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Probability Notebook

lecture #1 Quick Review

* Variables

[1] a. Determinastic [can be predicted]

ex: $X(t) = t^2 + 5t - 5$

↳ dependent ↳ independent

b. Non-determinastic [Unpredictable]

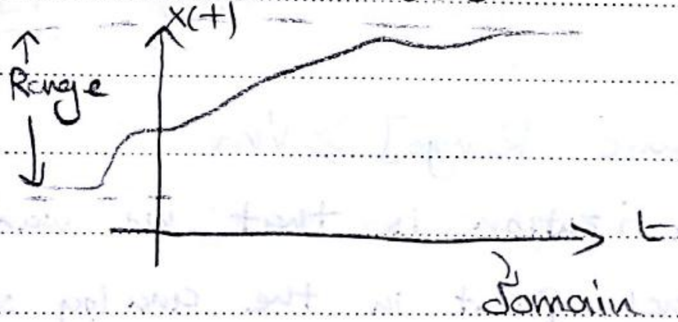
ex: flipping a coin ⇒ {T, F}

rolling the dice ⇒ {1, 2, 3, 4, 5, 6}

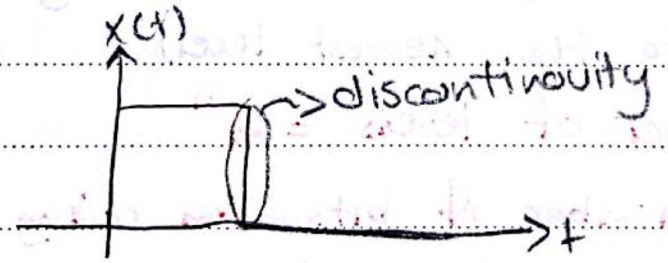
dealing a card from a deck ⇒ {A, K, Q, J, 2..10}

[2] a. discrete

b. continous



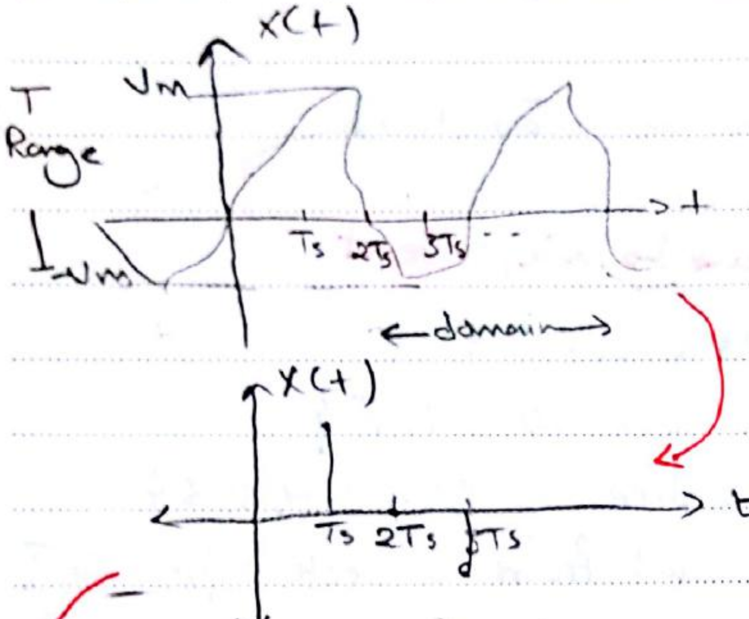
continous



continous time

How to convert analog to digital?

1. Sampling
2. Quantization
3. Coding



Sampling

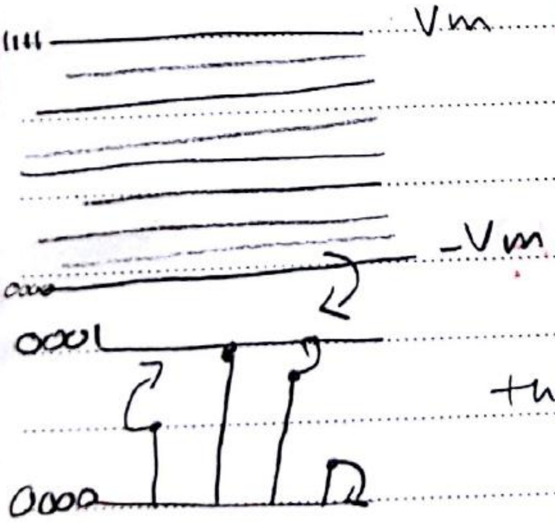
$T_s \rightarrow \quad f_s = \frac{1}{T_s}$

to get good sampling $f_s > 2f_m$ [fundemental frequency]

Quantization:

We define [Dynamic Range] $\geq V_m$

(the idea with quantization is that we want to define levels, so each point in the analog signal can be quantized to the nearest level)

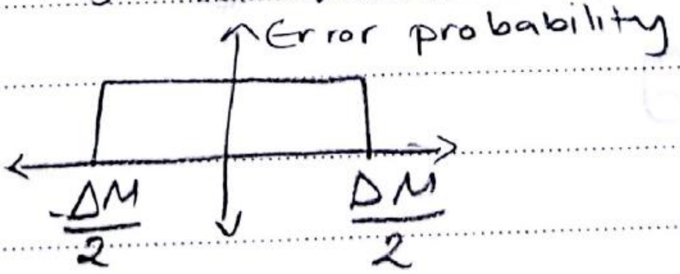


* Number of levels $= 2^n$

  n: number of bits I'm coding with

* Step size $\Delta M = \frac{2V_m}{2^n}$

this causes error



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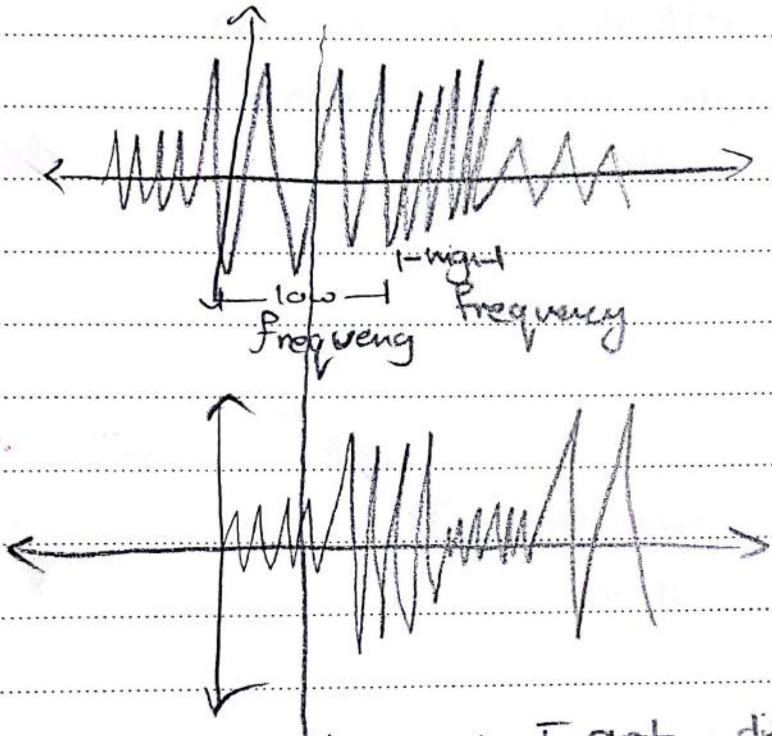
* Random processes

* Speach



t=1 m.s => I get different [Random] height.

* Doing the same experemint more then once, then collecting the data at a certain point of time Returns random Range.

lecture #2 11/10/2015

Ch # 1

1.1 set definitions

A set is a collection of objects (Capital letter) A

An element is an object in the set (small letter) a

Rolling a dice $A = \{1, 2, 3, 4, 5, 6\}$

$a \in A$ if the element belongs to A

$a \notin A$ if the element doesn't belong to A.

* To describe a set use:

a. Tabulor Method

assign numbers like $\{1, 2, 3 \cdots 6\}$

Elements are enumerated explicitly

e.g the set of all integers between 5 and 10

5, 10 not excluded

$A = \{6, 7, 8, 9\}$

b. Rule Method

$A = \{I : 5 < I < 10, I \text{ is an integer}\}$

$= \{I \mid 5 < I < 10, I \text{ is an integer}\}$

$= \{\text{integers between 5 and } 10\}$

Countable vs noncountable sets

$A = \{5 < a < 10\}$ non-countable / infinite

finite vs infinite

$A = \{1, 2, 3, 4, 5, 6\}$ finite

ends up to a limited number

null set denoted by φ, has no elements

$A = \{0\} \rightarrow$ not a null set

if set A has all its element in set B, with no element of B is not in A

$A \subseteq B$ We call A is a subset of B

$B \subseteq A$!?

$A = B$

if at least one element of B is not in A

$A \subset B$ We call A is a proper subset of B

ex: $A \{1, 2, 3\}$ $B \{1, 2, 3, 4\}$ A is proper subset

$A \{1, 2, 3\}$ $B \{1, 2, 3\}$ A is subset of B

Two sets are disjoint (mutal exclusive) if there is no element common among them.

ex: $A = \{2, 4, 6\}$ $B = \{1, 3, 5\}$

A and B are disjoint

Ex 1.1.1 in the book P.4

$A = \{1, 3, 5, 7\}$ $B = \{1, 2, 3, \cdots\}$

$C = \{0.5 < C \leq 8.5\}$ $D = \{0, 0\}$

$E = \{2, 4, 6, 8, 10, 12, 14\}$ $F = \{-5.0 < f \leq 12.0\}$

A countable, finite, tabulor

B tabulor, countable, infinite

C Ruled, infinite, non-countable

D tabulor, countable, finite

E finite, countable, tabulor

 F Ruled, non countable, infinite

/ /

T or F

1. A ⊂ B T

2. D ⊂ F T

3. A, D, E are mutally exclusive T

4. D, B " " "

Universal Set (S) contains all the possibilties of the experement

$S = \{\overbrace{1, 2, 3, 4, 5, 6}^{6}\}$ Rolling a dice

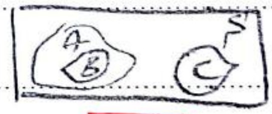
$A = \{2, 4, 6\}$ outcome even

$B = \{1, 3, 5\}$ odd

$C = \{1, 2\}$ less than 3

The total # of subsets that can be crea
from S is $= 2^N$

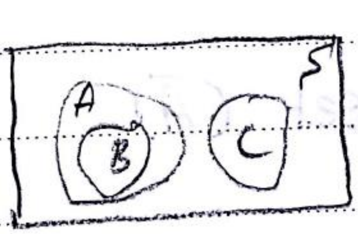
Bonus : list all possible 64 possibilties and discride how you got them.



Ven diagram

Universal set is represented by ▭

All subsets are represented by an area



A and C disjointed

B ⊂ A

Set operations :-

1. Defining the difference of two sets (A-B)

all element of A not in B

/ /

$A = \{ 0.6 < a \leq 1.6 \}$

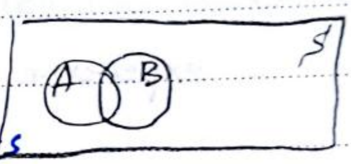
$B = \{ 1.0 \leq b \leq 2.5 \}$

$A - B = \{ 0.6 < c < 1.0 \}$

$B - A = \{ 1.6 < c \leq 2.5 \}$

$A - B \neq B - A$ surely desjointed

2. Union and intersection



* The union of two sets is given as $A \cup B$, all elements in A and B without repetition
* The intersection of two sets is given by $A \cap B$ represent all common elements

two sets that are disjoint have an intersection of φ

for N sets $A_1, A_2 \cdots A_N$

$$\bigcup_{n=1}^{N} A_n = A_1 \cup A_2 \cup \cdots \cup A_N$$

$$\bigcap_{n=1}^{N} A_N = A_1 \cap A_2 \cap A_3 \cdots \cap A_N$$

3. Complement of a set $(\overline{A})$

$\overline{A} = S - A$

$\overline{S} = \varphi$

$\overline{\varphi} = S$

$A \cup \overline{A} = S$

$A \cap \overline{A} = \varphi$

* Algebric laws

1. Commutative law

$A \cup B = B \cup A$

$A \cap B = B \cap A$

2. Distributive law

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

3. Associative law

$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C.$

$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$

4. De Morgan's law

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

Bonus: Prove them

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lecture #3

4 / 10 / 2015

Ex 1.2.2 /P8

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$A = \{2 < a \leq 16\}$ $\quad \bar{A} = \{16 < a \leq 24\}$

$B = \{5 < b \leq 22\}$ $\quad \bar{B} = \{2 < b \leq 5$ or $22 < b \leq 24\}$

$S = \{2 < s \leq 24\}$

$\bar{A} \cup \bar{B} = \{2 < x \leq 5 , 16 < x \leq 24\}$

$A \cap B = \{5 < x \leq 16\}$ $\quad \overline{A \cap B} = \{2 < x \leq 5, 16 < x \leq 24\}$

4. Duality Principle

* $\cap$, $\cup$
* S , φ

distributive law

* $A \cap (B \cup C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$

* ~~$A \cap B$~~ $\quad (A \cap B) \cup (A \cap B)$

1.3 Probability introduced through sets[1] and relative frequency[2]

sets $\Rightarrow$ $S = \{H, T\} \rightarrow H = \frac{1}{2}$

relative frequency $\Rightarrow$

1	2	3	4	5	6	7	8	9	10
H	H	H	T	T	T	H	T	H	H

$H = \frac{6}{10} \simeq \frac{1}{2}$ (less acurate)

(to increase accuracy increase the number of tries)

/ /

- A trial is the process of conducting an experement
- Outcome the output of a trial
- Possibe outcomes will give sample space S
- fair ≡ Unbalanced all outcomes have the same (chance/likelihood/probability) to occur

Rolling a dice — Probabilty sets

A = { even number } = { 3/6 }

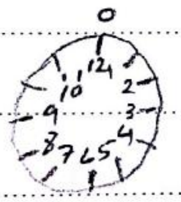
A = { 2, 4, 6 }

B = { odd number } — { 3/6 }

B = { 1, 3, 5 }

There would be two sets one set is the possible outcomes and other set is the likelihood of the outcomes.

→ Sample space is the universe ~~of~~ set S which has all possible outcomes



$S = \{ 0 < s \leq 12 \}$

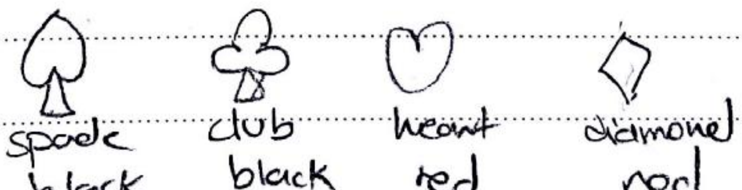
A = { odd numbers when spinning }

A = { 1, 3, 5, 7, 9, 11 } discrete

B = { numbers between 3, 5 } continuous finite

= { $3 < b < 5$ }

deck of cards



1 2 . . . 10 J Q K A // 13 Cards

$P(\text{spade}) = \frac{1}{4} = \frac{13}{52}$ //

* How many possible events can I create?
2^{52}

Certain event $P(\mathcal{S}) = 1$

impossible event $P(\varphi) = 0$

* Probability of an event $P(A)$, $P[A]$, $P\{A\} \geq 0$

$P[\mathcal{S}] = 1$

$P[\varphi] = 0$

* for N events

$A_1, A_2, A_3 \text{ ----- }, A_n \qquad n = 1, 2, \text{---}, N$

iff $A_i \cap A_j = \varphi \qquad i \neq j = 1, 2 \text{ --- } N$

$$P\left[\bigcup_{n=1}^{N} A_n\right] = \sum_{n=1}^{\infty} P[A_n]$$

because no element will be repeated more than once.

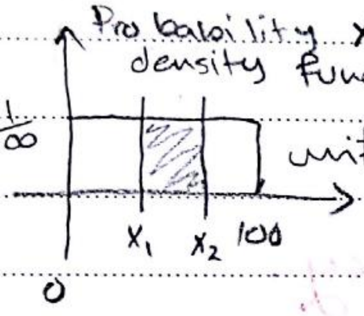
lecture #4 6 / 10 / 2015

Ex 1.3.1 P.11

fair wheel of chance labeled between 0-100

$\mathcal{S} = \{0 < s \le 100\}$

Find the probability of an event between x_1 and x_2 $\rightarrow$ $x_1 < x \le x_2$



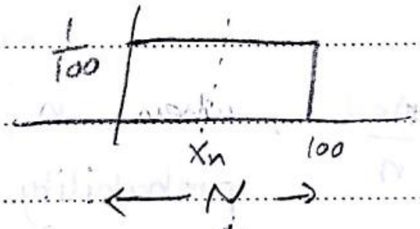
$P(\mathcal{S}) = 1$

$P(x) = \frac{1}{100}(x_2 - x_1)$ [Area under the curve]

What is the probability at a specific number ?!

Zero, in a cont. sample space

Why ?!



$(x_n) = \frac{n}{N}\,100$

$A_n = \{x_{n-1} \le x < x_{n+1}\}$

N↑ every section width ↑ gets less till it reaches 0.

Mathematical model of Experiment :-

Ex 1.3.2 P.12 Rolling two dices

1. Assign Sample Space

6^2 → #repc

↳ possible

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

A

B

C

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2. Define the event of intrest

$A = \{sum = 7\} = \{$ $P(A) = 6/36 = 1/6$

$B = \{ 8 < sum < 11 \}$ $P(B) = 9/36 = 1/4$

$C = \{ 10 < sum \}$ $P(C) = 3/36 = 1/12$

3. Make probability assignment

$P[B \cap C] = 2/36 = 1/18$

$P[B \cup C] = 10/36 = 5/18$

* Probability as a Relative frequency.

$n \to$ # of trials

$n_s \to$ # of success of a trial
↳ success

$P[A] = \frac{n_s}{n}$, when n is very large

probability as a relative frequency will reach the probability found using set theory.

→ Flipping a coin a 1000 time

$n_H = 490$ $p[H] = \frac{490}{1000} = \boxed{0.49}$

$n_T = 510$ $p[T] = \boxed{.51}$

→ = 1

Ex L3.3 / P 14

$18 \to 10\Omega$ $P(\text{drawing } 10\Omega) = 18/80$

$12 \to 22\Omega$ $P(\text{drawing } 22\Omega) = 12/80$

$33 \to 27\Omega$ $P(\text{drawing } 10\Omega \text{ given the first is } 22) = 18/79$

$17 \to 47\Omega$

80 resistors
↳ same size, shape
without return

1.4 Joint and Conditional Probability

$P(A \cap B) \neq 0$ events A and B are joint

$P(A \cap B) = P(A) + P(B) - P[A \cup B]$

$P[A \cup B] = P(A) + P(B) - P[A \cap B]$

$P(A \cup B) \leqslant P(A) + P(B)$

$P(A \cup B) = P(A \cup B)$ only when A, B are disjoint.

Conditional Probability

We need to find the probability of an event A given preior knowledge of the occurance of one another event B which affects the event A.

Role of a dice

$B = \{2, 4, 6\}$

$A = \{4\}$

$P(A \setminus B)$

⇒ lecture 5

lecture #5 4 / 10 / 2015

Conditional Probability

Event A occurance depends on the accurance of the occurance of event B.

$P[B]$ = prior probability

it is supposed to increase my probability

Ex: Rolling a dice $B = \{2, 4, 6\}$

$A = \{4\}$ $P[A] = \frac{1}{4}$

$P(A \backslash B) = \frac{P[A \cap B]}{P[B]} = \frac{1/6}{3/6} = \frac{1}{3}$

Definition:-

* $P[A \backslash B] = \frac{P[A \cap B]}{P[B]}$ $P[B] \neq 0$

- $P[A \backslash B] \geq 0$

when A and B are mutually exclusive

$P[A \backslash B] > 0$ because $P[B] > 0$ and $P[A \cap B] > 0$

- $P(S \backslash B) = 1 = \frac{P[S \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1$

* $P[\bigcup_{n=1}^{N} A_n] = \sum_{n=1}^{N} P[A_n]$ if $A_n \cap A_m = \emptyset$

$n \neq m$ $n = 1, 2 \cdots N$

$\Rightarrow$ For events A and C which are disjointed

$P(A \cap C) = 0$

$P(A \cup C \backslash B) = P(A \backslash B) + P(C \backslash B)$

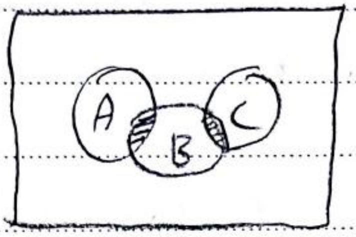
Proof:

Proof:

$$P(A \cup C \backslash B) = \frac{P[(A \cup C) \cap B]}{P[B]} = \frac{P((A \cap B) \cup (C \cap B))}{P(B)}$$

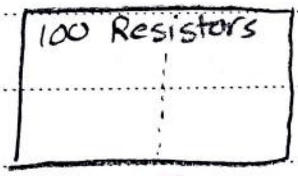
Since A and C are disjoin

$$= \frac{P(A \cap B) + P(C \cap B)}{P[B]}$$



$$= P(A \backslash B) + P(C \backslash B)$$

Example 1.4.1 / P16



R Ω	5 %	10 %	total
22	10	14	24
47	28	16	44
100	24	8	32
	62	38	100

$A = \{$ drawing a 47 Ω Resistor $\} = 44/100$

$B = \{$ drawing 5% tolerance resistor $\} = 62/100$

$C = \{$ drawing a 100 Ω Resistor $\} = 32/100$

$P(A \cap B) = 28/100$

$P(A \cap C) = 0$

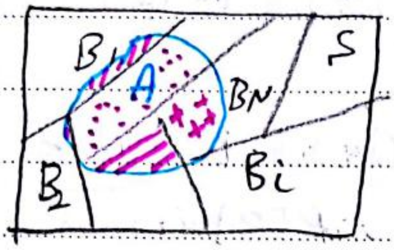
$P(B \cap C) = 24/100$

$$P(A \backslash B) = \frac{P(A \cap B)}{P[B]} = \frac{28/\cancel{100}}{62/\cancel{100}} = \frac{28}{62}$$

$P(A \backslash C) = 0$

$$P(B \backslash C) = \frac{24/100}{32/100} = \frac{24}{32}$$

Total Probability



$$S = \bigcup_{i=1}^{N} B_i$$

$P[A]$ = total Probability

$$P[A] = \sum_{n=1}^{N} \underbrace{P(A \backslash B_n)}_{\text{Priory Prob}} . \underbrace{P[B_n]}_{\text{Priory Prob.}}$$

Proof:

$$A = A \cap S$$

$$A = A \cap \left(\bigcup_{i=1}^{N} B_n \right)$$

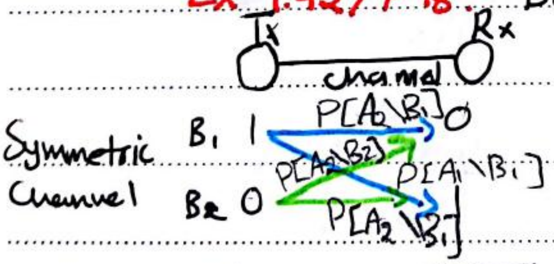
$$P[A] = P[(A \cap B_1) \cup (A \cap B_2) \cdots \cup (A \cap B_N)]$$

$$P[A] = P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_N)$$

But $\quad P[A \backslash B_n] = \dfrac{P[A \cap B_n]}{P[B_n]}$

$$P[A] = \sum_{n=1}^{N} P(A \backslash B_n) \, P[B_n]$$

Ex 1.42 / P18 Binary com. System (two level system 0,1)



Assume $\quad P[B_1] = 0.6$

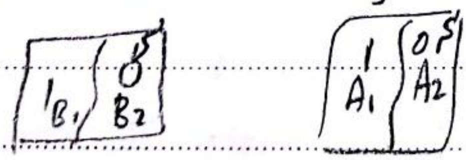
$P[B_2] = 0.4$

$P[A_1 \backslash B_1] = 0.9$

$P[A_2 \backslash B_2] = 0.9$

$P[A_1 \backslash B_2] = 0.1$

$P[A_2 \backslash B_1] = 0.1$



$$P[A_1] = P[A_1 \backslash B_1] \, . \, P[B_1] + P[A_1 \backslash B_2] \, . \, P[B_2]$$
$$= 0.9 \, . \, 0.6 + 0.1 \, . \, 0.4$$
$$= 0.58$$

$$P[A_2] = P[A_2 \backslash B_1] . P[B_1] + P[A_2 \backslash B_2] . P[B_2]$$
$$= 0.1 \cdot 0.6 + 0.9 \cdot 0.4$$
$$= 0.42$$

$P[B_1 \backslash A_1]$ We call this a posterior probability

$$= \frac{P[B_1 \cap A_1]}{P[A_1]} = \frac{P[A_1 \backslash B_1] \, P[B_1]}{P[A_1]} = \frac{0.9 \, . \, 0.6}{0.58}$$
$$= 0.931$$

$$P[B_2 \backslash A_2] = P[A_2 \backslash B_2] \, \frac{P[B_2]}{P[A_2]} = \frac{0.9 \, . \, 0.4}{0.42} = 0.857$$

Probability of System error

$$P[B_1 \backslash A_2] = \frac{P(A_2 \backslash B_1) \, P[B_1]}{P[A_2]} = \frac{0.1 \;\; 0.6}{0.42}$$

lecture #6 13 / 10 / 2015

* Conditional Probability

event A

event B

P[B] ~~prion probability~~

~~Condition Probability P(A|B) is the probability that event A occurs given periore knowledge that~~

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lecture #6 13 / 10 / 2015

1.5 Independent events

The occurence of event A doesn't depend on the occurence of event B.

$$P(A \cap B) = P(A)\,P(B)$$

$P(A \backslash B) = P(A)$ if A doesn't depend

$$P(A \backslash B) = \frac{P(A \cap B)}{0 \neq P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

$$P(A \cap B) = \underset{\neq 0}{P(A)} \cdot \underset{\neq 0}{P(B)}$$

∴ Two events can't be disjoint and independent at the same time.

∴ if two events are independent, they can't be disjoint.

Ex 1.5.1 / P21

52 cards

$A = \{$ Selecting a King $\}$ 4/52

$B = \{$ Selecting a jake or a queen $\}$ 8/52

$C = \{$ Selecting a heart $\}$ 13/52

$P(A \cap B) = 0$

$P(A \cap C) = 1/52$

$P(B \cap C) = 2/52$

A and C are independ.

$P(A \cap C) = P(A) \cdot P(C)$ to check.

$$\frac{1}{52} \stackrel{?}{=} \frac{4 \cdot 13}{52 \cdot 52}$$

$$\frac{1}{52} = \frac{1}{52}$$

Are A and B independent ?

$P(A \cap B) \stackrel{?}{=} P(A) \cdot P(B)$

$0 \neq \frac{4}{52} \cdot \frac{8}{52}$

disjoint, not independent

Are B and C independent ?

$P(C \cap B) \stackrel{?}{=} P(B) \cdot P(C)$

$\frac{2}{52} \stackrel{?}{=} \frac{13 * 8}{52}$

$\frac{2}{52} = \frac{2}{52}$ statiscally independent.

for Multi random variable ; they are statiscally independent if they satisfy set of equations.

$\Rightarrow$ # of equation $2^n - n - 1$

$\rightarrow$ if $A_1, A_2 \text{---} A_N$

* A, B, C

$\# = 2^3 - 3 - 1 = 4$

$P(A \cap B) = P(A) P(B)$

$P(A \cap C) = P(A) P(C)$

$P(B \cap C) = P(B) \cdot P(C)$

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ not sufficient for independence of three variables

/ /

Ex1.5.2 /P.23

52 card deck

drawing 4 cards with replacment

$\{A_1, A_2, A_3, A_4\} = \{$Ace on first time, 2nd, 3rd, 4th$\}$

$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)\, P(A_2)\, P(A_3)\, P(A_4)$

$= \left(\frac{4}{52}\right)^4 =$

Without replacment

$P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} =$

$= P(A_1) \cdot P(A_2 \backslash A_1) \cdot P(A_3 \backslash A_2 \cap A_1) \cdot P(A_4 \backslash A_1 A_2 A_3)$

Properties for statistically independent event.

Assume events

$A_1, A_2, A_3 \cdots, A_n, \cdots A_N$

are statistically independent, they will be independent from:

1. $A_i \cup A_j \quad i \neq j$
2. $A_i \cap A_j \quad i \neq j$
3. $\overline{A_i}$

* if there is two events A and B

A is independent of $\overline{B}$

$P(A \cap \overline{B}) = P(A) \cdot P(\overline{B})$

* events A, B, C

A is independent from $(B \cup C)$

$(B \cap C)$

/ /

$P(A \cap (B \cup C)) = P(A) \, . \, P(B \cup C)$

$= P(A) \, [\, P(B) + P(C) - p(A \cap C) \,]$

$= P(A) P(B) + P(A) P(C) - P(A) \, P(C)$

$= P(A) \, . \, P(B)$

$P(A \cap (B \cap C)) = P(A) \cdot P(B \cap C)$

$= P(A) \, . \, P(B) \, . \, P(C)$

1.6 Combined Experiments

flipcoin, Role dice

Subexperiments

→ Subexperiments are conducted simulantan
ouasly

→ Repeat experiments many times

Combined events: sample space

$S = S_1 \times S_2 \cdots S_n$ → An object has two elements (S_1, S_2)

$S' = S_1 \times S_2 = \{ H,1 \quad T,1 \quad H,2 \quad T,2 \quad H,3 \quad T,3 \quad H,4 \quad T,4 \quad H,5 \quad T,5 \quad H,6 \quad T,6 \}$

$s_1 \in S_1$

$s_2 \in S_2$

12 Samples $= 2 \times 6$

Flipping a coin two times

$\{ H,H \quad H,T \quad T,H \quad T,T \}$

lecture # 7   / /

Rolling a die and flipping a coin

ζ (1, H)
(1, T)
(2, H)
(2, T)
(3, H)
(3, T)
(4, H)
(4, T)
(5, H)
(5, T)
(6, H)
(6, T)

An element of S_1 is ζ_i
of S_2 is ζ_j

$\zeta = (\zeta_i, \zeta_j)$
↳ object/

⇒ For a combined experiment with events A and B defined on sample space S_1 and S_2 respectively

$C = A \times B$

$A \times S_2 = A$

$B \times S_1 = B$

Example: 1.6 / P.25

$S_1 = \{0 < x \leq 100\}$

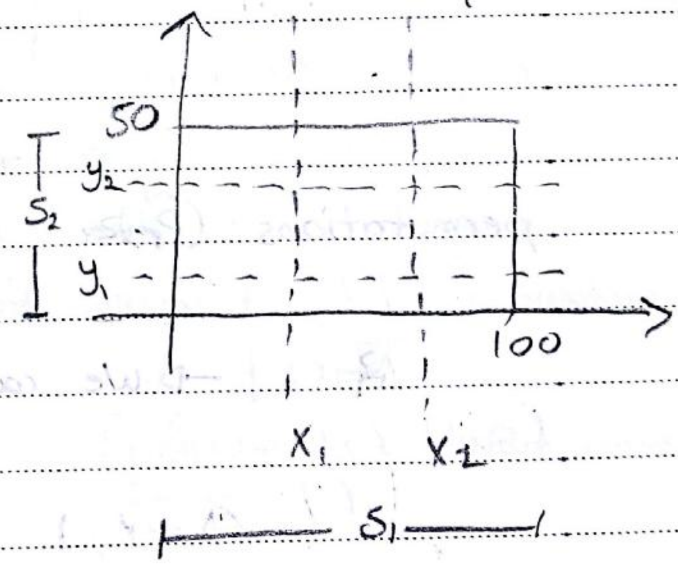
$S_2 = \{0 \leq y \leq 50\}$

define a combined event

$C = A \times B$

$A = \{x_1 < x < x_2\}$

$B = \{y_1 < y < y_2\}$



events A and B are independent
$\begin{cases} A \subset S_1 \\ B \subset S_2 \end{cases}$

$P(C) = P(A \times B) = P(A) \cdot P(B) =$

$P(A \times S_2) = P(A) \cdot P(S_2) = P(A)$

$P(S_1 \times B) = P(S_1) \cdot P(B) = P(B)$

/ /

Combined Exp. → Repeat exp. many times → permutations / → Combinations

order of objects

Exp. x y z u v people

picking 3 people:-

ordering isn't important

but 001 / 100 → ordering is important

* Ordering of elements → replacment / → no replacment

Without replacment for N element in tuple

First N

Second $N-1$

third $N-2$

⋮ ⋮

r th $N-r+1$

} Picking r elements

permutations ($\tilde{P}_r$) (1, 2, 3, 4, 5, 6) → ($\boxed{3, 4, 2}$, 5, 6, 1)

permutations $\tilde{P}_r = \frac{N!}{(N-r)!}$ → We care about order

Combinations $\binom{N}{r} = \frac{N!}{(N-r)!} \frac{1}{r!}$ → we don't care about order

Issues you take care of:

1) $r = ?$ 2) $N = ?$ 3) Replacment or not

4) Combination or permutation

5) Probability of event (r) [succes prob.]

/ /

Example 1.6.4 / P27

How many permutation for four cards taken from 52 cards deck. without replacment

$r=4 \quad N=52$

$\overset{N}{P}r = \frac{52!}{48!} = 6,497,400$

Exp 1.6 / P.28

5 Athletes , 3 team members.

How many teams can be chosen?

$$\binom{5}{3} = \frac{5!}{2!\,3!} = \underline{\underline{10}}$$

1.7 Bernoulli Trial

- Repeat exp. N times
- each trial has two possible outcomes
- Select R elements out of N elements
- define the prob. of success P

$P(A) = P$

Then probability of failure is $(1-P)$

$P(A') = 1-P$.

- all trials are statistically independent

$\left(P(A)^K \; (1-P(A))^{N-K} \right)$ with replacment.

↳ multiply by $(\;)$ or $\overset{N}{P}r$

/ /

Exp 1.7.1 / P29

You need two success to sink the carrier.

X X X not important

P(success) = 0.4 S S F $(0.4 \times 0.4 \times 0.6) \times 3$

P (fail) = 0.6 S S S + $(0.4 \times 0.4 \times 0.4) \times 1$

P(sinking) = 0.352

P (2 out of 3) or P(3 out 3)

$\binom{3}{2} = \frac{3!}{2!\,1!} = 3$

Ex 1.7.2 / P 30

6/100 X X X X → data point → 3 data point

↳ unspoilled.

no order SSSF order.

P(S) = 0.94

P(f) = 0.06

$\binom{4}{3} = \frac{4!}{3!} = 4$

4 . 0.94 . 0.94 . 0.94 . 0.06 = 0.98

3datapoint = P(S) . P(S) P(S) = 0.94

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lecture # 8 / /

A Quick Review of Bernolli trial :

1. We need two outcomes < succes / fail   < Head / tail . . .

2. Repeat the experiment N times

3. Define K (sucess) times

x x x x - - - x

$\xleftarrow{\quad N \quad}$

$\xleftarrow{\ K\ }\!\rightarrow \ \xleftarrow{\ N-K\ }\!\rightarrow$

4. Define $P(S) = P$

$P(F) = 1-P$

5. Decide → permutations
→ Combinations

6. Based on step 5. find number of success sequences.

$$\hat{P}_K = \frac{N!}{(N-K)!} \quad \text{For Comb Permutations}$$

$$\binom{N}{k} = \frac{N!}{K!\,(N-K)!}$$

7. Find the probability of every success sequence with replacment

$$P = (p)^K (1-p)^{N-k}$$

8. Multiply $P \propto$ # of sequence

If $N, N-k, K$ are large.

1. And P is relativly large, and $P.N \simeq k$
   Use De-Moivre-laplace approx

$$\binom{N}{K} P^K (1-P)^{N-K} \simeq \frac{1}{\sqrt{2\pi NP(1-P)}} e^{\frac{-(K-NP)^2}{2NP(1-P)}}$$

2. P is relativly small
   Use Poisson approx

$$\binom{N}{K} P^K (1-P)^K \simeq \frac{(NP)^K e^{-NP}}{K!}$$

Ex 1.7.3 / P.31 Firing bullets for 3 second, at a rate of 2400 bullet/min

$P(h) = 0.4 \quad P(m) = 0.6$

We need 50 suceeses

$N = \frac{\cancel{2400}^{40}}{\cancel{60}} \times 3 = 120$

$$\binom{120}{50} 0.4^{50}\, 0.6^{120-50} = 0.0689$$

↳ no approx

Using De-Moivre since $P.N \simeq K$
$48 \simeq 50$

$= 0.0693 \Rightarrow 6.9\%$

↳ not a good trial.

For chapter #1 : solve:

1.1.8

1.1.12

1.2.8

1.2.15

1.3.6

1.3.11

1.4.3

1.4.5

1.4.9

1.4.12

1.5.5

1.5.6

1.6.2

1.7.2

1.7.10

Ch#2

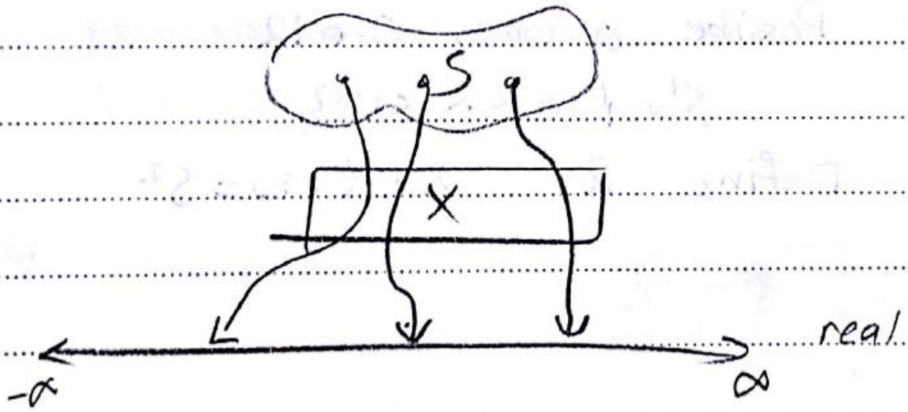
$y = f(x)$

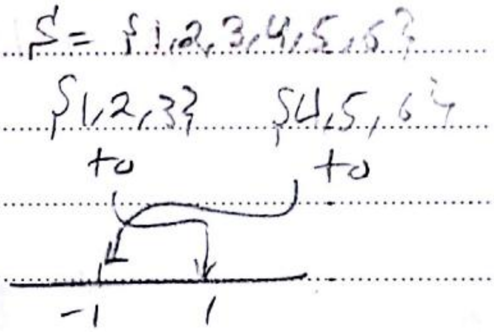
x: independent

y: dependent

In this chapter, we define function

X: is a random variable that maps the elements of the sample space of an experiment to real or complex plane using a specific function.





Mapping can be:

1. Point-to-Point [one element of S mapped to one number]
2. Multipoint-to-point
3. Point-to-multipoint

* only point-to-point mapping can be a random variable.

* Random variables can be real or complex

/ /

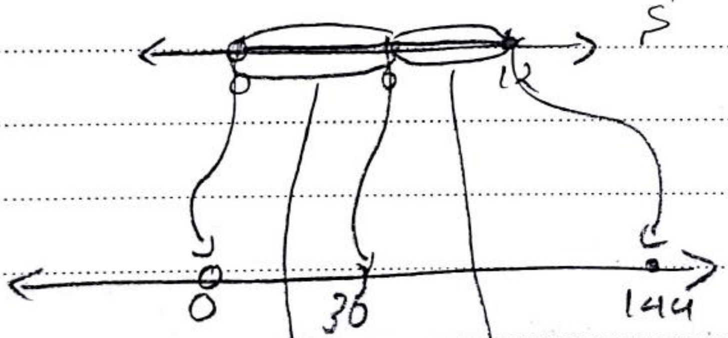
* The elements of sample space is S_i
While the elements of the random variable
Capital ~~Sma~~ X are x_i Small

* - Continous sample space → Cont. RV
→ discrete RV
→ a mix.
- Discrete sample space → discret RV

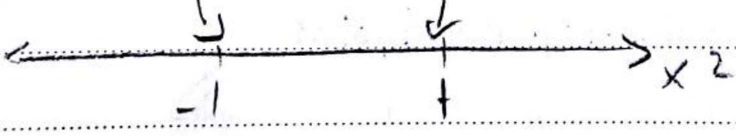
Ex 2.1.2 /P42 Passibe outcomes 0 → 12

$S' = \{ 0 < x \leq 12 \}$

Define RV $X = X(s) = s^2$



not a valid RV ⇐ Define $X_2 = \begin{cases} -1 , & 0 < s < 6 \\ 1 , & 6 \leq s \leq 12 \end{cases}$
multi-to-point



Conditions for random variable:

1. Map each point in S to a single point on the real line or complex plane [Point-to-Point mapping]

2. Event $\{X \leq x\}$ shall be an event for any real number less than or equal to x.

3. The probability of $\{X \leq x\}$ is the sum of probabilities of all events $\{X \leq x\}$ is given by $P\{X \leq x\}$

4. $P\{X = \infty\} = P\{X = -\infty\} = 0$

* Mapping to $\pm\infty$ is okay, but the prob must be 0.

lecture #9 20/ 10 /2015

* Conditions for R.V.

1. point-to-point mapping

2. set $\{X \quad x\}$ includes all real numbers less then x on real line. The probability of this even $P\{X \leq x\}$ is the sum of prob of all elents correspond to $\{X \leq x\}$

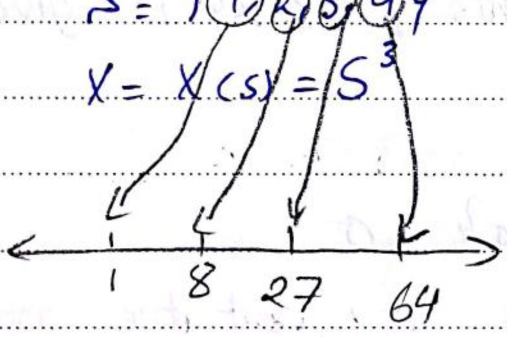
3. $P(-\infty) = P(\infty) = 0$

Ex 2.1.3 /P44

doing an exp 24 time.

$S' = \{1, 2, 3, 4\}$

$X = X(s) = S^3$



$P(1) = \frac{4}{24} \quad P(2) = \frac{3}{24}$

$P(3) = \frac{7}{24} \quad P(4) = \frac{10}{24}$

$P(X=1) = \frac{4}{24} \quad P(X=8) = \frac{3}{24} \quad P(X=27) = \frac{7}{24}$

$P(X < 32) = \frac{4}{24} + \frac{7}{24} + \frac{3}{24} = \frac{14}{24}$

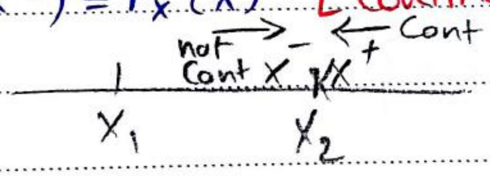
/ /

2.2 Distribution functions

→ CDF [(Comulative distribution function)
Distribution function (in our book)
$F_X(x) = P\{X \leq x\}$

→ PDF (Probability density function)
Density function (in our book)

1) $F(-\infty) = 0$ where as $F_X(-\infty) = 0$
2) $F_X(\infty) = 1$ where as $F(\infty) = 0$
3) $0 \leq F_X(x) \leq 1$
4) $x_1 \leq x_2$
$F_X(x_1) \leq F_X(x_2)$ → $F_X(x)$ is an increasing function
5) $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$
↳ no equality because
6) $F_X(x^+) = F_X(x)$ [Continous from the right]



* Check conditions 1 2 4 6 to vertify if CDF is a valid function

* Proof of 5.

set $\{X \leq x_1\}$ set $\{x_1 < X \leq x_2\}$ x_1 x_2

mutually exclusive

$P\{X \leq x_2\} = P\{X \leq x_1 \cup x_1 < X \leq x_2\}$

$P\{X \leq x_2\} = P\{X \leq x_1\} + P\{x_1 < X \leq x_2\}$

$F_X(x_2) = F_X(x_1) + P\{x_1 < X \leq x_2\}$

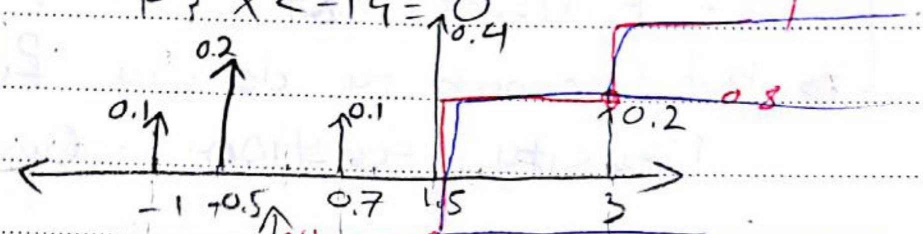
$P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

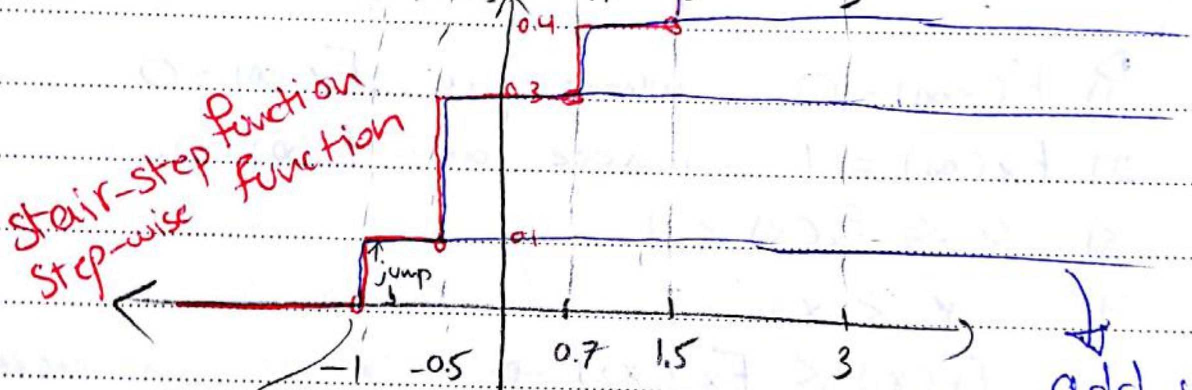
Ex 2.2.1 P46 X has values in the set $\{-1, -0.5, 0.7, 1.5, 3\}$

The corresponding prob. are $\{0.1, 0.2, 0.1, 0.4, 0.2\}$

1.) Check for CDF

$P\{X < -1\} = 0$





$F_X(-1)$ unidentified dis cont.

$P(X=-1) =$ jump $= 0.1$

$P(X=-0.5) = 0.2$

add up all these unit-step functions.

$F_X(x)$

$F_X(4) = 1 = P\{X \leq 4\}$

$$F_X(X) = P(X=-1)\, u(X+1) + P(X=-0.5)\, u(x+0.5) + \cdots$$

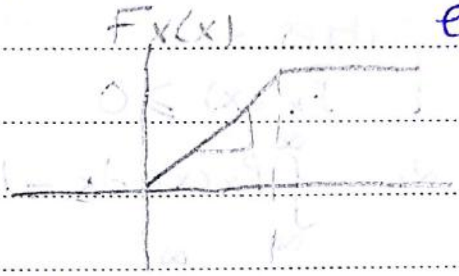
$$F_X(X) = \sum_{i=1}^{N} P(X_i)\, u(x - x_i)$$

↳ is the numbers at which X the RV is defined on

2.3 Density Function (PDF)

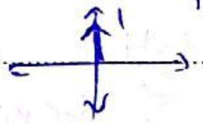
$f_X(x)$: represents the probability at that point

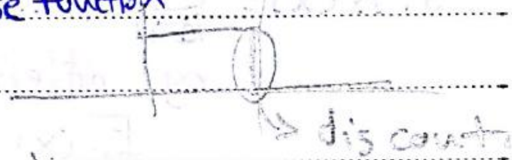
$f_X(x) = \frac{d}{dx} F_X(x)$, $F_X(x)$ derivative must exist

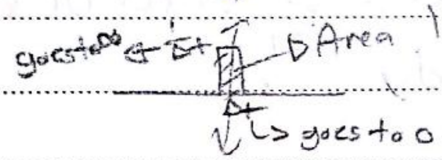


$\frac{d}{dx} u(x) = \delta(x)$

↳ unit impulse function







lecture #10 / /

* PDF

$$f_X(x) = \frac{\partial}{\partial x} F_X(x)$$

Properties :

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$\Big]\rightarrow$ check validty of pdf

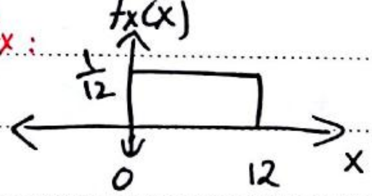
3. $f_X(x) = \sum_{i=1}^{\infty} P(x_i)\ \delta(x - x_i)$

by differinvtiation of

$$F_X(x) = \sum_{i=1}^{N} P(x_i)\ u(x - x_i)$$

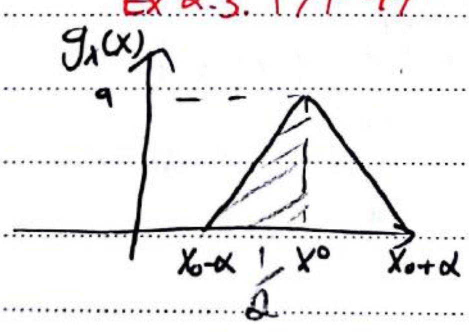
4. $F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy$

Ex :



5. $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)\, dx$

Ex 2.3.1 / P 44



Check validty:

1) $g_X(x)$ always +ve

2) $\int_{-\infty}^{\infty} g_X(x)\, dx \stackrel{?}{=} 1$

$$= \frac{1}{2} \times 2\alpha \times a \stackrel{?}{=} 1$$

$$\alpha a \stackrel{?}{=} 1$$

condition to get a pdf function. $\boxed{a = \frac{1}{\alpha}}$

find $G_X(x)$

$$G_X(x) = \int_{-\infty}^{x} g_X(y)\,dy$$

we need to find the equation of the lines:

$$y = mx + b$$

$$y_1 = \frac{a}{\alpha} x + b$$

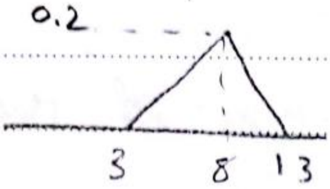
$$y_1 = \frac{1}{\alpha^2}\left[x - x_0 + \alpha\right]$$

$$G_X(x) = \begin{cases} 0 & x < x_0 - \alpha \\ \int_{-\infty}^{x} \frac{1}{\alpha^2}\left[y - x_0 + \alpha\right] dy & x_0 - \alpha \leq x < x_0 \\ \frac{1}{2} + \int_{x_0}^{x} \frac{1}{\alpha} - \frac{1}{\alpha^2}(y - x_0)\,dy & x_0 \leq x < x_0 + \alpha \\ 1 & x \geq x_0 + \alpha \end{cases}$$

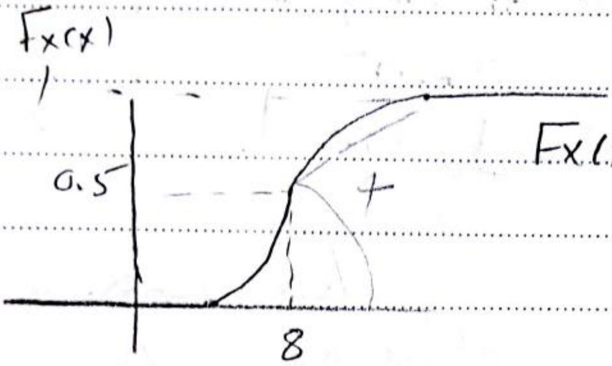
$$G_X(x) = \begin{cases} 0, & x < x_0 - \alpha \\ \frac{1}{2\alpha^2}\left(x - x_0 + \alpha\right)^2, & x_0 - \alpha \leq x < x_0 \\ \frac{1}{2} + \frac{1}{\alpha}\left[x - x_0\right] - \frac{1}{2\alpha^2}\left[x - x_0\right]^2, & x_0 \leq x < x_0 + \alpha \\ 1, & x \geq x_0 \end{cases}$$

/ /

Ex 2.3.2 / 50 $\quad X_0 = 8 \quad \alpha = 5 \quad a = \frac{1}{5} = 0.2$



$$f_X(x) = \begin{cases} 0 & , X < 3 \\ (X-3)/25 & , 3 \leq X < 8 \\ 0.2 - (X-8)/25 & , 8 \leq X < 13 \\ 0 & , X \geq 13 \end{cases}$$

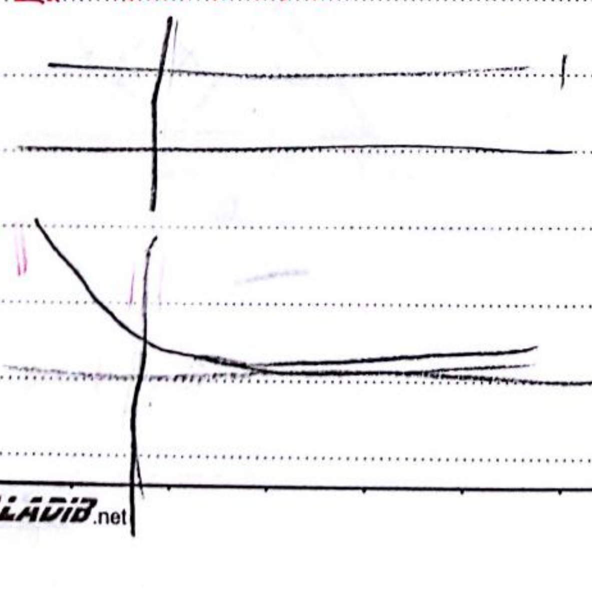


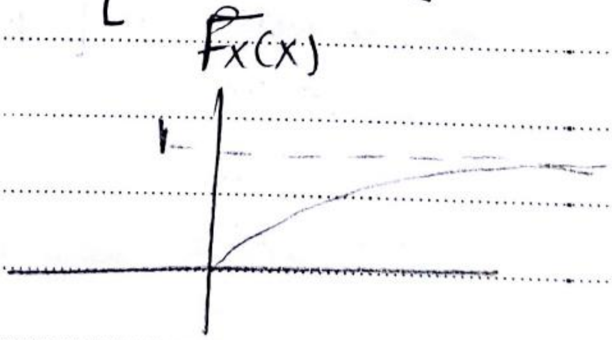
$$F_X(x) = \begin{cases} 0 \\ \frac{1}{50}(X-3)^2 \\ \frac{1}{2} + 0.2(X-8) - \frac{1}{50}(X-8)^2 \\ 1 \end{cases}$$

find $P\{4.5 < X \leq 6.7\}$ = Area under $f_X(x)$.

$$\int_{4.5}^{6.7} \frac{X-3}{25}\, dx = 0.2288$$

Ex 2.3.3 /P50 $\quad F_X(x) = U(x)\left[1 - e^{-\frac{x^2}{b}}\right], \quad b > 0$





Find $f_X(x)$

$$= \frac{d}{dx}F_X(x) = U(x)\frac{d}{dx}\left[1-e^{-\frac{x^2}{b}}\right] + \left[1-e^{-\frac{x^2}{b}}\right]\frac{d}{dx}(U(x))$$

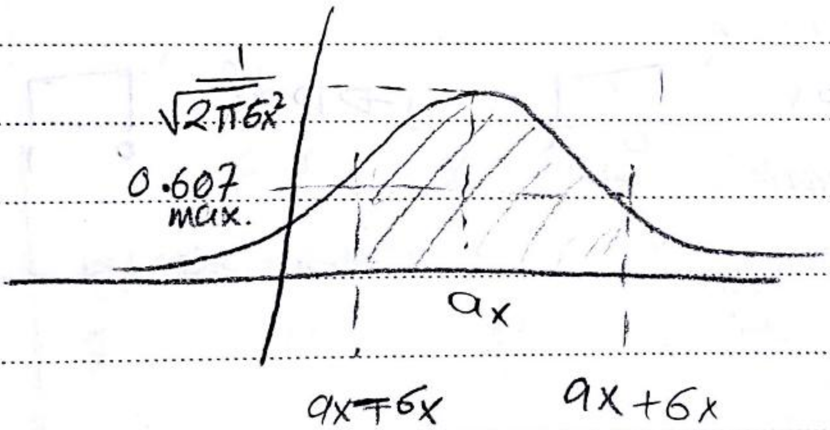
$$= U(x)\frac{2x}{b}e^{-\frac{x^2}{b}} + \cancel{\left(1-e^{-\frac{x^2}{b}}\right)\delta(x)}$$

$$f_X(x) = \frac{2x}{b}e^{-\frac{x^2}{b}}U(x)$$

$\frac{d}{dx}(y)=0$ $\quad 0+0=0$

2.4 Gaussian R.V

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \qquad -\infty < x < \infty$$



How far are the outcomes scattered around the mean.

deviation

σ_x^2 = varience.

CDF:

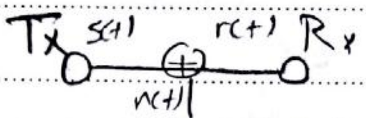
$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\int_{-\infty}^{x} e^{-\frac{(y-a_x)^2}{2\sigma_x^2}}\,dy$$

$P\{X$

Example of Gaussian distribution

1) Product line

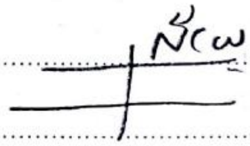
2) Noise



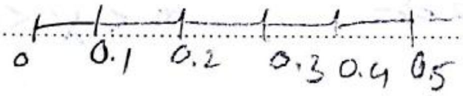
AWGN (Additive white Gaussian noise)

5 marks

$r(t) = s(t) + n(t)$

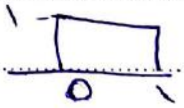


* Create normal 0—1 (1000 times)

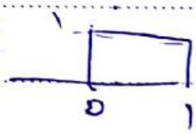


do the same again

you get X → pdf

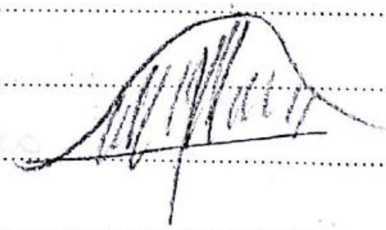


y → pdf



pdf(X + y) ⇒ Gaussian

normal   normal

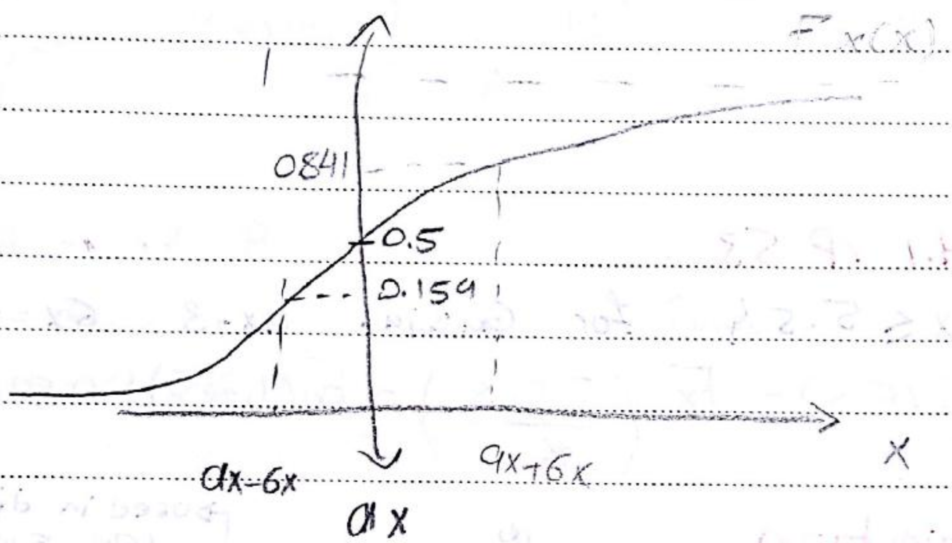


/ /

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(y-a_x)^2/2\sigma_x^2}\,dy$$ must be solved numerically

$P\{X \le x_i\} = F_X(x_i)$



$P\{X > x_i\} = 1 - P\{X < x_i\}$

by subsitution $= 1 - F_X(x_i)$

$U = \frac{y - a_x}{\sigma_x}$ $\qquad du = \frac{1}{\sigma_x} dy \Rightarrow dy = \sigma_x du$

$y = -\infty \qquad U = -\infty$

$y = x \qquad U = \frac{x - a_x}{\sigma_x}$

$$= \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma_x}} e^{-\frac{U^2}{2}} \cancel{\sigma_x}\,du$$

$$= F_X\left(\frac{x-a_x}{\sigma_x}\right) = \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{U^2}{2}}$$

$\Rightarrow$ normalized

$F_X(2) = F_X\left(\frac{2-3}{\ }\right)$

Appendix B / P418

X	.00	- - - -	0.09
0.0			
.			
⋮			
3.8			

Ex 2.4.1 /P 53

$P\{x \leq 5.5\}$ for Gaussian $a_X = 3$ $\sigma_X = 2$

$$F_X(5.5) = F_X\left(\frac{5.5-3}{2}\right) = F_X(1.25) = 0.8944$$

Q-function

used in digital com system

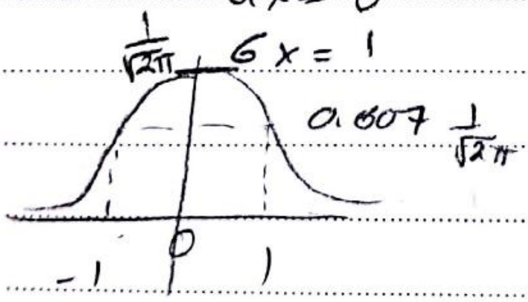
$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-y^2/2}\, dy$$

Compare with

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

$a_X = 0$

$\sigma_X = 1$



$$Q(x) = 1 - F_X(x)$$

Standard Gaussian Distribution function

/ /

Approximation of $Q(x)$

Bojresson and Sundberg $Q(x) \approx \dfrac{1 \cdot e^{-x^2/2}}{(1-a)x + a\sqrt{x^2+b}}$ $x \geq 0$

$a = 0.339$

$b = 5.510$

Abramowitz and Stegun. $Q(x) \approx \dfrac{1}{\sqrt{2\pi}\, x} e^{-x^2/2}$ $x > 3$

Ex 2.4.3/ P 54

$a_X = 7$ $\sigma_X = 0.5$ Gaussian

$P\{X \leq 7.3\}$

$F_X(x) = F_X(7.3) = F_X\left(\dfrac{7.3-7}{0.5}\right) = F_X(0.6)$

$= 0.727$ table

$= 1 - Q(x) = 1 - Q(0.6) =$

$\% \text{ error} = \left|\dfrac{\text{actual} - \text{approx}}{\text{actual}}\right| \times 100\%$

2.5 Other distribution and density function

Binomial —N→ Mainly used in radar!

× × × - - - ×

←—k—→←(N-k)→

$N = 5$
$k = 2$] → berrolli

$k=0 \to k=1 \to k=2 \dots k=N \Rightarrow$ bionomial

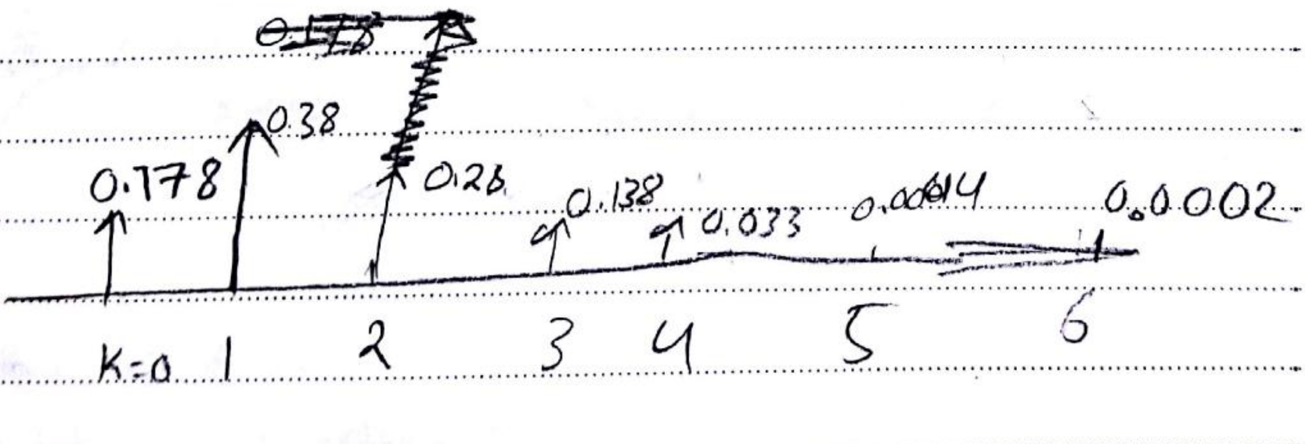
$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} U(x-k)$$

Ex $N = 6$ $p = 0.25$

$$f_X(x) = \sum_{k=0}^{6} \binom{6}{k} 0.25^k \, 0.75^{6-k} \delta(x-k)$$

$$= \binom{6}{0} 0.25^0 \, 0.75^6 \, \delta(x) + \binom{6}{1} + \dots$$



/ /

2.5 other distributions

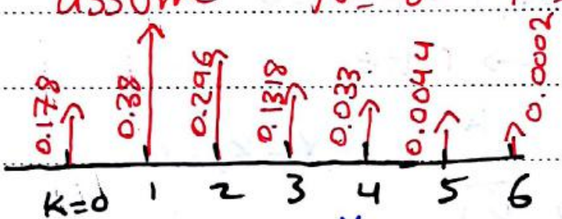
Bernolli trial vs Binomial distribution

Remember: Bernoli trial is one sample of binomial distribution

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

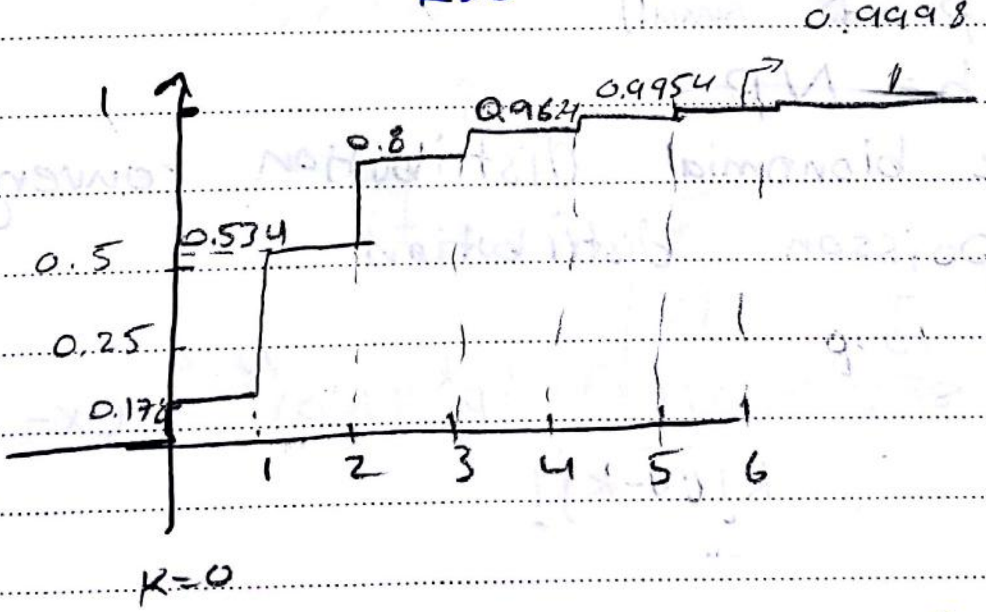
assume $N = 6$ $P = 0.25$

Pdf function



$$F_X(x) = \int_{-\infty}^{x} f_X(x)$$

$$= \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \, u(x-k)$$



0.964

$P\{k \le 4\} = F_x(4) = 0.9954$

/ /

Poisson [Discrete RV]

$|\xleftarrow{\quad T \quad}|$

counting defictive parts at a constant rate

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

↳ what I am looking for k defective parts.

$b > 0$

$b = \lambda T \rightarrow$ period

↳ rate

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

in bionomial pdf

$N \rightarrow$ large

$p \rightarrow$ small

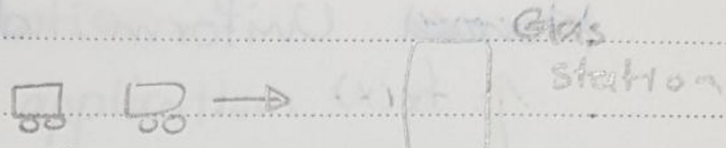
$b = NP$

the bionomial distribution converges to poisson distribution

$$f_X(x) = \sum_{k=0}^{1000} \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k} \delta(x-k)$$

Ex 2.5.1

P.56



$\lambda = 60$ cars / hour

$T = 1$ min to obtain fuel

$P\{$a waiting line occurs$\}$

When do we get a waiting line

1 car waiting

waiting

$P\{x > 1\} = P\{x \geq 2\}$

$b = \lambda T = \frac{50}{60} \times 1 \text{min} = \frac{5}{6}$

$P\{x > 1\} = 1 - P\{X \leq 1\}$
$x \geq 2$

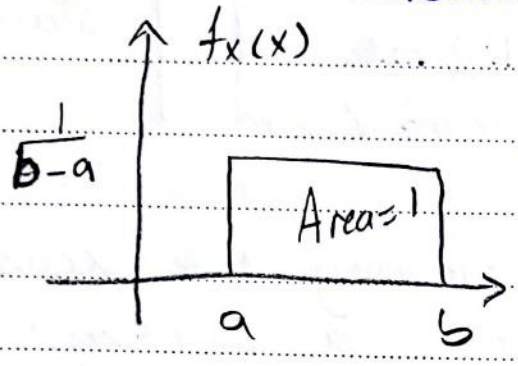
$= 1 - F_X(1)$

$$F_X(1) = \sum_{k=0}^{1} e^{-\frac{5}{6}} \frac{b^k}{k!} u(x-k)$$

$$= e^{-\frac{5}{6}} \left[\frac{\frac{5}{6}^0}{0!} + \frac{(5/6)^1}{1!} \right]$$

$P\{X > 1\} = 1 - F_X(1) = 0.2032$

~~Normal~~ Uniform distribution [Contionous RV]

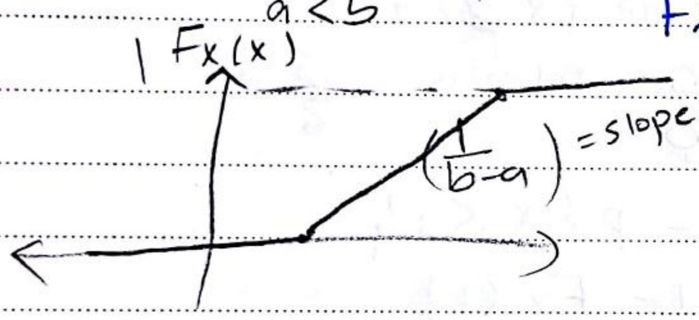


$$f_x(x) = \begin{cases} 0 & x<a \\ \frac{1}{b-a} & a \le x < b \\ 0 & x \geq b \end{cases}$$

$-\infty < a < \infty$

$-\infty < b < \infty$

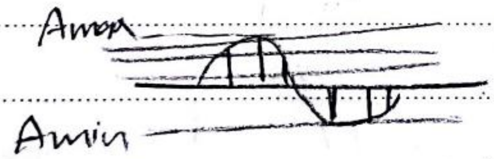
$a < b$

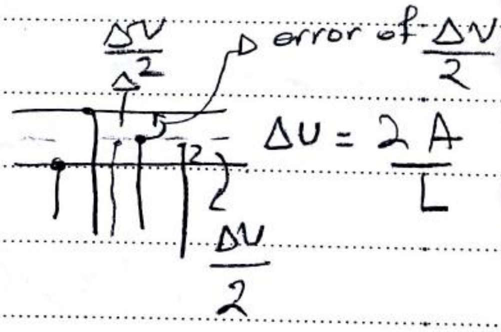


$$F_x(x) = \begin{cases} 0 & x<a \\ \frac{1}{b-a}(x-a) & a \le x < b \\ 1 & x \geq b \end{cases}$$

* quantization noise

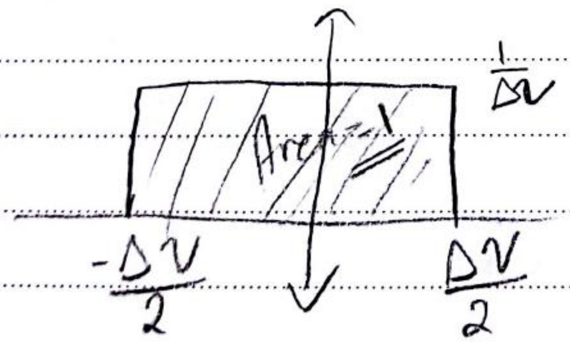
Analog $\rightarrow$ digital





$\Delta V = \frac{2A}{L}$

$L = 2^{n \rightarrow \text{bits}}$



Exponential distribution.

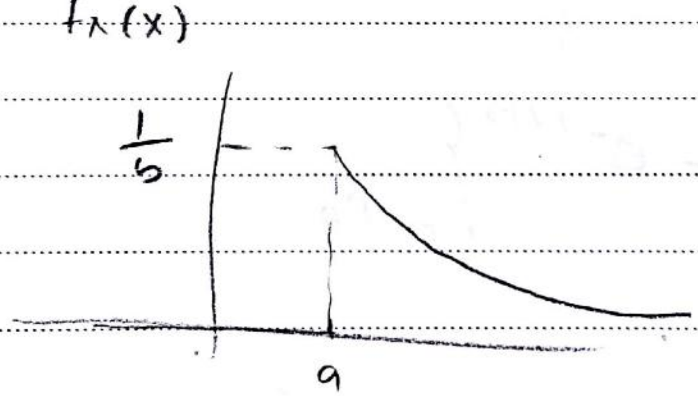
Used in radar application

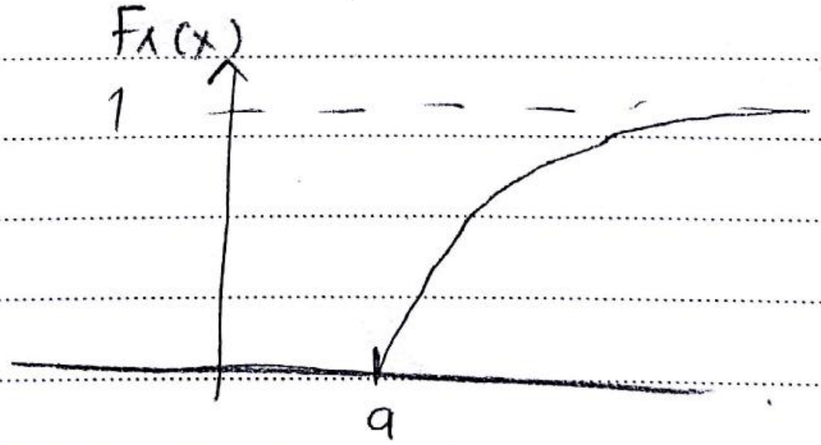
$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\left(\frac{x-a}{b}\right)} & , x \geq a \\ 0 & , x < a \end{cases}$$

$$-\infty < a < \infty$$

$$b > 0$$

$$F_X(x) = \begin{cases} 1 - e^{-\left(\frac{x-a}{b}\right)} & , x \geq a \\ 0 & , x < a \end{cases}$$





/ /

$$f_p(p) = \begin{cases} \frac{1}{P_0} e^{-P/P_0} & p \geq 0 \\ 0 & p < 0 \end{cases} \Rightarrow \begin{matrix} a = 0 \\ b = P_0 \end{matrix}$$

P_0 is the average power recieved

$$P\{\text{recieved powe} > P_0\}$$

$$= 1 - P\{p \leq P_0\}$$

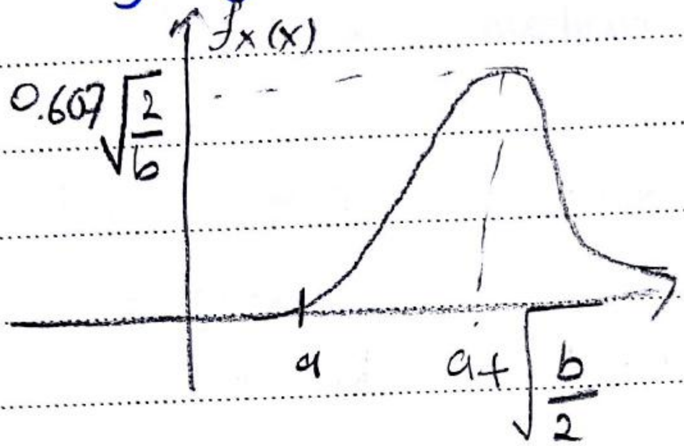
$$= 1 - F_p(P_0)$$

$$= 1 - \left\{1 - e^{-P/P_0}\right\}_{p = P_0}$$

$$= 0.368$$

/ /

Rayleigh distribution



$$f_X(x) = \begin{cases} \frac{2}{b} \\ \\ \end{cases}$$

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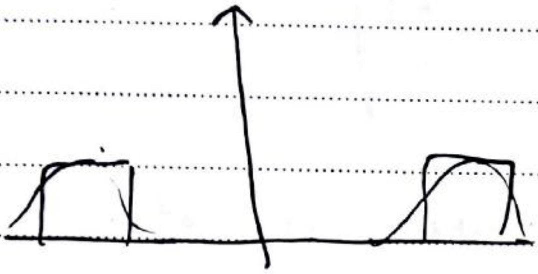
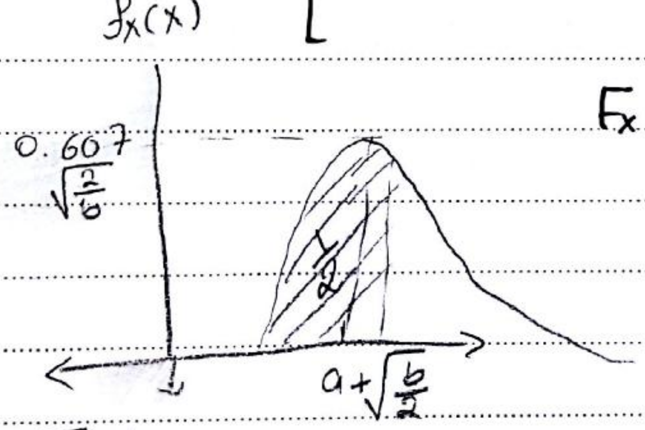
Rayleigh
AWGN

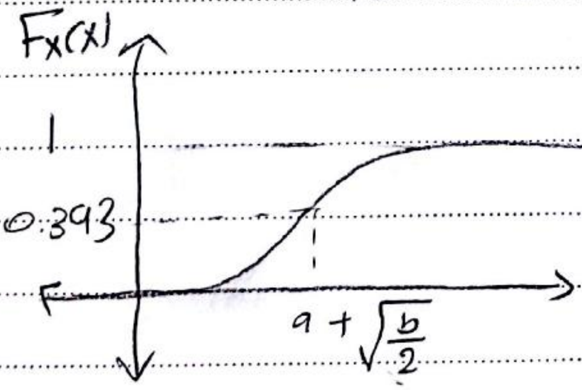


distribution can be obtained by passing noise into BPF

$$f(x) = \begin{cases} \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}}, & x \geq a \\ 0, & x < a \end{cases}$$



$$F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases}$$



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Ex 2.5.3 / P60

Find the median of R.V Rayleigh pdf?

$$P\{X \le x_0\} = P\{X > X_0\} = 0.5$$

$$= F_X(x_0)$$

$$1 - e^{-\frac{(x_0 - a)^2}{b}} = 0.5$$

$$e^{-\frac{(x_0 - a)^2}{b}} = 0.5$$

$$-\frac{(x_0 - a)^2}{b} = \ln 0.5$$

$$(x_0 - a)^2 = b \ln 2$$

$$x_0 - a = \sqrt{b \ln 2}$$

$$x_0 = a + \sqrt{b \ln 2}$$

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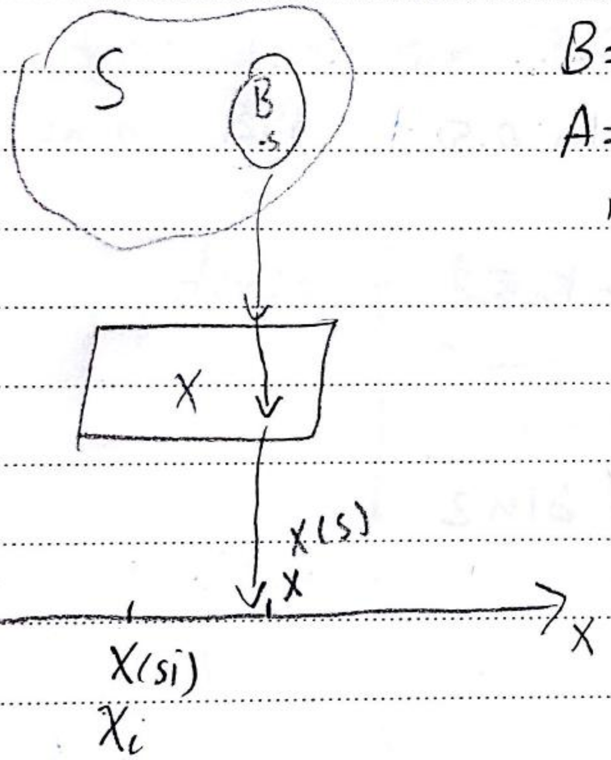
2.6 Conditional Distribution and density function

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

1) Conditional Distribution function

$P\{A|B\} = P\{\cancel{X} \le x \mid B\} = F_X(x/B)$

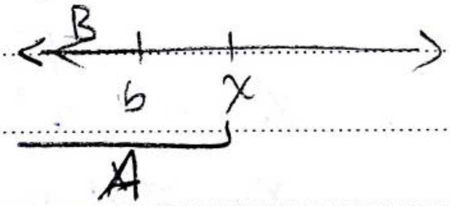


$B = \{X \le b\}$

$A = \{X \le x\}$

if $b < x$

$\{X \le b\} \subset \{X \le x\}$



if $b > \cancel{x}$

$\{X \le x\} \subset \{X \le b\}$

$$P\{A|B\} = \frac{P\{X \le x \cap X \le b\}}{P\{X \le b\}}$$

if $b < x$ $\quad = \dfrac{P\{X \le b\}}{P\{X \le b\}} = 1$

if $b > x$

$$= \frac{P\{X \le x\}}{P\{X \le b\}} =$$

$$F_X(x|B) = \frac{F_X(x)}{F_X(b)} = \frac{F_X(x)}{\int^{b} f_X(x)\, dx} \leftarrow \text{constant}$$

/ /

Same properties apply :-

$F_X(x)$ $\quad$ $F_X(x \backslash B)$

1) $F_X(-\infty) = 0$ $\quad$ $F_X(-\infty \backslash B) = 0$

2) $F_X(\infty) = 1$ $\quad$ $F_X(\infty \backslash B) = 1$

3) $x_1 < x_2$ $\quad$ $x_1 < x_2$

$F_X(x_1) < F_X(x_2)$ $\quad$ $F_X(x_1 \backslash B) < F_X\{x_2 \backslash B\}$

4) $F_X(x) = F_X(x^+)$ $\quad$ $F_X(x \backslash B) = F_X(x^+ \backslash B)$

5) $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

$P\{x_1 < X \leq x_2 \backslash B\} = F_X(x_2 \backslash B) - F_X(x_1 \backslash B)$

2. density function

$b > x$

$$F_X(x \backslash B) = \frac{F_X(x)}{F_X(b)}$$

↳ if b is constant, this term is constant

$$f_X(x) = \frac{1}{F_X(b)} f_X(x)$$

Ex 2.6.1

P61

	X	Box 1	Box 2	
R	1	5	80	85
G	2	35	60	95
B	3	60	10	70
		100	150	

1   2

$P\{B_2\} = 8/10$

$P\{B_1\} = 2/10$

$P(B_1 \cap B_2) = 0$

$P\{X = 1 \mid B = B_1\} = \frac{5}{100}$

$P\{X = 2 \mid B = B_1\} = \frac{35}{100}$

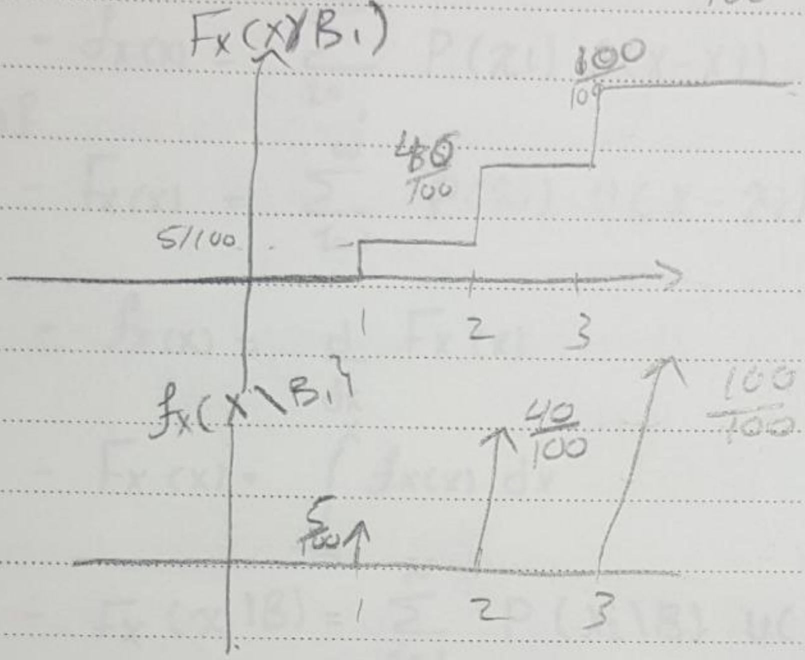
$B\{X = 3 \mid B_1\} = \frac{60}{100}$

$P\{X = 1 \mid B_2\} = \frac{80}{150}$

$P\{X = 2 \mid B_2\} = \frac{60}{150}$

$P\{X = 3 \mid B_2\} = \frac{10}{150}$

$$F_X(x \mid B_1) = \frac{5}{100}\,u(x-1) + \frac{35}{100}\,u(x-2) + \frac{60}{100}\,u(x-3)$$



total probability

$$P\{x=1\} = P\{x=1 \mid B_1\}P\{B_1\} + P\{x=1 \mid B_2\}P(B_2)$$

$$= \frac{5}{100}\times\frac{2}{10} + \frac{80}{150}\times\frac{8}{10} = 0.437$$

$$P\{x=2\} = \frac{35}{100}\times\frac{2}{10} + \frac{60}{150}\times\frac{8}{10} = 0.39$$

$$P\{x=3\} = 0.173$$

$$f_X(x) = 0.437\,\delta(x-1) + 0.398\,\delta(x-2) + 0.173\,\delta(x-3)$$

$$F_X(x) = 0.437\,u(x-1) + 0.398\,u(x-2) + 0.173\,u(x-3)$$

2.1.3
2.1.7
2.1.11
2.2.5
2.2.9
2.3.5
2.3.8
2.3.10
2.4.1
2.4.6
2.4.10
2.5.3
2.5.9
2.5.13
2.6.8

Pdf

- $f_X(x) = \sum_{i=1}^{\infty} P(x_i)\, \delta(x - x_i)$

cdf

- $F_X(x) = \sum_{i=1}^{\infty} P(x_i)\, u(x - x_i)$

- $f_X(x) = \frac{d}{dx} F_X(x)$

- $F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$

- $F_X(x|B) = \sum_{i=1}^{N} P(x_i|B)\, u(x - x_i)$

- $f_X(x|B) = \sum_{i=1}^{N} P(x_i|B)\, \delta(x - x_i)$

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Ch#3 : Operations on one RV   / /

- Expectation

related to averaging process of R.V

Can be called:

- The mathematical expectation of X
- The expected value of X
- The mean of X
- The statistical average of X

Denoted by:

$E[x]$

Ex 3.1.1 / P78

90 people 25¢ 10¢ 5¢ 1¢

if Σ of coins < 1$ record it.

if > 1$ record the fraction only

We recorded : 18¢, 45¢, 64¢, 72¢, 77¢, 95¢

how many times: 8 people, 12, 28, 22, 15, 5

Generally   in this class

$$E[x] = \bar{X} = \frac{\sum_{i=1}^{N} x_i}{N} = \sum_{i=1}^{N} x_i \, p(x_i)$$

to find fractional average 18¢ × 8 + ----

$$\bar{X} = \frac{18 \times 8}{90} + \frac{45 \times 12}{90} + \cdots$$

fraction   probability

if X : discrete R.V

$$E(x) = \sum_{i=1}^{N} x_i \, P(x_i)$$

if X: continous R.V

$$E[x] = \int_{-\infty}^{\infty} x \, P_x(x) \, dx$$

How to derive:

assume: $f_X(x) = \sum_{i=1}^{N} P(x_i) \, \delta(x - x_i)$

$$E(x) = \int_{-\infty}^{\infty} x \sum_{i=1}^{N} P(x_i) \, \delta(x - x_i)$$

$$= \sum_{i=1}^{N} p(x_i) \int_{-\infty}^{\infty} x \, \delta(x - x_i)$$

by sifting property

$$= \sum_{i=1}^{N} p(x_i) \, x_i$$

Ex 3.1.2 / P 79

find $\bar{x}$ of X with exponential distribution.

$$P_x(x) = f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x > a \\ 0 & x < a \end{cases}$$

$$E(x) = \int_{a}^{\infty} x \, \frac{1}{b} \, e^{-\frac{(x-a)}{b}} \, dx$$

/ /

$$\int x e^{ax} dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right]$$

$\boxed{a = -\frac{1}{b}}$

$$E(x) = \frac{1}{b} e^{a/b} \int_a^{\infty} x e^{-x/b} dx$$

$$= \frac{1}{b} e^{a/b} \left[e^{-x/b} \left[-\frac{bx}{a} - b^2 \right] \right]_a^{\infty}$$

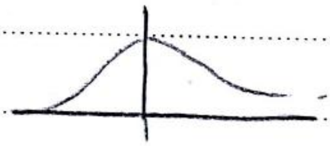
$$= \frac{e^{a/b}}{b} \left[e^{-a/b} (ab + b^2) \right]$$

$$= a + b$$

For symmetric R.V the E(x) "mean" will be the center point

$f_X(x+a) = f_X(-x+a)$

area left = area right.



/ /

Expected value of a function of R.V

assume $g(x)$ is function of X

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

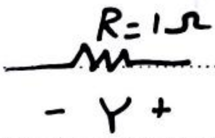
for example

$$E[ax + bx^2 + cx^3 + \dots]$$
$$= a\,E[x] + b\,E[x^2] + \dots$$

Ex 3.1.3 / P79

V described using a Rayleigh distribution

$$f_V(v) = \begin{cases} \frac{2}{b}(v-a)\, e^{-\frac{(v-a)^2}{b}} & v \geq a \\ 0 & v < a \end{cases}$$



$a=0 \quad b=5$

assume $Y = g(v) = V^2$ [Power]

find the averge power :- // find $E(g(v))$

$$= \int_{-\infty}^{\infty} V^2 f_V(v)\, dV = \int_0^{\infty} V^2 \frac{2}{5} V e^{-V^2/5}\, dv$$

$U = V^2 \rightarrow du = 2V\, dV \Rightarrow V dV = \frac{1}{2} dU$

$V = 0 \Rightarrow U = 0$

$$\frac{2}{2\times 5} \int_0^{\infty} U e^{-U/5}\, du = 5 = \underline{\underline{a+b}}$$

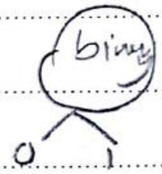
$V = \infty \Rightarrow U = \infty.$

* higher propability → less information
desert + rain example

* entropy is the averge of information.

Ex 3.1.4 /P80

L distinct symbols
$x_i \quad i = 1 : L$



$P(x_i)$ = probability of $x = x_i$

facts of information theory

① info $\propto \frac{1}{\text{Prob}}$

② information of two independent sources should add up.

③ info ≥ 0

④ info = 0 for certain events   أكيد → P(event) = 1

⑤ $L \geq 2$

all of these facts are satisfied for logarithmic function.

information about symbol x_i

$$I = -\log_2 P(x_i)$$

average info of the source

$$H = -\sum_{i=1}^{L} p(x_i) \log_2 (P x_i)$$

bits/symbol

/ /

2.3 Moment [about origin]

n^{th} moment $= m_n = E[X^n] = \int_{-\infty}^{\infty} X^n f_X(x)\,dx$

$m_0 = \int_{-\infty}^{\infty} x^0 f_X(x)\,dx = 1 \rightarrow$ zero moment

$m_1 = \int_{-\infty}^{\infty} x\, f(x)\,dx = \underbrace{E[x]}_{=\bar{x}} \rightarrow 1^{st}$ moment

→ Centeral moment (Refference about ~~the~~ $\bar{x}$)

nth central moment $= \mu_n = \int_{-\infty}^{\infty} (x-\bar{x})^n f_X(x)\,dx$

$= E[(X-\bar{x})^n]$

$\boxed{n=1}$ $\mu_{n=1} \Rightarrow \mu_1 = \int_{-\infty}^{\infty} (X-\bar{x})\, f_X(x)\,dx = E[x-\bar{x}] = 0$

$\int_{-\infty}^{\infty} x f_X(x)\,dx - \bar{x}\int_{-\infty}^{\infty} f_X(x)\,dx$

$\bar{x} \quad - \bar{x} \quad = 0$

$\boxed{n=2}$ $\mu_2 = E[(x-\bar{x})^2]$

$= E[x^2 - 2x\bar{x} - \bar{x}^2]$

$= E[x^2] - E[2x\bar{x}] - E[\bar{x}^2]$

$= m_2 - 2\bar{x}\,E[x] - \bar{x}^2$

$= m_2 - 2\bar{x}^2 - \bar{x}^2$

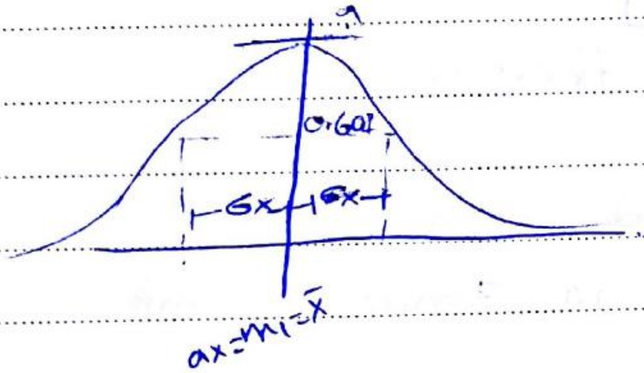
$E[\text{cont}] = \text{cosnt}$

$E[a\,x] = a\bar{x}$

$\mu_2 = m_2 - m_1^2 = m_2 - \bar{x}^2 \Rightarrow$ called varience $(\sigma_x)^2$

* Varience = $\sigma_x^2 = m_2 - m_1^2 = \mu_2$

$$\sqrt{\sigma_x^2} = \begin{matrix} +\sigma_x \\ -\sigma_x \end{matrix}$$ [Represents standard deviation]



Example 3.2.1 P.82 X with exponential density function

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} , & x > a \\ 0 , & x < a \end{cases}$$

$$\sigma_x^2 = E[(X-\bar{x})^2] = m_2 - m_1^2 =$$

Using tables

$$m_2 = \int_a^{\infty} \frac{x^2}{b} e^{-\frac{(x-a)}{b}} dx =$$

$$m_1 = \int_a^{\infty} \frac{x}{b} e^{-\frac{(x-a)}{b}} dx =$$

$$\int x^2 e^{ax} dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$

$$\boxed{\sigma_x^2 = b^2}$$

Skew $\Rightarrow$ μ_3 defines the measure of asymmetry of a density function

$$\mu_3 = E[(x-\bar{x})^3]$$

$$= \int_{-\infty}^{\infty} (x-\bar{x})^3 \, f_X(x) \, dx$$

OR expand.

$$E[(x-\bar{x})(x-\bar{x})^2] = m_3 - 3m_1m_2 + 2m_1^3$$

☆ Skewness / Skew coefficient $= \dfrac{\mu_3}{\sigma_x^3}$

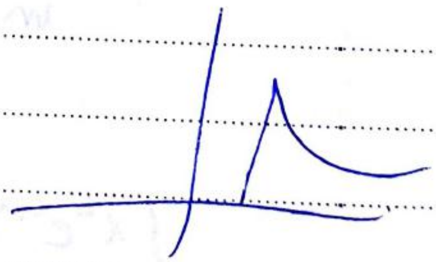
$$= \frac{m_3 - 3m_1m_2 + 2m_1^2}{m_2 - m_1^2}$$

Example 3.2.2 / P82

$$f_X(x) = \begin{cases} \dfrac{1}{b} e^{\frac{-(x-a)}{b}} & , \; x \geq a \\ 0 & , \; x < a \end{cases}$$

$\mu_3 = 2b^3$

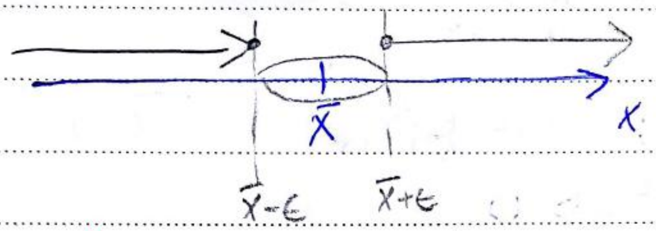
$\dfrac{\mu_3}{\sigma_x^3} = 2 \Rightarrow$ high asymmetry



Chebychev's inequality $P\{|X-\bar{X}| \geq \epsilon\} \leq \frac{\sigma_x^2}{\epsilon^2}$

$X > \bar{X}+\epsilon$

$X < \bar{X}-\epsilon$



* Proof:-

$$P\{|X-\bar{X}| \geq \epsilon\} = \int_{-\infty}^{\bar{X}-\epsilon} f_X(x)\,dx + \int_{\bar{X}+\epsilon}^{\infty} f_X(x)\,dx$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\bar{X}-\epsilon} (x-\bar{x})^2 f_X(x)\,dx + \int_{\bar{X}-\epsilon}^{\bar{X}+\epsilon} (X-\bar{X})^2 f_X(x)\,dx +$$

$$\int_{\bar{X}+\epsilon}^{\infty} (X-\bar{X})^2 f_X(x)\,dx =$$

$$\sigma_x^2 \geq \int_{-\infty}^{\bar{X}-\epsilon} \underbrace{(x-\bar{x})^2}_{>\epsilon^2} f_X(x)\,dx + \int_{\bar{X}+\epsilon}^{\infty} (X-\bar{X})^2 f_X(x)\,dx$$

$|X-\bar{X}| \geq \epsilon$

$|X-\bar{X}|^2 > \epsilon^2$

$$\frac{\sigma_x^2}{\epsilon^2} \geq \int_{-\infty}^{\bar{X}-\epsilon} \cancel{\epsilon^2} f_X(x)\,dx + \int_{\bar{X}+\epsilon}^{\infty} \cancel{\epsilon^2} f_X(x)\,dX$$

$$\frac{\sigma^2}{\epsilon^2} \geq P\{|x-\bar{x}| \geq \epsilon\}$$

$$P\{|X-\bar{X}|<\epsilon\} \geq 1 - \frac{\sigma_X^2}{\epsilon^2}$$

$$\epsilon \to 0$$

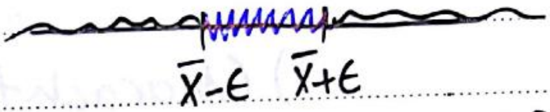
$$P\{|X-\bar{X}|<\epsilon\} = 1$$

$$\sigma_X^2 \to 0$$

$$P\{|X-\bar{X}|<\epsilon\} = 1$$

lecture　　　　/　/

Example 3.2.3 // P 83



$P\{ X < \bar{X} - 3\sigma_X \}$

$P\{ X > \bar{X} + 3\sigma_X \}$

$\hookrightarrow P\{ |X - \bar{X}| \geq 3\sigma_X \} \leq \frac{\sigma_X^2}{\epsilon^2}$

$P\{|X-\bar{X}| \geq \epsilon\} \leq \frac{\sigma_X^2}{\epsilon^2}$

$P\{|X-\bar{X}| \leq \epsilon\} \geq 1 - \frac{\sigma_X^2}{\epsilon^2}$

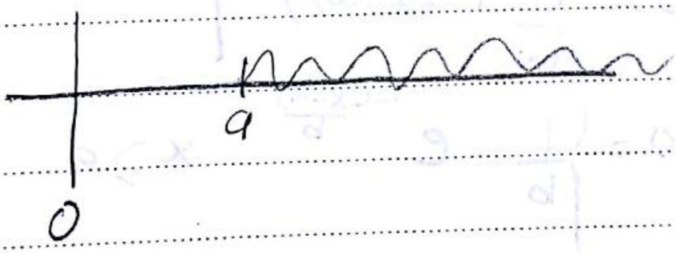
$\epsilon = 3\sigma_X$

$P\{|X-\bar{X}| \geq 3\sigma_X\} \leq \frac{\sigma_X^2}{9\sigma_X^2} = \frac{1}{9}$

* Markov's inequality [Sets an upper bound]

for non-negative random variable X

$P\{ X \geq a \} \leq \frac{1}{a} E[X]$



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reference to $\bar{X}$

reference to

3.3 Functions that give moments

1) Charachtaristic fun $\Phi_X(\omega)$

2) Moment Generating fun $M_X(v)$

↳ new = $(j\omega)$

$$\Phi_X(\omega) = E[e^{j\omega x}]$$ ⟶ flipped from $\mathcal{F}$

$$= \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega)\, e^{-j\omega x}\, dx$$

$$m_n = (-j)^n \frac{d}{d\omega}\left(\Phi_X(\omega)\right)\Big|_{\omega=0}$$

Ex 3.3.1 / P.85
$$f_X(x) = \begin{cases} \frac{1}{b}\, e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases}$$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx = \int_{a}^{\infty} \frac{1}{b}\, e^{-\frac{(x-a)}{b}}\, e^{j\omega x}\, dx$$

$$= \frac{e^{a/b}}{b} \int_{a}^{\infty} e^{-x(\frac{1}{b}-j\omega)}\, dx = \frac{e^{a/b}}{b} \frac{e^{-x(\frac{1}{b}-j\omega)}}{-\frac{1}{b}+j\omega}\Bigg|_{a}^{\infty}$$

$$= \frac{e^{j\omega a}}{1-j\omega b}$$

/ /

$$\frac{d\,\Phi_x(\omega)}{d\omega} = \frac{(1-j\omega b)(ja\,e^{j\omega a}) - e^{j\omega a}(-jb)}{(1-j\omega b)^2}$$

$$\frac{d}{d\omega}\Phi_x(\omega)\Big|_{\omega=0} = \frac{ja+jb}{1} =$$

$$m_1 = -j \times j(a+b) = a+b$$

Using Moment Generating Function

$$M_x(v) = \int_{-\infty}^{\infty} f_x(x)\, e^{vx}\, dx$$

the problem of this function is that it sometimes doesn't exist like $\sqrt{-\text{const}}$

$$m_n = \frac{d\, M_x(v)}{dv}\Big|_{v=0}$$

Chernoff's Inequality

$$P\{X \geq a\} \leq M_X(v)\, e^{-av}$$

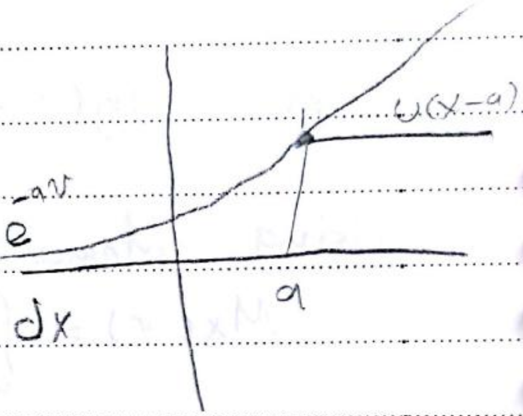
$-\infty < a < \infty$

Proof

$$u(x-a) \leq e^{v(x-a)}$$

$$f_X(x)\, u(x-a) \leq f_X(x)\, e^{v(x-a)}$$

$$\int_{-\infty}^{\infty} f_X(x)\, u(x-a)\, dx \leq \int_{-\infty}^{\infty} f_X(x)\, e^{v(x-a)}\, dx$$



$$\int_{a}^{\infty} f_X(x)\, dx \leq e^{-av} \underbrace{\int_{-\infty}^{\infty} f_X(x)\, e^{vx}\, dx}_{M_X(v)}$$

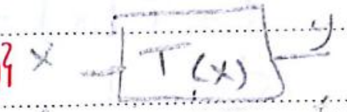
$$P\{X \geq a\} \leq e^{-av} M_X(v)$$

/ /

3.4 Transformation of random Variable ~~[illegible]~~

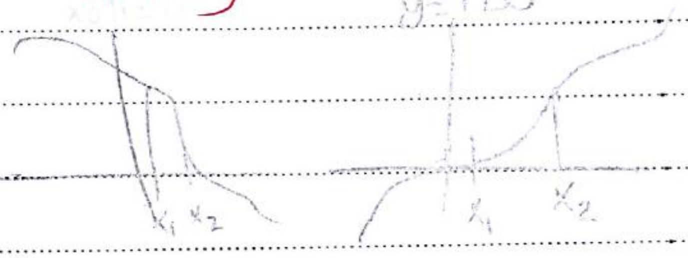
We will investegate

$P\{x\} = P\{y\}$ $x \to$ $\boxed{T(x)}$ $\to y$



$1 - x - 1 - y$ $f_x(x)$ $f_y(y)$

1. X cont. + T[x] cont → monotonic (always) → increasing / → decreasing
→ non-monotonic
m - x - 1 - y



$x_1 < x_2$	$x_1 < x_2$
$F[x_1] > F[x_2]$	$T[x_1] < T[x_2]$
-ve slope	+ve slope
monotonic decreasing //	monotonic increasing

2. X discrete + T[x] cont.

Proof: for monotonic increasing

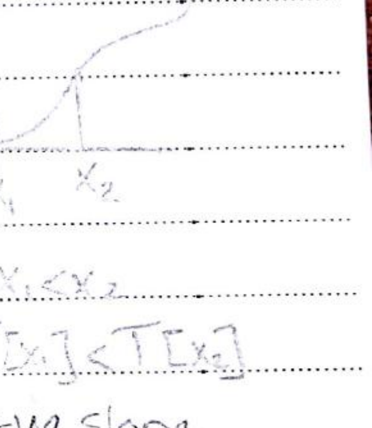
$$P\{Y \leq y_0\} = P\{X \leq X_0\}$$

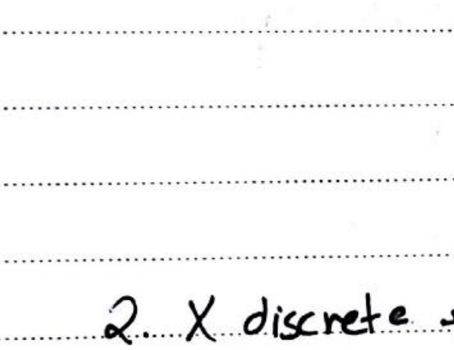
$$F_y(y_0) = F_x(x_0)$$

for monotonically decreasing

$$P\{Y \leq y_0\} = P\{X > X_0\}$$

$$F_y(y_0) = 1 - F_x(x_0)$$





$$f_y(y) = f_x(x) \left| \frac{dx}{dy} \right|$$

/ /

Ex 3.4.1 / P89

$Y = T[X] = aX + b$

$\left|\frac{dx}{dy}\right| = \frac{1}{|a|}$

Assume $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}$

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Ex 341 /P89

$Y = aX + b$ a,b are real numbers.

If X is gaussian function.

$$X = \frac{y-b}{a}$$

$$\frac{dx}{dy} = \frac{1}{|a|}$$

$$f_y(y) = f_x(x)\,\frac{dx}{dy}$$

$$= \frac{1}{|a|}\,\frac{1}{\sqrt{2\pi 6_x^2}}\, e^{\frac{-\left(\frac{y-b}{a} - a_x\right)^2}{2 6_x^2}}$$

$$= \frac{1}{\sqrt{2\pi a^2 6_x^2}}\, e^{\frac{-\left(y-b-a_x \cdot a\right)^2}{2a^2 6_x^2}}$$

$$= \frac{1}{\sqrt{2\pi\, 6_y^2}}\, e^{\frac{-\left(y-(b+a_x)\right)^2}{2 6_{y}^2}}$$

$$= \frac{1}{\sqrt{2\pi\, 6_y^2}}\, e^{\frac{-\left(y - a_y\right)^2}{2 6_y^2}}$$

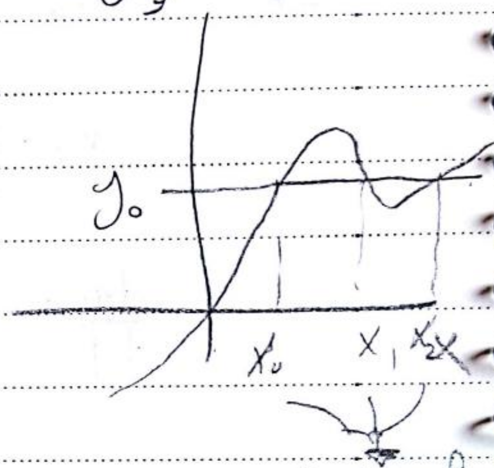
$\boxed{6_y^2 = a^2 6_x^2}$ $\boxed{a_y = b + a_x}$

/ /

non-monotonic [different slope $\pm$ve]

$P\{Y \le y_0\} = P\{X \mid y \le y_0\}$

$y = T[X]$



≡ vals the same y

$$F_Y(y_0) = \int_{X / y \le y_0} f_X(x)\, dx$$

derive with respect to y_0

$$f_Y(y_0) = \frac{d}{dy_0}\left(\int_{X \mid Y \le y_0} f_X(x)\, dx \right)$$

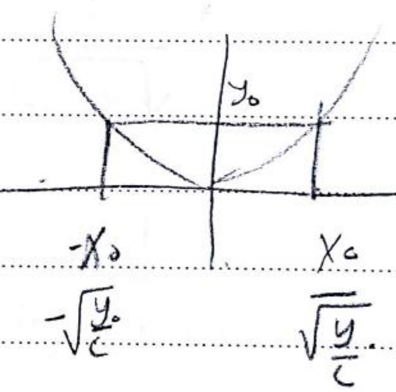
$$f_Y(y) = \sum_n \left(\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\Bigg|_{x = x_n} \right)$$

↳ roots of function

$Y = T[X] = cX^2$

$$\left|\frac{dy}{dx}\right| = |2cx|$$

$$X = \pm\sqrt{\frac{y}{c}}$$



$$f_Y(y) = \frac{f_X(X_0)}{\frac{dy}{dx}\Big|_{x=X_0}} + \frac{f_X(-X_0)}{\frac{dy}{dx}\Big|_{x=-X_0}}$$

$$= \frac{f_X\left(\sqrt{\frac{y_0}{c}}\right)}{\left|2c\sqrt{\frac{y_0}{c}}\right|} + \frac{f_X\left(-\sqrt{\frac{y_0}{c}}\right)}{\left|\left(-2c\sqrt{\frac{y_0}{c}}\right)\right|} =$$

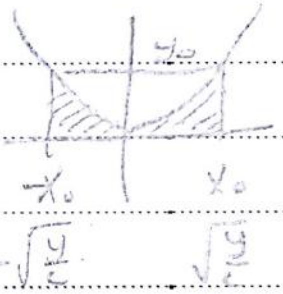
↳ $2\sqrt{c y_0}$

$$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{c}}\right) + f_X\left(-\sqrt{\frac{y}{c}}\right)}{2\sqrt{cy}}$$

OR

$$P\{Y \leq y\} = P\{X \mid Y \leq y\}$$

$$f_Y(y) = \int_{-\sqrt{\frac{y}{c}}}^{\sqrt{\frac{y}{c}}} f_X(x)\, dx$$



Leibniz's rule

$$f_Y(y) = f_X\left(\sqrt{\frac{y}{c}}\right) \frac{d\sqrt{\frac{y}{c}}}{dy} - f_X\left(-\sqrt{\frac{y}{c}}\right) \frac{d\left(-\sqrt{\frac{y}{c}}\right)}{dy}$$

$$\frac{d}{dy}\left(\frac{y}{c}\right)^{\frac{1}{2}} = \frac{1}{2c}\left(\frac{y}{c}\right)^{-\frac{1}{2}} = \frac{1}{2c\sqrt{\frac{y}{c}}} = \frac{1}{2\sqrt{cy}}$$

$=$

3) Transformation of discrete R.V

$$Y = T[x]$$

X is discrete

→ Same value of Y for more than one value of X.

→ Single Y for each X.

$$F_X(x) = \sum_n P(x_n)\, u(x - x_n)$$

$$f_X(x) = \sum_n p(x_n)\, \delta(x - x_n)$$

↳ for more than one value of x_n (same position) we'll get different amplitudes !!

* Point-to-point transformation

$$Y = T[x]$$

$$Y_n = T[x_n]$$

$$f_Y(y) = \sum_n P(y_n)\, u(y - y_n)$$

$$F_Y(y) = \sum_n p(y_n)\, \delta(y - y_n)$$

$$P(y_n) = \sum_n P(x_n)$$

/ /

Ex 3.4.1 / P.92

X	-1	0	1	2
$P(x_n)$	0.1	0.3	0.4	0.2
Y	$\frac{2}{3}$	2	$\frac{4}{3}$	$\frac{2}{3}$

$Y = 2 - X^2 + \frac{X^3}{3}$

$P(\frac{2}{3}) = 0.3 \quad P(2) = 0.3 \quad P(\frac{4}{3}) = 0.4$

$F_Y(y) = 0.3\, u(y - \frac{2}{3}) + 0.3\, u(y-2) + 0.4\, u(y - \frac{4}{3})$

3.5 Computer Generating of one R.V.

Same as the homework.

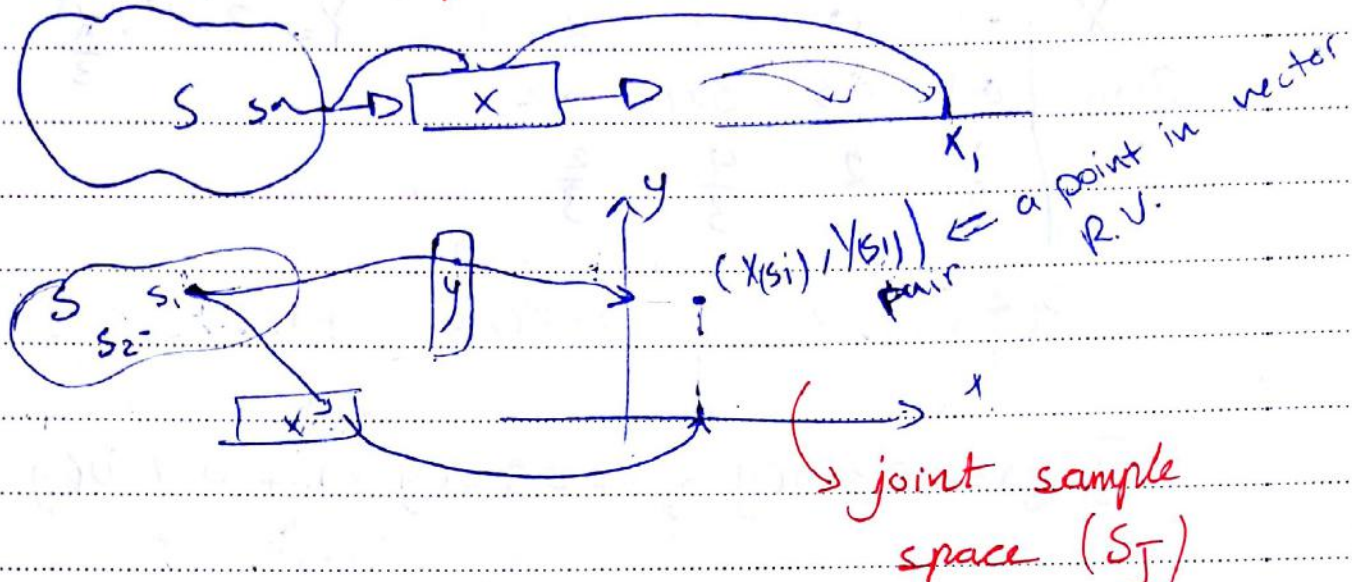
Solve:-

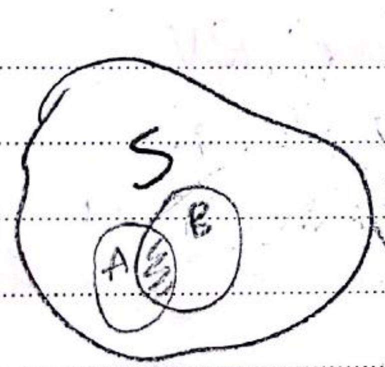
3.1.1 3.1.5 3.1.11

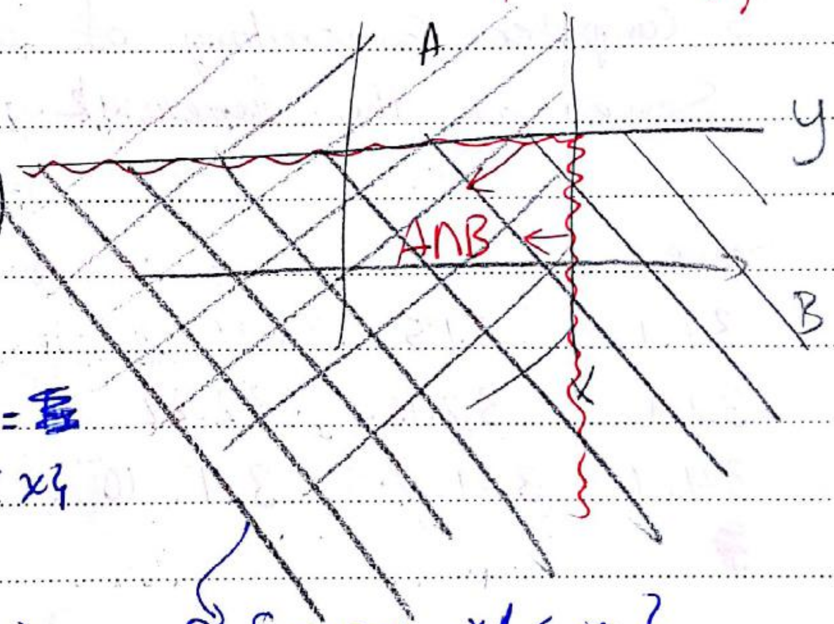
3.2.9 3.2.16 3.2.22 3.2.33 3.2.36

3.4.1 3.4.7 3.4.10 3.4.14

Ch#4 Multiple R.Vs.







$A = \{X \le x\}$
$P(A) = F_X(x)$

$B = \{Y \le y\}$
$P(B) = F_Y(y)$

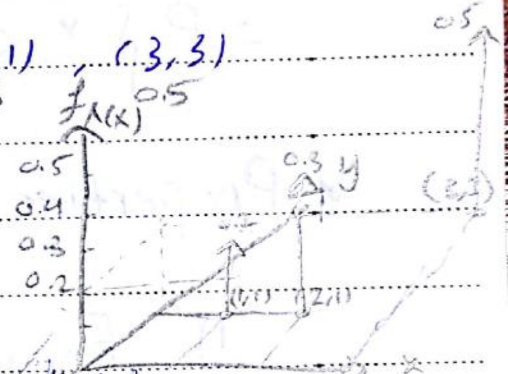
$P\{X \le x, Y \le y\}$
$= F_{XY}(x, y) \equiv$ joint distribution function

$P = A \cap B$

We will discuss discrete RV first, then we will go to cont.

Ex.4.2.1/P110 discrete RV X and Y

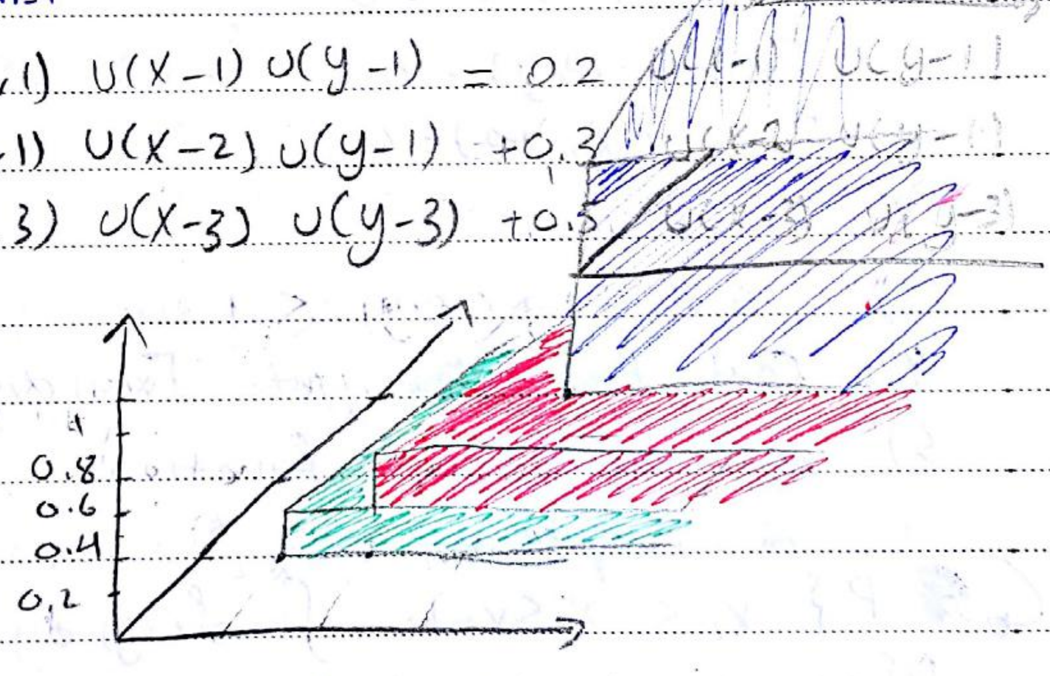
S_J has elements: (1,1), (2,1), (3,3)
probabilities. 0.2 0.3 0.5



$F_{X,Y}(x,y)$

$$= \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_n)\, u(x-x_n)\, u(y-y_n)$$

$$= P(1,1)\, u(x-1)\, u(y-1) = 0.2$$
$$+ P(2,1)\, u(x-2)\, u(y-1) + 0.3$$
$$+ P(3,3)\, u(x-3)\, u(y-3) + 0.5$$



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if we have N R.V.

$x_1, x_2, x_3, x_4 \ldots x_n \ldots x_N$

$F_{x_1, x_2 \ldots x_n}(x_1, x_2 \ldots x_n, x_N)$

$= P\{X_1 \leq x_1, X_2 \leq x_2 \ldots X_n \leq x_n \ldots X_N \leq x_N\}$

* Properties of Distribution function:-

1) $F_{x,y}(-\infty, -\infty) = 0$

$F_{x,y}(-\infty, y) = 0$

$F_{x,y}(x, -\infty) = 0$

2) $F_{x,y}(\infty, \infty) = 1$

3) $0 \leq F_{x,y}(x,y) \leq 1$

4) Cont from the right $F_{(x,y)}(x,y) = F_{x,y}(x^+, y^+)$

5) non-decreasing function

in chapter # 2

$P\{x_1 < x \leq x_2\} = \int_{x_1}^{x_2} f_x(y)\, dy = F_x(x_2) - F_x(x_1)$

$P\{x_1 < x < x_2, y_1 < y \leq y_2\} =$

$F_{x,y}(x_2, y_2) + F_{(x,y)}(x_1, y_1) - F_{x,y}(x_1, y_2) - F_{x,y}(x_2, y_2)$

6) $F_{x,y}(x, \infty) = F_x(x)$

$F_{x,y}(\infty, y) = F_y(y)$ $\Big\} \rightarrow$ marginal distribution function.

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To verify validity of $F_{X,Y}(x,y)$

① ② ⑤

Marginal distribution function

$$F_X(x) = F_{X,Y}(x, \infty)$$

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$$

$y \to \infty$

$B \to S$

$A \cap B = A$

$$P\{X \le x, Y \le \underset{\infty}{y}\} = P\{X \le x\} = F_X(x)$$

Ex 4.2.2 / P112

$$F_{X,Y}(x,y) = 0.2\, u(x-1)\, u(y-1) + 0.3\, u(x-2)\, u(y-1) + 0.5\, u(x-3)\, u(y-3)$$

$$F_X(x) = 0.2\, u(x-1) + 0.3\, u(x-2) + 0.5\, u(x-3)$$

$$F_Y(y) = 0.5\, u(x-1) + 0.5\, u(x-3)$$

if we have N R.Vs

to get K dimensional marginal CDF

you put N-K R.Vs to infinity

Lecture / /

4.3 , Joint Density and its properties

* For two variable: → for continous

$$f_{x,y}(x,y) = \frac{d\, F_{x,y}(x,y)}{dx\, dy}$$

* in discrete:

$$\sum_n p(x_i)\, u(x-x_i) \Rightarrow \sum_i \sum_j p(x_i, y_j)\, u(x-x_i)\, u(y-y_j)$$

↳ joint distribution function

$$\sum_i \sum_j p(x_i, y_j)\, \delta(x-x_i)\, \delta(y-y_j)$$

* For N variable

$$F_{x_1, x_2, \dots x_n, \dots x_N}(x_1, x_2 \dots x_n, \dots x_N)$$

$$f_{x_1, x_2, \dots x_N}(x, x_2 \dots x_N) = \frac{d^N}{dx_1\, dx_2 \dots dN}\left(F_{x_1 x_2 \dots x_N}(x_1, x_2 \dots x_N)\right)$$

$$F_{x,y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{x,y}(\xi_1, \xi_1)\, d\xi_1\, d\xi_2$$

* properities of density function:-

validity of pdf ←
1. $f_{x,y}(x,y) \geq 0$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{x,y}(x,y)\, dx\, dy = 1$

3. $F_{x,y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{x,y}(\xi_1, \xi_2)\, d\xi_2\, d\xi_1$

4. $F_x(x) \equiv$ marginal distribution function $\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{x,y}(\xi_1, \xi_2)\, d\xi_2\, d\xi_1$ ← keep the one you want!

$f_x(x) \equiv$ marginal density function

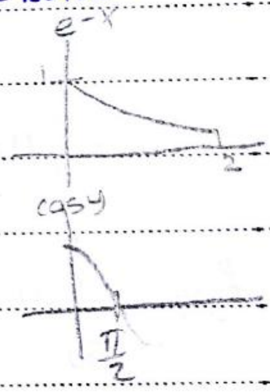
5. $P\{x_1 < X \le x_2, y_1 < Y \le y_2\} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dy\,dx$

6. $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dy$

OR $= \dfrac{d\,F_X(x)}{dx}$

Ex 4.3.1 / P115

$$g_{X,Y}(x,y) = \begin{cases} \overset{\text{constant}}{b}\, e^{-x}\cos y & 0 \le x \le 2,\ 0 \le y \le \pi/2 \\ 0 & \text{otherwise.} \end{cases}$$



$g_{X,Y}(x,y) \ge 0$

$$b\int_0^{\pi/2}\int_0^2 e^{-x}\cos y\,dx\,dy$$

$$b\int_0^{\pi/2}\cos y\,dy \int_0^2 e^{-x}\,dx$$

$$b\ \sin y\Big|_0^{\frac{\pi}{2}}\ \frac{e^{-x}}{-1}\Big|_0^2$$

$$b \times -1 \times [e^{-2} - 1] = b(1 - e^{-2}) \overset{\text{must}}{=} 1$$

$$b = 1/(1 - e^{-2})$$

/ /

Ex 4.3.21 P115

Find $f_X(x)$, $f_Y(y)$ of $f_{X,Y}(x,y) = U(x)U(y)\,x\,e^{-x(y+1)}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} U(x)\,U(y)\,x\,e^{-x(y+1)}\,dy$$

$$= x\,U(x)\,e^{-x} \int_{0}^{\infty} e^{-xy}\,dy$$

$$= x\,U(x)\,e^{-x} \cdot \frac{e^{-xy}}{-x}\Big|_{0}^{\infty}$$

$$= \frac{U(x)e^{-x}}{-1}\,[0-1] = U(x)e^{-x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} U(x)\,U(y)\,x\,e^{-x(y+1)}\,dx$$

$$= U(y)\int_{0}^{\infty} x\,e^{-x(y+1)}\,dx \qquad a = -(y+1)$$

$$\int x e^{ax}\,dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right]$$

$$= \frac{U(y)}{(1+y)^2}$$

/ /

⇒ N RV's

$X_1, X_2, \dots X_n, \dots X_N$

if we want K marginal pdf / cdf

$K < N$

$$f_{X_1 \dots X_K}(x_1, \dots x_K) = \underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_{(N-K)\text{ times}} f_{X_1, \dots X_N}(x_1, \dots x_N)\, dx_N\, dx_{N-1} \dots dx_{K+1}$$

4.4 Conditional distribution and density

$$F_X(x/B) = P\{X \le x \mid B\}$$

$B\{X \le b\}$

$$F_X(x \mid b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & x \ge b \end{cases}$$

$b < x$ x $b > x$

$$f_X(x \mid B) = \begin{cases} \dfrac{f_X(x)}{F_X(b) = \int_{-\infty}^{b} f_X(x)\,dx} & x < b \\ 0 & x \ge b \end{cases}$$

$B = \{Y \le y\}$ $P\{X \le x / B\}$

point conditioning: finding distribution of a R.V conditional on occurance of the other R.V with a specific point.

$y-\Delta y$ $y+\Delta y$ y $\Delta y \to 0$.

$B = \{y - \Delta y < y \le y + \Delta y$

$$F_X(x \mid B) = F_X(x \mid y - \Delta y < y \le y + \Delta y\}$$

$$= \int_{-\infty}^{x} \int_{y-\Delta y}^{y+\Delta y} f_{X,Y}(\xi_1, \xi_1)\, d\xi_2\, d\xi_1$$

$$\int_{y-\Delta y}^{y+\Delta y} f_Y(\xi)\, d\xi$$

4.4 Conditional distribution and density function

$$F_X(x/B) = \frac{P\{X \leq x \cap B\}}{P\{B\}}$$

- point conditioning

$$P\{X \leq x \mid Y = y_k\}$$

$$B = \{y - \Delta y \leq Y \leq y + \Delta y\}$$

$$A = \{X \leq x\}$$

$\Delta y \quad \Delta y$

$y-\Delta y \quad y \quad y+\Delta y$

$$P(A/B) = P\{X \leq x / y-\Delta y \leq Y \leq y+\Delta y\}$$

$$F_X(x/B) = \frac{P\{X \leq x ,\ y-\Delta y \leq Y \leq y+\Delta y\}}{P\{y-\Delta y \leq Y \leq y+\Delta y\}}$$

$$= \frac{\int_{-\infty}^{x} \int_{y-\Delta y}^{y+\Delta y} f_{x,y}(\xi_1, \xi_2)\, d\xi_2\, d\xi_1}{\int_{y-\Delta y}^{y+\Delta y} f_y(\xi)\, d\xi}$$

discrete case:

$$\begin{cases} X & x_i & i = 1, \dots N \\ Y & y_j & j = 1, \dots M \end{cases}$$

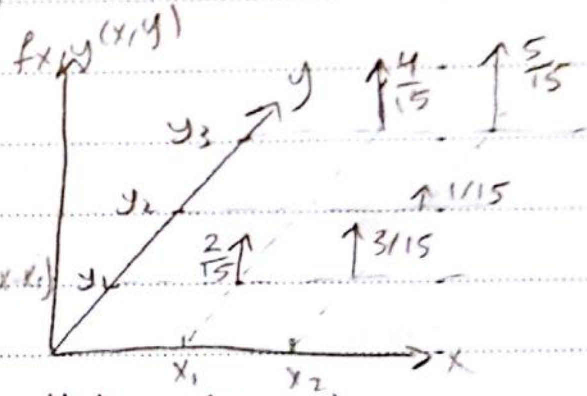
$$f_{x,y}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{M} P(x_i, y_j)\, \delta(x - x_i)\, \delta(y - y_j)$$

$$f_y(y) = \sum_{j=1}^{M} P(y_j)\, \delta(y - y_j)$$

$$F_X(x \mid y = y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} \, U(x - x_i)$$

$$f_X(x \mid Y = y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} \, \delta(x - x_i)$$

Ex: 4.4.1



$$P(y_3) = \frac{4}{15} + \frac{5}{15} = \frac{9}{15}$$

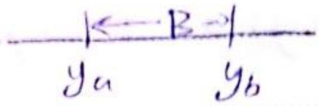
$$f_X(x \mid Y = y_3) = \sum_{i=1}^{N} \frac{P(x_i, y_3)}{P(y_3)} \, \delta(x - x_i)$$

$$= \frac{P(x_1, y_3)}{P(y_3)} \, \delta(x - x_1) + \frac{P(x_2, y_3)}{P(y_3)} \, \delta(x - x_2)$$

$$= \frac{\frac{4}{15}}{\frac{9}{15}} \, \delta(x - x_1) + \frac{\frac{5}{15}}{\frac{9}{15}} \, \delta(x - x_2)$$

$$= \frac{4}{9} \, \delta(x - x_1) + \frac{5}{9} \, \delta(x - x_2)$$

$B = \{y_a < Y \le y_b\}$



$P\{X \le x \mid y_a < Y \le y_b\}$

$= F_X(x \mid y_a < Y \le y_b\}$

→ the intersection.

$$F_X(x \le y_a < Y \le y_b) = \frac{\int_{y_a}^{y_b} \int_{-\infty}^{x} f_{X,Y}(\xi, y)\, d\xi\, dy}{\int_{y_a}^{y_b} f_Y(y)\, dy \neq 0}$$

↳ $F_Y(y_b) - F_Y(y_a)$

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} f_{X,Y}(\xi, y)\, d\xi$$

$\int_{y_a}^{y_b} F_{X,Y}(x,y)\, dy$

$F_{X,Y}(x$

$$= \frac{F_{X,Y}(x, y_b) - F_{X,Y}(x, y_a)}{F_Y(y_b) - F_Y(y_a) \neq 0}$$

Example 4.4.3 / P20 ~~Find ... given $f_X(x / Y < y)$~~

Find $f_X(x / Y \le y)$

For $f_{X,Y}(x,y) = u(x)\,u(y)\,x e^{x(y+1)}$

derive with respect to x to get $f_X(x)$ from $F_X(x)$

$$f_X(x \mid y_a \le Y \le y_b) = \frac{\int_{y_a}^{y_b} f_{X,Y}(x,y)\, dy}{\int_{y_a}^{y_b} f_Y(y)\, dy}$$

$$\boxed{1} \text{ find } f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$f_Y(y) = \frac{U(y)}{(1+y)^2}$$

$$\boxed{2} \text{ find } P\{Y \le y\} = \int_{-\infty}^{y} \frac{U(y)}{(1+y)^2}\,dy$$

$$\int_0^y \frac{1}{(1+y)^2}\,dy = \int_0^y (1+y)^{-2}\,dy$$

?

$$-2(1+y)^{-1}\Big|_0^y = \frac{-2 + 2(1+y)}{1+y} = \frac{y}{1+y} \quad y>0$$

$$\boxed{3} \text{ find } \int_{y_a}^{y_b} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{y} U(x)U(y)\,x e^{-x(y+1)}\,dy$$

$$= U(x)\,e^{-x}(1-e^{-xy})$$

$$f_X(x \mid Y \le y) = \frac{U(x)\,e^{-x}(1-e^{-xy})}{\frac{y}{y+1}}$$

interval conditioning

$$\boxed{= \frac{y+1}{y}\,U(x)\,e^{-x}(1-e^{-xy})}$$

point conditioning gives $f_X(x/y) = (y+1)^2 x e$

4.5 Statistically independent RVs

- In set theorem

event A   event B

$$P(A \cap B) = P(A)\,P(B)$$

- here

$A \rightarrow X$

$B - Y$

$$P\{X \leq x, Y \leq y\} = P\{X \leq x\} \cdot P\{Y \leq y\}$$

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

- In set theory

$$\left.\begin{aligned} P(A/B) &= P(A) \\ P(B/A) &= P(B) \end{aligned}\right\} \text{ if A and B are statiscally independent}$$

- $F_X(x / Y \leq y) = F_X(x)$
- $F_Y(y / X \leq x) = F_Y(y)$

Ex 4.5.2

/ /

→ For N R.Vs

$X_1, X_2 \cdots X_n, \cdots X_N$

$A_i = \{X \leq x_i\}$

the random variable are statiscally independent if and only if any ~~of~~ group of R.Vs are independent from any other R.Vs

$N=4$

X_1, X_2, X_3, X_4

$X_1, X_2 \rightarrow$

$X_1, X_3 \rightarrow$

$X_1, X_4 \rightarrow$

$X_2, X_3 \rightarrow$

$X_2, X_4 \rightarrow$

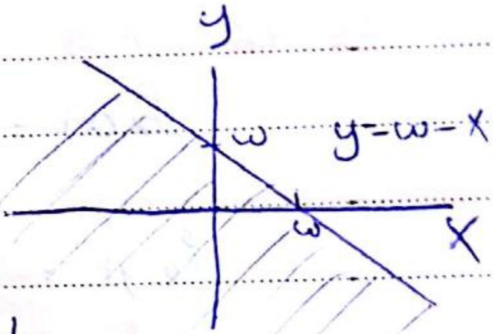
/ /

4.6 Distribution and Density of sum of R.Vs

Convolution ← statiscally independent
X, Y are independat Gaussian ← dependent

X
Y

$W = X + Y \Rightarrow Y = w - X$



$$F_W(w) = P\{W \le w\}$$
$$= P\{X + Y \le w\}$$

$$F_W(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_X(x)\,f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\int_{-\infty}^{w-y} f_X(x)\,dx\,dy$$

Leibniz's Rule

$$\cancel{\frac{d}{dw}F_W(w)} \qquad \frac{dG(u)}{du} = H(\beta(u),u) * \frac{d\beta(u)}{du} - H(\alpha(u),u)\,\frac{d\alpha(u)}{du} + \int^{\beta(u)} \frac{\partial H(x,u)}{\;} dx$$

$$G(u) = \int_{\boxed{\alpha(u)}}^{\boxed{\beta(u)}} H(x,u)\,dx \qquad \beta(u) = w - y \qquad \alpha(u) = \infty$$

$$f_W(w) = \frac{d}{dw}F_W(w) = \int_{-\infty}^{\infty} f_Y(y)\,f_X(w-y)\,dy$$

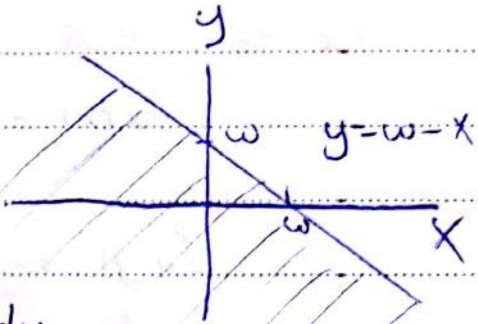
$$f_W(w) = f_X(x) * f_Y(y)$$

/ /

4.6 Distribution and Density of sum of R.Vs

Convolution ← statiscally independent
X $\quad$ X, Y are independat Gaussian ← dependent
Y

$W = X + Y \Rightarrow Y = w - X$



$F_W(w) = P\{ W \leq w \}$
$= P\{ X + Y \leq w \}$

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)\, dx\, dy$$

Leibniz's Rule

$\cancel{\frac{d}{dw} F_W(w)}$ $\quad \frac{d\,G(u)}{du} =$ $\quad G(u) = \int_{\boxed{\alpha(u)}}^{\boxed{\beta(u)}} H(x,u)\, dx$ $\quad \beta(u) = w - y$, $\quad \alpha(u) = \infty$

$$f_W(w) = \frac{d}{dw} F_W(w) =$$

$$\int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy$$

$$f_W(w) = f_X(x) * f_Y(y)$$

$$H(\beta(u), u) * \frac{d\,\beta(u)}{du}$$

$$- H(\alpha(u), u)\, \frac{d\,\alpha(u)}{du}$$

$$+ \int_{\alpha(u)}^{\beta(u)} \frac{\partial H(x,u)}{\partial u}\, dx$$

Lecture   / /

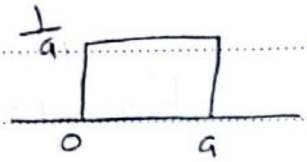
$$W = X + Y$$

$$f_W(w) = f_X(x) * f_Y(y)$$

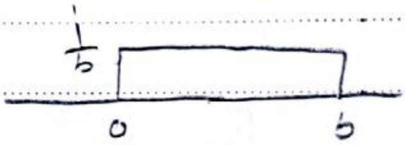
$$= \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\, dy$$

Ex 4.6.1 / P 123

$$f_X(x) = \frac{1}{a}\left[u(x) - u(x-a)\right]$$

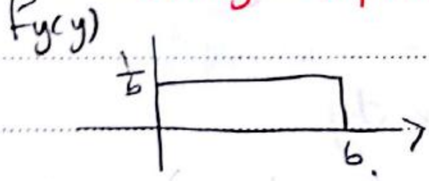


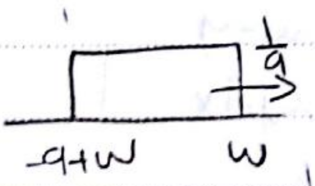
$$f_Y(y) = \frac{1}{b}\left[u(y) - u(y-b)\right]$$



$0 < a < b \qquad x > 0 \qquad y > 0 \qquad w > 0$

Solving Graphically





① flip around y-axis

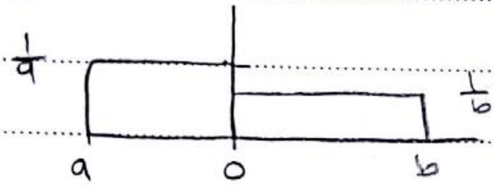
② Shift by w.

$w > 0$

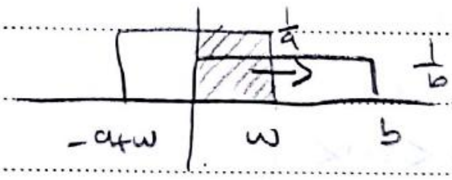
③ find area under the multiplied fun $f_Y(y) f_X(w-y)$

We define regions :

① $w<0$ $\qquad f_W(w)=0$



② $0<w<a$

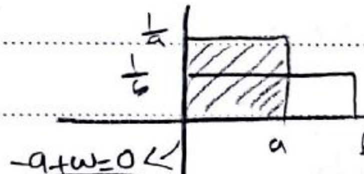


$$f_W(w)=\int_0^w \frac{1}{a}\,\frac{1}{b}\,dw$$

$$=\frac{1}{a}\,\frac{1}{b}\,w$$

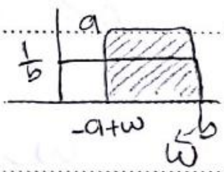
this overlap started at $w=0$ and continous till $-a+w=0$ $\boxed{w=a}$

③ $a<w<b$



$\boxed{w=a}$

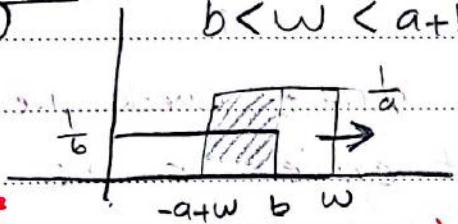
$\Rightarrow$



$=\frac{1}{a}\,\frac{1}{b}\times a=\frac{1}{b}$

$\Rightarrow \boxed{w=b}$

④ $b<w<a+b$

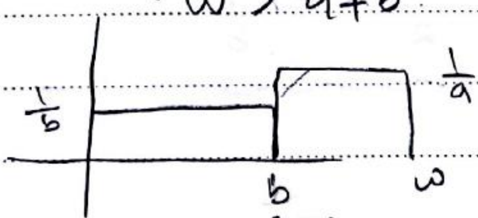


$$f_W(w)=\int_{-a+w}^{b}\frac{1}{a}\,\frac{1}{b}=\frac{1}{a}\,\frac{1}{b}\left[b+a-w\right]$$

this overlap started at $\boxed{w=b}$ and continous till $-a+w=b$

$\boxed{w=a+b}$

⑤ $w>a+b$

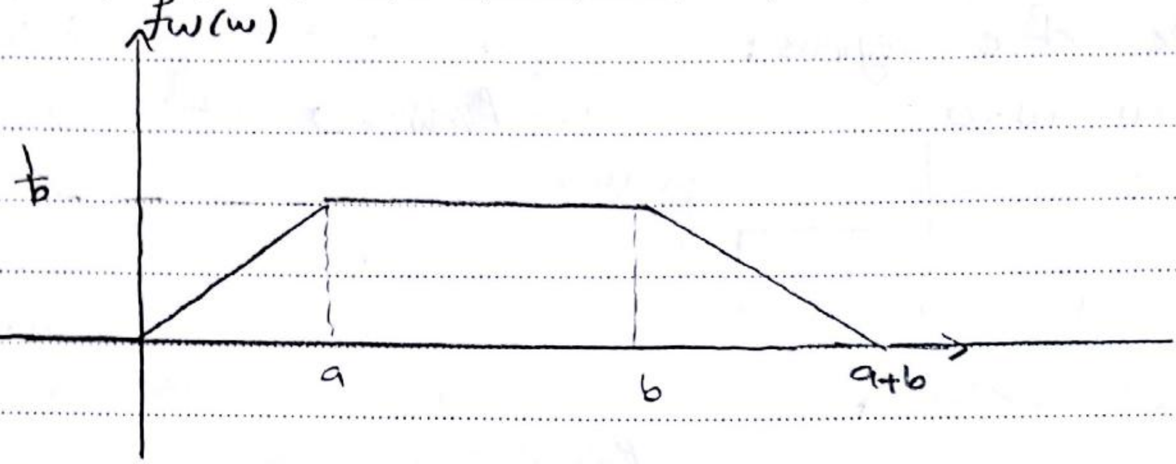


$f_W(w)=0$

$-a+w$

$-a+w=b$

$w=a+b$



$$f_W(w) = \begin{cases} 0 & w < 0 \\ \frac{w}{ab} & 0 < w < a \\ \frac{1}{b} & a < w < b \\ \frac{1}{ab}\left[a+b-w\right] & b < w < a+b \\ 0 & w > a+b \end{cases}$$

* to check if your answer is correct, check the area, since it's a pdf, it must equate to one

$$\therefore \underbrace{\frac{1}{2} \times a \times \frac{1}{b}} + \frac{b-a}{b} + \underbrace{\frac{1}{2} \times a \times \frac{1}{b}}$$

$$= 1 - \frac{a}{b} + \frac{a}{b} = \underline{\underline{1}}$$

/ /

Sum of several independent R.Vs

$X_1, X_2, \ldots X_n, \ldots X_N$

$$Y = X_1 + X_2 + \ldots X_N$$

$$f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * \ldots * f_{X_N}(x_N)$$

convolution is commutative

① $f_{X_1}(x_1) * f_{X_2}(x_2) = f_{Y_1}(y_1)$

② $f_{Y_1}(y_1) * f_{X_3}(x_3) = \ldots$

and continue till you reach $f_{X_N}(x_N)$

Solve:

4.1.2

4.2.1 4.2.4 4.2.8 4.2.11

4.3.2 4.3.10 4.3.15 4.3.19

4.4.1 4.4.8

4.5.3 4.5.5 4.5.10

4.6.2 4.6.4 4.6.10 4.6.13

Ch#5 Operations on Multiple RVs

$g(X, Y)$
$\; x \swarrow \quad y \swarrow$
$f_{X,Y}(x,y)$

$$\bar{g} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

Remember:

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

for $X_1, X_2, \dots X_N$

$$g(X_1, X_2, \dots X_N)$$

$$E[g(X_1, X_2 \dots X_N)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2, \dots x_N)\, f_{X_1,X_2,\dots X_N}(x_1, x_2, \dots x_N)\, dx_1\, dx_2 \dots dx_N$$

Example 5.1.1 / P.142

weighted R.V ⇒ multiplied by a factor.

$$g(X_1, X_2, \dots X_N) = \sum_{i=1}^{N} \alpha_i X_i$$

$$E\left[\sum_{i=1}^{N} \alpha_i X_i\right] = \sum_{i=1}^{N} E[\alpha_i X_i]$$

$$= \sum_{i=1}^{N} \alpha_i\, E[X_i]$$

$$= \sum_{i=1}^{N} \alpha_i\, \overline{X_i}$$

/ /

The expected value of sum of R.Vs is the sum of the expected value for each R.V.

special case : if $g(x_1, x_2 --- x_N) = g(x_1)$

$$\bar{g} = E[g(x_1)] = \int_{-\infty}^{\infty} g(x_1) f_{x_1}(x_1)\, dx_1$$

* Moments

Remember $m_n = E[X^n]$ ⇐ around origin

$$= \int_{-\infty}^{\infty} x^n f_x(x)\, dx$$

$$m_{nk} = E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^n f_{x,y}(x,y)\, dx\, dy$$

$$m_{n0} = E[X^n] = \int_{-\infty}^{\infty} x^n f_x(x)\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n f_{x,y}(x,y)\, dx\, dy$$

$$m_{0k} = E[Y^k]$$

the order of the joint moment is $n+k$

When the question asks for centeral of gravity, find both E[x] and E[y]

Order 2: 1. m_{02}
2. m_{20}
3. m_{11}
→ second moment of joint random variabe

* m_{01}, m_{10} : centeral of gravity of joint function $f_{x,y}(x,y)$

m_{11} : $E[XY]$ correlation of R.V X and Y

$$R_{X,Y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\, f_{x,y}(x,y)\, dx\, dy$$

if X and Y are statiscally independent

$m_{11} = R_{xy} = m_{10}\ m_{01} = E[x]\ E[Y]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X\,Y\ f_{x,y}(x,y)\ dx\,dy$$

if X,Y indepent:- $f_{x,y}(x,y) = f_x(x)\ f_y(y)$

$$\int_{-\infty}^{\infty} X f_x(x)\,dx \int_{-\infty}^{\infty} Y f_y(y)\,dy$$

$$R_{xy} = E[x]\ E[Y] = \overline{X}\ \overline{Y}$$

$\therefore$ this means x,Y are not correlated

* if RVs x,Y are statistically independent, they are uncorrelated. BUT if they are uncorrelated, they are not necessoraly ~~indepent~~ independent

* the R.V's X and Y are orthogonal if $R_{xy} = 0$

$R_{xy} = E[x]\ E[y]$

if this doesn't apply, they are definitly dependent. if it does, we can't judge thier independence.

$$m_{02} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^0 y^2 f_{x,y}(x,y)\, dx\, dy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{x,y}(x,y)\, dx\, dy$$

$$\int_{-\infty}^{\infty} y^2 dy \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dx$$

$$\int_{-\infty}^{\infty} y^2 f_y(y)\, dy$$

Ex 5.1.2 / P142

$\bar{X} = 3 \qquad \sigma_x^2 = 2 \qquad \mu_3 = m_2 - m_1^2$

$m_2 = E[x^2] = 2 + 3^2 = 11$

$Y = -6x + 22$

$\bar{Y} = E[-6x + 22]$

$= -6\, E[x] + 22$

$= -6 \times 3 + 22 = 4$

$R_{xy} = E[xY] = E[x(-6x + 22)]$

$= E[-6x^2] + E[22x]$

$= -6 \times 11 + 22 \times 3$

$= 0 \Rightarrow$ x, Y are orthogonal

$R_{xy} \overset{?}{=} E[x]\, E[Y] = 3 \times 4 \neq 0$

$\Rightarrow$ this means they are correlated

$\Rightarrow$ they are statiscally independent

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* Moments around origin

N RV's $\quad X_1, X_2, \dots X_n, \dots X_N$

$n_1, n_2, \dots n_n, \dots n_N$

$n_1 + n_2 + n_3 + \dots n_N$ order of the moments

$$m_{n_1, n_2, \dots n_N} = E[X_1^{n_1} X_2^{n_2} \dots X_N^{n_N}]$$

$$= \underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_{N \text{ times}} X_1^{n_1} X_2^{n_2} \dots X_N^{n_N} f_{X_1, X_2, \dots X_N}(x_1, x_2, \dots x_N) \, dx_1 \, dx_2 \dots dx_N$$

* Joint Centeral Moment μ_{nk}

$$\mu_{nk} = E[(X - \bar{X})^n (Y - \bar{Y})^k]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (X - \bar{X})^n (Y - \bar{Y})^k f_{X,Y}(x,y) \, dx \, dy$$

* varience of $X \Rightarrow \sigma_x^2 = \mu_{20} = E[(X - \bar{X})^2]$

$$= \int_{-\infty}^{\infty} (X - \bar{X})^2 f_X(x) \, dx$$

$\longrightarrow$ marginal

$$\sigma_y^2 = \int_{-\infty}^{\infty} (Y - \bar{Y})^2 f_Y(y) \, dy = E[(Y - \bar{Y})^2]$$

* co varience $= \mu_{11} = C_{xy} = E[(x - \bar{x})(y - \bar{y})]$

$= E[xy - \bar{y}x - \bar{x}y + \bar{x}\bar{y}] = R_{xy} - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y}$

$= R_{xy} - \bar{y}\bar{x} = R_{xy} - E[x]E[y]$

if X, Y are uncorrelated $R_{xy} = E[x] E[y]$

$C_{xy} = R_{xy} - R_{xy} = 0$

Normalized 2^{nd} order moment ρ

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{02}\,\mu_{20}}} = \frac{C_{xy}}{\sigma_x \sigma_y}$$

$$= E\left[\left(\frac{x-\bar{x}}{\sigma_x}\right)\left(\frac{y-\bar{y}}{\sigma_y}\right)\right]$$

→ normalization of gaussian function

$\boxed{0 < \rho \leq 1}$ always check this condition.

⇒ for N R.Vs $\quad X_1, X_2, X_3, \ldots X_n, \ldots X_N$

$\quad n_1, n_2, n_3, \ldots n_n, \ldots n_N$

$$\mu_{n_1, n_2, \ldots n_N} = E\left[(X_1-\bar{X}_1)^{n_1}(X_2-\bar{X}_2)^{n_2} \cdots (X_N-\bar{X}_N)^{n_N}\right]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (X_1-\bar{X}_1)^{n_1} \cdots (X_n-\bar{X}_n)^{n_N} f_{X_1 X_2 \ldots X_N}(x_1 x_2 \ldots x_N)\, dx_1 dx_2 \ldots$$

/ /

Correlation $R_{xy} = E[xy]$

Covarience $C_{xy} = R_{xy} - E[x]E[y]$

Orthogonal $R_{xy} = 0$; $C_{xy} = -E[x]E[y]$

Uncorrelated $R_{xy} = E[x]E[y]$

Ex 5.1.3/145

$$X = \sum_{i=1}^{N} \alpha_i x_i$$

find the varience of X $[\sigma x^2]$

$$\sigma x^2 = E[(x-\bar{x})^2] = E[x^2] - \bar{x}^2$$

$$E\left[\sum_{i=1}^{N} \alpha_i (x_i - \bar{x}_i) \sum_{j=1}^{N} \alpha_j (x_j - \bar{x}_j)\right]$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \, E[\underbrace{(x_i - \bar{x}_i)(x_j - \bar{x}_j)}_{C_{x_i x_j}}]$$

$i \neq j$ $i = j$

if they are uncorrelated $C_{x_i x_j} = 0$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \, E[(x_i - \bar{x}_i)^2]$$

$$\sigma x^2 = \sum_{i=1}^{N} \alpha_i \, \sigma x_i^2$$