Signals Notebook

Dr.Mahmoud Alhusari

By: Asma' Hakouz

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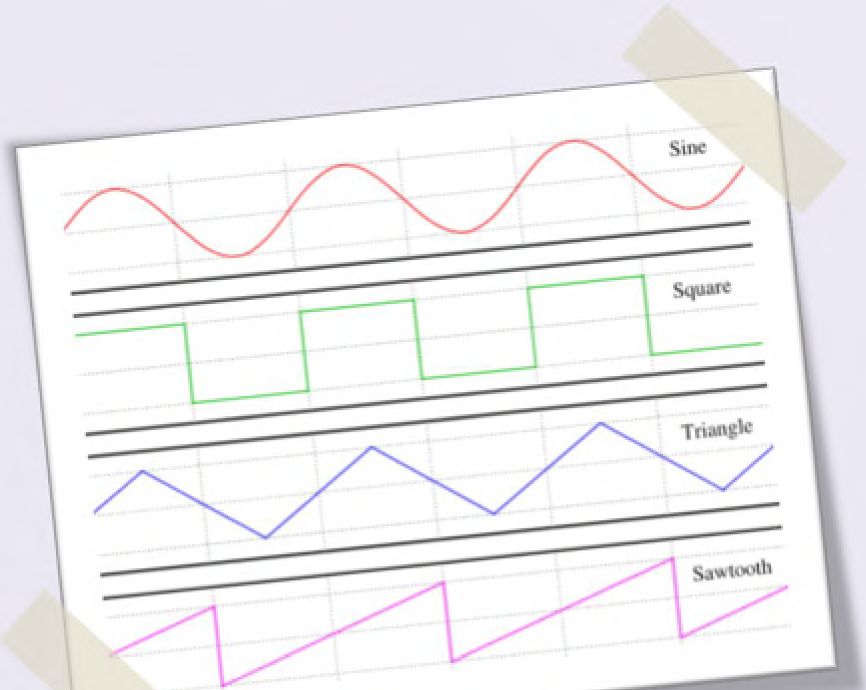
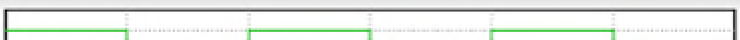


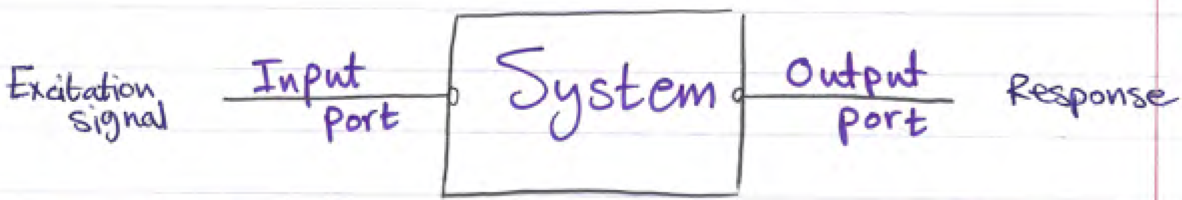


7/2/2013

Introduction ...

→ time
→ frequency

- A signal is defined as a function of one or more variable which conveys information on the nature of a physical phenomena.
- A System is defined as an intity that processes one or more signal to accomplish a function, thereby yeilding new signals.



└ Input Signal ┘ ↳ SISO └ Output Signal ┘

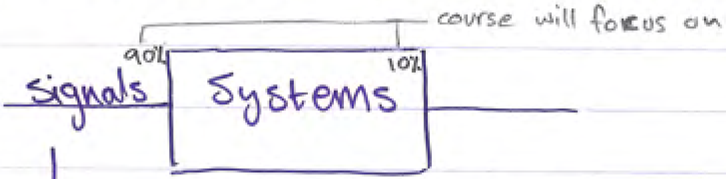
; single input single output.

↳ MIMO

; Multi Input Multi Output.

⋮

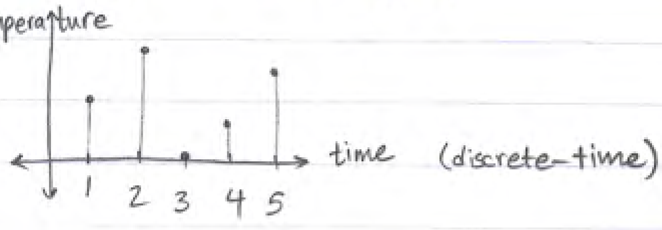
→ In this course we are concerned about SISO only!



↳ continuous-time (CT)

↳ Discrete-time (DT)

ex:



CT $\rightarrow x(t),\ t \in \mathbb{R}$ DT $\rightarrow x[n],\ n \in \mathbb{Z}$ (integers)

input		output
CT	$\rightarrow$	CT
DT	$\rightarrow$	DT
CT	$\rightarrow$	DT } hybrid (not considered in this course)
DT	$\rightarrow$	CT }

y

y-axis $\rightarrow \in \mathbb{R}$: analouge

y-axis $\rightarrow \in \mathbb{Z}$: digital

t

x-axis $\rightarrow \in \mathbb{R}$: continuous-time

x-axis $\rightarrow \in \mathbb{Z}$: discrete-time

Ch2 | $\boxed{\text{Ch 3}}$

- 1. Periodic signals, non-periodic (aperiodic).
- 2. Determinestic & Random.
- 3. Power & energy signals
- 4. Even & odd.

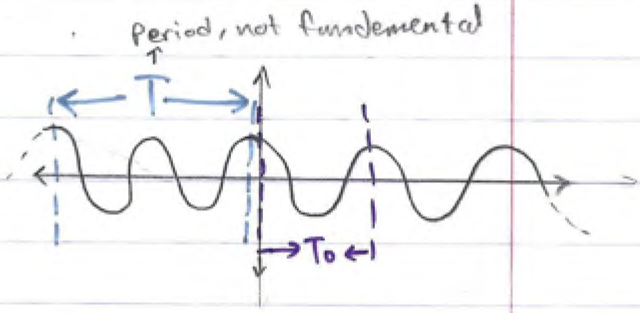
1. Periodic Signals:

$x(t) = x(t + nT)$

$n \in \mathbb{Z}$

$T > 0$

$-\infty < t < \infty$



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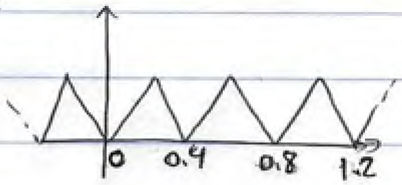
→ T_0 fundamental period (sec)

→ f_0 fundamental frequency [Hz] $\equiv \frac{1}{T_0}$

→ $\omega_0 = 2\pi f_0 \equiv \frac{2\pi}{T_0}$; fundamental angular freq. [rad/sec]

ex:



$T_0 = (0.4)$ sec.

$f_0 = \frac{1}{0.4} = (2.5)$ Hz.

$\omega_0 = 2\pi f_0 = (5\pi)$ rad/sec.

ex: $x(t) = A\sin(\omega_0 t + \phi)$

$x(t+T_0) = ?$

$x(t+T_0) = A\sin(\omega_0(t+T_0) + \phi)$

$= A\sin((\omega_0 t + \phi) + \underline{\omega_0 T_0})$

$\hookrightarrow \frac{2\pi}{\cancel{T_0}} \cancel{T_0}$

short cut:

$= A\sin(\underbrace{\omega_0 t + \phi}_{\alpha} + \underbrace{2\pi}_{\beta}) \rightarrow \equiv A\sin(\omega_0 t + \phi) \equiv x(t)$

→ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\omega_0 t + \phi + 2\pi) = \sin(\omega_0 t + \phi)\overset{1}{\cos 2\pi} + \cancel{\cos(\omega_0 t + \phi)\sin 2\pi}^{\,0}$

$x(t+T_0) = A\sin(\omega_0 t + \phi) \equiv x(t)$

* If $x_1(t)$ with period T_1

$x_2(t)$ with period T_2

& $x_3 = x_1 + x_2$, find $T_3 = ?!$

→ if $\frac{T_1}{T_2}$ is Rational → ; can be written as $\frac{a}{b}$ where a, b are Integers!

$\frac{6.2}{3.1}$ ✓ rational

$\frac{\pi}{3}$ ✗ irrational

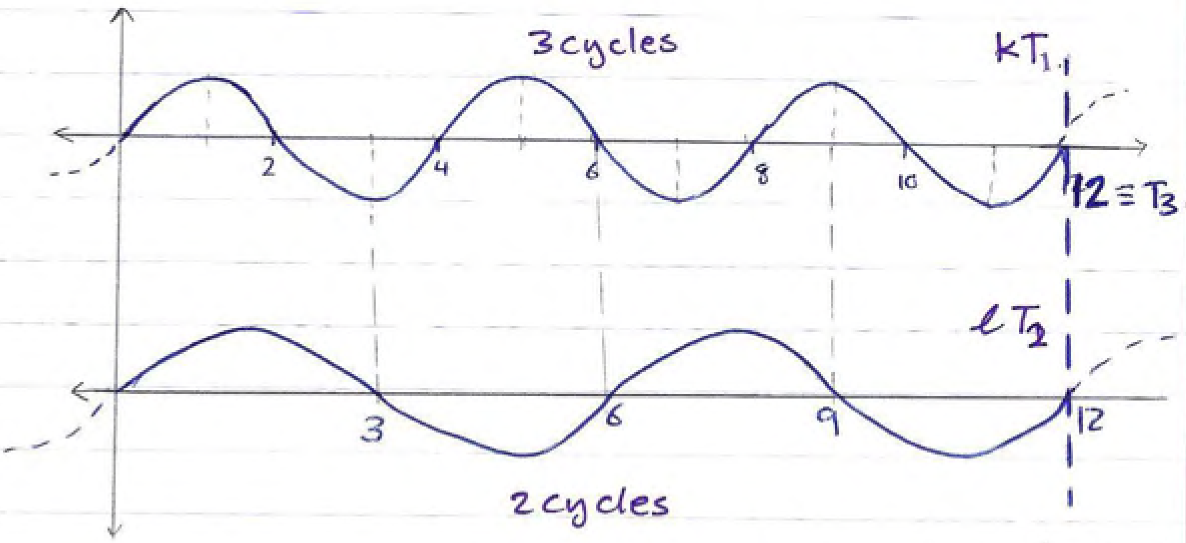
then: T_3 is the least common multiplier of T_1 & T_2.

ex: $x_1(t) = \cos\left(\underbrace{\frac{\pi}{2}}_{\omega_1} t\right)$, $x_2(t) = \cos\left(\underbrace{\frac{\pi}{3}}_{\omega_2} t\right)$

$T_1 = \frac{2\pi}{\omega_1}$ $\qquad$ $T_2 = \frac{2\pi}{\omega_2}$

$= 4$ sec $\qquad$ $= 6$ sec.

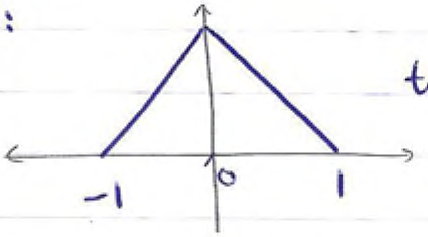
$\frac{T_1}{T_2} = \frac{4}{6}$ ✓rational $\therefore \boxed{T_3 = 12}$



we want to find when
$kT_1 = \ell T_2$ that's why
we look for the least common
multiplier

2. Determinestic & Random signals:
$\underline{\text{Random}}_{\ (\equiv \text{noise})}$

ex:



$$tri(t) = \begin{cases} 1-|t| , & -1 \leq t \leq 1 \\ 0 , & o.w \end{cases}$$

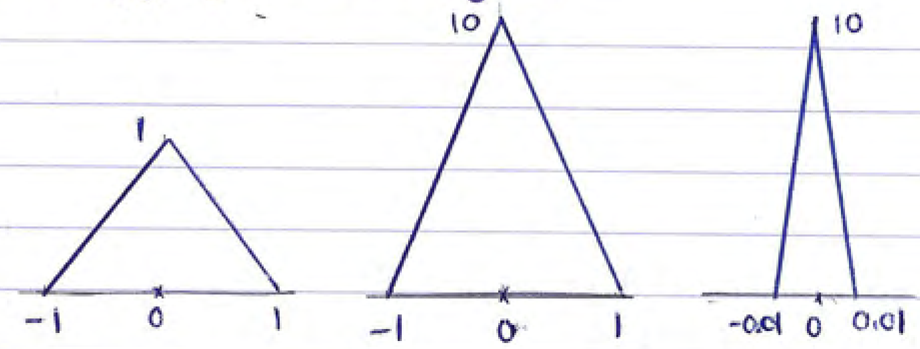
* Useful Trigonometric Identities:-

- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
- $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

- $\cos(a)\cos(b) = 1/2\,[\cos(a-b) + \cos(a+b)]$
- $\sin(a)\;\sin(b) = 1/2\,[\cos(a-b) - \cos(a+b)]$
- $\sin(a)\cos(b) = 1/2\,[\sin(a-b) + \sin(a+b)]$

- $\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$
- $\sin 2a = 2\sin(a)\cos(a)$
- $\cos^2(a) = \frac{1}{2}(1 + \cos 2a)$
- $\sin^2(a) = \frac{1}{2}(1 - \cos 2a)$

3. Energy & Power signals:



minimize the output ? on what sense?!!
first we have to define the size of the signal!

$$E_\infty = \int_{-\infty}^{\infty} x^2(t)\,dt \qquad 0 < E_\infty < \infty$$

; thus we can't classify periodic signals as Energy signals because E_∞ for them equals ∞.

⇝ E_∞ is a way to classify signals depending on their sizes.

I) in general for continuous-time functions

$$\boxed{E_\infty = \int_{-\infty}^{\infty} |x(t)|^2\,dt} \quad \text{---} \boxed{1}$$

↳ in case $x(t)$ is complex

$$\int_{-\infty}^{\infty} x(t) * \underbrace{x^*(t)}_{\text{conj.}}\,dt$$

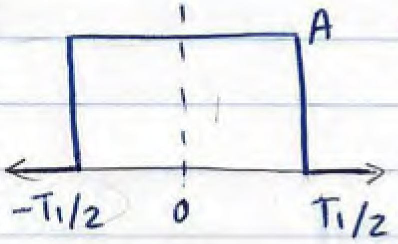
ex:- $e^{j\omega t} \cdot e^{-j\omega t} = e^0 = 1$

(12/2/2013

II)* E_∞ definition for discrete-time functions:

$$\boxed{E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2} \quad \text{---} \boxed{2}$$

ex: a)



→ continuous-time signal.

$E_\infty =$ a) $A T_1$

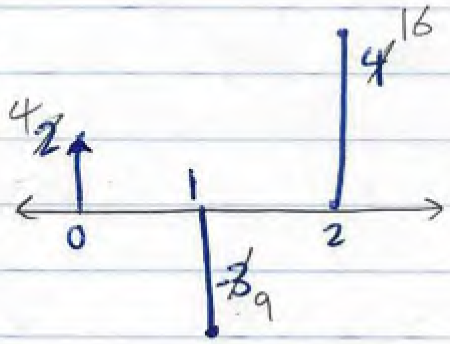
$\boxed{\text{b) } A^2 T_1}$ ✓

c) $(A T_1)^2$

d) none of the above

; E_∞ equals the area under the square of the function, not the square of the area under the function!

b)



→ Discrete-time signal

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 = 2^2 + (-3)^2 + 4^2 = 29$$

→ Previous 2 signals were "**time-limited functions**", which occur over a finite interval & are zero otherwise. & they're considered to be energy signals since $P=0$.

⇝ for infinite signal

$$P_\infty = \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} |x(t)|^2 dt}$$, $0 < P_\infty < \infty$ Power signals

* Energy signals have 0 power. ⇒ $\frac{\text{finite value}\ (E_\infty)}{\infty} = 0$!

so, Power Signals $\rightarrow E_\infty = \infty$

Energy signals $\rightarrow P_\infty = 0$

- Periodic functions:

$\rightarrow$ since it's periodic, no need for taking the limit.

$$\boxed{P_\infty = \frac{1}{T} \int_T |x(t)|^2 dt}$$ ...; P_∞ for periodic signals.

$\hookrightarrow$ integrate over 1 period

<u>Def.</u> $\boxed{r.m.s = \sqrt{P_\infty}}$

<u>ex:</u> $x(t) = A\cos(\omega_0 t + \phi)$

find r.m.s. ?

Sol: $P_\infty = \frac{1}{T} \int_T A^2 \cos^2(\omega_0 t + \phi) dt$

$= \frac{A^2}{T} \int_T \frac{1}{2}(1 + \cos(2\omega_0 t + 2\phi) dt$

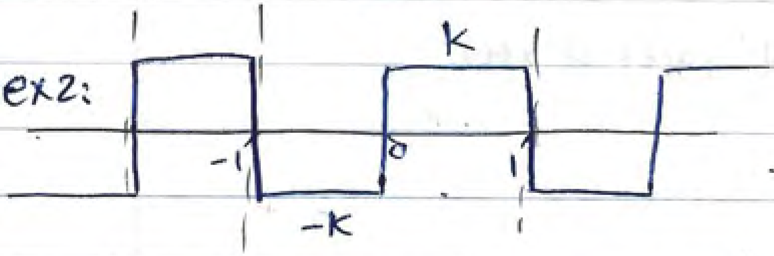
$= \frac{A^2}{2T} \int_T 1 dt + \underbrace{\frac{A^2}{2T} \int_T \cos(2\omega_0 t + 2\phi) dt}_{0}$

$P_\infty = \frac{A^2}{2\cancel{T}} \cdot \cancel{T} = \frac{A^2}{2}$

$\rightarrow r.m.s = \sqrt{P_\infty} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}}$

* for Sinusoidal signals $\boxed{r.m.s = \frac{A}{\sqrt{2}}}$ $\rightarrow$ amplitude.

$T=2$

ex2: 

find r.m.s?

$$P_\infty = \frac{1}{2}\left(\int_{-1}^{0} (-K)^2 dt + \int_{0}^{1} K^2 dt\right)$$

$$= \frac{1}{2}K^2 + \frac{1}{2}K^2 = K^2$$

$\therefore$ $r.m.s = \sqrt{K^2} = K$

** r.m.s isn't dependant on the frequency nor the phase shift, it's dependant on the amplitude.

* Power in a resistor: $P_R = \int_{-\infty}^{\infty} \frac{V^2(t)}{R} dt = \frac{1}{R}\int_{-\infty}^{\infty} V^2(t) dt$

$$P_R = \frac{1}{R} \underbrace{E_\infty}_{\text{Energy of the signal}} \Rightarrow [E_\infty] = V^2.sec$$

of the signal

Energy delivered by the signal to the load (absorbed by the load).

* Energy's unit depends on the nature of the signal.

Signals
- Power, $E = \infty$
  ex: periodic,
  $\Rightarrow \int_T (f)^2 dt$ is infinite!
- Energy, $P = 0$
  ex: time-limited func.
  $\int_T (f)^2 dt$ is +ve & finite.

*a signal can't be power and energy togther, but it can be neither of them.

4. Even & odd functions:

→ Even:

$x(t) = x(-t) \ \forall t$ ... mathematically
for all

whatever's happening in (+ve) t is ... physically
symmetrically happening in −ve t

→ Odd:

$x(t) = -x(-t) \ \forall t$

x(t) may be neither even nor odd.

** Any function can be decomposed into an ~~one~~ even part & odd part.

$x(t) = x_{even}(t) + x_{odd}(t)$ ... ①

$x(-t) = x_e(-t) + x_o(-t)$

$\rightarrow x(-t) = x_e(t) - x_o(t)$ ... ②

① + ② ⇝ $\frac{x(t) + x(-t)}{2} = x_e(t)$

① - ② ⇝ $\frac{x(t) - x(-t)}{2} = x_o(t)$

that's as long as we're dealing with REAL func.

→ In complex functions we deal with "conjugate symmetry"

$x(t) = x^*(-t)$

ex: $x(t) = e^{jwt}$

$x(-t) = e^{-jwt}$

$x^*(t) = e^{-jwt}$

$$e^{jw_0t} \rightarrow x_{even} = \frac{e^{jw_0t} + e^{-jw_0t}}{2} = \cos w_0t$$

$$\rightarrow x_{odd} = \frac{e^{jw_0t} - e^{-jw_0t}}{2} = j \sin w_0t$$

$$\boxed{e^{jw_0t} = \cos w_0t + j\sin w_0t} \quad \text{.... Euler's eq.}$$

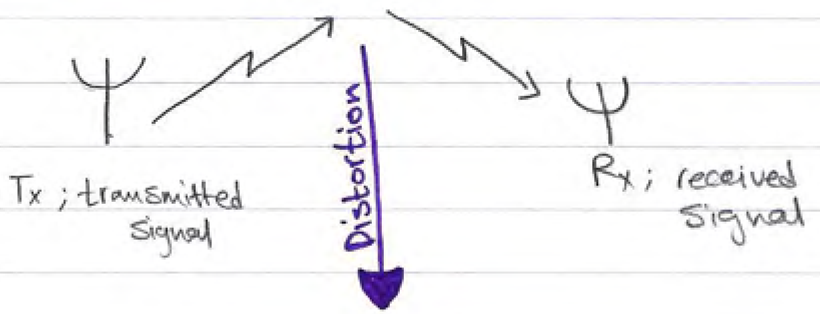
++ $\boxed{2.10}$

- even + even = even
- odd + odd = odd
- odd + even = neither odd nor even
- odd * odd = even $\quad x(-t) = -x_1(t) \times -x_2(t) = x(t)$
- even * even = even $\quad x(-t) = x_1(t) \times x_2(t) = x(t)$
- even * odd = odd $\quad x(-t) = x_1(t) \times -x_2(t) = -x(t)$

ال odd هو السالب وال even هو الموجب

14/2/2013

Signals operations:



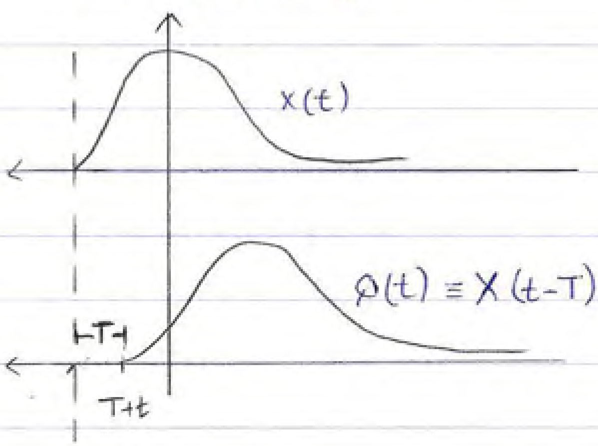
1) Shifing → delayed
↳ advanced; it's practically impossible, but we'll represent it mathematically only.

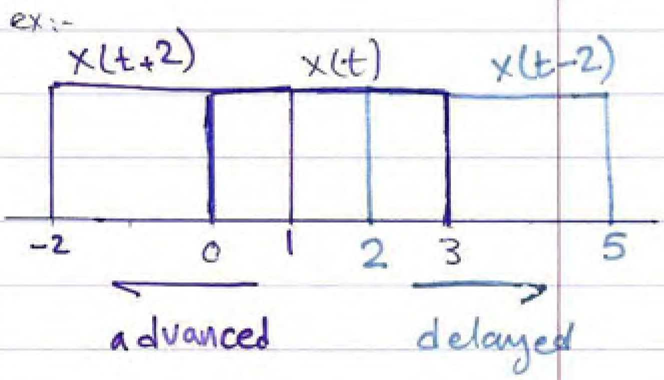
2) Scaling → Compressed
↳ expanded / Stretched

3) Reflection

These types of distortion can happen with respect to x-axis **(time), or y-axis **(amplitude)**.
↳ the independant variable ↳ the dependant variable

1. Time shifting:





** x-axis

shift right (−) delayed →

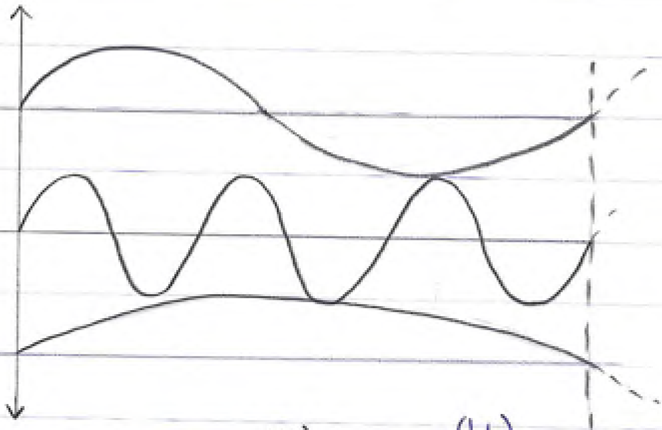
shift left (+) advanced ←

$$\boxed{\phi(t) = x(t \pm T)}$$

2. time-Scaling

$$\boxed{Q(t) = x(at)}$$; a is +ve $\to$ >1 $\to$ compression
$\hookrightarrow$ <1 $\to$ expansion

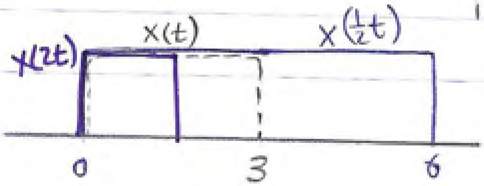
ex:



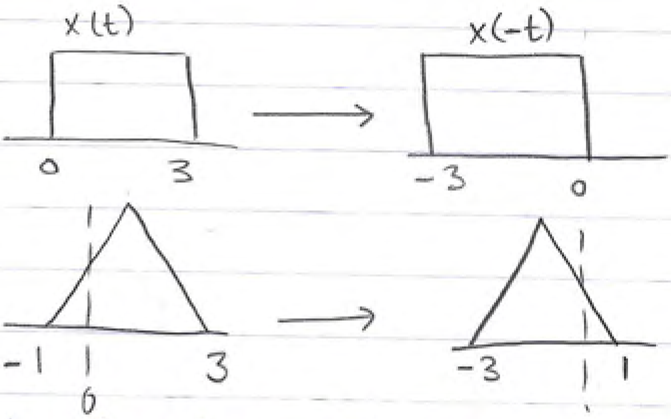
$\sin(t)$

$\sin(3t)$ — comprison by a factor of 3

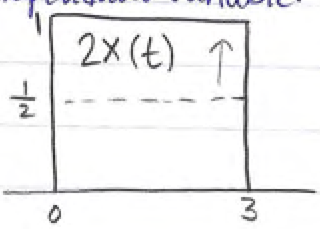
$\sin(\frac{1}{2}t)$ — expansion by a factor of 2



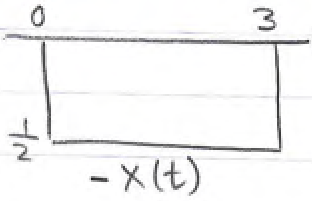
3. time-reflection :



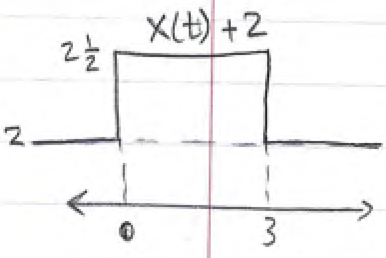
$\leadsto$ transformation with respect to the amplitude (y-axis); the dependant variable.



Scaling



reflection



Shifting

given $x(t)$, find $x\left(\frac{t-t_0}{a}\right)$ أي تغيير بيكون بدل (t) فقط

$$x(t) \xrightarrow[\text{}]{\text{Switch } t \to t-t_0} x(t-t_0) \xrightarrow{t \to t/a} x\left(\frac{t}{a}-t_0\right) \times$$

sol: $x(t) \xrightarrow{t \to t/a} x(t/a) \xrightarrow{t \to t-t_0} x\left(\frac{t-t_0}{a}\right)$ *

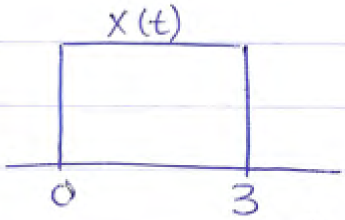
or $x(t) \xrightarrow{t \to t-\frac{t_0}{a}} x\left(t-\frac{t_0}{a}\right) \xrightarrow{t \to t/a} x\left(\frac{t-t_0}{a}\right)$ **

in the previous case it's better to scale first then shift.

this may differ in other cases;

ex2: $x(at+t_0)$ ⇝ here shifting first is much better.

ex: Given

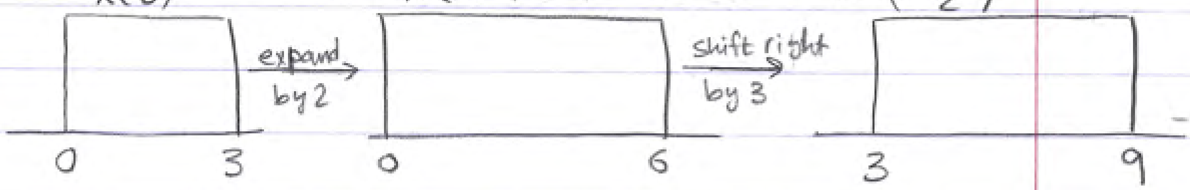


, find:

a) $x\left(\frac{t-3}{2}\right)$

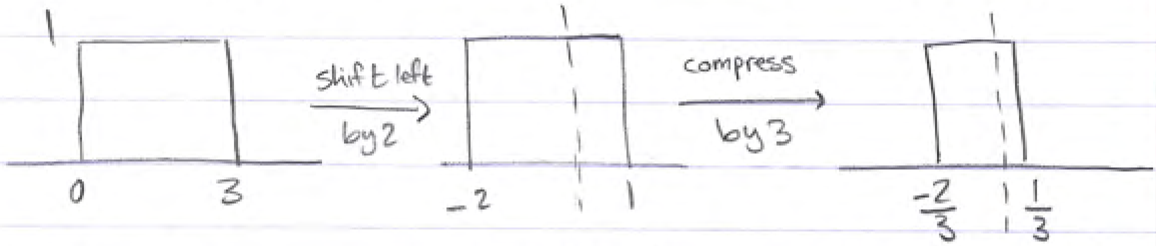
اخترنا القسمة على 2 أول لأنه القسمة شاملة للـ t والـ 3

$$x(t) \xrightarrow{t \to t/2} x(t/2) \xrightarrow{t \to t-3} x\left(\frac{t-3}{2}\right)$$



b) $x(3t+2)$

$$x(t) \xrightarrow{t \to t+2} x(t+2) \xrightarrow{t \to 3t} x(3t+2)$$



* the question may give as $x\left(\frac{t-3}{2}\right)$ and asks for $x(t)$...

→ we follow same steps in reverse

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* Useful functions:

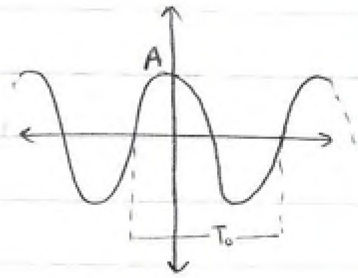
1. Sinusoids & complex functions.
2. Unit step
3. Signum.
4. Unit Ramp
5. Unit impulse.
6. Rectangler (Rect)
7. Sinc (sampling).

1. Sinusoids:

$$x(t) = A\cos(\underbrace{\omega_0}_{\text{rad/sec}} t + \phi)$$

$$\equiv A\cos((2\pi f_0)t + \phi)$$

$$\equiv A\cos\left(\left(\frac{2\pi}{T_0}\right)t + \phi\right)$$



2. Exponential Complex functions:-

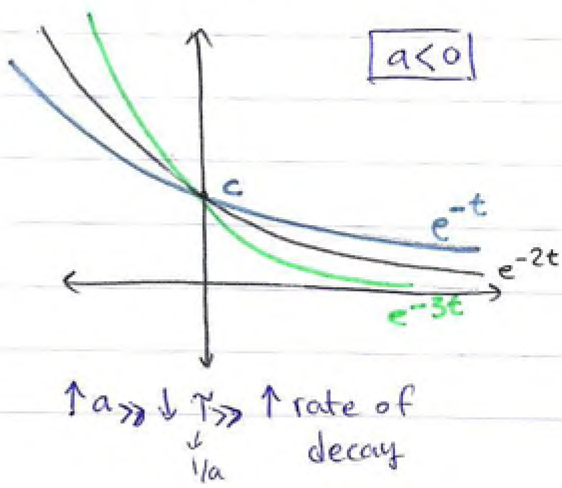
$$x(t) = Ce^{at} \qquad (a,c) \rightarrow 3 \text{ cases!!}$$

$\boxed{\text{Case 1}}$ $c, a \in \mathbb{R}$ (Real).

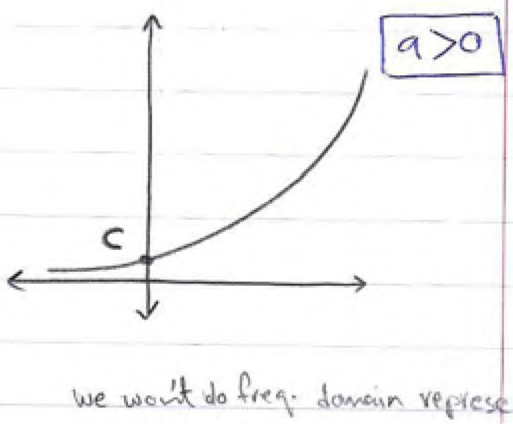
ex: $x(t) = 2e^{-2t}$

$= 3e^{-3t}$

$\boxed{a=0}$; constant $(y=c)$



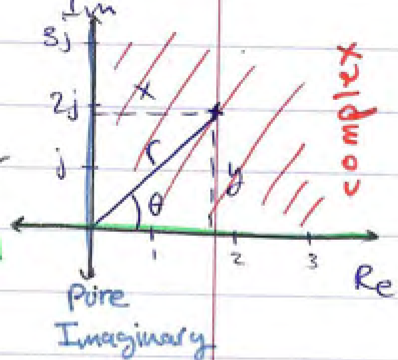
$\uparrow a \gg \downarrow \underset{1/a}{T} \gg \uparrow$ rate of decay



we won't do freq. domain representation for this case in this course! needs laplace!

** Revision of complex numbers:

ways of representing pts in the complex plane:-



1. $p = x + jy$ "Cartesian form" / rectangular

2. $p = r\angle\theta$ "Polar form"

3. $p = re^{j\theta}$ "exponential form"

; $x = r\cos\theta$

$y = r\sin\theta$

plug in Cartesian...

$$p = r\cos\theta + jr\sin\theta$$
$$= r(\cos\theta + j\sin\theta) \leftarrow \text{Euler's identity!}$$
$$= re^{j\theta}$$

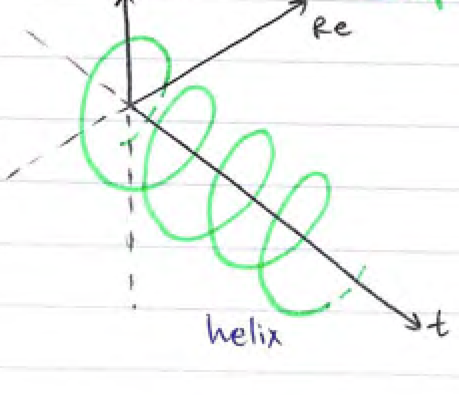
$\boxed{\text{Case 2}}$ c : complex , a : Imaginary.

$$x(t) = \underbrace{Ae^{j\theta}}_{c} \, e^{\overbrace{j\omega_0}^{a} t}$$

$$= Ae^{j(\omega_0 t + \theta)}$$

$$x(t) = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta)$$

"complex sinusoids"



views: from above $(Re/t) \to \cos$; $Re\{x(t)\}$

" the side $(Im/t) \to \sin$; $Im\{x(t)\}$

$\boxed{\text{Case 3}}$ a, c are complex

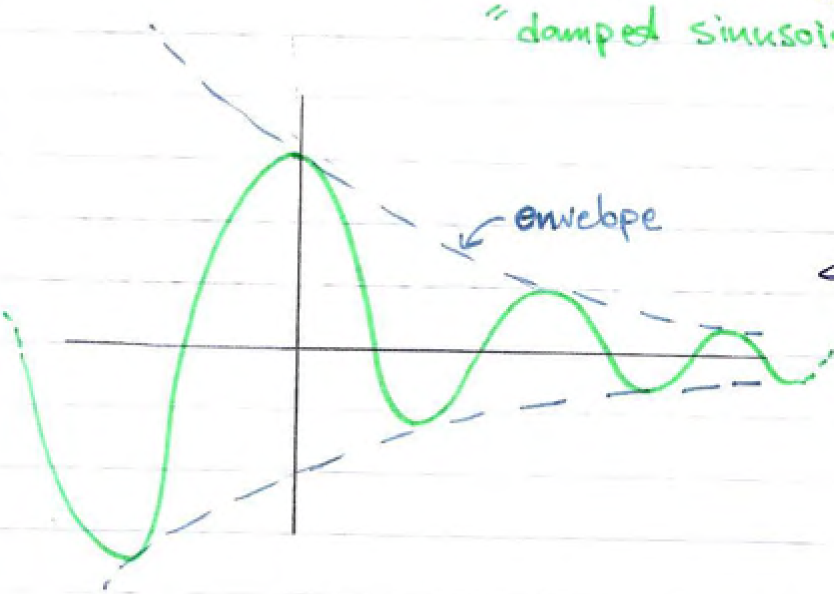
$$x(t) = \underbrace{Ae^{j\phi}}_{c}\, e^{\overbrace{(\delta + j\omega_0)}^{a}t}$$

$$= A\, e^{j\phi}\, e^{\delta t}\, e^{j\omega_0 t}$$

$$= A\, e^{\delta t}\, e^{j(\omega_0 t + \phi)}$$

$$= A e^{\delta t}\cos(\omega_0 t + \phi) + j\, A e^{\delta t}\sin(\omega_0 t + \phi)$$

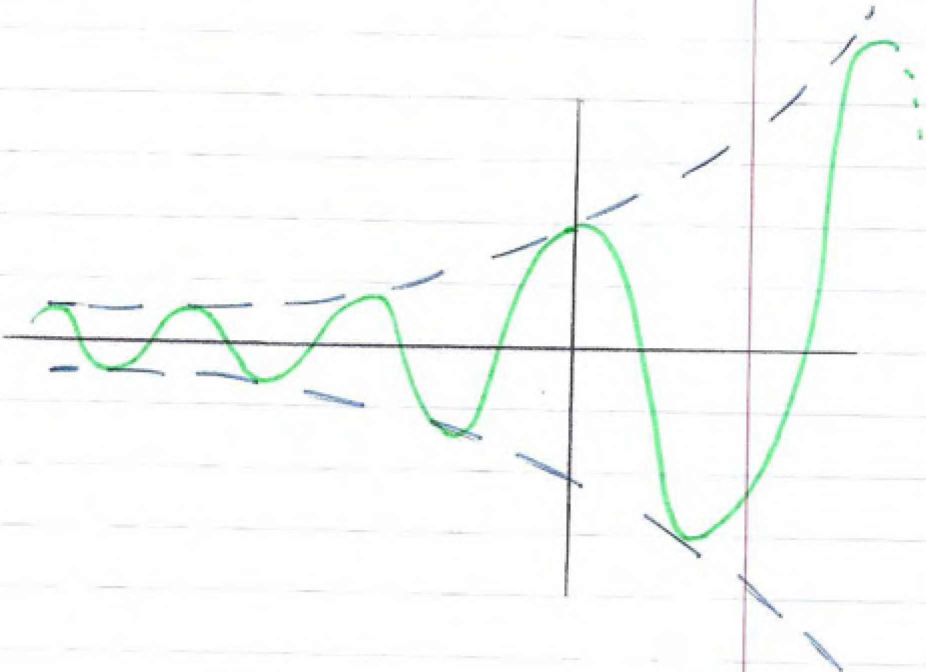
"damped sinusoids"



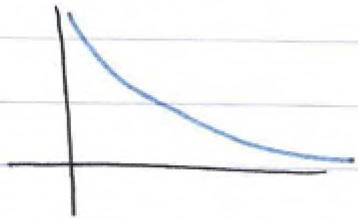
$\boxed{\delta < 0}$

δ: damping coefficient
exponential

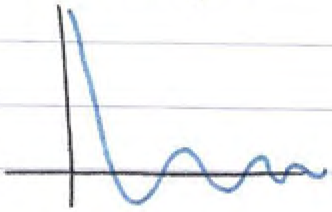
$\boxed{\delta > 0}$



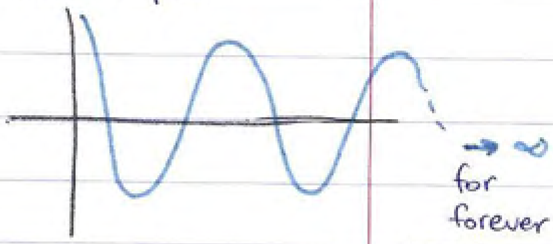
Over damped



Underdamped

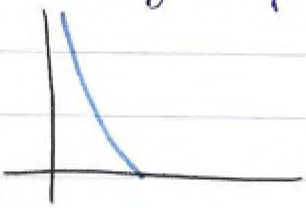


Undamped



$(\delta = 0)$

critically damped



19/2/2013

$$x(t) = e^{j(\omega_0 t + \phi)}$$

a) is it periodic?

$$\equiv \; x(t) \overset{?}{=} x(t + nT)$$

sol: $x(t+nT) = e^{j(\omega_0 (t+nT) + \phi)}$

$$= e^{j(\omega_0 t + \omega_0 nT + \phi)}$$

$$= e^{j(\omega_0 t + \phi)} \, e^{\omega_0 nTj} \longrightarrow \omega_0 nT = \left(\tfrac{2\pi}{T}\right) nT = 2\pi n$$

$$\therefore x(t+nT) = x(t)\, e^{2\pi n j}$$

$$= x(t) * 1$$

$$x(t+nT) = x(t)$$

$\therefore x(t)$ is periodic!

* $e^{j 2\pi n}$
  $\to \theta = 2\pi n \equiv 0$
  $r = 1$
* $e^{-j\pi/2} = -j$ (on the Im axis!)
* $e^{-j\pi} = -1$ (on the Re axis)

b) $I_m \{ e^{j(\omega_0 t + \phi)} \} = \sin(\omega_0 t + \phi)$

$Re \{ e^{j(\omega_0 t + \phi)} \} = \cos(\omega_0 t + \phi)$

c) $e^{j(w_0 t+\varnothing)} = e^{jw_0\left(t+\frac{\varnothing}{w_0}\right)}$ ↳ shift in t!

d) is it a power signal?

let $x(t) = Ae^{j(w_0 t+\varnothing)}$, A is const.

$$P_\infty = \frac{1}{T}\int_{-T/2}^{T/2} A\cancel{e^{j(w_0 t+\varnothing)}} \cdot A\cancel{e^{-j(w_0 t+\varnothing)}}\,dt = A^2$$

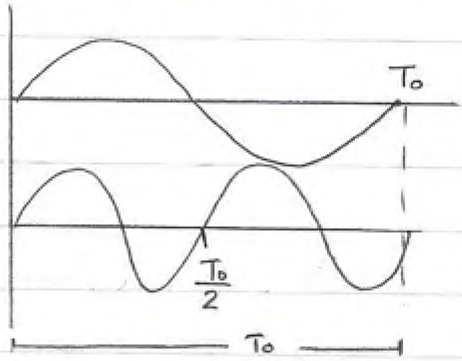
1st / fundamental harmonic!

$$\left\{\ldots, e^{-j2w_0 t}, e^{-jw_0 t}, 1, \underset{T_0}{e^{jw_0 t}}, \underset{\frac{T_0}{2}}{e^{j2w_0 t}}, \ldots, \underset{\frac{T_0}{|K|}}{e^{jKw_0 t}}, \ldots\right\}$$

2nd harmonic

$$\varnothing_K(t) = e^{jKw_0 t} \quad ; \quad K \in \{0, \pm1, \pm2, \ldots\}$$

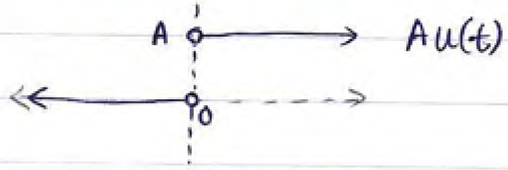
* this set is called "harmonically related"; integer multiples of some fundamental frequency.
* Each function of this set is periodic with Period (T_0)



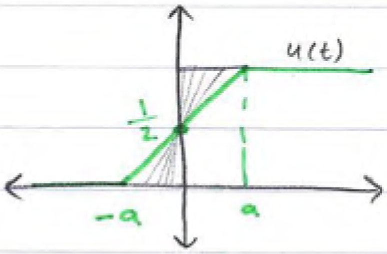
it's a period but not fundamental!

* if we sum them we'll get a new <u>periodic</u> function ($\frac{T_1}{T_2}$: rational $\frac{int}{int}$).

3. Unit Step function (test signal):



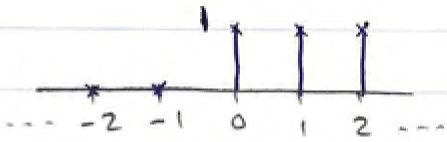
$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$$



$\lim_{a \to 0} u(t) = \frac{1}{2}$; Generalized Unit step func.

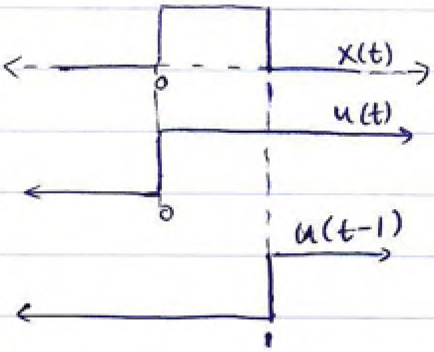
* u(t) in discrete time :

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



ex: represent x(t) using unit step function:-

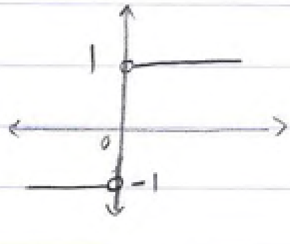
sol:-



$\Rightarrow x(t) = u(t) - u(t-1)$

4. Signum function:

$$\operatorname{sgn}(t) = \begin{cases} -1, & t < 0 \\ 1, & t > 0 \end{cases}$$



* $\operatorname{sgn}(t) = 2u(t) - 1$

* $u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$

*** Sgn is an odd function

$\boxed{2.5}$

Given :- $x(t) = \underset{t>-2}{4(t+2).\underline{u(t+2)}} - \underset{t>0}{4t.\underline{u(t)}} - \underset{t>2}{4.\underline{u(t-2)}} - \underset{t>4}{4(t-4).\underline{u(t-4)}} + \underset{t>5}{4(t-5)\,\underline{u(t-5)}}$

find & sketch $x(2t-4)$

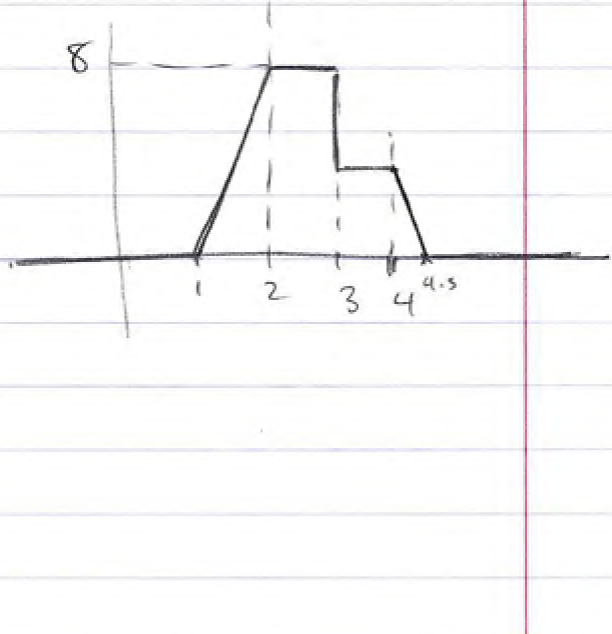
$\Rightarrow$ think of $u(t-a)$ as a switch , which turns on whe $t>a$ then the function which is multiplied bu $u(t-a)$ is elemenated when $t<a$, because $u(t-a)$ will equall zero

then :-

$$x(t) = \begin{cases} 0, & t<-2 \\ 4(t+2), & -2<t<0 \\ 4(t+2)-4t, & 0<t<2 \\ 4(t+2)-4t-4, & 2<t<4 \\ 4(t+2)-4t-4-4(t-4), & 4<t<5 \\ 4(t+2)-4t-4-4(t-4)+4(t-5), & t>5 \end{cases} = \begin{cases} 0, & t<-2 \\ 4t+8, & -2<t<0 \\ 8, & 0<t<2 \\ 4, & 2<t<4 \\ 20-4t, & 4<t<5 \\ 0, & t>5 \end{cases}$$

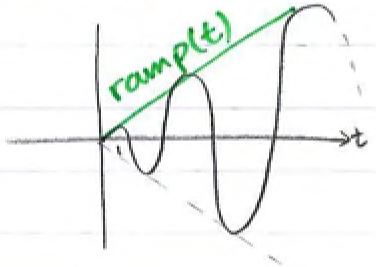
عوض عن (2t-4) مكان كل t حتى نحدد الاقتران :)

$$x(2t-4) = \begin{cases} 0, & 2t-4<-2 \Rightarrow t<1 \\ 8t-8, & 1<t<2 \\ 8, & 2<t<3 \\ 4, & 3<t<4 \\ 36-8t, & 4<t<4.5 \\ 0, & t>4.5 \end{cases}$$



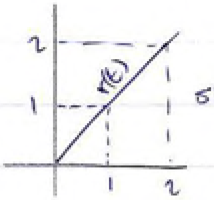
$\boxed{2.6}$

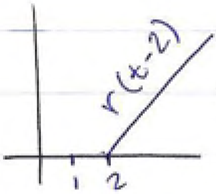
5) Unit Ramp function:-



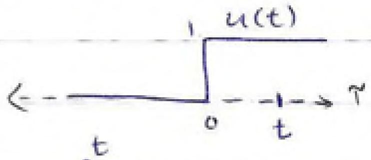
linear envelope!

$$ramp(t) / r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t > 0 \end{cases} = t\,u(t)$$





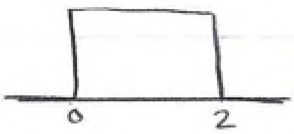
** another approach:-



$$\int_{-\infty}^{t} u(\tau)\,d\tau = \begin{cases} 0, & t \leq 0 \\ t, & t > 0 \end{cases} \equiv r(t)$$

$$\boxed{r(t) = \int_{-\infty}^{t} u(\tau)\,d\tau}$$

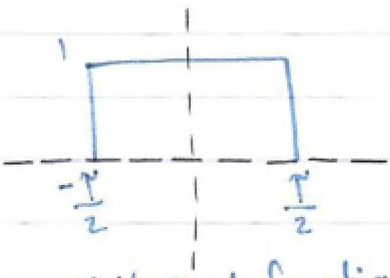
6) Rectangular function (Rect) $\Pi(t)$



$= u(t) - u(t-2) = rect\left(\frac{t-1}{2}\right)$

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1, & |t| < \frac{\tau}{2} \\ 0, & o.w \end{cases}$$

$-\frac{\tau}{2} < t < \frac{\tau}{2}$

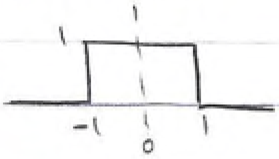


→ Unit Rect function

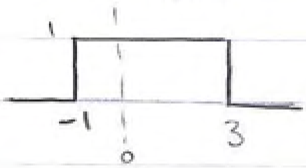
$\text{rect}\left(\frac{t-t_0}{T}\right)$ $\xrightarrow{}$ center of the rect.
$\xrightarrow{}$ width " " "

ex:

a) $\text{rect}\left(\frac{t}{2}\right)$



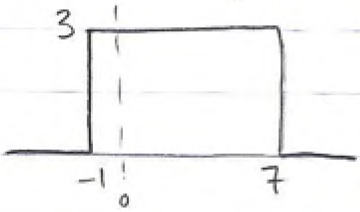
b) $\text{rec}\left(\frac{t-1}{4}\right)$



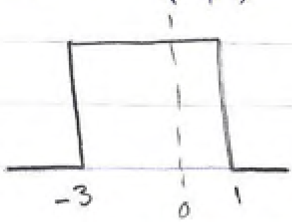
** (Rect) is an even function

$\therefore\ \text{rect}\left(\frac{1-t}{4}\right) = \text{rect}\left(\frac{t-1}{4}\right)$

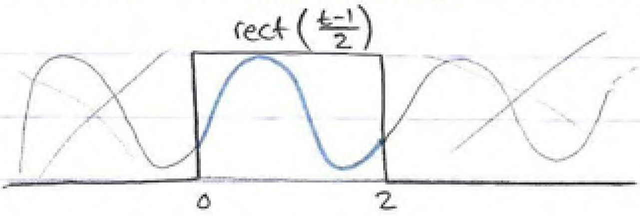
c) $3\,\text{rect}\left(\frac{t-3}{8}\right)$



d) $3\text{rect}\left(\frac{t+1}{4}\right)$



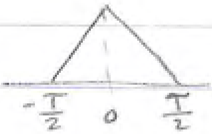
* Rect is also called "window function"



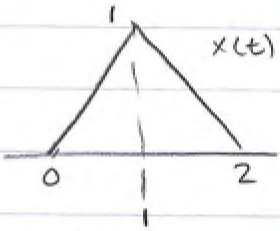
let $x(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & \text{o.w} \end{cases} = \cos t * \text{rect}\left(\frac{t-\pi}{2\pi}\right)$

** when you want a part of a func. over some period, you can always think of it as the function multiplied by Rect function

** $tri(t/\tau)$



<u>ex</u>:- express $x(t)$ in terms of: a) rect function & b) Unit step func.



a)

$$x(t) = t * \left[rect\left(\frac{t-\frac{1}{2}}{1}\right)\right] + (2-t) * \left[rect\left(\frac{t-1.5}{1}\right)\right]$$

b)

$$x(t) = t * \left[u(t) - u(t-1)\right] + (2-t) * \left[u(t-1) - u(t-2)\right]$$

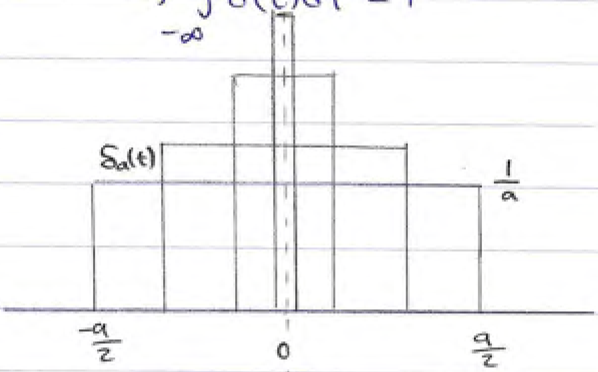
7) Unit Impulse function: / Delta function ($\delta(t)$)

$$\delta(t) = 0, \quad t \neq 0$$

$$\& \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\boxed{\delta(t) = \delta(-t)}$$

an even function!



$$\delta_a(t) = \begin{cases} \frac{1}{a}, & |t| < \frac{a}{2} \\ 0, & o.w \end{cases}$$

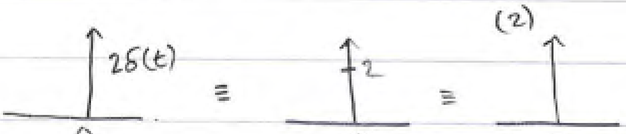
$$\lim_{a \to 0} \delta_a(t) = \delta(t) \text{ ... unit impulse func.}$$

" Shock ", OFF-ON-OFF in zero time!

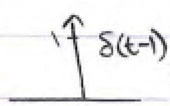
when we apply $\delta(t)$ to a system, its behavior will depend on its own dynamics only!

* Unit impulse has a strength & weight of (1).

** Strength / weight comes from the area!



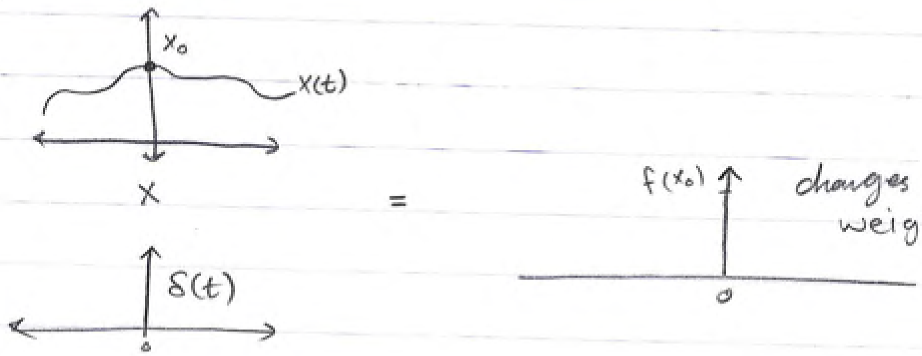
different notations :)



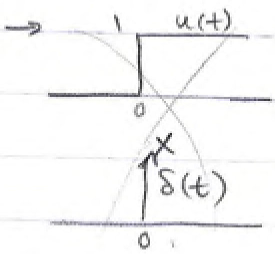
$\ast \; \delta(2t) = \frac{1}{2}\delta(t)$
$\delta(\frac{1}{2}t) = 2\,\delta(t)$ } from area!

26/2/2013 Properties:-

1. Sampling:

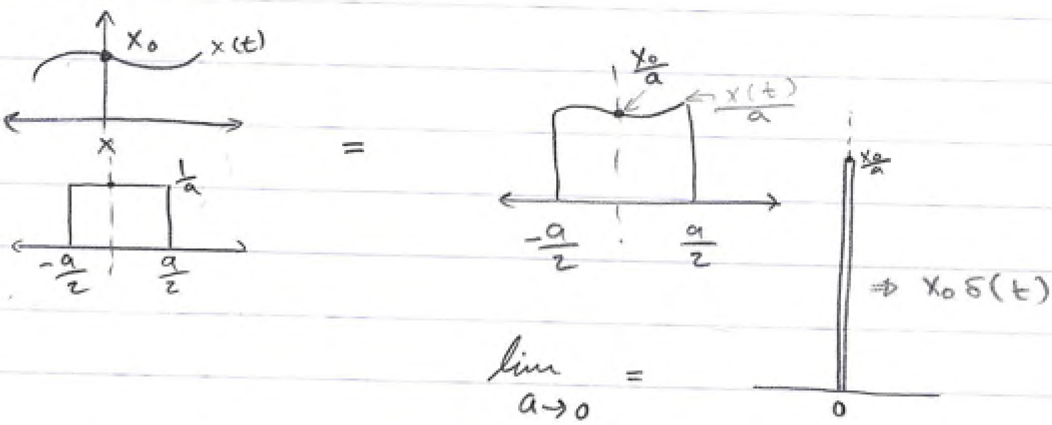


changes the strenghth/weight of δ (area)



not valid!!
because $x(t)$ @ $t=0$ must be continuous & finite!

The limit & area sense →



in general: $\boxed{f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)}$, where $f(t)$ is continuous at $t=t_0$!

2. Sifting Property:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = \int_{\infty}^{-\infty} x(0)\,\delta(t)\,dt = x(0)\overbrace{\int_{-\infty}^{\infty}\delta(t)\,dt}^{1} = x(0)$$

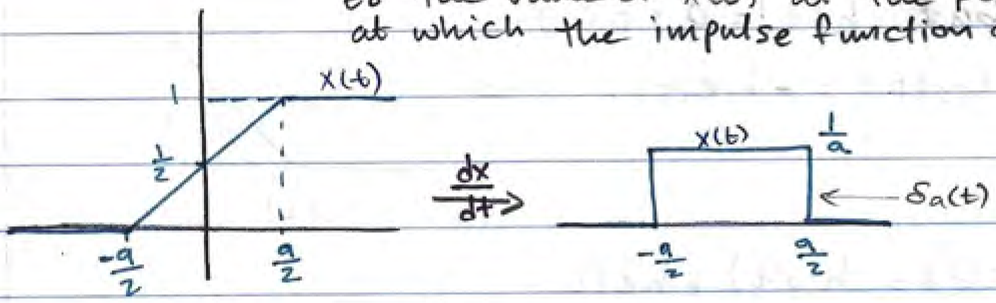
- the General case of sifting:-

$$\boxed{\int_{t_1}^{t_2} x(t)\,\delta(t-t_0)\,dt = x(t_0)} \quad , t_0 \in (t_1, t_2)$$

o.w it'll become 0

* the value of the integral is equal to the value of $x(t)$ at the point at which the impulse function occurs.

$$\int_{-\infty}^{\infty} x(t-t_1)\,\delta(t-t_2) = x(t_2-t_1)$$



$$\lim_{a\to 0} x(t) = u(t) \xrightarrow{\frac{du}{dt}} \lim_{a\to 0} x(t) = \delta(t)$$

Generalized derivative of $u(t)$ $\boxed{\frac{du}{dt} = \delta(t)}$ only when we use the limit sense, while in general $\frac{du}{dt}$ isn't valid because $u(t)$ isn't defined at 0!

but... if that's true, then it must satisfy sifting, so,

$$\int_{-\infty}^{\infty} x(t)\underbrace{\left(\frac{du}{dt}\right)}_{\delta(t)}dt \overset{?}{=} x(0) \quad \text{is it true?}$$

by parts

$$= x(t)u(t)\Big]_{-\infty}^{\infty} - \int_{\cancel{-\infty}\ 0}^{\infty} \cancel{u(t)}\,x(t)\,dt = x(\infty)u(\infty) - x(-\infty)\cancel{u(-\infty)}^{0} - x(\infty) + x(0)$$

$$= x(0)$$

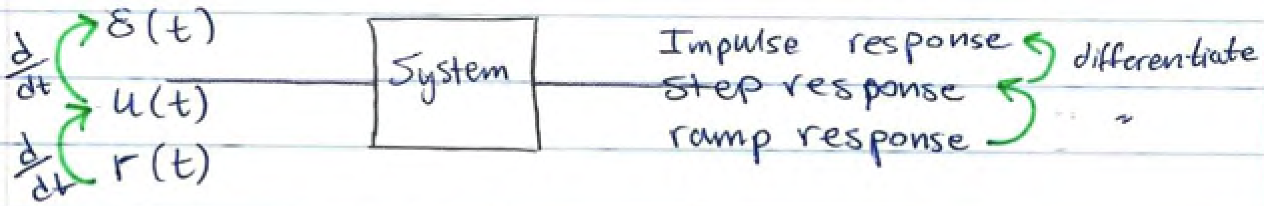
$u(t), t<0 = 0$

so the integration will equal 0 as well!

$$\boxed{\int_{-\infty}^{\infty} x(t)\,\cancel{u(t-t_0)}\,dt \equiv \int_{t_0}^{\infty} x(t)\,dt}$$

$$\boxed{\frac{du}{dt} = \delta(t)} \qquad \boxed{\int_{-\infty}^{t} \delta(t)\,dt = u(t)}$$

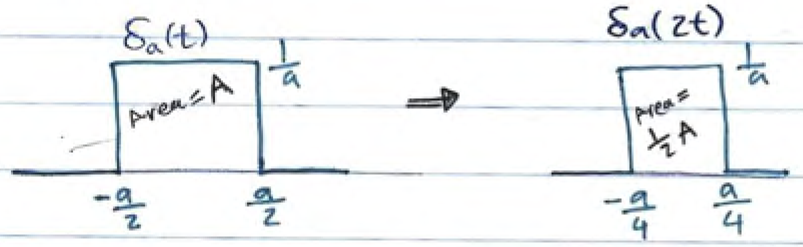
$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases} = u(t)$$



3. Scaling property:

$\delta(2t)$, $\delta(t/2)$ } time scaling — it doesn't have an effect on t, it affects area (strength & weight).

we have to go back to δ_a inorder to know what's happening t $\delta(t)$



$$\therefore \ \delta(2t) = \frac{1}{2}\delta(t)$$
$$\delta\left(\frac{t}{2}\right) = 2\,\delta(t)$$

trying to see it calculus wise:-

$$\int_{-\infty}^{\infty} \delta(\alpha t)dt \quad \ldots (1)$$

$$\left[\begin{array}{l} \text{let } x = \alpha t \\ dx = \alpha dt \end{array}\right.$$

$$\frac{1}{\alpha}\int_{-\infty}^{\infty} \delta(x)dx = \frac{1}{\alpha}$$

$$\boxed{\delta(\alpha t - \beta) = \frac{1}{|\alpha|}\delta\left(t - \frac{\beta}{\alpha}\right)} \qquad \underline{\text{Scaling}}$$

<u>ex</u>:- $\delta(\frac{t}{2} - 1) = 2\delta(t-2)$

$\alpha = \frac{1}{2}$

$\beta = 1$

<u>ex2</u>: $\int_{-2}^{4} \underbrace{(t+t^2)}_{x(t)} \delta(t-3)dt$

<u>sol</u>: $\Rightarrow$ $3 \in [-2, 4]$

$= 3+(3)^2 = 12$

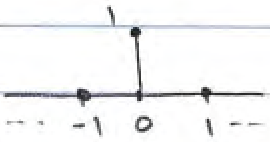
<u>ex3</u>: $\int_{0}^{3} e^{(t-2)} \delta(2t-4)dt$

<u>sol</u>: $= \int_{0}^{3} e^{t-2}\left(\frac{1}{2}\delta(t-2)\right)dt = \frac{1}{2}e^{(2)-2} = \frac{1}{2}e^{0} = \frac{1}{2}$

$$\boxed{\int_{-\infty}^{\infty} x(t-t_0)\,\delta(t) = x(-t_0)}$$

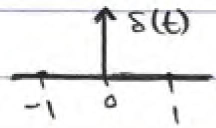
- Discrete-time unit impulse function:-

$\boxed{DT}$ * $\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$



$\boxed{CT}$ * $\delta(t) = 0 \ , \ t \neq 0$

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$



* $\sum_{m=-\infty}^{n} \delta[m] = \begin{cases} 0, & n<0 \\ 1, & n\geq 0 \end{cases} = u[n]$ * $\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$

* $x[n]\,\delta[n] = x[0]\,\delta[n]$

  $x[n]\,\delta[n-k] = x[k]\,\delta[n-k]$

* $\delta[\alpha n] = \delta[n]$

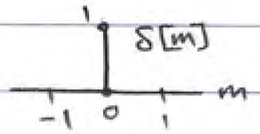
  if you land on time which is an integer!

* $x(t)\,\delta(t) = x(0)\,\delta(t)$

  $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

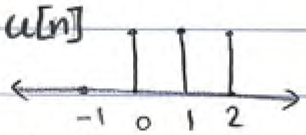
* $\delta(\gamma t) = \frac{1}{\gamma}\,\delta(t)$

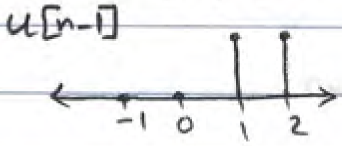
ex:-



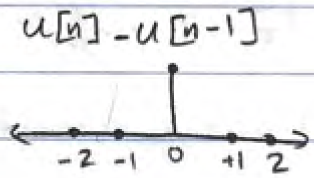
define it in terms of $u(t)$:

sol



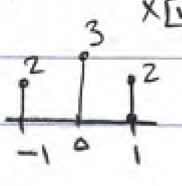


$\Rightarrow$

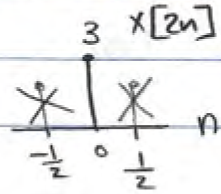


→ when you compress a function in Discrete-Time, you don't lose any sample if you land on time (n) which is an integer, else you have to wipe out the samples because $n \in \mathbb{Z}$!

ex:-



$\Rightarrow$

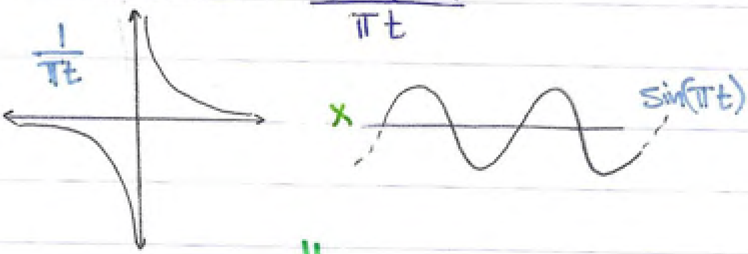


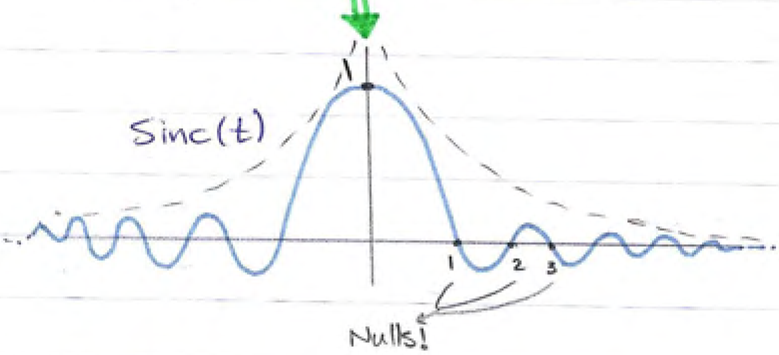
$n \in \mathbb{Z}$!!

we don't have the same problem in expansion!

8. Sinc function :-

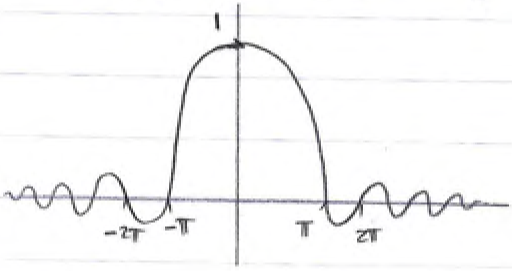
$$\text{Sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$





$$\lim_{t \to 0} \text{sinc}(t) = \lim_{t \to 0} \frac{\sin(\pi t)}{\pi t} = \lim_{t \to 0} \frac{\pi \cos(\pi t)}{\pi} = 1$$

also... $S_a = \frac{\sin(t)}{t}$



++ * excersize time transformation on sinc! & identify possition of 1^{st}, 2^{nd}, ... nulls!

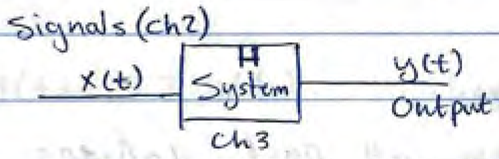
when Given $\text{sinc}(t) \Rightarrow \frac{\sin(\pi t)}{\pi t}$

" " $S_a(t) \Rightarrow \frac{\sin(t)}{t}$

Sinc in t-domain $\rightleftharpoons$ rect in freq domain & vice versa!

Chapter 3 :- Systems:

* Notation used for systems :-



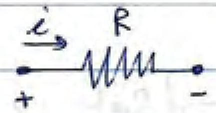
1^{st} notation : $y(t) = H\{x(t)\}$

2^{nd} " : $x(t) \to \boxed{H} \to y(t)$

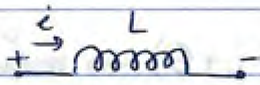
OR

$x(t) \xrightarrow{H} y(t)$

ex:-



$\underset{\text{Output}}{V} = \underset{\text{Input}}{I}R \rightarrow$ system!



$\underset{O/p}{V} = \underset{\text{System}}{L\frac{d}{dt}}\underset{I/p}{i}$

mathematical models of systems!

Classes of systems:-

1. memoryless / with memory.
2. Invertable / non-invertable.
3. Causal / (anti/non) - causal.
4. Stability.
5. time-invarient & time varying.
6. linear / non-linear.

1. → dynamic (with memory)
   ↳ Static (without " / memoryless).

the Resistor is an example of memoryless System.

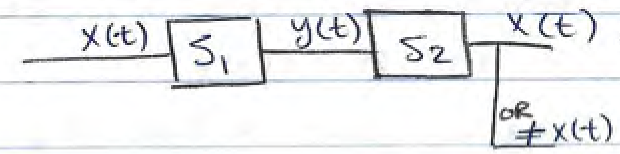
Capacitors & inductors are " " " dynamic (with memory) systems

→ if the output $y(t_0)$ depends on input values other than $x(t_0)$ then it's with memory.

memoryless System ⇒ $\boxed{y(t) = k\,x(t)}$ I/p & O/p are at the same time.

ex: $y(t) = 3.2\,x(t)$, they are amplifiers or attenuators

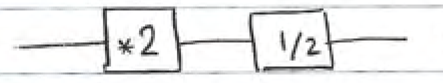
2. Invertablity:



: System S_1 is invertable and S_2 is its inverse!

system is non invertable.

ex:-



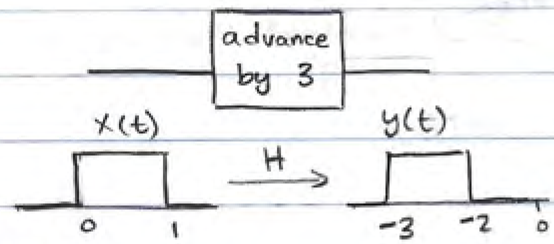
invertable

non-invertable $x(t) = \pm\sqrt{y(t)}$ non-invertable

the system is invertable if distinct inputs result in distinct o/ps

3. Causality:

* no output could be taken (Practically) without exciting the system with some input. 'Cause & effect'.



↳ non-causal / anticipative / non-reliable.

practically * we can't build a system that gives a response for something that didn't happen yet!!

the system is said to be causal when its output depends on present or past inputs only $(t \leq t_0)$

3/3/2013

$y(t) = x(t+2) \longrightarrow$ non-causal (advanced by 2)! depends on Future I/P

$y(t) = x(t-2) \longrightarrow$ causal (delayed by 2)! depends on Past I/P

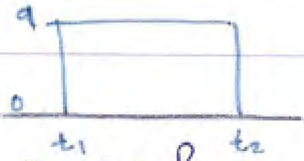
$y(t) = k\,x(t) \longrightarrow$ memoryless, causal ; depends on present input.

4. Stability:

- Def: a signal is Bounded if:

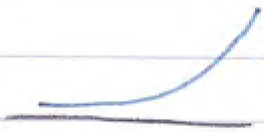
$$|x(t)| < \infty \quad \forall t$$

- Bounded $\neq$ time-limited.
  ↳ occurs over a finite period of time, & equals zero otherwise.



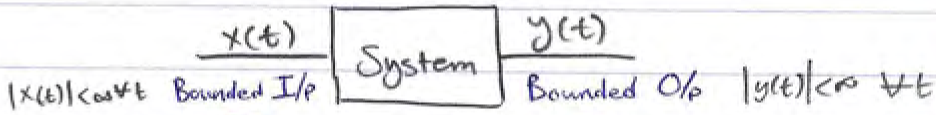
Bounded & time-limited.

Bounded, but time unlimited.



not Bounded nor time limited

- Def: when we apply a bdd (bounded) input to a system & have a bdd output, then system is stable.



(BIBO) stable.

5. Time-invariance.

$x(t) \xrightarrow{\#} y(t)$

$x(t-t_0) \xrightarrow{\#} y(t-t_0)$, time-invarient (shift in input = shift in O/P)

$\quad \hookrightarrow$ o.w , time-varient

ex: $x(t) \longrightarrow y(t+5)$

$x(t-t_0) \longrightarrow y(t-t_0+5)$ $\Big] \longrightarrow$ time-invarient

* I only care about the shift in (t), not the value of the output.

** Time-invarience test (TI test).

$$\begin{tikzcd}
 & \mathrm{I} & x(t) \arrow[r] \arrow[r, bend left, "H"] & y(t) \arrow[d, bend left, "\text{delay by } t_o"] \\
\mathrm{II} & x(t) \arrow[r, "\text{delay by } t_o"'] & x(t-t_o) \arrow[r] \arrow[r, bend right, "H"'] & y(t-t_o)
\end{tikzcd}$$

} T-invarient system

** if the resulting output from I equals that from II, then then system is time-invarient.

ex:- check TI for:-

a) $y(t) = \cos(x(t))$

I: $x(t) \rightarrow \boxed{\cos(\ )} \xrightarrow{y(t)=\cos(x(t))} \boxed{\text{delay by } t_o} \rightarrow y(t-t_o) = \cos(x(t-t_o))$

$\parallel$

II: $x(t) \rightarrow \boxed{\text{delay by } t_o} \xrightarrow{y(t-t_o)=x(t-t_o)} \boxed{\cos(\ )} \rightarrow y(t-t_o) = \cos(x(t-t_o))$

o/p I = o/p II ∴ time-invarient

b) $y(t) = x(t)\cos t$

I: $x(t) \rightarrow \boxed{\cos t \ H} \xrightarrow{y(t)=x(t)\cos t} \boxed{\text{delay by } t_o} \rightarrow y(t-t_o) = x(t-t_o)\cos(t-t_o)$

II: $x(t) \rightarrow \boxed{\text{delay by } t_o} \xrightarrow{y(t-t_o)=x(t-t_o)} \boxed{\cos t \ H} \rightarrow y(t-t_o) = x(t-t_o)\cos t$

o/p I ≠ o/p II ∴ time-varient

6. linearity:

for a system to be linear, it must satisfy 2 things:-

1. Scaling (homogeneousity).
2. Additivity.

1. $\underline{K}\, x(t) = \underline{k} y(t)$

homogenous.

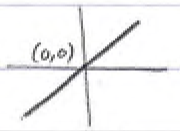
2. $x_1 \longrightarrow y_1$

$x_2 \longrightarrow y_2$

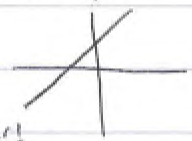
$x_1 + x_2 \longrightarrow y_1 + y_2$ ✓ satisfies additivity.

if both ① & ② are satisfied by a system then it's considered to be linear.

++



linear



non-linear!

piecewise linear

→ we can apply Super position on it.

ex:- $a\ddot{y}(t) + by^2(t) = x(t)$, is it linear?

$$x_1(t) = a\ddot{y}_1(t) + by_1^2(t)$$
$$x_2(t) = a\ddot{y}_2(t) + by_2^2(t)$$

$$x_1(t) + x_2(t) = a\ddot{y}_1(t) + a\ddot{y}_2(t) + by_1^2(t) + by_2^2(t) \quad \text{—①}$$

but let $y = y_1 + y_2 \Rightarrow$

$$x(t) = a\ddot{y}_1(t) + a\ddot{y}_2(t) + b(y_1 + y_2)^2 \neq x_1 + x_2$$

$\therefore$ it's non-linear

- Extra Examples

examples on 1. memory:-

$$y(t) = k \int_{-\infty}^{t} x(\tau)\, d\tau$$

$y(t)$ here depends on all past values of the input $x(t)$! so it's <u>with memory</u>!

$y(t) = k\, x(t)$
& $y(t) = x^2(t)$ } are memory less → $y(t_0)$ depends only on $x(t_0)$!

$y(t) = x(t+5)$ → with memory
because $y(t_0)$ depends on $x(\underline{t_0+5})$! 5 units in time ahead t_0!

examples on 2. invertability:-

- $y(t) = k\, x(t) \Rightarrow x(t) = \frac{1}{k} y(t)$ ✓ invertable

$y(t) = x^2(t) \Rightarrow x(t) = \pm\sqrt{y(t)}$ ✗ non-invertable
we have to find a unique $x(t)$ for each $y(t)$ to consider the system to be invertable.

$y = |x(t)|$ → not invertable
$y = \cos(x(t))$ → " " ; $\cos\pi$, $\cos 3\pi$ have the same value

$y = \frac{\cos(t)}{t} \cdot x(t)$ → " " ; we can't get an output value at $t=0$
(can be considered <u>almost</u> invertable.)

-Extra Examples /2

When a System is linear
we can apply super position. on it
هو تطبيقه وليس شرط ;

$\boxed{2.27}$ d) $y(t) = e^{t\,x(t)}$

① memory less / ② $x(t) = \frac{\ln y(t)}{t}$; we can get values of x back except at t=0 | so it's considered not invertable

③ causal ④ not stable → e^{tM} ; $t \to \infty$ so $y \to \infty$

⑤ non-linear

⑥ $x(t) \to e^{t\,x(t)} \to e^{(t-t_0)\,x(t-t_0)}$
$x(t) \to x(t-t_0) \to e^{t\,x(t-t_0)}$ } ≠ ∴ it's time varient

e) $y(t) = 7x(t) + 6$

* memory less / * causal / * stable $|y(t)| \leq 7M + 6$
* invertable $x(t) = \frac{y(t) - 6}{7}$ / * non-linear
* $x(t) \xrightarrow{H} 7x(t) + 6 \to 7x(t-t_0) + 6$
  $x(t) \xrightarrow{t} x(t-t_0) \xrightarrow{H} 7x(t-t_0) + 6$ } = time invarient

f) $y(t) = \int_{-\infty}^{t} x(5\tau)\,d\tau$

* with memory / Not stable / linear.
* Non causal ; it depends on past values of t ($-\infty \to t$) & also depends on $5t > t$ for $t > 0$!!)
* invertable ⇒ $\frac{d\,y(t)}{dt} = x(5t)$
  $\therefore x(t) = \left.\frac{d\,y(t)}{dt}\right|_{t = \frac{t}{5}}$

* $x(t) \to \int_{-\infty}^{t} x(5\tau)\,d\tau \to \int_{-\infty}^{t-t_0} x(5(\tau - t_0))\,d\tau$
  $x(t) \to x(t-t_0) \to \int_{-\infty}^{t} x(5(\tau - t_0))\,d\tau$ ≠ time varient.

-Extra Examples /3

$\boxed{2.26}$ $y(t) = \int_{t}^{t+1} x(\tau - \alpha)\, d\tau$

a) * with memory ; $y(t_0)$ doesn't depend only on $x(t_0)$!

* not invertable
* if $|x(t)| \leq M \Rightarrow y(t) = \int_{t}^{t+1} M\, d\tau = M(\cancel{t}+1-\cancel{t}) = M$ $\therefore$ <u>stable</u>
* linear $Kx(t) \Rightarrow Ky(t)$ & يحقق الـ Additivity لأنّ التكامل يتجزأ على الجمع
* time invarient ! $x(t) \xrightarrow{H} \int_{t}^{t+1} x(\tau-\alpha)\,d\tau \xrightarrow{\text{delay by } t_0} \int_{t-t_0}^{t-t_0+1} x(\tau-\alpha)\,d\tau$
$x(t) \xrightarrow[t_0]{\text{delay by}} x(t-t_0) \rightarrow \int_{t}^{t+1} x(\tau - t_0)\,d\tau = \int_{t-t_0}^{t-t_0+1} x(\tau-\alpha)\,d\tau$

b) for what value of α is the system causal?

$t+1-\alpha \Rightarrow 1-\alpha \leq 0$ $\boxed{-ve}$ shift to right (delayed)

$\therefore \boxed{\alpha \geq 1}$

$\boxed{2.27}$ a) $y(t) = \cos(x(t-1))$

* with memory ; $y(t_0)$ depends on $x(t_0 - 1)$
* Not invertable ; since $x(t)$ & $x(t+2\pi)$ has the same output!!
* causal , it depends on past I/p values $x(t-1)$ delayed
"Output at time $\underline{t}$ doesn't depent on input at times greater than t!"
* Stable, because $|y(t)| \leq 1$ for any value of input. cosine!
* time invarient
$x(t) \xrightarrow{H} \cos(x(t-1)) \xrightarrow{\text{delay by } t_0} \cos(x(t-t_0-1))$
$x(t) \xrightarrow[t_0]{\text{delay by}} x(t-t_0) \xrightarrow{H} \cos(x(t-t_0-1))$ $=$ time in-varient!
* non-linear ; $Ky(t) \neq \cos(kx(t-1))$

c) $y(t) = \ln(x(t))$

* memoryless / * invertable $x(t) = e^{y(t)}$ / * causal / * non stable

* $x(t) \rightarrow \ln(x(t)) \rightarrow \ln(x(t-t_0))$
$x(t) \rightarrow x(t-t_0) \rightarrow \ln(x(t-t_0))$ } time invarient

* non-linear ($\ln x_1 + \ln x_2 \neq \ln(x_1 + x_2)$)

5/3/2013

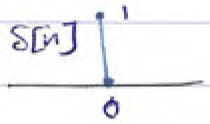
in $\boxed{DT}$ Discrete time

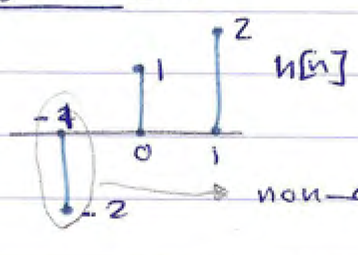
linear time-invarient

$x[n] \rightarrow \boxed{LTi} \rightarrow$?!

* inorder to know the output, we need the impulse response.

$x[n] = \delta[n] \rightarrow \boxed{LTi} \rightarrow y[n] = h[n]$

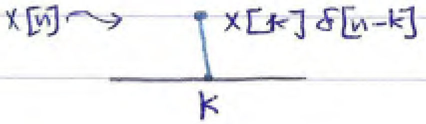




non-causal

* knowing the impulse response for an LTi, we become able to determine the output for any input.

In general:

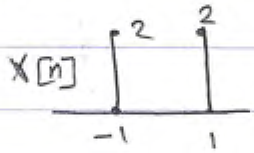


$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$ ... sifting property in DT

then:-

I/p		O/p
$\delta[n]$	$\xrightarrow{LTi}$	$h[n]$
$2\delta[n+1]$	$\xrightarrow{LTi}$	$2h[n+1]$

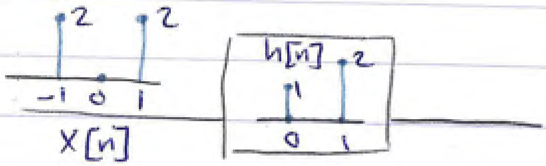
ex:-



find the output $y[n]$?

$x[n] = 2\delta[n+1] + 2\delta[n-1] \xrightarrow{LTi} y[n] = 2h[n+1] + 2h[n-1]$

ex2 :-

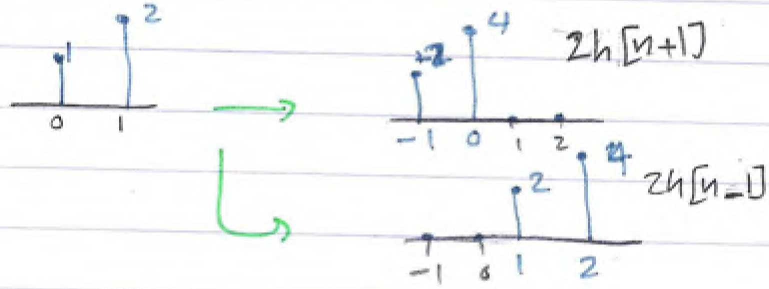


$y[n]$?!

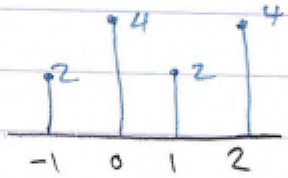
Sol:- $x[n] = 2\delta[n+1] + 2\delta[n-1]$

$\xrightarrow{LTI} y[n] = 2h[n+1] + 2h[n-1]$

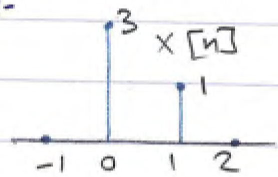
$\Rightarrow h[n]$



$y[n] \rightarrow$



ex3:-

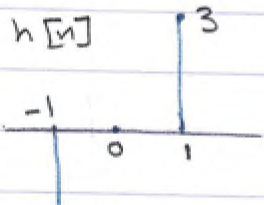


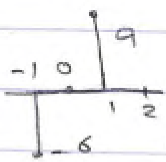
find $y[n]$?!

$x[n] = 3\delta[n] + \delta[n-1]$

$\rightarrow y[n] = 3h[n] + h[n-1]$

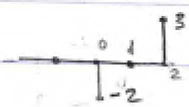
h[n]





$3h[n]$

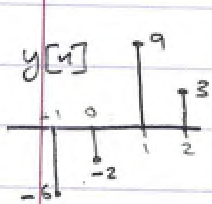
+



$h[n-1]$

=

y[n]



$x[n] = \delta[n] \xrightarrow{LTI} y[n] = h[n]$

in general:

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$ ... Convolution Sum.

$\rightarrow y[n] = x[0]h[n-0] + x[1]h[n-1] + x[2]h[n-2] + \dots$

$y[n] = x[n] * h[n]$

$*$ ↓ convolution

convolution (time-domain) $\longrightarrow$ multiplication (freq. domain).

Properties of convolution:-

1. Commutative:

$$x[n] * h[n] = h[n] * x[n]$$

proofs in notes...

$$\therefore y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

→ we always chose the function with the lowest # of samples to be $h[n]$.

2. Distributive:

$$x[n] * (h[n] + z[n]) = x[n] * h[n] + x[n] * z[n]$$

3. Associative:

$$x[n] * (h[n] * z[n]) = (x[n] * h[n]) * z[n]$$

7/3/2013

4. Shifting Property

$$x[n] \xrightarrow{h[n]} y[n]$$

$$x[n-m] \xrightarrow{h[n]} y[n-m]$$

$$x[n-m] \xrightarrow{h[n-q]} y[n-m-q]$$

5. Convolution with $\delta[n]$:

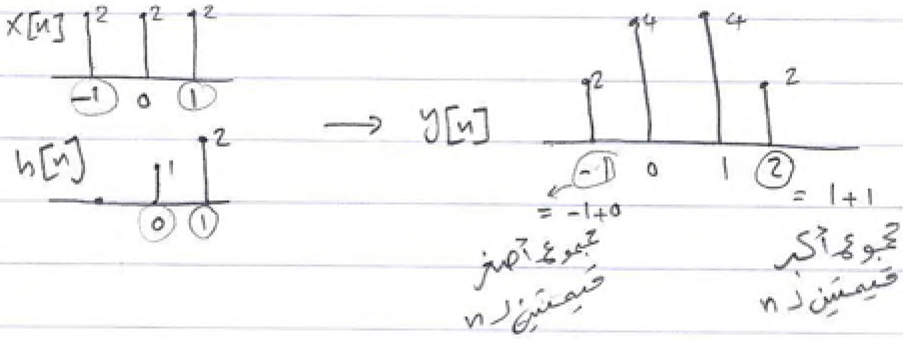
$$x[n] * \delta[n] = x[n]$$

$h[n] = \delta[n]$

$$x[n] * \delta[n-1] = x[n-1]$$

$$x[n+1] * \delta[n-1] = x[n]$$

6. Width property:



* Graphical method of convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k], \quad \text{find } y[-1]?$$

$$y[-1] = \sum_{k=-\infty}^{\infty} x[k]\, h[-1-k], \quad \text{how to find it?}$$

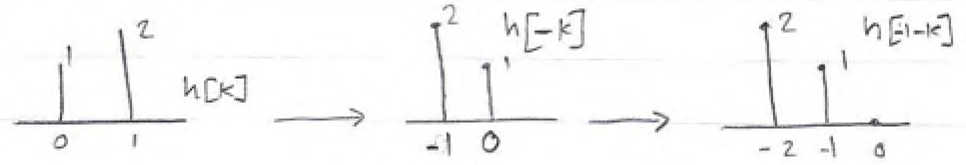
$$h[-1-k] \Rightarrow \quad h[k] \xrightarrow{k \to -k} h[-k] \xrightarrow{k \to k-n} h[-(k-n)]$$

$(= h[n-k]$

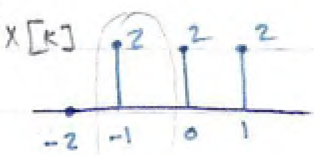
k → +ve → shift right

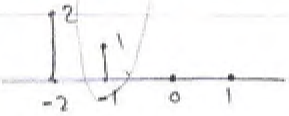
→ -ve → shift left

ex:-

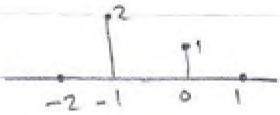


Method #1:

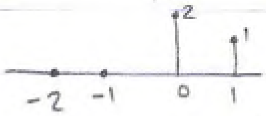




$h[-1-k]$ $\quad 1 \times 2 = 2$ at $n=-1$

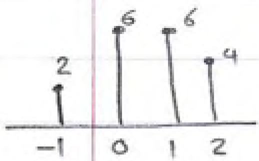


$h[0-k]$ $\quad 2\times 2 + 1\times 2 = 6$ at $n=0$



$h[1-k]$ $\quad 2\times 2 + 1\times 2 = 6$ at $t=1$

$h[2-k]$ $\quad 2\times 2 = 4$ at $t=2$



*if we fixed $h[k]$ & shifted $x[n]$, it'll be the same because $(*)$ is commutative.

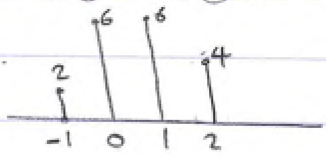
applicable in CT & DT (discrete and cont. time).

method #2:

$$\overset{x[n]}{\left(2x^{-1}+2x^{0}+2x^{1}\right)} * \overset{h[n]}{\left(x^{0}+2x^{1}\right)}$$

$$= 2x^{-1}+4x^{0}+2x^{0}+4x^{1}+2x^{1}+4x^{2}$$

$$= 2x^{-1}+6x^{0}+6x^{1}+4x^{2}$$



it's not applicable in CT!!

method #3:-

k	-2	-1	0	1	2
x[k]	0	2	2	2	0
h[k]				1	2
h[-1-k]	2	1			
h[0-k]		2	1		
h[1-k]			2	1	
h[2-k]				2	1

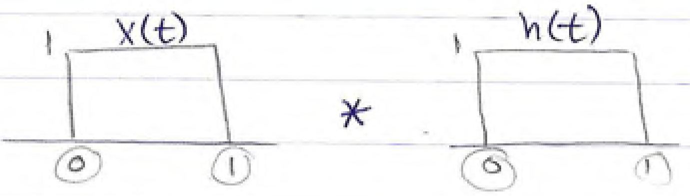
$y[-1] = 2\times0 + 1\times2 = 2$

$y[0] = 2\times2 + 1\times2 = 6$

$y[1] = 2\times2 + 1\times2 = 6$

$y[2] = 2\times2 = 4$

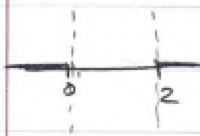
Convolution ~~in~~ between continuous time (CT) signals :



Step 1

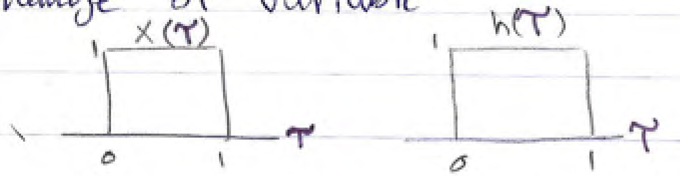
Determine the range of the output:- $0+0 \longrightarrow 1+1$

$0 \longrightarrow 2$



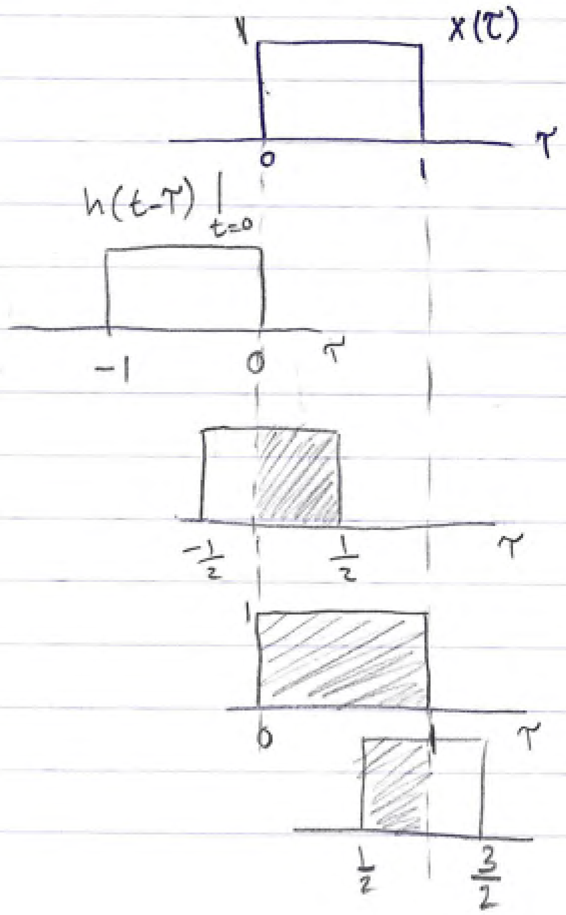
Step 2

Change of variable

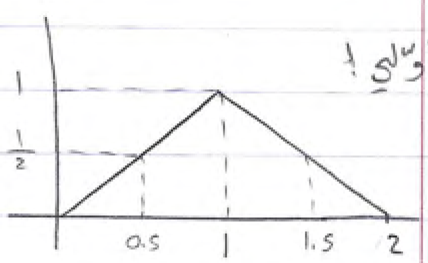


Step 3

fix one of them while shifting the other.



$\Rightarrow$



حل أولي!

10/3/2013

Convolution in $\boxed{CT}$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\delta(t) \longrightarrow h(t)$$
$$\delta(t-\tau) \longrightarrow h(t-\tau)$$ because of time-invarient

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, d\tau \longrightarrow \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore \boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau}$$ ... convolution integral

$$y(t) = x(t) * h(t)$$

while in $\boxed{DT}$

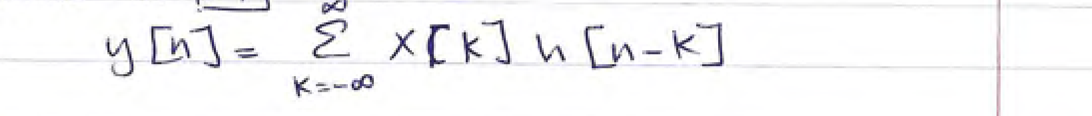
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Properties:- (in CT)

1) $x(t) * h(t) = h(t) * x(t)$

$$y(t) \equiv \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

2) $x(t) * \delta(t) = x(t)$



if it's an invertable system then $y(t) = x(t)$

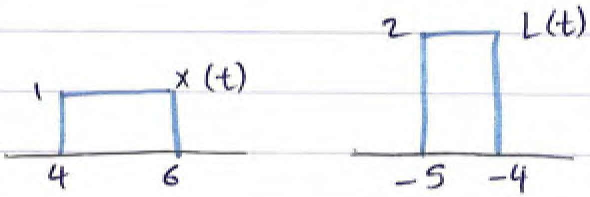
$\therefore y(t) = x(t) * \underline{h_1(t) * h_2(t)} = x(t)$

then $h_1(t) * h_2(t)$ must equal $\delta(t)$!

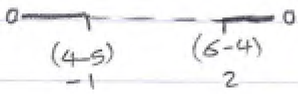
& all other properties also apply

* Width of the function that results from the convolution is the sum of width 1 + width 2 + ...

ex:-

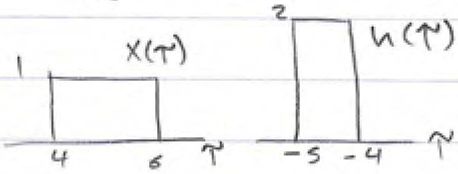


1st step: the range of the output ⇒



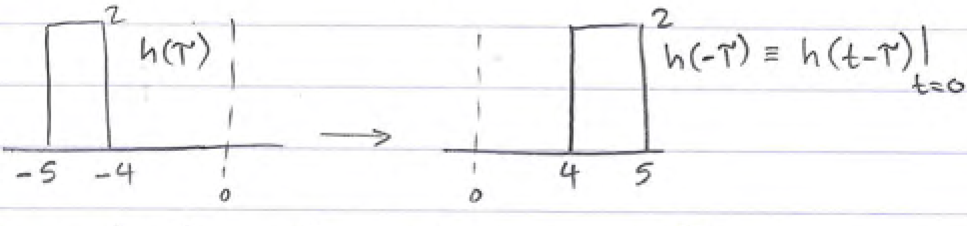
2nd Step: Change of variables

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \quad \underline{\underline{\text{or}}} \quad y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$$

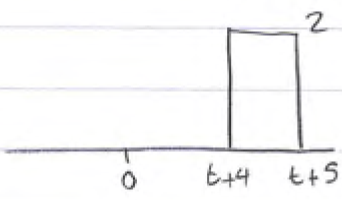


3rd Step: chose (h) to be the one with the lowest # of samples.

4th Step: find $h(t-\tau)\big|_{t=0}$

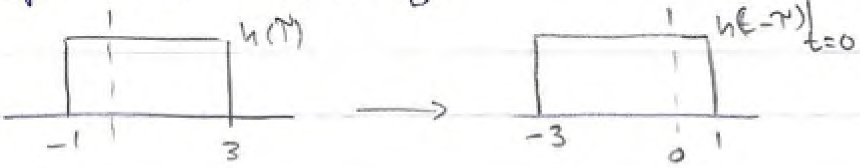


5th Step: Lable the edges correctly



always consider $t=0$

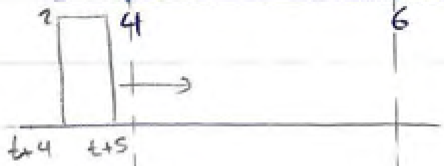
examples on labeling:



* we do it to define the limits of our integral, since we want to move the function cotinuously.

5^{th} Step: fix $x(\tau)$ & move $h(\tau)$: (t, is constant).

Case 1:



$y(t)=0 \quad , t+5<4$

$\boxed{y(t)=0 \ , \ t<-1}$

Case 2:

Part of (h) overlaps $x(\tau)$

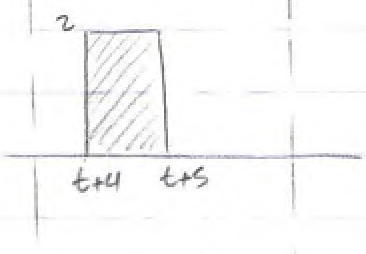
$4<t+5<5 \Rightarrow \boxed{-1<t<0}$

$y=\int_4^{t+5} 2 \, d\tau = \boxed{2t+2}$

our function

Case 3:

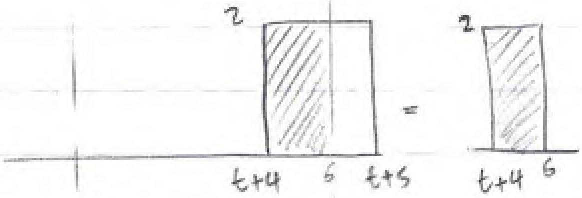
the whole function overlaps $x(t)$



$5<t+5<6 \Rightarrow \boxed{0<t<1}$

$\boxed{y(t)=2}$

Case 4:



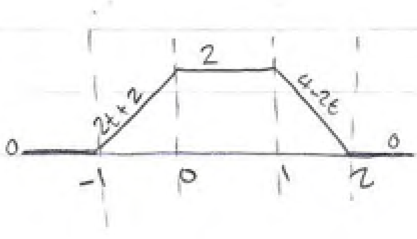
$5<t+4<6$

$\boxed{1<t<2}$

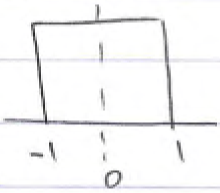
$y(t)=\int_{t+4}^{6} 2 \, d\tau = 2(6-t-4)$

$\boxed{y(t)=4-2t}$

$$y(t)=\begin{cases} 0 & , \quad t<-1 \\ 2t+2 & , -1<t<0 \\ 2 & , \ 0<t<1 \\ 4-2t & , \ 1<t<2 \\ 0 & , \quad t>2 \end{cases}$$

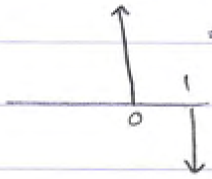


ex2: $x(t) =$



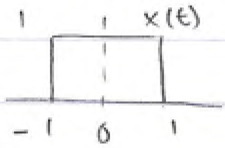
$*$

$h(t) =$

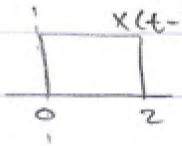


$\Rightarrow \delta(t) - \delta(t-1)$

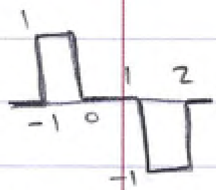
$$y(t) = x(t) * h(t)$$
$$= x(t) * (\delta(t) - \delta(t-1))$$
$$= x(t) * \delta(t) - x(t) * \delta(t-1) \quad \text{...from properties}$$
$$= x(t) - x(t-1)$$



$\rightarrow$

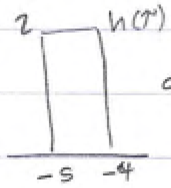


$\Rightarrow x(t) - x(t-1) \Rightarrow$

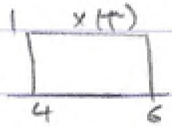


you can always check your answer by taking area.

from ex1:-

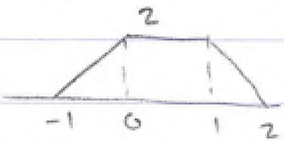


area = 2



area = 2

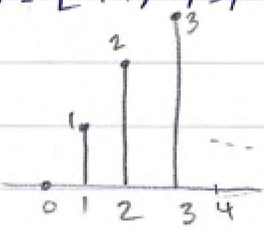
(area of h) x (area of x) = area of y



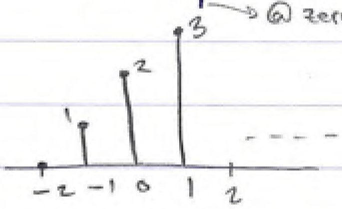
area of $y = 1 + 2 + 1 = 4$ ⋇

notation in exam →

$x[n] = [0, 1, 2, 3, 4, \ldots]$



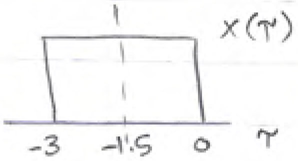
$x[n] = [0, 1, 2, 3, 4, \ldots]$ → @ zero.

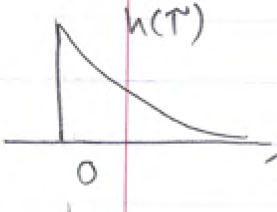


ex2: $x(t) = \text{rect}\left(\frac{t+1.5}{3}\right) * h(t) = e^{-t}u(t)$

Sol: * range of output $[-3, \infty)$

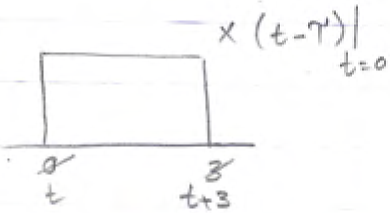
* change of variables:-



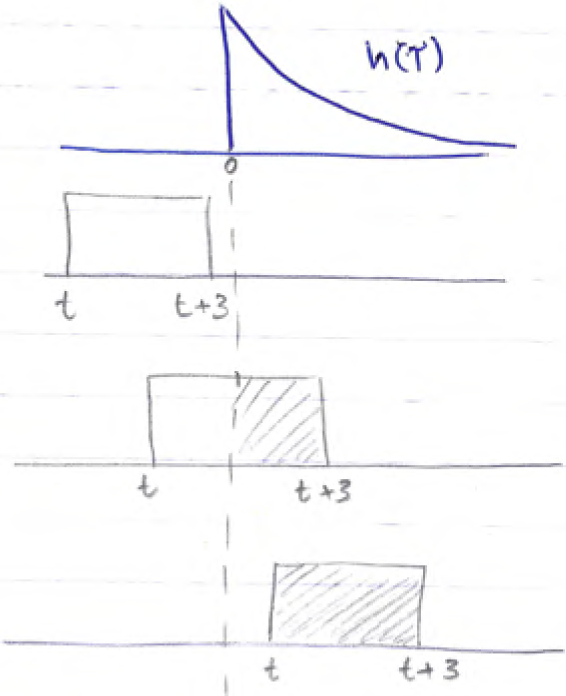


* we chose $x(\tau)$ to be moved while fixing $h(\tau)$

*



← label the edges correctly

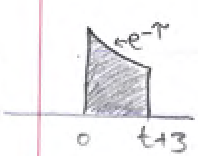


$t+3 > 0 \Rightarrow t > -3$

$y(t) = 0$

$0 < t+3 < 3 \Rightarrow -3 < t < 0$

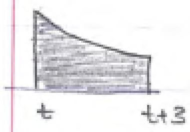
$y(t) = \int_0^{t+3} e^{-\tau}\, d\tau = 1 - e^{-(t+3)}$



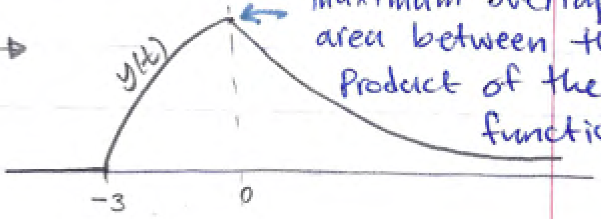
$t+3 > 3 \Rightarrow t > 0$

$y(t) = \int_t^{t+3} e^{-\tau}\, d\tau = e^{-t} - e^{-3}e^{-t}$

$= e^{-t}(1 - e^{-3})$



$$y(t) = \begin{cases} 0 & , t < -3 \\ 1 - e^{-(t+3)} & , -3 < t < 0 \\ e^{-t}(1 - e^{-3}) & , t > 0 \end{cases} \rightarrow$$



at maximum overlaping area between the Product of the 2 functions

12/3/2013

Systems classes in terms of the impulse response:

** We can obtain impulse response $h(t)$, by applying an impulse signal $\delta(t)$ as an input to the system.

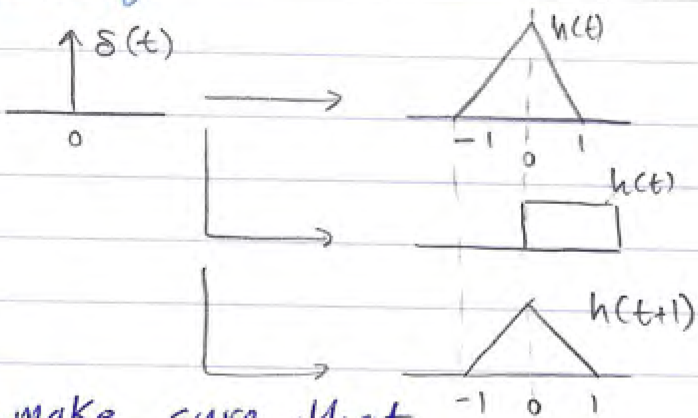
1. Any system with impulse response ~~is~~ is linear & time invarient
   الImpulse response ماله معنى إذا ما كان LTi

2. memory:

$$h(t) = \frac{1}{2}\delta(t) \longrightarrow \text{LTi \& memory less}$$

$$h(t) = \frac{1}{2}\delta(t-1) \longrightarrow \text{LTi \& with memory.}$$

3. Causality:



non-causal (response has started before the input !!)

causal

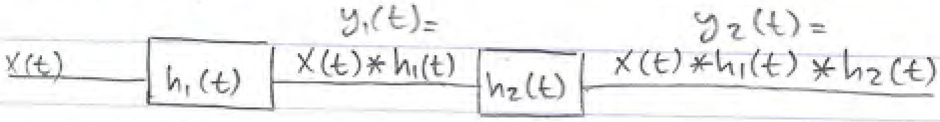
causal

** make sure that you are looking at $h(\underline{t})$ when:

$h(t) \neq 0,\ t<0$ ... non-causal

$h(t) = 0,\ t<0$ ... causal

4. Invertability:



if $h_1(t) * h_2(t) = \delta(t)$ then $y_2(t) = x(t)$ & $h_2(t)$ is the inverse of system $h_1(t)$!

5. Stability:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

using the triangular inequality $\rightarrow$ $$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau$$

$\rightarrow$ to test for stability I must have a bounded input & output!

1^{st}: assume that I/p is bounded

$\therefore |x(t)| < \infty \;\; \forall t$

لما يكون المنعكس Bounded دوما
كمان shifting بضل Bounded

consider the Bound of $x(t)$ is B since it's finite

$\therefore |x(t)| < B$

$$\rightarrow |y(t)| \leq B \int_{-\infty}^{\infty} |h(\tau)\, d\tau|$$

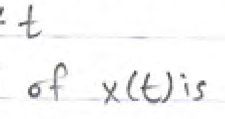
for the system to be stable:

$|y(t)| < \infty$

then.. $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$; $h(t)$ must be "Absoulutely Integrable"

(the area under $h(t)$ is finite).

ex:- $h(t) = e^{-3t} u(t)$



is it stable?

$\int_{0}^{\infty} e^{-3t}\, dt = \frac{1}{3}$ $\therefore$ it's stable!

Summary:

Signals & Systems

time domain
Ch(1-3)

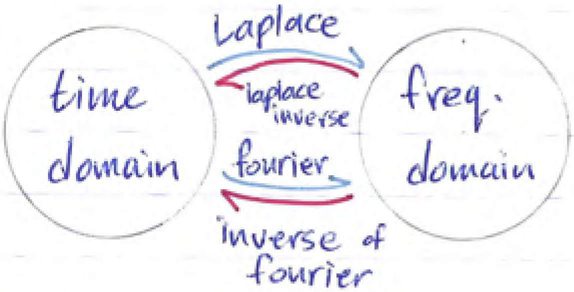
frequency domain
Ch 4

1. Introduction
2. CT & DT
3. time transformations (Shifting, Scaling, Reflection)
4. Useful signals.
5. Systems → classes
6. Convolution (the only action in t-domain).
   (how to find the output of any LTi system knowing its impulse response).

Ch #4:

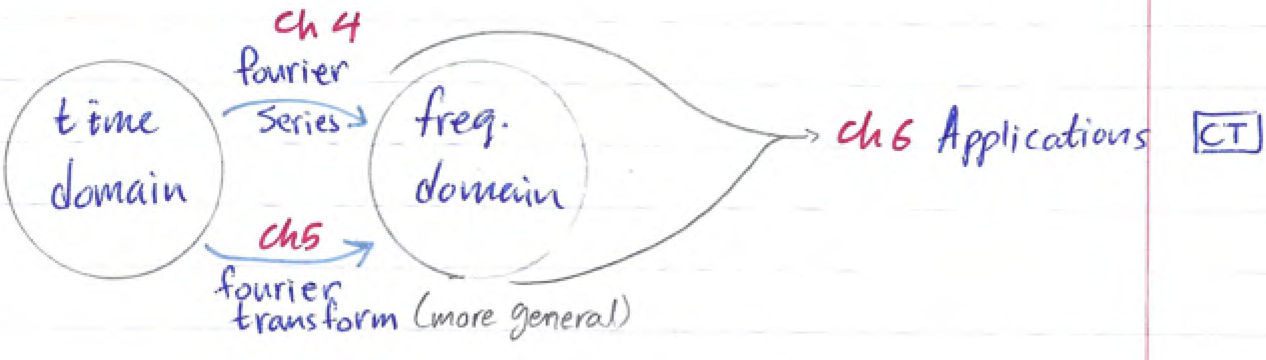
frequency domain:

* It's a virtual domain, it's been created to help in analysis!



* the most important thing about laplace & fourier is that they have "closed formulas" to get you back to t-domain.

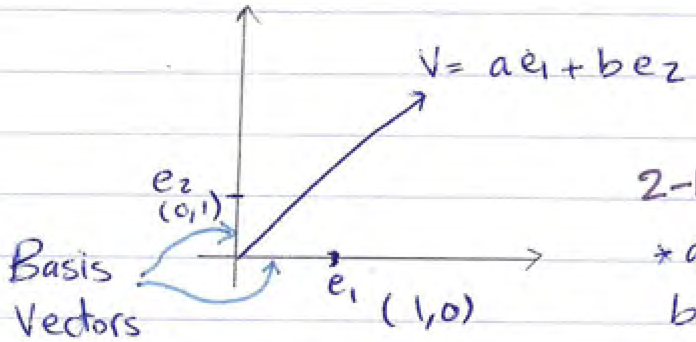
- **fourier Analysis:**



14/3/2013

* Vector space:

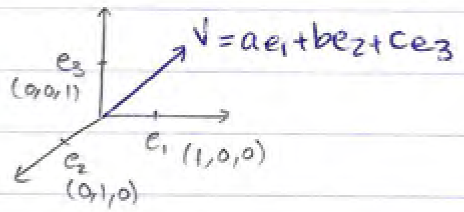
- It's the space that the vector resides in.



2-D orthogonal vector space.

* any vector (or point) in this space can be represented using the two orthogonal basis vectors.

3-D orthogonal vector space.



* we can mathematically imagine an infinite dimensional vector space. with an infinite set of orthogonal basis vectors

* how to know if the set is orthogonal?

$\{\dots, e_1, e_2, \dots, e_n, e_m, \dots\}$

do an inner product between e_n & e_m (two general terms)

$$\langle e_m, e_n \rangle = e_m^{T} e_n \quad T \rightarrow \text{transpose}$$

$m \neq n$ $\quad \langle e_1, e_2 \rangle = [1\ 0\ 0]\begin{bmatrix}0\\1\\0\end{bmatrix} = 0 \ \boxed{m \neq n} \quad e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \ e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

- if $\langle e_m, e_n \rangle = 0$ for $\forall\, m \neq n$ then this set is an orthogonal set.

$m = n$ $\quad \langle e_1, e_1 \rangle = [1 0 0]\begin{bmatrix}1\\0\\0\end{bmatrix} = 1$

- if $\langle e_m, e_n \rangle = 1$ for $m = n$ then it's an OrthoNormal Set.

* You can always extract the coeficients of any ~~basis~~ vector by doing an inner product between the vector (V) & e_n.

ex:- $\langle V, e_1 \rangle = [a \ b \ c] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a$

⇒ now, apply this on Signals!

let $f(t)$, $g(t)$ be two functions over a finite interval of time then...

$$\langle f, g \rangle = \int_a^b f(t) \cdot g(t)\,dt$$

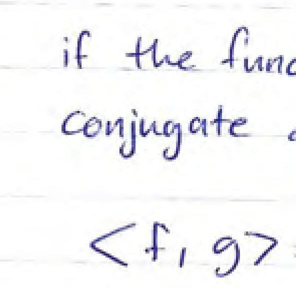
if the functions are complex we take the conjugate of one of them

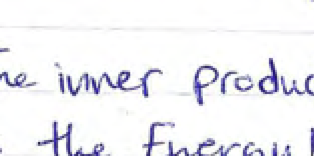
$$\langle f, g \rangle = \int_a^b f(t)\, g^*(t)\,dt$$

* The inner product of the function with itself is the Energy!

$$\langle f, f \rangle = \int_a^b f(t)\, f^*(t)\,dt = \text{Energy!}$$

ex:-



$\int_0^2 \Big($  $\Big) = 0$ ∴ then $f(t)$ & $g(t)$ are orthogonal.

* Can I have a set that has an infinite # of orthogonal function?!

$\phi_i(t)$; $i = 0, \pm 1, \pm 2, \ldots$ — the orthogonal basis signals.

$\{\ldots, \phi_{-2}(t), \phi_{-1}(t), \phi_0(t), \phi_1(t) \ldots, \phi_m(t), \phi_n(t), \ldots\}$ (general terms).

could be real or complex.

if $\int_a^b \phi_m(t)\, \phi_n^*(t)\, dt = \begin{cases} 0, & m \neq n \\ E, & m = n \end{cases}$ then, ϕ_i is an orthogonal set!

else if $= \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \rightarrow$ orthonormal set.

to make the set orthonormal we divide it by the energy $\Rightarrow$ eg $\left\{ \frac{\phi_i}{\sqrt{E_i}} \right\}$

* But can we create an infinite dimensional signal space from these infinite ~~sig~~ Basis signals? yes:)

- Examples of orthogonal sets :

ex1: $\phi_m(t) = \sin(mt)$; $m = 1, 2, 3, \ldots$

$[\sin(t), \sin(2t), \sin(3t), \ldots \sin(mt), \sin(nt), \ldots]$

over the period $(-\pi, \pi)$

is this set orthogonal?

Sol: $\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\, dt = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)t\, dt - \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)t\, dt$

$= \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

* the same set might not be orthogonals if we change the interval.

$\Rightarrow \therefore \phi_m = \sin(mt)$ over $(-\pi, \pi)$ is orthogonal.

& $\phi = \dfrac{\sin(mt)}{\sqrt{\pi}}$ is orthonormal (over the same period).

<u>ex2:-</u>

$$\phi_K(t) = e^{jK\omega_0 t} \quad ; K = 0, \pm 1, \pm 2, \dots \quad \text{over } (0,T)$$

$$\{\dots, e^{-j2\omega_0 t}, e^{-j\omega_0 t}, 1, e^{j\omega_0 t}, \dots, e^{jm\omega_0 t}, e^{jn\omega_0 t}, \dots\}$$

<u>Sol:-</u>

$$\int_0^T e^{jm\omega_0 t} \cdot \underbrace{e^{-jn\omega_0 t}}_{\text{conjugate}} dt = \int_0^T e^{j(m-n)\omega_0 t} dt.$$

we test for $m=n$, $m \neq n$ before integrating

$$* \text{when } m=n \Rightarrow \int_0^T e^0 dt = T$$

$$* \text{when } m \neq n \Rightarrow \int_0^T e^{j(m-n)\omega_0 t} dt = \left.\frac{e^{j(m-n)\omega_0 t}}{j(m-n)\omega_0}\right]_0^T$$

$$= \frac{1}{j(m-n)\omega_0}\left(e^{j(m-n)\overbrace{\omega_0}^{=2\pi/\cancel{T}}\cancel{T}} - 1\right)$$

$$= \frac{1}{j(m-n)\omega_0}\left(\underbrace{e^{jk2\pi}}_{=1} - 1\right) = 0$$

$$\therefore \int_0^T e^{j\omega_0 t} e^{-j\omega_0 t} dt = \begin{cases} 0, & m \neq n \\ T, & m = n \end{cases} \quad \text{over } (0,T)$$

$\therefore$ ϕ_K is an orthogonal set.

$\{\ldots, \phi_{-1}(t), \phi_0(t), \phi_1(t), \ldots\}$ basis signals.

$\{\ldots, c_{-1}, c_0, c_1, \ldots\}$ coeficients for each basis signal of the set ϕ.

$$\Rightarrow x(t) = \{\ldots + c_{-1}\phi_{-1} + c_0\phi_0 + c_1\phi_1 + \ldots\}$$

$$x(t) = \sum_{i=-\infty}^{\infty} c_i \phi_i(t)$$ Generalized fourier series

* Any function - not necessarily periodic - defined over a $\boxed{\text{finite}}$ interval of time can be represented using orthogonal basis signals.

→ Extracting Coeficients:

$$c_i = \int_a^b x(t)\,\phi_i^*(t)\,dt \longrightarrow$$ inner product; this is true (: [illegible]

but...

$$\int_a^b x(t)\,\phi_k^*(t)\,dt = \int_a^b \sum_{-\infty}^{\infty} c_i \phi_i(t)\,\phi_k^*(t)\,dt$$

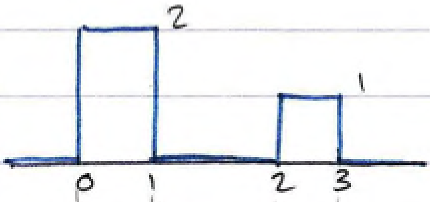
$$= \sum_{-\infty}^{\infty} c_i \int_{-\infty}^{\infty} \phi_i(t)\,\phi_k^*(t)\,dt = \begin{cases} 0, & i \neq k \\ E_k, & i = k \end{cases}$$

$$\therefore = c_k E_k \quad (\text{when } i = k)$$

then

$$c_i = \frac{\int_a^b x(t)\,\phi_i(t)\,dt}{E_i}$$

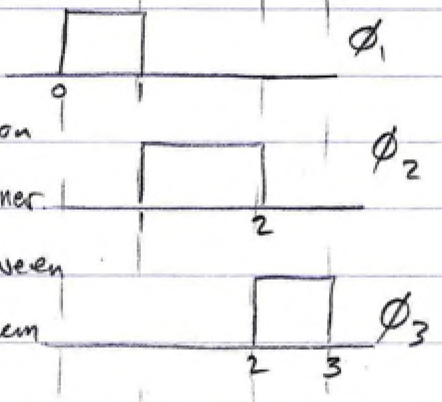
ex:-



$x(t)$ over $(0,3)$

1st we find basis function where the inner Product between any 2 of them is 0!

we have infinite possibilities



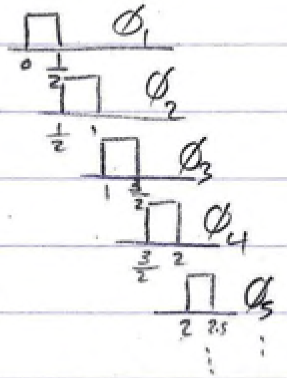
orthonormal & orthogonal set

$$\boxed{\hookrightarrow x(t) = 2\phi_1 + \phi_3}$$

$$\hookrightarrow c_1 = \frac{1}{\underset{=1}{E_1}} \int_0^3 x(t)\,\phi_1(t)\,dt$$

$$= \frac{1}{1}\int_0^3 x(t)\,\phi_1(t)\,dt = 2 \text{ (area)}$$

e.g.




first
exam

Test No. 3

UNIVERSITY OF JORDAN
DEPARTMENT OF ELECTRICAL ENGINEERING
SIGNALS AND SYSTEMS
20 MARKS

FIRST EXAM 20/3/2013

DR. MAHMOUD AL-HUSARI NAME: ______________________ SERIAL NO. ____

Please write your name in arabic

SERIAL NO. ☐ SECTION:
- ☐ SECTION 1, SUN, TUE, THUS 09:00-10:00
- ☒ SECTION 2, SUN, TUE, THUS 11:00-12:00
- ☐ SECTION 3, MON, WED 09:30-11:00

Write your answers here													
1	b	2	c	3	e	4	b	5	e	6	a	7	a
8	a	9	a	10	d	11	c	12	e	13	b	14	b
15	d	16	a										

1. The impulse response of the continuous-time cascade system shown is

$$\longrightarrow \boxed{h_1(t) = 3u(t)} \longrightarrow \boxed{h_2(t) = e^{-3t}u(t)} \longrightarrow \boxed{h_3(t) = \delta(t-2)} \longrightarrow$$

   a. $3e^{-6}\delta(t-2)$
   b $u(t-2) - e^{-3(t-2)}u(t-2)$
   c. none of these
   d. $3u(t) + e^{-3t}u(t) + \delta(t-2)$
   e. $3e^{-3t}u(t)\delta(t-2)$

2. The LTI system with $h(t) = \delta(t-1) + e^{t}u(-t-1)$ is
   a. causal and nonstable
   b causal and stable
   c. noncausal but stable
   d. need more information
   e. noncausal and nonstable

3. The output of a system is given by $y(t) = (ax(t)+1)^2 - x(t)^2 - b$. For which values of a and b is the system linear.
   a. $a = \pm 1$, $b = 0$
   b need more information

c. $a = -1$, $b = 1$

d. $a = 1$, $b = 1$

e. $a = \pm 1$, $b = 1$

4. The input $x(t) = u(t+2) - u(t-2)$ is applied to an LTI system with impulse response $h(t) = t$. The value of the output signal at $t = 3$ is $y(3) =$

a. zero

b 12

c. 15/2

d. 25/2

e. none of these

5. Consider a continuous-time LTI system with impulse response $h(t) = u(t) - 2u(t-2) + u(t-4)$. If the input signal $x(t) = u(t)$, then the response $y(t)$ at $t = 4$ is

a. 1

b -1

c. 2

d. none of these

e. zero

6. The integral $\int_{-\infty}^{\infty} (t+2)\delta(4-2t)dt$ has the value

a. 2

b 4

c. 8

d. none of these

e. zero

7. Which of the following statements is FALSE for convolution?

a. $x(t) * [y(t)z(t)] = [x(t) * y(t)]z(t)$

b all are correct

c. $x(t) * \delta(t - t_0) = x(t - t_0)$

d. $x(t) * y(t) = y(t) * x(t)$

e. $x(t) * [y(t) + z(t)] = [x(t) * y(t)] + [x(t) * z(t)]$

8. Consider the following statements

S1: The set of the two functions $\{1/2, (t - 1/2)\}$ is orthogonal over interval $(0, 1)$.
S2: The set of the two functions $\{1, (2t - 1)\}$ is orthonormal over interval $(0, 1)$.

a. S1 is true but S2 is false

b S1 and S2 are false

c. none of these

d. S1 and S2 are true

e. S1 is false but S2 is true

9. An LTI system has input $x[n] = \delta[n-2]$ which gives the output $y[n] = u[n-3]$. The impulse response is

a. $h[n] = u[n-1]$
b $h[n] = u[n-3] - u[n-2]$
c. $h[n] = u[n-3] - u[n-1]$
d. $h[n] = u[n-2]$
e. $h[n] = u[n]$

10. Calculate the output $y[n]$ when the sequence $x[n] = \{1, -1, 1\}$ is input to a linear time-invariant system that has an impulse response given by $h[n] = \{3, 2, 1\}$.

a. none of these
b $y[n] = \{3, -2, 1\}$
c. $y[n] = \{3, 8, 14, 8, 3\}$
d. $y[n] = \{3, -1, 2, 1, 1\}$
e. $y[n] = \{-1, 8, 5, 8, 3\}$

11. Given the function $x(t) = \mathrm{Sa}\left(\frac{t-\pi}{3}\right)$. The second null will occur at $t =$

a. 6π
b 4π
c. 7π
d. 3π
e. none of these

12. Calculate the energy in the signal $x(t) = 5t$, for $0 \le t < 1$ and $x(t) = 0$ otherwise.

a.. 25
b . infinity
c.. none of these
d.. 25/4
e. 25/3

13. The continuous-time signal $x(t) = 4\sin(4\pi t) - 6\cos(6\pi t)$

a. is periodic with fudamental period $T_0 = 2$
b is periodic with fundamental period $T_0 = 1$
c. is periodic with fundamental period $T_0 = 1/2$
d. is not periodic
e. none of these

14. Evaluate the following integral, $\int_{-\infty}^{\infty} u(\tau - 1)u(t - \tau)d\tau$.

a. $tu(t)$
b $(t-1)u(t-1)$
c. $t - 1$
d. $(t-1)u(t)$

e. $tu(t-1)$

15. If $x(t) = u(t+1)u(1-t) - 2r(t-3)$, then the generalized derivative of $x(t)$ is

a. none of these

b $\delta(t+1)\delta(1-t) - 2$

c. $\delta(t+1)\delta(1-t) - 2u(t-3)$

d. $\delta(t+1) - \delta(t-1) - 2u(t-3)$

e. $\delta(t)u(1-t) + u(t+1)\delta(-t) - 2$

16. Given $f(t) = rect\left(\frac{2t-1}{2}\right)$ and $g(t) = rect\left(\frac{t-4}{2}\right)$. $g(t)$ can be expressed as

a. $g(t) = f(t/2 - 3/2)$

b none of these

c. $g(t) = f(2t-3)$

d. $g(t) = f(2t - 3/2)$

e. $g(t) = f(t/2 - 3)$

test No. 3 first exam.

1) $\overset{h_1 * h_2}{3\int_{-\infty}^{\infty} e^{-3\tau} u(\tau)\, u(t-\tau)\, d\tau}$ by definision

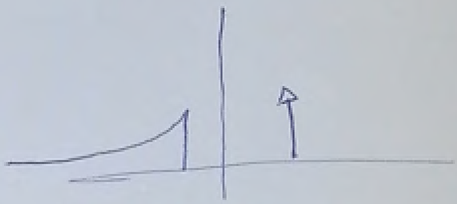
$3\int_0^t e^{-3\tau}\, d\tau = -e^{-3\tau}\Big/_0^t = \left[1 - e^{-3t}\right] u(t)$

$(h_1 * h_2) * h_3$

$\left(u(t) - e^{-3t} u(t)\right) * \delta(t-2) = u(t-2) - e^{-3(t-2)} u(t-2)$

2) $h(t) \neq 0$ for $t<0$

so it's noncausal



$\int_{-\infty}^{\infty} h(t) \leq \infty \Longrightarrow$ stable

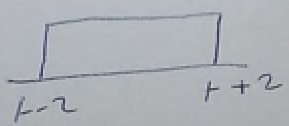
3) $y(t) = (a\,x(t) + 1)^2 - x(t)^2 - b$

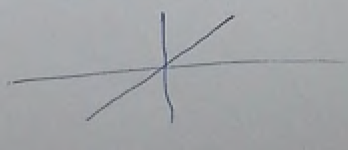
$= a^2 x(t)^2 + 2a\,x(t) + 1 - x(t)^2 - b$

so to be linear $\boxed{a = \pm 1 \quad b = 1}$

4) $x(t) = u(t+2) - u(t-2)$

$h(t) = t$

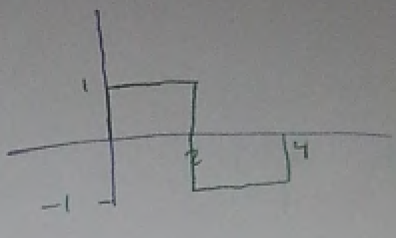




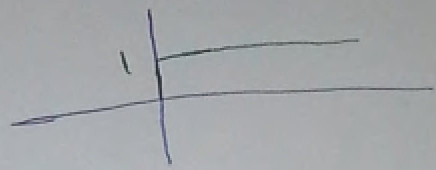
$y(t) = \int_{t-2}^{t+2} \tau\, d\tau$

$y(3) = \int_1^5 \tau\, d\tau = \boxed{12}$

5) $h(t) = u(t) - 2u(t-2) + u(t-4)$

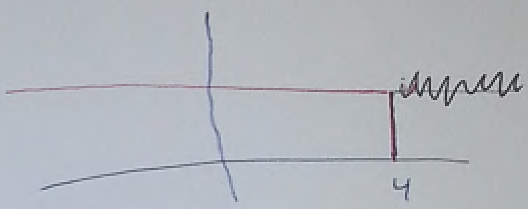


$x(t)$



$y(t) = h(t) * x(t)$

$y(4) = \int_0^2 1 + \int_2^4 -1 = 0$



6) $\int_{-\infty}^{\infty} (t+2)\,\delta(4-2t)\,dt$

$\int_{-\infty}^{\infty} (t+2)\,\delta(-2(t-2))\,dt$

$\frac{1}{2}\int_{-\infty}^{\infty} (t+2)\,\delta(t-2)\,dt = \frac{1}{2}(2+2) = \boxed{2}$

7) $x(t) * [y(t)\,z(t)] = [x(t) * y(t)]\,z(t)$

this is false.

8) S1: $\int_0^1 \frac{1}{2}(t-\frac{1}{2})\,dt = 0$ so it's orthogonal.

S2: $\int_0^1 (2t-1)\,dt = 0$ so it's orthogonal and not orthonormal

$\boxed{S_1 \text{ is true but } S_2 \text{ is false}}$

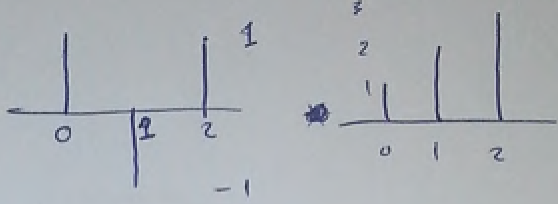
9) $x[n] = \delta[n-2]$

$y[n] = u[n-3]$

$u[n-3] = \delta[n-2] * \underbrace{h[n]}$

$\boxed{h[n] \text{ must be } u[n-1]}$

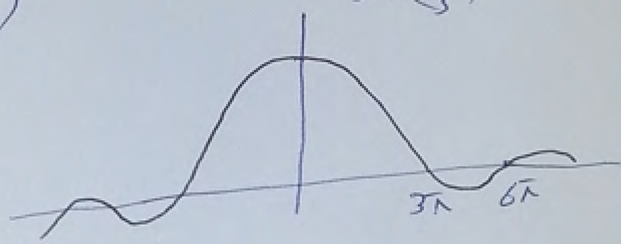
10) d) $\{3, -1, 2, 1, 1\}$



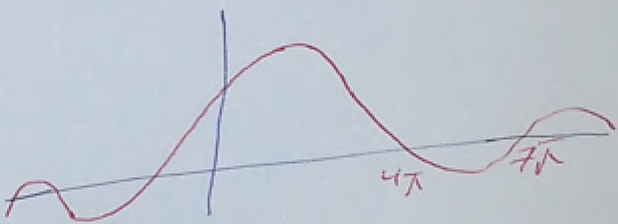
11) $x(t) = S_a\left(\frac{t-\pi}{3}\right)$

first $S_a\left(\frac{t}{3}\right)$ $\longrightarrow$

then delay by π



the second null will occur at $\boxed{t = 7\pi}$



12) $E = \int_0^1 (5t)^2 \, dt = \boxed{\frac{25}{3}}$

13) $T_1 = \frac{2\pi}{4\pi} = \frac{1}{2}$

$T_2 = \frac{2\pi}{6\pi} = \frac{1}{3}$

$\frac{T_1}{T_2} = \frac{3}{2}$ $\qquad$ $\boxed{T_0 = 2T_1 = 1}$

14) $\int_{-\infty}^{\infty} u(\tau-1)\, u(t-\tau)\, d\tau$

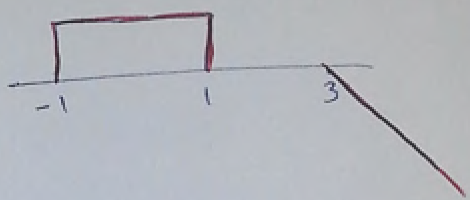
$= \int_{1}^{t} d\tau = (t-1)\, u(t-1)$

---

15) $x(t) = u(t+1)\, u(1-t) - 2r(t-3)$

derivative of $x(t)$ is

$\delta(t+1) - \delta(t-1) - 2u(t-3)$



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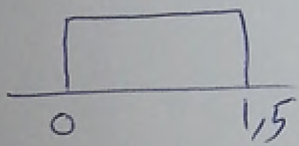
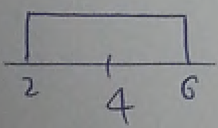
16) $f(t) = \text{rect}\left(\frac{2(t-0{,}5)}{2}\right) = \text{rect}(t-0{,}5)$



$g(t) = \text{rect}\left(\frac{t-4}{2}\right)$



so $g(t)$ is expanded by 2 and shifted right by 3

$g(t) = f\left(\frac{1}{2}(t-3)\right) = f(t/2 - 3/2)$

4.2

17/3/2013

- Any function $x(t)$ (not neccesairily periodic), defined over a finite interval of time can be represented as:

$$\boxed{x(t) = \sum_{i=-\infty}^{\infty} C_i \phi_i(t)}$$

* ϕ_i is an orthogonal set, it has infinite possibilites.
  example, $\phi_k(t) = e^{jk\omega_0 t}$; $k = 0, \pm1, \pm2, \dots$
  → Trigonometric Basis functions.
  $\phi_m(t) = \sin m\omega_0 t$, $m = 0, 1, 2, 3, \dots$

to extract coeficients:

$$\boxed{C_k = \frac{1}{E_k} \int_a^b x(t)\, \phi_k^*(t)\, dt}$$

$[a,b]$ is the interval, where $x(t)$ is defined.

→ This is called the "Generalized Fourier Series".

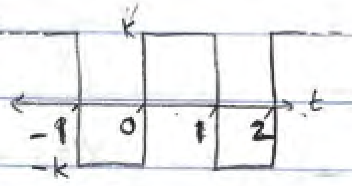
- Example 1:
  1) Exponential F.S:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \underbrace{e^{jn\omega_0 t}}_{\phi_n(t)}$$

$$C_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \quad \dots$$ fourier coeficients/ or Spectral coeficients

Energy of ϕ_n ← T

ex:-



- determine the exponential fourier series representation of this function.

Sol:- $x(t) = \dots C_{-2} e^{-j2\omega_0 t} + C_{-1} e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + \dots$

- first, we have to extract the coeficients:

$T = 2 \rightarrow \therefore \omega_0 = \frac{2\pi}{T} = \pi$

* We always start with $n=0$, (C_0) ← لأنه حكم خلال التكامل n=0 تكون في المقام فدائماً بنحسبها لحالها!

$$C_0 = \frac{1}{\underset{T=}{2}} \underbrace{\int_{-1}^{1} x(t)\, d\ }_{\text{average value}} = 0 \longrightarrow \boxed{\text{Dc component}}$$

(أو $0 \to 2$)

* Calculate C_n: (the general case).

$$C_n = \frac{1}{2}\int_{-1}^{0} -k\, e^{-jn\omega_0 t}\, dt + \frac{1}{2}\int_{0}^{1} k\, e^{-jn\omega_0 t}\, dt$$

($\omega_0 = \pi$)

$$= \frac{1}{2}\left(\frac{-k\, e^{-jn\pi t}}{-jn\pi}\Big]_{-1}^{0}\right) + \frac{1}{2}\left(\frac{k\, e^{-jn\pi t}}{-jn\pi}\Big]_{0}^{1}\right)$$

$$= \frac{k}{2jn\pi}\left(1 - e^{jn\pi}\right) - \frac{k}{2jn\pi}\left(e^{-jn\pi} - 1\right)$$

$$= \frac{k}{2jn\pi}\Big(2 - \underbrace{\left(e^{jn\pi} + e^{-jn\pi}\right)}_{2\cos n\pi}\Big)$$

$$= \frac{k}{jn\pi}\left(1 - \cos n\pi\right)$$

remember:

- $\dfrac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$
- $\dfrac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$

$$= \frac{k(1 - \cos n\pi)}{jn\pi}$$

→ if $n=0$ ⇒ $\dfrac{k(1-\cos 0)}{0} = \dfrac{0}{0}$ L'Hospital

that's why we took $n=0$ at the begining.

↳ (n) even → $C_n = 0$

↳ (n) odd → $C_n = \dfrac{2k}{jn\pi}$

$$\therefore C_n = \begin{cases} 0, & n=0 \\ 0, & n \text{ even} \\ \dfrac{2k}{jn\pi}, & n \text{ odd} \end{cases}$$

$x(t)$ is an odd function so it has only odd components!

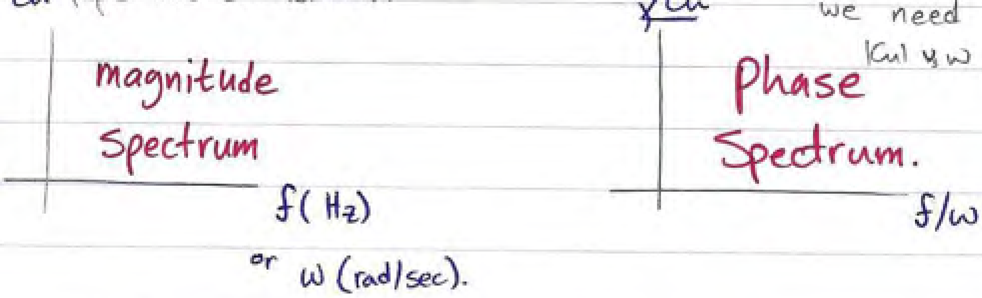
$$\therefore x(t) = \ldots\ \frac{2k}{-j3\pi} e^{-j3\pi t} + \frac{2k}{-j\pi} e^{-j\pi t} + \overset{C_0}{0} + \frac{2k}{j\pi} e^{j\pi t} + \ldots$$

↳ time domain function.

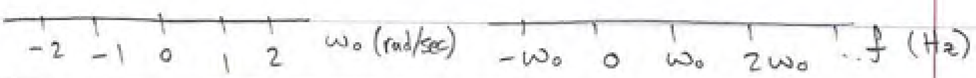
- Frequency domain representation:-

C_n (spectral coeficients).

← if C_n is complex we need 2 plots: $|C_n|$ vs ω / $\angle C_n$ vs ω



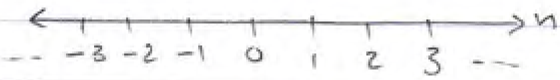
* there are two ways to deal with the x-axis:-



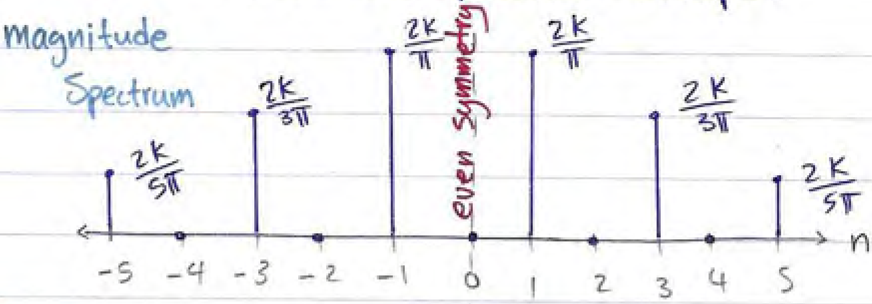
we only have integer multiples of ω_0!

∴ This spectrum is discrete! & it's called 'line Spectrum'.

→ another way to represent the x-axis is to use the harmonic number (n).



→ for the previous example:-

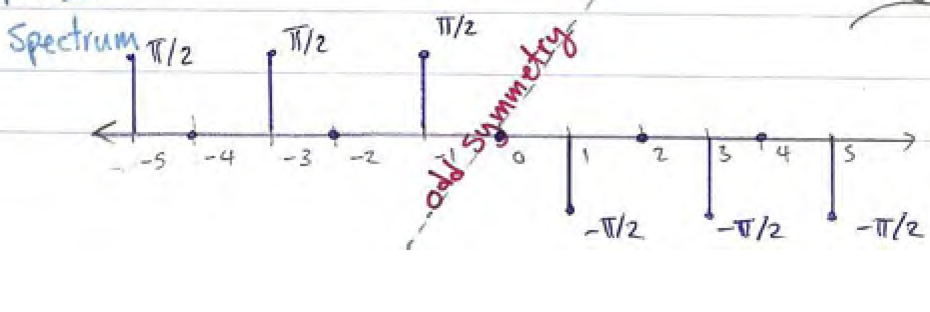


* when $n=1$ or $\omega=\omega_0$

$|C_n| = \frac{2K}{\pi}$ & $\angle C_n = -\frac{\pi}{2}$

* when $n=-1$ or $\omega=-\omega_0$

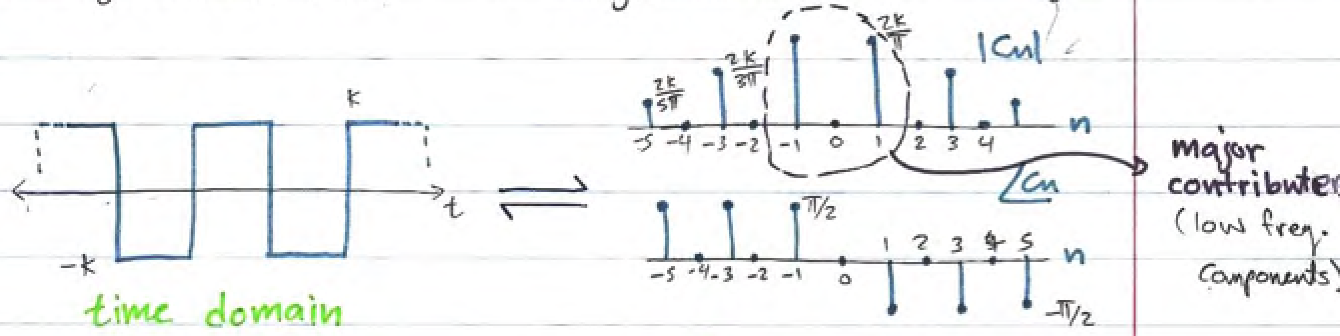
$|C_n| = \frac{2k}{\pi}$, $\angle C_n = \frac{\pi}{2}$



→ this is called a 'double-sided line Spectrum', has +ve & -ve (n).

* line spectrum (discrete) → double-sided (±ve n/frequency).
→ Single-sided (+ve n/frequency).

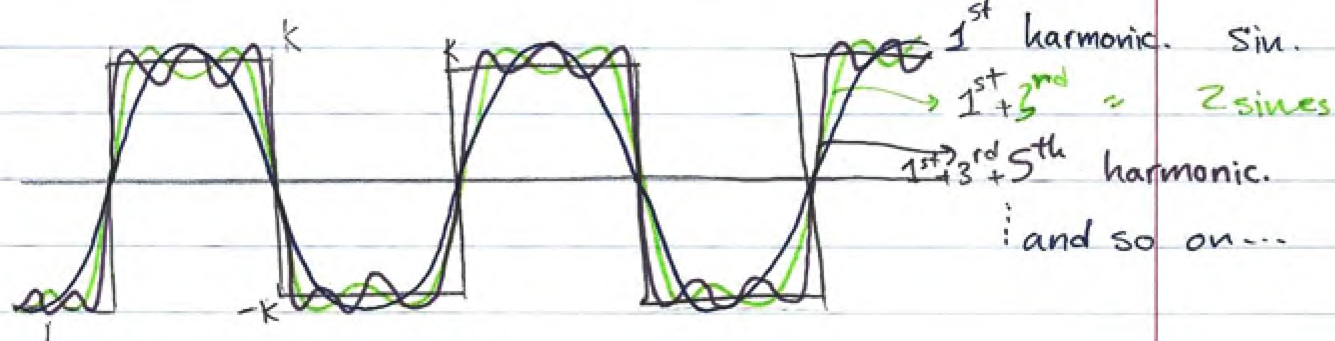
* exponential F.S has a double-sided & discrete spectrum.
* trigonometris F.S 〃 〃 Single-sided 〃 〃



time domain

frequency domain

from freq. domain I can see that this function is made up of low frequency components.



→ as n goes to ∞
it converges to the time domain function.

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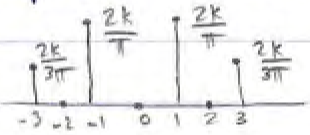
* going to frequency domain using Fourier series representation you'll always end up with a discrete spectrum in the freq. domain. (that's because we need w_0 & integer multiples of it <u>only</u> to represent the function).

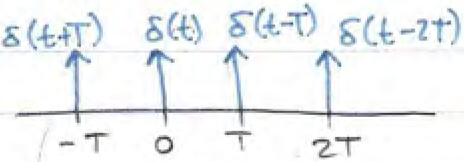
↳ then, you can always represent the line spectrum using impulse function $\delta[n]$



$x[n] = \frac{2k}{\pi}\delta[n-1] + \frac{2k}{3\pi}\delta[n-3] + \dots$

<u>Example 1</u>:-

$x(t) = \delta_T(t)$ ... "Train of impulses"



$C_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\,dt = \frac{1}{T}$

← ** in impulses don't confuse your self by taking $\int_0^T$ because $\int_0^\infty \delta(t)\,dt = \frac{1}{2}$ so we take it from $-\frac{T}{2} \to \frac{T}{2}$

$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} \underbrace{\delta_T(t)}_{x(t)} e^{-jnw_0t}\,dt$

by <u>sifting</u>:-

$= \frac{1}{T} e^{-jnw_0t}\Big|_{t=0} = \frac{1}{T} \quad \forall n$

$\therefore C_n = \frac{1}{T} \quad \forall n \qquad \therefore$ it's always <u>Real</u> & <u>+ve</u>

<u>NOTE</u> * Phase spectrum

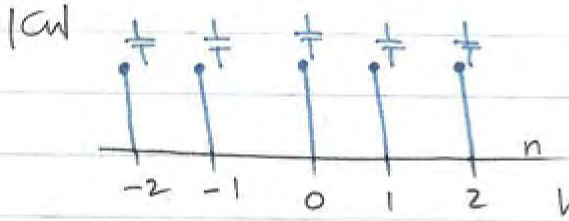
↳ if, C_n Real & +ve ⟶ phase = 0 <u>always</u>.

↳ if, C_n pure imaginary ⟶ phase = ±90° only.

↳ if, C_n Real & ±ve ⟶ Phase = 0, ±π only.

↳ if, C_n Complex ⟶ phase can take any value

→ back to the example:-

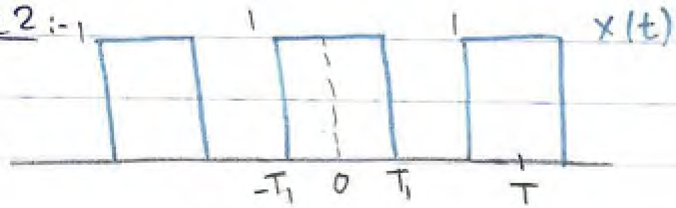


$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_0 t}$$

here it's a periodic magnitude spectrum

* Magnitude spectrum in freq. domain will always be even if your time domain function is real! but it's not necessarily periodic.
& the Phase Spectrum will be odd for real x(t).

Example 2:-



sol:-

→ $\omega_0 = \frac{2\pi}{T}$

→ $C_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot dt = \frac{2T_1}{T}$ ←

** You can always get the DC component by taking the area over T (average Value from the graph)!

→ $C_n = \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot e^{-jn\omega_0 t} dt$

$C_n = \frac{1}{jTn\omega_0}\left(\underbrace{e^{jn\omega_0 T_1} - e^{-jn\omega_0 T_1}}_{2j\sin(n\omega_0 T_1)}\right)$

$C_n = \frac{2\sin(n\omega_0 T_1)}{Tn\omega_0} \times \frac{T_1}{T_1}$

$C_n = \frac{2T_1}{T}\left(\frac{\sin(n\omega_0 T_1)}{n\omega_0 T_1}\right)$

$C_n = \frac{2T_1}{T} Sa(n\omega_0 T_1)$ ← Sinc ; since the time domain function is made of Rects

→

→ I want C_n in terms of $\text{Sinc}(t)$

$$\rightarrow C_n = \frac{2T_1}{T} \frac{\sin\left(\frac{2nT_1}{T}\pi\right)}{\frac{2nT_1}{T}\pi} \quad \text{--- Since } \omega_0 = \frac{2\pi}{T}$$

$$\therefore C_n = \frac{2T_1}{T}\,\text{sinc}\left(\frac{2nT_1}{T}\right)$$

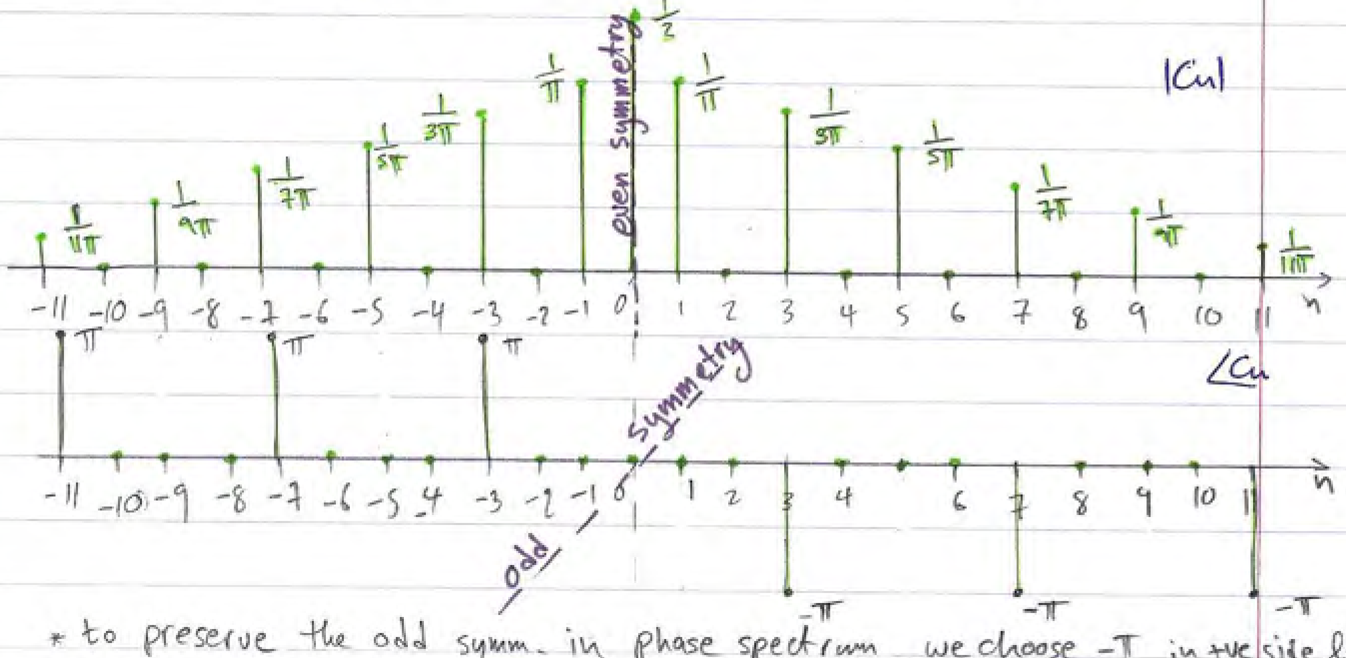
$$\Rightarrow X(t) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{2T_1}{T}\,\text{sinc}\left(\frac{2nT_1}{T}\right)}_{C_n} \underbrace{e^{jn\omega_0 t}}_{\phi_n(t)}$$

⇝ let $T_1 = \frac{\pi}{2}$, $T = 2\pi$

$$C_0 = \frac{2T_1}{T} = \frac{1}{2}$$

$$C_n = \frac{1}{2}\,\text{sinc}\left(\frac{n}{2}\right)$$

** nulls of $\text{sinc}(t)$ occurs at integer values of t, then $\text{sinc}\left(\frac{n}{2}\right)$'s nulls will be at integer values of $\frac{n}{2}$; ∴ when n is even $\text{sinc}\left(\frac{n}{2}\right) = 0$

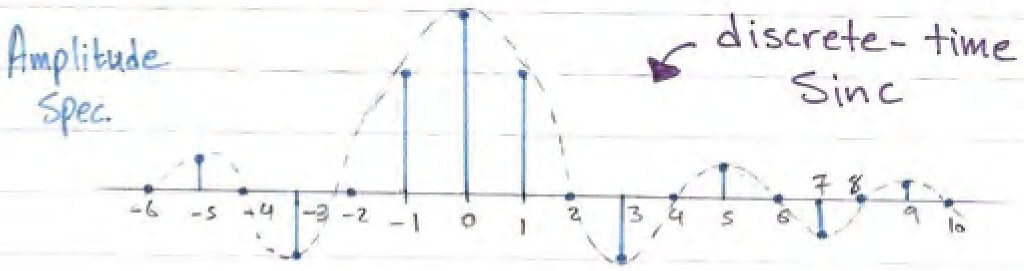


* to preserve the odd symm. in phase spectrum we choose $-\pi$ in +ve side & $+\pi$ in -ve side
* Don't confuse an even number n with even symmetry of the spectrum

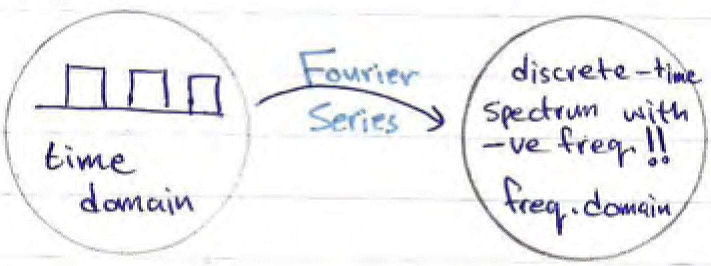
even function = even component + even comp. + even + ...
odd " = odd " + odd " + ...

* In phase spectrum we can choose $\pm\pi$, but in order to preserve the odd symmetry of the phase spectrum when $x(t)$ is real : we choose $-\pi$ in the +ve side (n) & $+\pi$ in the -ve side (n)

- In magnitude spectrum you're not allowed to have any negative value, because sign will be represented in phase Spectrum.
- If we want to represent the sign it'll be called the (Amplitude spectrum)



Explenation of -ve frequency:-



but -ve frequencies doesn't exist in Real life!

$\cos(+t) = \cos(-t)$

$\sin(+t) = -\sin(-t)$

so, why do we need -ve freq. in freq. domain?!

→ you'll only need -ve frequency when using the exponential F.S, because $\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$

we have to use -ve freq.

for an example I have $C_7 e^{j7\omega_0 t}$ & $C_{-7} e^{-j7\omega_0 t}$
& because the series will be added, it'll be

$$C_7 e^{j7\omega_0 t} + C_{-7} e^{-j7\omega_0 t} = 2\cos(7\omega_0 t)$$ +ve freq.

CT in t-domain $\xrightarrow[\text{fourier series}]{F.S}$ DT spectrum in freq. domain

CT in t-domain $\xrightarrow[\text{Fourier transform}]{F.T}$ CT spectrum in freq. domain

DT in t-domain $\xrightarrow{F.S}$ DT spectrum ~ ~ ~

DT in t-domain $\xrightarrow{F.T}$ CT spectrum ~ ~ ~

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Example 2: Trigonometric Fourier Series:

in general: $x(t) = \sum_{i=-\infty}^{\infty} c_i \phi_i(t)$

$$C_k = \frac{1}{E_k} \int_a^b x(t)\, \phi_k^*(t)\, dt$$

Trig. F.S

$$\phi_{i(t)} = \{1, \cos(w_0 t), \cos(2w_0 t), \cos(3w_0 t), \dots, \sin(w_0 t), \sin(2w_0 t), \sin(3w_0 t), \dots\}$$

- proving orthogonality:

$$\int_T \cos(nw_0 t)\cos(mw_0 t)\,dt = \begin{cases} 0, & m \neq n \\ \frac{T}{2}, & m = n \neq 0 \end{cases}$$

$m=n=0$ is the DC component حالة خاصة.

$$\int_T \sin(nw_0 t)\sin(mw_0 t)\,dt = \begin{cases} 0, & m \neq n \\ \frac{T}{2}, & m = n \neq 0 \end{cases}$$

- Energy of the DC component $= T$
- We only have +ve frequencies. $\cos(-t) = \cos(+t)$

- let a_n be the coeficient of (cosines) &
  b_n " " " " (Sines), then:

$$X(t) = a_0 + a_1\cos(w_0 t) + a_2\cos(2w_0 t) + \dots + a_n\cos(nw_0 t) + b_1\sin(w_0 t) + b_2\sin(2w_0 t) + \dots + b_n\sin(nw_0 t) + \dots$$

Trig. F.S: $X(t) = x_0 + \sum_{n=1}^{\infty} \left(a_n\cos(nw_0 t) + b_n\sin(nw_0 t)\right)$

Energy for DC component = T

- $a_0 = \frac{1}{T} \int_T x(t)\, dt$ $\boxed{b_0 = 0}$ because $\sin(0) = 0$

- $a_n = \frac{2}{T} \int_T x(t) \cos(n w_0 t)\, dt$

- $b_n = \frac{2}{T} \int_T x(t) \sin(n w_0 t)\, dt$

if $x(t)$ is Real:- if it's imaginary $x(t) = -x^*(t)$

$$x(t) = x^*(t)$$

→ The Relation between exp. F.S & Trig. F.S:-

for exponential F.S:

$$C_n = \frac{1}{T} \int_T x(t)\, e^{-jnw_0 t}\, dt$$

$$C_n^* = \frac{1}{T} \int_T x^*(t)\, e^{jnw_0 t}\, dt = \frac{1}{T} \int_T x(t)\, e^{jnw_0 t}\, dt = C_{-n}$$

هذه الحالة فقط إذا كان x(t) real

since $x(t)$ is Real then : $x(t) = x^*(t)$

$\therefore$ $\boxed{C_{-n} = C_n^*}$, $x(t)$ is Real

* if $x(t)$ is pure imaginary $C_n^* = -C_{-n}$ $(x(t) = -x^*(t))$

↳ $C_n = \alpha + j\beta$ or $C_n = |C_n|\, e^{j\angle C_n}$

↓ C_n^* ↓ C_n^*

$C_n^* = \alpha - j\beta$ $C_n^* = |C_n|\, e^{j\angle -C_n}$

but $C_{-n} = |C_{-n}|\, e^{j\angle C_{-n}}$

since $C_{-n} = C_n^*$

$\therefore$

$|C_n| = |C_{-n}|$ even symmetry (magnitude spectrum)

$\angle -C_n = -\angle C_{-n}$ odd symmetry (phase spectrum)

* if $x(t)$ is pure imaginary $\quad x(t) = -x^*(t)$

$|C_n| = -|C_{-n}| \rightarrow$ odd symmetry in magnitude spec

$\angle -C_n = \angle C_{-n} \rightarrow$ even ~ ~ phase spectrum.

• let's derive the raltion ship between C_n & b_n/a_n

$$x(t) = \underbrace{\sum_{n=-\infty}^{\infty} C_n e^{jnw_0t}}_{\text{exp. F.S}} \Longrightarrow \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos(nw_0t) + b_n \sin(nw_0t)}_{\text{trig. F.S}}$$

to do this transformation:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0t}$$

$$x(t) = C_0 + \sum_{n=-\infty}^{-1} C_n e^{jnw_0t} + \sum_{n=1}^{\infty} C_n e^{jnw_0t}$$

$$\leftarrow \sum_{n=-\infty}^{-1} C_n e^{jnw_0t} \quad + C_0 + \quad \sum_{n=1}^{\infty} C_n e^{jnw_0t} \rightarrow$$

$-\infty \leftarrow$ —— -1 —— C_0 —— 1 —— $\rightarrow \infty \quad (n)$

→ toget red of -ve frequency

change of variables in the -ve part :- بدل كل (n) ب(-n)

$$\sum_{-\infty}^{-1} = \sum_{\infty}^{1} = \sum_{1}^{\infty}$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_{-n} e^{-jnw_0t} + \sum_{n=1}^{\infty} C_n e^{jnw_0t}$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} \left(C_{-n} e^{-jnw_0t} + C_n e^{jnw_0t} \right)$$

$$C_n = \alpha + j\beta \;/\; C_{-n} = C_n^* = \alpha - j\beta$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} \left((\alpha - j\beta) e^{-jnw_0t} + (\alpha + j\beta) e^{jnw_0t} \right)$$

$$\rightarrow x(t) = C_0 + \sum_{n=1}^{\infty} \alpha\left(e^{-jnw_0t} + e^{jnw_0t}\right) + j\beta\left(e^{jnw_0t} - e^{-jw_0nt}\right)$$

$$= C_0 + \sum_{n=1}^{\infty} \alpha\left(2\cos(nw_0t)\right) + j\beta\left(2j\sin(nw_0t)\right)$$

$$= \underbrace{(C_0)}_{a_0} + \sum_{n=1}^{\infty} \underbrace{(2\alpha)}_{a_n}\cos(nw_0t) + \underbrace{(-2\beta)}_{b_n}\sin(nw_0t)$$

↳ $a_0 = C_0$; $C_n = \alpha + j\beta$ f $x(t)$ is Real

↳ $a_n = 2\alpha = 2\,Re\{C_n\}$

↳ $b_n = -2\beta = -2\,Im\{C_n\}$

$$\therefore C_n = \frac{1}{2}a_n - j\frac{b_n}{2}$$

- if C_n is pure imaginary $\rightarrow a_n = 0$
  $x(t)$ will be made of sines $\rightarrow$ odd
- if C_n is pure real $\rightarrow b_n = 0$
  $x(t)$ will be made of cosines $\rightarrow$ even

$\therefore$ The nature of (C_n) gives the symmetrical properties of $x(t)$.

Another way:-

Using Euler:- e^{-jnw_0t}

$$C_n = \frac{1}{T}\int_T x(t)\left[\cos(nw_0t) - j\sin(nw_0t)\right]dt$$

$$\alpha + j\beta = \underbrace{\frac{1}{T}\int_T x(t)\cos(nw_0t)\,dt}_{\alpha} \underbrace{- \frac{j}{T}\int_T x(t)\sin(nw_0t)\,dt}_{+j\beta}$$

but... $a_n = \frac{1}{T}\int_T x(t)\cos(nw_0t)\,dt = 2\alpha$

& $b_n = \frac{1}{T}\int_T x(t)\sin(nw_0t)\,dt = -2\beta$

* Plotting the trig. F.S :

in the exp. F.S we plot $|C_n|$ Vs ω (mag. sepectrum)
$\angle C_n$ Vs ω (phase spectrum).

but in trig F.S we need 4 plots if we want to do the sam
$|a_n|$ Vs ω
$|b_n|$ Vs ω
$\angle a_n$ Vs ω $\propto$ but that's not the case
$\angle b_n$ Vs ω

so, in order to minimize the number of Plots we do the following:-

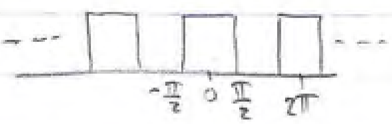
$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \underline{A_n} \cos(n\omega_0 t + \underline{\theta_n})$$

$$\boxed{\begin{aligned} &\hookrightarrow A_n = \sqrt{(a_n)^2 + (b_n)^2} \\ &\hookrightarrow \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \end{aligned}}$$

then, we plot (A_n) Vs (ω) $\rightarrow$ magnitude spectrum.
(θ_n) Vs (ω) $\rightarrow$ phase spectrum.

ex:- $C_n = \frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) \rightarrow$ Real & even $\rightarrow \therefore \beta = 0$, $b_n = 0 \rightarrow$ Even Symmetry



** if we want to construct an even and periodic function from a series, we have to use even & harmonically Related functions.
$\hookrightarrow$ (periodic + periodic + ...) = periodic only if they're harmonically related.

Sol:- $c_n = \frac{1}{2}\,\text{sinc}\left(\frac{n}{2}\right)$

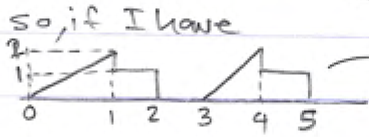
$$\therefore a_n = 2\alpha$$

$$= 2\left(\frac{1}{2}\,\text{sinc}\left(\frac{n}{2}\right)\right)$$

$$\boxed{a_n = \text{Sinc}\left(\frac{n}{2}\right)} \quad \ldots (1)$$

$$a_0 = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.\,dt = \frac{1}{2} \longrightarrow$$ ** I can also get this value from $\left(\frac{\text{area}}{T}\right)$

$\Longrightarrow$ so, if I have



$\rightarrow$ $T = 3$, area $= 2$] $a_0 = \frac{\text{area}}{T} = \frac{2}{3}$

Short Cut 1)

$$a_n = \frac{2}{(2\pi)}\int_{-\frac{\pi}{2}}^{\pi/2} 1.\cos(nt)\,dt = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n = 1, 5, 9, 11, \ldots \\ \frac{-2}{n\pi}, & n = 3, 5, 7, \ldots \end{cases}$$

$$b_n = \frac{2}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.\,\sin(nt)\,dt$$

$$\therefore x(t) = \overset{a_0}{\frac{1}{2}} + \frac{2}{\pi}\left(\cos(t) - \frac{1}{3}\cos(3t) + \frac{1}{5}\cos(5t) - \frac{1}{7}\cos(7t) + \ldots\right)$$

to write it in compact trig. form:-

we can add π but default is $\boxed{-\pi}$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\cos t + \frac{1}{3}\cos(3t - \pi) + \frac{1}{5}\cos(5t) + \frac{1}{7}\cos(7t - \pi) + \ldots\right)$$

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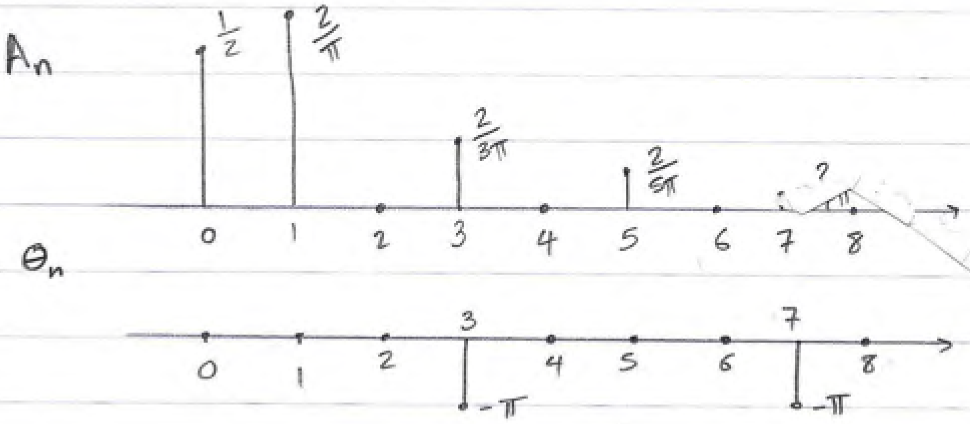
cont. →



Single-sided spectrum.

$$C_n = \frac{1}{2}a_n - j\frac{1}{2}b_n = |C_n| \measuredangle C_n = |C_n| e^{j\measuredangle C_n}$$

علاقة ال single-sided

بال double-sided ...

$$|C_n| = \sqrt{\left(\frac{a_n}{2}\right)^2 + \left(\frac{b_n}{2}\right)^2} = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = \frac{1}{2}A_n$$

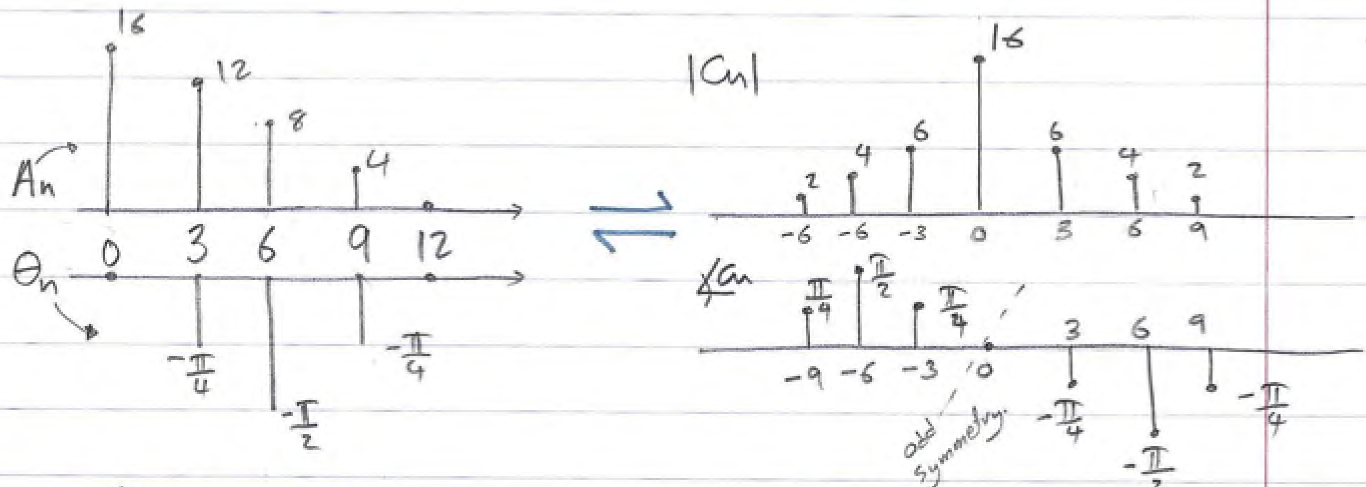
$$\measuredangle C_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \theta_n \quad \text{... for +ve (n)}$$

for −ve (n):-

$$C_{-n} = C_n^* \Rightarrow \therefore \boxed{\measuredangle C_n = -\theta_n}$$

- $|C_n| = \frac{1}{2}A_n$
- $C_0 = a_0$
- $C_n = \theta_n$ (+ve n)
  $C_n = -\theta_n$ (−ve n)

(Single ⇌ double)_ sided transformation:



Single-sided   double-sided

** Only if x is Real...

* from the single-sided spectrum:-

$$x(t) = 16 + 12\cos\left(3\omega_0 t - \frac{\pi}{4}\right) + 8\cos\left(6\omega_0 t - \frac{\pi}{2}\right) + 4\cos\left(9\omega_0 t - \frac{\pi}{4}\right)$$

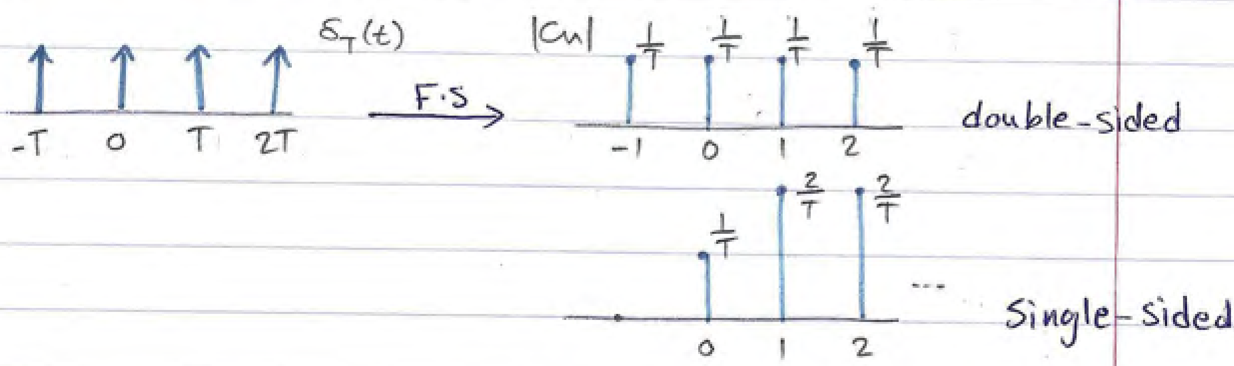
* from the double-sided spectrum:-

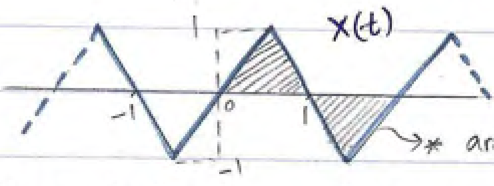
$$x(t) = 16 + \begin{bmatrix} 6e^{-j\frac{\pi}{4}} e^{j3\omega_0 t} \\ +6e^{j\frac{\pi}{4}} e^{-j3\omega_0 t} \end{bmatrix} + \begin{bmatrix} 4e^{-j\frac{\pi}{2}} e^{j6\omega_0 t} \\ +4e^{j\frac{\pi}{2}} \cdot e^{-j6\omega_0 t} \end{bmatrix} + \begin{bmatrix} 2e^{-j\frac{\pi}{4}} e^{j9\omega_0 t} \\ +2e^{j\frac{\pi}{4}} e^{-j9\omega_0 t} \end{bmatrix}$$

$$= 16 + 6\left(2\cos\left(3\omega_0 t - \frac{\pi}{4}\right)\right) + 4\left(2\cos\left(6\omega_0 t - \frac{\pi}{2}\right)\right) + 2\left(2\cos\left(9\omega_0 t - \frac{\pi}{4}\right)\right)$$

$$x(t) = 16 + 12\cos\left(3\omega_0 t - \frac{\pi}{4}\right) + 8\cos\left(6\omega_0 t - \frac{\pi}{2}\right) + 4\cos\left(9\omega_0 t - \frac{\pi}{4}\right)$$ ✻

ex:





✻ area=0 → $a_0 = C_0 = 0$

$$x(t) = \begin{cases} 2t, & |t| < \frac{1}{2} \\ 2(1-t), & \frac{1}{2} < t < \frac{3}{2} \end{cases} \qquad T=2 \qquad \omega_0 = \pi$$

* Since $x(t)$ has odd symmetry → $a_n = 0$ (made of sines).

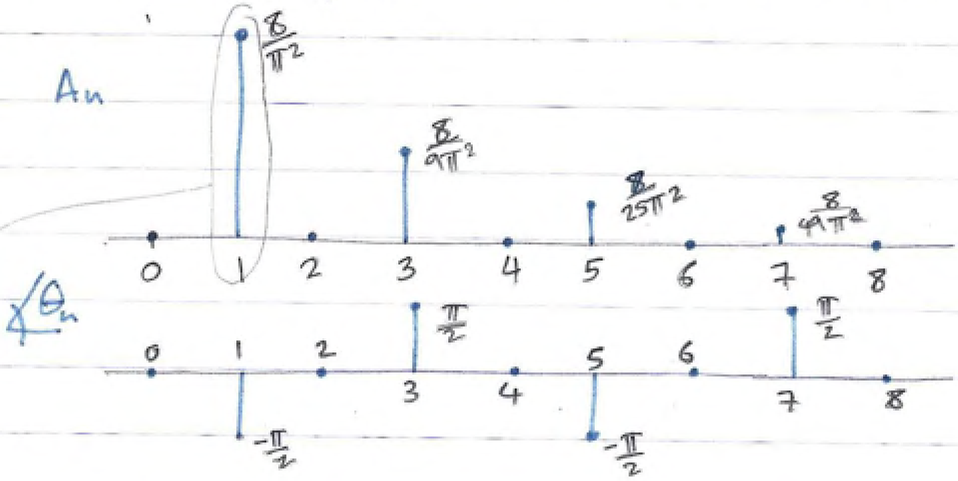
$$* \; b_n = \frac{2}{T}\int_{-1/2}^{1/2} 2t\sin(n\pi t)\,dt + \frac{2}{T}\int_{1/2}^{3/2} 2(1-t)\sin(n\pi t)\,dt$$

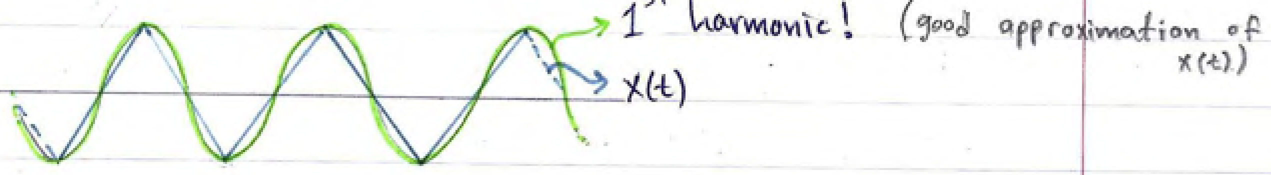
$$b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{8}{n^2\pi^2}, & n = 1, 5, 9, 13 \ldots \\ \frac{-8}{n^2\pi^2}, & n = 3, 7, 11, \ldots \end{cases}$$

$\therefore \quad x(t) = \frac{8}{\pi^2}\left[\sin \pi t - \frac{1}{9}\sin 3\pi t + \frac{1}{25}\sin 5\pi t - \frac{1}{49}\sin 7\pi t + \cdots\right]$

in the compact trig. form:-

$\rightarrow x(t) = \frac{8}{\pi^2}\left[\cos\left(\pi t - \frac{\pi}{2}\right) + \frac{1}{9}\cos\left(3\pi t + \frac{\pi}{2}\right) + \frac{1}{25}\cos\left(5\pi t - \frac{\pi}{2}\right) + \cdots\right]$





ex: $x(t) = 2 + 7\cos\left(\frac{t}{2} + \theta_1\right) + 3\cos\left(\frac{2t}{3} + \theta_2\right) + 5\cos\left(\frac{7}{6}t + \theta_3\right)$

find ω_0 ?!

$\frac{1}{2} = k\omega_0$

$\frac{2}{3} = m\omega_0 \quad \rightsquigarrow \quad \omega_0 = \frac{1}{6}$

$\frac{7}{6} = n\omega_0 \quad\quad \rightarrow k = 3,\ m = 4,\ n = 7$ (integer multiples).

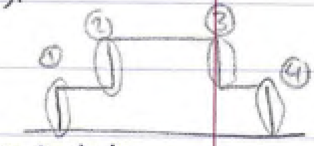
$$X(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$ exponential. F.s

$$\text{or} = a_0 + \sum_{n=1}^{\infty} A_n \cos(nw_0 t + \theta_n)$$ ~~Trigonometric~~ F.S.

* dirichlet conditions:

The Previous summations will converge if:-

1. $x(t)$ is absolutely integrable (bounded).
   $$\int_T |x(t)|\, dt < \infty$$
2. Number of discontinuities $< \infty$! in one period.

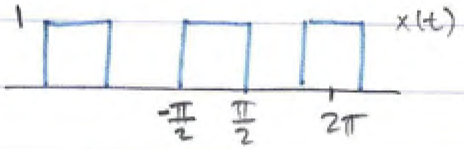


3. $x(t)$ has a finite number of maxima & minima in one period.

26/3/2013 →

* Convergance:-

Applications on convergance → $$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$



$T = \pi, \quad w_0 = 1$

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)$$

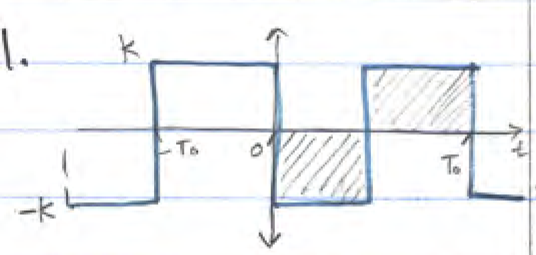
$x\left(\frac{\pi}{2}\right) = \frac{1}{2}$ ... in limit sense there's a values $= \frac{0+1}{2} = \frac{1}{2}$

$$x(0) = \frac{1}{2} + \frac{2}{\pi}\underbrace{\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right)}_{\text{converges to } \boxed{\frac{\pi}{4}}} = \frac{1}{2} + \frac{2}{\pi} \times \frac{\pi}{4} = 1$$

Fourier Series For Common Signals:-

Wave form	C_0	$C_n,\ n \neq 0$

1. Square wave



Square wave

$C_0 = 0$

from the $\left(\frac{area}{T}\right)$

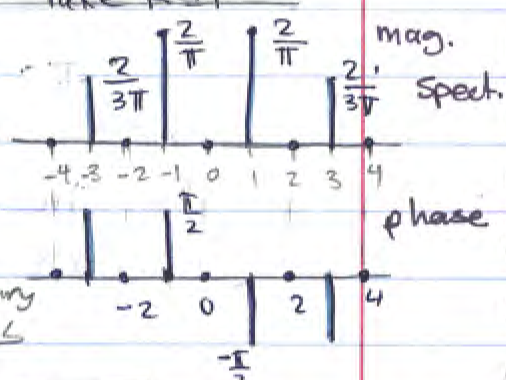
$$C_n = \begin{cases} 0, & n = \text{even} \\ \dfrac{-2K}{\pi n} j, & n\ \text{odd} \end{cases}$$

$C_n \to$ pure imaginary

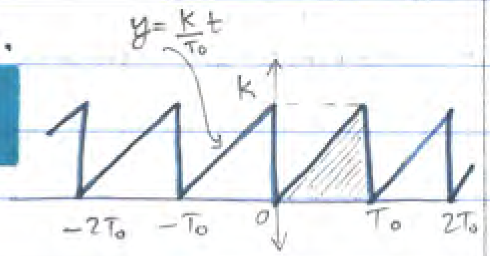
كتر ٢ ضرب sines وطلع

$b_n = 2\,Img\{C_n\}$ odd func.

take K=1



2. Sawtooth



Sawtooth

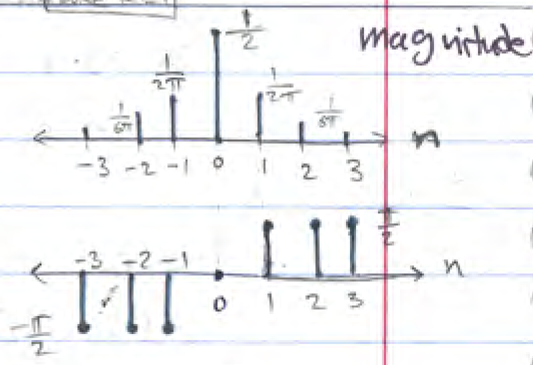
$C_0 \neq 0$

$\frac{area}{T_0} = \frac{\frac{1}{2} K\,T_0}{T_0}$

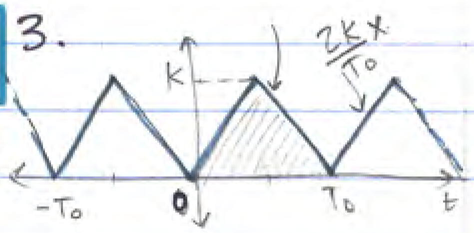
$$\boxed{C_0 = \frac{K}{2}}$$

take k=1

$$C_n = \frac{jK}{2\pi n}$$



3. Triangular wave



Triangular wave

$C_0 \neq 0$

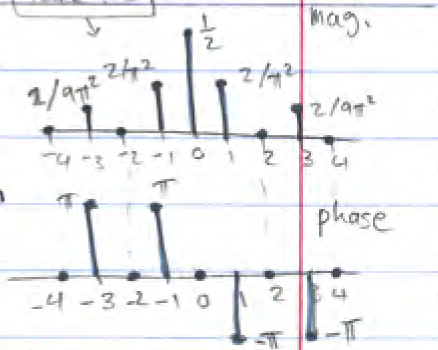
$\frac{area}{T_0} = \frac{\frac{1}{2} \times T_0 K}{T_0}$

$$\boxed{C_0 = \frac{K}{2}}$$

$$C_n = \frac{-2K}{(\pi n)^2}, \quad n: \text{odd}$$

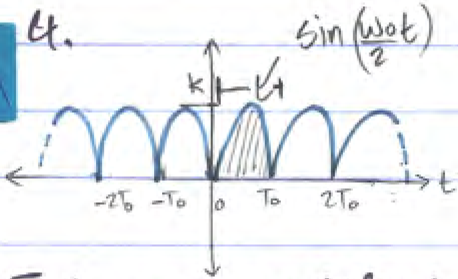
$$0, \quad n: \text{even}$$

take k=1



Be smart when you choose your limits $(-T/2 \to T/2)$

4. Full-wave rectified



Full-wave rectified

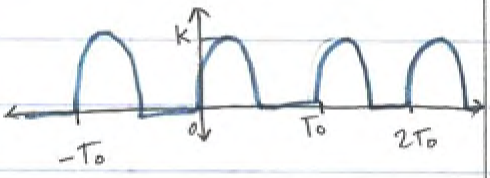
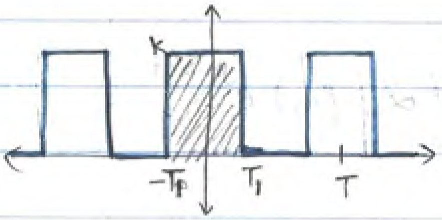
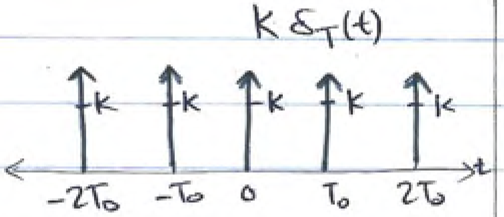
$$C_0 = \frac{2k}{\pi}$$

$$C_n = \frac{-2K}{\pi(4n^2 - 1)}$$

$\pm\frac{1}{2}$ (أصفار المقام)

← كسور ∴ لا تقسيم n

→ F.S for common signals (continued)..

wave form	C_0	$C_n \, , n \neq 0$
 <u>Half-wave rectified</u>	$C_0 = \frac{K}{\pi}$	$C_n = \begin{cases} \frac{-K}{\pi(n^2-1)} & , \text{ n is even} \\ 0 & , \text{ n is odd except } n=\pm 1) \\ -j\frac{K}{4} & , n=1 \\ j\frac{K}{4} & , n=-1 \end{cases}$ لأنهم أصفار تمام لازم احفظهم كال!
 <u>Rectangular wave</u>	$C_0 = \frac{2T_1}{T} K$ $\frac{\text{area}}{T_0}$ $\frac{2T_1 \times K}{T_0}$	$C_n = \frac{2KT_1}{T} \text{sinc}\left(\frac{2nT_1}{T}\right)$
 <u>Train of Impulses</u>	$C_0 = \frac{K}{T_0}$	$C_n = \frac{K}{T_0}$

• Properties of Fourier Series:

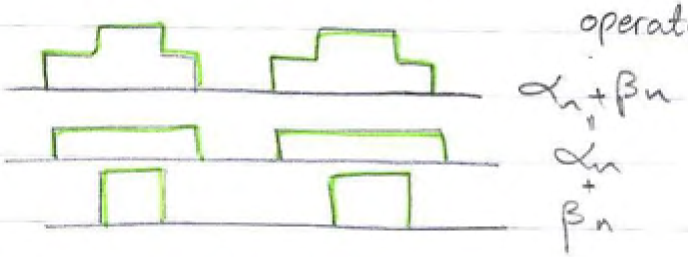
1. linearity:

$x(t) \xrightarrow{F.S} \alpha_n$

$y(t) \xrightarrow{F.S} \beta_n$

$k\,x(t) + k\,y(t) \xrightarrow{F.S} k\,\alpha_n + k\,\beta_n$

because the series is a linear operation (summation is a linear operator)



$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t}$$

$+$

$$y(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega_0 t}$$

$$\Rightarrow \quad z(t) = \sum_{n=-\infty}^{\infty} \underbrace{(\alpha_n + \beta_n)}_{\gamma_n} e^{jn\omega_0 t} \Leftarrow x(t) + y(t)$$

2. time shifting

$x(t) \xrightarrow{F.S} C_n$

$x(t \pm T) \xrightarrow{F.S} C_n\, e^{\pm jn\omega_0 T}$ phase shift.

$$\frac{1}{T}\int_T x(t-T) e^{-jn\omega_0 t}\, dt$$

let $\delta = t - T$

$d\delta = dt$

$$= \frac{1}{T}\int_T x(\delta)\, e^{-jn\omega_0(\delta + T)}\, d\delta$$

$$= e^{-jn\omega_0 T} \underbrace{\left(\frac{1}{T}\int_T x(\delta)\, e^{-jn\omega_0 \delta}\, d\delta\right)}_{C_n}$$

تمثيل الـ phase shift على الـ phase spect.

لا يكون بالإزاحة يمين ويسار بل بالدوران

$$= e^{-jn\omega_0 T} C_n$$ ... shift in the phase of C_n!

مع أو عكس عقارب الساعة!

$C_n = |C_n| e^{j\angle C_n}\, e^{\pm jn\omega_0 t}$

(لأنه الإزاحة زاوية).

ولو حركناها على محور السينات تكون الإزاحة فيها لسين فيه θ!

3.

$$x(t) \xrightarrow{F.S} C_n$$

$$e^{wot}\, x(t) \xrightarrow{F.S} C_{(n-m)} \quad \text{(frequency shifting).}$$

$$; \quad \frac{1}{T}\int_T x(t)\, e^{jmw_0t}\, e^{-jnw_0t}\, dt$$

$$= \frac{1}{T}\int_T x(t)\, e^{-j(n-m)w_0t}\, dt$$

$$= C_{(n-m)}$$ shift in frequancy (shift left and right in magnitude spectrum).

so...

$$x(t) \xrightarrow{F.S} C_n$$

$$x(t \pm T) \xrightarrow{F.S} C_n\, e^{\pm jnw_0T}$$

$$e^{\pm jmw_0t}\, x(t) \xrightarrow{F.S} C_{(n\mp m)}$$

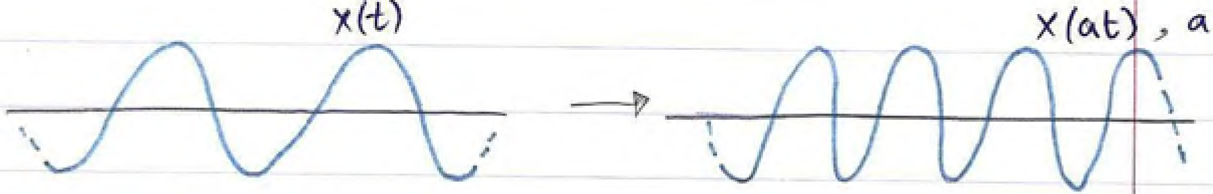
Duality!

4. time scaling:

$$x(t) \xrightarrow{F.S} C_n$$

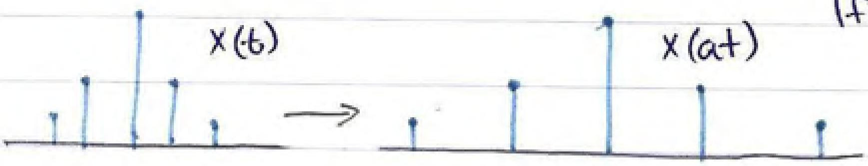
$$x(at) \xrightarrow{F.S} ?$$

$$x(-t) \xrightarrow{F.S} C_{-n}$$



comprision in t-domain (frequency increases)

|||



(expansion in freq-domain)

$$\rightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jnw_0 t}$$

$$x(at) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn(w_0 a)t}$$

→ expansion in w

(wo changes, & so does the spacing between samples).
harmonic # (n) isn't affected.

5. differantiation:

$$x(t) \xrightarrow{F.S} C_n$$

$$\frac{dx(t)}{dt} \xrightarrow{F.S} jnw_0 C_n$$

$$; \quad x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} \underbrace{C_n (jnw_0)}_{d_n} e^{jnw_0 t}$$

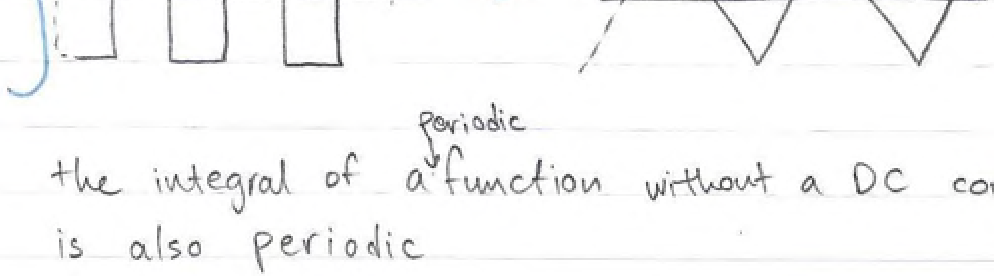
differantiation in t-domain ⇌ (x jnwo) in freq-domain

6. Integration:

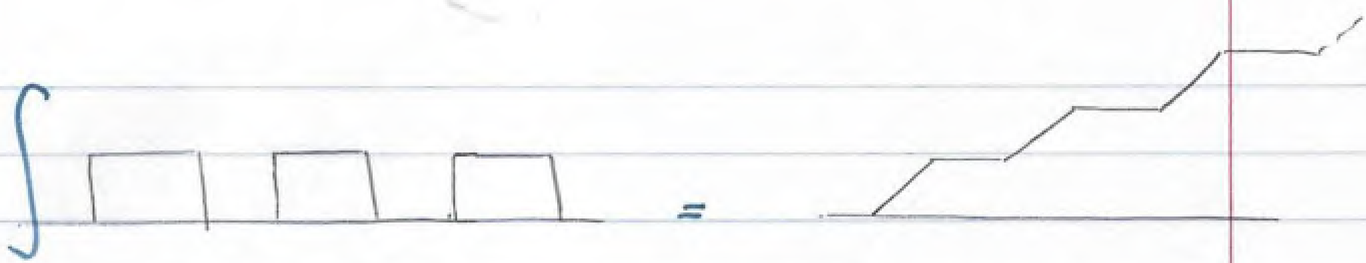
$$x(t) \xrightarrow{F.S} C_n$$

$$\int_{-\infty}^{t} x(\tau) d\tau \xrightarrow{F.S} \frac{C_n}{jnw_0} \quad , n \neq 0$$

we might have some problems with the DC component!



the integral of a periodic function without a DC componant is also periodic



when $C_0 \neq 0$, the integration produces a component that increases linearly with time, ∴ the resulting signal won't be periodic.

28/3/2013

7) Multiplication

let $x(t)$, $y(t)$ be periodic signals with the same period, $z(t) = x(t) \cdot y(t)$

$$x(t) \xrightarrow{F.S} \alpha_n$$
$$y(t) \xrightarrow{F.S} \beta_n$$
$$z(t) \xrightarrow{F.S} \frac{1}{T}\int_T x(t)\, y(t)\, e^{-jn\omega_0 t}\, dt \quad \ldots ①$$

also...

$$y(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega_0 t} = \sum_{m=-\infty}^{\infty} \beta_m e^{jm\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t} = \sum_{m=-\infty}^{\infty} \alpha_m e^{jm\omega_0 t}$$

back to ①

change of variables

$$\frac{1}{T}\int_T x(t) \sum_{m=-\infty}^{\infty} \beta_m e^{jm\omega_0 t}\, e^{-jn\omega_0 t}$$

$$= \sum_{m=-\infty}^{\infty} \beta_m \underbrace{\frac{1}{T}\int_T x(t)\, e^{-j(n-m)\omega_0 t}\, dt}_{\alpha_{n-m}}$$

$$= \underbrace{\sum_{m=-\infty}^{\infty} \beta_m \alpha_{n-m}}_{\text{convolution sum!}} = \alpha_n * \beta_n$$

$\therefore$

$$\boxed{\begin{aligned} &x(t) \xrightarrow{F.S} \alpha_n \\ &y(t) \xrightarrow{F.S} \beta_n \\ &\bullet\ z(t) = x(t) \cdot y(t) \xrightarrow{F.S} \alpha_n * \beta_n \end{aligned}}$$

8) Convolution: (is there duality?)

- $z(t) = x(t) \circledast y(t) \xrightarrow{F.S} \alpha_n \cdot \beta_n$

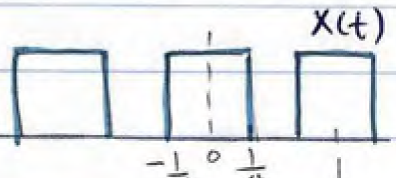
$\circledast$: Circular convolution ; a special form of convolution for periodic signals with the same period

$$z(t) = x(t) \circledast y(t) = \frac{1}{T} \int_T x(\tau)\, y(t-\tau)\, d\tau$$

↳ it's tedious to do it in t-domain ... so we can calcute it using $(\alpha_n \cdot \beta_n)$; fourier series coefficients of the convolution!

ex:

X(t) $\boxed{T=1}$



if $T_1 \neq T_2$ it's outside the scope of this course

$\boxed{T=1}$

$$y(t) = 2\cos(\underbrace{2\pi t}_{\omega_0}) + \sin(\underbrace{4\pi t}_{2\omega_0})$$

(1st harmonic) (2nd harmoic)

sol:

$$\alpha_n = \frac{1}{2}\,\text{Sinc}\left(\frac{n}{2}\right)$$

$$\beta_n = 2\left(\frac{e^{j2\pi t} + e^{-j2\pi t}}{2}\right) + \left(\frac{e^{j4\pi t} - e^{-j4\pi t}}{2j}\right)$$

$$= \underbrace{-\frac{1}{2j}}_{C_{-2}} e^{-j4\pi t} + \underbrace{1}_{C_{-1}} e^{-j2\pi t} + \underbrace{1}_{C_1} e^{j2\pi t} + \underbrace{\frac{1}{2j}}_{C_2} e^{j4\pi t}$$

$$\beta_n = \begin{cases} 1, & n = \pm 1 \\ \frac{1}{j2}, & n = 2 \\ \frac{-1}{j2}, & n = -2 \\ 0, & \text{o.w} \end{cases} \quad \longrightarrow \quad \boxed{\alpha_n \cdot \beta_n = \gamma_n}$$

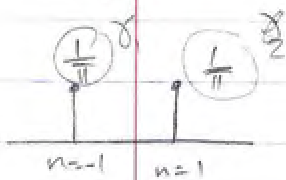
• The only γ_n(s) I'm interested in are when $n = \pm1, \pm2$!
because β_n otherwise $= 0$!

↳ $$\begin{aligned} \cancel{\gamma_{-2}} &\cancel{= \alpha_{-2} \cdot \beta_{-2}} \\ \gamma_{-1} &= \alpha_{-1} \cdot \beta_{-1} \\ \gamma_1 &= \alpha_1 \cdot \beta_1 \\ \cancel{\gamma_2} &\cancel{= \alpha_2 \cdot \beta_2} \end{aligned}$$ → we don't need them because $\alpha_{-2} = \alpha_2 = 0$!

↳ $\alpha_n = \frac{1}{2}\operatorname{sinc}\left(\frac{n}{2}\right) \longrightarrow \frac{1}{2}\frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi}$

↳ $\gamma_{-1} = \alpha_{-1}\beta_{-1} = \frac{1}{\pi} \times 1 = \frac{1}{\pi}$

$\gamma_1 = \alpha_{+1}\beta_{+1} = \frac{1}{\pi} \times 1 = \frac{1}{\pi}$

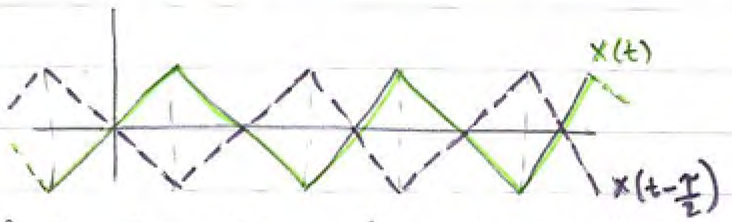


$\xrightarrow[\text{to t-domain}]{\text{back}}$ $\underline{\text{conv}} = \frac{1}{\pi}e^{-j2\pi t} + \frac{1}{\pi}e^{j2\pi t}$ ← $z(t) = \sum_{n=-\infty}^{\infty} \gamma_n e^{jn\omega_0 t}$

$x(t) \circledast y(t) = \frac{2}{\pi}\cos(2\pi t)$

*Effects of Symmetry:

1. $x(t) = x(-t)$ ... even
2. $x(t) = -x(-t)$ ... odd
3. half wave symmetry! $x(t - \frac{T}{2}) = -x(t)$



when $x(t) = -x(t - \frac{T}{2})$ it's a half wave symmetry

*if the two halves of one period of a periodic signal are of identical shape except that one is the negative of the other, then the signal is said to have a half wave symmetry.

- $a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} x(t)\,dt$
- $a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n w_0 t)\,dt$
- $b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n w_0 t)\,dt =$

Symmetry of x(t)	a_0	a_n	b_n
even	$a_0 \neq 0$	$a_n \neq 0$	$b_n = 0$
odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$
$\frac{1}{2}$ wave Symmetry.	$a_0 = 0$	$a_{2n} = 0$ $a_{2n+1} \neq 0$	$b_{2n} = 0$ $b_{2n+1} \neq 0$

*NOTE:-

$\int_{-T}^{T} \underbrace{x(t)}_{\text{even}}\,dt = 2\int_{0}^{T} x(t)\,dt$

$\int_{-T}^{T} \underbrace{x(t)}_{\text{odd}}\,dt = 0$

↳ it makes life much much easier!:)

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n w_0 t) + b_n \sin(n w_0 t)$$

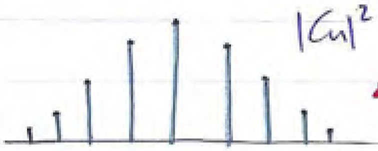
exp. → $C_n = \frac{a_n}{2} - j\frac{b_n}{2}$ → if C_n is pure imaginary → $x(t)$ is odd
〃 〃 〃 〃 real → 〃 〃 even

*Parseval's Theorem:

- Periodic functions are power signals

time domain — frequency domain

$P_\infty = \frac{1}{T}\int_T |x(t)|^2\,dt \longrightarrow P_\infty = \sum_{-\infty}^{\infty} |C_n|^2$

$\sum_{n=-\infty}^{\infty}$  $|C_n|^2$ ↙ "Power Spectrum". $= P_\infty$

- In time domain, you can only see the total power! but in frequency domain you can see the contribution to the power by each harmonic (the power distribution with respect to frequency).

- if $x(t) = C_n e^{jn\omega_0 t}$ (one component).

$$P_\infty = \frac{1}{T}\int_T C_n \cancel{e^{jn\omega_0 t}} \; C_n^* \cancel{e^{-jn\omega_0 t}} \, dt$$

$$P_\infty = C_n C_n^* = |C_n|^2$$

- but does $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

has $P_\infty = \sum_{n=-\infty}^{\infty} |C_n|^2$?

let's take

$$\underbrace{C_1 e^{j\omega_0 t}}_{P_\infty \downarrow} + \underbrace{C_3 e^{j3\omega_0 t}}_{P_\infty \downarrow}$$

$$|C_1|^2 + |C_3|^2 \overset{?}{=} P_\infty$$

the power relation linear only if they were harmonically related (Periodic)

↳ this is true only if the summation is periodic.

* $\cos 2\pi t + 2\cos 4\pi t$

$$\frac{(1)^2}{\cancel{2}}\frac{1}{2} + \frac{(2)^2}{\cancel{2}}2 = P_\infty$$ ✓ harmonically related

* $\cos(2\pi t) + 2\cos(1.43t)$

doesn't work any more! (summation of them isn't period)

* **Mathematically Speaking...**

let $X(t) = \sum_{n=\infty}^{\infty} C_n e^{jn w_0 t}$

then $P_\infty = \frac{1}{T} \int_T \underbrace{\left(\sum_{m=-\infty}^{\infty} C_m e^{jm w_0 t} \right)}_{X(t)} \cdot \underbrace{\left(\sum_{n=-\infty}^{\infty} C_n^* e^{-jn w_0 t} dt \right)}_{X^*(t)}$

I have to change the index from $n \rightarrow m$

e.g if I have $(1+2+3+4+5)(1+2+3+4+5)$

if $n,n \rightarrow 1\times1,\ 2\times2,\ 3\times3 \ldots$

therefore we use $m,n \rightarrow 1\times1,\ 1\times2,\ 1\times3, \ldots, 2\times1,\ 2\times2, \ldots$

$$P_\infty = \sum_{m=-\infty}^{\infty} C_m \sum_{n=-\infty}^{\infty} C_n^* \frac{1}{T} \int e^{-j(n-m) w_0 t} dt \Big\} \Rightarrow \text{if } m \neq n =$$

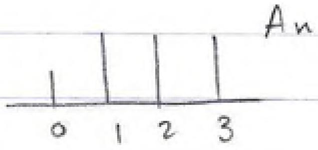
because of orthogonality!

$$P_\infty = \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad *$$

in power spectrum

$C_n = \frac{a_n}{2} - j\frac{b_n}{2}$

$|C_n|^2 = \frac{1}{2} A_n^2$



Single sided

$$P_\infty = \underbrace{a_0^2}_{\text{Dc alone.}} + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2$$

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$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$x(t) \longrightarrow \boxed{LTi} \longrightarrow y(t)$$

in general...

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

Convolution...

if $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \longrightarrow y = ?!$

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow ? \qquad y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

constant

$$= e^{j\omega t} \underbrace{\boxed{\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}}_{} \rightarrow H(\omega)$$

"Frequency Response"

$$\therefore \Longrightarrow \quad e^{j\omega t} \longrightarrow \boxed{LTi} \longrightarrow e^{j\omega t} H(\omega)$$

$$\cos(\omega t) \longrightarrow \boxed{LTi} \longrightarrow A\cos(\underline{\omega t} + \emptyset)$$

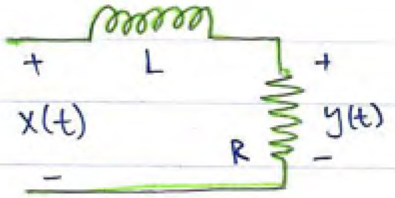
(ω) mustn't be changed because the system is linear.

$$\equiv$$

$$Re\{e^{j\omega t}\} \longrightarrow \boxed{LTi} \longrightarrow Re\{e^{j\omega t} H(\omega)\}$$

$$\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \longrightarrow \boxed{LTi} \longrightarrow \sum_{n=-\infty}^{\infty} C_n H(n\omega_0)\, e^{jn\omega_0 t}$$

ex:



if $x(t) = 4\cos t - 2\cos 2t$

what's $y(t)$?

Sol:

$$y'(t) + \frac{R}{L}\, y(t) = \frac{R}{L}\, x(t) \quad \ldots ①$$

if $x(t) = e^{j\omega t} \longrightarrow y(t) = H(\omega)\, e^{j\omega t}$

plug x, y, y' in ①

$$j\omega \cancel{e^{j\omega t}}\, H(\omega) + \frac{R}{L}\, H(\omega)\, \cancel{e^{j\omega t}} = \frac{R}{L}\, \cancel{e^{j\omega t}}$$

$$H(\omega) = \frac{R/L}{R/L + j\omega} = \frac{1}{1 + j\omega}$$

$$H(n\omega_0) = \frac{1}{1 + jn\omega_0}$$

→ now I'm interested in the output,

first, we should write x in terms of the exponentials
$(e^{j\omega t})$

$$x = 4\cos t - 2\cos 2t$$

$$= \overset{2}{\cancel{4}} \left(\frac{e^{jt} + e^{-jt}}{\cancel{2}} \right) - \cancel{2} \left(\frac{e^{j2t} + e^{-j2t}}{\cancel{2}} \right)$$

$$= -e^{-j2t} + 2e^{-jt} + 2e^{jt} - e^{j2t}$$

let's take a case where $\omega_0 = 1$

$$H(n\omega_0) = H(n) = \frac{1}{1 + jn}$$

$$y(t) = -e^{-j2t} H(-2) + 2e^{-jt} H(-1) + 2e^{jt} H(1) - e^{j2t} H(2) \ldots ②$$

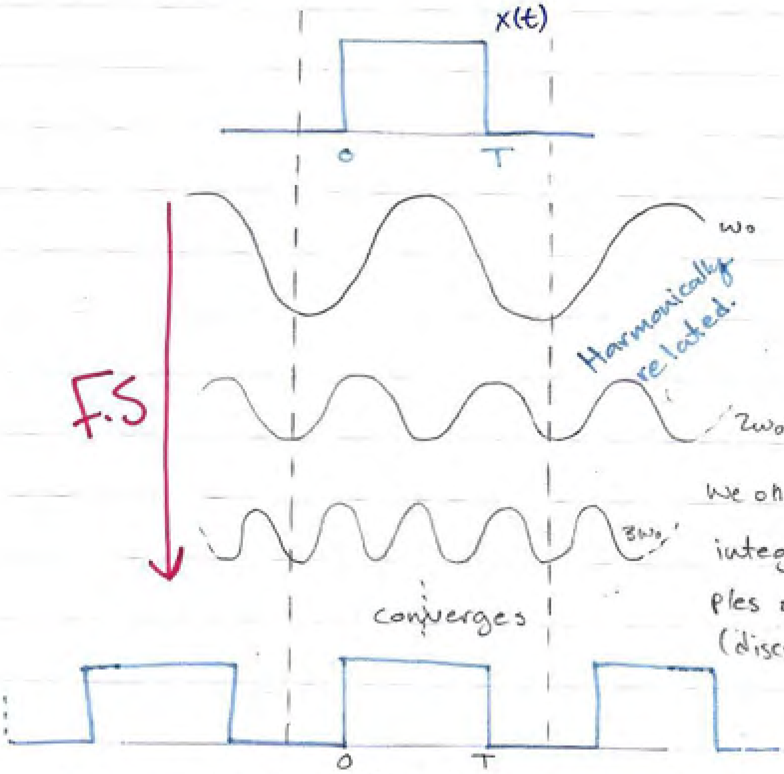
$$H(-2) = \frac{1}{1 - j2} = |H(-2)|\, e^{j\angle H(-2)} = \frac{1}{\sqrt{5}} \angle \tan^{-1}\left(\frac{-2}{1}\right) = \frac{1}{\sqrt{5}} \angle 63.43^\circ$$

----

* freq. response changes input's magnitude & phase!

Ch#5: Fourier Transform:

- previously (in F.s), we were interested of $x(t)$ over a finite interval of time ! only. وليش دائماً periodic !



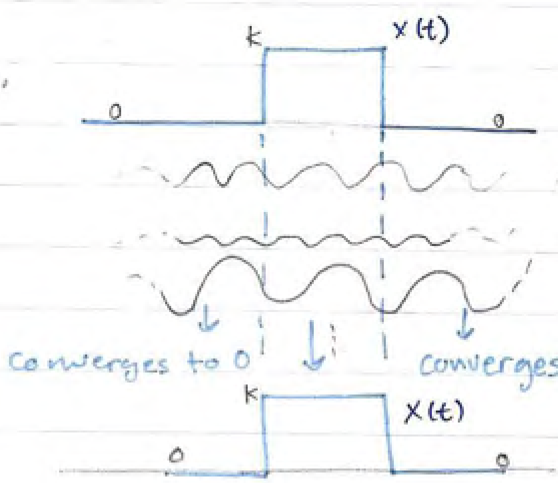
← limitation of fourier series!

we only need integer multiples of wo (discrete spectrum).

Since I'm adding harmonically related periodic functions then the resulting function must be periodic! :)

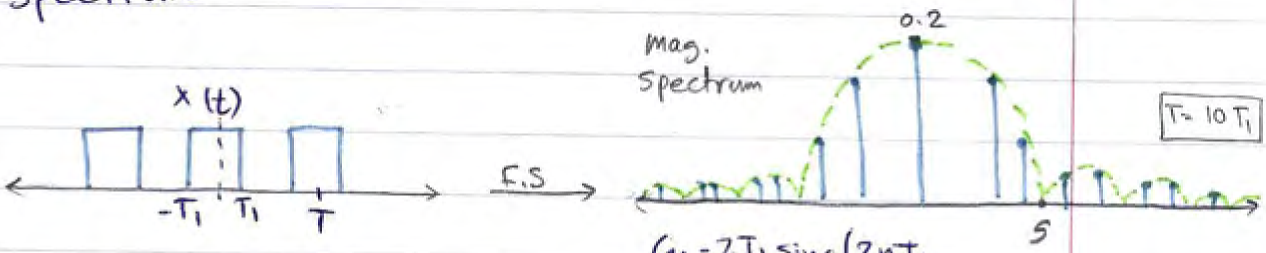
it's periodic, but I only care about the period ($0 \to T$) → (finite interval of time).

- in F.T we are interested of an infinite interval



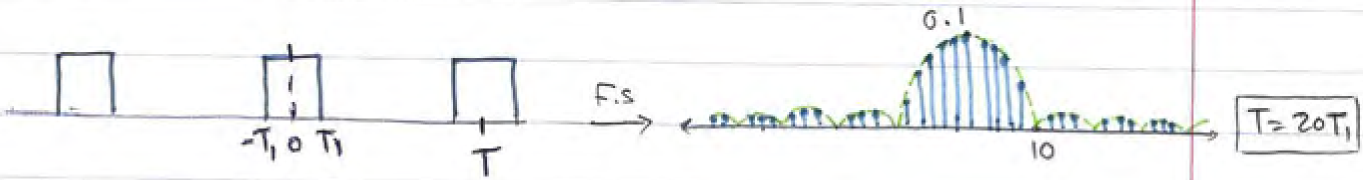
we need wo & <u>all</u> of it's multiples (continuous spectrum).

* The effect of changing T on the frequency spectrum



$C_n = \frac{2T_1}{T} \operatorname{sinc}\left(\frac{2nT_1}{T}\right)$

→ when T increases, the amplitude of each harmonic ~~is~~ decreases as well as the fundamental frequency $(w_0) = \frac{2\pi}{T}$, ~~the~~ the spacing between samples will decrease!



as $T \rightarrow \infty$
$a \rightarrow 0$ (amplitude)
spacing $\rightarrow 0$

we'll end up with nothing! yet we have everything!!

سحر عالم ال continuous

you have to deal with everything as a density!

non-periodic

→ freq domain?

← this can be represented using F.S! then take $\lim_{T\to\infty}$!

bring the periodic version of the function x(t) and choose ~~&~~ T to be large enough to avoid overlapping.

$$\lim_{T\to\infty} x_T(t) = x(t)!$$

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad \ldots ①$$

$$\& \; C_n = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)\, e^{-jn\omega_0 t}\, dt \quad \ldots ② \; ; \quad \omega_0 = \frac{2\pi}{T}$$

let $\boxed{X(n\omega_0) \triangleq T C_n}$ ... a

$$\therefore \; x_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} X(n\omega_0)\, e^{jn\omega_0 t} \quad \ldots ①'$$

$$X(n\omega_0) = \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t}\, dt \quad \ldots ②'$$

* when we multiply C_n by T before plotting it, the amplitude won't go to zero, while ω_0 goes to zero when T approaches ∞. so the spacing between adjacent lines in the line spectrum is spaced at zero, so we change ω_0 with $\underset{\text{Spacing}}{\Delta\omega}$

from ①'

$$x_T(t) = \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi}\right) X(n\Delta\omega)\, e^{jn\Delta\omega t}$$

$$x(t) = \lim_{T\to\infty} x_T(t) = \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Delta\omega\, X(n\Delta\omega)\, e^{jn\Delta\omega t}$$

as $T \to \infty$, $\Delta\omega \to 0$ and the spectrum becomes continuous

$$\therefore \; \boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega}$$

↳ Inverse F.T

the "Fourier transform Pair"

also

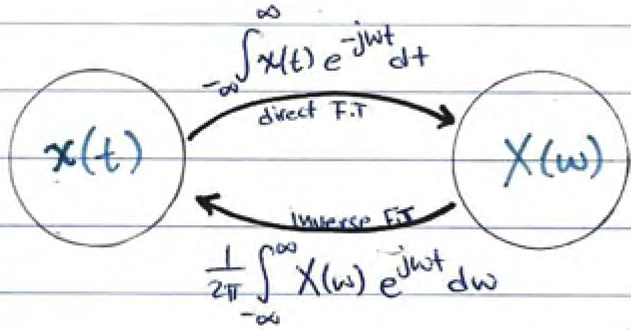
$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt} \quad \text{from ②'}$$

↳ direct F.T

$$X(\omega) = \mathcal{F}[x(t)] \qquad x(t) = \mathcal{F}^{-1}[X(\omega)]$$

$$\& \; \boxed{C_n = \frac{1}{T} X(\omega)\Big|_{\omega = n\omega_0}} \quad \text{from ⓐ}$$

2/4/2013

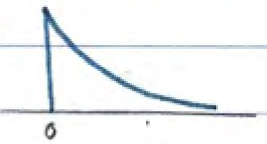


closed path :)

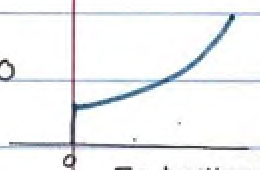
time d $\xrightarrow{F.S}$ Discrete frequency Spectrum.

time d $\xrightarrow{F.T}$ Continuous "Density" Spectrum.

ex:- ① $x(t) = e^{-at}\, u(t)$, if $a>0$



if $a<0$



F.s Lo/116*

- it doesn't satisfy dirichlet's conditions (area under the curve $=\infty$).

Sol:- $X(\omega) = \int_{\cancel{-\infty}\, 0}^{\infty} e^{-at}\, \cancel{u(t)}\, e^{-j\omega t}\, dt$

$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$

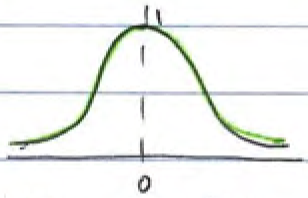
$= \dfrac{-e^{-(a+j\omega)t}}{a+j\omega} \Big|_{0}^{\infty}$

$= \dfrac{e^{-at}\, e^{-j\omega t}}{-(a+j\omega)} \Big|_{0}^{\infty}$ this will only converge if $a>0$.

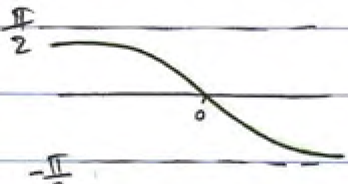
$= 0 - \left(\dfrac{1}{-(a+j\omega)}\right)$

$\boxed{X(\omega) = \dfrac{1}{a+j\omega}} = \dfrac{1}{\sqrt{a^2+\omega^2}}\, e^{-j\tan^{-1}\left(\frac{\omega}{a}\right)}$

let $a=1$



Mag. spectrum



phase spectrum

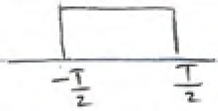
② $x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \underbrace{e^{-j\omega t}\Big|_{t=0}}_{\text{sifting}} = 1$$

③ $x(t) = \delta(t-T)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t-T) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=T} = e^{-j\omega T}$$

④ $x(t) = \text{Rect}\left(\frac{t}{T}\right)$

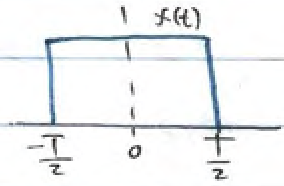


$$X(\omega) = \int_{-\frac{T}{2}}^{T/2} 1 . e^{-j\omega t} dt = -\frac{1}{j\omega}\left(e^{+j\omega \frac{T}{2}} - e^{+j\omega\frac{T}{2}}\right)$$

$$= T \,\text{sinc}\left(\frac{\omega T}{2\pi}\right)$$

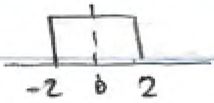
another way...

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} (\cos\omega t - j \underbrace{\sin\omega t}_{\int \text{odd} = 0 \;\to 0}) dt \quad \ldots$$

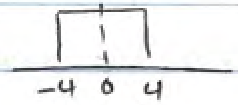


$\xrightarrow{F.T}$ $X(\omega) = T \,\text{sinc}\left(\frac{\omega}{\pi} . \frac{T}{2}\right)$

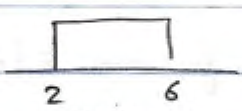
ex:



$\xrightarrow{F.T}$ $X(\omega) = 4 \,\text{sinc}\left(\frac{\omega}{\pi} . 2\right)$

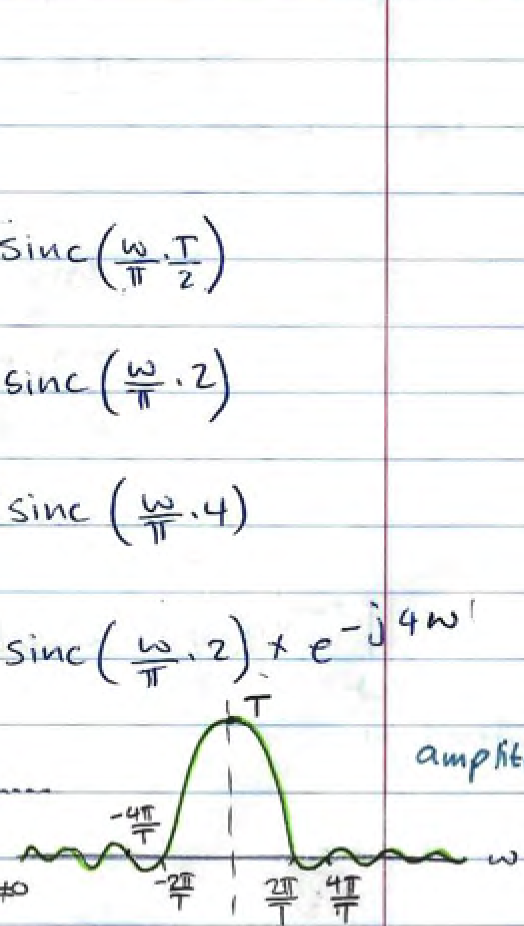


$\xrightarrow{F.T}$ $X(\omega) = 8 \,\text{sinc}\left(\frac{\omega}{\pi} . 4\right)$



$\xrightarrow{F.T}$ $X(\omega) = 4\,\text{sinc}\left(\frac{\omega}{\pi} . 2\right) \times e^{-j4\omega}$

Shifted!

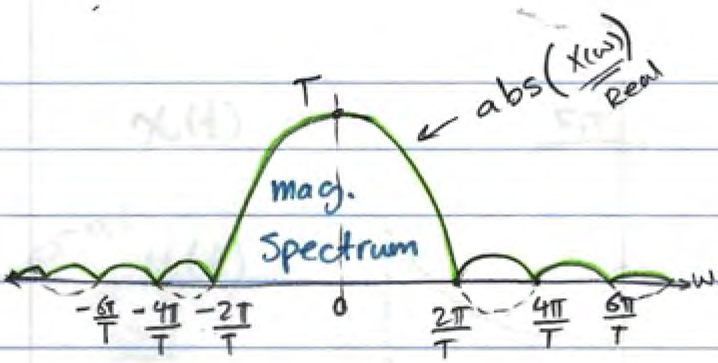


amplitude

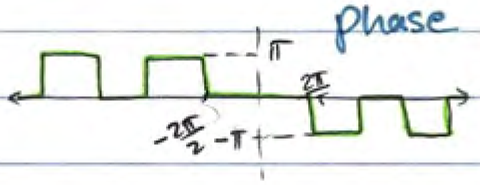
nulls of $X(\omega)$ at sinc(1), sinc(2), ...

$\text{sinc}(1) = \text{sinc}(2) = \text{sinc}(-1) = \underbrace{\text{sinc}(n)}_{\text{integer} \neq 0} = 0$

$\therefore$ nulls at $\omega = \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \pm\frac{6\pi}{T}, \ldots$



phase spectrum.



⑤ $x(t) = 1$

$$X(\omega) = \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-\infty}^{\infty} \Rightarrow \text{doesn't converge!}$$

$$\lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot e^{-j\omega t}\, dt = \lim_{\substack{T\to\infty \\ \omega\to 0}} T \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right)$$

= impulse! & its strength is the area under the sinc.

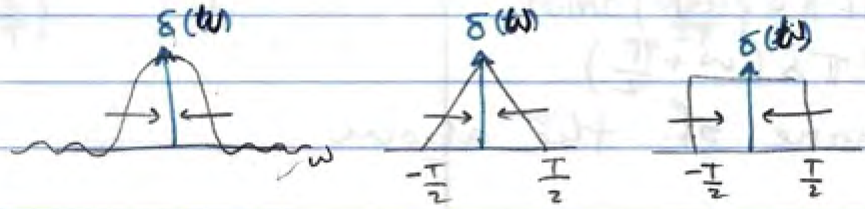
"Generalized F.T"

$$\boxed{\text{area under the sinc} = (\text{heigh} \times \text{first null})}$$

$T \times \frac{2\pi}{T} = \underline{\underline{2\pi}}$ strength (area) of the impulse

$x(t) \xrightarrow{F.T} X(\omega)$

→ $1 \longrightarrow 2\pi\delta(\omega)$



→ inverse of $\delta(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{2\pi}{2\pi} e^{j\omega t}\Big|_{\omega=0}$$

Be careful!

$$x(t) = \frac{1}{2\pi} \times 2\pi = 1$$

(6) $x(t) \xrightarrow{F.T} X(\omega)$

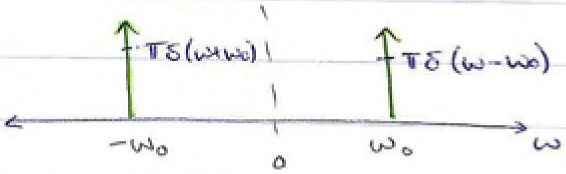
$jc \xrightarrow{F.T} 2\pi\,\delta(\omega-\omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega = e^{j\omega t}\Big|_{\omega=\omega_0} = e^{j\omega_0 t}$$

(7) $x(t) = e^{-j\omega_0 t} \xrightarrow{F.T} X(\omega) = 2\pi\,\delta(\omega+\omega_0)$

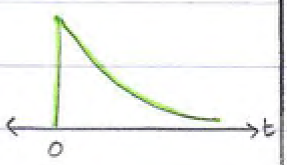
(8) $\cos(\omega_0 t) = \frac{e^{j\omega_0 t}}{2} + \frac{e^{-j\omega_0 t}}{2}$ Euler ...

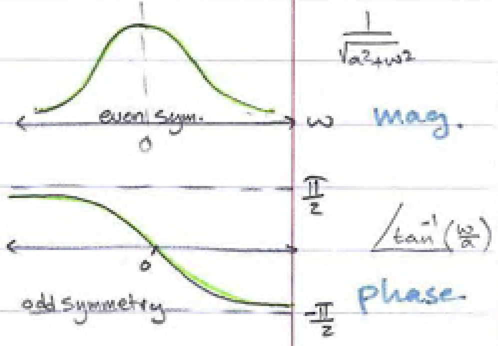
$$\xrightarrow{F.S} \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$



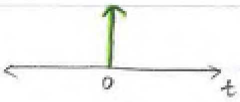
$x(t) \xrightarrow{F.T} X(\omega)$

$e^{-at}u(t)$, $a>0$ $\quad\longrightarrow\quad$ $\dfrac{1}{a+j\omega}$

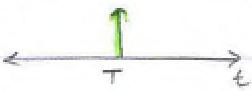


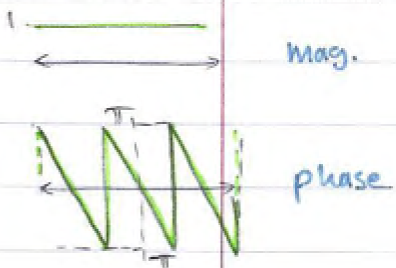


$\delta(t)$ $\quad\longrightarrow\quad$ 1

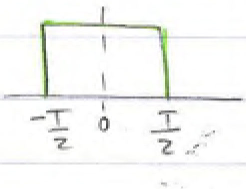


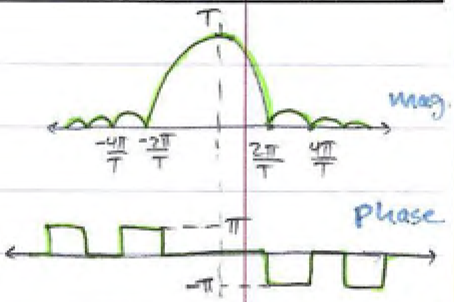
$\delta(t-T)$ $\quad\longrightarrow\quad$ $e^{-j\omega T}$





$Rect\left(\frac{t}{T}\right)$ $\quad\longrightarrow\quad$ $T\operatorname{sinc}\left(\frac{\omega T}{2\pi}\right)$





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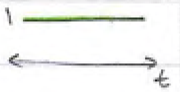
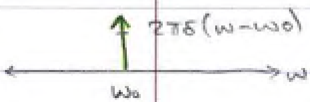
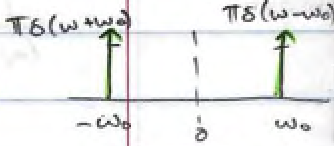
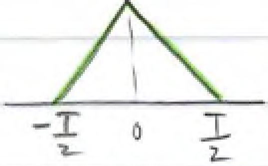
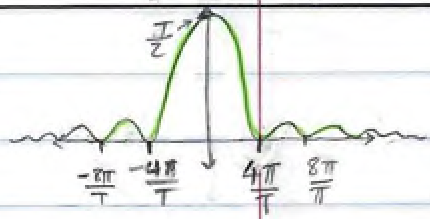
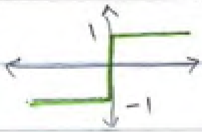
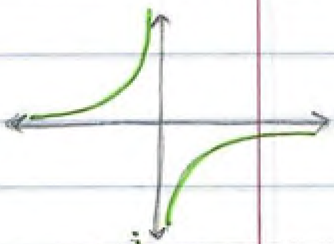
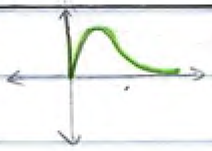
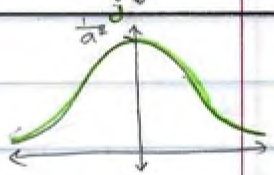
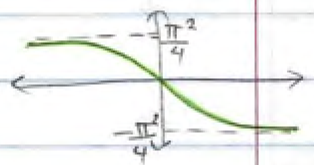
1 $\quad\longrightarrow\quad$ $2\pi\delta(\omega)$



$e^{j\omega_0 t}$ $\quad\longrightarrow\quad$ $2\pi\delta(\omega-2\omega_0)$



$e^{-j\omega_0 t}$ $\quad\longrightarrow\quad$ $2\pi\delta(\omega+\omega_0)$

$x(t)$	$\xrightarrow{F.T}$ $X(\omega)$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$ 
$\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$	$\sum_{n=-\infty}^{\infty} 2\pi C_n \cdot \delta(\omega-n\omega_0)$
→ to transform line spectrum to density spectrum.	
$tri\left(\frac{t}{T}\right)$ 	$\frac{T}{2}\, sinc^2\left(\frac{\omega T}{4\pi}\right)$ 
$sgn(t)$ 	$\frac{2}{j\omega}$, $DC=0$ $= -j\left(\frac{2}{\omega}\right)$ → 
$te^{-at}u(t)$ 	$\frac{1}{(a+j\omega)^2}$  
$\frac{j}{\pi t}$	$sgn(\omega)$
$e^{-\lvert t\rvert}$	$\frac{2}{1+\omega^2}$  real & even phase=0

$u(t) \xleftrightarrow{\mathcal{L}} \pi\delta(\omega)+\frac{1}{j\omega}$

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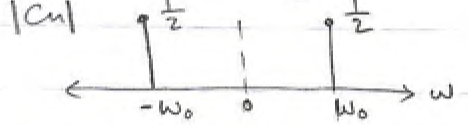
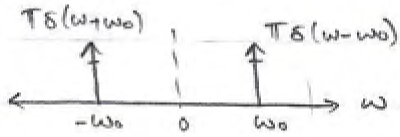
$$\cos(\omega_0 t) \xrightarrow{F.S} \underset{C_{-1}}{\frac{1}{2}} e^{-j\omega_0 t} + \underset{C_1}{\frac{1}{2}} e^{j\omega_0 t}$$



discrete

$\xrightarrow{F.T}$ $\pi\delta(\omega-\omega_0) + \pi\delta(\omega-\omega_0)$



continuous

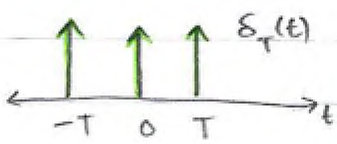
- for any periodic function $x(t)$

$\rightarrow x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

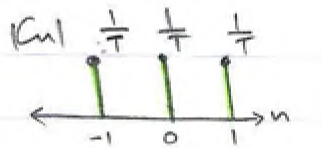
$$\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \xrightarrow{F.T} \sum_{n=-\infty}^{\infty} 2\pi C_n \delta(\omega-\omega_0)$$

* Use it to transform the double sided line spectrum to a density spectrum

ex:-

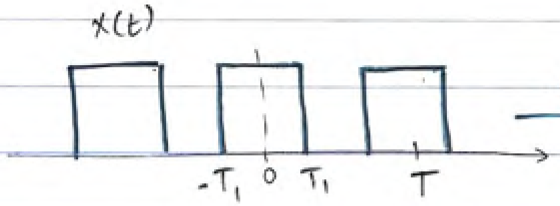


$\xrightarrow{F.S}$ $C_n = \frac{1}{T}$



$\downarrow$ F.T $\begin{cases} C_n \times 2\pi \\ \delta[n] \longrightarrow \delta(\omega) \end{cases}$

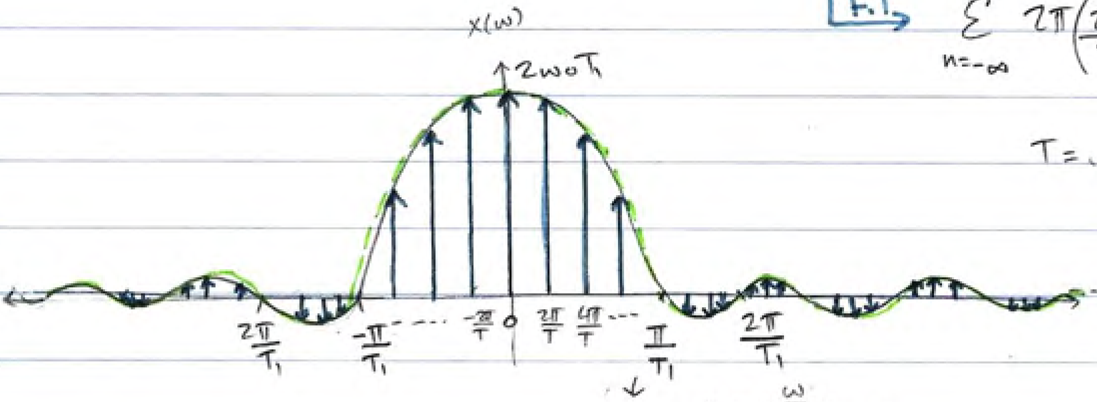
the density spectrum for Fourier transform of periodic functions only exists at $(n\omega_0)$ but yet it's still continuous (to conserve power).



$\xrightarrow{F.S}$ $C_n = \frac{2T_1}{T} \operatorname{Sinc}\left(\frac{2T_1 n}{T}\right)$

$\xrightarrow{F.T.}$ $\sum_{n=-\infty}^{\infty} 2\pi\left(\frac{2T_1}{T}\operatorname{sinc}\left(\frac{2T_1 n}{T}\right)\right)\delta(\omega - n\omega_0)$

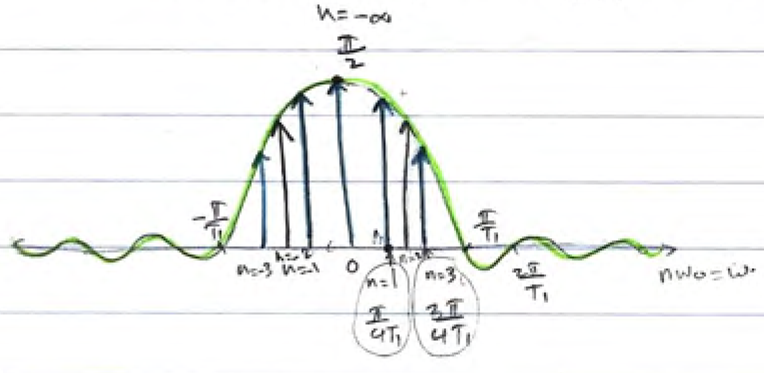
$T = \frac{2\pi}{\omega_0}$



$\operatorname{Sinc}\left(\frac{T_1 \overset{\omega}{\omega_0 n}}{\pi}\right) = 0$

let's take a perticular example $T = 8T_1$

$\therefore X(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi}{2}\operatorname{sinc}\left(\frac{n}{4}\right)\delta(\omega - n\omega_0)$

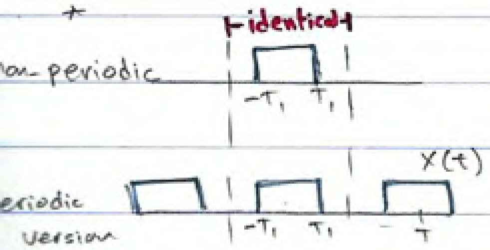


$\delta\left(\omega - n\frac{2\pi}{\cancel{8}T_1}\right) = \delta\left(\omega - \frac{n\pi}{4T_1}\right)$

$n = 1 \rightarrow \omega_0 = \frac{\pi}{4T_1}$

$n = 3 \rightarrow \omega_0 = \frac{3\pi}{4T_1}$

*

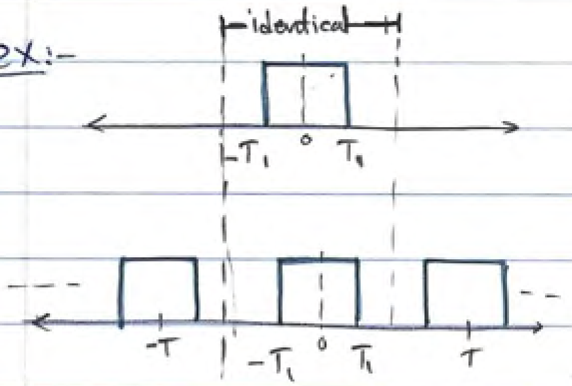


F.S

$$\boxed{\begin{aligned} X(n\omega_0) &\triangleq C_n T \\ \frac{1}{T} X(\omega)\Big|_{\omega = n\omega_0} &= C_n \end{aligned}}$$

* & from the fourier transform of the non-periodic version I can always get the fourier series for x(t)

ex:-



$\xrightarrow{F.T} X(\omega) = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$

$\downarrow$ F.S

$\longleftarrow \quad C_n = \frac{1}{T} X(\omega)\Big|_{w = n w_0}$

$= \frac{1}{T} \cdot 2T_1 \operatorname{sinc}\left(\frac{n w_0 T_1}{\pi}\right)\Big|_{w_0 = \frac{2\pi}{T}}$

$= \frac{2T_1}{T} \operatorname{sinc}\left(\frac{2nT_1}{T}\right)$

- You can always determine the fourier transform of the fourier series coeficients if the two functions are identical
  the periodic $x(t)$ & the non-periodic version of it.

* Properties of Fourier Transform:

1. linearity:

$x_1(t) \xrightarrow{F.T} X_1(\omega)$

$x_2(t) \xrightarrow{F.T} X_2(\omega)$

- $\alpha x_1(t) + \beta x_2(t) \xrightarrow{F.T} \alpha X_1(\omega) + \beta X_2(\omega)$

Proof:

$$\int_{-\infty}^{\infty} [\alpha x_1(t) + \beta x_2(t)]\, e^{-j\omega t}\, dt$$

integration is a linear operation

$$\therefore \alpha \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt + \beta \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{\infty} [\alpha x_1(t) + \beta x_2(t)]\, e^{-j\omega_0 t}\, dt = \alpha X_1(\omega) + \beta X_2(\omega) \quad \#$$

ex.:- $e^{j\omega_0 t} \xrightarrow{F.T} 2\pi\,\delta(\omega-\omega_0)$

$e^{-j\omega_0 t} \xrightarrow{F.T} 2\pi\,\delta(\omega+\omega_0)$

$$\underbrace{\frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}}_{\cos(\omega_0 t)} \xrightarrow{F.T} \frac{1}{2}\cdot 2\pi\,\delta(\omega-\omega_0) + \frac{1}{2}\cdot 2\,\delta(\omega+\omega_0)$$

2. Time Shifting:

$x(t) \xrightarrow{F.T} X(\omega)$

$x(t \pm t_0) \xrightarrow{F.T} e^{\pm j\omega t_0}\, X(\omega)$

proof:-

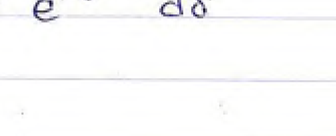
$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(\underbrace{t-t_0}_{\delta})\, e^{-j\omega t}\, dt$$

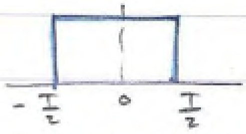
$$= \int_{-\infty}^{\infty} x(\delta)\, e^{-j\omega(\delta+t_0)}\, d\delta$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\delta)\, e^{-j\omega\delta}\, d\delta$$

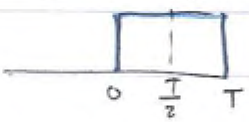
$$= e^{-j\omega t_0}\, X(\delta)$$

ex.:- $\mathrm{rect}\left(\frac{t - T/2}{T}\right)$





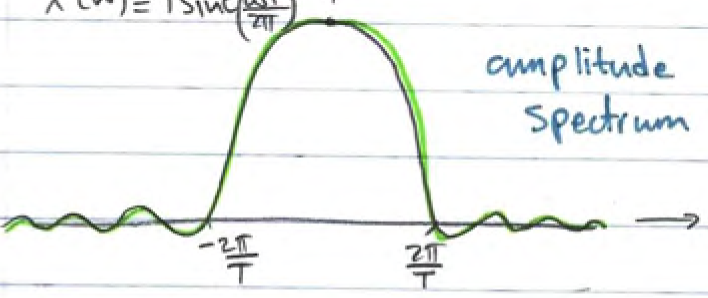
$\xrightarrow{F.T} T\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$

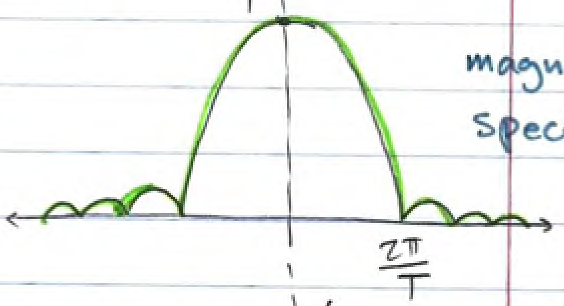


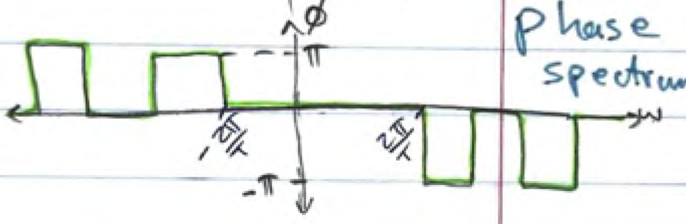
$\xrightarrow{F.T} \underbrace{T\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)}_{\text{mag.}}\, e^{-j\frac{\omega T}{2}}$ $\varnothing$

no shift in magnitude

for $X(w) = T\,\mathrm{sinc}\left(\frac{wT}{2\pi}\right)$

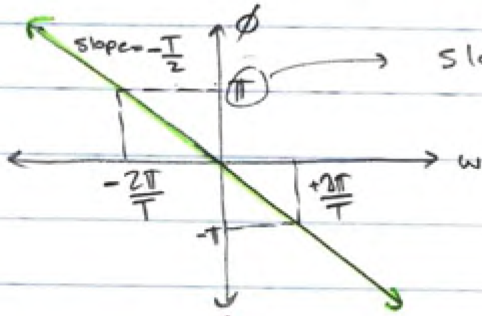






for $X(w) = T\,\mathrm{sinc}\left(\frac{wT}{2\pi}\right) e^{-j\frac{wT}{2}}$

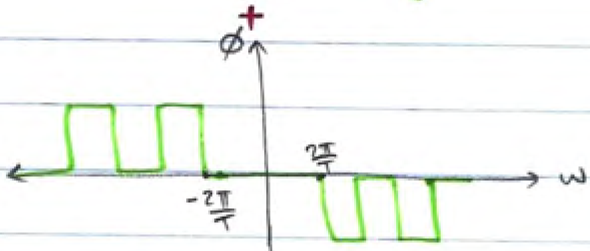
* mag. spectum doesn't change
* $\phi = -\frac{T}{2}\omega$; linear relationship with ω



$\text{slope} = \frac{\Delta y}{\left(\frac{2\pi}{T}\right)} = -\frac{T}{2}$

$\Delta x \quad \therefore y = \pi$

+



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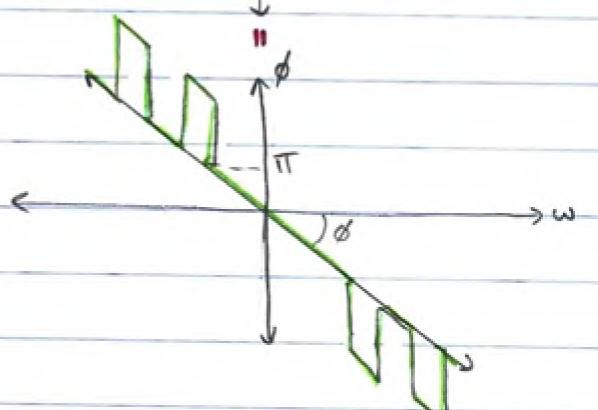


* shifting in time will always introduce a linear relationship with ω.

7/4/2013

3. Frequency Shifting (Modulation).

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

duality

$$x(t \pm t_0) \xleftrightarrow{F.T} e^{\pm j\omega_0 t} X(\omega) \qquad \text{time shifting}$$

$$e^{\mp j\omega_0 t} x(t) \xleftrightarrow{F.T} X(\omega \pm \omega_0) \qquad \text{Frequency shifting}$$

Proof:

$$\int_{-\infty}^{\infty} \left(x(t)\, e^{j\omega_0 t}\right) e^{-j\omega t}\, dt =$$

$$\int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt = X(\omega - \omega_0)$$

since $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$X(\omega-\omega_0) = \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

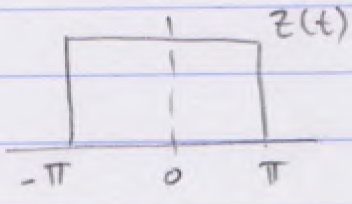
how to use this property

ex:

$$x(t) = \begin{cases} e^{j10t}, & |t| \leq \pi \\ 0, & o.w \end{cases} \quad , \text{find } \mathcal{F}\{x(t)\}?$$

$$x(t) = e^{j10t} \underbrace{\mathrm{Rect}\left(\frac{t}{2\pi}\right)}_{z(t)}$$

$$e^{j10t} z(t) \xleftrightarrow{F.T} Z(\omega - 10) = 2\pi\, \mathrm{sinc}(\omega - 10)$$



$$\xleftrightarrow{F.T} Z(\omega) = 2\pi\, \mathrm{Sinc}\left(\frac{\omega}{\pi}\cdot\pi\right)$$

$$Z(\omega) = 2\pi\, \mathrm{sinc}(\omega)$$

- Amplitude modulation (AM): is changing the amplitude of one signal by multiplying it by another one.

ex: $\frac{1}{2}e^{jw_0t}\,x(t) \xleftrightarrow{F.T} \frac{1}{2}X(w-w_0)$ ...①

because of linearity

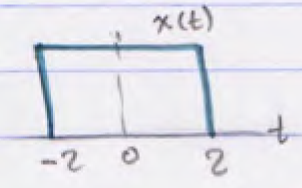
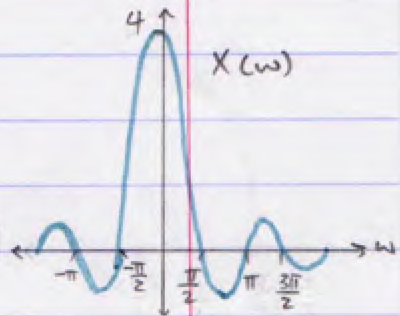
$\frac{1}{2}e^{-jw_0t}\,x(t) \xleftrightarrow{F.T} \frac{1}{2}X(w+w_0)$ ...②

① + ②

$$\boxed{\cdot\ \cos(w_0t)\,x(t) \xleftrightarrow{F.T} \frac{1}{2}X(w+w_0)+\frac{1}{2}X(w-w_0)}$$

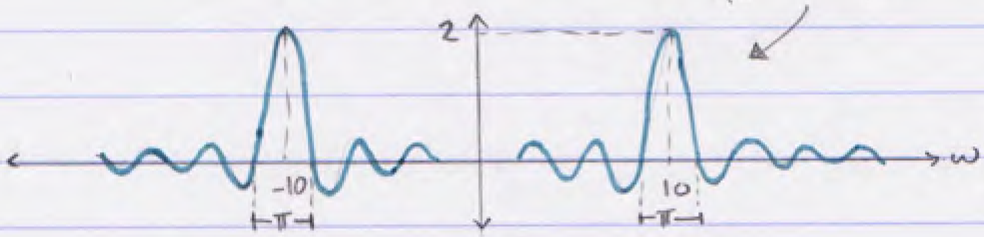
ex2: $\mathcal{F}\{\underbrace{Rect\left(\frac{t}{4}\right)}_{x(t)}\cos(\underbrace{10}_{w_0}t)\} = ?$

sol:

rect$\left(\frac{t}{4}\right)$  $\xleftrightarrow{F.T}$ $4\,\text{sinc}\left(\frac{2w}{\pi}\right)$ 

but.. $x(t)\cos w_0t \xleftrightarrow{F.T} \frac{1}{2}X(w+w_0)+\frac{1}{2}X(w-w_0)$

$\therefore$ rect$\left(\frac{t}{4}\right)\cos(10t) \xrightarrow{F.T} 2\,\text{Sinc}\left(\frac{2}{\pi}(w+10)\right)+2\,\text{sinc}\left(\frac{2}{\pi}(w-10)\right)$



4- time scaling:

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

- $x(at) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ for any real valued scaling constant (a)

Proof:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

let $u = at$
$du = a\,dt$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(u)\, e^{-j\left(\frac{\omega}{a}\right)u}\, du \quad , a > 0$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right) \quad * \quad , \text{for } a>0 \ (+ve)$$

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt \quad , a<0$$

$u = at$, a is -ve ∴ limits on the integral will be reversed
$du = a\,dt$

$$= \frac{1}{a}\int_{\infty}^{-\infty} x(u)\, e^{-j\left(\frac{\omega}{a}\right)u}\, du$$

س عكس حدود التكامل

$$= -\frac{1}{a} X\left(\frac{\omega}{a}\right) \quad , \text{since } a \text{ is negative}$$

$\left(-\frac{1}{a}\right)$ will be always +ve

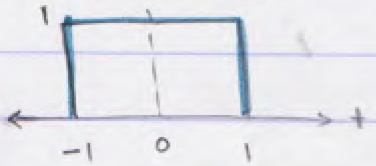
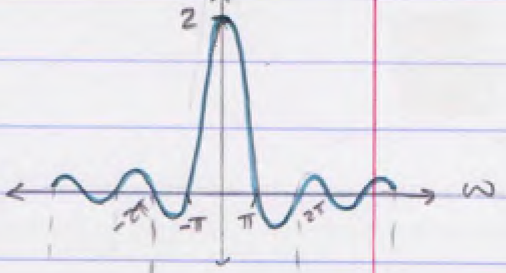
$$\therefore \mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \quad , \text{for all values of (a)}$$

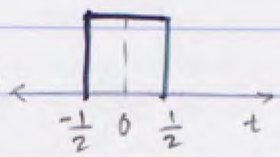
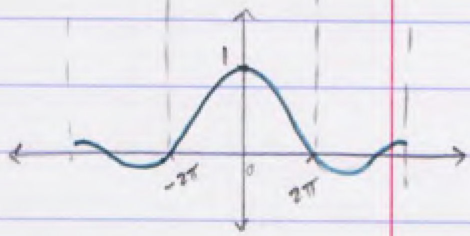
↳ Notice that the amplitude also gets scaled.

also...

$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \xleftrightarrow{F.T} X(a\omega) \quad \text{... frequency scaling.}$$

ex:

$x(t) = \text{rect}\left(\frac{t}{2}\right)$  $\xleftrightarrow{F.T}$ $2\,\text{sinc}\left(\frac{\omega}{\pi}\right)$ 

$x(2t) = \text{rect}(t)$  $\xleftrightarrow{F.T}$ $\frac{1}{2}\left(2\,\text{sinc}\left(\frac{\omega}{2\pi}\right)\right)$ 

$\text{sinc}\left(\frac{\omega}{2\pi}\right)$

5. time reflection:

a special case of time scaling is when $a=-1$

- $x(-t) \xleftrightarrow{F.T} X(-\omega) = \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right)$

reflection in t-domain $\longleftrightarrow$ reflection in freq-domain

6. time Differentiation

$x(t) \xleftrightarrow{F.T} X(\omega)$

- $\frac{d\,x(t)}{dt} \xleftrightarrow{F.T} j\omega\, X(\omega)$
- $\frac{d^n x(t)}{dt^n} \xleftrightarrow{F.T} (j\omega)^n X(\omega)$

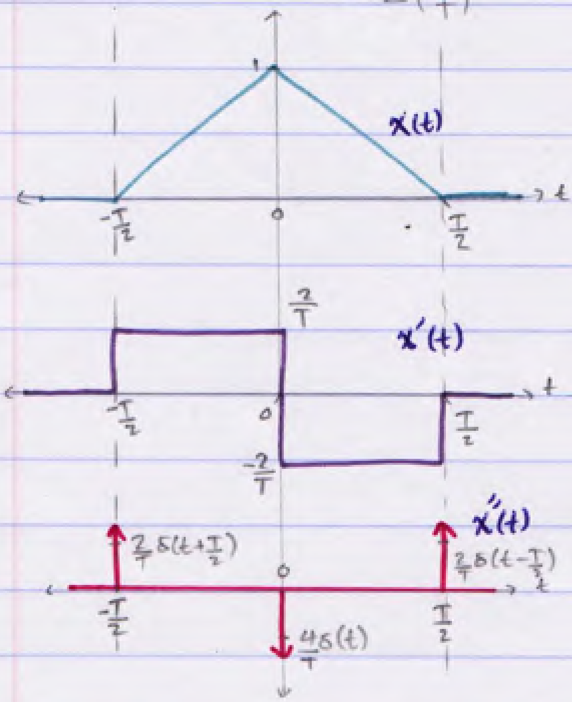
proof:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\underbrace{\left(\frac{d\,x(t)}{dt}\right)}_{z(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{j\omega\, X(\omega)}_{Z(\omega)}\, e^{j\omega t}\, d\omega$$

ex:- Using the differentiation property, find $\mathcal{F}\{\mathrm{tri}(\frac{t}{T})\}$
$\equiv \Delta(\frac{t}{T})$

* methods used to solve examples might not be the best nor the only ways to solve them; sometimes they are only used to highlight the properties!



$$x(t) = \mathrm{tri}\left(\frac{t}{T}\right)$$

$$x'(t) = \frac{2}{T}\left[u(t+\tfrac{T}{2}) - u(t)\right] - \frac{2}{T}\left[u(t) - u(t-\tfrac{T}{2})\right]$$

$$x''(t) = \frac{2}{T}\left[\delta(t+\tfrac{T}{2}) - \delta(t) - \delta(t) + \delta(t-\tfrac{T}{2})\right]$$

$$= \frac{2}{T}\delta(t+\tfrac{T}{2}) - \frac{4}{T}\delta(t) + \frac{2}{T}\delta(t-\tfrac{T}{2})$$

$$\mathcal{F}\{x''(t)\} = \mathcal{F}\left\{\frac{2}{T}\delta(t+\tfrac{T}{2}) - \frac{4}{T}\delta(t) + \frac{2}{T}\delta(t-\tfrac{T}{2})\right\}$$

$$-\omega^2 X(\omega) = \frac{2}{T}e^{j\omega\frac{T}{2}} - \frac{4}{T} + \frac{2}{T}e^{-j\frac{T}{2}\omega}$$

solving for $X(\omega)$...

$$X(\omega) = -\frac{1}{\omega^2}\cdot\frac{2}{T}\Big[\underbrace{e^{j\omega\frac{T}{2}} - 2 + e^{-j\omega\frac{T}{2}}}_{(2\cos(\omega\frac{T}{2}) - 2) \equiv -2\left[1-\cos(\frac{\omega T}{2})\right]}\Big]$$

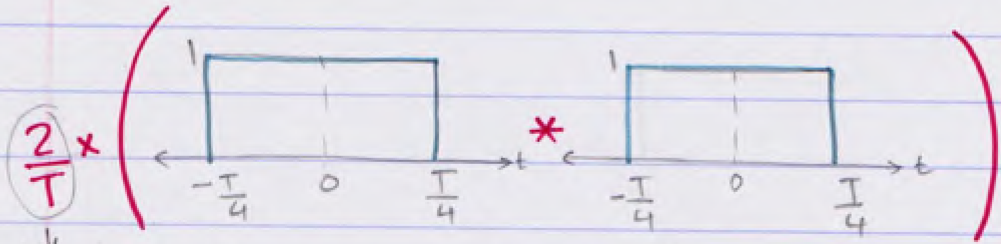
** $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$X(\omega) = \frac{4}{\omega^2 T}\left[2\sin^2\left(\frac{\omega T}{4}\right)\right] = \frac{8}{\omega^2 T}\sin^2\left(\frac{\omega T}{4}\right)\times\frac{(4)^2 T}{(4)^2 T} = \frac{T}{2}\frac{\sin^2(\frac{\omega T}{4})}{(\frac{\omega T}{4})^2}$$

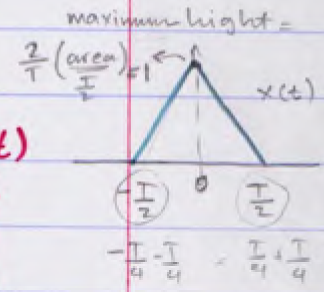
$$X(\omega) = \frac{T}{2}\mathrm{sinc}^2\left(\frac{\omega T}{4\pi}\right)$$

another way :- easier :).

* we can write $x(t)$ as a result of a convolution between 2 rects :



$= x(t)$



to adjust the maximum height to 1

F.T

F.T of the 1st rect   F.T of the 2nd rect

$$\frac{2}{T}\left[\frac{T}{2}\,\text{sinc}\left(\frac{wT}{4\pi}\right) \times \frac{T}{2}\,\text{sinc}\left(\frac{wT}{4\pi}\right)\right]$$

* in t-domain ≡ × in freq. domain.

F.T

$$\frac{T}{2}\,\text{sinc}^2\left(\frac{wT}{4\pi}\right) = \mathcal{F}\{x(t)\}$$ ✱

$\rightarrow$ $X(w) = \mathcal{F}\{x(t)\}$

& $X(w) = \frac{1}{jw}\,\mathcal{F}\{x'(t)\}$ ....①

& $X(w) = -\frac{1}{w^2}\,\mathcal{F}\{x''(t)\}$

but what happens when $w=0$?

* The DC component must appear at $w=0$.
* Differentiating $x(t)$ destroys its DC component !
  that's why equ ① applies only to signals with DC component $=0$ (zero average value).
* functions with non-zero DC component have to be modified before applying the differentiation property.
* in the previous example, the sinc is well defined at $w=0$ so we didn't lose the DC component.

ex2 :-

$u(t) \xleftrightarrow{FT} \mathcal{U}(\omega) = ?!$

sol :-

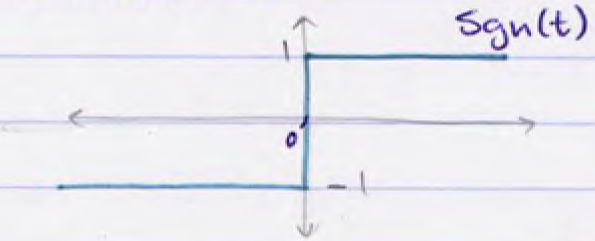
$\mathcal{F}\left\{\frac{d\,u(t)}{dt}\right\} = \mathcal{F}\{\delta(t)\}$

$j\omega\, \mathcal{U}(\omega) = 1$

$\mathcal{U}(\omega) = \frac{1}{j\omega}$, $\mathcal{U}(0)$ is undefined !!

** don't be tempted to use this property unless the DC component = 0 , or it's well defined at zero.
$X(\omega)$

so, in this case we have to make some modification:



$sgn(t) = 2u(t) - 1$ ... ⓐ

* Since sgn(t) is an odd function, DC comp. = 0

$\underset{x(t)}{sgn(t)} = 2u(t) - 1$

$\mathcal{F}\left\{\frac{d\,x(t)}{dt}\right\} = \mathcal{F}\{2\delta(t)\}$

$j\omega\, X(\omega) = 2$

$\boxed{X(\omega) = \frac{2}{j\omega}}$, & we know that @ $\omega = 0$
$X(\omega) = 0$ (because $x(t)$ is odd)

from equ. ⓐ :

$$\mathcal{F}\{u(t)\} = \mathcal{F}\{\frac{1}{2} + \frac{1}{2}\,\text{sign}(t)\}$$

$$= \frac{\cancel{2}\pi}{\cancel{2}}\,\delta(\omega) + \frac{1}{j\omega}$$

$$\boxed{U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}}$$

*** The DC component here appeared as an impulse at $w=0$!

In general DC component in freq. domain appears as:-

① an impulse at $w=0$.

② a value!

خميس
7/4/2013

7. frequency differentiation:-

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

$$\frac{d\,x(t)}{dt} \xleftrightarrow{F.T} j\omega X(\omega)$$

- $$t\,x(t) \xleftrightarrow{F.T} j\,\frac{d\,X(\omega)}{d\omega}$$

Proof:-

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\underbrace{\frac{d\,X(\omega)}{d\omega}}_{Z(\omega)} = \int_{-\infty}^{\infty} \underbrace{-jt\,x(t)}_{Z(t)}\, e^{-j\omega t}\, dt$$

$$\left(-jt\,x(t) \xleftrightarrow{F.T} \frac{d\,X(\omega)}{d\omega} \right) \times j$$

$$\boxed{t\,x(t) \xleftrightarrow{F.T} j\,\frac{d\,X(\omega)}{d\omega}} \quad *$$

in general:

$$\boxed{t^n x(t) \xleftrightarrow{F.T} j^n\,\frac{d^n X(\omega)}{d\omega^n}}$$

ex: $\mathcal{F}\{\underbrace{t}_{t}\, \underbrace{e^{-at}\,u(t)}_{x(t)}\} = ?$ $a>0.$

$$\mathcal{F}\{e^{-at}\,u(t)\} = \boxed{\frac{1}{a+j\omega}} \leftarrow X(\omega)$$

since $\mathcal{F}\{t\,x(t)\} = j\,\dfrac{d\,X(\omega)}{d\omega}$

$$\therefore \mathcal{F}\{t e^{-at}\,u(t)\} = j\,\frac{d}{dt}\left[(a+j\omega)^{-1}\right]$$

$$= \frac{1}{(a+j\omega)^2}$$

8. time integration:

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

- $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F.T} \dfrac{X(\omega)}{j\omega} + \underbrace{\pi X(0)\,\delta(\omega)}_{\text{constant (DC) as an impulse.}}$

Proof: $$\int_{-\infty}^{\infty} \boxed{\int_{-\infty}^{\cancel{t}\ \infty} x(\tau)\,d\tau\ u(t-\tau)}\ e^{-j\omega t}\,dt \quad \ldots \circledast$$

** Idea: Use $u(t-a)$ to change the limits of your integral!

$$\hookrightarrow \int_{-\infty}^{\infty} x(\tau)\,u(\tau)\,d\tau = \int_{0}^{\infty} x(\tau)\,d\tau$$

$$\hookrightarrow \int_{-\infty}^{\infty} x(\tau)\,u(-\tau)\,d\tau = \int_{-\infty}^{0} x(\tau)\,d\tau$$

back to $\circledast$

$$\int_{-\infty}^{\infty} x(\tau)\Big(\underbrace{\int_{-\infty}^{\infty} u(t-\tau)\,e^{-j\omega t}\,dt}_{\substack{\text{F.T of a shifted } u(t) \\ = e^{-j\omega\tau}U(\omega)}}\Big)\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\,\underbrace{U(\omega)}_{\text{constant}}\,e^{-j\omega\tau}\,d\tau$$

$$= U(\omega)\cdot X(\omega)$$

$$= \Big(\pi\delta(\omega) + \frac{1}{j\omega}\Big)\,X(\omega)$$

$$= \frac{X(\omega)}{j\omega} + \Big(\underbrace{\pi X(0)\,\delta(\omega)}_{\text{Sifting}}\Big) \quad \text{DC comp.}$$

ex: Using the integration property, find $\mathcal{F}\{u(t)\} = ?$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\int_{-\infty}^{t} \underbrace{\delta(\tau)}_{x(t)\,\rightarrow\,X(\omega)=1} d\tau\right\}$$

$$= \frac{1}{j\omega} + \pi \overset{1}{\cancel{X(0)}}\, \delta(\omega)$$

$$= \frac{1}{j\omega} + \pi\delta(\omega) \quad \divideontimes$$

9. Convolution :

$x(t) \xleftrightarrow{F.T} X(\omega)$

$h(t) \xleftrightarrow{F.T} H(\omega)$

- $x(t) * h(t) \xleftrightarrow{F.T} X(\omega) \cdot H(\omega)$

proof:

$$\int_{-\infty}^{\infty} \underline{x(t) * h(t)}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt\right) d\tau$$

$$= H(\omega) \cdot X(\omega) \quad \divideontimes$$

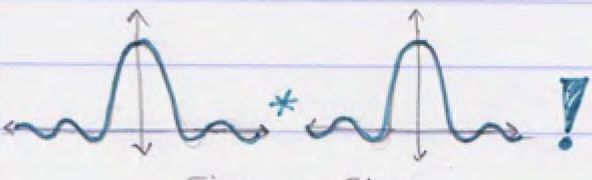
$x(t)$ → $\boxed{h(t)}$ → $y(t)$

e.g: Sinc(t) — Sinc(t) — Sinc(t) * Sinc(t)

$\Longleftrightarrow$

$X(\omega)$ → $\boxed{H(\omega)}$ → $Y(\omega)$

Rect — Rect — Rect . Rect



Sinc * Sinc
in t-domain

Rect x Rect
in freq-domain

Note:

integrator

$$x(t) \longrightarrow \boxed{h(t)=u(t)} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\cancel{\infty}\,t} x(\tau)\ \cancel{u(t-\tau)}\, d\tau$$

$\Rightarrow$ When you have the unit step function as an impulse response, it acts as an integrator.

Proof:-

$\rightarrow$ $x(t) * u(t) \xleftrightarrow{F.T} X(\omega) \cdot U(\omega)$

$\qquad\qquad\qquad\qquad\quad X(\omega) \cdot \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$

$\qquad |||$

$\rightarrow$ $\int_{-\infty}^{\infty} x(\tau)\,d\tau \xleftrightarrow{F.T} X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$

10. Multiplication:

$$x(t) \xleftrightarrow{F.T} X(\omega)$$
$$h(t) \xleftrightarrow{F.T} H(\omega)$$
$$x(t) * h(t) \xleftrightarrow{F.T} X(\omega) \cdot H(\omega)$$

- $x(t) \times p(t) \xleftrightarrow{F.T} \frac{1}{2\pi} X(\omega) * P(\omega)$

Proof:

$$\int_{-\infty}^{\infty} x(t)\, \underline{p(t)}\, e^{-j\omega t}\, dt$$

p(t) can be also written as

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\delta)\, e^{j\delta t}\, d\delta$$

$\omega = \delta$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} P(\delta) e^{j\delta t}\, d\delta\right) e^{-j\omega t}\, dt$$

$e^{-j(\omega-\delta)t}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\delta)\left(\int_{-\infty}^{\infty} x(t)\, \underbrace{e^{-j\omega t}\, e^{j\delta t}\, dt}_{X(\omega-\delta)}\right) d\delta$$

$$= \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} P(\delta)\, X(\omega-\delta)\, d\delta}_{\text{convolution}} = \frac{1}{2\pi} P(\omega) * X(\omega) \quad *$$

frequency shifting properties.

from

ex: $x(t)\cos(\omega_0 t) \overset{F.T}{\longleftrightarrow} \frac{1}{2}X(\omega+\omega_0)+\frac{1}{2}X(\omega-\omega_0)$ ← AM from before

another way $x(t).\underbrace{\cos(\omega_0 t)}_{p(t)} \overset{F.T}{\longleftrightarrow} \frac{1}{2\cancel{\pi}}X(\omega) * [\cancel{\pi}\delta(\omega+\omega_0)+\cancel{\pi}\delta(\omega-\omega_0)]$

$= \frac{1}{2}X(\omega+\omega_0)+\frac{1}{2}X(\omega-\omega_0)$ *

* Symmetrical Properties:

* assume $x(t)$ is real → ∴ $x(t)=x^*(t)$ ... ①

$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$$X^*(\omega)=\int_{-\infty}^{\infty}\underbrace{x^*(t)}_{x(t)\text{ from ①}}e^{+j\omega t}dt$$

$X^*(\omega)=X(-\omega)$ ...② , when $x(t)$ is real

- $X^*(\omega)=|X(\omega)|e^{-j\angle X(\omega)}$
- $X(-\omega)=|X(-\omega)|e^{j\angle X(-\omega)}$

but from ②

$X^*(\omega)=X(-\omega)$

$|X(\omega)|e^{-j\angle X(\omega)}=|X(-\omega)|e^{j\angle X(-\omega)}$

↳ $|X(\omega)|=|X(-\omega)|$ even symmetry in magnitude spectrum.

$-\angle X(\omega)=\angle X(-\omega)$ odd symmetry in phase spectrum.

* assume $x(t)$ is pure imaginary → ∴ $x(t)=-x^*(t)$ ...①'

$|X(\omega)|=-|X(-\omega)|$ odd ...

$\angle X(\omega)=\angle X(-\omega)$ even ...

- assume $x(t)$ is Real & Even

  ↳ ∴ $x(t) = x^*(t)$

  & ↳ $x(t) = x(-t)$

  $X^*(\omega) = X(-\omega)$ because $x(t)$ is real (from before).

$$X^*(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

→ let $T = -t$

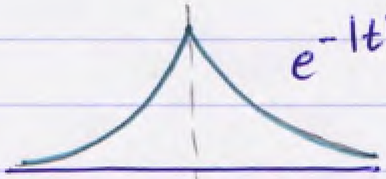
$$= -\int_{\infty}^{-\infty} X(-T)\, e^{-j\omega T}\, dT$$

$$= \int_{-\infty}^{\infty} x(T)\, e^{-j\omega T}\, dT = X(\omega)$$

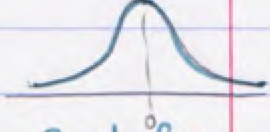
∴ $X^*(\omega) = X(\omega) = X(-\omega)$ when $x(t)$ is Real & even.

$x(t)$	$X(\omega)$
• Real & Even $x(t) = x^*(t)$ & $x(t) = x(-t)$	Real & Even $X^*(\omega) = X(\omega) = X(-\omega)$
• Real & Odd $x(t) = -x^*(t)$ $x(t) = -x(-t)$	Imaginary & odd
• Imag. & Even $x(t) = -x^*(t)$ $x(t) = x(-t)$	Imaginary & even
• Imag. & odd $x(t) = -x^*(t)$ $x(t) = -x(t)$	Real & odd

ex:-



$e^{-|t|} \xleftrightarrow{F.T} \dfrac{2}{1+w^2}$



Real & Even

Real & even

Out of curiosity:

what's the trigonometric form of the Fourier transform?

→ in Fourier series:-

- $C_n = \frac{1}{T} \int_T x(t)\, e^{-jnw_0 t}\, dt$ → exponential.
- $a_n = \frac{2}{T} \int_T x(t) \cos(nw_0 t)\, dt$

  $b_n = \frac{2}{T} \int_T x(t) \sin(nw_0 t)\, dt$ trigonometric

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(nw_0 t + \theta_n) \quad , x(t) \text{ is real}$$

→ in Fourier transform:-

↳ when $x(t)$ is real → $X^*(w) = X(-w)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw$$

نوزع التكامل

$-\infty \to 0 \,/\, 0 \to \infty$

$$= \underbrace{\frac{1}{2\pi} \int_{-\infty}^{0} X(w) e^{jwt}\, dw}_{\text{switch } (w) \text{ with } (-w)} + \int_0^{\infty} X(w) e^{jwt}\, dw$$

$$= \frac{1}{2\pi} \left(\int_0^{\infty} X(-w)\, e^{-jwt}\, dw \right) + \int_0^{\infty} X(w) e^{jwt}\, dw$$

$X(w) = |X(w)|\, e^{j \angle X(w)}$

$X(-w) = |X(-w)|\, e^{j \angle X(-w)}$

$X^*(w) = |X(w)|\, e^{-j \angle X(w)}$

$$= \frac{1}{2\pi} \int_0^{\infty} |X(-\omega)| e^{j\angle X(-\omega)} e^{-j\omega t} d\omega + \int_0^{\infty} |X(\omega)| e^{j\angle X(\omega)} e^{j\omega t} d\omega$$

$\Rightarrow$ because $x(t)$ is real:

$X(-\omega) = X^*(\omega)$

$\hookrightarrow |X(\omega)| = |X(-\omega)|$

$\underline{\measuredangle X(\omega)} = -\underline{\measuredangle X(-\omega)}$

$$= \frac{1}{2\pi} \int_0^{\infty} \left[|X(\omega)| e^{-\underline{\measuredangle X(\omega)}} e^{-j\omega t} d\omega + |X(\omega)| e^{j\underline{\measuredangle X(\omega)}} e^{j\omega t} \right] d\omega$$

$$= \frac{1}{2\pi} \int_0^{\infty} |X(\omega)| \left[e^{-j(X(\omega) + \omega t)} + e^{j(\underline{\measuredangle X(\omega)} + \omega t)} \right] d\omega$$

$$x(t) = \boxed{\frac{1}{\cancel{2}\pi} \int_0^{\infty} |X(\omega)| \cdot \cancel{2} \cos(\omega t + \underline{\measuredangle X(\omega)}) \, d\omega}$$

Trigonometric fourier transform
inverse

11. Duality:

$x(t) \xleftrightarrow{F.T} X(\omega)$

$X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)$

e.g: $\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{F.T} T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right)$

$T\,\text{sinc}\left(\frac{tT}{2\pi}\right) \xleftrightarrow{F.T} 2\pi\,\text{rect}\left(\frac{\omega}{T}\right)$ → the -ve cancels because rect is an even function.

Proof:-

- replace (ω) with (δ)
- replace (t) with (-ω)

$$2\pi x(\overset{-\omega}{\cancel{t}}) = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} X(\delta) e^{-j\delta \overset{\omega}{\cancel{t}}} d\delta$$

$$2\pi x(-\omega) = \underbrace{\int_{-\infty}^{\infty} X(\overset{t}{\cancel{\delta}}) e^{-j\overset{t}{\cancel{\delta}}\omega} d\overset{t}{\cancel{\delta}}}_{\mathcal{F}\{X(\overset{t}{\cancel{\delta}})\}}$$

$$\mathcal{F}\{X(t)\} = 2\pi\, x(-\omega) \quad \#$$

ex: $x(t) = \frac{2}{1+t^2}$, $\mathcal{F}\{x(t)\} = ?!$

Sol: $\mathcal{F}\{\underbrace{e^{-|t|}}_{x(t)}\} = \underbrace{\frac{2}{1+\omega^2}}_{X(\omega)}$;

- $x(t) = e^{-|t|} \longrightarrow$ • $x(-\omega) = e^{-|\omega|}$
- $X(\omega) = \frac{2}{1+\omega^2} \longrightarrow$ • $X(t) = \frac{2}{1+t^2}$

$$\therefore \mathcal{F}\{\underbrace{\frac{2}{1+t^2}}_{X(t)}\} = 2\pi \underbrace{e^{-|\omega|}}_{x(\omega)}$$

ex2: $X(\omega) = \mathrm{sgn}(\omega)$, $\mathcal{F}^{-1}\{X(\omega)\} = ?$

Sol:- $\mathrm{sgn}(t) \overset{F.T}{\longleftrightarrow} \frac{2}{j\omega}$

$$\frac{1}{2\pi} \cdot \frac{2}{jt} \overset{F.T}{\longleftrightarrow} \frac{\cancel{2\pi}}{\cancel{2\pi}} \mathrm{sgn}(-\omega) \longrightarrow \text{sgn is odd}$$

$\mathrm{sgn}(-\omega) = -\mathrm{sgn}(\omega)$

$$-\frac{j}{\pi t} \overset{F.T}{\longleftrightarrow} \mathrm{sgn}(\omega)$$

Summary of F.T Properties:-

$x(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$X(\omega)$	
$\alpha x_1(t) + \beta x_2(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$\alpha X_1(\omega) + \beta X_2(\omega)$	linearity
$x(t \pm t_0)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$e^{\pm j\omega t_0} X(\omega)$	time-shifting
$e^{\mp j\omega_0 t} x(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$X(\omega \pm \omega_0)$	freq. shifting
$\cos(\omega_0 t)\, x(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$	AM; Amplitude Modulation
$x(at)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$	time scaling
$\frac{1}{\lvert a\rvert} x\left(\frac{t}{a}\right)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$X(a\omega)$	freq. Scaling
$x(-t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$X(-\omega)$	Reflection.
$\frac{d^n x(t)}{dt^n}$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$(j\omega)^n X(\omega)$	time differentiation
$t^n x(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$(j)^n \frac{d^n X(\omega)}{d\omega}$	freq. differentiation
$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$\frac{X(\omega)}{j\omega} + \underbrace{\pi X(0)\,\delta(\omega)}_{\text{The DC component}}$	time integration
$x(t) * h(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$X(\omega) \cdot H(\omega)$	convolution
$x(t) \times p(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$\frac{1}{2\pi} X(\omega) * P(\omega)$	Multiplication

duality

$$\boxed{\begin{array}{lll} x(t) & \overset{\mathcal{F}}{\longleftrightarrow} & X(\omega) \\ X(t) & \overset{\mathcal{F}}{\longleftrightarrow} & 2\pi x(-\omega) \end{array}}$$

Duality

power signals

- Parseval's Thm. for Periodic signals (in F.S)

$$P_\infty = \frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2$$

→ for non-periodic signals (F.T):

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt \longrightarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, dt$$

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, d\omega\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\left(\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right) d\omega$$

$$\boxed{E_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega}$$

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega}$$

↳ Parseval's relation for energy functions.

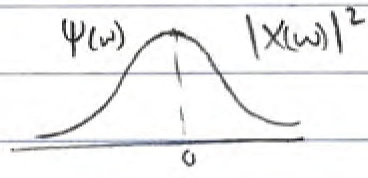
ex:- find $\displaystyle\int_{-\infty}^{\infty} \frac{2}{|j\omega+2|^2}\, d\omega = 2\times\int_{-\infty}^{\infty}\left(\frac{1}{j\omega+2}\right)^2 d\omega$

sol:- $X(\omega) = \dfrac{1}{j\omega+2} \longrightarrow x(t) = e^{-2t}u(t)$

$$2\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = 2\left(2\pi\int_{-\infty}^{\infty} |x(t)|^2\, dt\right)$$

$$= 4\pi\int_{0}^{\infty} e^{-4t}\, dt = \pi$$

Energy Spectral Density:



← Energy Spectrum / or energy spectral density

$$E_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega$$

$$\Delta E_\infty = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} \psi(\omega)\, d\omega$$

* we can see the distribution of Energy with respect to ω.

* The only thing you can get in time domain is the total energy.

ex: $x(t) = e^{-t}u(t)$, find E_∞?

Sol: $E_\infty = \int_{0}^{\infty} e^{-2t}\,dt = \frac{1}{2}$ ⋇

↳ $u(t)$ $\int_{-\infty}$ ∘

*if I want E_∞ from $(\omega = -4 \to 4)$

$$\Delta E_\infty = \frac{1}{2\pi}\int_{-4}^{4} |X(\omega)|^2\, d\omega \quad \cdots ①$$

$$X(\omega) = \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1+j\omega}$$

plug in ①

$$\Delta E_\infty = \frac{1}{2\pi}\int_{-4}^{4} \frac{1}{1+j\omega}\cdot\frac{1}{1-j\omega}\, d\omega$$

↓ Since $x(t)$ is real ∴ $X(\omega)$ has an even symmetry.

$$\Delta E_\infty = \frac{\cancel{2}}{\cancel{2}\pi}\int_{0}^{4} \frac{1}{1+\omega^2}\, d\omega$$

$$= \frac{1}{\pi}\tan^{-1}\Big]_0^4 = 0.422 \,!$$

means that 84.4 % of energy is distributed over (ω) from $-4 \to 4$

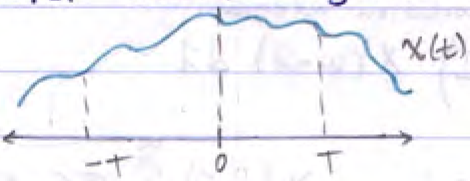
time-domain frequency domain 9/4/2013

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

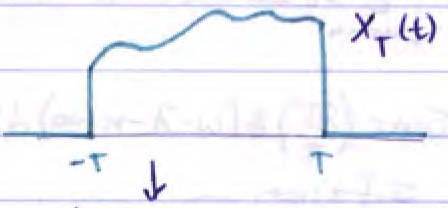
$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega)\, d\omega \quad ; \quad \Psi(\omega) = |X(\omega)|^2$$

→ For Power Signals:-



$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$ x(t) is non-periodic



$$\rightarrow \quad x_T(t) = \begin{cases} x(t), & |t| < T \\ 0, & o.w \end{cases} = x(t)\,\mathrm{rect}\left(\frac{t}{2T}\right)$$

↓ truncated version of x(t)

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\left(\int_{\cancel{-T}\,-\infty}^{\cancel{T}\,\infty} |x_T(t)|^2 dt\right)$$

→ because it's a truncated version.

using parseval's thm.

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\cdot\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{2T}\, d\omega$$

$$\boxed{P_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega} \quad \ldots (A) \; ; \quad \boxed{S(\omega) = \lim_{T\to\infty}\left[\frac{|X_T(\omega)|^2}{2T}\right]}$$ for Power signals.

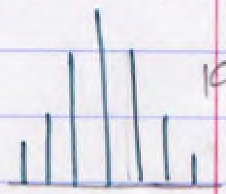
Power Spectral Density (PSD)

→ PSD is a function that describes the distribution of the average power of the signal as a function of frequency.

to obtain the power information for a periodic function:- (PSD for Power Signals).

- $\underset{\text{Periodic}}{x(t)} \xrightarrow{F.S} C_n$

Line Spectrum

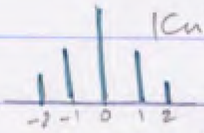


Power Spectrum

$$P_\infty = \sum_{n=-\infty}^{\infty} |C_n|^2$$

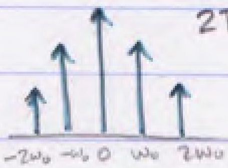
- $$\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \overset{F.T}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} 2\pi C_n \, \delta(\omega - n\omega_0)$$

Proof in notes!



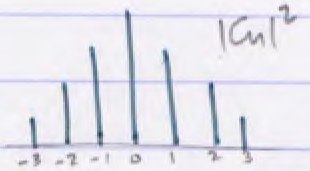
line Spectrum

F.S coeficients



$X(\omega)$ (F.T)

in the same manner, we can introduce the PSD :-



powe line spectrum

power spectral Density (PSD) for Periodic functions:-

$$\sum_{n=-\infty}^{\infty} 2\pi |C_n|^2 \, \delta(\omega - n\omega_0) = S(\omega)$$

to check it's true:-

from (A) $\longrightarrow$

$$P_\infty = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left(\sum_{n=-\infty}^{\infty} \cancel{2\pi} |C_n|^2 \, \delta(\omega - n\omega_0) \right)}_{S(\omega)} d\omega$$

$$= \sum_{n=-\infty}^{\infty} |C_n|^2 \underbrace{\int_{-\infty}^{\infty} \delta(\omega - n\omega_0) \, d\omega}_{1}$$

$$= \sum_{n=-\infty}^{\infty} |C_n|^2$$

- Calculating Power Information:- (4 different ways)

let $x(t) = A\cos(\omega_0 t + \phi)$

1. from time domain:

$$P_\infty = \frac{1}{T}\int_T \underbrace{\overbrace{A^2\cos^2(\omega_0 t+\phi)}^{|x(t)|^2}}_{\frac{1}{2}(1+\cos(2(\omega_0 t+\phi))} \, dt$$

$$P_\infty = \frac{A^2}{2}$$ total power from t-domain.

2. frequency-domain (F.S)

using Eulers identity: (Find F.S)

$$x(t) = \underbrace{\frac{A}{2}}_{C_1} e^{j(\omega_0 t+\phi)} + \underbrace{\frac{A}{2}}_{C_{-1}} e^{-j(\omega_0 t+\phi)}$$

$$P_\infty = \sum_{n=-\infty}^{\infty} |C_n|^2$$

$$= (|C_1|)^2 + (|C_{-1}|)^2 = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2} *$$

3. frequency-domain (F.T)

$$S(\omega) = \sum_{n=-\infty}^{\infty} 2\pi |C_n|^2 \delta(\omega - n\omega_0)$$

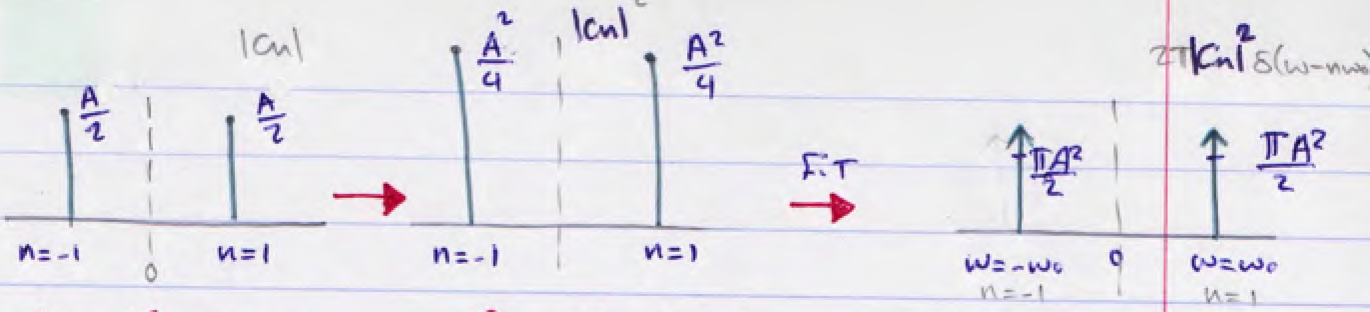
$$= 2\pi|C_{-1}|^2\delta(\omega+\omega_0) + 2\pi|C_1|^2\delta(\omega-\omega_0)$$

$$S(\omega) = \frac{\cancel{2}\pi A^2}{\cancel{4}\,2}\delta(\omega+\omega_0) + \frac{\cancel{2}\pi A^2}{\cancel{4}\,2}\delta(\omega-\omega_0)$$

$$P_\infty = \frac{1}{2\cancel{\pi}}\int_{-\infty}^{\infty} \overbrace{\left(\frac{\cancel{\pi}A^2}{2}\delta(\omega+\omega_0) + \frac{\cancel{\pi}A^2}{2}\delta(\omega-\omega_0)\right)}^{S(\omega)} d\omega$$

$$= \frac{1}{2}\left(\frac{A^2}{2}\times 1 + \frac{A^2}{2}\times 1\right) = \frac{A^2}{2} *$$

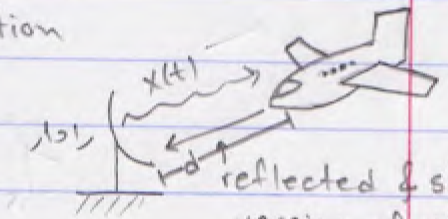
4. Using Auto correlation (will be discussed later)



Line Spectrum Power line Spectrum P.S.D

↳ power information in freq. domain

- Correlation functions:-
  - ↳ cross-correlation between x, y
  - ↳ Auto-correlation between x, x → application



reflected & shifted version of x(t)!

Auto-correlation between these 2 signal helps to calculate the distance (d).

- The definition of correlation depends on the nature of the signal.
  - **Energy Signals**

R_{xy} ; cross- correlation

R_{xx} ; Auto- 〃

$$R_{xy} = \int_{-\infty}^{\infty} x(T)\, y^*(\underbrace{T-t}_{\text{no reflection of } T})\, dT$$

$(t-T)$ for convolution

$$\triangleq \int_{-\infty}^{\infty} x(T+t)\, y^*(T)\, dT$$

• <u>Power Signals</u>

$$R_{xy}(t) = \lim_{T\to\infty} \frac{1}{T} \int_T x(\tau)\, y^*(\tau - t)\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_T x(\tau + t)\, y^*(\tau)\, d\tau$$

if x & y are periodic we ommit the limit

$$R_{xy}(t) = x(t) * y(-t)$$

$\therefore$ $R_{xy}(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \times Y(-\omega)$

$X(\omega) \times Y^*(\omega)$, for real $x(t), y(t)$

for real valued functions $Y(-\omega) = Y^*(\omega)$

$\therefore$

$R_{xx}(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \times X^*(\omega)$, if $x(t)$ is real

$= |X(\omega)|^2$

$= \Psi(\omega)$

• Interesting Situation of Auto-correlation :)

• $R_{xx}(0) = \int_{-\infty}^{\infty} x(\tau)\, x^*(\tau)\, d\tau$ for energy signals

$= \int_{-\infty}^{\infty} |x(\tau)|^2\, d\tau$ = Energy

$\boxed{R_{xx}(0) = \text{Energy}}$; for E signals

• $R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_T x(\tau)\, x^*(\tau)\, d\tau$ = total power for power Signals

→ so, if you have the Auto-correlation function R_{xx} for a (power / Energy) signal, you can find the total (power / Energy) just by looking at $t=0$, $R_{xx}(0)$

another **ex**:

- $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

  $X(0) = \int_{-\infty}^{\infty} x(t)\, dt$ = DC component (avg).

  area under x(t) in t-domain

- $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{+j\omega t}\, d\omega$

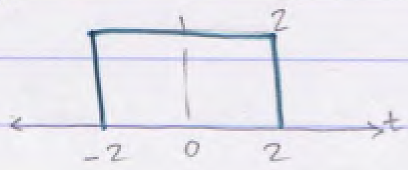
  $x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, d\omega$ → area under $X(\omega)$ in freq-domain. $/2\pi$

$\therefore$ $R_{xx}(0)$ = Energy / power حسب نوع ال Signal

$\underset{\text{freq. domain}}{X(0)}$ = DC component (area under $x(t)$)

$\underset{\text{time domain}}{x(0)}$ = $\frac{\text{Area under } X(\omega)}{2\pi}$

ex:

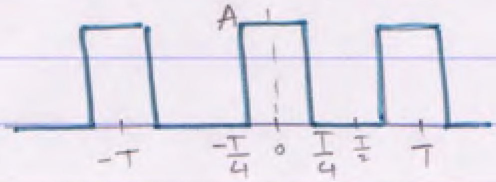


, Evaluate $\int_{-\infty}^{\infty} X(\omega)\, d\omega$?

$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, d\omega$

$\therefore \int_{-\infty}^{\infty} X(\omega)\, d\omega = 2\pi\, x(0) = 2\pi \times 2 = 4\pi$

ex 2:-

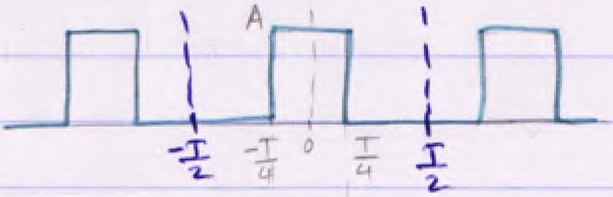


, find the Auto-correlation $R_x(t)$?

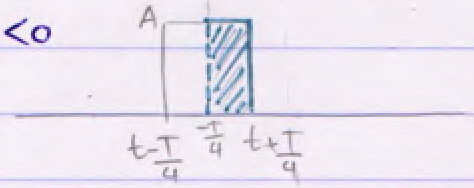
Sol:-

Since it's a periodic func.

$$R_x(t) = \frac{1}{T} \int_T X(T)\, X(T-t)\, dT$$



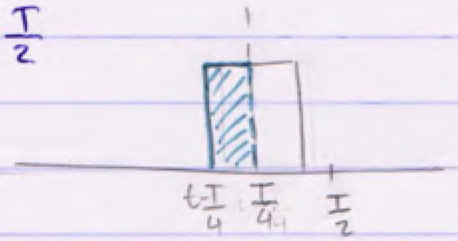
$-\frac{T}{2} < t < 0$



$$R_x(t) = \frac{1}{T} \int_{-\frac{T}{4}}^{t+\frac{T}{4}} A^2\, dT = A^2\left(\frac{1}{2} + \frac{t}{T}\right)$$

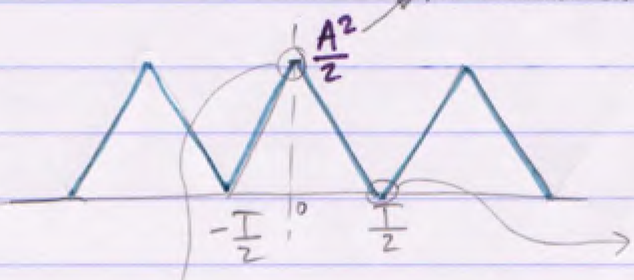
for $-\frac{T}{2} < t < 0$

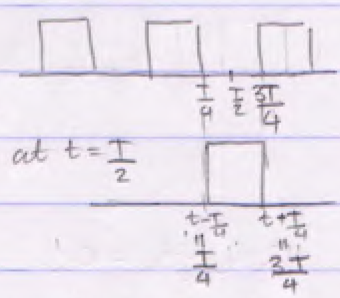
$0 < t < \frac{T}{2}$



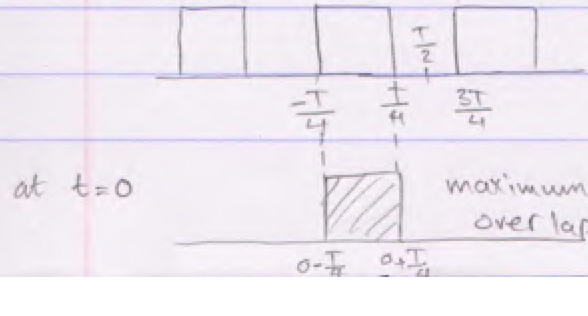
$$R_x(t) = \frac{1}{T} \int_{t-\frac{T}{4}}^{\frac{T}{4}} A^2\, dt = A^2\left(\frac{1}{2} - \frac{t}{T}\right)$$

Maximum over lap = $R_x(0)$ = total power





التداخل ينعدم = 0!
/التطابق



maximum over lap.

إضافي

9/4/2013

- Auto-correlation functions & Fourier Transform:

→ Energy signals:

$R_x(t) \xleftrightarrow{\mathcal{F}}$?!

$x(t) * x(-t) \xleftrightarrow{\mathcal{F}} X(\omega) \cdot X(-\omega)$

for real valued $x(t)$: $X(-\omega) = X^*(\omega)$

$= X(\omega)\, X^*(\omega)$

$= |X(\omega)|^2 = \Psi(\omega)$; energy spectral density

Proof:

$$R_x(t) = \int_{-\infty}^{\infty} x(T)\, x^*(T-t)\, dT \overset{\text{or}}{=} \int_{-\infty}^{\infty} x(T+t)\, x^*(T)\, dT$$

$$\mathcal{F}\{R_x(t)\} = \int_{-\infty}^{\infty} \Bigg(\underbrace{\int_{-\infty}^{\infty} x(T+t)\, x^*(T)\, dT}_{R_x(t)}\Bigg) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty}\Bigg(\int_{-\infty}^{\infty} \underbrace{x(t+T)\, e^{-j\omega t}\, dt}_{X(\omega)\, e^{j\omega T}}\Bigg) X^*(T)\, dT$$ (time-shifting).

$$= \int_{-\infty}^{\infty} \underbrace{X(\omega)}_{\text{constant}} \underbrace{e^{j\omega T}\, x^*(T)\, dT}_{X^*(\omega)}$$

$$\mathcal{F}\{R_{xx}\} = X(\omega) \cdot X^*(\omega) = |X(\omega)|^2 = \Psi(\omega)$$

** Energy $R_x(t) \xleftrightarrow{\mathcal{F}} \Psi(\omega)$

Power $R_x(t) \xleftrightarrow{\mathcal{F}} S(\omega)$ Proof in notes.

Properties:

- $R_x(t) = x(t) * x(-t)$

  $R_x(-t) = x(-t) * x(t)$

  $\therefore R_x(t)$ is an even function $R_x(t) = R_x(-t)$

→ R_x can be used to calculate Power:

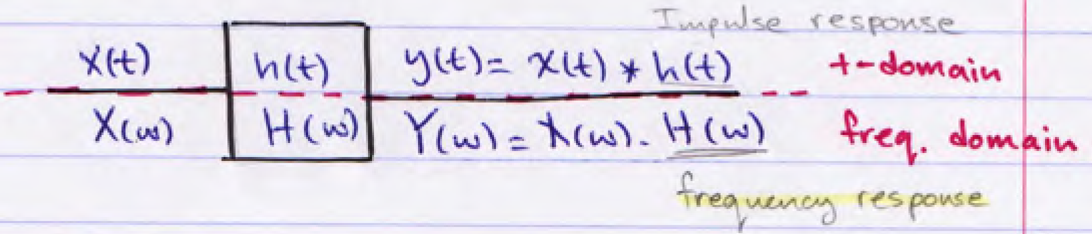
$$R_x(t) \xleftrightarrow{\mathcal{F}} S(\omega) \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = P$$

Chapter # 6: Applications:

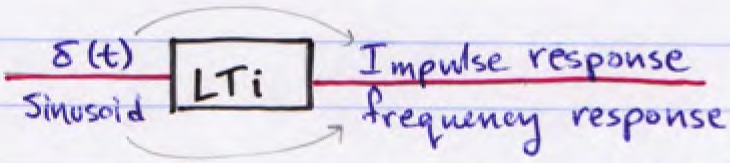
1. Filtering:
    - Ideal Filters.
    - Practical Filtering.

   Designing Filters in Freq-domain.
2. Bandwidth.
3. Amplitude modulation (AM).
4. Sampling. converting from continuous to discrete without loosing the info. contained in the cont. form.

1) Filters:



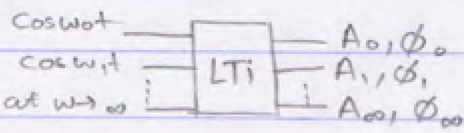
→ Frequency response:



remains the same

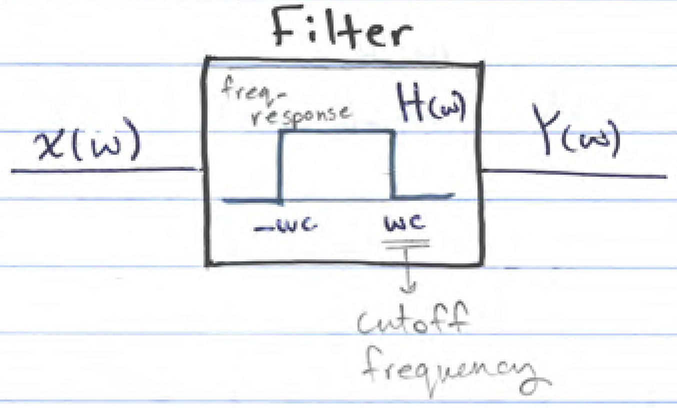
$$\begin{aligned}
w_0 \quad & \cos w_0 t \xrightarrow{LTi} A_0 \cos(w_0 t + \phi_0) \\
& \cos w_1 t \xrightarrow{LTi} A_1 \cos(w_1 t + \phi_1 \\
& \qquad \vdots \\
\infty \quad & \cos W_n t \xrightarrow{LTi} A_n \cos(w_n t + \phi_n)
\end{aligned}$$

$\equiv$

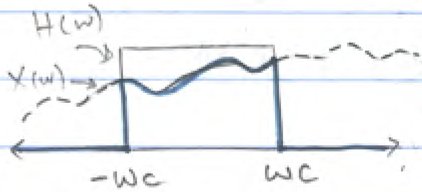


→ I'm looking at how does the phase changes with w (that's why it's called frequency response).

- we can know the mathematical model of a system from the frequency response
- "Filter" is a fancy way of saying "frequency Response".



$$Y(\omega) = X(\omega) \cdot \underline{\underline{H(\omega)}}$$
Rect (window function)



$$Y(\omega) = \begin{cases} X(\omega) & , -wc \longleftrightarrow wc \\ 0 & , o.w \end{cases}$$

<u>ex</u>:- $y(t) = \int_0^\infty e^{-T} x(t-T)\, dT$ ... ①

find the inverse system:

* first we have to find $h(t)$, so we substitute $x = \delta(t)$, then $h(t) = \int_0^\infty e^{-T} \delta(t-T) dT$ ⋮

→ <u>Short cut:</u>

$$y(t) = \int_{-\infty}^{\infty} e^{-T} u(T)\; x(t-T) dT$$

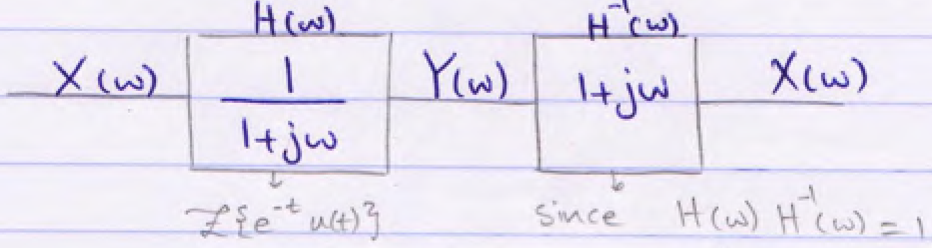
= convolution integral

$\therefore$ $\boxed{h(t) = e^{-t} u(t)}$

Since $h(t) = e^{-t} u(t)$

$\therefore \quad e^{-t} u(t) * h^{-1}(t) = \delta(t)$ in t-domain!

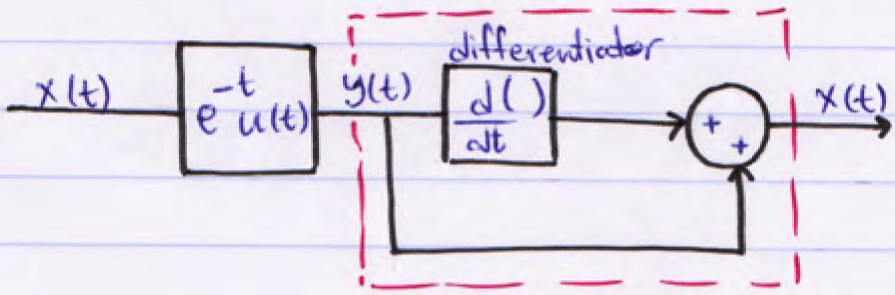
على طول، ورح على ال freq- domain



back to t-domain $\longrightarrow$ $Y(\omega) \cdot (1+j\omega) = X(\omega)$

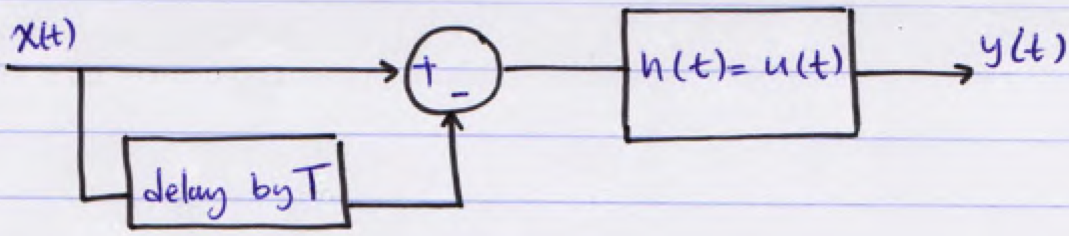
$Y(\omega) + Y(\omega)\, j\omega = X(\omega)$ F-domain

$y(t) + \dfrac{d\, y(t)}{dt} = x(t)$ t-domain



inverse system

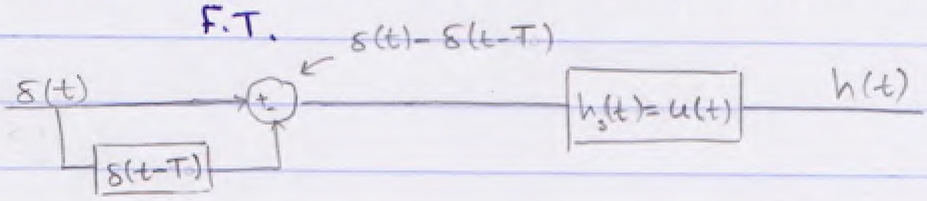
ex:



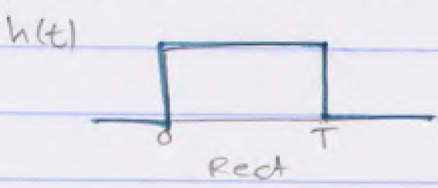
forward path.

find the overall frequency response.

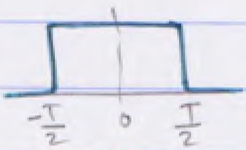
* method #1: find the impulse response, then find it's F.T.



$$h(t) = [\delta(t) - \delta(t-T)] * u(t)$$
$$= u(t) - u(t-T)$$



$\xleftrightarrow{F.T}$?!



$\xleftrightarrow{F.T}$ $T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right)$

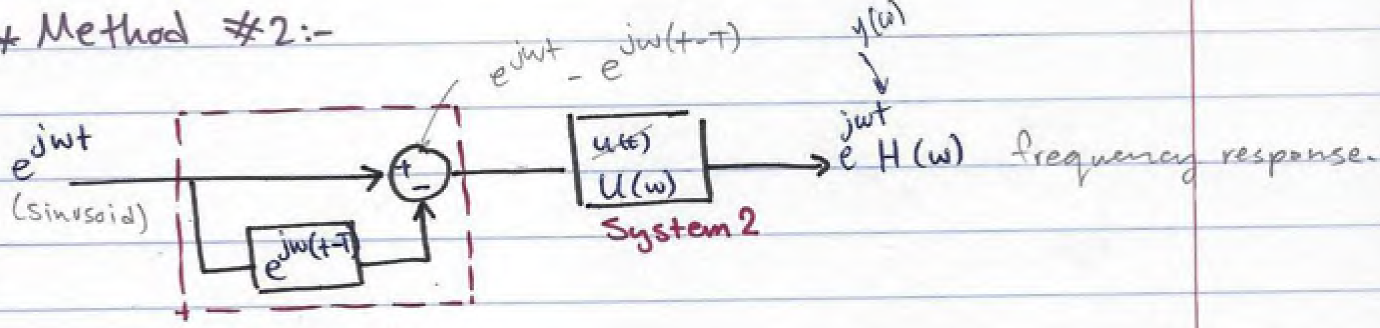
h(t) 

$\xleftrightarrow{F.T}$ $T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right)e^{-j\omega\frac{T}{2}} = H(\omega)$

$$\boxed{H(\omega) = \mathcal{F}\{h(t)\}}$$

cont. → Method 2

* Method #2:-



System 1

find the over all $H(\omega)$:

output of system 1 is:-

$$e^{j\omega t} - e^{j\omega t} e^{-j\omega T}$$

$$= e^{j\omega t} \underbrace{(1 - e^{-j\omega T})}_{} \quad \text{.... (A)}$$

$H_1(\omega)$; the freq. response of system 1.

Since $e^{j\omega t} \rightarrow \boxed{h(t)} \rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)}$ t-domain

or $= e^{j\omega t} H(\omega)$ freq. domain

$$\therefore \boxed{H_1(\omega) = 1 - e^{-j\omega T}} \quad \text{.... (1)}$$

$h_2(t) = u(t)$

$H_2(\omega) = \mathcal{F}\{h_2(t)\}$

$= \mathcal{F}\{u(t)\}$

$= U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$

$$\therefore \boxed{H_2(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}} \quad \text{-- (2)}$$

$\therefore$ the over all system freq. response $H(\omega)$ equals:-

$Y(\omega) = e^{j\omega t} H_1(\omega) H_2(\omega)$

$H(\omega) = H_1(\omega) . H_2(\omega) \longrightarrow$

$$H(w) = \underline{\underline{\left(1 - e^{jwT}\right)}} \cdot \left(\pi\delta(w) + \frac{1}{jw}\right)$$

also can be written as:-

$$\underbrace{\left(e^{jw\frac{T}{2}} - e^{-jw\frac{T}{2}}\right)}_{2j\sin\left(w\frac{T}{2}\right)} \cdot e^{jw\frac{T}{2}} \quad \cdots \circledast$$

$$2j\sin\left(w\tfrac{T}{2}\right)\, e^{jw\frac{T}{2}}$$

0 by sifting.

$$= \left(1 - e^{jwT}\right) \cdot \pi\delta(w) + \frac{1}{jw}\left(1 - e^{jwT}\right)$$

from $\circledast$

$$= \frac{1}{jw} \underbrace{\left(e^{jw\frac{T}{2}} - e^{-jw\frac{T}{2}}\right)}_{2j\ \sin\left(w\frac{T}{2}\right)} e^{jw\frac{T}{2}}$$

$$= e^{-jw\frac{T}{2}} \frac{\left(\sin\left(\frac{wT}{2}\right) \cdot 2\cancel{j}\right)}{\cancel{j}w \times \frac{T}{2}} \times \frac{T}{\cancel{2}}$$

$$= T e^{-jw\frac{T}{2}} \frac{\sin\left(\frac{wT}{2}\right)}{\left(\frac{wT}{2}\right)}$$

$$H(w) = Te^{-jw\frac{T}{2}} \operatorname{sinc}\left(\frac{wT}{2\pi}\right) \quad \#$$

*Note:- $H(w) \times H^{-1}(w) = 1$ freq- domain

$h(t) * h^{-1}(t) = \delta(t)$ time domain

ex: $\frac{d\,y(t)}{dt} + y(t) = 2x(t)$, find frequency response.

$\hookrightarrow Y(w)\,jw + Y(w) = 2X(w)$

$(1 + jw)\,Y(w) = 2\,(X(w)) \quad \cdots ①$

$X(w) \longrightarrow \boxed{H(w)} \xrightarrow{Y(w)} \quad \Rightarrow$

$Y(w) = X(w) \cdot H(w)$

$\therefore H(w) = \frac{Y(w)}{X(w)}$

from ①

$H(w) = \frac{2}{1 + jw}$

$h(t) = 2e^{-t}u(t)$ (inverse F.T of $H(w)$)


second
exam

UNIVERSITY OF JORDAN
DEPARTMENT OF ELECTRICAL ENGINEERING
SIGNALS AND SYSTEMS

SECOND EXAM — 30 MARKS — 23/4/2013

DR. MAHMOUD AL-HUSARI — NAME: ______________ — SERIAL NO. 34

Please write your name in arabic — 0110483

SERIAL NO. ☐ · SECTION:
- ☐ SECTION 1, SUN, TUE, THUS 09:00-10:00
- ☑ SECTION 2, SUN, TUE, THUS 11:00-12:00
- ☐ SECTION 3, MON, WED 09:30-11:00

21 x 1,5

Write your answers here													
1	e ✓	2	c ✓	3	b ✓	4	c ✓	5	a ✓	6	e ✓	7	a ✓
8	d ✓	9	b ✓	10	c ✓	11	b ✓	12	b ✓	13	b ✓	14	a ✓
15	a ✓	16	b ✓	17	c ✓	18	d ✓	19	c ✓	20	d ✓	21	d ✓
22	b ✓												

1. Given that $X(\omega) = \dfrac{\cos(4\omega)\sin(2\omega)}{\omega}$, find the value of $\int_{-\infty}^{\infty} x(t)dt$

   a. 1
   b. zero
   c. 1/2
   d. infinity
   e. 2

2. A signal $x(t)$ can be written as

$$x(t) = \frac{1}{2} + \cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots$$

   What is the power in the signal UP TO the fundamental harmonic?

   a. 5/4
   b. 5/8
   c. 3/4
   d. 3/8
   e. none of these

3. Let $x(t) = \begin{cases} t, & 0 \le t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$, be a periodic function with fundamental period equal to 3. The average value of the signal is

   a. 3
   b. 1
   c. none of these
   d. 2
   e. 6

4. A periodic function is given over one period by $x(t) = |t|$, $-\pi < t < \pi$. Which of the following statements is correct for the trignometric Fourier series of $x(t)$

   a. $a_n = 0$ for $n = 0, 1, 2, \cdots$.
   b. $a_n = 0$ for all odd integers n, but not for any even integers n.
   c. $a_n = 0$ for all even integers n, but not for any odd integers n.
   d. $a_n = 0$ for $n = 1, 2, 3, \cdots$, but $a_0 \neq 0$.
   e. none of these

5. Consider the signal $x(t) = \text{rect}\left(\frac{t}{2}\right) + \text{rect}\left(\frac{t}{4}\right)$. Evaluate the integral $\int_{-\infty}^{\infty} X(\omega)e^{-j\frac{3}{2}\omega}d\omega$

   a. 2π
   b. 2
   c. none of these
   d. 1
   e. 4π

6. Consider the following statements regarding the signal $x(t) = \sin(3t) + e^{j\frac{2}{7}t}$

S1: The exponential Fourier coefficients $c_{\pm 21} = \frac{1}{2j}$.

S2: The fundamental frequency of $x(t)$ is $\omega_0 = 2/7$

a. none of these
b. S1 and S2 are true
c. S1 is false but S2 is true
d. S1 is true but S2 is false
e. S1 and S2 are false

7. Find the Fourier transform of $x(t) = \delta(t+1) + \delta(t-1)$

a. $2\cos(\omega)$
b. $2\pi\cos(\omega)$
c. $\cos(\omega)$
d. none of these
e. $\pi\cos(\omega)$

8. When the signal

$$x(t) = 5 + \cos(12t + \pi/4)$$

is applied to a system, the output is

$$y(t) = 6\sin(12t) + \cos(24t + \pi/4)$$

Which of the following statements is true

S1: The system is LTI with $H(0) = 0$ and $|H(j12)| = 6$

S2: The system is not LTI because the cos function becomes a sin function

S3: The system is not LTI because the constant term disappears at the output

a. All are false
b. S1 is false but both S2 and S3 are true
c. S2 only is true
d. S3 only is true
e. S1 only is true

9. The Fourier transform of $x(t) = \text{sgn}(2t)$ is

a. none of these
b. $2/(j\omega)$
c. $2/(2j\omega)$
d. $1/(2j\omega)$
e. $1/(j\omega)$

10. For the trignometric Fourier series of $x(t)$ defined over one period as $x(t) = t$, $-2 \le t \le 2$. the coefficient $b_1 =$

a. $2/\pi$

b. zero

c. $4/\pi$

d. 1

e. $\pi/4$

11. The Fourier transform of the signal $x(t) = u(t-1) * \delta(t-1)$ is

a. $e^{-j\omega}/(j\omega) + \pi\delta(\omega - 1)$

b. $e^{-j2\omega}/(j\omega) + \pi\delta(\omega)$

c. $e^{-j2\omega}/(j\omega) + \pi\delta(\omega - 1)$

d. none of these

e. $e^{-j\omega}/(j\omega) + \pi e^{-j\omega}\delta(\omega)$

12. Consider the function $x(t) = \begin{cases} 2, & -2 \leq t \leq -1 \\ |t|, & -1 < t < 1 \\ -2, & 1 \leq t \leq 2 \end{cases}$. At $t = 1$ the Fourier series converges to

a. none of these

b. $-1/2$

c. zero

d. -2

e. -1

13. Given a periodic function $x(t) = \delta(t)$ with period π. The exponential Fourier series coefficient of $y(t) = x(t - \pi/2)$ is

a. $\frac{1}{\pi}e^{j\pi n}$

b. $\frac{1}{\pi}e^{-j\pi n}$

c. $\frac{1}{\pi}e^{-j\frac{\pi}{2}n}$

d. none of these

e. $\frac{2}{\pi}e^{-j\frac{\pi}{2}n}$

14. The auto-correlation function of a rectangular pulse of duration 2 seconds is

a. a triangular pulse of duration 4 seconds

b. a rectangular pulse of duration 2 seconds

c. a triangular pulse of duration 2 seconds

d. a rectangular pulse of duration 4 seconds

e. need more information

15. Consider the signal $x(t) = \sin 4\pi t + \cos 6\pi t$. One possible representation of the Fourier series coefficients is

a. $c_n = \frac{1}{2}\delta[n+3] - \frac{1}{2j}\delta[n+2] + \frac{1}{2j}\delta[n-2] + \frac{1}{2}\delta[n-3]$

b. $c_n = \frac{1}{2}\delta[n+6\pi] + \frac{1}{2j}\delta[n+4\pi] + \frac{1}{2j}\delta[n-4\pi] + \frac{1}{2}\delta[n-6\pi]$

c. none of these

d. $c_n = \frac{1}{2}\delta[n+2] - \frac{1}{2j}\delta[n+1] + \frac{1}{2j}\delta[n-1] + \frac{1}{2}\delta[n-2]$

e. $c_n = \frac{1}{2}\delta[n+3\pi] - \frac{1}{2j}\delta[n+2\pi] + \frac{1}{2j}\delta[n-2\pi] + \frac{1}{2}\delta[n-3\pi]$

16. The value of phase (in radians) at $\omega = 1$ rad/s of the signal $x(t) = 10\delta(t-2)$ is

a. -0.5

b. -2

c. 1

d. 10

e. -0.2

17. The inverse Fourier transform of $X(\omega) = \frac{1}{1+\omega^2}$ is

a. e^{-t}

b. none of these

c $\frac{1}{2}e^{-|t|}$

d. $\frac{1}{2}e^{|t|}$

e. $\frac{1}{2}e^{-|t|}u(t)$

18. The input signal $x(t) = \cos(t)$ is applied to a system with frequency response $H(\omega) = \frac{2}{1+j\omega}$, the output signal will be

a. $2\cos(t)$

b. $\sqrt{2}\cos(t)$

c. $\sqrt{2}\cos(t+\pi/4)$

d. $\sqrt{2}\cos(t-\pi/4)$

e. none of these

19. The frequency response function of a system described by the diffrential equation

$$2\frac{d^2y(t)}{dt^2} + 3y(t) = 4\frac{dx(t)}{dt}$$

is

a. $\frac{4}{2j\omega^2+3}$

b. none of these

c. $\frac{4j\omega}{3-2\omega^2}$

d. $\frac{4j\omega}{2j\omega^2+3}$

e. $\frac{4}{2(j\omega)^2+3}$

20. If the Fourier transform of $x(t)$ is $X(\omega)$, then the Fourier transform of $4x(8-4t)$ is

a. none of these

b. $X\left(\frac{-w}{4}\right) e^{j\omega 8}$

c. $X\left(\frac{w}{8}\right) e^{j\omega 4}$

d $X\left(\frac{-w}{4}\right) e^{-j\omega 2}$

e. $X\left(\frac{w}{4}\right) e^{-j\omega 8}$

21. Let the signal $x(t)$ be periodic with period $T = 4$ and the Fourier coefficients $c_n = e^{jn}$, $-3 \leq n \leq 3$ and zero otherwise. What is the power of the signal $x(t)$?

a. 7/4

b. 3

c. 4

d. 7

e. 3/4

22. Consider the system with impulse response $h(t) = te^{-3t}u(t)$. What is the average value of the output if the input is

$$x(t) = 3 + 2\cos(t + \pi/3) + \sin(2t + \pi/4)$$

a. zero

b. 1/3

c. none of these

d. 1/9

e. 1

Signals second exam test No. 2

1- $\int_{-\infty}^{\infty} x(t)\,dt = X(w)\Big|_{w=0}$

$\dfrac{\cos(4(0))\sin(2(0))}{0} = \dfrac{0}{0}$!

using LH $\dfrac{2\cos(4w)\cos(2w) + -4\sin(4w)\sin(2w)}{1}$

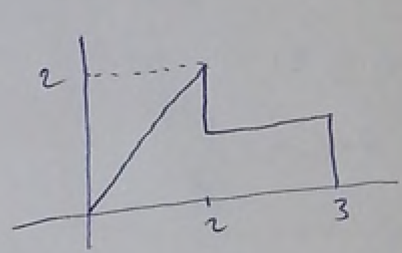
~~sub~~ sub $w=0$ $\boxed{\rightarrow 2}$

2- up to the fundamental harmonic means the first harmonic

the power $= \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \boxed{\frac{3}{4}}$

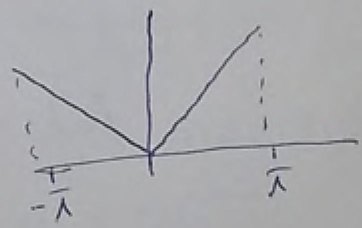
3- the average value

$= \dfrac{\text{area under the curve}}{T}$



$= \dfrac{3}{3} = \boxed{1}$

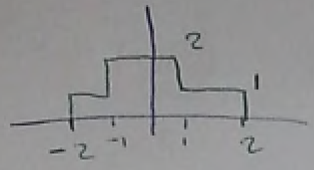
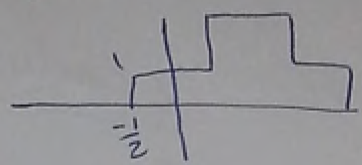
4- this function is even and and has halfwave symmetry



so $a_n = 0$ for all even integers n, but not for any odd integers n.

5) $\int_{-\infty}^{\infty} X(w)\, e^{-j\frac{3}{2}w}\, dw = 2\pi\, x(t)\Big|_{t=0}$

the shifted version of $x(t)$ by $\frac{-3}{2}$

$x(t) =$  $\xrightarrow[\text{by } \frac{-3}{2}]{\text{delay}}$ 

$x(t)_{\text{shifted}}\Big|_{t=0} = 1$

So the ans $= \boxed{2\pi}$

6) S_1 and S_2 are false

$$\boxed{\begin{array}{l} c_{\pm 21} = \pm\frac{1}{2j} \\ w_0 = \frac{1}{7} \end{array}}$$

7) $\delta(t+1) + \delta(t-1) \xrightarrow{\mathcal{F}} e^{jw} + e^{-jw} = \boxed{2\cos(w)}$

8) All are false, the system is not LTI because a new frequency appeared in the output.

9) $x(t) = \text{sgn}(2t)$

if we scaled a sgn function it will still the same

$\text{sgn}(2t) = \text{sgn}(t) \xrightarrow{\mathcal{F}} \boxed{\frac{2}{jw}}$

10) $w = \frac{2\pi}{4} = \frac{\pi}{2}$

$$b_n = \frac{2}{4}\int_{-2}^{2} x \sin\left(\frac{n\pi}{2}t\right)$$

$$b_1 = \frac{1}{2}\int_{-2}^{2} x \sin\left(\frac{\pi}{2}t\right) = \boxed{\frac{4}{\pi}}$$

11) $x(t) = u(t-1) * \delta(t-1) = u(t-2)$

$$u(t-2) \xrightarrow{F} \left[\frac{1}{jw} + \pi\delta(w)\right] e^{-j2w}$$

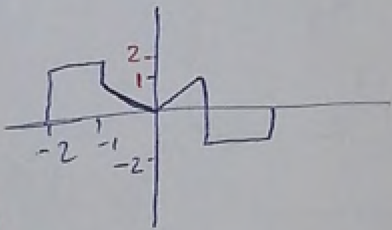
$$= \frac{e^{-2jw}}{jw} + \underbrace{\pi e^{-2jw}\delta(w)}_{\text{by sampling property}}$$

$$\pi e^{-2j(0)}\delta(w) = \pi\delta(w)$$

$$\Rightarrow \boxed{\frac{e^{-2jw}}{jw} + \pi\delta(w)}$$

12) at $t=1$ the fourier series converges to

$$\frac{x(1^-) + x(1^+)}{2} = \frac{-2+1}{2} = \boxed{\frac{-1}{2}}$$



13) $\delta(t) \xrightarrow{F.S} \frac{1}{T} = \frac{1}{\pi}$ $\qquad w_0 = \frac{2\pi}{\pi} = 2$

$$\delta\left(t - \frac{\pi}{2}\right) \xrightarrow{F.S} \frac{1}{\pi} e^{-jnw_0 t} = \frac{1}{\pi} e^{-jn2\left(\frac{\pi}{2}\right)} = \boxed{\frac{1}{\pi} e^{-j\pi n}}$$

14) the auto corrleation of $x(t) = x(t) * x(-t)$

because rect is even function $x(-t) = x(t)$

the auto correlation $= x(t) * x(t)$

~~and~~ and from width property the duration $\underline{\underline{4}}$

and we know the convolution between two rect gives $\underline{\Delta(t)}$

so ~~the~~ it will be a triangular of duration 4

15) $w_0 = 2\pi$

using Euler $\sin 4\pi t = \dfrac{e^{j4\pi t} - e^{-j4\pi t}}{2j}$

$\cos 6\pi t = \dfrac{e^{j6\pi t} + e^{-j6\pi t}}{2}$

$$\left.\begin{aligned} n_{-3} &= \frac{1}{2} \\ n_{-2} &= \frac{-1}{2j} \\ n_2 &= \frac{1}{2j} \\ n_3 &= \frac{1}{2} \end{aligned}\right\}$$

$$C_n = \frac{1}{2}\delta[n+3] - \frac{1}{2j}\delta[n+2] + \frac{1}{2j}\delta[n-2] + \frac{1}{2}\delta[n-3]$$

18) $x(t) = 10\,\delta(t-2) \xrightarrow{\mathcal{F}} 10\,e^{\underline{-j2w}}$

this is the phase

at $w = 1$

the phase e^{-2j}

$\boxed{-2}$

17) $e^{-|t|} \xrightarrow{\mathcal{F}} \frac{2}{1+w^2}$

$\boxed{\frac{1}{2} e^{-|t|}} \xrightarrow{\mathcal{F}} \frac{1}{1+w^2}$

---

18) $x(t) = \frac{e^{jt} + e^{-jt}}{2} \qquad w_0 = 1$

$H(nw) = \frac{2}{1+jn}$

$y(t) = \sum c_n H(nw) e^{jnwt}$

$= \frac{2}{1+j} e^{jt} + \frac{2}{1-j} e^{-jt}$

$= \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}} e^{jt} + \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}} e^{-jt}$

$= \boxed{\sqrt{2} \cos\left(t - \frac{\pi}{4}\right)}$

---

19) $2 \frac{d^2 y(t)}{dt^2} + 3y(t) = \frac{4\, x(t)}{dt}$

$x(t) = e^{jwt}$

$-2w^2 e^{jwt} H(w) + 3 e^{jwt} H(w) = 4jw\, e^{jwt}$

$\boxed{H(w) = \frac{4jw}{3 - 2w^2}}$

---

20) $x(t) \xrightarrow{\mathcal{F}} X(w)$

$x(8+t) \xrightarrow{\mathcal{F}} X(w) e^{jw8}$

$x(8+-4t) \xrightarrow{\mathcal{F}} \frac{1}{4} X\left(-\frac{w}{4}\right) e^{-\frac{jw8}{4}}$

$4x(8-4t) \xrightarrow{\mathcal{F}} \boxed{X\left(-\frac{w}{4}\right) e^{-j2w}}$

21) $c_n = e^{-3j} + e^{-2j} + e^{-j} + 1 + e^{j} + e^{2j} + e^{3j}$

$$\text{power} = \sum |c_n|^2 = \boxed{7}$$

22) $$Y(w) = X(w) \cdot H(w)$$

$$\underbrace{Y(0)}_{\text{Avg of } y} = \underbrace{X(0)}_{\text{Avg of } X \to X(0)=3} \cdot H(0)$$

$$H(w) = \frac{1}{(3+Jw)^2} \Rightarrow H(0) = \frac{1}{9}$$

$$\therefore Y(0) = 3 * \frac{1}{9} = \frac{1}{3}$$

23/4/2013

Ideal Filters :-

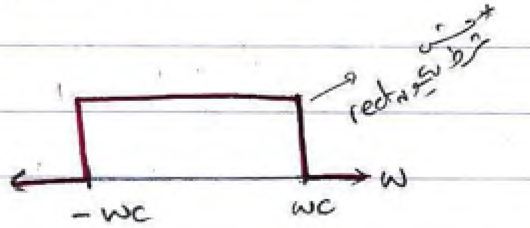
we'll cover:

↳ 4 types of filters:-

1. Low Pass Filters.
2. High Pass Filter.
3. Band pass filter.
4. Band stop filter.

- Design in frequency domain:

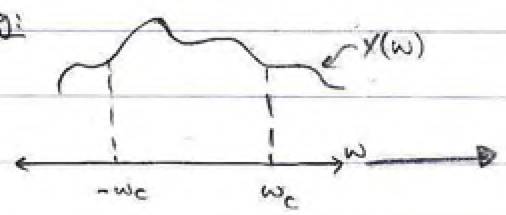
[I] Low Pass Filter :

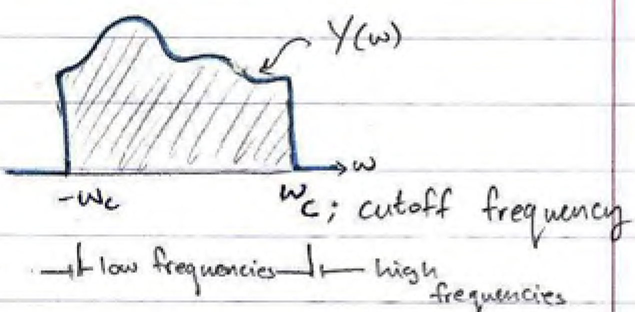


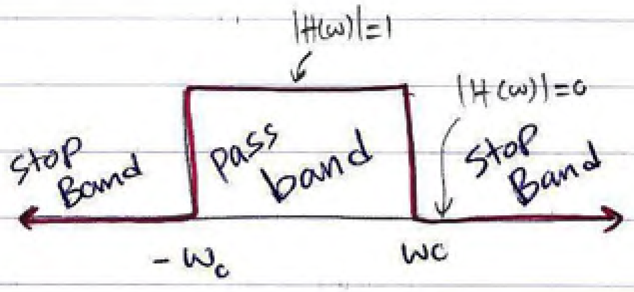
* Remember: "Filter" is a fancy way to describe the "Frequency response".

$$\boxed{Y(w) = H(w) . X(w)}$$

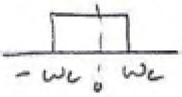
e.g:







* a phase $\angle H(w)$ might exist!

* It's called an ideal filter, because you can never build a system that has a frequency response of this type  .

Since:-

$$\mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} T\,\mathrm{sinc}\left(\frac{wT}{2\pi}\right)$$

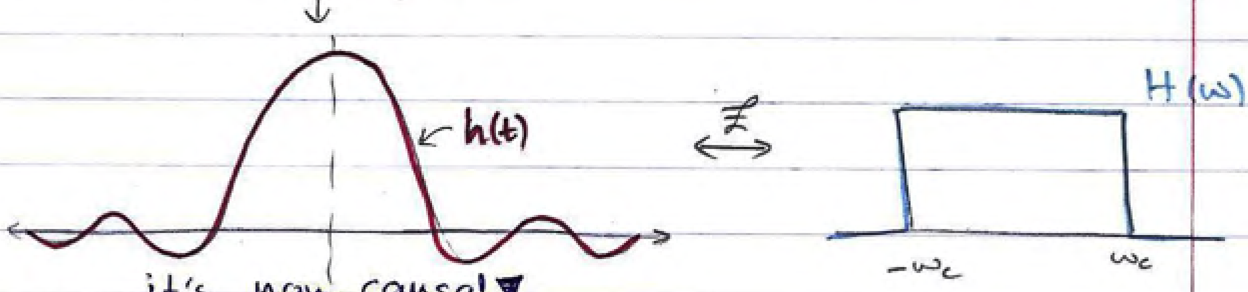
$$T\,\mathrm{sinc}\left(\frac{tT}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\mathrm{rect}\left(\frac{-w}{T}\right)$$

Duality

$T = 2w_c$

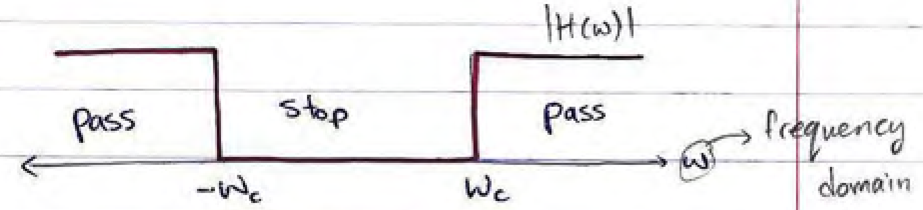
$$\underbrace{\frac{2w_c}{2\pi}\,\mathrm{Sinc}\left(\frac{t\,w_c}{\pi}\right)}_{\text{h(t) for the ideal filter.}} \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{\mathrm{rect}\left(\frac{\mp w}{2w_c}\right)}_{H(w)}$$

since rect is an even f-n.



it's non-causal!
$h(t) \neq 0$, for $t < 0$
$\therefore$ it can't be built!
practically!

2. High Pass Filter:-

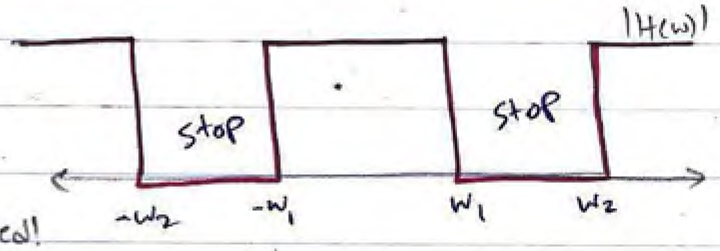


3. Band Pass Filter:-

→ only certain range of frequencies ($\omega_1 \to \omega_2$) passes!

4. Band Stop Filter:

→ only certain range of frequencies is stoped!

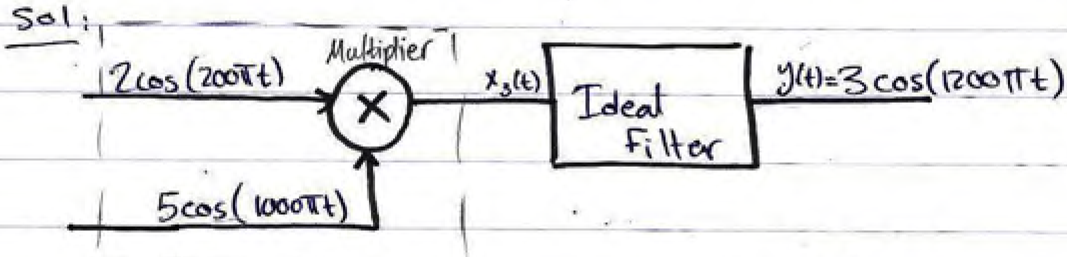


→ let's design a filter:-

ex:- Given $x_1 = 2\cos(200\pi t)$ → you need to design a filter that

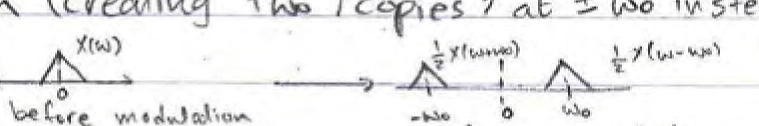
$x_2 = 5\cos(1000\pi t)$ outputs $3\cos(1200\pi t)$:

Sol:



Modulation (creating two copies scaled ($\times\frac{1}{2}$) of $X(\omega)$ at $\pm\omega_0$ instead of the origin)!

e.g:



* we need to find $x_3(t)$:

* using modulation Property:-

$$\underbrace{10\cos(200\pi t)\cos(1000\pi t)}_{\text{multiplication ناتج ال}} \stackrel{\mathcal{F}}{\longleftrightarrow} 5X(\omega-1000\pi) + 5X(\omega+1000\pi)$$

OR (another way to do the modulation)

$$\overset{x(t)}{\underline{10\cos(200\pi t)}}\left[\frac{1}{2}e^{j1000\pi t}+\frac{1}{2}e^{-j1000\pi t}\right] \quad \ldots \text{ t-domain}$$

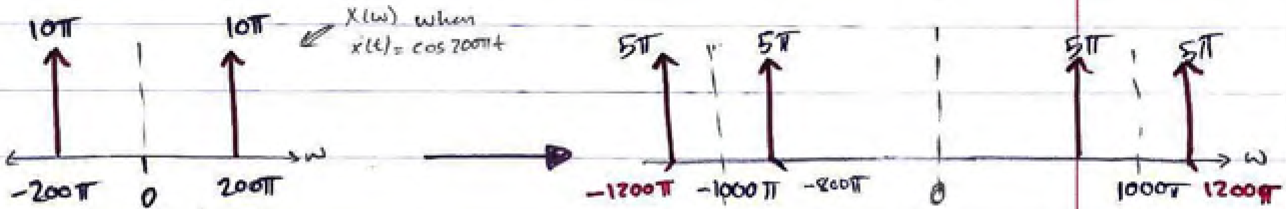
$\Big\updownarrow \mathcal{F}$ using frequency shifting property!

$$5X(\omega-1000\pi)+5X(\omega+1000\pi) \quad \cdots \circledast \quad \text{. freq.-domain}$$

$$x(t)=10\cos(200\pi t) \xleftrightarrow{\mathcal{F}} X(\omega)=\pi\delta(\omega-200\pi)+\pi\delta(\omega+200\pi)$$

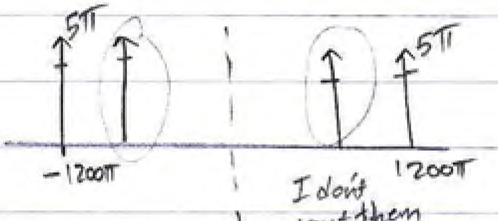
plug $X(\omega)$ in $\circledast$

$$5\pi\delta(\omega-1200\pi t)+5\pi\delta(\omega-800\pi)+5\pi\delta(\omega+800\pi)+5\pi\delta(\omega+1200\pi)$$

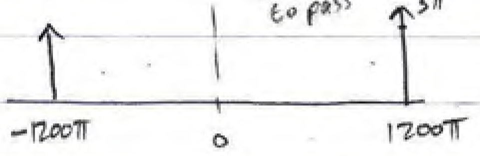


After Modulation.

I have this :

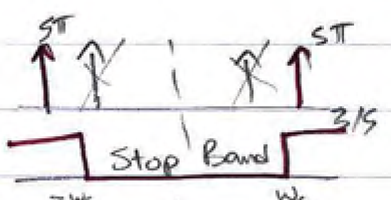


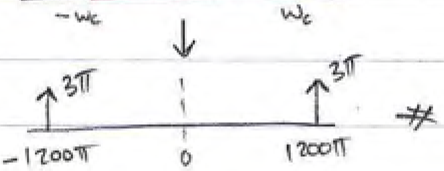
but I need this:-



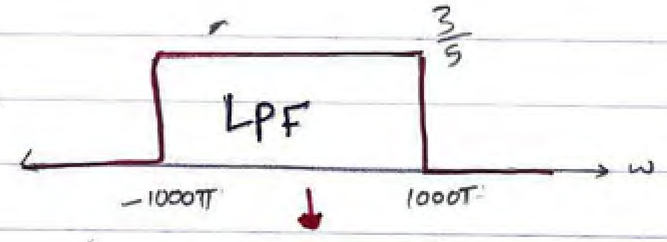
So, I can use a high pass filter with w_c between:

$800\pi < w_c < 1200\pi$, & amplitude $= \frac{3}{5}$





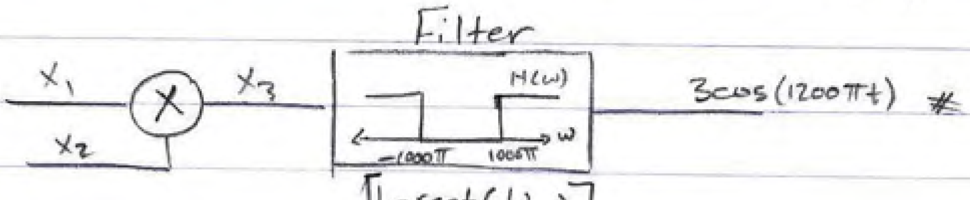
بدنا نطلّع معادلة الفلتر :-



$\leftarrow \text{rect}\left(\frac{\omega}{2000\pi}\right)$

$\leftarrow 1-\text{rect}\left(\frac{\omega}{2000\pi}\right)$

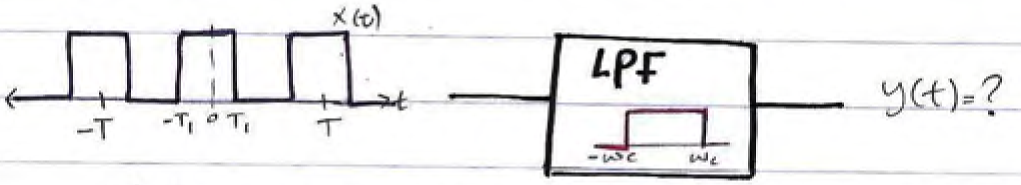
* writting the HPF using an LPF !



$\left[1-\text{rect}\left(\frac{\omega}{2000\pi}\right)\right]$

* its much easier to design a filter in frequency domain.

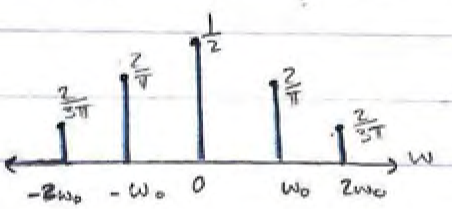
ex2 :-



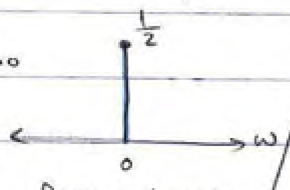
** قبل ما ندخل أي signal على ال filter بنحولها لل freq.-domain ؟

take a General case :-

any periodic function is represented in frequency domain as a train of impulses :- ex: $x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t + \dots\right)$

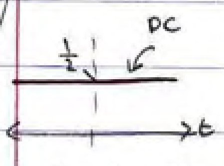


1) if $0 < \omega_c < \omega_0$



freq.-domain

$x(t) = 1$

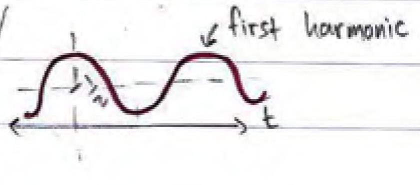


only the DC component

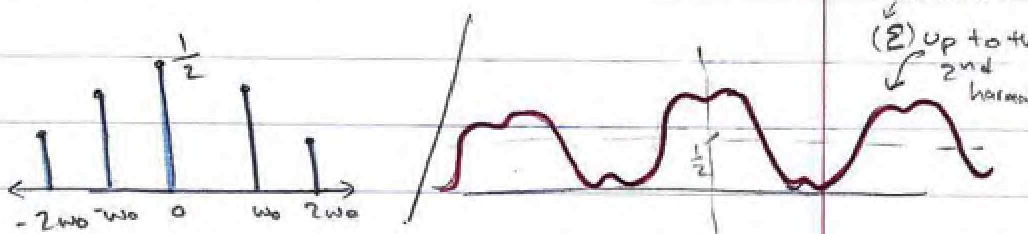
t-domain

2) if $\omega_0 < \omega_c < 2\omega_0$

$x(t) = \frac{1}{2} + \frac{2}{\pi}\cos t$



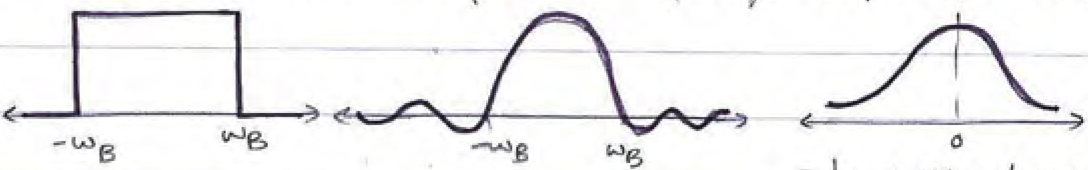
3) if $2\omega_0 < \omega_c < 3\omega_0$



$$x(t) = \frac{1}{2} + \frac{2}{\pi}\cos t - \frac{2}{3\pi}\cos 3t$$

Bandwidth: it's a term to describe the width of the filter (also know as frequency response).

* we have different shapes of (freq. response/filter)



↳ width here is ∞!

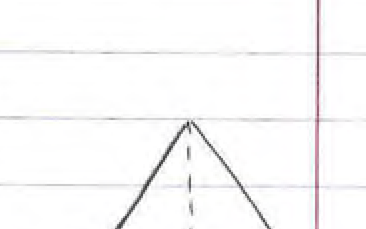
↳ here I have nulls!

now, how to describe the range of frequencies depending on the freq. response's shape? we have 3 popular ways:

1. Absolute Bandwidth.
2. half Power / 3dB Bandwidth.
3. Null-to-Null Bandwidth.

A) Absolute Bandwidth:-

- it's used when the signal is (**Band limited**); it has a value over a finite interval of frequencies & it equals zero otherwise.

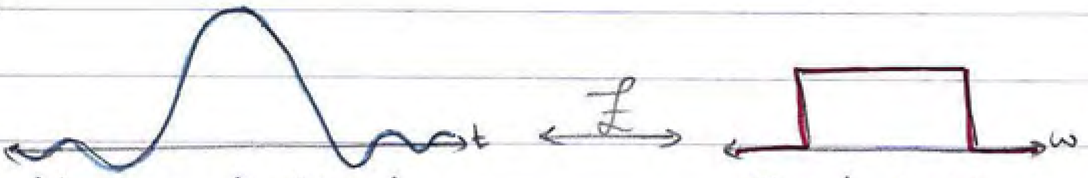


a-1: Base Band (if ceneter is at zero)

a-2: Band Pass (if 〃 isn't 〃 〃)

* don't associate "time-limited" & "Band limited" functions!

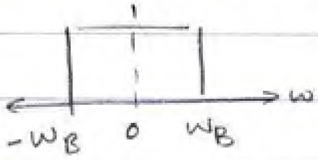
ex:



time unlimited — Band limited

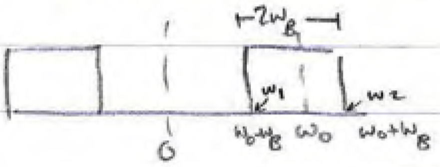
time limited — Band unlimited

* Describing the width of Band limited: * we use absolute value, & only the +ve frequency.



this function is Band limited with an <u>absolute</u> band width = ω_B (+ve فقط)

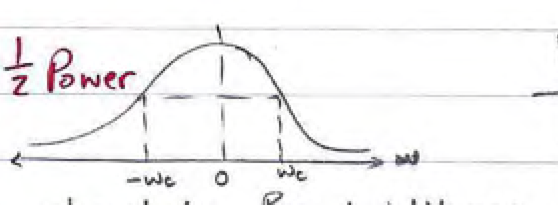
→ for Band - pass



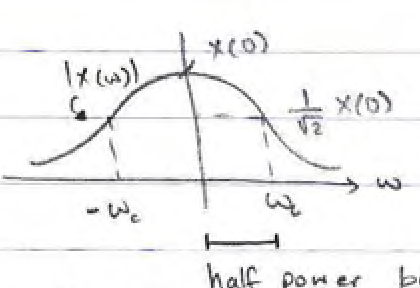
absolute Band Width = $2W_B = \omega_2-\omega_1$

↳ this is one of the draw backs in modulation; it increases the Bandwidth of the signal (moves the signal from the Base band to the pass band) which increases the costs!

B) $\frac{1}{2}$ Power



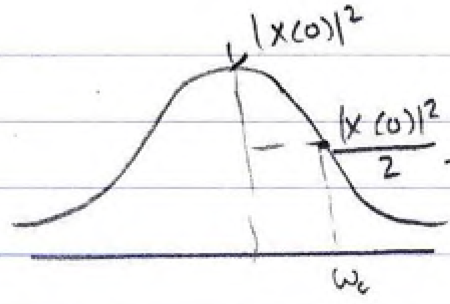
absolute Bandwidth = ∞



half power band width

for Base Band signal:-

$$|X(\omega_c)| = \frac{1}{\sqrt{2}}|X(0)|$$



Energy spectrum

it's called 3dB, because $20\log\left(\frac{1}{\sqrt{2}}\right) \simeq -3\,dB$

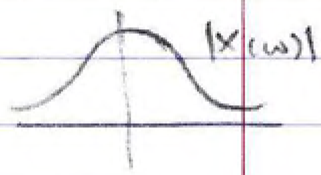
power is $\frac{1}{2}$ maximum power

, I describe the width using $\frac{1}{2}$ Power

ex:- $x(t) = e^{-t/T}u(t)$, what's the $\frac{1}{2}$ power Band width?

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$$

$$e^{-t/T}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{\left(\frac{1}{T}\right)+j\omega}$$



* I can do it with power!

$$|X(\omega)|^2 = X(\omega)\,X^*(j\omega)$$

$$= \frac{1}{\left(\frac{1}{T}\right)^2 + \omega^2}$$

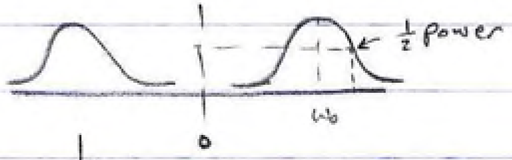
$$|X(0)|^2 = T^2$$

$$|X(\omega_1)|^2 = \left(\frac{1}{\left(\frac{1}{T}\right)^2 + (\omega_1)^2} = \frac{1}{2}T^2\right)$$

$$\downarrow$$

$$\omega_1 = \frac{1}{T}$$

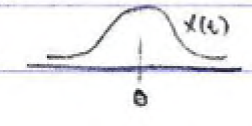
* Now, if it was in Band pass region?



$\downarrow$

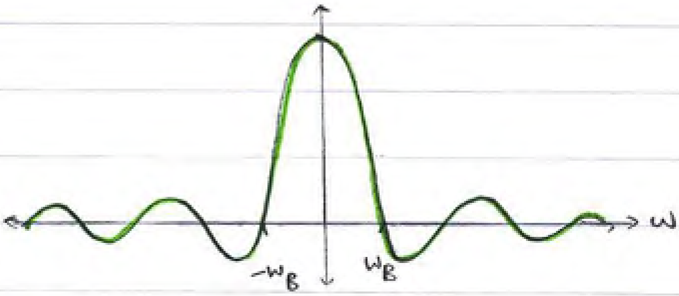
$x(t)\cos(\omega_0 t)$

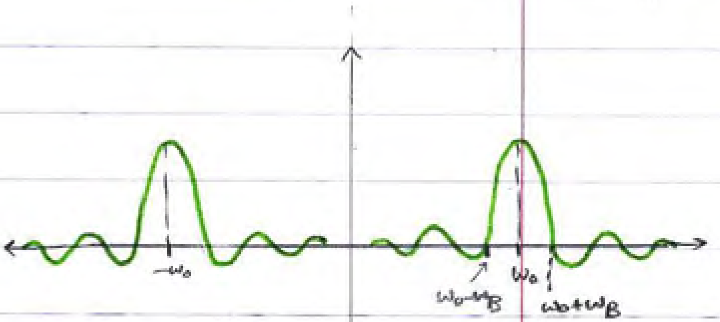
where $x(t) \longrightarrow$



25/4/2013

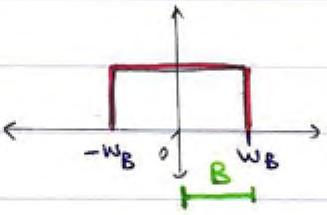
c) Null-to-Null /Zero crossing Bandwidth:



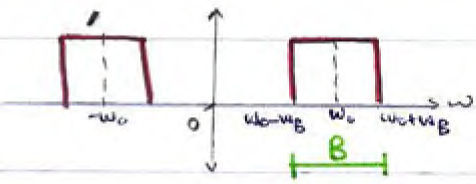


- Base Band signal.
  - * use the first null (ω_B) to define the width "zero crossing"
- Band pass signal / function!
  - Bandwidth $= 2\omega_B$ "null-to-null"

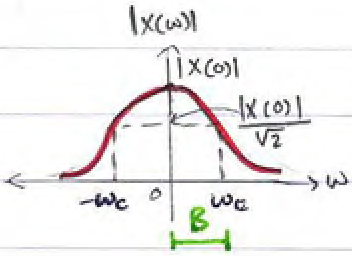
Bandwidth determination (summary):



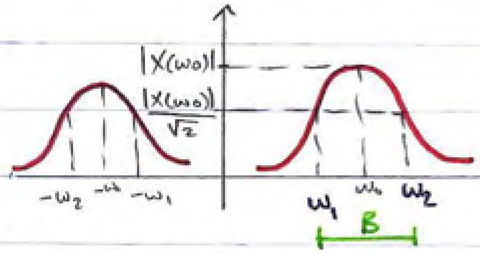
- Band limited
- Base Band
- $B = \omega_B$
"Absolute B"



- Band limited
- Band pass
- $B = 2\omega_B$
"Absolute B"



- Base Band
- $B = \omega_c$
"$\frac{1}{2}$-power/ 3-dB Bandwidth"

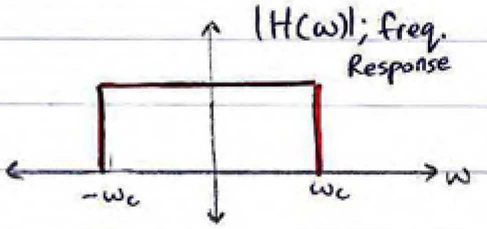


- Band pass
- $B = \omega_2 - \omega_1$
"$\frac{1}{2}$ power / 3dB Bandwidth".

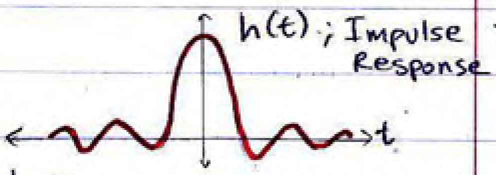
Null-to-Null & Zero crossing are mentioned above!

Practical Filters:

- Why can't we build an ideal Filter?



$\overset{\mathcal{F}}{\longleftrightarrow}$

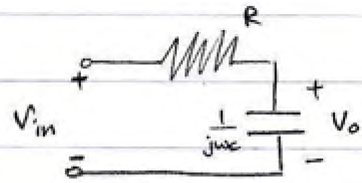


Ideal LPF

↳ the impulse response is a non-causal system (it can't be built Practically)!

$h(t) \neq 0$, when $t<0$

- Design a Practical Filter:-

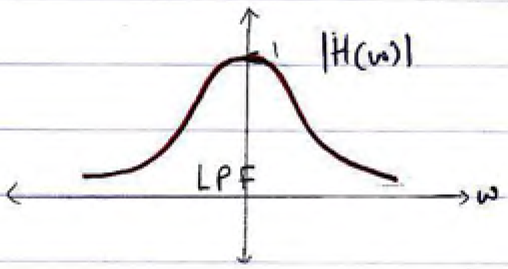


$$\frac{V_o}{V_i} = \frac{1}{1+j\omega RC} = \frac{1/RC}{1/RC + j\omega} = H(\omega)$$

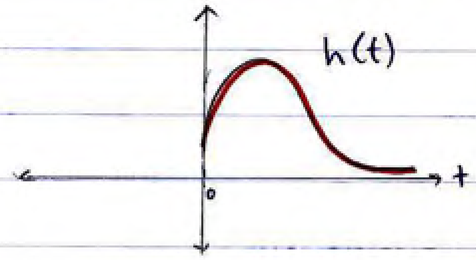
$$\therefore h(t) \overset{\mathcal{F}^{-1}(H(\omega))}{=} \frac{1}{RC} e^{-t/RC} \underline{\underline{u(t)}}$$

→ now it's causal!

$h(t)=0$, $t<0$ ✓



$\overset{\mathcal{F}}{\longleftrightarrow}$



* Practically speaking; all frequencies will pass but high frequencies will be suppressed/attenuated!

Cutoff frequency for Practical Filters:

* use $\frac{1}{2}$-power band width

from Previous LPF $\quad H(\omega) = \frac{1}{1 + j\omega RC}$

$$|H(\omega)|^2 = \frac{1}{1 + \omega^2 (RC)^2}$$

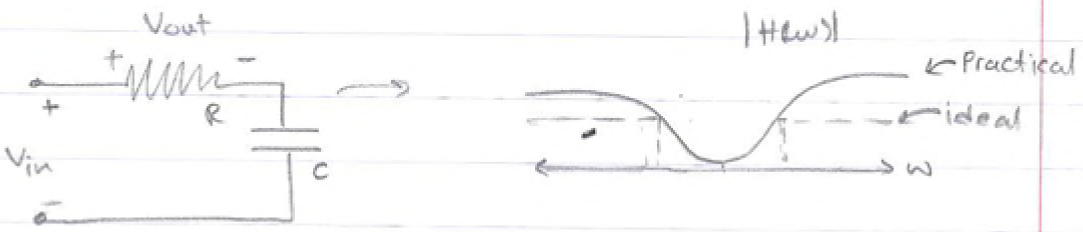
$$|H(0)|^2 = \frac{1}{1} = 1$$

$$\therefore \; |H(\omega_1)|^2 = \frac{1}{2} |H(0)|^2$$

$$\frac{1}{1 + \omega_1^2 (RC)^2} = \frac{1}{2} \times 1$$

$$\boxed{\omega_1 = \frac{1}{RC}}$$

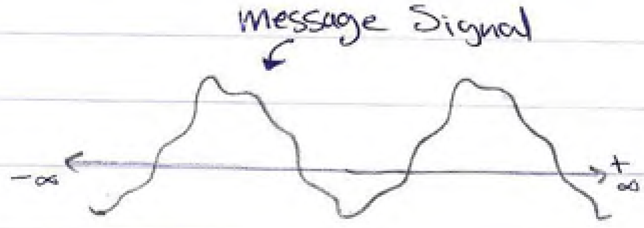
→ HPF design:-



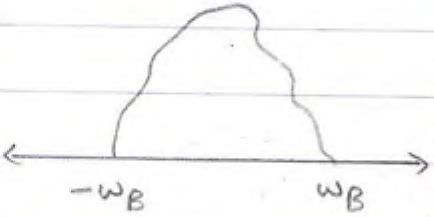
** $\boxed{HPF = 1 - LPF}$ $\quad \omega_c$ remains the same

e.g

Amplitude Modulation (AM)



$\xleftrightarrow{\mathcal{F}}$



4 kHz (for human beings).

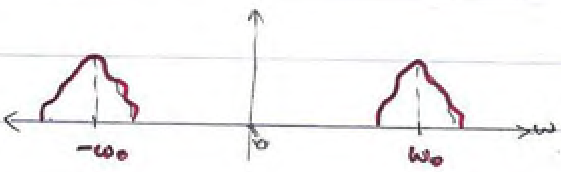
time unlimited

Band limited

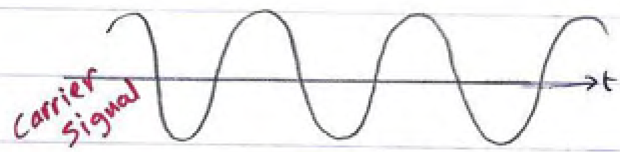
* if we want to increase the frequency! we use modulation.

[$X_m \times \cos \omega_0 t$]

→ in freq. domain

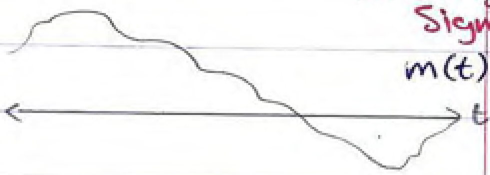


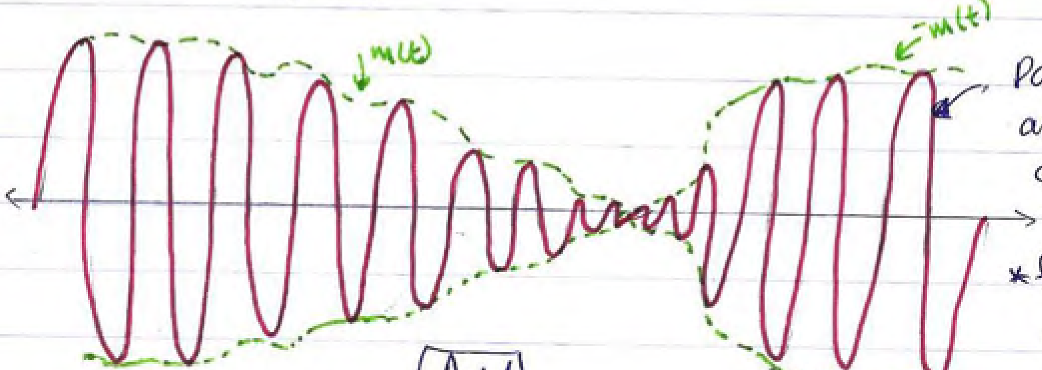
→ in time domain :-



$\cos(\omega_0 t)$

×



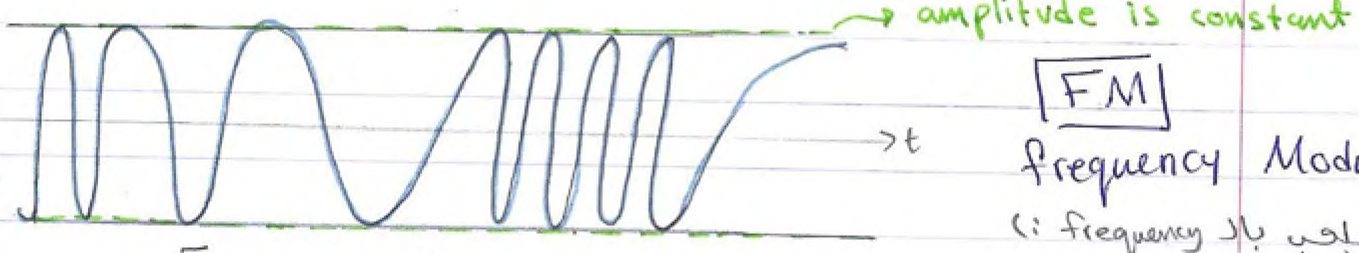


AM

Data is stored in the amplitude of the carrier signal.

* frequency of the carrier doesn't change.

حطينا ال message على ال carrier



Frequency Modulated Signal

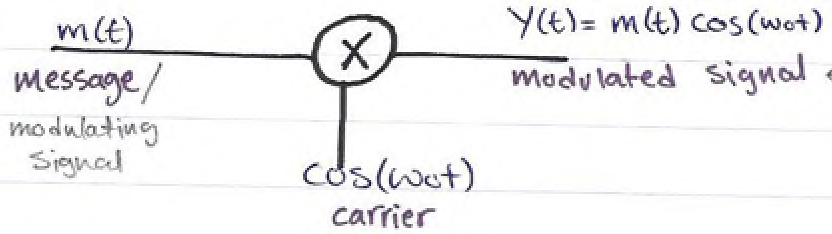
$\boxed{FM}$

frequency Modulation

بلعب بال frequency :)

data is stored in the freq. of the carrier.

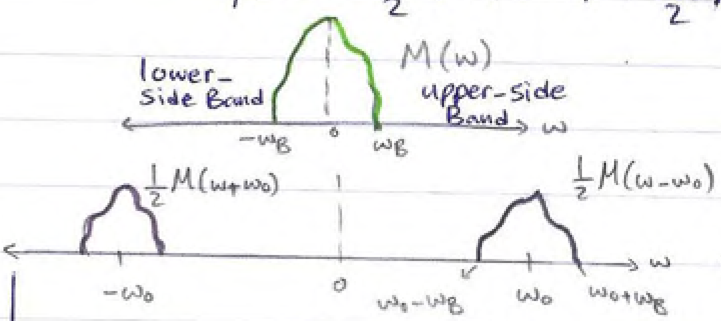
Back to <u>AM</u>



← this signal I want to transmit!

in frequency-domain :-

$$Y(\omega) = \frac{1}{2} M(\omega - \omega_0) + \frac{1}{2} M(\omega + \omega_0)$$



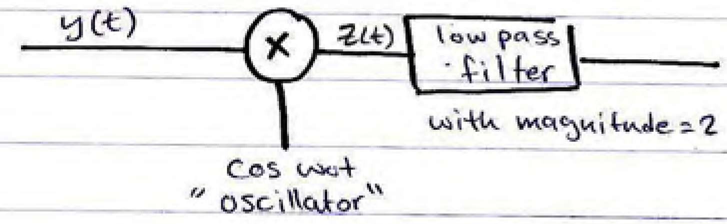
after modulation

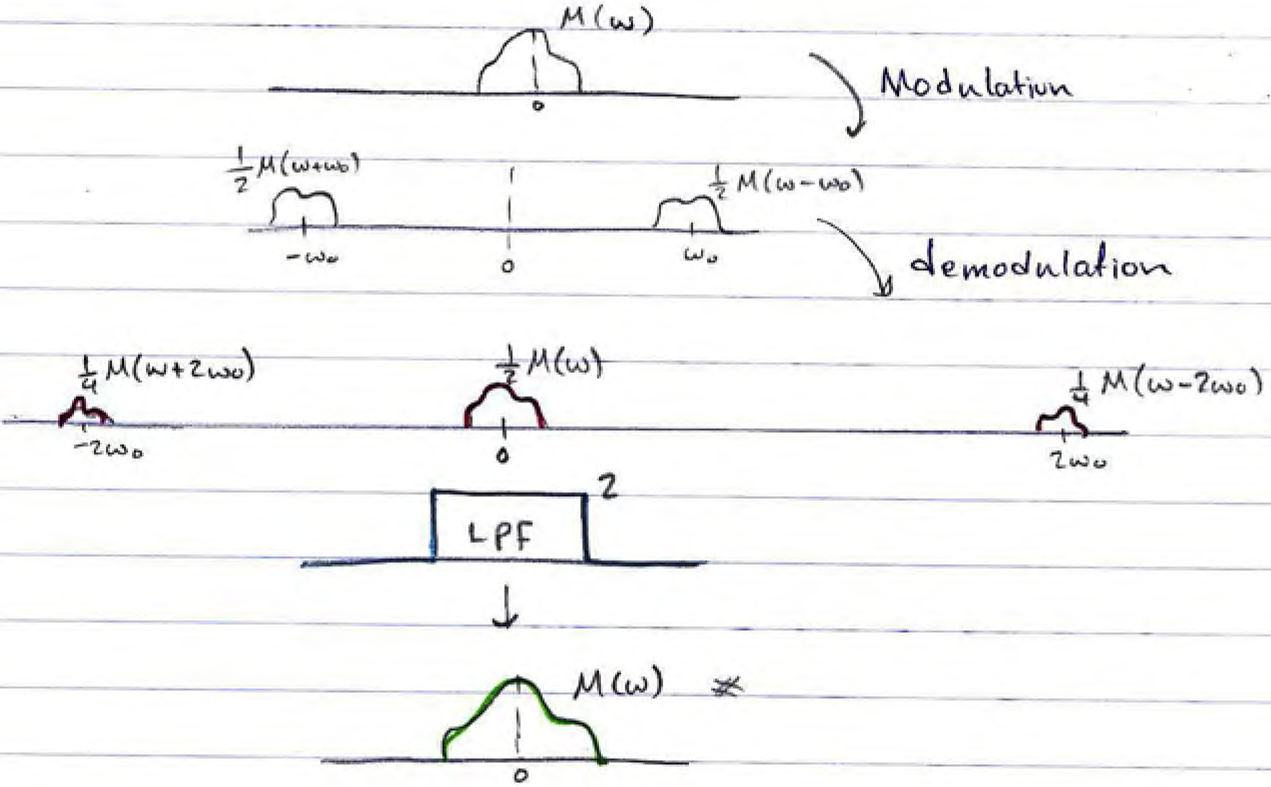
↳ this is called "Modulation" or (DSB/SC) ; Double-Side-Band <u>Suppressed</u> Carrier.

لأن ال carrier = cos المفروض يظهر impulses لكنه اختفى!

- Demodulation / synchronous detection:

to extract the message signal from the modulated signal.





$\rightarrow z(t) = y(t)\cos(\omega_0 t)$

$$Z(\omega) = \frac{1}{2}Y(\omega+\omega_0) + \frac{1}{2}Y(\omega-\omega_0)$$

$$= \frac{1}{2}\left[\frac{1}{2}M(\omega+2\omega_0) + \frac{1}{2}M(\omega)\right] + \frac{1}{2}\left[\frac{1}{2}M(\omega) + \frac{1}{2}M(\omega-2\omega_0)\right]$$

$\rightarrow$ the previous AM was theoritical, & it has many problems

first ; you double the bandwidth which increases the cost

second ; any delay in signal will introduce a phase shift, the cos may become sine! and when you demodulate the signal and apply LPF, it'll vanish!

Sampling:

* we want to construct a discrete time function from a cont. time function without losing the information contained in the (CT) world! so that we can always revert to the CT function from the samples.

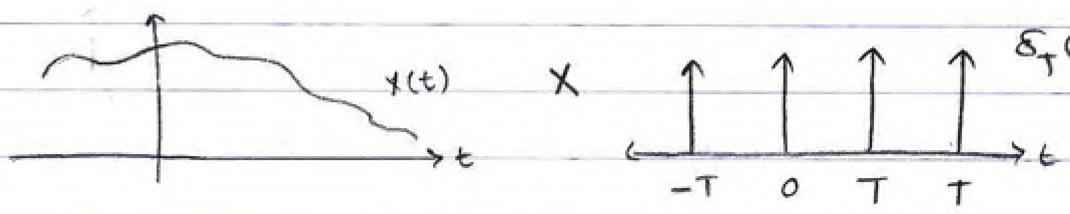
* **The Sampling Theorem:-** (shanon sampling Thm)
  - this Thm. states that a band limited signal $x(t)$ can be reconstructed exactly, from its samples if the samples are taken at a rete $w_s = \frac{2\pi}{T}$; where $w_s = 2w_B$ <u>at least</u>; $\therefore w_s \geq 2w_B$

  when $w_s = 2w_B$ it's called the "Nyquist Rate"

28/4/2013

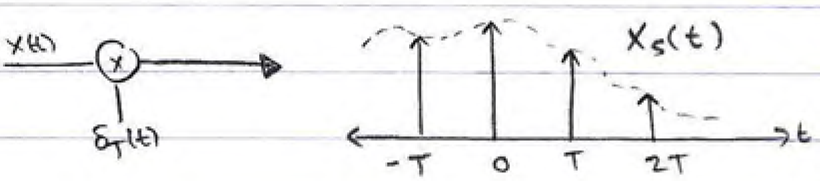
* There are two ways of sampling:-
  1) Ideal Sampling
  2) Practical sampling

1. Ideal Sampling / Impulse modulation model



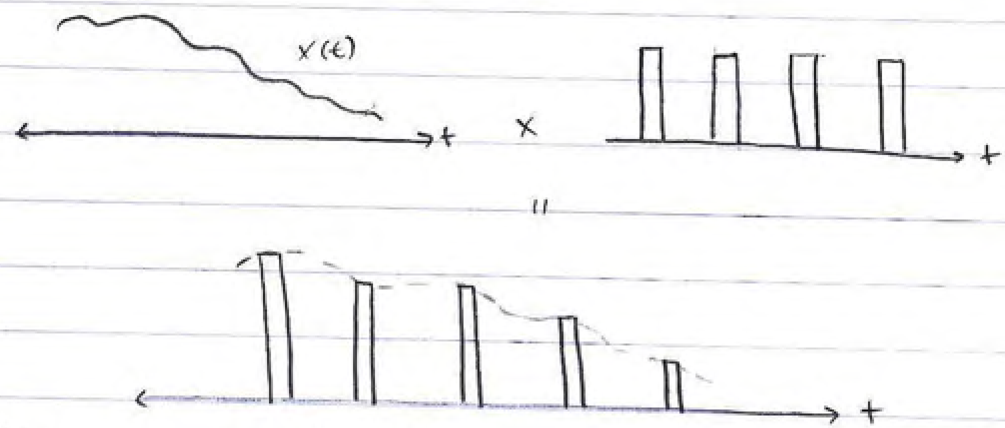
$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

=



2. Practical Sampling:

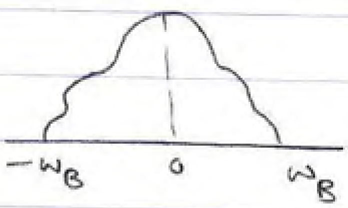
* because in real life we can't generate $\delta_T(t)$, then we replace it by thin rects.



this was Practical & ideal sampling in general, more details will be discussed later.

* why ω_s has to be $\geq 2\omega_B$?

if $X(\omega)$ is a Band limitited signal



- You have to sample at atleast $2\omega_B$
- if the sampling rate $= 2\omega_B$ it's called "Nyquist rate"

* Why $2\omega_B$?

↳ The ideal way :- (more details)

** In sampling, we use Hz instead of rad/s

$$x(t) \longrightarrow \otimes \longrightarrow x_s(t) \quad \xleftrightarrow{\mathcal{F}} \quad X(\omega) \longrightarrow \circledast \longrightarrow X_s(\omega)$$

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \qquad\qquad \sum_{n=-\infty}^{\infty} \frac{2\pi}{T}\delta(\omega-n\omega_s)$$

صارت الـ (*)

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

$$= \frac{1}{T} X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

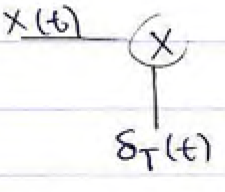
$$= \frac{1}{T}\left[\cdots + X(\omega+\omega_s) + X(\omega) + X(\omega-\omega_s) + \cdots \right]$$

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)$$

** Sampling rate:-

- in time-domain = T_{sec} ; (I take a sample every (T) sec)

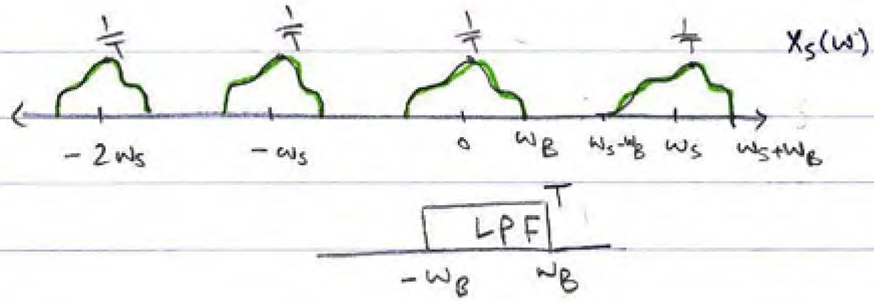
or - in freq.- " = ω_s [Hz]



$x_s(t)$

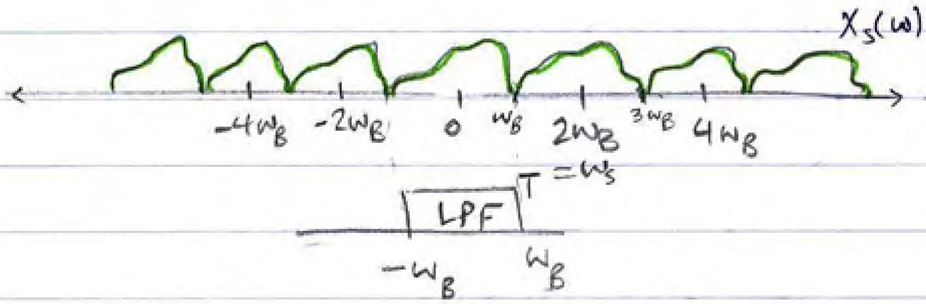
$$\hookrightarrow x_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

∴ it creates infinite copies at $(n\omega_s)$



$\omega_s > \omega_B$

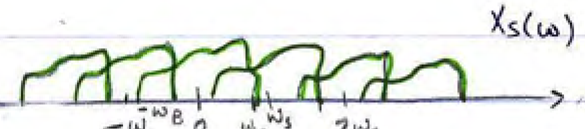
* Don't over sample!
(more storage area, higher frequency!)
so, don't use it unless you need it!



$\omega_s = \omega_B$

Nyquist rate

That's why we have to sample at atleast twice ω_B.



$\omega_s < 2\omega_B$

Problem!

Aliasing!

تداخل!

the signal is lost!

*if the function wasn't band limited, Aliasing must happen

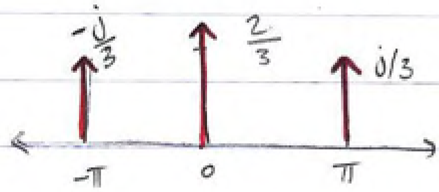
!

$\omega_B \rightarrow$ maximum frequency

ex:- $x(t) = \frac{1}{3\pi} + \frac{1}{3\pi}\cos\left(\pi t + \frac{\pi}{2}\right) = \frac{1}{3\pi} + \frac{1}{3\pi}\sin(\pi t)$

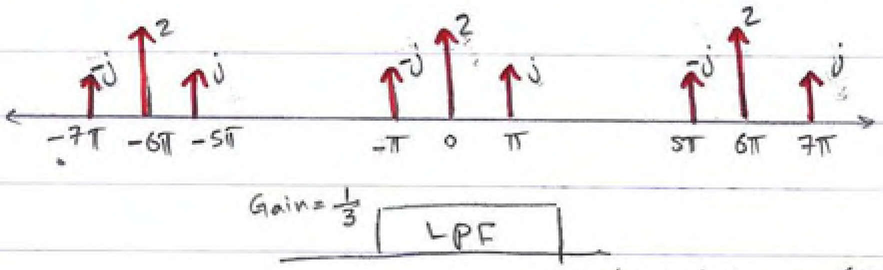
$X(\omega) = \frac{2}{3}\delta(\omega) + \frac{j}{3}\delta(\omega - \pi) - \frac{j}{3}\delta(\omega + \pi)$ ← Band limited function

- Nyquist rate $= 2\omega_B = 2\times\pi = 2\pi$



let $\omega_s = 6\pi$ (3 times the nyquist rate).

$\omega_s = \frac{2\pi}{T}$, $T = \frac{1}{3}$

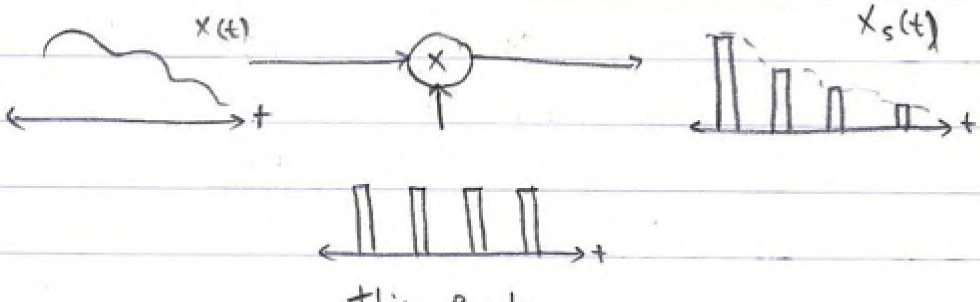


any where between $(\pi, 5\pi)$

* whenever you have impulses in freq-domain , at nyquist rate aliasing always happen!

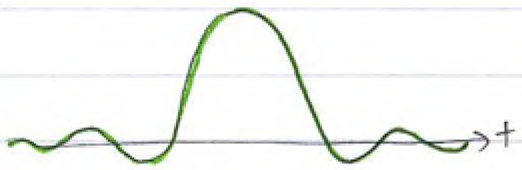
↳ Practical Sampling

time domain:



thin Rects

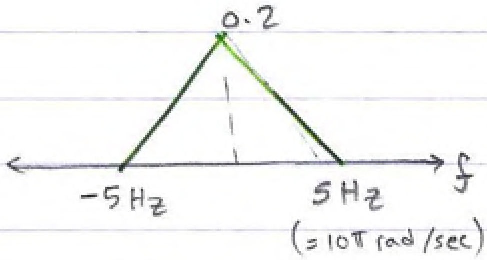
ex:- $x(t) = \text{sinc}^2(5t)$



t-domain

$$\Delta\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \frac{T}{2}\,\text{sinc}^2\left(\frac{\omega T}{2\pi}\right)$$

$$\text{sinc}^2(5t) \xleftrightarrow{\mathcal{F}} \frac{2}{10}\,\Delta\left(\frac{\omega}{20\pi}\right)$$, Band limitid



freq.-domain

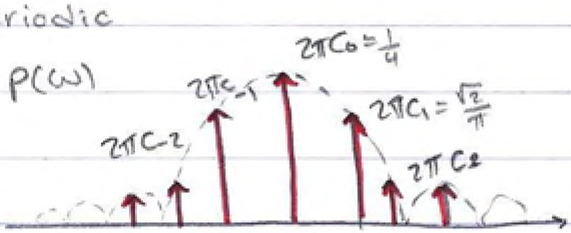
I want to sample at the nyquist rate = 2×5 = 10 Hz

يعني sample بكل 0.1 ثانية !

practical sampling function (rects)

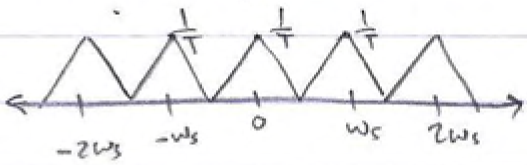
p(t) is periodic

$p(t) \xrightarrow{\mathcal{F}} P(\omega)$



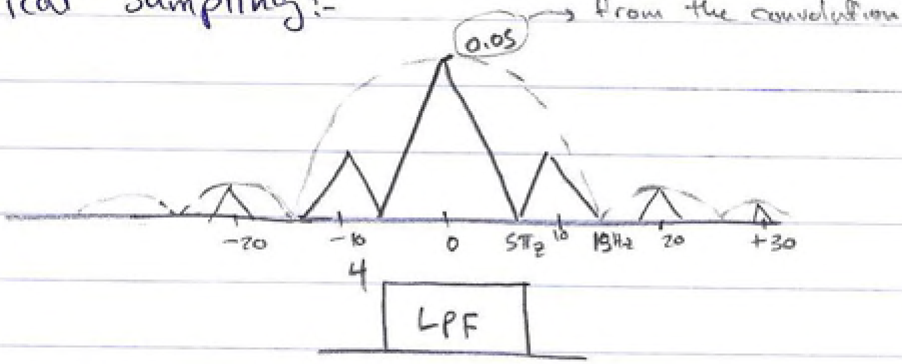
$$X_s(\omega) = \frac{1}{2\pi}\left[0.2\,\Delta\left(\frac{\omega}{20\pi}\right) * P(\omega)\right]$$

in ideal sampling:-



exact copies at ws & it's multiples.

- Practical Sampling:-



<u>DT</u>:-

$\cos 2t$ ✓ periodic

$\cos[2n]$ ✗ not periodic!

we have to test the periodicity for complex exponentials & sinusoids in DT.

* $x[n] = C\alpha^n$

$= C(e^\beta)^n \quad ; \quad \alpha = e^\beta$

$\beta = \ln \alpha$

ex:- $5^n = e^{0.609n}$

$e^{3n} = (e^3)^n = 20.086^n$

30/4/2013

Discrete Time DT:

- complex exponentials:-

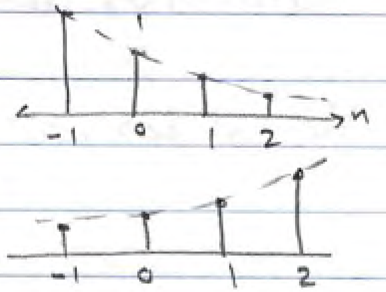
* exponentials in general:-

$$X[n] = C\alpha^n$$

$$X[n] = C(e^{\beta})^n$$

$\alpha^n \rightarrow |\alpha| < 1$

$\alpha^n \rightarrow |\alpha| > 1$



* complex exp.:-

capital omega!

$$X[n] = e^{j\Omega n}$$

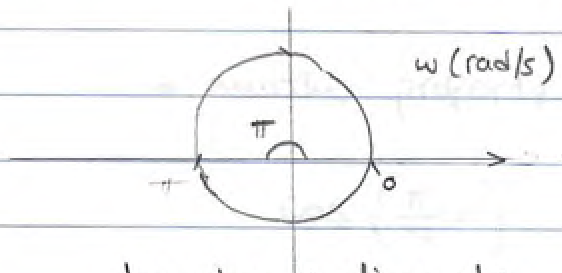
$re^{j\theta}$; $r = 1$, $\theta = \Omega n$

in CT:-

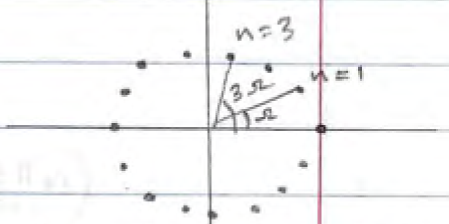
$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

in DT:-

$e^{jn\Omega} = \cos(n\Omega) + j\sin(n\Omega)$



traveling continuously



traveling discretely

→ test of periodicity:

for periodic function:-

$$X[n] = X[n+N]$$

for complex exp:-

$$e^{j\Omega(n)} \overset{?}{=} e^{j\Omega(n+N)} = e^{j\Omega n} e^{j\Omega N}$$

↳ in order to have a periodic function $e^{j\Omega N} = 1$

$\therefore$ $\boxed{\Omega N = 2\pi m}$ integer the physical meaning of this test:-

→ I need to get back to the same starting point.

$$\boxed{\frac{\Omega}{2\pi} = \frac{m}{N}}$$ ← test of periodicity

rational

in CT → $\cos 2t$ ✓ periodic

in DT → we have to test for periodicity.

- $\cos[2n]$

$\frac{2}{2\pi}$ is irrational! ∴ $\cos 2n$ isn't periodic.

- $\cos\left[\frac{4\pi}{17}n\right]$

$\left(\frac{4\pi}{17}\right) / 2\pi = \frac{2}{17}$ periodic with $N = 17$ (period)

- $e^{j\left(\frac{7\pi}{9}\right)n}$

$\frac{\left(\frac{7\pi}{9}\right)}{2\pi} = \frac{7}{18}$, periodic with $N=18$

- another property of DT:-

$\cos\left(\frac{\pi}{4}n\right)$ $\qquad$ $\cos\left(\frac{7\pi}{4}n\right)$

$\Omega_1 = \frac{\pi}{4}$ $\qquad$ $\Omega_2 = \frac{7\pi}{4}$ higher freq.

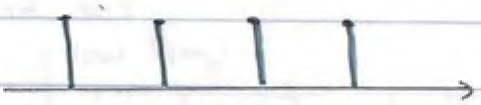
$\left(\frac{\pi}{4}\right)/2\pi = \frac{1}{8}$ ✓ periodic $\qquad$ $\left(\frac{7\pi}{4}\right)/2\pi = \frac{7}{8}$ ✓ periodic

$N_1 = 8 \longleftrightarrow N_2 = 8$!

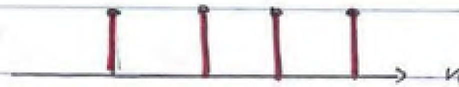
have the same period!

∴ high & low frequencies look the same on DT!

low frequencies | high frequencies!

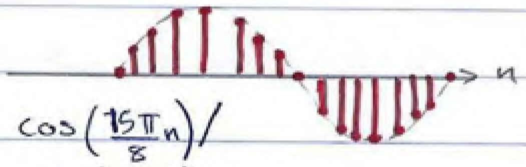


$\cos(0 * n)$

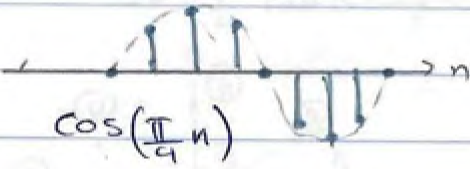


$\cos(2\pi n)$ / $\cos(4\pi n)$ ...

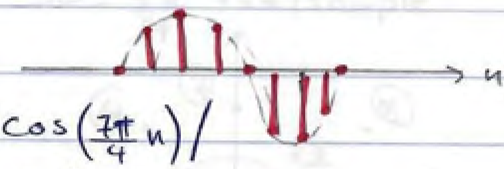
$\cos\left(\frac{\pi}{8}n\right)$



$\cos\left(\frac{15\pi}{8}n\right)$ / $\cos\left(\frac{31\pi}{8}n\right)$ ...

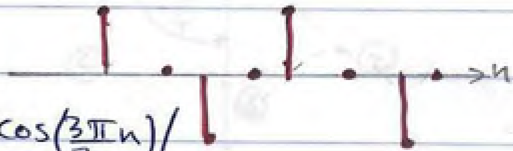


$\cos\left(\frac{\pi}{4}n\right)$

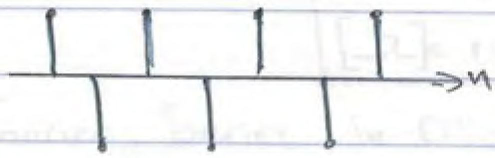


$\cos\left(\frac{7\pi}{4}n\right)$ / $\cos\left(\frac{15\pi}{4}n\right)$ ...

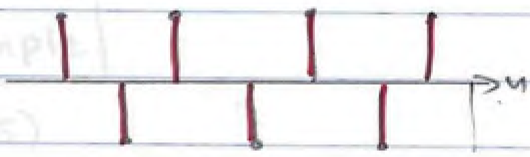
$\cos\left(\frac{\pi}{2}n\right)$



$\cos\left(\frac{3\pi}{2}n\right)$ / $\cos\left(\frac{5\pi}{2}n\right)$ ...



$\cos(\pi n)$



$\cos(3\pi n)$ / $\cos(5\pi n)$ ...

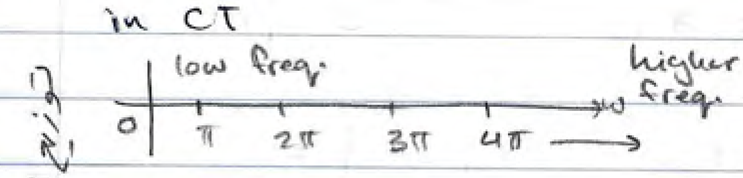
high frequencies appear as low frequencies in DT!

$$\cos\left(\frac{3\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right)$$

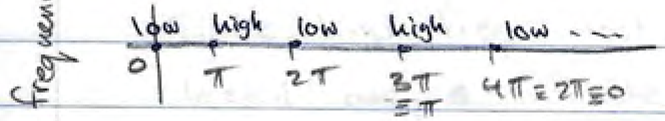
$$\cos\left(\frac{7\pi n}{4}\right) = \cos\left(\frac{\pi}{4}n\right)$$

$$\cos(2\pi n) = \cos(0 \times n) \quad \ldots$$

in CT

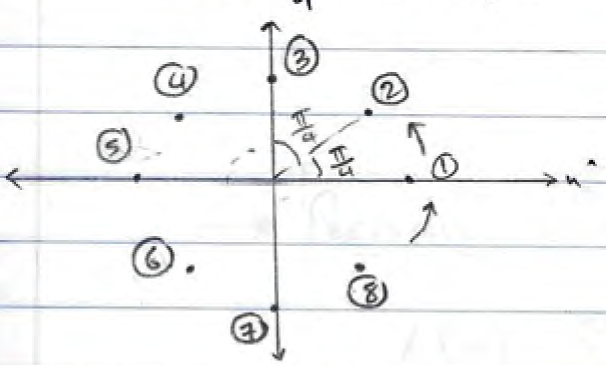


in DT



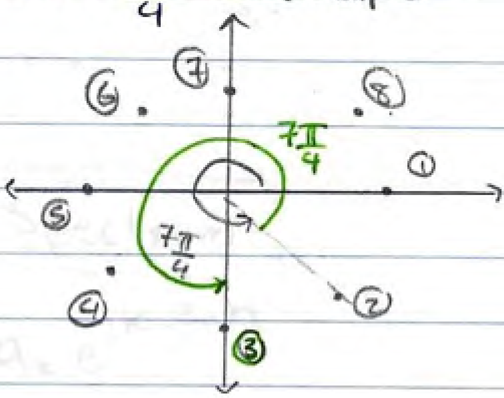
Explanation :-

$\Omega = \frac{\pi}{4}$ rad/sample



بعد لفة واحدة يرجع لنفس النقطة

$\Omega = \frac{7\pi}{4}$ rad/sample



بعد 8 لفات يرجع لنفس النقطة!

$\boxed{[\Omega] = \text{rad/sample}}$

Fourier Series in DT :- (DTFS)

- in CT :-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\hookrightarrow C_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t}\,dt$$

- in DT :-

because the (a_k)s are not distinct; they keep repeating theirselves; (periodic line spectrum)

$$x[n] = \sum_{\cancel{k=-\infty}}^{\cancel{\infty}} a_k e^{jk\Omega_0 n} \qquad k = \langle N \rangle$$

$$\hookrightarrow a_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n}$$

لأنه a_k يكرر نفسه فما احتاج $-\infty \to \infty$

← summation over N samples

* بعمل (Σ) من $0 \to N-1$

Be careful

$e^{jk\Omega_0 n}$, $k = 0, \pm 1, \pm 2, \ldots$

$= \{\ldots, e^{-j\Omega_0 n}, 1, e^{j\Omega_0 n}, \ldots, e^{jk\Omega_0 n}, \ldots e^{j(k+N)\Omega_0 n}, \ldots\}$

since high freq. looks lik low freq. in DT, I don't need an ~~e~~ infinite ~~\#~~ of exponentials to represent my function.

$$e^{j(k+N)\Omega_0 n} = e^{jk\Omega_0 n} \cdot e^{jN \overset{\frac{2\pi}{N}}{\Omega_0} n}$$

$$e^{j2\pi n} = 1$$

$$e^{j(k+N)\Omega_0 n} = e^{jk\Omega_0 n}$$

$\longrightarrow$ $\boxed{\text{Periodic line Spectrum}}$

$$\therefore \; X[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}$$

5/5/2013

DTFS:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} \quad ; \quad a_k = a_{k+N}$$

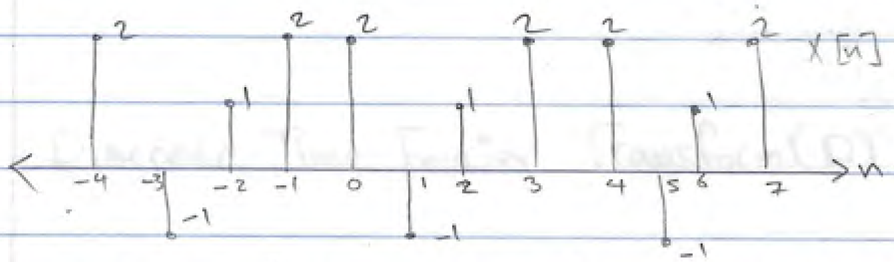
$k = \langle N \rangle$: over N samples (from 0 to N-1)

t-domain — **freq-domain**

- CT, periodic $\xrightarrow{F.S}$ discrete & non-periodic
- CT, non-periodic $\xrightarrow{F.T}$ continuous & non-periodic

- DT, periodic $\xrightarrow{DTFS}$ Discrete & periodic
- DT, non-periodic $\xrightarrow{DTFT}$ continuous & "

ex:- $x[n] = \{2, -1, 1, 2\}$, periodic function



$\boxed{N=4}$

$\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{2}$

since it has a periodic line spectrum, I only need to calculate a_0, a_1, a_2, a_3. $\left(a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\Omega_0 n} \right)$

- $a_0 = \frac{1}{4} \sum_{n=0}^{3} x[n] = \frac{2+1-1+2}{4} = 1$ (average value)/DC
- $a_1 = \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-j\frac{\pi}{2}n}$ $= (j)^{-n}$

$$= \frac{1}{4} x[0] + \frac{1}{4} x[1] \overset{-j}{\cancel{e^{-j\frac{\pi}{2}}}} + \frac{1}{4} x[2] \overset{-1}{\cancel{e^{-j\pi}}} + \frac{1}{4} x[3] \overset{j}{\cancel{e^{-j\frac{3\pi}{2}n}}}$$

$$= \frac{1}{\cancel{4}_2} \times \cancel{2} + \frac{1}{4} \times -1 \times -j + \frac{1}{4} \times 1 \times -1 + \frac{1}{4} \times 2 \times j = \frac{1}{2} - \frac{1}{4} + \frac{j}{4} + \frac{j}{2} = \boxed{\frac{1}{4} + j\frac{3}{4}}$$

$$a_2 = \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-j\pi n} \equiv (-1)^n$$

$$= \frac{1}{4} x[0] + \frac{1}{4} x[1] \times (-1) + \frac{1}{4} x[2] \times (1) + \frac{1}{4} x[3] \times (-1)$$

$$= \frac{1}{2}$$

$$a_3 = \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-j\frac{3\pi}{2} n} \equiv e^{j\frac{\pi}{2} n} \equiv (j)^n$$

$$= \frac{1}{4} x[0] + \frac{1}{4} x[1] (j) + \frac{1}{4} x[2] (-1) + \frac{1}{4} x[3] (-j)$$

$$= \frac{1}{2} + \frac{j}{4} - \frac{1}{4} - \frac{j}{2} = \boxed{\frac{1}{4} - j\frac{3}{4}}$$

→ * whenever you have a complex coeficient, you'll find it's conjugate since you are dealing with real time func

$$\boxed{a_1 = a_3^*}$$

Discrete Time Fourier Transform (DTFT):

- In CT:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{direct F.T}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \qquad \text{Inverse F.T}$$

- In DT:-

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \qquad \text{DTFT}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega \qquad \text{IDTFT}$$

inverse

$X(\Omega)$:

- it's continuous, since $\Omega \in \mathbb{R}$
- test of periodicity:

we chose 2π because $e^{-j2\pi n} = 1$

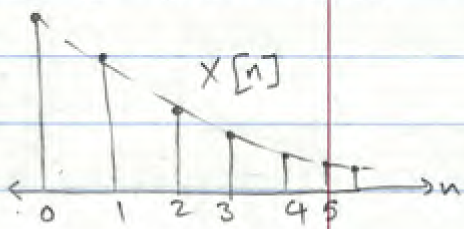
$$X(\Omega) \overset{?}{=} X(\Omega + 2\pi) \, ?$$

$$X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j(\Omega + 2\pi) n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \cdot \underbrace{e^{-j2\pi n}}_{1} = X(\Omega)$$

$\therefore$ $X(\Omega)$ is periodic with priod $= 2\pi$

ex:- $x[n] = \alpha^n u[n]$, $\boxed{\alpha < 1}$ equiveland to $e^{-at} u(t)$ in CT



$$X(\Omega) = \sum_{n=0}^{\infty} \alpha^n \cancel{u[n]} \, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \left(\alpha e^{-j\Omega}\right)^n$$

$$X(\Omega) = \frac{1}{1 - \alpha \underbrace{(\cos\Omega - j\sin\Omega)}_{e^{-j\Omega}}}$$

since:- $\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \text{ if } r<1}$

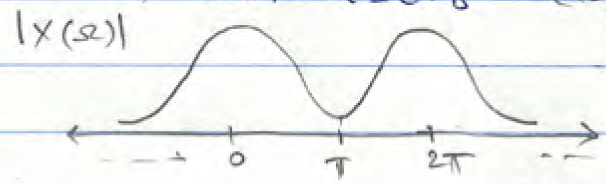
convergance of the series!

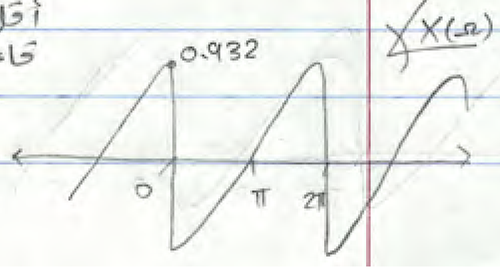
$$\hookrightarrow |X(\Omega)| = \frac{1}{\sqrt{(1 - \alpha\cos\Omega)^2 + (\alpha\sin\Omega)^2}}$$

$$\hookrightarrow \measuredangle X(\Omega) = \tan^{-1}\left(\frac{\alpha\sin\Omega}{1 - \alpha\cos\Omega}\right)$$

let $\alpha = 0.8$

أقل ما فيّ نطبق
قاعدة الـ convergance





ex 2: $x[n] = \delta[n]$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = e^{-j\Omega n}\Big|_{n=0} = 1$$

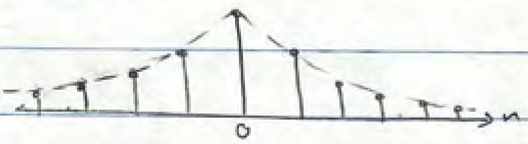
$x[n] = \delta[n-k]$

$$X(\Omega) = e^{-j\Omega n}\Big|_{n=k} = e^{-j\Omega k}$$

* Tricks!

$$\sum_{n=1}^{\infty}\left(\begin{matrix}\text{some func.}\\ \text{of } n\end{matrix}\right) = \sum_{n=0}^{\infty}\left(\begin{matrix}\text{the same}\\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{قيمة الـ summation}\\ \text{لما نعوض } \boxed{n=0}\end{matrix}\right).$$

ex:- $x[n] = \alpha^{|n|}$



$$X(\Omega) = \sum_{n=-\infty}^{-1}\left(\alpha^{-n} e^{-j\Omega n}\right) + \sum_{n=0}^{\infty}\left(\alpha^{n} e^{-j\Omega n}\right)$$

عوّض (-n) مكان (n)

عكسهم عادي لأنه عملية الجمع تبديلية!

$$= \sum_{n=1}^{\infty} \alpha^{n} e^{j\Omega n} + \sum_{n=0}^{\infty}\left(\alpha^{n} e^{-j\Omega n}\right)$$

$$= \sum_{n=0}^{\infty} \alpha^{n} e^{j\Omega n} - \left(\cancel{\alpha^{(0)} e^{j\Omega(0)}}\right) + \sum_{n=0}^{\infty} \alpha^{n} e^{-j\Omega n}$$

$$= -1 + \sum_{n=0}^{\infty}\left[\left(\alpha e^{j\Omega}\right)^{n} + \left(\alpha e^{-j\Omega}\right)^{n}\right]$$

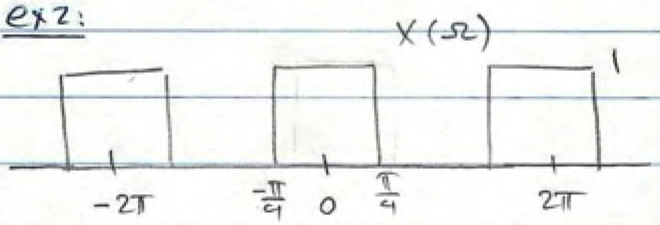
$$= -1 + \frac{1}{1-(\alpha e^{j\Omega})} + \frac{1}{1-(\alpha e^{-j\Omega})}$$

$$= \frac{-1(1-\alpha e^{j\Omega})(1-\alpha e^{-j\Omega}) + 1(1-\alpha e^{-j\Omega}) + 1(1-\alpha e^{j\Omega})}{(1-\alpha e^{j\Omega})(1-\alpha e^{-j\Omega})}$$

$$= \frac{2 - 1 + \cancel{\alpha e^{-j\Omega}} + \cancel{\alpha e^{j\Omega}} - \alpha^2 - \cancel{\alpha e^{-j\Omega}} - \cancel{\alpha e^{j\Omega}}}{1 - \alpha\left(e^{-j\Omega} + e^{j\Omega}\right) + \alpha^2}$$

$$= \frac{1 - \alpha^2}{1 - 2\alpha\cos\Omega + \alpha^2}$$

ex2:



$\xrightarrow{IDTFT}$

$x[n]$?

sol: $$x[n] = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} 1 \cdot e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi j n} e^{j\Omega n}\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{4}\sin\left(\frac{n}{4}\right)$$

→ DTFT (for periodic functions):

- In CT:

$$X(\omega) = \sum_{n=-\infty}^{\infty} 2\pi |C_n| \, \delta(\omega - n\omega_0)$$

- In DT:-

$$X(\Omega) = \sum_{K=-\infty}^{\infty} 2\pi a_K \, \delta(\Omega - K\Omega_0)$$

ex: $x[n] = 2\cos\left(\frac{3\pi}{8n} + \frac{\pi}{3}\right) + 4\sin\left(\frac{\pi}{2}n\right)$

** whenever you see sines & cosines ↑ in (DT), Be sure to test for periodicity first!

$$x[n] = 2\cos\Big(\underbrace{\frac{3\pi}{8n} + \frac{\pi}{3}}_{\Omega_1}\Big) + 4\sin\Big(\underbrace{\frac{\pi}{2}n}_{\Omega_2}\Big)$$

$$\frac{\left(\frac{3\pi}{8}\right)}{2\pi} = \frac{3}{16}$$ ✓ periodic with period $N_1 = 16$

$$\frac{\left(\frac{\pi}{2}\right)}{2\pi} = \frac{1}{4}$$ ✓ periodic with period $N_2 = 4$

∴ both functions and their sum are periodic

$$\rightarrow \frac{3\pi}{8} = k\Omega_0 \qquad \frac{\pi}{2} = m\Omega_0$$

$$\hookrightarrow \boxed{\Omega_0 = \frac{\pi}{8}}$$

$k = 3$, $m = 4$; third & fourth harmonics

$k = \pm 3, \pm 4$

$$a_k = \begin{cases} -\frac{2}{j}, & k = -4 \\ -3, & k = -3 \\ 3, & k = 3 \\ \frac{2}{j}, & k = 4 \\ 0, & \text{o.w} \end{cases}$$

over one period (N)

Since $x[n]$ is periodic & $\Omega_0 = \frac{\pi}{8}$

$$\therefore \frac{\frac{\pi}{8}}{2\pi} = \frac{m}{N} = \frac{1}{16}$$

$$\therefore \boxed{N = 16} \longrightarrow -7 \leq k \leq 8$$

zero is included!

$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\Omega - k\Omega_0)$$

$$\rightarrow X(\Omega) = -\frac{4\pi}{j}\delta\left(\Omega+\frac{\pi}{2}\right) + 2\pi e^{-j\frac{\pi}{3}}\delta\left(\omega+\frac{3\pi}{8}\right) + 2\pi e^{j\frac{\pi}{3}}\delta\left(\omega-\frac{3\pi}{8}\right) + \frac{4\pi}{j}\delta\left(\Omega-\frac{\pi}{2}\right)$$

$\hookrightarrow$ this $X(\Omega)$ over one period (2π) $\rightarrow -\pi \leq \Omega \leq \pi$

- Properties of The DTFT:

1. Periodic $\quad X(\Omega) = X(\Omega + 2\pi)$
2. linear

$$x_1[n] \xrightarrow{DTFT} X_1(\Omega)$$

$$x_2[n] \xrightarrow{DTFT} X_2(\Omega)$$

$$\Rightarrow \alpha x_1 + \beta x_2 \xrightarrow{DTFT} \alpha X_1(\Omega) + \beta X_2(\Omega)$$

3. Shifting

$$x[n] \xrightarrow{DTFT} X(\Omega)$$

- $x[n-n_0] \xrightarrow{DTFT} X(\Omega)e^{-jn_0\Omega}$
- $e^{j\Omega_0 n}x[n] \xrightarrow{DTFT} X(\Omega-\Omega_0)$

4. $\quad n\,x[n] \xrightarrow{DTFT} j\frac{dX(\Omega)}{d\Omega}$
5. $\quad x[n] * h[n] \xrightarrow{DTFT} X(\Omega) \cdot H(\Omega)$

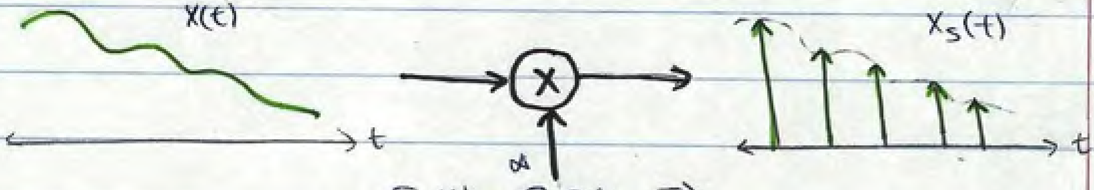
6. $\quad$ in CT:- $\displaystyle\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$

in DT:- $\displaystyle\sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|X(\Omega)|^2 d\Omega$

7/5/2013

Sampling:-

- In CT: Calculate $X_s(\omega)$ using Multiplication Property:-



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

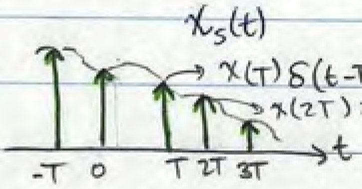
$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * \mathcal{F}\{\delta_T(t)\}$$

$$\boxed{X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)} \quad \text{...①}$$

↳ see the proof in the previous chapter.

; infinite copies of $X(\omega)$ scaled ~~with~~ by a factor of $\left(\frac{1}{T}\right)$ & placed at ω_s & its integer multiples.

* Calculate $X_s(\omega)$ using time Shifting property:-

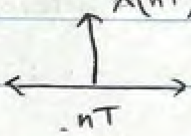


$x(T)\delta(t-T) \xrightarrow{\mathcal{F}} x(T)\, e^{-jT\omega}$

$x(2T)\delta(t-2T) \xrightarrow{\mathcal{F}} x(2T)\, e^{-j2T\omega}$

& so on...

in general

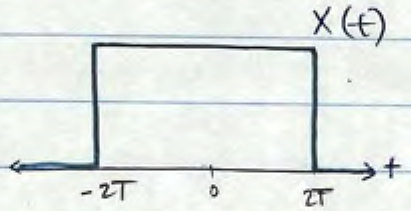


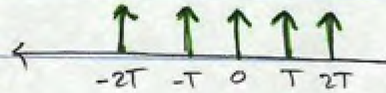
$x(nT)\delta(t-nT) \xleftrightarrow{\mathcal{F}} X(nT)\, e^{-jnT\omega}$

from linearity

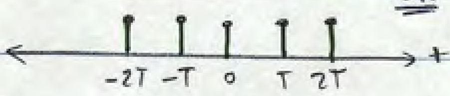
$$\boxed{X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-jnT\omega}} \quad \text{...②}$$

eq ① = ② !



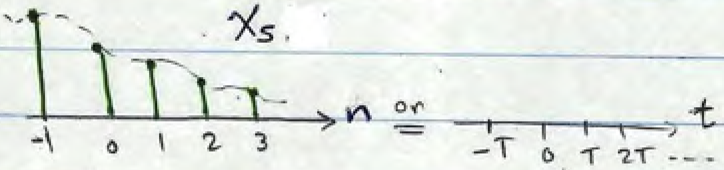

Sampling using continuous Train of impulses ⇒ X_s cont.

OR


" " Discrete " " " ⇒ X_s discrete

ex:

Sampling using DT train of impulses



now, take the DTFT for $X_s[n]$

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} X_s[n] \, e^{-j\Omega n}$$

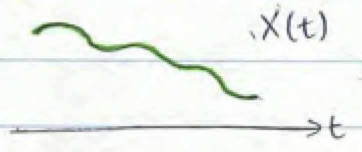
but from 2

$$\text{|||}$$

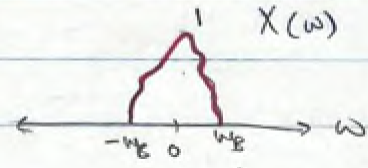
$$X_s(\omega) = \sum_{n=-\infty}^{\infty} X(nT) \, e^{-jnT\omega}$$

$$\therefore \; X_s(\Omega) = X_s(\omega) \quad \text{when} \quad \boxed{\omega = \frac{\Omega}{T}}$$

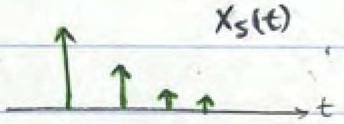
* but what does this mean?



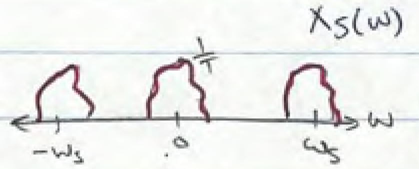
$\xrightarrow{F.T}$



the ideal sampling

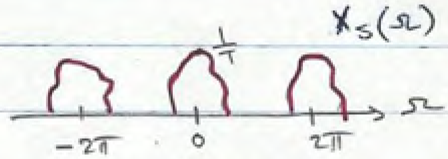


$\xrightarrow{F.T}$



$\boxed{\omega_s = \frac{2\pi}{T}}$ → Ω_0

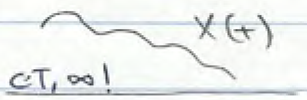
$\xrightarrow{DTFT}$



$X_s(\omega)$ & $X_s(\Omega)$

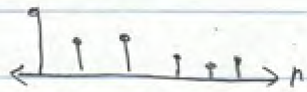
* they have the same frequency representation, except that the frequency-axis is bieng scaled!

so, we're interested in F.T of $x(t)$!



$\xrightarrow{F.T}$?!

* computationaly I can't find F.T of the train of impulses.



$\xrightarrow{DTFT}$ $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$

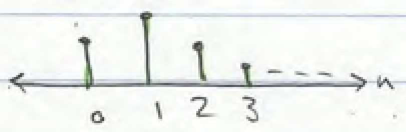
↳ if $x(t)$ isn't time limited

↳ 1^{st} Problem: infinite summation!

↳ 2^{nd} Problem: $\Omega \in \mathbb{R}$ infinite values of Ω!

Solution:

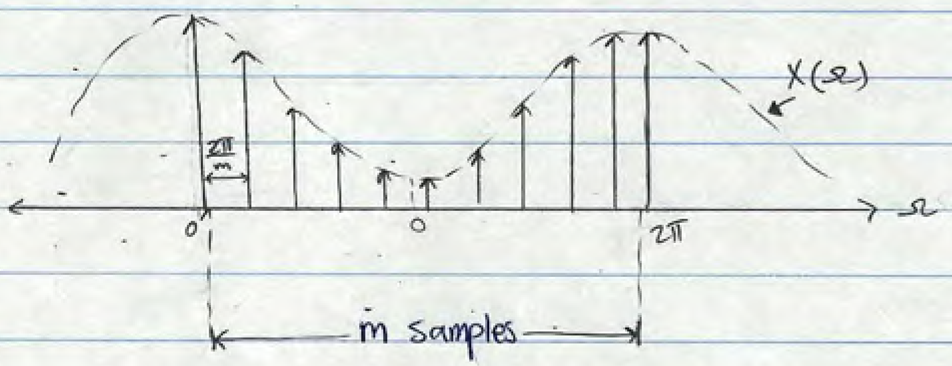
→ Truncation: (solving the 1^{st} prob.)



$\xrightarrow{DTFT}$ $\sum_{n=0}^{N-1} x[n]\, e^{-j\Omega n}$ → Still have a Problem!

take (N) Samples

→ sampling $X(\Omega)$! (solving the 2^{nd} prob.)



$\Omega_1 = \frac{2\pi}{m}$

$\Omega_2 = \frac{2\pi}{m} \times 2$

$\Omega_3 = \frac{2\pi}{m} \times 3$

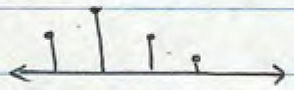
$\vdots$

$\Omega_k = \frac{2\pi}{m} \times k \quad , K = 0, 1, 2, \ldots\ldots, m-1 \quad = \boxed{\frac{2\pi}{N} K}$

$\boxed{m = N}$, عادةً نختا

عدد الـsamples اللي اخترناها بالـtruncation

* now, it'll be called DFT instead of DTFT!



(N samples) of $x[n]$! (truncated version) $\xrightarrow[\text{Discrete Fourier Transform.}]{\text{DFT}}$ $X[\Omega_k] = \sum_{n=0}^{N-1} x[n]\, e^{-j \overbrace{\frac{2\pi}{N} K}^{\Omega_k} n}$

periodic with period = 2π

(DFT) is the discrete (sampled) version of (DTFT)
& $\underset{\text{DFT}}{\text{it's}}$ periodic with period (N)

* $X[K] = X[K+N]$

* $\underline{N}$-point DFT , $\left(\sum_{n=0}^{N-1}\right)$

عدد الـsamples !

Inverse DFT:-

- IDTFT : $x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$

  Inverse Discrete Time Fourier Transform

- IDFT : $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$

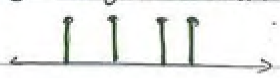
  Inverse Discrete Fourier Transform.

Summary: my ultimate goal is to be able to find F.T computationly (e.g using MATLAB)

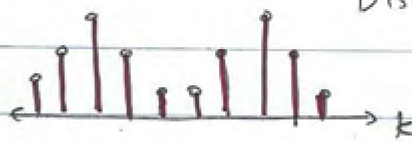
* For time-limited Signals:

F.T → ?

Sampling using DT train of impulses



DFT →

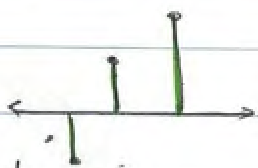


Discrete & Periodic F.T representation

* For time-unlimited signal $x(t)$

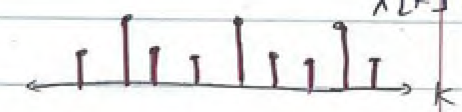
* we take truncated version of the Discrete sampled version of $x(t)$.

is sampled using DT train of impulses



DFT →

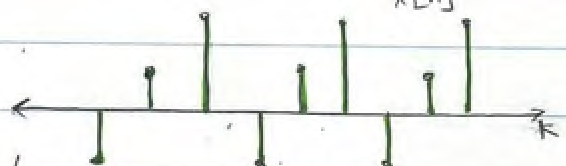
$X[k]$



N-samples

IDFT

$x[n]$

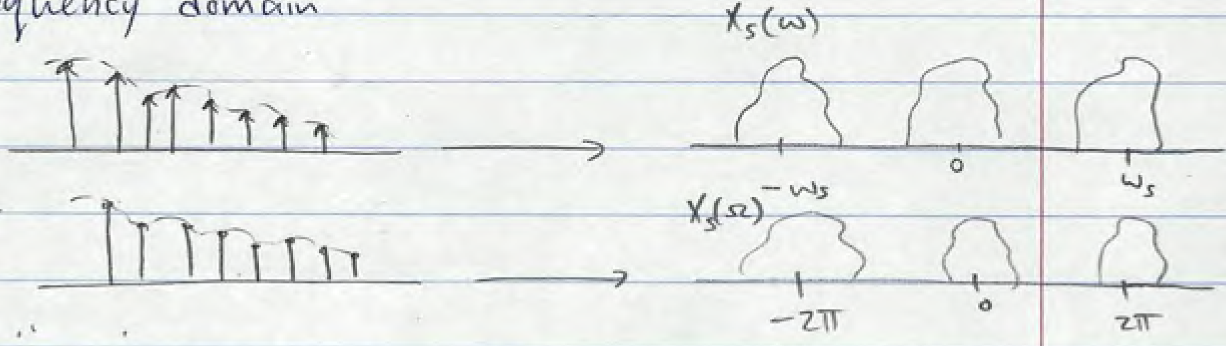


$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$.

$x[n+N] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}k(n+N)} = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}\, \cancel{e^{j2\pi}}$

The Output of the IDFT:

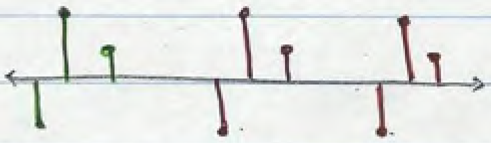
- sampling in time domain creates a periodic function in frequency domain



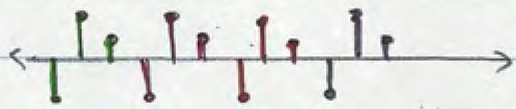
from duality!

- sampling in frequency domain will create a periodic function in time domain!

ω_s ; the sampling rate (in t-domain) $\equiv m$ (in f-domain)

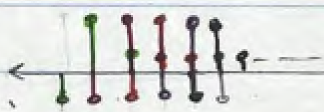


$m > N$



$m = N$

↳ as if I'm sampling @ the nyquest rate.



$\underline{m} < N$ Aliased Version!

m: # of samples in f-domain

N: # of samples of the truncated version of $x_s(t)$!

DFT & DTFS:-

DTFS:-

$$x[n] = \sum_{k=0}^{N-1} a_k \, e^{jk\Omega_0 n}$$

$$\hookrightarrow a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{-jk\Omega_0 n}.$$

لو حطناها هناك ما بتفرق

it's only scaling
بتفرقش وبيه يحطوها!

DTFS: $\therefore\ a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\Omega_0 n}$

DFT: $X[\Omega_k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$

$$\therefore \boxed{DFT = DTFS} \quad \text{where} \left(\Omega_0 = \frac{2\pi}{N}\right)$$

" it's just to complete the circle" :)

* بالامتحان لا تستخدم هذه القاعدة!

ex: $x[n] = \{1, 2, 2, 1\} \xrightarrow{DFT} ?$

it's a 4-points DFT / $N=4$

$$X[k] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn}$$

$$= x[0] + x[1]\, e^{-j\frac{\pi}{2}k} + x[2]\, e^{-j\pi k} + x[3]\, e^{-j\frac{3\pi}{2}k}$$

$$= 1 + 2e^{-j\frac{\pi}{2}k} + 2e^{-j\pi k} + e^{-j\frac{3\pi}{2}k}$$

I only need $X[0], X[1], X[2], X[3]$, then it'll repeat!

$X[0] = 1+2+2+1 = 6$

$X[1] = 1 + 2e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}} = -1-j$

$X[2] = 1 + 2e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi} = 0$

$X[3] = 1 + 2e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}} = -1+j$

conjugates!

$e^{-j\frac{\pi}{2}} = -j$

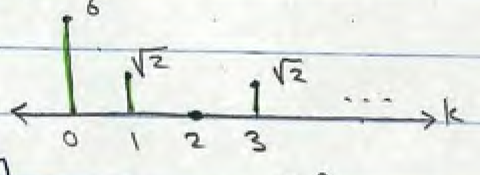
$e^{j\frac{\pi}{2}} = j$

$e^{\pm j\pi} = -1$

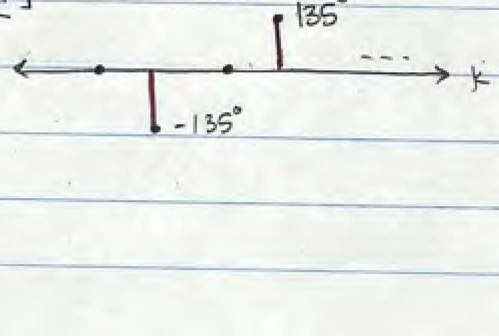
$e^{\pm j2\pi} = 1$

9/5/2013

$|X[k]|$



$\angle X[k]$



ex/cont. what's the IDFT of $X[k]$?

$$x[n] = \frac{1}{4} \sum_{k=0}^{3} X[k] e^{j \frac{2\pi}{4} kn}$$

$$= \frac{1}{4} \left[X[0] + X[1] e^{j\frac{\pi}{2}n} + X[2] e^{j\pi n} + X[3] e^{j\frac{3\pi}{2}n} \right]$$

$$x(0) = \frac{1}{4}\left[X[0] + X[1] + X[2] + X[3] \right] = \frac{1}{4}(6 - 1 - \cancel{j} - 1 + \cancel{j} + 0)$$
$$= \frac{4}{4} = 1$$

$$x(1) = \frac{1}{4}\left[X[0] + X[1] e^{j\frac{\pi}{2}} + X[2] e^{j\cdot\pi} + X[3] e^{j\frac{3\pi}{2}} \right]$$

$$= \frac{1}{4}\left[6 + (-1-j)(+j) + 0 \times (-1) + (-1+j)(-j) \right] = 2$$

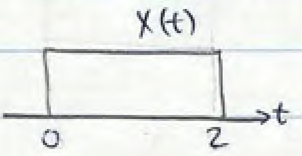
$$x(2) = \frac{1}{4}\left[X[0] + X[1] e^{j\pi} + X[2] e^{j2\pi} + X[3] e^{j3\pi} \right] = 2$$

$$x(3) = \frac{1}{4}\left[X[0] + X[1] e^{j\frac{3\pi}{2}} + X[2] e^{j3\pi} + X[3] e^{j\frac{9\pi}{2}} \right] = 1$$

$x[n] = \{1, 2, 2, 1\}$ over $\underline{\underline{N}}$

الناتج صورة periodic version من الاقتران الأصلي.

Let's Wrap it up :)



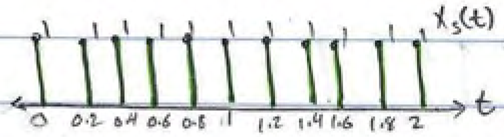
$\xrightarrow{F.T}$ $2 \operatorname{sinc}\left(\frac{w}{\pi}\right) e^{jw}$

* Unfortunately, computers can't do this !

$\therefore$ we have to sample!

let the sampling period $= (0.2)$ sec

$\therefore W_s = 5\,kHz$ (sampling rate).

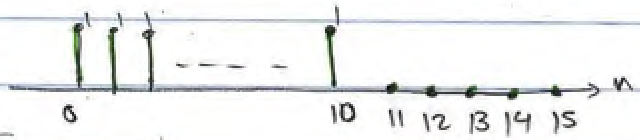


(11 samples)

* In MATLAB, the command FFT (Fast Fourier Transform). is the same as DFT, but it contains algorithms to speed up DFT!

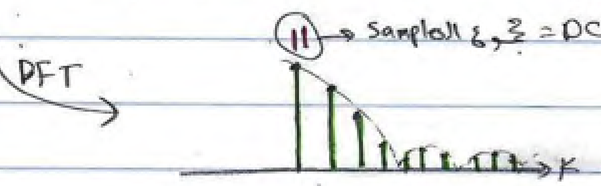
powers of 2

* Usually # of samples $= 2^n$



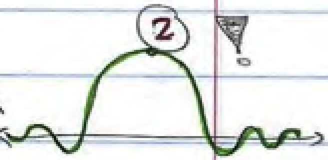
(16 samples)

but, is DFT a good approximation of F.T?



but

$\xrightarrow{F.T}$ $2 \operatorname{sinc}\left(\frac{w}{\pi}\right) e^{jw}$



the reason of the differance in magnitude in the DFT is the scaling Done by sampling (sampling the t-domain function scales the f-domain function by $\left(\frac{1}{T}\right)$)!

T here equals 5

$\frac{11}{T} = \frac{11}{5} \approx 2$ good approximation