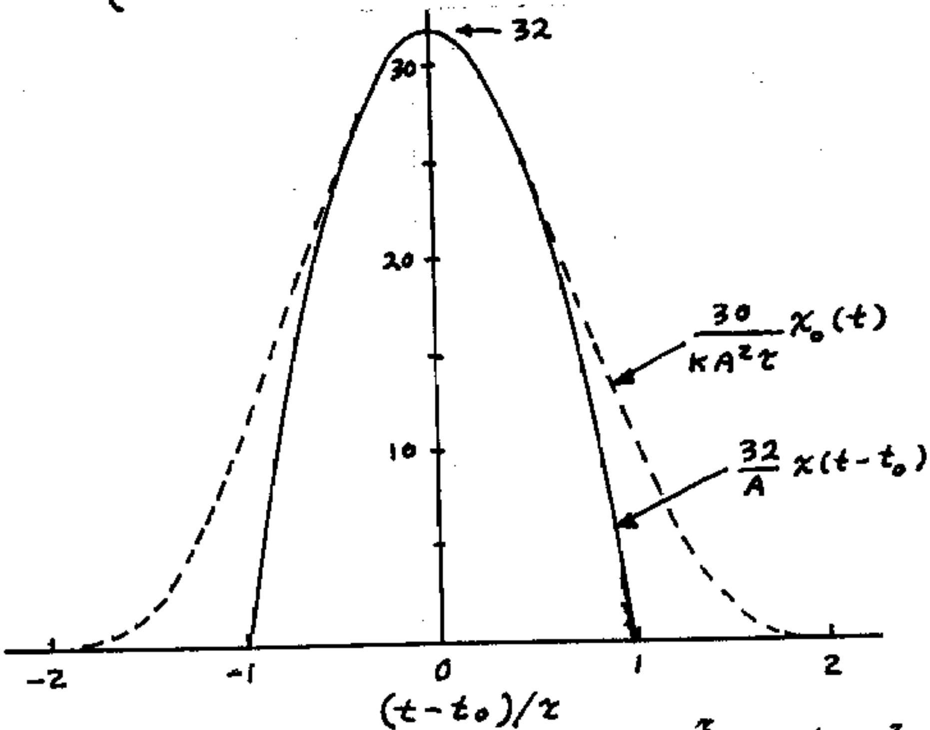
$= KA\,\mathrm{rect}\left[\frac{t-t_0}{2\tau}\right]\left[1-\left(\frac{t-t_0}{\tau}\right)^2\right]$. The filter's output is

$x_0(t)=\int_{-\infty}^{\infty} x(\xi)\,h_{opt}(t-\xi)\,d\xi = KA^2\int_{-\infty}^{\infty}\mathrm{rect}\left(\frac{\xi}{2\tau}\right)\mathrm{rect}\left[\frac{t-\xi-t_0}{2\tau}\right]$

$\cdot\left[1-\left(\frac{\xi}{\tau}\right)^2\right]\left[1-\left(\frac{t-\xi-t_0}{\tau}\right)^2\right]d\xi$. Evaluation follows two cases: $0\le t-t_0\le 2\tau$ and $-2\tau\le t-t_0<0$. In these cases straightforward reduction gives

$$x_0(t)=\frac{KA^2\tau}{30}\begin{cases}32-40\left(\frac{t-t_0}{\tau}\right)^2-20\left(\frac{t-t_0}{\tau}\right)^3+\left(\frac{t-t_0}{\tau}\right)^5, & -2\tau\le t-t_0<0\\ 32-40\left(\frac{t-t_0}{\tau}\right)^2+20\left(\frac{t-t_0}{\tau}\right)^3-\left(\frac{t-t_0}{\tau}\right)^5, & 0\le t-t_0\le 2\tau.\end{cases}$$

(b)



(c) $(\hat{S}_o/N_o)_{max}=\frac{2}{\mathcal{N}_0}\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{2A^2}{\mathcal{N}_0}\int_{-\tau}^{\tau}\left[1-\left(\frac{t}{\tau}\right)^2\right]^2dt$

$=32A^2\tau/15\mathcal{N}_0$. (d) From (9.1-14): $|H(0)|K=K4A\tau\left[\frac{\sin(\omega\tau)-\omega\tau\cos(\omega\tau)}{\omega^3\tau^3}\right]\Big|_{\omega=0}=\frac{4}{3}KA\tau\overset{\text{must}}{=}1$ so $K=3/4A\tau$.

(9.1-24) (Continued)

(d) A sketch of $h_{opt}(t)$ shows that the filter is causal if $t_0 \geq \tau$.

★(9.1-25.) (a) Response $= \int_{-\infty}^{\infty} \psi_R(\xi) h_{opt}(t-\xi) d\xi = \int_{-\infty}^{\infty} \psi(\xi) e^{-j\omega_d \xi} \cdot \psi^*(t_0 - t + \xi) d\xi$ so

$$\chi(t_0 - t, \omega_d) = \int_{-\infty}^{\infty} \psi(\xi) \psi^*(t_0 - t + \xi) e^{-j\omega_d \xi} d\xi = \text{Response}.$$

(b) Volume $= V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(t_0 - t, \omega_d)|^2 dt\, d\omega_d$ ← $\alpha = t_0 - t$, $d\alpha = -dt$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\alpha, \omega_d)|^2 d\alpha\, d\omega_d = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\xi) \psi^*(\xi + \alpha) \int_{-\infty}^{\infty} \psi^*(z) \cdot \psi(z + \alpha) \underbrace{\int_{-\infty}^{\infty} e^{j\omega_d (z - \xi)} d\omega_d}_{2\pi \delta(z - \xi)} dz\, d\alpha\, d\xi =$

$2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\xi) \psi^*(\xi) \psi^*(\xi + \alpha) \psi(\xi + \alpha) \cdot d\alpha\, d\xi = 2\pi \int_{-\infty}^{\infty} |\psi(\xi)|^2 \int_{-\infty}^{\infty} |\psi(\xi + \alpha)|^2 d\alpha\, d\xi$. But $\chi(0,0) = \int_{-\infty}^{\infty} |\psi(\eta)|^2 d\eta$ so $V = 2\pi |\chi(0,0)|^2$ which is independent of the form of $\psi(t)$. (c) $\chi(t_0 - t, \omega_d) = \int_{-\infty}^{\infty} \psi(\xi) \psi^*(t_0 - t + \xi) \cdot e^{-j\omega_d \xi} d\xi = \int_{-\infty}^{\infty} v(\xi) e^{j\omega_0 \xi} v^*(t_0 - t + \xi) e^{-j\omega_0 (t_0 - t + \xi)} \cdot e^{-j\omega_d \xi} d\xi = e^{-j\omega_0 (t_0 - t)} \int_{-\infty}^{\infty} v(\xi) v^*(t_0 - t + \xi) e^{-j\omega_d \xi} d\xi$.

★(9.1-26.) (a) Use Schwarz's inequality and results of solution to part (b) of Problem 9.1-25: $|\chi(\tau, \omega_d)|^2 = \left| \int_{-\infty}^{\infty} \psi(\xi) \psi^*(\xi + \tau) e^{-j\omega_d \xi} d\xi \right|^2 \leq \int_{-\infty}^{\infty} |\psi(\xi)|^2 d\xi \cdot \int_{-\infty}^{\infty} |\psi^*(\xi + \tau) e^{-j\omega_d \xi}|^2 d\xi = \int_{-\infty}^{\infty} |\psi(\xi)|^2 d\xi \int_{-\infty}^{\infty} |\psi(\xi + \tau)|^2 d\xi = |\chi(0,0)|^2$ so $|\chi(\tau, \omega_d)|^2 \leq |\chi(0,0)|^2$.

(b) $\chi(\tau, \omega_d) = \int_{-\infty}^{\infty} \psi(\xi) \psi^*(\xi + \tau) e^{-j\omega_d \xi} d\xi$

★ 9.1-26. (Continued)

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega) e^{j\omega\xi} d\omega \, \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi^*(\alpha) e^{-j\alpha(\xi+\tau)} d\alpha \, e^{-j\omega_d \xi} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega) \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi^*(\alpha) e^{-j\alpha\tau} \underbrace{\int_{-\infty}^{\infty} e^{-j(\alpha+\omega_d-\omega)\xi} d\xi}_{2\pi\delta(\alpha+\omega_d-\omega)} d\alpha \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega)\Psi^*(\omega-\omega_d) e^{-j(\omega-\omega_d)\tau} d\omega \quad \leftarrow \begin{array}{l} \xi = \omega-\omega_d \\ d\xi = d\omega \end{array}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi^*(\xi)\Psi(\xi+\omega_d) e^{-j\xi\tau} d\xi \quad \leftarrow \xi = \omega, \; d\xi = d\omega$$

$$\chi(\tau,\omega_d) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi^*(\omega)\Psi(\omega+\omega_d) e^{-j\omega\tau} d\omega .$$

(c) From $\chi(\tau,\omega_d) = \int_{-\infty}^{\infty} \psi(\xi)\psi^*(\xi+\tau) e^{-j\omega_d\xi} d\xi$

$$\chi(\tau,\omega_d) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi^*(\omega)\Psi(\omega+\omega_d) e^{-j\omega\tau} d\omega$$

we set $\omega_d = 0$ or $\tau = 0$ to get the desired results.

(d) $\chi(-\tau,-\omega_d) = \int_{-\infty}^{\infty} \psi(\xi)\psi^*(\xi-\tau) e^{j\omega_d\xi} d\xi$. $\chi^*(-\tau,-\omega_d) = \int_{-\infty}^{\infty} \psi^*(\xi)\psi(\xi-\tau) e^{-j\omega_d\xi} d\xi \leftarrow \begin{array}{l} \eta = \xi-\tau \\ d\eta = d\xi \end{array}$

$$= e^{-j\omega_d\tau}\int_{-\infty}^{\infty} \psi(\eta)\psi^*(\eta+\tau) e^{-j\omega_d\eta} d\eta = e^{-j\omega_d\tau}\chi(\tau,\omega_d).$$

Thus, $\chi(\tau,\omega_d) = e^{j\omega_d\tau}\chi^*(-\tau,-\omega_d)$.

9.1-27. (a) Instantaneous frequency $= \omega(t) = \frac{d}{dt}[\omega_0 t + \frac{\mu}{2}t^2] = \omega_0 + \mu t$. But $\omega(\frac{T}{2}) - \omega(\frac{-T}{2}) \overset{\text{must}}{=} \Delta\omega =$ $\omega_0 + \mu(T/2) - \omega_0 - \mu(-T/2) = \mu T$ so $\mu = \Delta\omega/T$.

(b) $H_{opt}(\omega_0) = \int_{-\infty}^{\infty} h_{opt}(t) e^{-j\omega_0 t} dt = K\int_{-T/2}^{T/2} e^{-j\mu t^2/2} dt$

$= 2K\int_0^{T/2} [\cos(\mu t^2/2) - j\sin(\mu t^2/2)] dt$

$= 2K\sqrt{\frac{\pi}{\mu}}\left\{ C\left(\sqrt{\frac{\mu}{\pi}}\,\frac{T}{2}\right) - jS\left(\sqrt{\frac{\pi}{\mu}}\,\frac{T}{2}\right)\right\}$. As $\mu \to \infty$ we

9.1-27. (Continued)

get $|H_{opt}(\omega_0)| = 2K\sqrt{\frac{\pi}{\mu}}\left|\left\{\frac{1}{2} - j\frac{1}{2}\right\}\right| = \sqrt{\frac{2\pi}{\mu}}\left|e^{-j\pi/4}\right| \underline{\underline{\text{must}}}\ 1$

or $K = \sqrt{\mu/2\pi}$. (c) $x_o(t) = \int_{-\infty}^{\infty} x(\xi) h_{opt}(t-\xi)\, d\xi$

$= Ke^{j\omega_0 t - j\frac{\mu}{2}t^2} \int_{-\infty}^{\infty} \text{rect}\left(\frac{\xi}{T}\right) \text{rect}\left(\frac{t-\xi}{T}\right) e^{j\mu t\xi}\, d\xi$. There

are two cases: (1) $t > 0$ and (2) $t \le 0$. For $0 < t < T$:

$$x_o(t) = Ke^{j\omega_0 t - j\frac{\mu}{2}t^2} \int_{t-\frac{T}{2}}^{T/2} e^{j\mu t \xi}\, d\xi$$

$$= K(T-t)\, \text{Sa}\left[\frac{\mu t}{2}(t-T)\right] e^{j\omega_0 t}, \quad 0 < t < T.$$

For $-T < t \le 0$: $x_o(t) = Ke^{j\omega_0 t - j\frac{\mu}{2}t^2} \int_{-T/2}^{t+\frac{T}{2}} e^{j\mu t\xi}\, d\xi$

$$= K(T+t)\, \text{Sa}\left[\frac{\mu t}{2}(T+t)\right] e^{j\omega_0 t}, \quad -T < t \le 0.$$

Thus, for any t:

$$x_o(t) = KT\, \text{rect}\left(\frac{t}{2T}\right)\left(1 - \frac{|t|}{T}\right) \text{Sa}\left[\frac{\mu |t| T}{2}\left(1 - \frac{|t|}{T}\right)\right] e^{j\omega_0 t}.$$

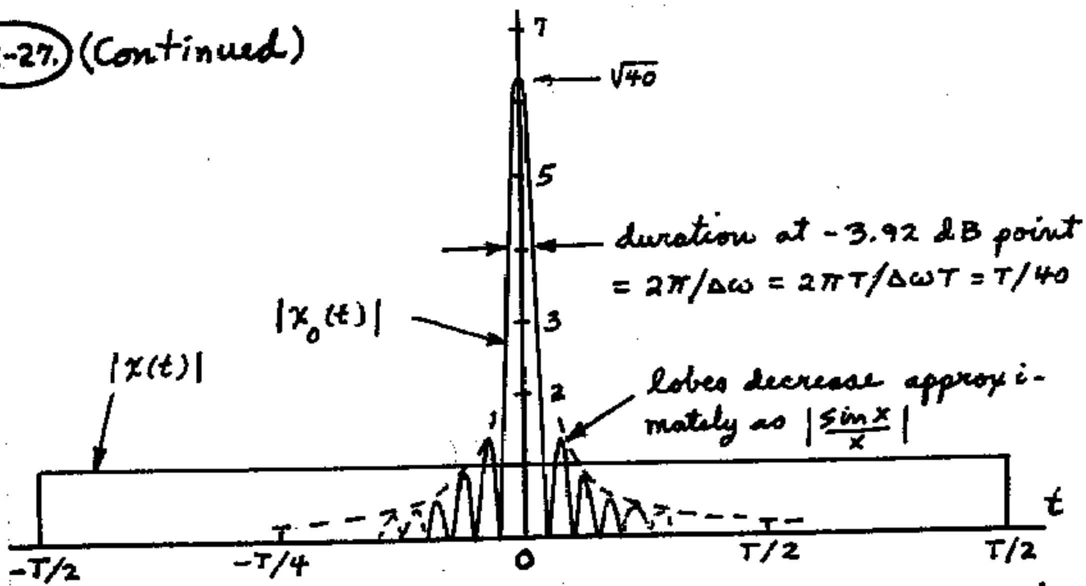
Since $\mu = \Delta\omega/T$ and $K = \sqrt{\mu/2\pi} = \sqrt{\Delta\omega/2\pi T}$:

$$x_o(t) = \sqrt{\frac{\Delta\omega T}{2\pi}}\, \text{rect}\left(\frac{t}{2T}\right)\left(1 - \frac{|t|}{T}\right) \text{Sa}\left[\frac{\Delta\omega |t|}{2}\left(1 - \frac{|t|}{T}\right)\right] e^{j\omega_0 t}.$$

For $\Delta\omega T = 80\pi$ this becomes approximately

$$|x_o(t)| \approx \sqrt{40}\, |\text{Sa}(\Delta\omega |t|/2)| .$$

(9.1-27.) (Continued)



Observations: (1) $|\chi_0(0)|/|\chi(0)| = \sqrt{40}$ so output is much larger than input. (2) Output signal's duration is much smaller (by factor of 1/40). (3) Sidelobes show up in the response that extend from $-T/2$ to $T/2$.

★(9.1-28.) (a) $H_{opt}(\omega) = \int_{-\infty}^{\infty} h_{opt}(t)\, e^{-j\omega t}\, dt$

$= K\int_{-T/2}^{T/2} e^{-j(\omega-\omega_0)t \; - j(\mu t^2/2)}\, dt$. Complete the square in the exponent to get $H_{opt}(\omega) =$

$K e^{j(\omega-\omega_0)^2/2\mu} \int_{-T/2}^{T/2} e^{j\left[\sqrt{\frac{\mu}{2}}\, t \, + \, \frac{(\omega-\omega_0)}{\sqrt{2\mu}}\right]^2} dt$. Next, define

★ 9.1-28. (Continued) $x = \sqrt{\frac{\mu}{\pi}}\, t + \frac{(\omega-\omega_0)}{\sqrt{\pi\mu}}$, $dx = \sqrt{\frac{\mu}{\pi}}\, dt$, so

$$H_{opt}(\omega) = K e^{j(\omega-\omega_0)^2/2\mu} \int_{-x_1}^{x_2} e^{-j\pi x^2/2}\, dx \sqrt{\frac{\pi}{\mu}} \quad \text{where}$$

$-x_1 \triangleq \frac{(\omega-\omega_0)}{\sqrt{\pi\mu}} - \sqrt{\frac{\mu}{\pi}}\frac{T}{2}$, $x_2 \triangleq \frac{(\omega-\omega_0)}{\sqrt{\pi\mu}} + \sqrt{\frac{\mu}{\pi}}\frac{T}{2}$. In terms of Fresnel integrals (Problem 9.1-27):

$$H_{opt}(\omega) = K\sqrt{\frac{\pi}{\mu}}\, e^{j(\omega-\omega_0)^2/2\mu}\Big\{[C(x_2)+C(x_1)] - j[S(x_2)+S(x_1)]\Big\}.$$

Because $\mu = \Delta\omega/T$ we obtain

$$x_1 = \sqrt{\frac{\Delta\omega T}{2\pi}}\left\{1 - \frac{2(\omega-\omega_0)}{\Delta\omega}\right\}\frac{1}{\sqrt{2}}$$

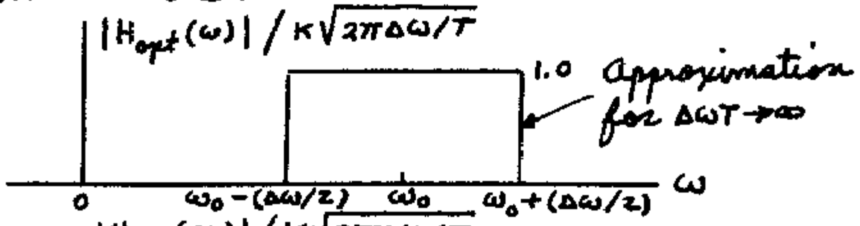
$$x_2 = \sqrt{\frac{\Delta\omega T}{2\pi}}\left\{1 + \frac{2(\omega-\omega_0)}{\Delta\omega}\right\}\frac{1}{\sqrt{2}}.$$

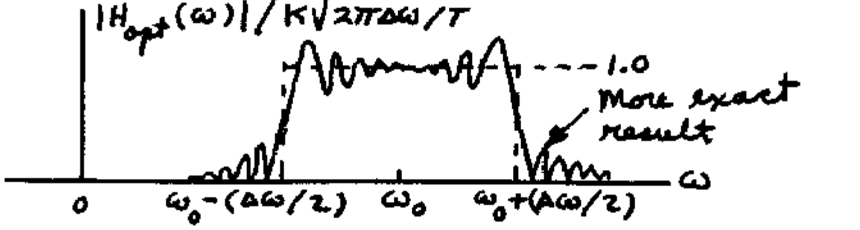
(b) For $\Delta\omega T$ large $C(x_2)$ and $S(x_2)$ abruptly change from $-1/2$ to $+1/2$ at $\omega_0 - (\Delta\omega/2)$. Similarly, $C(x_1)$ and $(S(x_1)$ abruptly change from $1/2$ to $-1/2$ as ω goes through $\omega_0 + (\Delta\omega/2)$. Thus,

$$|H_{opt}(\omega)| \approx K\sqrt{\pi\mu}\,\sqrt{2}\cdot \text{rect}\left(\frac{\omega-\omega_0}{\Delta\omega}\right)$$

$$= K\sqrt{2\pi\Delta\omega/T}\ \text{rect}\left(\frac{\omega-\omega_0}{\Delta\omega}\right),\ \Delta\omega T \to \infty.$$

Rough sketches are:





9.1-29. $\left(\frac{\hat{S}_o}{N_o}\right)_{max} = \frac{2}{\mathcal{N}_0}\int_{-\infty}^{\infty}|x(t)|^2 dt = 14$ so $\frac{\mathcal{N}_0}{2} = \frac{1}{14}\int_0^{3/2}\frac{1}{4}e^{-t/3}dt$

$= \frac{3}{56}(1-e^{-1/2}) \approx 0.02108.$

9.1-30. $\left(\frac{\hat{S}_o}{N_o}\right)_{max} = \frac{2}{\mathcal{N}_0}\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{24\pi}{10^{-8}}\int_0^{2/W} W^2 t^2 e^{-2Wt} dt = 120\pi(1-13e^{-4})$

$\approx 287.23.$

~~**9.1-31.** $\left(\frac{\hat{S}_o}{N_o}\right)_{max} = \frac{2}{\mathcal{N}_0}\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{24\pi}{10^{-8}}\int_0^{\infty} W^2 t^2 e^{-2Wt} dt = 120\pi \approx 376.99.$~~

★ **9.1-32.** (a) From (9.1-15) with $K=1$: $h_{opt}(t) = x_a(t_o - t)$ or

$h_{opt}(t) = u(t_o - t)\, W(t_o - t)\, e^{-W(t_o-t)}$. Since $x(t) = [u(t) - u(t-\frac{2}{W})]\, Wt$

$\cdot e^{-Wt}$, then $x_o(t) = \int_{-\infty}^{\infty} x(\xi) h_{opt}(t-\xi) d\xi = \int_{-\infty}^{\infty}[u(\xi) - u(\xi - \frac{2}{W})] W\xi$

$\cdot e^{-W\xi}\, u(t_o - t + \xi)\, e^{-W(t_o-t) - W\xi} d\xi$

$= W^2 e^{-W(t_o-t)}\int_0^{2/W}[(t_o-t)\xi + \xi^2]\, e^{-2W\xi}\, u(t_o - t + \xi)\, d\xi.$

There are 3 cases to consider: (1) $0 < t - t_o < 2/W$

(2) $t - t_o < 0$

(3) $2/W < t - t_o$.

In case (3) the function is zero so only cases (1) and (2) are to be evaluated. For case (1)

$$x_o(t) = W^2 e^{-W(t_o-t)}\int_{t-t_o}^{2/W}[(t_o-t)\xi e^{-2W\xi} + \xi^2 e^{-2W\xi}]d\xi$$

$$= \frac{e^{\alpha}}{4W}\left\{e^{-4}(5\alpha - 13) + e^{-2\alpha}(\alpha+1)\right\}, \quad 0 \le \alpha \triangleq W(t-t_o) < 2 \qquad \text{(A)}$$

from (C-46) and (C-47). For case (2):

$$x_o(t) = W^2\int_0^{2/W}\xi(t_o - t + \xi)\, e^{-W(t_o - t + 2\xi)} d\xi$$

$$= \frac{e^{\alpha}}{4W}\left\{e^{-4}(5\alpha - 13) - (\alpha - 1)\right\}, \quad \alpha \triangleq W(t-t_o) < 0 \qquad \text{(B)}$$

On combining (A) and (B) we have

★ 9.1-32. (Continued)

$$x_o(t) = \begin{cases} 0, & 2 < \alpha < \infty \\ \dfrac{e^{\alpha}}{4W}\left\{e^{-4}(5\alpha-13) + e^{-2\alpha}(\alpha+1)\right\}, & 0 \le \alpha \le 2 \\ \dfrac{e^{\alpha}}{4W}\left\{e^{-4}(5\alpha-13) - (\alpha-1)\right\}, & \alpha < 0 \end{cases}$$

(b) Initial evaluation (via sketches) shows the maximum of $x_o(t)$ occurs when $\alpha < 0$. Thus,

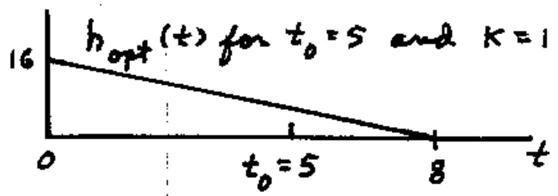
$$\frac{d}{d\alpha}\left\{e^{\alpha}\left[e^{-4}(5\alpha-13) - (\alpha-1)\right]\right\} = e^{\alpha}\left[5e^{-4}-1\right] + e^{\alpha}\left[(5\alpha-13)e^{-4} - \alpha + 1\right] = 0$$

when $\alpha = 8e^{-4}/(5e^{-4}-1) \approx -0.1613$ or, since $\alpha \triangleq W(t-t_o)$, $t_o = 0.1613/W$ when $t=0$. (c) $|x_o(t)|^2_{max} = \dfrac{e^{-2(0.1613)}}{(4W)^2}\left\{e^{-4}\left[5(-0.1613)-13\right] - (-0.1613-1)\right\}^2 = 0.03736/W^2$. (d) $H_a(\omega) = \mathfrak{F}\{x_a(t)\}$ so $x_a(t) = u(t)Wt\,e^{-Wt} \longleftrightarrow H_a(\omega) = \dfrac{W}{(W+j\omega)^2}$ from pair 16, and $|H_a(\omega)|^2 = \dfrac{W^2}{(W^2+\omega^2)^2}$. Noise power becomes $N_o = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\frac{\mathcal{N}_0}{2}|H_a(\omega)|^2 d\omega$

$$= \frac{\mathcal{N}_0}{4\pi}\int_{-\infty}^{\infty}\frac{W^2\,d\omega}{(W^2+\omega^2)^2} = \frac{\mathcal{N}_0}{8W}$$ from (C-28). Finally,

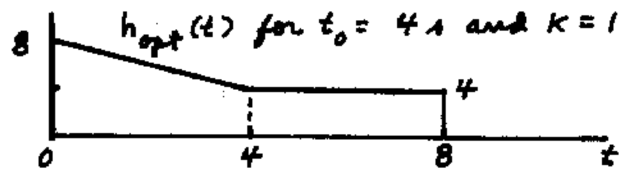
$$\left(\frac{\hat{S}_o}{N_o}\right)_{max} = \frac{0.03736/W^2}{\mathcal{N}_0/(8W)} = \frac{0.03736(4)\,24\pi}{5(10^6)\,10^{-8}} = 225.35.$$

9.1-33. (a) $t_o \ge 5$ s. (b)



(c) $\left(\dfrac{\hat{S}_o}{N_o}\right)_{max} = \dfrac{2}{\mathcal{N}_0}\displaystyle\int_{-3}^{5}(6+2t)^2 dt = \dfrac{10(16^3)}{6} \approx 6,826.7.$

9.1-34. (a) $t_o \ge 4$ s. (b)



(c) $\left(\dfrac{\hat{S}_o}{N_o}\right)_{max} = \dfrac{2}{\mathcal{N}_0}\displaystyle\int_{0}^{4}(8-t)^2 dt + \dfrac{2}{\mathcal{N}_0}\int_{4}^{8}4^2 dt = \dfrac{6400}{3} = 2,133.3.$

(9.1-35.) $H_{opt}(\omega) = K X^*(\omega) e^{-j\omega t_0}$. From (C-45) and (C-46):

$X(\omega) = \int_{-A/B}^{A/B} (A+Bt) e^{-j\omega t} dt = \frac{2A}{j\omega}\{ Sa(\omega A/B) - e^{-j\omega A/B} \}$, so

$H_{opt}(\omega) = \frac{2KA}{j\omega}\{ e^{j\omega A/B} - Sa(\omega A/B)\} e^{-j\omega t_0}$.

(9.1-36.) $h(t) = u(t) W e^{-Wt} \longleftrightarrow H(\omega) = \frac{W}{W+j\omega}$ from pair 15, Table E-1.

(a) $x_o(t) = \int_{-\infty}^{\infty} x(\xi) h(t-\xi) d\xi = \frac{A}{\tau}\int_{-\infty}^{\infty} \xi [u(\xi) - u(\xi-\tau)] u(t-\xi) W e^{-W(t-\xi)}$ $\cdot d\xi$. This output is zero for $t \le 0$. For $t>0$ two cases are of interest: $0 < t \le \tau$ and $\tau < t$.

For $0<t\le\tau$ — $x_o(t) = \int_0^t \frac{A}{\tau}\xi W e^{-W(t-\xi)} d\xi$ $= \frac{A}{W\tau} u(t)\{Wt - 1 + e^{-Wt}\}$, $0<t\le\tau$. For $t>\tau$ — $x_o(t) = \frac{AW}{\tau}e^{-Wt}\int_0^\tau \xi$ $\cdot e^{W\xi} d\xi = \frac{A}{W\tau}\{W\tau - 1 + e^{-W\tau}\} e^{-W(t-\tau)}$, $\tau<t$. Thus,

$$x_o(t) = \begin{cases} 0, & t \le 0 \\ \frac{A}{W\tau} u(t)\{Wt - 1 + e^{-Wt}\}, & 0<t\le\tau \\ \frac{A}{W\tau}\{W\tau - 1 + e^{-W\tau}\} e^{-W(t-\tau)}, & \tau<t. \end{cases}$$

$E[N^2(t)] = N_o$ = noise power $= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{N}_0}{2} \frac{W^2 d\omega}{(W^2+\omega^2)} = \frac{\mathcal{N}_0 W}{4}$. Thus,

$$\left(\frac{S_o}{N_o}\right) = \begin{cases} 0, & t \le 0 \\ \frac{4A^2}{\mathcal{N}_0 W^3 \tau^2}\{Wt - 1 + e^{-Wt}\}^2, & 0<t\le\tau \\ \frac{4A^2}{\mathcal{N}_0 W^3\tau^2}\{Wt - 1 + e^{-Wt}\}^2 e^{-2W(t-\tau)}, & \tau < t \end{cases}$$

for any time t. A sketch shows this result is maximum when $t = t_0 = \tau$. (b) At this time $(S_o/N_o)|_{t=\tau} = \frac{4A^2}{\mathcal{N}_0 W^3\tau^2}(W\tau - 1 + e^{-W\tau})^2$. Maximum with W occurs when

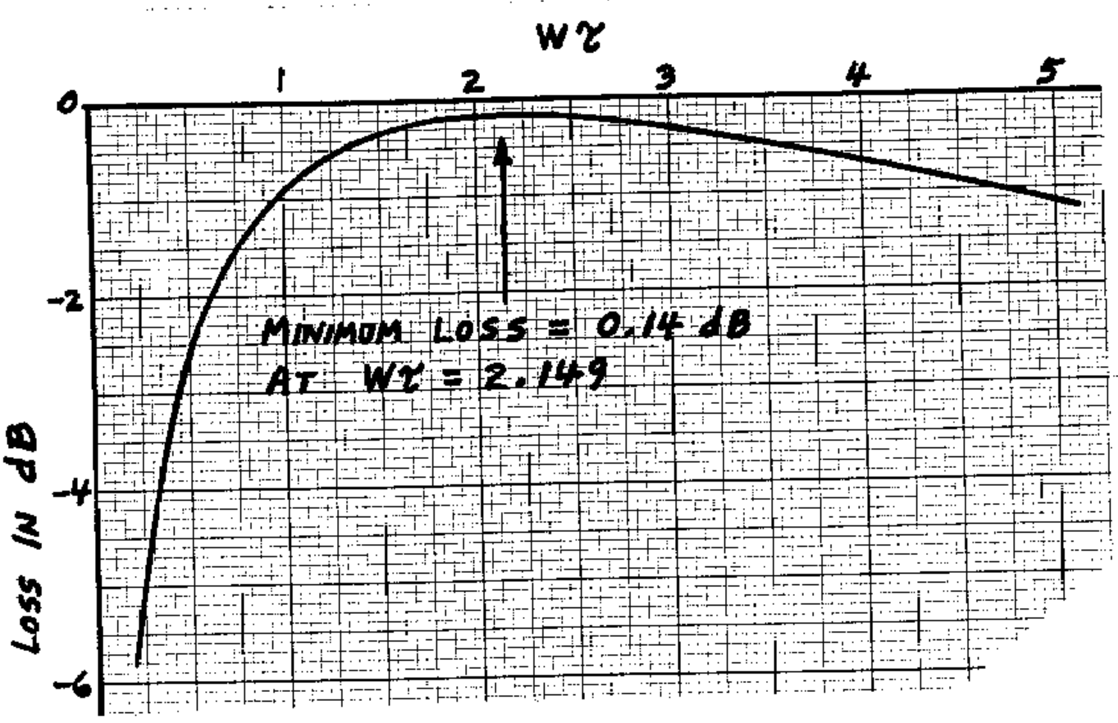
9.1-36. (Continued)

$$\frac{d(S_o/N_o)|_{t=\tau}}{dW} = \frac{4A^2}{\mathcal{N}_o\tau^2}\left\{\frac{W^3[\tau+e^{-W\tau}(-\tau)]2[W\tau-1+e^{-W\tau}]-[W\tau-1+e^{-W\tau}]^2 3W^2}{W^6}\right\}$$

$=0$, which occurs when $[1+2e^{-W\tau}][3+2W\tau]=9$. By trial and error we find $W=2.149/\tau$. This value gives $(S_o/N_o)_{max} = \frac{4A^2\tau}{\mathcal{N}_o(2.149)^3}[2.149-1+e^{-2.149}]^2 = 0.6456\,A^2\tau/\mathcal{N}_o$. (c) For a matched filter $(\hat{S}_o/N_o)_{max} = \frac{2}{\mathcal{N}_o}\int_{-\infty}^{\infty}|x(t)|^2\,dt = \frac{2}{\mathcal{N}_o}\int_0^{\tau}\frac{A^2}{\tau^2}t^2\,dt = \frac{2A^2\tau}{3\mathcal{N}_o}$. Hence,

$$\frac{(S_o/N_o)|_{t=t_o=\tau}}{(\hat{S}_o/N_o)_{max}} = \frac{6}{(W\tau)^3}[W\tau-1+e^{-W\tau}]^2.$$

This function (in dB) is plotted below for $W\tau$ up to 5. Minimum loss is 0.14 dB when $W\tau = 2.149$.



9.2-1. (a) From pairs 1 and 19 of Appendix E to get

$$R_{NN}(\tau) = (\mathcal{N}_0/2)\,\delta(\tau) \longleftrightarrow (\mathcal{N}_0/2) = \mathcal{S}_{NN}(\omega)$$

$$R_{XX}(\tau) = \frac{PW}{2} e^{-W|\tau|} \longleftrightarrow \frac{PW}{2}\,\frac{2W}{W^2+\omega^2} = \frac{PW^2}{W^2+\omega^2} = \mathcal{S}_{XX}(\omega).$$

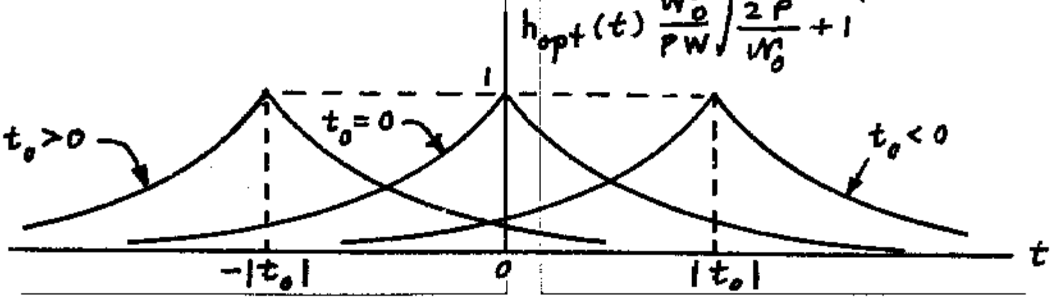
From these results and (9.2-20):

$$H_{opt}(\omega) = \frac{(2PW^2/\mathcal{N}_0)}{\left(\frac{2P}{\mathcal{N}_0}+1\right)W^2+\omega^2}\, e^{j\omega t_0}.$$

(b) By using (D-6) and pair 19 of Appendix E:

$$\frac{PW}{\mathcal{N}_0\sqrt{\frac{2P}{\mathcal{N}_0}+1}}\, e^{-W\sqrt{\frac{2P}{\mathcal{N}_0}+1}\,|t+t_0|} \longleftrightarrow \frac{(2PW^2/\mathcal{N}_0)}{W^2\left(\frac{2P}{\mathcal{N}_0}+1\right)+\omega^2}\, e^{j\omega t_0}$$

$$h_{opt}(t) = \frac{PW}{\mathcal{N}_0\sqrt{\frac{2P}{\mathcal{N}_0}+1}} \exp\left\{-W\sqrt{\frac{2P}{\mathcal{N}_0}+1}\,|t+t_0|\right\}.$$



9.2-2. Substitute $\mathcal{S}_{NN}(\omega)$ and $\mathcal{S}_{XX}(\omega)$ from the solution to Problem 9.2-1 into (9.2-22) to get

$$E[\varepsilon^2(t)]_{min} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\dfrac{\mathcal{N}_0 PW^2/2}{W^2+\omega^2}}{\dfrac{\mathcal{N}_0}{2}+\dfrac{PW^2}{W^2+\omega^2}}\, d\omega$$

$$= \frac{PW^2}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{W^2\left(\frac{2P}{\mathcal{N}_0}+1\right)+\omega^2} \underset{\text{from (C-25)}}{=} \frac{PW}{2\sqrt{\frac{2P}{\mathcal{N}_0}+1}}$$

9.2-3. From the solution to Problem 9.2-1: $\mathcal{S}_{XX}(\omega) = PW^2/(W^2+\omega^2)$. From pair 19 of Appendix E:

$$R_{NN}(\tau) = W_N e^{-W_N|\tau|} \longleftrightarrow \mathcal{S}_{NN}(\omega) = \frac{2W_N^2}{W_N^2+\omega^2}.$$

(a) Substitute into (9.2-20):

$$H_{opt}(\omega) = \frac{\dfrac{PW^2}{W^2+\omega^2}\, e^{j\omega t_0}}{\dfrac{PW^2}{W^2+\omega^2} + \dfrac{2W_N^2}{W_N^2+\omega^2}}$$

$$= \frac{P}{\left[P+2\left(\frac{W_N}{W}\right)^2\right]} \cdot \frac{(W_N^2+\omega^2)\, e^{j\omega t_0}}{\left[\dfrac{W_N^2(P+2)}{P+2\left(\frac{W_N}{W}\right)^2} + \omega^2\right]}.$$

(b) Write $H_{opt}(\omega)$ as

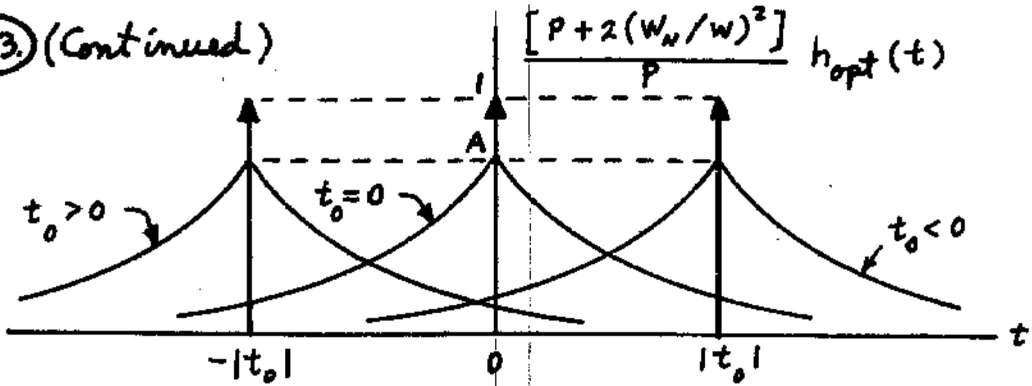
$$H_{opt}(\omega) = \frac{P}{\left[P+2\left(\frac{W_N}{W}\right)^2\right]} \left\{ 1 + \frac{\dfrac{2W_N^2\left[\left(\frac{W_N}{W}\right)^2 - 1\right]}{P+2\left(\frac{W_N}{W}\right)^2}}{\dfrac{W_N^2(P+2)}{P+2\left(\frac{W_N}{W}\right)^2} + \omega^2} \right\} e^{j\omega t_0}.$$

Next, use pairs 1 and 19 of Appendix E to obtain

$$h_{opt}(t) = \frac{P}{\left[P+2\left(\frac{W_N}{W}\right)^2\right]} \left\{ \delta(t+t_0) \right.$$

$$\left. + \frac{W_N\left[\left(\frac{W_N}{W}\right)^2 - 1\right] e^{-W_N\sqrt{\frac{P+2}{P+2\left(\frac{W_N}{W}\right)^2}}\,|t+t_0|}}{\sqrt{\dfrac{P+2}{P+2\left(\frac{W_N}{W}\right)^2}}\left[P+2\left(\frac{W_N}{W}\right)^2\right]} \right\}.$$

9.2-3. (Continued)



$$A = \frac{W_N[(W_N/W)^2-1]}{\sqrt{\dfrac{P+2}{P+2(W_N/W)^2}}} \cdot \frac{1}{\left[P+2\left(\frac{W_N}{W}\right)^2\right]}.$$

9.2-4. Substitute $\mathcal{S}_{XX}(\omega)$ and $\mathcal{S}_{NN}(\omega)$ from the solution to Problem 9.2-3 into (9.2-22):

$$E[\varepsilon^2(t)]_{min} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\dfrac{PW^2}{W^2+\omega^2}\cdot\dfrac{2W_N^2}{W_N^2+\omega^2}}{\dfrac{PW^2}{W^2+\omega^2}+\dfrac{2W_N^2}{W_N^2+\omega^2}}\,d\omega$$

$$= \frac{PW^2W_N^2}{\pi}\int_{-\infty}^{\infty}\frac{d\omega}{[(P+2)W^2W_N^2]+[PW^2+2W_N^2]\,\omega^2}$$

$$= PWW_N/\sqrt{P+2}\sqrt{PW^2+2W_N^2} \text{ from (C-25).}$$

9.2-5. (a) From (9.2-20):

$$H_{opt}(\omega) = \frac{\dfrac{9}{9+\omega^4}\,e^{j\omega t_0}}{\dfrac{9}{9+\omega^4}+\dfrac{3}{6+\omega^4}} = \frac{3(6+\omega^4)\,e^{j\omega t_0}}{27+4\omega^4}.$$

(b) From (9.2-22):

$$E[\varepsilon^2(t)]_{min} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\dfrac{9}{9+\omega^4}\cdot\dfrac{3}{6+\omega^4}}{\dfrac{9}{9+\omega^4}+\dfrac{3}{6+\omega^4}}\,d\omega = \frac{9}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{27+4\omega^4}$$

9.2-5. (Continued) $= \dfrac{1}{4(3)^{1/4}} \approx 0.19$ from (C-37).

9.2-6. (a) Use (9.2-20):

$$H_{opt}(\omega) = \frac{\dfrac{A}{W^2+\omega^4}}{\dfrac{A}{W^2+\omega^4} + \dfrac{\mathcal{N}_0}{2}} e^{j\omega t_0} = \frac{2A}{\mathcal{N}_0} \frac{e^{j\omega t_0}}{\left[\left(W^2 + \dfrac{2A}{\mathcal{N}_0}\right) + \omega^4\right]} .$$

(b) Use (9.2-22):

$$E[\varepsilon^2(t)]_{min} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\dfrac{A}{W^2+\omega^4} \cdot \dfrac{\mathcal{N}_0}{2}}{\dfrac{A}{W^2+\omega^4} + \dfrac{\mathcal{N}_0}{2}} d\omega = \frac{A}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\left(\dfrac{2A}{\mathcal{N}_0} + W^2\right) + \omega^4}$$

$$= \frac{A}{2\sqrt{2}\left(\dfrac{2A}{\mathcal{N}_0} + W^2\right)^{3/4}} \quad \text{from (C-37).}$$

9.2-7. (a) Use (9.2-20): $H_{opt}(\omega) = \dfrac{2\sqrt{2} P_{XX} W_X \omega^2 e^{j\omega t_0}}{2\sqrt{2} P_{XX} W_X \omega^2 + \frac{\mathcal{N}_0}{2}(W_X^4 + \omega^4)}$

or

$$H_{opt}(\omega) = \frac{(4\sqrt{2} P_{XX}/\mathcal{N}_0) W_X \omega^2 \exp(j\omega t_0)}{W_X^4 + \left(\dfrac{4\sqrt{2} P_{XX} W_X}{\mathcal{N}_0}\right)\omega^2 + \omega^4} .$$

(b) Use (9.2-22): $E[\varepsilon^2(t)]_{min} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{(2\sqrt{2} P_{XX} W_X)\ \omega^2 d\omega}{W_X^4 + \left(\frac{4\sqrt{2} P_{XX} W_X}{\mathcal{N}_0}\right)\omega^2 + \omega^4}$.

Use the integral of the hint with $a_0 = 1$, $a_2 = W_X^2$, $b_1 = 0$, $b_0 = -1$ and $a_1 = \{2W_X^2 + (4\sqrt{2} P_{XX} W_X/\mathcal{N}_0)\}^{1/2}$:

$$E[\varepsilon^2(t)]_{min} = P_{XX}\Big/\sqrt{1 + \frac{2\sqrt{2} P_{XX}}{\mathcal{N}_0 W_X}} .$$

(c) $E[\varepsilon^2(t)]_{min} = 2\Big/\sqrt{1 + [2\sqrt{2}(2)/0.2(15)]} = 1.1774.$

(9.2-8.) $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A\,d\omega}{(W_X^2+\omega^2)^2} = \frac{A}{4W_X^3}$ so $A = 4P_{XX}W_X^3$.

(a) From (9.2-20):

$$H_{opt}(\omega) = \frac{(8P_{XX}W_X^3/\mathcal{N}_0)\,e^{j\omega t_0}}{\left(W_X^4 + \frac{8P_{XX}W_X^3}{\mathcal{N}_0}\right) + 2W_X^2\omega^2 + \omega^4}.$$

(b) $$E[\epsilon^2(t)]_{min} = \frac{4P_{XX}W_X^3}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\left(W_X^4 + \frac{8P_{XX}W_X^3}{\mathcal{N}_0}\right) + 2W_X^2\omega^2 + \omega^4},$$

from (9.2-22). From the hint with $b_0 = 0$, $b_1 = 1$, $a_0 = 1$, $a_2 = W_X^2\sqrt{1 + \frac{8P_{XX}}{\mathcal{N}_0 W_X}}$, and $a_1 = \sqrt{2}\,W_X\left\{1 + \sqrt{1 + \frac{8P_{XX}}{\mathcal{N}_0 W_X}}\right\}^{1/2}$:

$$E[\epsilon^2(t)]_{min} = \frac{\sqrt{2}\,P_{XX}}{\sqrt{1 + \frac{8P_{XX}}{\mathcal{N}_0 W_X}}\sqrt{1 + \sqrt{1 + \frac{8P_{XX}}{\mathcal{N}_0 W_X}}}}.$$

(c) $$E[\epsilon^2(t)]_{min} = \frac{\sqrt{2}(2)}{\sqrt{1 + \frac{8(2)}{0.2(15)}}\sqrt{1 + \sqrt{1 + \frac{8(2)}{0.2(15)}}}} = 0.5993.$$

(9.2-9.) (a) From (9.2-20):

$$H_{opt}(\omega) = \frac{(16^2+\omega^2)^2\,e^{j\omega t_0}}{16^4 + [2(16^2) + 20(10^2)]\omega^2 + 21\omega^4}.$$

(b) From (9.2-22):

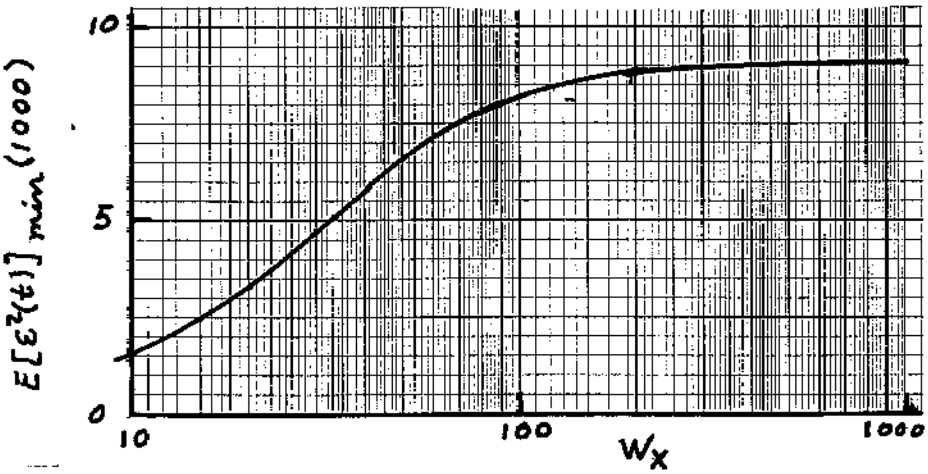
$$E[\epsilon^2(t)]_{min} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2\,d\omega}{16^4 + [2(16^2) + 20(10^2)]\omega^2 + 21\omega^4}.$$

Use the integral of Problem 9.2-7 with $b_1 = 0$, $b_0 = -1$, $a_2 = 16^2$, $a_0 = \sqrt{21}$, and $a_1 = \sqrt{2000 + 2(16^2)[1+\sqrt{21}]}$:

$$E[\epsilon^2(t)]_{min} = \frac{1}{2\sqrt{21}\sqrt{2000 + 2(16^2)[1+\sqrt{21}]}} = 1.5654(10^{-3}).$$

★ **9.2-10.** From (9.2-22) $$E[\varepsilon^2(t)]_{min} = \frac{W_X^2}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2\, d\omega}{16^4 W_X^2 + (2W_X^2 16^2 + 2000\, W_X^2)\omega^2 + (2000 + W_X^2)\omega^4}$$

$$= \frac{1}{8\sqrt{1+\frac{2000}{W_X^2}}\sqrt{157+32\sqrt{1+\frac{2000}{W_X^2}}}}$$ from the integral given in Problem 9.2-7 with $b_1 = 0$, $b_0 = -1$, $a_2 = 16^2 W_X$, $a_0 = W_X\sqrt{1+(2000/W_X^2)}$, and $a_1^2 = 16W_X^2\left[157 + 32\sqrt{1+(2000/W_X^2)}\right]$.



In a physical sense, increasing W_X amounts to broadening the bandwidth of the noise while keeping the in-band noise density constant; total noise power also increases linearly with W_X.

9.2-11. (a) From (9.2-22): $$E[\varepsilon^2(t)]_{min} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2P_{XX}W_X 4P_{NN}W_N^3\, d\omega}{2P_{XX}W_X(W_N^2+\omega^2)^2 + 4P_{NN}W_N^3(W_X^2+\omega^2)}$$

$$= \frac{4P_{NN}W_N^3}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\left[W_N^4 + 2\frac{P_{NN}}{P_{XX}}W_N^3 W_X\right] + \left[2W_N^2 + 2\frac{P_{NN}}{P_{XX}}\frac{W_N^3}{W_X}\right]\omega^2 + \omega^4}$$

9.2-11. (Continued)

$$= \frac{\sqrt{2}\,P_{NN}}{\sqrt{\left(1+2\frac{P_{NN}}{P_{XX}}\frac{W_X}{W_N}\right)\left(1+\frac{P_{NN}}{P_{XX}}\frac{W_N}{W_X}+\sqrt{1+2\frac{P_{NN}}{P_{XX}}\frac{W_X}{W_N}}\right)}}$$ where the integral of Problem 9.2-7 has been used with $b_0=0$, $b_1=1$, $a_0=1$, $a_2=W_N^2\sqrt{1+2\frac{P_{NN}}{P_{XX}}\frac{W_X}{W_N}}$, and $a_1^2 = 2W_N^2\left\{1+\frac{P_{NN}}{P_{XX}}\frac{W_N}{W_X}+\sqrt{1+2\frac{P_{NN}}{P_{XX}}\frac{W_X}{W_N}}\right\}$. (b) as $P_{XX}\to\infty$ $E[\varepsilon^2(t)]_{min}\to P_{NN}$ which means the signal dominates at the input. As $P_{NN}\to\infty$ $E[\varepsilon^2(t)]_{min}\to P_{XX}$ which is the case of input noise dominant. (a) A graphical sketch of $E[\varepsilon^2(t)]_{min}/P_{NN}$ for $P_{NN}/P_{XX}=8$ as a function of W_X/W_N shows the function $\to 0$ for $W_X/W_N\to 0$ or for $W_X/W_N\to\infty$. It also shows a maximum at $W_X/W_N = 0.325$ of 0.10713. Thus, $W_X/W_N = 0.325$ is a value to be avoided.

9.2-12. (a) From (9.2-20): $H_{opt}(\omega) = \dfrac{P_{XX}\,4W_X\,\omega^2\,e^{j\omega t_0}}{P_{XX}\,4W_X\,\omega^2+\frac{\mathcal{N}_0}{2}(W_X^2+\omega^2)^2}$.

(b) From (9.2-22): $E[\varepsilon^2(t)]_{min} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\frac{(P_{XX}\,4W_X\,\mathcal{N}_0/2)\,\omega^2\,d\omega}{\frac{\mathcal{N}_0}{2}W_X^4+(P_{XX}4W_X+\mathcal{N}_0W_X^2)\omega^2+\frac{\mathcal{N}_0}{2}\omega^4}$

Use I_3 of Problem 9.2-7 with $a_0=1$, $a_2=W_X^2$, $b_0=-1$, $b_1=0$, and $a_1=2W_X\sqrt{1+(2P_{XX}/\mathcal{N}_0W_X)}$. $E[\varepsilon^2(t)]_{min} = P_{XX}\Big/\sqrt{1+(2P_{XX}/\mathcal{N}_0W_X)}$.

(c) $E[\varepsilon^2(t)]_{min} = 2\Big/\sqrt{1+[4/0.2(15)]} \simeq 1.3093$.

(9.2-13.) Write (9.2-20) as

$$H_{opt}(\omega) = \frac{e^{j\omega t_0}}{1 + \dfrac{\mathcal{S}_{NN}(\omega)}{\mathcal{S}_{XX}(\omega)}}.$$

If $\mathcal{S}_{XX}(\omega)/\mathcal{S}_{NN}(\omega) > 1$, then $\frac{1}{2} < |H_{opt}(\omega)| \leq 1$

If $0 < \mathcal{S}_{XX}(\omega)/\mathcal{S}_{NN}(\omega) \leq 1$, then $0 < |H_{opt}(\omega)| \leq \frac{1}{2}$.

Thus, $|H_{opt}(\omega)|$ increases as $\mathcal{S}_{XX}(\omega)/\mathcal{S}_{NN}(\omega)$ increases.

(9.2-14.) (a) $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{N}_X}{2} \text{rect}\left(\frac{\omega}{W_X}\right) d\omega = \frac{\mathcal{N}_X W_X}{4\pi}$.

(b) From (9.2-20): $H_{opt}(\omega) = \left(\frac{\mathcal{N}_X}{\mathcal{N}_X + \mathcal{N}_0}\right) \text{rect}\left(\frac{\omega}{W_X}\right) e^{j\omega t_0}$.

(c) From (9.2-22): $\frac{E[\varepsilon^2(t)]_{min}}{P_{XX}} = \frac{4\pi}{\mathcal{N}_X W_X} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\mathcal{N}_X}{\mathcal{N}_X + \mathcal{N}_0}\right) \frac{\mathcal{N}_0}{2} \text{rect}\left(\frac{\omega}{W_X}\right) d\omega$

$= \frac{1}{1 + (\mathcal{N}_X/\mathcal{N}_0)}$. For $\mathcal{N}_X/\mathcal{N}_0 = 16$: $\frac{E[\varepsilon^2(t)]_{min}}{P_{XX}} = \frac{1}{1+16} \approx 0.0588$.

(9.3-1.) $x(t) = A\cos(\omega_0 t) = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$.

Define the output signal as $x_o(t)$. Then

$$x_o(t) = \frac{A}{2} e^{j\omega_0 t} H(\omega_0) + \frac{A}{2} e^{-j\omega_0 t} H(-\omega_0)$$

$$= \frac{A}{2}\left\{\frac{W}{W + j\omega_0} e^{j\omega_0 t} + \frac{W}{W - j\omega_0} e^{-j\omega_0 t}\right\}$$

$$= \frac{AW}{\sqrt{W^2 + \omega_0^2}} \cos(\omega_0 t - \theta) \quad \text{where}$$

$\theta = \tan^{-1}(\omega_0/W)$. Since the average power in any sinusoid of peak voltage V is $V^2/2$ (in an assumed one ohm impedance), we have

$$S_o = E[x_o^2(t)] = \frac{A^2 W^2/2}{W^2 + \omega_0^2}.$$

For output average noise power use (8.4-6):

9.3-1. (Continued)

$$N_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\mathcal{N}_0/2)W^2}{W^2+\omega^2}\,d\omega = \mathcal{N}_0 W/4 \quad \text{from (C-25).}$$

$$\left(\frac{S_0}{N_0}\right) = \frac{2A^2W/\mathcal{N}_0}{W^2+\omega_0^2}.$$ Differentiate to find W for which (S_0/N_0) is maximum: $\frac{d(S_0/N_0)}{dW} = \frac{2A^2}{\mathcal{N}_0}\left\{\frac{(W^2+\omega_0^2)-W(2W)}{(W^2+\omega_0^2)^2}\right\} = 0$ when $W=\omega_0$. A sketch of (S_0/N_0) proves there is a maximum at $W=\omega_0$ and not a minimum.

9.3-2. Here $H(\omega) = W^2/(W+j\omega)^2$, $|H(\omega)|^2 = W^4/(W^2+\omega^2)^2$. Repeat the procedures of the solution to Problem 9.3-1:

$$x_0(t) = \frac{A}{2}\left\{\frac{W^2}{(W+j\omega_0)^2}e^{j\omega_0 t} + \frac{W^2}{(W-j\omega_0)^2}e^{-j\omega_0 t}\right\}$$

$$= \frac{AW^2}{(W^2+\omega_0^2)}\cos(\omega_0 t-\theta) \quad \text{where } \theta = \tan^{-1}\left(\frac{2W\omega_0}{W^2-\omega_0^2}\right)$$

$$S_0 = E[x_0^2(t)] = (A^2W^4/2)/(W^2+\omega_0^2)^2.$$

$$N_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\mathcal{N}_0/2)W^4}{(W^2+\omega^2)^2}\,d\omega = \mathcal{N}_0 W/8 \quad \text{from (C-28).}$$

$$\left(\frac{S_0}{N_0}\right) = \frac{4A^2}{\mathcal{N}_0}\cdot\frac{W^3}{(W^2+\omega_0^2)^2}, \quad \frac{d(S_0/N_0)}{dW} = \frac{4A^2}{\mathcal{N}_0}\left\{\frac{(W^2+\omega_0^2)^2(3W^2) - W^3 2(W^2+\omega_0^2)2W}{(W^2+\omega_0^2)^4}\right\} = 0 \quad \text{when}$$

(9.3-2.) (Continued)

$W=\sqrt{3}\,\omega_0$. A sketch of (S_o/N_o) versus W proves a maximum results when $W=\sqrt{3}\,\omega_0$.

(9.3-3.) This problem is identical to Problem 9.3-1 except for signal characteristics. From Example 6.2-1 and pair 11 of Appendix E:

$$R_{XX}(\tau)=\frac{A^2}{2}\cos(\omega_0\tau)\leftrightarrow\frac{A^2\pi}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]=\mathcal{S}_{XX}(\omega).$$

From (8.4-6): $S_o=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{A^2\pi}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$

$\cdot\frac{W^2}{W^2+\omega^2}\,d\omega = A^2W^2/2(W^2+\omega_0^2)$. Since S_o and N_o are both the same as Problem 9.3-1, (S_o/N_o) is the same and is maximum for the same value of W, namely $W=\omega_0$.

(9.3-4.) This a problem in parameter selection. From pair 19 of Appendix E:

$$R_{XX}(\tau)=W_X e^{-W_X|\tau|}\longleftrightarrow\mathcal{S}_{XX}(\omega)=\frac{2W_X^2}{W_X^2+\omega^2}.$$

(a) Since we are estimating $x(t)$, $t_o=0$. From (9.3-1) using (9.3-3):

$$E[\varepsilon^2(t)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\mathcal{S}_{XX}(\omega)|1-H(-\omega)|^2+\mathcal{S}_{NN}(\omega)|H(\omega)|^2\right\}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{2W_X^2}{(W_X^2+\omega^2)}\cdot\frac{\omega^2}{(W^2+\omega^2)}+\frac{\mathcal{N}_0W^2/2}{(W^2+\omega^2)}\right\}d\omega.$$

Expand the first integrand as follows:

(9.3-4.) (Continued)

$$\frac{\omega^2}{(W_x^2+\omega^2)(W^2+\omega^2)} = \left\{\frac{-W_x^2}{W_x^2+\omega^2} + \frac{W^2}{W^2+\omega^2}\right\}\frac{1}{(W^2-W_x^2)}.$$

Thus, $E[\varepsilon^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{2W_x^2}{W^2-W_x^2}\left[\frac{-W_x^2}{W_x^2+\omega^2}+\frac{W^2}{W^2+\omega^2}\right]\right.$ $\left.+\frac{\mathcal{N}_0 W^2/2}{W^2+\omega^2}\right\}d\omega = \frac{W_x^2}{W+W_x} + \frac{\mathcal{N}_0 W}{4}$ after using (C-25). Differentiate to find W that gives minimum mean-squared error:

$$\frac{dE[\varepsilon^2(t)]}{dW} = \frac{-4W_x^2+\mathcal{N}_0(W+W_x)^2}{4(W+W_x)^2} = 0 \quad \text{when}$$

$W = W_x\{-1 \pm (2/\sqrt{\mathcal{N}_0})\}$. Only $W>0$ is of interest so $E[\varepsilon^2(t)]$ is minimum when $W = W_x\left(\frac{2}{\sqrt{\mathcal{N}_0}}-1\right)$, which can occur only when $(\mathcal{N}_0/2)<2$.

(b) By substitution of the minimizing value of W: $E[\varepsilon^2(t)]_{min} = \frac{W_x^2}{W_x\frac{2}{\sqrt{\mathcal{N}_0}}} + \frac{\mathcal{N}_0}{4}W_x\left(\frac{2}{\sqrt{\mathcal{N}_0}}-1\right)$

$$= \frac{W_x\sqrt{\mathcal{N}_0}}{4}\left(4-\sqrt{\mathcal{N}_0}\right).$$

★(9.3-5.) (a) From (9.3-1) using (9.3-3):

$$E[\varepsilon^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{2W_x^2}{W_x^2+\omega^2}\left|1-\frac{GW}{W-j\omega}\right|^2 + \frac{\mathcal{N}_0}{2}\frac{G^2W^2}{(W^2+\omega^2)}\right\}d\omega$$

$$= \frac{W_x^2W^2(1-G)^2}{\pi}\int_{-\infty}^{\infty}\frac{d\omega}{(W_x^2+\omega^2)(W^2+\omega^2)} \longleftarrow \text{Term 1}$$

★9.3-5. (Continued)

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{2W_X^2}{(W_X^2+\omega^2)}\cdot\frac{\omega^2}{(W^2+\omega^2)}+\underbrace{\frac{G^2\mathcal{N}_0W^2/2}{(W^2+\omega^2)}\right\}d\omega}_{\text{Term 2}}.$$

Term 2 is identical to the integral in the solution of Problem 9.3-4 except for the factor G^2. The term's solution is therefore the same as in Problem 9.3-4 if we replace $\mathcal{N}_0$ before by $G^2\mathcal{N}_0$ here. Thus,

$$E[\varepsilon^2(t)]=\frac{W_X^2W^2(1-G)^2}{\pi}\int_{-\infty}^{\infty}\frac{d\omega}{(W_X^2+\omega^2)(W^2+\omega^2)}+\frac{W_X^2}{W+W_X}+\frac{G^2\mathcal{N}_0W}{4}. \quad (1)$$

Next, expand the integrand as follows:

$$\frac{1}{(W_X^2+\omega^2)(W^2+\omega^2)}=\left[\frac{1}{W_X^2+\omega^2}+\frac{-1}{W^2+\omega^2}\right]\frac{1}{(W^2-W_X^2)}$$

By substituting this expansion into (1) and evaluating via (C-25) we obtain

$$E[\varepsilon^2(t)]=\frac{W_XW(1-G)^2}{W+W_X}+\frac{W_X^2}{W+W_X}+\frac{G^2\mathcal{N}_0W}{4}$$

so

$$\frac{\partial E[\varepsilon^2(t)]}{\partial G}=\frac{\mathcal{N}_0W}{2}G-\frac{W_XW}{W+W_X}2(1-G)=0$$

*9.3-5. (Continued) when $G = \dfrac{4W_X}{\mathcal{N}_0 W + (\mathcal{N}_0 + 4)W_X}$. (2)

also

$$\frac{\partial E[\varepsilon^2(t)]}{\partial W} = \frac{G^2\mathcal{N}_0}{4} + \frac{(W+W_X)W_X(1-G)^2 - [W_X^2 + W_X(1-G)^2 W]}{(W+W_X)^2} = 0.$$

By solving for W that makes this result zero we have

$$W = W_X\left\{-1 \pm \frac{2}{\sqrt{\mathcal{N}_0}}\sqrt{\frac{2-G}{G}}\right\}.$$

For a real solution $W > 0$ we require $0 < G < 2$ and $2\sqrt{(2-G)/G/\mathcal{N}_0} > 1$ or $\mathcal{N}_0/2 < 2(2-G)/G$. (Note that $0 < G < 2$ means that $0 < \mathcal{N}_0/2 < \infty$ as we might suspect.) Assume these two conditions are true; then

$$W = W_X\left\{\frac{2}{\sqrt{\mathcal{N}_0}}\sqrt{\frac{2-G}{G}} - 1\right\} \qquad (3)$$

must hold. Substitution of (2) into (3) gives

$$W = W_X\sqrt{(\mathcal{N}_0+4)/\mathcal{N}_0}. \qquad (4)$$

Substitution of (4) into (2) gives

$$G = \frac{4}{(\mathcal{N}_0+4) + \sqrt{\mathcal{N}_0(\mathcal{N}_0+4)}}. \qquad (5)$$

(b) By substitution of G and W, $E[\varepsilon^2(t)]_{min}$ is found to be

$$E[\varepsilon^2(t)]_{min} = \frac{2W_X}{1+\sqrt{\dfrac{4+\mathcal{N}_0}{\mathcal{N}_0}}} = \frac{2\sqrt{\mathcal{N}_0}\,W_X}{\sqrt{\mathcal{N}_0} + \sqrt{4+\mathcal{N}_0}}.$$

CHAPTER 10

10.1-1. (a) $R_{AM}(t,t+\tau) = E[s_{AM}(t)\,s_{AM}(t+\tau)] = E\{[A_0+X(t)]$
$\cdot\cos(\omega_0 t+\Theta_0)[A_0+X(t+\tau)]\cos(\omega_0 t+\Theta_0+\omega_0\tau)\}$
$= E\{[A_0^2 + A_0X(t+\tau) + A_0X(t) + X(t)X(t+\tau)]\,\frac{1}{2}[\cos(\omega_0\tau)$
$+ \cos(2\omega_0 t + 2\Theta_0 + \omega_0\tau)]\}$. Take expectation with respect to Θ_0 first and then with respect to $X(t)$:
$R_{AM}(t,t+\tau) = E[A_0^2 + A_0X(t) + A_0X(t+\tau) + X(t)X(t+\tau)]\,\frac{1}{2}$
$\cdot\cos(\omega_0\tau) = \left[\frac{A_0^2}{2} + \frac{R_{XX}(\tau)}{2}\right]\cos(\omega_0\tau) = R_{AM}(\tau)$.
(b) Expand $\cos(\omega_0\tau) = (1/2)e^{j\omega_0\tau} + (1/2)e^{-j\omega_0\tau}$ and use (D-7) and pair 2 of Appendix E to Fourier transform $R_{AM}(\tau)$: $\mathcal{S}_{AM}(\omega) = \frac{A_0^2\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$+ \frac{1}{4}[\mathcal{S}_{XX}(\omega-\omega_0) + \mathcal{S}_{XX}(\omega+\omega_0)]$.

10.1-2. Since power is given by the autocorrelation function evaluated at the origin the signal and carrier powers are $R_{XX}(0)/2$ and $A_0^2/2$ from Problem 10.1-1. Hence $\eta_{AM} = \dfrac{R_{XX}(0)/2}{(A_0^2/2) + [R_{XX}(0)/2]} = \dfrac{R_{XX}(0)}{A_0^2 + R_{XX}(0)}$
$= \dfrac{\overline{X^2(t)}}{A_0^2 + \overline{X^2(t)}}$. Next, since $R_{XX}(0) = \overline{X^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega$ we have
$$\eta_{AM} = \frac{\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega}{2\pi A_0^2 + \int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega}.$$

10.1-3. From Problem 10.1-2 $\eta_{AM} = \dfrac{\overline{X^2(t)}}{A_0^2 + \overline{X^2(t)}} =$
$$\frac{\overline{X^2(t)}/|X(t)|^2_{max}}{A_0^2/|X(t)|^2_{max} + \overline{X^2(t)}/|X(t)|^2_{max}} = \frac{1}{1 + K_{cr}^2 A_0^2/|X(t)|^2_{max}}.$$

(10.1-3) (Continued) But $|X(t)|^2_{max} \le A_0^2$ so $\eta_{AM} \le \frac{1}{1+K_{cr}^2}$.

Next, $\overline{X^2(t)} = E[A_m^2 \cos^2(\omega_m t + \Theta_m)] = \frac{A_m^2}{2} E[1 + \cos(2\omega_m t + 2\Theta_m)] = \frac{A_m^2}{2}$, so $\eta_{AM} = \frac{\overline{X^2(t)}}{A_0^2 + \overline{X^2(t)}} = \frac{A_m^2/2}{A_0^2 + (A_m^2/2)} = \frac{1}{1 + 2\frac{A_0^2}{A_m^2}}$.

For no overmodulation $A_m \le A_0$ so $\eta_{AM} \le 1/3$.

(10.1-4.) $\overline{S_R^2(t)} = G_{ch}^2 \overline{[A_0 + X(t)]^2 \cos^2(\omega_0 t + \Theta_0)}$

$$= G_{ch}^2 \overline{[A_0^2 + \underbrace{2A_0 X(t)}_{\substack{0 \text{ because} \\ \overline{X(t)}=0}} + X^2(t)] \frac{1}{2} [1 + \underbrace{\cos(2\omega_0 t + \Theta_0)}_{\substack{0 \text{ when averaged} \\ \text{over } \Theta_0}}]}$$

$\overline{S_R^2(t)} = G_{ch}^2 [A_0^2 + \overline{X^2(t)}]/2$. On using (10.1-14):

$\left(\frac{S_i}{N_i}\right)_{AM} = \frac{\overline{S_R^2(t)}}{\overline{N^2(t)}} = \frac{\pi G_{ch}^2 [A_0^2 + \overline{X^2(t)}]}{\mathcal{N}_0 W_{rec}}$. From (10.1-15)

we get $\left(\frac{S_o}{N_o}\right)_{AM} = \frac{\pi G_{ch}^2 [A_0^2 + \overline{X^2(t)}]}{\mathcal{N}_0 W_{rec}} \cdot \frac{2\overline{X^2(t)}}{[A_0^2 + \overline{X^2(t)}]} = 2\eta_{AM}\left(\frac{S_i}{N_i}\right)_{AM}$

from Problem 10.1-2

(10.1-5.) (a) $S_i = G_{ch}^2$ (transmitted power) $= 18(10^{-6})10^3 = 18(10^{-3})$ W. (b) $(S_i/N_i)_{AM} = 18(10^{-3})/10^{-5} = 1800$ (or 32.55 dB).

(c) From Problem 10.1-4: $\eta_{AM} = (S_o/N_o)_{AM}/2(S_i/N_i)_{AM}$ $= 180/2(1800) = 0.05$ (or 5%).

(10.1-6.) (a) From Problem 10.1-2: $\eta_{AM} = \frac{\overline{X^2}}{A_0^2 + \overline{X^2}} = \frac{1}{1 + K_{cr}^2}$ or $K_{cr}^2 = \frac{1}{\eta_{AM}} - 1$ so $K_{cr} = \sqrt{(1/0.159) - 1} = 2.2998$. (b) $A_0 = |X(t)|_{max} = K_{cr}\sqrt{\overline{X^2(t)}}$ $= 2.2998\sqrt{0.1} = 0.7273$ V. (c) From Problem 10.1-4: $(S_i/N_i)_{AM} = (1/2\eta_{AM})(S_o/N_o)_{AM} = 5000/2(0.159) = 15{,}723.3$. (d) From (10.1-15): $(S_o/N_o)_{AM} = \frac{2\pi}{\mathcal{N}_0 W_{ch}} G_{ch}^2 \overline{X^2(t)} = \frac{1}{N_i} G_{ch}^2 \overline{X^2(t)}$ or $G_{ch} = \sqrt{[N_i/\overline{X^2(t)}](S_o/N_o)_{AM}} = \sqrt{10^{-4}(5000)/0.1} = 2.2361$.

10.1-7. (a) $\eta_{AM} = \overline{X^2}/(A_0^2+\overline{X^2}) = 1/[1+(|X|^2_{max}/\overline{X^2})] = 1/(1+K_{cr}^2)$ $= 1/(1+24) = 0.04$. (b) $S_i = G_{ch}^2\,\overline{S_{AM}^2} = 9(10^{-12})2000 = 18(10^{-9})$ W. (c) From Problem 10.1-4: $(S_0/N_0)_{AM} = 2\eta_{AM}(S_i/N_i)_{AM} = 2(0.04)5(10^{-5})$ $= 4(10^{-4})$.

10.2-1. (a) $s_{FM}(t) = A\cos[\omega_0 t + \Theta_0 + k_{FM}\int x(t)\,dt] =$ $A\cos[\omega_0 t + \Theta_0 + \dfrac{k_{FM}A_m}{\omega_m}\sin(\omega_m t)]$. But $\Delta\omega =$ $k_{FM}|x(t)|_{max} = k_{FM}A_m$ so $\beta_{FM} = k_{FM}A_m/\omega_m$ and $s_{FM}(t) = A\cos[\omega_0 t + \Theta_0 + \beta_{FM}\sin(\omega_m t)] = A\cos[\phi(t)]$. Instantaneous frequency $= \dfrac{d\phi(t)}{dt} = \omega_0 + \beta_{FM}\omega_m\cos(\omega_m t)$. (b) $W_{FM} \approx 2\Delta\omega = 2\dfrac{\Delta\omega}{\omega_m}\omega_m = 2\beta_{FM}\omega_m$. (c) Here $A_m =$ 0.1, $\omega_m = 10^3$ so $\beta_{FM} = \Delta\omega/\omega_m = W_{FM}/2\omega_m = 200\pi$.

Also, since $k_{FM} = \Delta\omega/A_f$ then $k_{FM} = 200(10^3)2\pi/2(0.1) = 200\pi(10^4)$.

10.2-2. $R_{FM}(t,t+\tau) = E[S_{FM}(t)S_{FM}(t+\tau)] = E\{A^2\cos[\omega_0 t + \Theta_0$ $+ k_{FM}\int_{-\infty}^{t} X(\xi)d\xi]\cos[\omega_0 t + \Theta_0 + \omega_0\tau + k_{FM}\int_{-\infty}^{t+\tau}X(\xi)\,d\xi]\}$

$$= \frac{A^2}{2}E\left\{\cos[\omega_0\tau + k_{FM}\int_{-\infty}^{t+\tau}X(\xi)d\xi - k_{FM}\int_{-\infty}^{t}X(\xi)\,d\xi]\right.$$
$$\left. + \cos[2\omega_0 t + 2\Theta_0 + \omega_0\tau + k_{FM}\int_{-\infty}^{t+\tau}X(\xi)d\xi + k_{FM}\int_{-\infty}^{t}X(\xi)\,d\xi]\right\}$$

The second term becomes zero when averaged over Θ_0. Thus

$$R_{FM}(t,t+\tau) = \frac{A^2}{4}E\left\{e^{j\omega_0\tau + j\Gamma(t+\tau) - j\Gamma(t)} + e^{-j\omega_0\tau - j\Gamma(t+\tau) + j\Gamma(t)}\right\}$$

On recognizing that the expectations are character-

(10.2-2) (Continued)

istic functions, we have

$$R_{FM}(t,t+\tau) = \frac{A^2}{4}\left\{ e^{j\omega_0\tau}\Phi_{\Gamma_1,\Gamma_2}(+1,-1) + e^{-j\omega_0\tau}\Phi_{\Gamma_1,\Gamma_2}(-1,+1)\right\} \quad (1)$$

where $\Phi_{\Gamma_1,\Gamma_2}(\omega_1,\omega_2)$ is the characteristic function of random variables $\Gamma_1 = \Gamma(t+\tau)$ and $\Gamma_2 = \Gamma(t)$, which are gaussian. If $C_{\Gamma_1,\Gamma_2}(\tau)$ denotes the covariance function of Γ_1 and Γ_2 then, from Problem 5.2-4,

$$\Phi_{\Gamma_1,\Gamma_2}(\omega_1,\omega_2) = e^{-\frac{1}{2}\left[\sigma_\Gamma^2\omega_1^2 + 2C_{\Gamma_1,\Gamma_2}(\tau)\omega_1\omega_2 + \sigma_\Gamma^2\omega_2^2\right]} . \quad (2)$$

Here σ_Γ^2 represents the variances of Γ_1 and Γ_2 which are equal. On using (2) in (1) we have

$$R_{FM}(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)\exp\{-\sigma_\Gamma^2[1-\rho(\tau)]\} \quad \text{where } \rho(\tau) = \frac{C_{\Gamma_1,\Gamma_2}(\tau)}{\sigma_\Gamma^2} .$$

(10.2-3.) (a) Use (10.2-7): $S_i = \frac{A^2}{2}G_{ch}^2 = P_{FM}G_{ch}^2 = 10^4(10^{-12}) = 10^{-8}$ W.

From (10.2-9): $N_i = \frac{\mathcal{N}_0}{2}\frac{2\Delta\omega}{\pi} = \frac{5}{3}(10^{-15})\frac{2\pi(150)10^3}{\pi} = 50(10^{-11})$ W.

$(S_i/N_i)_{FM} = 10^{-8}/50(10^{-11}) = 20$ (or 13.01 dB).

(b) Use (10.2-20): $25(10^3) = \frac{6}{K_{cr}^2}\left(\frac{\Delta\omega}{W_X}\right)^3\left(\frac{S_i}{N_i}\right)_{FM} = \frac{6}{16}\left(\frac{\Delta\omega}{W_X}\right)^3 20$

or $(\Delta\omega/W_X)^3 = \frac{60(10^{-3})}{8(25)} = 3(10^{-4})$, $W_X = \{3(10^{-4})(2\pi(75)10^3)^3\}^{1/3}$

$= 2\pi(5020.75)$ rad/s.

(10.2-4.) (a) $\Delta\omega = k_{FM}|X(t)|_{max} = k_{FM}K_{cr}\sqrt{\overline{X^2(t)}} = 1.06(10^6)4.2\sqrt{0.02} =$ $6.2961(10^5)$ rad/s. (b) $(S_o/N_o)_{FM} = (6/K_{cr}^2)(\Delta\omega/W_X)^3(S_i/N_i)_{FM}$

$= \frac{6}{4.2^2}\left(\frac{629,607.88}{\pi(10^4)}\right)^3 10 = 27.379(10^3)$.

10.2-5. (a) $\left(\frac{S_o}{N_o}\right)_{FM} = 4200 = \frac{6}{K_{cn}^2}\left(\frac{\Delta\omega}{W_x}\right)^3\left(\frac{S_i}{N_i}\right)_{FM} = \frac{6}{(3.2)^2}\left(\frac{\Delta\omega}{W_x}\right)^3 14$ so $W_x =$ $\Delta\omega\left[\frac{6(14)}{(3.2)^2\,4200}\right]^{1/3} = 8\pi(10^4)\left[1.9531(10^{-3})\right]^{1/3} = \pi(10^4)$ rad/s.

(b) $\left(\frac{S_i}{N_i}\right)_{FM} = \pi G_{ch}^2 A^2/(2\mathcal{N}_0\Delta\omega)$, $G_{ch}A = \sqrt{2\mathcal{N}_0\Delta\omega(S_i/N_i)_{FM}/\pi} =$ $\sqrt{2(10^{-11})8\pi(10^4)14/\pi} = 4.733(10^{-3})$ W. (c) Yes, because $(S_i/N_i)_{FM}$ $= 14 > 10$.

10.2-6. (a) $G_{ch}A = G_{ch}(2000) = 0.1$, so $G_{ch} = 5(10^{-5})$. (b) $N_i = S_i/\left(\frac{S_i}{N_i}\right)_{FM}$ $= \frac{(0.1)^2}{2}\frac{1}{20} = 2.5(10^{-4}) = \mathcal{N}_0\Delta\omega/\pi$, so $\mathcal{N}_0 = \frac{2.5\pi(10^{-4})}{(10^5)\pi} = 2.5(10^{-9})$ W/Hz. (c) $(S_o/N_o)_{FM} = \frac{6}{K_{cn}^2}\left(\frac{\Delta\omega}{W_x}\right)^3\left(\frac{S_i}{N_i}\right)_{FM} = \frac{6}{9}\left(\frac{\pi(10^5)}{\pi(10^4)}\right)^3 20 = 13{,}333.33$. (d) $k_{FM} =$ $\Delta\omega/|x(t)|_{max} = \pi(10^5)/1.4 = 224{,}399.5$ $\frac{\text{rad/s}}{\text{V}}$.

10.3-1. (a) $H_1(\omega)H_2(\omega) = \frac{K_1W_1}{j\omega(W_1+j\omega)}$ has no phase less than $-\pi$ for any $\omega < \infty$ so there are no values of K_1 and/or W_1 that make the loop unstable.

(b) $H(\omega) = \frac{H_1(\omega)}{1+H_1(\omega)H_2(\omega)} = \frac{j\omega K_1W_1}{(K_1W_1-\omega^2)+j\omega W_1}$. Thus,

$$|H(\omega)|^2 = \frac{K_1^2W_1^2\omega^2}{K_1^2W_1^2+(W_1^2-2K_1W_1)\omega^2+\omega^4} = \frac{8000^2\omega^2}{(8000)^2+24000\omega^2+\omega^4}.$$

Output noise power is $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{N}_0}{2}|H(\omega)|^2 d\omega =$ $\frac{10^{-4}}{2\pi}\int_{-\infty}^{\infty}\frac{8000^2\omega^2\,d\omega}{(8000)^2+24000\omega^2+\omega^4} = 16$ from the integral of Problem 9.2-7 with $a_0 = 1$, $a_1 = 200$, $a_2 = 8000$, $b_0 = -8000$ and $b_1 = 0$.

(10.3-2.) Write (9.2-20) as

$$H_{opt}(\omega) = \frac{\overbrace{[\mathcal{S}_{XX}(\omega)/\mathcal{S}_{NN}(\omega)]\, e^{j\omega t_0}}^{H_1(\omega)}}{1 + \underbrace{\frac{\mathcal{S}_{XX}(\omega)}{\mathcal{S}_{NN}(\omega)}\, e^{j\omega t_0}}_{H_1(\omega)} \underbrace{e^{-j\omega t_0}}_{H_2(\omega)}} = \frac{H_1(\omega)}{1+H_1(\omega)H_2(\omega)} = H(\omega).$$

(10.3-3.) (a) $H(\omega) = \dfrac{H_1(\omega)}{1+H_1(\omega)H_2(\omega)} = \dfrac{Kj\omega}{(W_1+j\omega)(W_2+j\omega) + Kj\omega} = \dfrac{Kj\omega}{(W_1W_2-\omega^2)+j\omega(K+W_1+W_2)}$

$$|H(\omega)|^2 = \frac{K^2\omega^2}{W_1^2W_2^2 + [(K+W_1+W_2)^2 - 2W_1W_2]\,\omega^2 + \omega^4}.$$ Use integral of Problem 9.2-7 with $b_1 = 0$, $b_0 = -\mathcal{N}_0 K^2/2$, $a_2 = W_1W_2$, $a_1 = K+W_1+W_2$, and $a_0 = 1$.

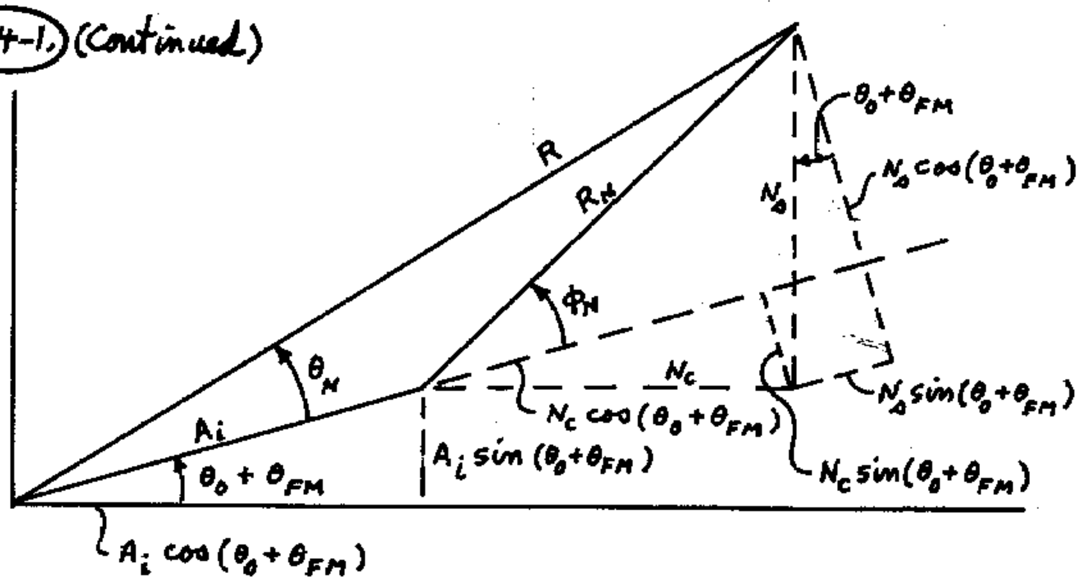
$$\text{Noise power} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{N}_0}{2} |H(\omega)|^2 d\omega = \frac{-b_0}{2a_0a_1} = \frac{\mathcal{N}_0 K^2}{4(K+W_1+W_2)}.$$

(b) $\text{Noise power} = \dfrac{10^{-3}(100)}{2(10+5+25)} = 1.25(10^{-3})$ W.

★(10.4-1.) Signal + Noise $= A_i\cos[\omega_0 t + \theta_0 + \theta_{FM}] + N_c\cos(\omega_0 t)$
$- N_s\sin(\omega_0 t) = A_i\cos(\theta_0+\theta_{FM})\cos(\omega_0 t) - A_i\sin(\theta_0+\theta_{FM})$
$\cdot\sin(\omega_0 t) + N_c\cos(\omega_0 t) - N_s\sin(\omega_0 t)$
$= [A_i\cos(\theta_0+\theta_{FM}) + N_c]\cos(\omega_0 t)$
$\qquad - [A_i\sin(\theta_0+\theta_{FM}) + N_s]\sin(\omega_0 t).$

Use a phasor diagram to suppress rotation at rate ω_0 by defining: Noise $= N(t) = R_N\cos(\omega_0 t+\phi_N)$ where $R_N = [N_c^2 + N_s^2]^{1/2}$ and $\phi_N = \tan^{-1}(N_s/N_c)$:

★ 10.4-1. (Continued)



From the diagram:

$$R = \left\{ \left[N_c + A_i\cos(\theta_0+\theta_{FM})\right]^2 + \left[N_s + A_i\sin(\theta_0+\theta_{FM})\right]^2 \right\}^{1/2}$$

$$\theta_N = \tan^{-1}\left\{ \frac{N_s\cos(\theta_0+\theta_{FM}) - N_c\sin(\theta_0+\theta_{FM})}{A_i + N_c\cos(\theta_0+\theta_{FM}) + N_s\sin(\theta_0+\theta_{FM})} \right\}.$$

★ 10.4-2. (a) $R_{\Theta_N\Theta_N}(t,t+\tau) = E[\Theta_N(t)\,\Theta_N(t+\tau)]$

$$= E\left\langle \left\{ \frac{N_s(t)}{A_i}\cos[\theta_0+\Theta_{FM}(t)] - \frac{N_c(t)}{A_i}\sin[\theta_0+\Theta_{FM}(t)] \right\} \cdot \left\{ \frac{N_s(t+\tau)}{A_i}\cos[\theta_0+\Theta_{FM}(t+\tau)] - \frac{N_c(t+\tau)}{A_i}\sin[\theta_0+\Theta_{FM}(t+\tau)] \right\} \right\rangle$$

$$= \frac{1}{A_i^2} R_{N_sN_s}(\tau)\, E\left\{ \cos[\theta_0+\Theta_{FM}(t)]\cos[\theta_0+\Theta_{FM}(t+\tau)] \right\}$$

$$+ \frac{1}{A_i^2} R_{N_cN_c}(\tau)\, E\left\{ \sin[\theta_0+\Theta_{FM}(t)]\sin[\theta_0+\Theta_{FM}(t+\tau)] \right\}$$

$$- \frac{1}{A_i^2} R_{N_cN_s}(\tau)\, E\left\{ \sin[\theta_0+\Theta_{FM}(t)]\cos[\theta_0+\Theta_{FM}(t+\tau)] \right\}$$

$$- \frac{1}{A_i^2} R_{N_sN_c}(\tau)\, E\left\{ \cos[\theta_0+\Theta_{FM}(t)]\sin[\theta_0+\Theta_{FM}(t+\tau)] \right\}$$

*(10.4-2) (Continued)

But $R_{N_sN_s}(\tau) = R_{N_cN_c}(\tau)$ and $R_{N_sN_c}(\tau) = -R_{N_cN_s}(\tau)$ from Section 8.6, so

$$R_{\Theta_N\Theta_N}(t,t+\tau) = \frac{1}{A_i^2} R_{N_cN_c}(\tau) E\left\{\cos[\Theta_{FM}(t+\tau) - \Theta_{FM}(t)]\right\}$$

$$+ \frac{1}{A_i^2} R_{N_cN_s}(\tau) E\left\{\sin[\Theta_{FM}(t+\tau) - \Theta_{FM}(t)]\right\}.$$

But $\Theta_{FM}(\alpha) = k_{FM}\int_{-\infty}^{\alpha} X(\xi)\,d\xi$ so

$$R_{\Theta_N\Theta_N}(t,t+\tau) = \frac{1}{A_i^2} R_{N_cN_c}(\tau) E\left\{\cos\left[k_{FM}\int_t^{t+\tau} X(\xi)\,d\xi\right]\right\}$$

$$+ \frac{1}{A_i^2} R_{N_cN_s}(\tau) E\left\{\sin\left[k_{FM}\int_t^{t+\tau} X(\xi)\,d\xi\right]\right\}. \quad (1)$$

(b) For noise broadband relative to $X(t)$ correlations $R_{N_cN_c}(\tau)$ and $R_{N_cN_s}(\tau)$ will have their principal variations over values of τ for which $X(\xi)$ stays nearly constant. This means the arguments of $\cos(\cdot)$ and $\sin(\cdot)$ in (1) are small so $\cos(\cdot) \approx 1$ and $\sin(\cdot) \approx 0$ giving

$$R_{\Theta_N\Theta_N}(t,t+\tau) \approx \frac{1}{A_i^2} R_{N_cN_c}(\tau) = R_{\Theta_N\Theta_N}(\tau).$$

(c) In a less extreme case but still for small τ (1) becomes $R_{\Theta_N\Theta_N}(t,t+\tau) \approx \frac{1}{A_i^2} R_{N_cN_c}(\tau) E\{\cos[k_{FM}X(t)\tau]\}$ $+ \frac{1}{A_i^2} R_{N_cN_s}(\tau) E\{\sin[k_{FM}X(t)\tau]\} = \frac{1}{2A_i^2}\Big(R_{N_cN_c}(\tau)\,\overline{e^{jk_{FM}X(t)\tau}}$ $+ R_{N_cN_c}(\tau)\,\overline{e^{-jk_{FM}X(t)\tau}} + R_{N_cN_s}(\tau)\,\overline{\frac{e^{jkX(t)\tau}}{j}}$

* **10.4-2.** (Continued)

$$-R_{N_cN_s}\frac{\overline{e^{-jk_{FM}x(t)\tau}}}{j}\Big\}.$$ For gaussian $x(t)$:

$$\overline{e^{jk_{FM}x(t)\tau}} = \int_{-\infty}^{\infty} f_x(x)e^{jk_{FM}x\tau}dx = \Phi_x(k_{FM}\tau) =$$

$$e^{-\sigma_x^2(k_{FM}\tau)^2/2} = \Phi(-k_{FM}\tau).$$ Hence:

$$R_{\Theta_N\Theta_N}(t,t+\tau) = \frac{R_{N_cN_c}(\tau)}{2A_i^2}\Big\{\Phi_x(k_{FM}\tau)+\Phi_x(-k_{FM}\tau)\Big\}$$

$$+\frac{R_{N_cN_s}(\tau)}{2jA_i^2}\Big\{\Phi_x(k_{FM}\tau)-\Phi_x(-k_{FM}\tau)\Big\}$$

$$= R_{N_cN_c}(\tau)\Phi_x(k_{FM}\tau)\frac{1}{A_i^2} = \frac{1}{A_i^2}\exp\left[\frac{-\sigma_x^2k_{FM}^2\tau^2}{2}\right]R_{N_cN_c}(\tau).$$

10.4-3. With $\zeta=1/2$: $H(\omega)=\dfrac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+j\left(\frac{\omega}{\omega_n}\right)}$ and

$|H(\omega)|^2 = 1\Big/\left[1-\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{\omega}{\omega_N}\right)^4\right]$ so $N_0=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\frac{\mathcal{N}_0\omega^2|H(\omega)|^2}{A_i^2k_v^2}d\omega$

$$=\frac{\mathcal{N}_0\omega_n^4}{A_i^2k_v^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2 d\omega}{\omega_n^4-\omega_n^2\omega^2+\omega^4} = \frac{\mathcal{N}_0\,\omega_n^3}{2A_i^2k_v^2} = \frac{\mathcal{N}_0 W_X^3}{2A_i^2k_v^2}$$

from Problem 9.2-7 with $b_0=-1$, $b_1=0$, $a_0=1$, $a_1=\omega_n$, ω_n^2 and $\omega_n=W_X$. By comparison: N_0(here)/ N_0 (Example 10.4-1) $=\sqrt{2}$, a degradation, not an improvement.

10.4-4. (a) By substitution into (10.4-13): $|H(\omega)|^2 =$

$$\frac{(KW_1W_3)^2(W_2^2+\omega^2)}{\left\{W_2^2\omega^6+[W_2^2(W_1+W_3)^2-2W_1W_2W_3(K+W_2)]\omega^4 +[(W_1W_3)^2(W_2+K)^2-2KW_1W_2^2W_3(W_1+W_3)]\omega^2+(KW_1W_2W_3)^2\right.}.$$

Thus, by use of the hint, $N_0=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\frac{\mathcal{N}_0}{A_i^2k_v^2}\omega^2|H(\omega)|^2d\omega$

(10.4-4) (Continued)

$$= \frac{\mathcal{N}_0 K^2 W_1 W_3 [W_2^3 + W_1 W_3 (W_2 + K)]}{A_i^2 k_v^2 \, 2 W_2^2 [(W_1 + W_3)(W_2 + K) - K W_2]} \quad (1)$$

(b) With the parameters specified (1) will reduce to

$$N_0 = \frac{\mathcal{N}_0 K W_x^3 [51 K + 10 W_x]}{A_i^2 k_v^2 [45 K^2 + 15 K W_x + W_x^2]}$$

(c) For K very large $N_0 \to \dfrac{51 \mathcal{N}_0 W_x^3}{45 A_i^2 k_v^2}$.

(10.5-1.) From pair 15, Appendix E:

$h(t) = W_L u(t) e^{-W_L t} \longleftrightarrow H(\omega) = W_L / (W_L + j\omega)$.

The input waveform is $x(t) = -A + 2A\,u(t)$.

The output waveform is $y(t) = \int_{-\infty}^{\infty} x(\xi) h(t-\xi) d\xi$

$= \int_{-\infty}^{\infty} [-A + 2A\,u(\xi)] W_L u(t-\xi) e^{-W_L(t-\xi)} d\xi$

$= -A + 2A(1 - e^{-W_L t}) u(t)$. Thus $y(0) = -A$ and

$y(t_r) = A - 2A e^{-W_L t_r} = 0.9A$. The solution for

W_L gives $W_L \geq (20/T_b) \ln(20) = 59.9146/T_b$

when we must have $t_r \leq 0.05\, T_b$.

★(10.6-1.) For convenience let $\phi = \theta - \theta_0$. Then

$$f_{\Theta}(\theta) = \int_{-\infty}^{\infty} f_{R,\Theta}(r,\theta)\, dr = \frac{1}{2\pi\sigma^2} \int_0^{\infty} r e^{-\frac{r^2 - 2rA_0 \cos(\phi) + A_0^2}{2\sigma^2}} dr$$

Next, complete the square in r

$$f_{\Theta}(\theta) = \frac{e^{\frac{A_0^2 \cos^2(\phi)}{2\sigma^2} - \frac{A_0^2}{2\sigma^2}}}{2\pi\sigma^2} \int_0^{\infty} r e^{-\frac{r^2 - 2rA_0 \cos(\phi) + A_0^2 \cos^2(\phi)}{2\sigma^2}} dr$$

★ 10.6-1. (Continued)

$$= \frac{e^{-A_0^2[1-\cos^2(\phi)]/2\sigma^2}}{2\pi\sigma^2}\int_0^\infty r\, e^{-[r-A_0\cos(\phi)]^2/2\sigma^2}\, dr.$$

Let $\xi = [r - A_0\cos(\phi)]/\sigma$, $d\xi = dr/\sigma$ so

$$f_\Theta(\theta) = \frac{e^{-A_0^2\sin^2(\phi)/2\sigma^2}}{2\pi}\int_{-(A_0/\sigma)\cos(\phi)}^{\infty}\left[\xi + \frac{A_0\cos(\phi)}{\sigma}\right]e^{-\xi^2/2}\, d\xi.$$

The first integral readily evaluates to give

$$f_\Theta(\theta) = \frac{e^{-A_0^2/2\sigma^2}}{2\pi} + \frac{A_0\cos(\phi)}{\sigma\sqrt{2\pi}}\, e^{-A_0^2\sin^2(\phi)/2\sigma^2}\int_{-(A_0/\sigma)\cos(\phi)}^{\infty}\frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\, d\xi.$$

On recognizing that the integral equals $F\left[\frac{A_0}{\sigma}\cos(\phi)\right]$ we replace ϕ by $\theta-\theta_0$ to finally obtain

$$f_\Theta(\theta) = \frac{e^{-A_0^2/2\sigma^2}}{2\pi} + \frac{A_0\cos(\theta-\theta_0)}{\sigma\sqrt{2\pi}}\, e^{-\frac{A_0^2}{2\sigma^2}\sin^2(\theta-\theta_0)}\, F\left[\frac{A_0}{\sigma}\cos(\theta-\theta_0)\right].$$

★ 10.6-2. For convenience, let $x = A_0/\sigma$ in (10.6-13):

$$f_\Theta(\theta) = \frac{e^{-x^2/2}}{2\pi} + \underbrace{\frac{x\cos(\theta-\theta_0)}{\sqrt{2\pi}}}_{F_1}\,\underbrace{e^{-\frac{x^2}{2}\sin^2(\theta-\theta_0)}}_{F_2}\,\underbrace{F[x\cos(\theta-\theta_0)]}_{F_3}.$$

Sketches of F_1, F_2, and F_3 show that two cases are of interest. When $\theta-\theta_0$ is any multiple of 2π we have F_1, F_2, and F_3 all relatively large. When $\theta-\theta_0$ is near π we have F_1 large in magnitude while F_2 and F_3 are small. For $\theta-\theta_0 \approx \pi$ then $F(-\alpha) = 1 - F(\alpha)$

★ 10.6-2 (Continued)

$$\approx e^{-\alpha^2/2}/\alpha\sqrt{2\pi} \quad \text{so} \quad f_{\Theta}(\theta) \approx \frac{e^{-x^2/2}}{2\pi} - \frac{x\cos(\theta-\theta_0)}{\sqrt{2\pi}}$$

$$\cdot e^{-(x^2/2)\sin^2(\theta-\theta_0)}\left[\frac{e^{-(x^2/2)\cos^2(\theta-\theta_0)}}{x\cos(\theta-\theta_0)\sqrt{2\pi}}\right]$$

$$= \frac{e^{-x^2/2}}{2\pi} - \frac{e^{-x^2/2}}{2\pi} = 0.$$ For $\theta-\theta_0 \approx 0$ (or 2π) $F(\alpha) \approx$

$$1 - \left[e^{-\alpha^2/2}/\alpha\sqrt{2\pi}\right] \quad \text{so} \quad f_{\Theta}(\theta) \approx \frac{e^{-x^2/2}}{2\pi} + \frac{x\cos(\theta-\theta_0)}{\sqrt{2\pi}}$$

$$\cdot e^{-(x^2/2)\sin^2(\theta-\theta_0)}\left[1 - \frac{e^{-(x^2/2)\cos^2(\theta-\theta_0)}}{x\cos(\theta-\theta_0)\sqrt{2\pi}}\right]$$

$$= \frac{x\cos(\theta-\theta_0)}{\sqrt{2\pi}}\, e^{-(x^2/2)\sin^2(\theta-\theta_0)} \approx \left(\frac{x}{\sqrt{2\pi}}\right) e^{-\pi(\theta-\theta_0)^2\left(\frac{x}{\sqrt{2\pi}}\right)^2}$$

Thus $\lim_{A_0/\sigma\to\infty} f_{\Theta}(\theta) \approx \lim_{\frac{A_0}{\sigma}\to\infty} \left(\frac{A_0}{\sigma\sqrt{2\pi}}\right) e^{-\pi(\theta-\theta_0)^2\left(\frac{A_0}{\sigma\sqrt{2\pi}}\right)^2}$

$= \delta(\theta-\theta_0)$ where the equation of Problem 7.1-7 (a) is used.

10.6-3. Use (F-82) where $b=\sigma$, $a=A_0$, $k^2 = 2(A_0^2/2)/\sigma^2 = 4$, and $x = R$. $\bar{R}/\sigma = \sqrt{\pi/2}\; e^{-4/4}\left[\left(1+\frac{4}{2}\right)I_0(1) + \frac{4}{2}I_1(1)\right] = \sqrt{\pi/2}\; e^{-1}[3I_0(1)+2I_1(1)]$ $= \sqrt{\pi/2}\; e^{-1}[3(1.26607)+2(0.56516)] = 2.27238$. Use (F-83): $\sigma_R^2/\sigma^2 = 2 + k^2 - (\bar{R}/\sigma)^2 = 2+4-(2.27238)^2 = 0.8363$.

10.7-1. Here $f_0(w) = f_R(r) = \frac{u(r)}{\sigma^2}\, r e^{-r^2/2\sigma^2}$ so

$$P_{fa} = \int_{W_T}^{\infty} \frac{r}{\sigma^2}\, e^{-r^2/2\sigma^2}\, dr = e^{-W_T^2/2\sigma^2}$$

10.7-2. From (10.7-7): $W_T = K\left[\{2\sigma^2 \ln(1/P_{fa})\}^{1/2}\right]^2$

so $W_T/2K\sigma^2 = -\ln(P_{fa})$ and $P_{fa} = e^{-W_T/2K\sigma^2}$.

10.7-3. From (10.7-7): $0.7 = \frac{1}{4}\sqrt{2\sigma^2 \ln[1/4(10^{-7})]}$

so $\sigma^2 = (2.80)^2/2\ln[2.5(10^6)] = 2.661(10^{-1})$ W.

10.7-4. From (10.7-7): $0.7 = \frac{1}{4}\{2\sigma^2 \ln[1/4(10^{-7})]\}$

so $\sigma^2 = 1.4/\ln[2.5(10^6)] = 95.03(10^{-3})$ W.

10.7-5. From (10.7-11): $P_d = 0.9901 = F\left(\frac{A_0}{\sigma} - \sqrt{2\ln(10^8)}\right)$ occurs at $\frac{A_0}{\sigma} - \sqrt{2\ln(10^8)} = 2.33$ from Table B-1 (Appendix B). Thus, $\frac{A_0}{\sigma} = 2.33 + \sqrt{2\ln(10^8)}$ and

$$\left(\frac{S_i}{N_i}\right) = \frac{A_0^2}{2\sigma^2} = \frac{1}{2}\left[2.33 + \sqrt{2\ln(10^8)}\right]^2 = 35.2776$$

(or 15.48 dB).

★10.7-6. $f_R(r) = \frac{u(r)}{\sigma^2} r I_0\left(\frac{rA_0}{\sigma^2}\right) e^{-(r^2+A_0^2)/2\sigma^2}$.

Let $w = Kr^2$ so $r = \sqrt{w/K}$ and $dw = 2Kr\,dr$:

$$f_W(w) = f_R[r(w)]\frac{dr(w)}{dw}$$

$$= \frac{u(w)}{\sigma^2} I_0\left(\frac{A_0}{\sigma^2}\sqrt{\frac{w}{K}}\right) e^{-\left(\frac{w}{K}+A_0^2\right)/2\sigma^2} \frac{1}{2K} \quad \text{so}$$

$$f_W(w) = \frac{u(w)}{2K\sigma^2} I_0\left(\frac{A_0}{\sigma^2}\sqrt{\frac{w}{K}}\right) e^{-\left(\frac{w}{K}+A_0^2\right)/2\sigma^2}.$$

★10.7-7. This is a Bernoulli trials problem with $p = P_{fa_1}$ or P_{d_1} where Prob. exactly k detections in N trials $= [N!/k!(N-k)!]\,p^k(1-p)^{N-k}$. Thus,

$$P = \sum_{k=n}^{N} \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k}$$ with $P = P_{fa}$ if $p = P_{fa_1}$ and $P = P_d$ if $p = P_{d_1}$.

10.7-8. (a) $R_T = \sqrt{3W_T} = \sqrt{3(0.92)}$, $P_{fa} = e^{-R_T^2/2\sigma^2} = e^{-3(0.92)/2(0.33)^2} = 3.1372(10^{-6})$. (b) From Appendix B: $P_d = 0.9015 \approx F\left[\frac{A_0}{\sigma} - \sqrt{2\ln(1/P_{fa})}\right] = F\left[\frac{A_0}{\sigma} - \frac{R_T}{\sigma}\right]$ occurs when $\left[\frac{A_0}{\sigma} - \frac{R_T}{\sigma}\right] = 1.29$ or $(A_0/\sigma) = 1.29 + \sqrt{3(0.92)}/0.33 = 6.3243$. Thus, $A_0^2/(2\sigma^2) = 19.9985$ (or 13.01 dB).

10.7-9. (a) $R_T = \sqrt{2\sigma^2 \ln(1/P_{fa})} = [2(0.01)\ln(10^8)]^{1/2} = 0.60697$. $W_T = 0.4 R_T^3 = 0.4(0.60697)^3 = 0.08945$ V. (b) From Fig. 10.7-2: $10\log_{10}\left(\frac{A_0^2}{2\sigma^2}\right) = 16$ so $A_0 = \sqrt{2\sigma^2(10^{1.6})} = 0.8923$ V.

10.7-10. (a) $R_T = \sqrt{2\sigma^2 \ln(1/P_{fa})} = \sqrt{2(0.03)\ln(10^4)} = \sqrt{0.5526}$. $W_T = 0.4 R_T^2 = 0.4(0.5526) = 0.221$ V. (b) From Fig. 10.7-2 at $P_{fa} = 10^{-4}$ and $P_d = 0.92$: $10\log_{10}\left(\frac{A_0^2}{2\sigma^2}\right) = 12$ so $A_0^2 = 2\sigma^2\, 10^{1.2} = 0.9509$ and $A_0 = 0.9752$ V.

APPENDIX

D

D-1. $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} A e^{-j\omega t} dt$

$= A\tau \dfrac{\sin(\omega\tau/2)}{\omega\tau/2}$ after using (C-45).

D-2. $X(\omega) = \int_{-\tau/2}^{\tau/2} A \cos(\omega_0 t + \theta_0) e^{-j\omega t} dt$

$$= \frac{A}{2}\int_{-\tau/2}^{\tau/2}\left[e^{j(\omega_0-\omega)t + j\theta_0} + e^{-j(\omega_0+\omega)t - j\theta_0}\right] dt$$

$$= \frac{A\tau}{2}\left\{e^{j\theta_0}\frac{\sin[(\omega_0-\omega)\tau/2]}{(\omega_0-\omega)\tau/2} + e^{-j\theta_0}\frac{\sin[(\omega_0+\omega)\tau/2]}{(\omega_0+\omega)\tau/2}\right\}$$

after using (C-45).

D-3. $X(\omega) = \int_{-\tau}^{\tau} A\left[1 - \frac{|t|}{\tau}\right] e^{-j\omega t} dt$

$$= A\int_{-\tau}^{\tau} e^{-j\omega t} dt - \frac{A}{\tau}\int_{0}^{\tau} t\, e^{-j\omega t} dt + \frac{A}{\tau}\int_{-\tau}^{0} t\, e^{-j\omega t} dt.$$

By use of (C-45) and (C-46) these integrals reduce to

$$X(\omega) = A\tau\left[\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right]^2.$$

D-4. $X(\omega) = \int_{-\tau}^{\tau} A\cos\left(\frac{\pi t}{2\tau}\right) e^{-j\omega t} dt = \frac{A}{2}\int_{-\tau}^{\tau}\left[e^{j\left(\frac{\pi}{2\tau}-\omega\right)t} + e^{-j\left(\frac{\pi}{2\tau}+\omega\right)t}\right] dt = \dfrac{(A\pi/\tau)\cos(\omega\tau)}{(\pi/2\tau)^2 - \omega^2}$ after using (C-45).

D-5. From pair 5 of Appendix E and (D-8):
$A\,\mathrm{rect}(t/2\tau) \longleftrightarrow 2A\tau \dfrac{\sin(\omega\tau)}{\omega\tau}$. From pair 11:
$\cos(\pi t/2\tau) \longleftrightarrow \pi\left[\delta(\omega - \frac{\pi}{2\tau}) + \delta(\omega + \frac{\pi}{2\tau})\right]$. From (D-19): $x(t) = A\,\mathrm{rect}(t/2\tau)\cos(\pi t/2\tau) \longleftrightarrow X(\omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2A\tau \frac{\sin(\xi\tau)}{\xi\tau}\,\pi\left[\delta(\xi+\omega-\frac{\pi}{2\tau}) + \delta(\xi+\omega+\frac{\pi}{2\tau})\right] d\xi$$

$$= A\tau\left\{\frac{\sin(\omega\tau-\frac{\pi}{2})}{\omega\tau-\frac{\pi}{2}} + \frac{\sin(\omega\tau+\frac{\pi}{2})}{\omega\tau+\frac{\pi}{2}}\right\} = \frac{(A\pi/\tau)\cos(\omega\tau)}{\left(\frac{\pi}{2\tau}\right)^2-\omega^2}$$

after using (A-2).

D-6. $Y(\omega) = \int_{-\infty}^{\infty} y(t)\,e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t}\,e^{-j\omega t}\,dt$

$= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty}\left[e^{jn2\pi t/T}\right] e^{-j\omega t}\,dt$. From pair 9 of Appendix E the transform of $\exp(jn2\pi t/T)$ is $2\pi\delta(\omega - \frac{n2\pi}{T})$ so

$$Y(\omega) = 2\pi\sum_{n=-\infty}^{\infty} c_n\,\delta(\omega - \frac{n2\pi}{T}).$$

* **D-7.** Let $x(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi t/T}$

since $x(t)$ is periodic. From Problem D-6:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,e^{-jn2\pi t/T}\,dt = \frac{1}{T}\int_{-T/2}^{T/2}\sum_{m=-\infty}^{\infty}\delta(t-mT)\,e^{-j\frac{n2\pi t}{T}}\,dt$$

$= \frac{1}{T}$ since the integral has a value only when $m=0$. Thus,

$$x(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn2\pi t/T}.$$

★ D-7. (Continued) Next, since $\exp(jn2\pi t/T) \leftrightarrow 2\pi\delta(\omega - \frac{n2\pi}{T})$ from pair 9 of Appendix E,

$$X(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - \frac{n2\pi}{T}) \text{ and}$$

$$\sum_{n=-\infty}^{\infty}\delta(t-nT) \leftrightarrow \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - \frac{n2\pi}{T}).$$

★ D-8. From (D-6): $Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\,dt$

$$= \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(t-nT)\, e^{-j\omega t}\,dt = \sum_{n=-\infty}^{\infty} X(\omega)\, e^{-j\omega nT}$$

$$= X(\omega)\sum_{n=-\infty}^{\infty} e^{-jn\omega T} = \frac{2\pi}{T}X(\omega)\sum_{n=-\infty}^{\infty}\delta(\omega - \frac{n2\pi}{T}).$$

Now since $\delta(\omega - \frac{n2\pi}{T})$ "exists" only at $\omega = n2\pi/T$, this result is equivalent to

$$Y(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} X\left(\frac{n2\pi}{T}\right)\delta(\omega - \frac{n2\pi}{T}).$$

D-9. From (D-7) with $x(t) = u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$ from pair 3 of Appendix E:

$$u(t)\,e^{j\omega_0 t} \leftrightarrow \pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}.$$

D-10. Since $x(t) \leftrightarrow A\tau\, Sa(\omega\tau/2)$ from the solution to Problem D-1, where $Sa(\cdot)$ is defined by (E-3), (D-5) gives

$$Y(\omega) = A\tau\sum_{k=-N}^{N} Sa(\omega\tau/2)\,e^{-j\omega kT} = A\tau\, Sa(\tfrac{\omega\tau}{2})\sum_{k=-N}^{N} e^{-jk\omega T}.$$

Let $n = N + k$:

(D-10.) (Continued)

$$Y(\omega) = A\tau e^{j\omega NT} Sa(\omega\tau/2) \sum_{n=0}^{2N} e^{jn(-\omega T)}.$$

From (C-60) with $\theta = 0$ and $\phi = -\omega T$:

$$Y(\omega) = A\tau \frac{\sin[(2N+1)\omega T/2]}{\sin(\omega T/2)} Sa(\omega\tau/2).$$

(D-11.) $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^{\tau} A t^2 e^{-j\omega t} dt$

$= \frac{A}{(j\omega)^3}\left\{2 - e^{-j\omega\tau}[2 + 2j\omega\tau - \omega^2\tau^2]\right\}$ after using (C-47).

(D-12.) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-W}^{W} K e^{j\omega t} d\omega$

$= \frac{KW}{\pi} \frac{\sin(Wt)}{Wt} = \frac{KW}{\pi} Sa(Wt)$ after using (C-45).

(D-13.) Write $H(\omega)$ in the form

$$H(\omega) = K_0 \, \text{rect}\left(\frac{\omega}{2W}\right) + \sum_{n=1}^{N} K_n \, \text{rect}\left(\frac{\omega}{2W}\right)\left[e^{j\frac{n\pi\omega}{W}} + e^{-j\frac{n\pi\omega}{W}}\right]$$

from (E-2) and (C-7). From (D-7) and pair 6 of Appendix E :

$$x(t) = \frac{K_0 W}{\pi} Sa(Wt) + \sum_{n=1}^{N} \frac{K_n W}{\pi}\left\{Sa\left[W\left(t + \frac{n\pi}{W}\right)\right] + Sa\left[W\left(t - \frac{n\pi}{W}\right)\right]\right\}$$

or

$$x(t) = \frac{W}{\pi}\sum_{n=-N}^{N} K_n \, Sa\left[W\left(t - \frac{n\pi}{W}\right)\right]$$

with $K_{-n} = K_n$, $n = 1, 2, \cdots, N$.

D-14. (a) From (D-6): $x(t-2) \leftrightarrow X(\omega)\, e^{-j\omega 2}$. From (D-7):

$$x(t-2)\, e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)\, e^{-j2(\omega-\omega_0)}.$$

(b) From (D-10) and (D-7):

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

$$\frac{dx(t)}{dt}\, e^{j\omega_0 t} \leftrightarrow j(\omega-\omega_0)\, X(\omega-\omega_0).$$

Thus,

$$\frac{dx(t)}{dt}\, e^{j\omega_0 (t-3)} \leftrightarrow j(\omega-\omega_0)\, X(\omega-\omega_0)\, e^{-j3\omega_0}.$$

(c) From (D-6) and (D-8) with $\alpha = 2$:

$$x(t-3) \leftrightarrow X(\omega)\, e^{-j3\omega}$$

$$x(2t) \leftrightarrow \frac{1}{2} X\left(\frac{\omega}{2}\right).$$

Thus,

$$x(t-3) - 3x(2t) \leftrightarrow X(\omega)\, e^{-j3\omega} - \frac{3}{2} X\left(\frac{\omega}{2}\right).$$

D-15. (a) From (D-7): $x(t)\, e^{-j\omega_0 t} \leftrightarrow X(\omega+\omega_0)$.
Define $X_1(\omega) = X(\omega+\omega_0)$ and $X_2(\omega) = X(\omega)$ and use (D-18):

$$\int_{-\infty}^{\infty} x^*(\tau)\, x(\tau+t)\, e^{j\omega_0 \tau}\, d\tau \leftrightarrow X(\omega)\, X^*(\omega+\omega_0).$$

(b) From (D-7) and (D-11):

$$x(t)\, e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$$

$$(-jt)\, x(t) \leftrightarrow dX(\omega)/d\omega.$$

Use (D-16):

$$\int_{-\infty}^{\infty} (t-\tau)\, x(\tau)\, x(t-\tau)\, e^{j\omega_0 \tau}\, d\tau \leftrightarrow j X(\omega-\omega_0)\, \frac{dX(\omega)}{d\omega}.$$

D-15. (Continued) (c) From (D-14): $x^*(t) \leftrightarrow X^*(-\omega)$.

From (D-5):

$$x(t) + x^*(t) = 2\,\mathrm{Re}[x(t)] \longleftrightarrow X^*(-\omega) + X(\omega)$$

where Re[·] represents the real part.

D-16. Instantaneous power at any time t is $x^2(t)/R$. Energy at time t in an interval dt is $x^2(t)\,dt/R$. Total energy is

$$E = \int_{-\infty}^{\infty} \frac{x^2(t)}{R}\,dt = \frac{1}{R}\int_{-\infty}^{\infty} x^2(t)\,dt .$$

Apply Parseval's theorem (D-21):

$$E = \frac{1}{R}\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi R}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega .$$

D-17. Apply (D-9) to get

$$\frac{2\alpha}{\alpha^2+t^2} \longleftrightarrow 2\pi e^{-\alpha|-\omega|} = 2\pi e^{-\alpha|\omega|} .$$

Let $\alpha = 2$ and scale by $3/2$ to get

$$y(t) = \frac{6}{4+t^2} \longleftrightarrow 3\pi e^{-2|\omega|} = Y(\omega).$$

D-18. Let $x(t) = \delta(t) \leftrightarrow X(\omega)$. Then

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt = e^{-j\omega(0)} = 1$$

from (A-2), so $\delta(t) \leftrightarrow 1$.

D-19. (a) Since $\delta(t) \leftrightarrow 1$ from Problem D-18, use (D-9) to get $1 \leftrightarrow 2\pi\delta(-\omega) = 2\pi\delta(\omega)$ because $\delta(\omega)$ behaves as though it were a function with even symmetry. From (D-5): $A \leftrightarrow 2\pi A\,\delta(\omega)$.

(b) $\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$. Define $x(t) = 1$ and write $\cos(\omega_0 t) = \frac{x(t)}{2}e^{j\omega_0 t} + \frac{x(t)}{2}e^{-j\omega_0 t}$. Now use (D-7) and the fact that $1 \leftrightarrow 2\pi\delta(\omega)$ to get $\cos(\omega_0 t) \leftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$.

D-20. Define $x_1(t) = u(t)e^{-\alpha t} \leftrightarrow X_1(\omega) = \frac{1}{\alpha + j\omega}$

$$x_2(t) = \cos(\omega_0 t) \leftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)].$$

From (D-17):

$$x(t) = u(t)e^{-\alpha t}\cos(\omega_0 t) \leftrightarrow X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\xi)X_2(\omega-\xi)\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(\alpha+j\xi)}\,\pi[\delta(\omega-\xi-\omega_0) + \delta(\omega-\xi+\omega_0)]\,d\xi$$

$$= \frac{\alpha + j\omega}{(\alpha^2+\omega_0^2-\omega^2) + j2\alpha\omega} \quad \text{after using (A-2).}$$

D-21. For (D-6): $x(t-t_0) \leftrightarrow \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}\,dt$.

Let $\xi = t - t_0$, $d\xi = dt$. $\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}\,dt$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\xi)e^{-j\omega\xi}\,d\xi = X(\omega)e^{-j\omega t_0}.$$

For (D-10) differentiate $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

D-21. (Continued) n times with respect to t:

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j\omega)^n X(\omega).$$

★**D-22.** Write the left side of (D-12) as

$$\int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{\infty} u(t-\tau)\, x(\tau)\, d\tau,$$

where $u(t) \longleftrightarrow \pi\delta(\omega) + (1/j\omega)$ from pair 3 of Appendix E. From (D-16):

$$\int_{-\infty}^{\infty} u(t-\tau)\, x(\tau)\, d\tau \longleftrightarrow X(\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

or

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \pi X(0)\,\delta(\omega) + \frac{X(\omega)}{j\omega}.$$

D-23. $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^*(\tau)\, x_2(\tau+t)\, d\tau$

$\cdot\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x_1^*(\tau) \int_{-\infty}^{\infty} x_2(\tau+t)\, e^{-j\omega t}\, dt\, d\tau$

Let $\xi = \tau + t$, $d\xi = dt$.

$$X(\omega) = \int_{-\infty}^{\infty} x_1^*(\tau)\, e^{j\omega\tau}\, d\tau \int_{-\infty}^{\infty} x_2(\xi)\, e^{-j\omega\xi}\, d\xi$$

$$= \left[\int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau\right]^* \int_{-\infty}^{\infty} x_2(\xi)\, e^{-j\omega\xi}\, d\xi = X_1^*(\omega) X_2(\omega).$$

D-24. $X(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2)\, e^{-j\omega_1 t_1 - j\omega_2 t_2}\, dt_1\, dt_2$

$= A\int_{-\tau_2}^{\tau_2} e^{-j\omega_2 t_2}\, dt_2 \int_{-\tau_1}^{\tau_1} e^{-j\omega_1 t_1}\, dt_1 =$

$= 4A\tau_1\tau_2\, \dfrac{\sin(\omega_2\tau_2)}{(\omega_2\tau_2)}\, \dfrac{\sin(\omega_1\tau_1)}{(\omega_1\tau_1)}$ after using (E-45).

D-25. Use (C-46): $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^{\tau} At e^{-j\omega t} dt = \frac{A\tau}{j\omega} e^{-j\omega\tau/2} \{ Sa(\omega\tau/2) - e^{-j\omega\tau/2} \}$.

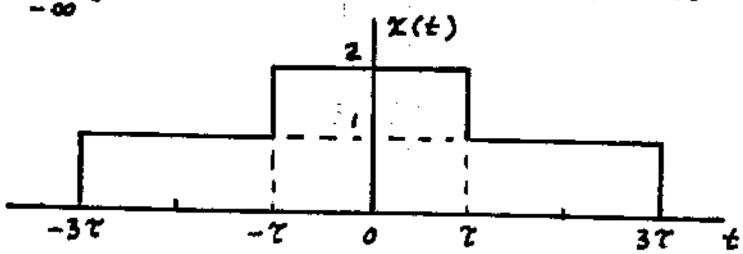
D-26. We know that $1/2\pi \longleftrightarrow \delta(\omega)$ so we transform $1/2\pi$ to get $\delta(\omega) = \mathcal{F}\{\frac{1}{2\pi}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t} dt$.

D-27. We know that $\delta(t) \longleftrightarrow 1$ so we inverse transform 1 to get
$$\delta(t) = \mathcal{F}^{-1}\{1\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega.$$

D-28. First, $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = A\int_{-\infty}^{0} e^{Wt} e^{-j\omega t} dt + A\int_0^{\infty} e^{-Wt} e^{-j\omega t} dt$
$= \frac{A}{W - j\omega} + \frac{A}{W + j\omega} = \frac{2AW}{W^2 + \omega^2}$ (checks with pair 19.) Second,
$y(t) = x(t)\cos(\omega_0 t) = \frac{x(t)}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$, so $Y(\omega) = \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0) = \frac{AW}{W^2 + (\omega - \omega_0)^2} + \frac{AW}{W^2 + (\omega + \omega_0)^2}$.

D-29. Since $X(t) \longleftrightarrow 2\pi x(-\omega)$ then
$$\frac{6}{(\alpha + jt)^4} \longleftrightarrow 2\pi u(-\omega)(-\omega)^3 e^{-\alpha(-\omega)} = -2\pi u(-\omega)\omega^3 e^{\alpha\omega}.$$

D-30. (a) $x(t) = \int_{-\infty}^{t} y(\xi)d\xi = u(t+3\tau) + u(t+\tau) - u(t-\tau) - u(t-3\tau)$.



(b) Since $Y(0) = 1 + 1 - 1 - 1 = 0$, (D-12) gives $\int_{-\infty}^{t} y(\xi)d\xi = x(t) \longleftrightarrow X(\omega) = \frac{Y(\omega)}{j\omega} = \frac{e^{j\omega\tau} - e^{-j\omega\tau}}{j\omega} + \frac{e^{j3\omega\tau} - e^{-j3\omega\tau}}{j\omega} = 2\tau\{Sa(\omega\tau) + \frac{1}{3}Sa(3\omega\tau)\}$.

D-31. (a) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{\omega_0-(W/2)}^{\omega_0+(W/2)} A\cos^2\left[\frac{\pi(\omega-\omega_0)}{W}\right] e^{j\omega t} d\omega$

$+ \frac{1}{2\pi}\int_{-\omega_0-(W/2)}^{-\omega_0+(W/2)} A\cos^2\left[\frac{\pi(\omega+\omega_0)}{W}\right] e^{j\omega t} d\omega$. Use $\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$.

$$x(t) = \frac{A}{4\pi}\int_{\omega_0-(W/2)}^{\omega_0+(W/2)} \left\{1 + \cos\left[\frac{2\pi(\omega-\omega_0)}{W}\right]\right\} e^{j\omega t} d\omega$$

$$+ \frac{A}{4\pi}\int_{-\omega_0-(W/2)}^{-\omega_0+(W/2)} \left\{1 + \cos\left[\frac{2\pi(\omega+\omega_0)}{W}\right]\right\} e^{j\omega t} d\omega$$

Use $\cos(\alpha) = \frac{1}{2}e^{j\alpha} + \frac{1}{2}e^{-j\alpha}$ and then use (C-45):

$$x(t) = \frac{AW}{2\pi}\cos(\omega_0 t)\left\{Sa(Wt/2) + \frac{1}{2}Sa[(Wt+2\pi)/2] + \frac{1}{2}Sa[(Wt-2\pi)/2]\right\}$$

On further reduction, we get another form:

$$x(t) = \frac{AW}{4\pi}\left\{2Sa\left(\frac{Wt}{2}\right) + \frac{-\sin\left(\frac{Wt}{2}\right)}{\left(\frac{Wt}{2}+\pi\right)} + \frac{-\sin\left(\frac{Wt}{2}\right)}{\left(\frac{Wt}{2}-\pi\right)}\right\}\cos(\omega_0 t)$$

$$= \frac{AW\pi}{2} Sa\left(\frac{Wt}{2}\right)\frac{\cos(\omega_0 t)}{[\pi^2 - (Wt/2)^2]}.$$

(b) $E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \frac{2}{2\pi}\int_{\omega_0-(W/2)}^{\omega_0+(W/2)} A^2\cos^4\left[\frac{\pi(\omega-\omega_0)}{W}\right] d\omega$. Use (C-16):

$$E = \frac{A^2}{\pi}\int_{\omega_0-(W/2)}^{\omega_0+(W/2)} \left\{\frac{3}{8} + \frac{1}{2}\cos\left[\frac{2\pi(\omega-\omega_0)}{W}\right] + \frac{1}{8}\cos\left[\frac{4\pi(\omega-\omega_0)}{W}\right]\right\} d\omega = \frac{3A^2W}{8\pi}.$$

D-32. $\int_{-\infty}^{\infty} x_1^*(t)x_2(t)\,dt = \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1^*(\omega)e^{-j\omega t}d\omega\,\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(\xi)e^{j\xi t}d\xi\,dt$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1^*(\omega)\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(\xi)\underbrace{\int_{-\infty}^{\infty}e^{j(\xi-\omega)t}dt}_{2\pi\delta(\xi-\omega)\text{ from (A-23)}}d\xi}_{X_2(\omega)\text{ from (A-16)}}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1^*(\omega)X_2(\omega)d\omega$$

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